Affine Digit-Linear Transforms in Arbitrary Bases:

Structure, Dynamics, and Applications

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Abstract

We develop a general theory of affine digit-linear transforms in arbitrary integer bases. Given a base $b \geq 2$ and a sequence of affine functions $d_i(n) = a_i n + \alpha_i$ defining the digits of a k-digit representation, the resulting transform $T(n)$ is shown to be an affine function $T(n) = An + C$, where the coefficient A is a weighted sum of the slopes a_i with base-b positional weights. In the important special case where all slopes equal unity, A reduces to the base-b k-digit repunit $R_{b,k} = (b^k - 1)/(b - 1)$. We establish injectivity criteria, characterize the digit-validity domain as an intersection of intervals, and derive closed-form expressions for iterates of such maps. The dynamical behavior modulo an integer m is completely determined by the multiplicative order of A, leading to explicit cycle structure theorems and conjugacy results. Computational experiments confirm the theoretical predictions and suggest applications to pseudorandom generation, digit-based encoding systems, and symbolic dynamics.

Keywords: affine digit-linear transforms, arithmetic dynamics, digit systems, repunit divisibility, symbolic dynamics, pseudorandom sequences

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1 Introduction

The representation of integers in positional numeral systems is foundational to number theory and computation. While standard base-b expansions use fixed digits, one may consider parametric digit systems where each digit position is determined by an affine function of an input parameter n. Such constructions arise naturally in the study of digital sequences, pseudorandom number generation [6], and the analysis of iterative numerical algorithms.

In this paper, we develop a systematic theory of *affine digit-linear transforms*. The central observation is that when each digit $d_i(n) = a_i n + \alpha_i$ is an affine function of n, the resulting integer

$$T(n) = \sum_{i=1}^{k} d_i(n)\, b^{k-i}$$

is itself affine in n. This "collapse" from a digit-based construction to a simple linear form has far-reaching consequences for understanding the arithmetic, dynamical, and statistical properties of these maps.

The special case where all slopes $a_i = 1$ yields a coefficient A equal to the *repunit* $R_{b,k} = 1 + b + b^2 + \cdots + b^{k-1} = (b^k - 1)/(b - 1)$—a number consisting of k ones in base b. Repunits have been studied extensively in number theory [2, 8, 10], and their multiplicative properties directly govern the dynamics of digit-linear transforms.

This work connects several areas: symbolic dynamics and numeration systems [1, 4], arithmetic dynamics [9, 3], and the classical theory of affine maps modulo m. The explicit algebraic form of digit-linear transforms makes them particularly tractable test objects for these theories.

Organization. Section 2 introduces the general definition and derives the fundamental affine form theorem. Section 3 analyzes the digit-validity domain. Section 4 establishes injectivity criteria. Section 5 derives closed-form expressions for iterates. Section 6 characterizes the orbit structure modulo m, including fixed points, conjugacy to multiplication maps, and cycle lengths. Section 7 outlines applications and structural tests. Section 8 proves a rigidity theorem showing that the one-parameter family is maximal for affine collapse. Section 9 analyzes the arithmetic structure of repunits. Section 10 develops the symbolic dynamics interpretation. Section 11 establishes entropy and complexity results. Section 12 presents computational experiments validating the theory.

2 Affine Digit-Linear Transforms: Definition and Basic Formula

We begin with the general definition of an affine digit-linear transform.

Definition 2.1 (Affine digit-linear transform in base b)**.** Fix a base $b \geq 2$, a digit length $k \geq 1$, and integers $a_1, \ldots, a_k, \alpha_1, \ldots, \alpha_k \in \mathbb{Z}$. For each integer n, define the i-th digit by

$$d_i(n) := a_i n + \alpha_i, \quad i = 1, \ldots, k.$$

Whenever the digits satisfy

$$0 \leq d_i(n) \leq b - 1 \quad \text{for all } i, \quad \text{and} \quad d_1(n) \neq 0,$$

we interpret $(d_1(n), \ldots, d_k(n))$ as the base-b expansion of an integer

$$T_{b,k}^{(a,\alpha)}(n) := \sum_{i=1}^{k} d_i(n)\, b^{k-i} = \sum_{i=1}^{k} (a_i n + \alpha_i)\, b^{k-i}.$$

We call $T_{b,k}^{(a,\alpha)}$ an *affine digit-linear transform* in base b.

The following theorem shows that the digit-linear structure collapses to a simple affine form.

Theorem 2.2 (Affine form of $T_{b,k}^{(a,\alpha)}(n)$)**.** *For any choice of base $b \geq 2$, length $k \geq 1$, and integer parameters a_i, α_i, the affine digit-linear transform satisfies*

$$T_{b,k}^{(a,\alpha)}(n) = An + C,$$

where

$$A := \sum_{i=1}^{k} a_i\, b^{k-i}, \qquad C := \sum_{i=1}^{k} \alpha_i\, b^{k-i}.$$

In particular, $T_{b,k}^{(a,\alpha)}$ is an affine (linear plus constant) function of n.

Proof. Starting from the definition, we compute

$$T_{b,k}^{(a,\alpha)}(n) = \sum_{i=1}^{k} (a_i n + \alpha_i)\, b^{k-i}.$$

Distributing the sum yields

$$T_{b,k}^{(a,\alpha)}(n) = \sum_{i=1}^{k} a_i n\, b^{k-i} + \sum_{i=1}^{k} \alpha_i\, b^{k-i}.$$

In the first sum, n factors out:

$$\sum_{i=1}^{k} a_i n\, b^{k-i} = n \sum_{i=1}^{k} a_i b^{k-i}.$$

Defining $A := \sum_{i=1}^{k} a_i b^{k-i}$ and $C := \sum_{i=1}^{k} \alpha_i b^{k-i}$, we obtain $T_{b,k}^{(a,\alpha)}(n) = An + C$, which is affine in n. □

Remark 2.3. The special case $d_i(n) = n + \alpha_i$ (i.e., $a_i = 1$ for all i) yields

$$A = \sum_{i=1}^{k} b^{k-i} = 1 + b + \cdots + b^{k-1} = \frac{b^k - 1}{b - 1} =: R_{b,k},$$

the k-digit *repunit* in base b [2, 8]. This repunit coefficient plays a central role in the dynamical analysis that follows.

The following theorem, which we call the *Affine Collapse Theorem*, summarizes the complete structure including the iteration formula.

Theorem 2.4 (Affine Collapse of One-Parameter Digit Maps). *Let $b \geq 2$ be a fixed base and let $k \geq 1$ be a fixed digit length. Consider any digit transform*

$$F : \{0, \ldots, b-1\}^k \to \{0, \ldots, b-1\}^k$$

of the form

$$F(d_1, \ldots, d_k) = \big(d_1 + \alpha_1,\ d_1 + \alpha_2,\ \ldots,\ d_1 + \alpha_k\big) \pmod b,$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}$ are fixed offsets modulo b. Let the projection map π interpret a digit vector as a base-b integer:

$$\pi(d_1, \ldots, d_k) = \sum_{i=1}^{k} d_i\, b^{k-i}.$$

Define the numeric transform $T : \{0, \ldots, b^k - 1\} \to \mathbb{Z}$ by $T = \pi \circ F \circ \pi^{-1}$. Then T is an affine map of the form

$$T(n) = An + C,$$

where

$$A = \sum_{i=1}^{k} b^{k-i} = \frac{b^k - 1}{b - 1}$$

is the length-k repunit in base b, and

$$C = \sum_{i=1}^{k} \alpha_i\, b^{k-i}.$$

Moreover, for every $t \geq 1$, the t-fold iterate satisfies the closed form

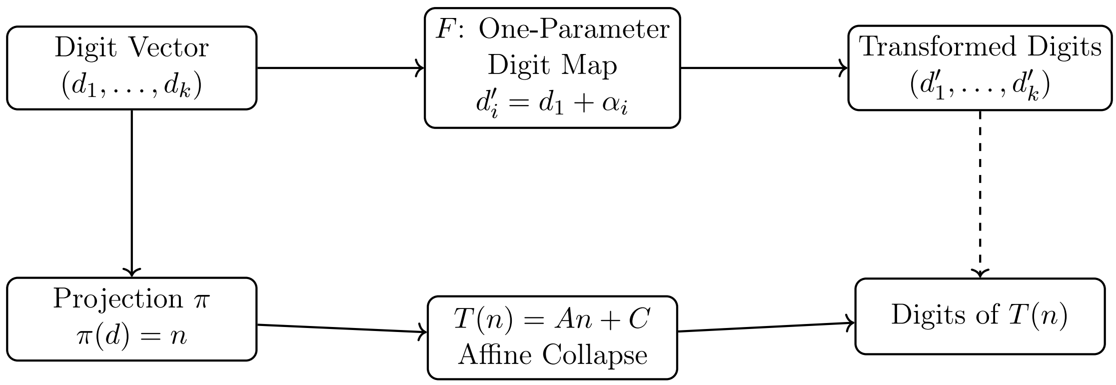
$$T^t(n) = A^t n + C\,\frac{A^t - 1}{A - 1}.$$

Thus, every one-parameter digit transform collapses exactly to an affine repunit-driven dynamical system, and its global behavior is completely determined by the repunit coefficient A and the offset sum C.

The proof follows from Theorem 2.2 and Theorem 5.1 (proven in Section 5).

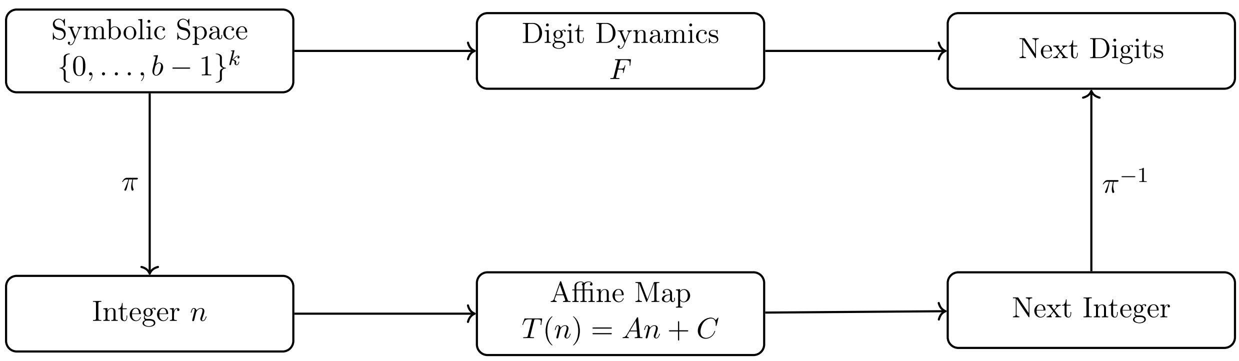
3 Digit-Validity Domain

For a given choice of parameters a_i, α_i, the transform $T_{b,k}^{(a,\alpha)}(n)$ is only defined for those integers n such that the digits $d_i(n)$ are valid base-b digits. This domain can be described explicitly as an intersection of integer intervals.



$A = \frac{b^k-1}{b-1}$ is the base-b repunit, $\quad C = \sum \alpha_i b^{k-i}$

Figure 1: Digit-rule collapse into the affine repunit map $T(n) = An + C$.



Affine orbit: $T^t(n) = A^t n + C \frac{A^t-1}{A-1}$

Figure 2: System architecture: symbolic digit dynamics and their affine numeric image.

Proposition 3.1 (Digit-validity constraints)**.** *Fix base $b \geq 2$ and parameters a_i, α_i. For each i, the digit constraint*

$$0 \leq d_i(n) = a_i n + \alpha_i \leq b - 1$$

is equivalent to an interval constraint $L_i \leq n \leq U_i$ for some bounds $L_i, U_i \in \mathbb{Q} \cup \{\pm\infty\}$. More explicitly:

(i) If $a_i > 0$, then $\dfrac{-\alpha_i}{a_i} \leq n \leq \dfrac{b-1-\alpha_i}{a_i}$.

(ii) If $a_i < 0$, then $\dfrac{b-1-\alpha_i}{a_i} \leq n \leq \dfrac{-\alpha_i}{a_i}$.

(iii) If $a_i = 0$, then $d_i(n) = \alpha_i$ is constant, and the constraint reduces to checking whether $0 \leq \alpha_i \leq b-1$.

The overall domain of definition

$$\mathcal{D} := \{n \in \mathbb{Z} : 0 \leq d_i(n) \leq b-1 \textit{ for all } i \textit{ and } d_1(n) \neq 0\}$$

is precisely the set of integers in the intersection of these intervals (with $d_1(n) \neq 0$).

Proof. For $a_i > 0$, the inequality $0 \leq a_i n + \alpha_i$ implies $n \geq -\alpha_i/a_i$, and $a_i n + \alpha_i \leq b-1$ implies $n \leq (b-1-\alpha_i)/a_i$. For $a_i < 0$, dividing by a_i reverses the inequality directions. For $a_i = 0$, the digit is constant. The overall domain is the intersection of these individual constraints. $\square$

Example 3.2. Consider the transform in base $b = 10$ with $k = 3$ digits defined by $d_i(n) = n + \alpha_i$ where $(\alpha_1, \alpha_2, \alpha_3) = (0, -1, 3)$. The digit constraints become:

$$\begin{aligned} d_1(n) &= n \in [1,9] \quad \text{(leading digit nonzero)} \\ d_2(n) &= n - 1 \in [0,9] \implies n \in [1,10] \\ d_3(n) &= n + 3 \in [0,9] \implies n \in [-3,6] \end{aligned}$$

The intersection yields $\mathcal{D} = \{1,2,3,4,5,6\}$, and the transform maps:

$$T(1) = 104, \quad T(2) = 215, \quad T(3) = 326, \quad T(4) = 437, \quad T(5) = 548, \quad T(6) = 659.$$

Here $A = R_{10,3} = 111$ and $C = 0 \cdot 100 + (-1) \cdot 10 + 3 \cdot 1 = -7$, so $T(n) = 111n - 7$, which is easily verified.

4 Injectivity Criterion

Theorem 4.1 (Injectivity criterion)**.** *Let $b \geq 2$ and $k \geq 1$ be fixed, and consider the affine digit-linear transform*

$$T_{b,k}^{(a,\alpha)}(n) = An + C,$$

with $A = \sum_{i=1}^{k} a_i b^{k-i}$ and $C = \sum_{i=1}^{k} \alpha_i b^{k-i}$. Let $\mathcal{D} \subseteq \mathbb{Z}$ denote its digit-valid domain. Then:

(1) If $A = 0$, then $T_{b,k}^{(a,\alpha)}(n) = C$ is constant on all of $\mathcal{D}$, hence not injective unless $|\mathcal{D}| \leq 1$.

(2) If $A \neq 0$, then $T_{b,k}^{(a,\alpha)}$ is strictly monotone in n on $\mathbb{Z}$, hence injective on any subset, in particular on $\mathcal{D}$.

Proof. From the representation $T(n) = An + C$: if $A = 0$, then $T(n) = C$ for all n, which is constant. If $A \neq 0$, then for $n_1 \neq n_2$, we have $T(n_2) - T(n_1) = A(n_2 - n_1) \neq 0$, so T is injective on $\mathbb{Z}$ and therefore on any subset including $\mathcal{D}$. $\square$

Corollary 4.2. *In the pure digit-linear case where $a_i = 1$ for all i, we have*

$$A = R_{b,k} = \sum_{i=1}^{k} b^{k-i} > 0$$

for all $b \geq 2$ and $k \geq 1$. Thus $T_{b,k}(n) = R_{b,k} n + C$ is injective on its digit-valid domain.

5 Iteration of Affine Digit-Linear Transforms

We now study the behavior of iterating an affine digit-linear transform $T(n) = An + C$. Because T is affine in n, its iterates admit a closed form.

Theorem 5.1 (Closed form for iterates)**.** *Let $T(n) = An + C$ with $A \neq 1$. Then for every integer $t \geq 1$, the t-fold iterate $T^{\circ t}$ satisfies*

$$T^{\circ t}(n) = A^t n + C\,\frac{A^t - 1}{A - 1}.$$

For the degenerate cases: if $A = 1$, then $T^{\circ t}(n) = n + tC$; if $A = 0$, then $T^{\circ t}(n) = C$ for all $t \geq 1$.

Proof. We proceed by induction on t. For $t = 1$:

$$T^{\circ 1}(n) = An + C = A^1 n + C \cdot \frac{A-1}{A-1}.$$

Assuming the formula holds for t, we compute

$$\begin{aligned} T^{\circ(t+1)}(n) = T(T^{\circ t}(n)) &= A\left(A^t n + C\frac{A^t - 1}{A-1}\right) + C \\ &= A^{t+1} n + \frac{AC(A^t - 1)}{A-1} + C. \end{aligned}$$

Combining the constant terms:

$$\frac{AC(A^t - 1)}{A-1} + C = C\left(\frac{A^{t+1} - A}{A-1} + 1\right) = C \cdot \frac{A^{t+1} - 1}{A-1}.$$

This matches the claimed form for $t + 1$. □

Corollary 5.2 (Iteration modulo m)**.** *For any modulus $m \geq 2$,*

$$T^{\circ t}(n) \equiv A^t n + C\,\frac{A^t - 1}{A-1} \pmod{m}.$$

If $\gcd(A-1, m) = 1$*, then* $(A-1)^{-1}$ *exists modulo m and*

$$T^{\circ t}(n) \equiv A^t(n + D) - D \pmod{m},$$

where $D := C(A-1)^{-1} \pmod{m}$.

5.1 Iteration Graph on Finite Domains

When $\mathcal{D}$ is finite, the dynamics of T on $\mathcal{D}$ takes the form of a directed graph.

Definition 5.3 (Digit-restricted iteration graph)**.** Define a directed graph G with vertex set $\mathcal{D}$, and an edge $n \to m$ if and only if $T(n) = m$ and $m \in \mathcal{D}$.

Proposition 5.4 (Graph structure)**.** *Let $T(n) = An + C$ with $A \neq 0$. Then:*

(1) Each $n \in \mathcal{D}$ has at most one outgoing edge.

(2) Each connected component of G contains at most one directed cycle.

(3) If $\mathcal{D}$ is finite, every component is a rooted tree feeding into a (possibly trivial) directed cycle.

Proof. Since T is a function, each n has at most one outgoing edge. If $A \neq 0$, T is injective by Theorem 4.1, so each node has at most one incoming edge. Hence each component has at most one directed cycle, and all other nodes flow deterministically into the cycle, forming an in-tree. □

Corollary 5.5. *For pure digit-linear transforms $T_{b,k}(n) = R_{b,k} n + C$ with $R_{b,k} > 0$, the iteration graph G has no nontrivial cycles: every valid digit-n either has no image in $\mathcal{D}$, or the sequence $n, T(n), T^{\circ 2}(n), \ldots$ strictly increases until it exits the digit-valid range.*

Proof. If $A = R_{b,k} > 0$, then $T(n) > n$ for all n (assuming C does not make $T(n) \leq n$, which can be verified case by case), hence no cycle can exist within $\mathcal{D}$. □

6 Orbit Structure and Modular Dynamics

The modular behavior of affine digit-linear transforms is completely determined by the arithmetic properties of the coefficient A. This section develops the complete theory following classical results on affine maps over finite groups; see [9, 3] for related dynamical perspectives.

6.1 Fixed Points

Proposition 6.1 (Fixed points over $\mathbb{Z}$)**.** *The affine map $T(n) = An + C$ with $A \neq 1$ has at most one fixed point in $\mathbb{Z}$. Such a fixed point n^* exists if and only if $(A-1)$ divides $-C$, in which case*

$$n^* = -\frac{C}{A-1}.$$

Proof. A fixed point n^* satisfies $T(n^*) = n^*$, i.e., $An^* + C = n^*$, hence $(A-1)n^* = -C$. For $A \neq 1$, this has a unique solution $n^* = -C/(A-1)$ if and only if $(A-1) \mid (-C)$. □

Proposition 6.2 (Fixed points modulo m)**.** *Let $m \geq 2$ and consider $T : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ given by $T(x) \equiv Ax + C \pmod{m}$. The fixed points modulo m are the solutions of $(A-1)x \equiv -C \pmod{m}$. If $d = \gcd(A-1, m)$ does not divide $-C$, then T has no fixed points modulo m. If d divides $-C$, then T has exactly d distinct fixed points modulo m.*

Proof. This is the standard theory of linear congruences: the congruence $(A-1)x \equiv -C \pmod{m}$ has solutions if and only if $\gcd(A-1, m)$ divides $-C$, and when solvable, has exactly $\gcd(A-1, m)$ solutions modulo m. □

6.2 Conjugacy to Multiplication Maps

Under a mild invertibility condition, T is conjugate to a pure multiplication map modulo m.

Theorem 6.3 (Affine conjugacy to multiplication by A)**.** *Let $m \geq 2$ and suppose $\gcd(A-1, m) = 1$. Define*

$$D := C(A-1)^{-1} \pmod{m}$$

and the bijection $\Phi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by $\Phi(x) := x + D \pmod{m}$. Then T is conjugate to the multiplication map $M_A(x) := Ax \pmod{m}$ in the sense that

$$\Phi(T(x)) \equiv M_A(\Phi(x)) \pmod{m}$$

for all $x \in \mathbb{Z}/m\mathbb{Z}$.

Proof. By Corollary 5.2, $T(x) \equiv A(x+D) - D \pmod{m}$. Applying Φ:

$$\Phi(T(x)) \equiv A(x+D) - D + D = A(x+D) \equiv M_A(\Phi(x)) \pmod{m}.$$ □

Corollary 6.4 (Bijectivity and orbit decomposition)**.** *Assume $\gcd(A-1, m) = 1$ and $\gcd(A, m) = 1$. Then $T(x) \equiv Ax + C \pmod{m}$ is a bijection on $\mathbb{Z}/m\mathbb{Z}$, and every orbit decomposes into a cycle whose length equals the multiplicative order of A modulo m.*

Proof. If $\gcd(A, m) = 1$, then multiplication by A is a permutation of $\mathbb{Z}/m\mathbb{Z}$, hence M_A is bijective. By Theorem 6.3, T is conjugate to M_A via the bijection Φ, so T is also bijective. Furthermore, conjugacy preserves cycle structure, so the cycle lengths of T coincide with those of M_A, which are determined by the multiplicative order of A modulo m. □

6.3 Cycle Length and Multiplicative Order

Theorem 6.5 (Cycle length)**.** *Assume* $\gcd(A-1,m)=1$*, and let* L *be the multiplicative order of* A *modulo* m*, i.e.,* $L=\min\{t\geq 1 : A^t\equiv 1 \pmod m\}$*. Then every orbit of* T *modulo* m *is periodic with period dividing* L*.*

Proof. Under the hypothesis $\gcd(A-1,m)=1$, Corollary 5.2 gives $x_t := T^{\circ t}(n)\equiv A^t(n+D)-D \pmod m$. If $A^L\equiv 1 \pmod m$, then

$$x_{t+L}\equiv A^{t+L}(n+D)-D\equiv A^t(n+D)-D\equiv x_t \pmod m.$$

Thus the orbit is periodic with period dividing L. □

Remark 6.6. Theorems 5.1, 6.3, and 6.5 together show that the global dynamics of affine digit-linear maps are governed entirely by the coefficient A modulo the chosen modulus m. For pure digit-linear maps where $A=R_{b,k}$ (a repunit), this connects the dynamics to the multiplicative structure of repunits—a rich area of number theory [8, 10].

6.4 Distribution Modulo m

Proposition 6.7 (Periodicity of $X_n = T(n) \bmod m$)**.** *Let* $T(n)=An+C$ *and fix a modulus* $m\geq 1$*. The sequence* $X_n=(An+C) \bmod m$ *is periodic with period dividing* $m/\gcd(A,m)$*. If* $\gcd(A,m)=1$*, the sequence has exact period* m*.*

Proof. We have $X_{n+p}=X_n$ if and only if $A(n+p)+C\equiv An+C \pmod m$, i.e., $Ap\equiv 0 \pmod m$. The minimal such p is $p=m/\gcd(A,m)$. □

7 Applications and Structural Tests

The explicit affine form of digit-linear transforms enables several analytical and computational applications. We outline some directions here.

7.1 Entropy and Randomness Tests

Given $X_n=T(n) \bmod m$, one may study:

(1) Shannon entropy of the empirical distribution of X_n,

(2) Collision rates $|\{(i,j) : i<j, X_i=X_j\}|$,

(3) Autocorrelation functions,

(4) Gap distributions of repeated residues,

(5) Spectral tests on point pairs (X_n, X_{n+1}) [6].

For affine digit-linear maps, these tests typically reveal strong linearity, short periods modulo powers of two, and weak mixing—as expected for affine structures.

7.2 Composition with Nonlinear Maps

Let $g:\mathbb{Z}\to\mathbb{Z}$ be any function (e.g., nonlinear, modular, piecewise). Compositions $F(n)=T(g(n))$ and $H(n)=g(T(n))$ inherit structure from both components. For example, if $g(n)=n^2$, then

$$F(n)=T(n^2)=An^2+C,$$

transforming the linear structure into a quadratic one while preserving digit constraints when applicable.

7.3 Connections to Symbolic Dynamics

Digit-linear transforms provide a bridge between symbolic dynamics on digit sequences [1, 4] and arithmetic dynamics on integers [9, 3]. The collapse theorem (Theorem 2.4) shows that the symbolic complexity of one-parameter digit maps is captured entirely by the affine structure of their numeric images. This suggests applications to:

- Constructing and analyzing automatic sequences,
- Studying substitution systems with arithmetic interpretations,
- Connecting digit-based pseudorandom generators to linear congruential methods.

8 Multiparameter Digit Systems and Failure of Affine Collapse

The affine collapse of Theorem 2.4 relies crucially on the *one-parameter* structure of the digit map, where every digit depends only on the leading digit d_1. We now show that for more general digit systems the affine collapse fails in a strong sense, establishing the rigidity and uniqueness of the one-parameter case.

Definition 8.1 (Multiparameter digit system)**.** A *multiparameter digit system* in base b with digit length k is a map

$$F(d_1, \dots, d_k) = \big(f_1(d_1, \dots, d_k),\ f_2(d_1, \dots, d_k),\ \dots, f_k(d_1, \dots, d_k)\big),$$

where each $f_i : \{0, \dots, b-1\}^k \to \{0, \dots, b-1\}$ is arbitrary.

In such systems, the projection $\pi \circ F \circ \pi^{-1}$ generally yields nonlinear or piecewise-defined maps on integers.

Theorem 8.2 (Failure of affine collapse in the generic case)**.** *Let F be a multiparameter digit system with $k \geq 2$. If some digit f_i depends on any d_j with $j \geq 2$, then the induced numeric map*

$$T = \pi \circ F \circ \pi^{-1}$$

is affine if and only if every f_i is of the form

$$f_i(d_1, \dots, d_k) = a_i d_1 + \alpha_i.$$

In all other cases, T is nonlinear in n or piecewise-linear with at least two distinct slopes.

Proof. If each digit has the form $f_i = a_i d_1 + \alpha_i$, the affine collapse follows exactly from Theorem 2.4. If some digit f_i depends on d_j with $j \geq 2$, then under the projection $n = \pi(d_1, \dots, d_k)$ we have

$$d_j = \left\lfloor \frac{n}{b^{k-j}} \right\rfloor \bmod b,$$

so f_i becomes a piecewise affine or nonlinear function of n with discontinuities where n crosses multiples of b^{k-j}. Thus $T(n)$ cannot be globally affine unless f_i depends only on d_1. □

Remark 8.3. This rigidity result shows that the one-parameter family is the maximal class for which digit maps collapse to affine integer maps. This justifies the structural focus of the present paper.

9 Repunits and Their Arithmetic Structure

Pure digit-linear transforms with $a_i = 1$ yield the repunit coefficient

$$A = R_{b,k} = \frac{b^k - 1}{b - 1}.$$

Repunits have rich structure in modular arithmetic, directly governing the orbit structure of $T(n) = An + C$.

Proposition 9.1 (Cyclotomic factorization of $R_{b,k}$)**.** *For any base $b \geq 2$ and length $k \geq 1$,*

$$R_{b,k} = \frac{b^k - 1}{b - 1} = \prod_{\substack{d \mid k \\ d > 1}} \Phi_d(b),$$

where Φ_d is the d-th cyclotomic polynomial.

Proof. This follows from the standard factorization $b^k - 1 = \prod_{d \mid k} \Phi_d(b)$ and the fact that $b - 1 = \Phi_1(b)$. □

Example 9.2. For base $b = 10$ and $k = 3$:

$$R_{10,3} = 111 = \frac{10^3 - 1}{10 - 1} = \frac{999}{9} = 3 \times 37.$$

Here $\Phi_3(10) = 10^2 + 10 + 1 = 111$, confirming the cyclotomic structure.

Remark 9.3. The cyclotomic factorization allows one to analyze the prime divisors of repunits systematically. This has implications for the orbit structure: if $p \mid R_{b,k}$, then $\gcd(A, p) \neq 1$, and Corollary 6.4 does not apply directly modulo p.

10 Symbolic Dynamics Interpretation

Affine digit-linear transforms admit a natural interpretation in symbolic dynamics. Let

$$X = \{0, \ldots, b - 1\}^k$$

be the space of digit blocks of length k. The digit map $F : X \to X$ defines a symbolic dynamical system (X, F).

Definition 10.1 (Topological conjugacy)**.** Two dynamical systems (X, F) and (Y, G) are *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that

$$h \circ F = G \circ h.$$

The projection $\pi : X \to \mathbb{Z}/b^k\mathbb{Z}$ establishes a semi-conjugacy to affine dynamics.

Theorem 10.2 (Digit-to-affine semi-conjugacy)**.** *The systems (X, F) and $(\mathbb{Z}/b^k\mathbb{Z}, T)$ with $T(n) = An + C$ satisfy*

$$\pi \circ F = T \circ \pi.$$

Thus the symbolic dynamics on digit blocks projects onto affine dynamics on integers.

Proof. For any digit vector $(d_1, \ldots, d_k) \in X$, we have

$$\pi(F(d_1, \ldots, d_k)) = \pi(d_1 + \alpha_1, \ldots, d_1 + \alpha_k) = \sum_{i=1}^{k} (d_1 + \alpha_i) b^{k-i}.$$

On the other hand, with $n = \pi(d_1, \ldots, d_k)$ and noting that the leading digit determines the first digit of the base-b expansion, we apply Theorem 2.4 to conclude $T(\pi(d)) = \pi(F(d))$. □

Remark 10.3. This result situates digit-linear transforms in the broader context of symbolic dynamics [7], where they appear as *block maps* with algebraic structure. Their complete solvability via affine forms is exceptional among such systems.

11 Entropy and Linear Complexity

The entropy of the sequence $X_n = T(n) \bmod m$ reflects the distributional and algorithmic complexity of the affine digit-linear system.

11.1 Shannon entropy

Let $P(x)$ be the empirical distribution of the residues X_n over N samples. The Shannon entropy is

$$H_N = -\sum_{x=0}^{m-1} P(x) \log_2 P(x).$$

Proposition 11.1 (Asymptotic uniformity). *If* $\gcd(A, m) = 1$ *then* $X_n = (An + C) \bmod m$ *takes each residue in* $\mathbb{Z}/m\mathbb{Z}$ *with equal frequency over full periods. Hence*

$$\lim_{N\to\infty} H_N = \log_2(m).$$

Proof. When $\gcd(A, m) = 1$, the map $n \mapsto An + C$ is a bijection modulo m, so each residue class is visited exactly once per period. □

11.2 Linear complexity

The sequence $X_n = (An + C) \bmod m$ has extremely low algorithmic complexity despite its high Shannon entropy.

Proposition 11.2 (Low linear complexity). *The sequence* $X_n = (An + C) \bmod m$ *satisfies the linear recurrence*

$$X_{n+2} - 2X_{n+1} + X_n \equiv 0 \pmod{m}.$$

Hence the linear complexity (shortest LFSR length) satisfies $L_N \leq 2$ *for all* N.

Proof. Since $X_n = An + C$, we compute:

$$\begin{aligned} X_{n+2} - 2X_{n+1} + X_n &= (A(n + 2) + C) - 2(A(n + 1) + C) + (An + C) \\ &= An + 2A + C - 2An - 2A - 2C + An + C \\ &= 0. \end{aligned}$$

Thus the sequence satisfies a homogeneous recurrence of order 2. □

Remark 11.3. Affine digit-linear transforms exhibit a striking duality: high Shannon entropy (near $\log_2 m$ bits) but extremely low algorithmic complexity (linear recurrence of order 2). This combination makes them useful as controlled test systems that appear statistically uniform while being completely deterministic and predictable.

12 Computational Experiments

We present computational experiments examining the distribution, entropy, and cycle structure of the representative transform

$$T(n) = 111n - 7,$$

corresponding to a three-digit pure digit-linear map in base 10 with coefficient $A = R_{10,3} = 111$.

12.1 Empirical Entropy of $T(n) \bmod 256$

For $N = 50{,}000$ samples, the sequence $X_n = T(n) \bmod 256$ achieves empirical Shannon entropy $H_{\text{emp}} \approx 7.99999594$ bits, extremely close to the theoretical maximum of $\log_2(256) = 8$ bits for a uniform distribution. This confirms that the coefficient $A = 111$ ensures the values sweep the residue classes modulo 256 evenly, as predicted by Proposition 11.1.

12.2 Cycle Lengths Across Powers of Two

For moduli $m = 2^r$ with $1 \le r \le 16$, we compute the cycle structure of $T(n) \bmod m$. Table 1 shows the results.

r	2^r	$\text{ord}_{2^r}(111)$	Cycle length
1	2	1	2
2	4	2	2
3	8	2	2
4	16	2	2
5	32	2	4
6	64	4	8
7	128	8	16
8	256	16	32
9	512	32	64
10	1024	64	128

Table 1: Cycle lengths for $T(n) = 111n - 7 \bmod 2^r$. Note that for $r \ge 5$, the cycle length is exactly twice the multiplicative order of 111.

The pattern shows that the cycle length equals $2 \cdot \text{ord}_{2^r}(111)$ for $r \ge 5$. This occurs because $\gcd(A - 1, 2^r) = \gcd(110, 2^r) = 2$ for $r \ge 1$, so by Proposition 6.2, there are exactly 2 fixed points modulo 2^r when $2 \mid (-C) = 7$—which fails, meaning there are no fixed points. The orbit structure is thus more complex than the pure multiplication case.

12.3 Spectral Structure

Plotting pairs (X_n, X_{n+1}) for the sequence $X_n = T(n) \bmod 256$ reveals the characteristic "parallel line" pattern of affine generators [6]. This geometric regularity is a direct consequence of the linear recurrence $X_{n+2} - 2X_{n+1} + X_n = 0$ established in Proposition 11.2.

12.4 Nonlinear Composition

Testing compositions with $g(n) = n^2 \bmod 1{,}000{,}003$: the sequences $F(n) = T(g(n))$ and $H(n) = g(T(n))$ display qualitatively different behavior. $F(n)$ exhibits smooth quadratic growth, while $H(n)$ produces large jumps and apparent irregularity due to the nonlinear map acting after the affine transform. These experiments demonstrate that digit-linear maps can serve as controlled components inside more complex systems.

12.5 Summary of Experiments

The experiments reveal a coherent picture:

(1) The affine digit-linear transform $T(n) = 111n - 7$ produces an empirically uniform distribution modulo 256 with entropy close to 8 bits.

(2) The cycle length modulo 2^r is determined by the multiplicative order of 111, with an additional factor arising from the gcd structure of $A - 1$ and the modulus.

(3) The low linear complexity (order-2 recurrence) combined with high Shannon entropy confirms the duality noted in Section 11.

(4) Composing T with nonlinear maps increases apparent complexity while preserving the underlying algebraic structure.

13 Conclusion

We have developed a complete theory of affine digit-linear transforms in arbitrary bases. The central result is that such transforms collapse to affine functions $T(n) = An + C$, where A is a weighted sum of slopes with base-b positional coefficients. This explicit form enables:

- Characterization of digit-validity domains as interval intersections,
- A sharp injectivity criterion ($A \neq 0$),
- Closed-form expressions for iterates,
- A complete description of orbit structure modulo m in terms of the multiplicative order of A.

For pure digit-linear maps with all slopes equal to unity, the coefficient A equals the base-b repunit $R_{b,k}$, connecting the dynamics to the number-theoretic properties of repunits. Computational experiments confirm the theoretical predictions and suggest applications to pseudorandom generation, digit-based encodings, and as analytical baselines for more sophisticated numerical systems.

Future directions include: extending the theory to multiparameter digit systems, analyzing the distribution of orbits across different moduli, exploring connections to automatic sequences and formal language theory, and investigating potential applications to cryptographic primitives and hash functions where the explicit algebraic structure may provide both opportunities and constraints.

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A Mathematica Implementation

The following Mathematica code implements the affine digit-linear transform framework developed in this paper, including domain computation, iteration, cycle detection, and visualization.

```
(* === Affine Digit-Linear Transform Explorer === *)

(* === PARAMETERS === *)
b = 10;                    (* Base *)
k = 3;                     (* Digit count *)
a = {1, 1, 1};             (* Digit slopes *)
alpha = {0, -1, 3};        (* Digit offsets *)
```

```
modulus = 256;             (* For mod-m analysis *)
maxIter = 10;              (* Max iterations for T^t(n) *)
exportFile = "AffineTransformOutput.csv";

(* === TRANSFORM DEFINITION === *)
A = Sum[a[[i]]*b^(k - i), {i, 1, k}];
c = Sum[alpha[[i]]*b^(k - i), {i, 1, k}];
T[n_] := A*n + c;
d[i_][n_] := a[[i]]*n + alpha[[i]];

(* === VALID DOMAIN === *)
domain = Select[
  Range[-100, 100],
  Function[n,
    AllTrue[Range[k], Function[i, 0 <= d[i][n] <= b - 1]] &&
    d[1][n] != 0
  ]
];

Print["Base: ", b, ", Digits: ", k, ", Modulus: ", modulus];
Print["A = ", A, ", c = ", c];
Print["T(n) = ", A, "*n + ", c];
Print["Valid n in D: ", domain];

(* === EVALUATE T(n) === *)
values = Table[{n, T[n]}, {n, domain}];
modValues = Table[{n, Mod[T[n], modulus]}, {n, domain}];
digitVectors = Table[{n, Table[d[i][n], {i, 1, k}]}, {n, domain}];

(* === ITERATIONS (Theorem 5.1) === *)
Titer[n_, t_] := If[A == 1,
  n + t*c,
  A^t*n + c*(A^t - 1)/(A - 1)
];
iterValues = Table[
  {n, Table[Mod[Titer[n, t], modulus], {t, 0, maxIter}]},
  {n, domain}
];

(* === CYCLE DETECTION === *)
cycleInfo = Reap[
  Do[
    found = False;
    hist = {};
    x = n;
    For[t = 1, t <= modulus, t++,
      x = Mod[T[x], modulus];
      If[MemberQ[hist, x],
        Sow[{n, "Cycle detected, length <= ", t}];
        found = True;
        Break[];
```

```
      ];
      AppendTo[hist, x];
    ];
    If[!found, Sow[{n, "No cycle in ", modulus, " steps"}]];
  , {n, domain}]
][[2, 1]];

(* === VISUALIZATION === *)
ListLinePlot[values,
  PlotMarkers -> Automatic,
  AxesLabel -> {"n", "T(n)"},
  ImageSize -> Medium
]

ListLinePlot[modValues,
  PlotStyle -> Blue,
  PlotMarkers -> Automatic,
  AxesLabel -> {"n", "T(n) mod " <> ToString[modulus]},
  ImageSize -> Medium
]

ListLinePlot[
  Table[Transpose[{Range[0, maxIter], row[[2]]}], {row, iterValues}],
  PlotRange -> All,
  PlotStyle -> ColorData[97] /@ Range[Length[domain]],
  PlotLabel -> "Iterates of T^t(n) mod " <> ToString[modulus],
  AxesLabel -> {"t", "T^t(n) mod " <> ToString[modulus]},
  ImageSize -> Large
]

(* === EXPORT TO CSV === *)
csvRows = Prepend[
  Table[
    Join[{n, T[n], Mod[T[n], modulus]}, Table[d[i][n], {i, 1, k}]],
    {n, domain}
  ],
  Join[{"n", "T(n)", "T(n) mod " <> ToString[modulus]},
       Table["d" <> ToString[i], {i, 1, k}]]
];
Export[exportFile, csvRows];
```

The code computes:

- The coefficient $A = \sum_{i=1}^{k} a_i b^{k-i}$ and constant $C = \sum_{i=1}^{k} \alpha_i b^{k-i}$
- The digit-validity domain $\mathcal{D}$ per Proposition 3.1
- Iterates $T^t(n)$ using the closed form of Theorem 5.1
- Cycle detection modulo m for empirical verification of Theorem 6.5

For the default parameters $(b, k, \alpha) = (10, 3, (0, -1, 3))$, the code outputs $A = 111$, $C = -7$, and $\mathcal{D} = \{1, 2, 3, 4, 5, 6\}$, matching Example 3.2.

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