The Recursive Division Tree and Collatz Conjecture: Independent Measures of Integer Complexity

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Abstract

We investigate the relationship between two integer complexity measures: Recursive Division Tree (RDT) depth and Collatz stopping time. Through systematic multi-scale analysis, we demonstrate that these measures are fundamentally independent: partial correlation controlling for $\log(n)$ yields $r = 0.002$, confirming orthogonality. The observed raw correlation exhibits scale-dependent decay (from $r \approx 0.20$ for $n < 10^3$ to $r \approx 0.03$ for $n > 5 \times 10^4$), but this correlation is entirely spurious—arising from both measures exhibiting some dependence on integer magnitude. RDT strongly tracks size ($r = 0.76$ with $\log n$) due to its doubly logarithmic design, while Collatz weakly tracks size ($r = 0.22$ with $\log n$) through general growth trends. After controlling for this confounding variable, true independence emerges. This establishes RDT and Collatz as orthogonal dimensions in a multi-scale complexity landscape: RDT captures logarithmic magnitude complexity, while Collatz captures arithmetic path complexity. The study demonstrates the critical importance of controlling for confounding variables when comparing complexity measures.

1 Introduction

Integer complexity can be characterized through recursive reduction algorithms. Two prominent examples are:

1. **The Collatz Conjecture** (3n+1 problem): Apply $n \to n/2$ if even, $n \to 3n+1$ if odd, until reaching 1. The number of steps is the *stopping time* $T_C(n)$.
2. **The Recursive Division Tree (RDT)**: Iteratively divide by $d = \max(2, \lfloor (\log n)^\alpha \rfloor)$ until reaching 1. The number of steps is the *RDT depth* $R_\alpha(n)$.

Both measures quantify recursive depth, yet they employ fundamentally different strategies: Collatz uses fixed arithmetic operations (multiplication by 3, division by 2), while RDT uses adaptive logarithmic division. A natural question arises: do these measures correlate, suggesting they capture the same underlying complexity, or are they independent, indicating orthogonal structural properties?

Previous analyses of individual complexity measures often assume scale invariance. However, finite-size effects, saturation phenomena, and asymptotic behavior may all influence correlations between measures. This paper demonstrates that the RDT-Collatz relationship exhibits *scale-dependent decorrelation*: weak correlation at small n transitions to essential independence at large n.

1.1 The Collatz Conjecture

The Collatz conjecture asserts that every positive integer eventually reaches 1 under:

$$C(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2, \\ 3n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The stopping time $T_C(n)$ exhibits irregular behavior with no known closed form, though computational verification extends to $n \approx 2^{68}$.

1.2 The Recursive Division Tree

The RDT algorithm defines:

$$x_0 = n, \quad x_{i+1} = \left\lfloor \frac{x_i}{d_i} \right\rfloor, \quad d_i = \max\left(2, \lfloor (\log x_i)^\alpha \rfloor\right).$$

The depth $R_\alpha(n)$ grows as $O(\log \log n)$ and exhibits high stability under perturbations. We fix $\alpha = 1.5$ throughout.

2 Methodology

2.1 Data Collection

We computed $R(n)$ and $T_C(n)$ for all integers $n \in [2, 10^5]$. To investigate scale-dependence, we analyzed correlation across nested ranges:

- $[2, 10^3]$ (small integers)
- $[2, 10^4]$ (moderate range)
- $[2, 5 \times 10^4]$ (large range)
- $[2, 10^5]$ (full dataset)
- $[5 \times 10^4, 6 \times 10^4]$ (high-magnitude slice)

2.2 Statistical Analysis

For each range, we computed:

- Pearson correlation r (linear relationship)
- Spearman correlation ρ (monotonic relationship)
- Sample size and confidence intervals

3 Results

3.1 Scale-Dependent Correlation Decay

Table 1: Correlation between RDT depth and Collatz stopping time across ranges.

Range	n	Pearson r	R^2
$[2, 10^3]$	998	0.2022	0.0409
$[2, 10^4]$	9,998	0.1611	0.0260
$[2, 5 \times 10^4]$	49,998	0.1444	0.0209
$[2, 10^5]$	99,998	0.0869	0.0075
$[5 \times 10^4, 6 \times 10^4]$	10,000	0.0256	0.0007

Observation 1 (Systematic Decorrelation with Scale)**.** *The correlation between RDT depth and Collatz stopping time decreases monotonically as the range increases. For $n < 10^3$, a weak positive correlation exists ($r \approx 0.20$). As n increases beyond 5×10^4, correlation approaches zero ($r \approx 0.03$), indicating asymptotic independence.*

This pattern is inconsistent with a fundamental, scale-invariant relationship. Instead, it suggests that apparent correlation at small scales arises from finite-size effects.

3.2 Detailed Analysis at Large Scale

For the full range $[2, 10^5]$:

Table 2: Summary statistics for $n \leq 10^5$.

Measure	Min	Max	Mean	Std. Dev.
RDT Depth $R(n)$	1	6	5.13	0.45
Collatz Time $T_C(n)$	1	350	107.54	51.36

The compressed range of RDT (1–6) versus the wide range of Collatz (1–350) hints at their different scaling behaviors.

3.3 Stratified Analysis

Examining Collatz statistics within each RDT depth class:

Table 3: Collatz stopping time conditioned on RDT depth ($n \leq 10^5$).

$R(n)$	Count	Mean T_C	Std. Dev.	Range
1	2	4.0	3.0	[1, 7]
2	11	12.1	6.0	[2, 20]
3	159	45.9	39.2	[3, 127]
4	4,056	78.9	45.3	[6, 243]
5	78,241	108.4	51.2	[12, 350]
6	17,530	111.1	51.3	[18, 339]

Observation 2 (Massive Within-Class Variance)**.** *For fixed RDT depth (e.g., $R = 5$), Collatz times vary by a factor of 29 (from 12 to 350). This enormous spread demonstrates that RDT provides minimal predictive power for Collatz behavior at large scales.*

3.4 Robustness Testing

To verify the scale-dependent pattern is not a sampling artifact, we conducted subsampling stability tests:

Table 4: Correlation from 10 random samples of 5000 integers from $[2, 10^5]$.

Trial	Pearson r
1–10	[0.072, 0.107]
Mean	0.093
Std. Dev.	0.011

Low variance confirms the result is robust to sampling, while the mean $r \approx 0.09$ matches our full-dataset finding.

3.5 Controlling for Size-Dependence

Since both RDT and Collatz may depend on integer magnitude, we investigated whether their correlation is spurious—arising solely from shared size-dependence rather than direct structural relationship.

For a representative range $[2, 2 \times 10^4]$, we computed:

Table 5: Correlations with $\log(n)$ reveal shared size-dependence.

Measure Pair	Pearson r
RDT $\leftrightarrow$ Collatz (raw)	0.167
RDT $\leftrightarrow$ $\log(n)$	0.760
Collatz $\leftrightarrow$ $\log(n)$	0.218
RDT $\leftrightarrow$ Collatz (partial, controlling for $\log n$)	**0.002**

Observation 3 (True Independence After Size Control)**.** *The raw correlation of $r = 0.167$ between RDT and Collatz drops to $r = 0.002$ after controlling for $\log(n)$ via partial correlation. This near-zero partial correlation confirms that RDT and Collatz are truly independent; their apparent correlation is entirely attributable to both measures exhibiting some dependence on integer magnitude.*

The strong correlation between RDT and $\log(n)$ ($r = 0.76$) reflects RDT's doubly logarithmic nature: $R(n) \approx \log \log n$. The weaker Collatz-$\log(n)$ correlation ($r = 0.22$) reflects the tendency for larger integers to require longer Collatz sequences on average, though with high variance.

4 Interpretation

4.1 True Independence: Size as Confounding Variable

The partial correlation analysis (Section 3.4) reveals the complete picture: RDT and Collatz are *truly independent* when accounting for their shared dependence on integer magnitude. The observed raw correlation is a classic case of spurious correlation induced by a confounding variable—here, $\log(n)$.

Both measures exhibit some relationship with size:

- **RDT**: Strong dependence on magnitude ($r = 0.76$ with $\log n$) due to its doubly logarithmic design: $R(n) \sim \log \log n$
- **Collatz**: Weak dependence on magnitude ($r = 0.22$ with $\log n$) reflecting the general tendency for larger integers to have longer (though highly variable) stopping times

When both measures track the same underlying variable (size), they appear correlated—but this correlation vanishes ($r = 0.002$) when size effects are removed. This confirms they capture fundamentally orthogonal structural properties.

4.2 Source of Small-n Correlation

At small n, both measures are in their "growth phase":

- RDT depths increase from 1 to 4 as n grows from 2 to 1000
- Collatz times also increase on average with n
- Both track the underlying size variable, creating spurious correlation

This is analogous to observing correlation between height and vocabulary in children: both grow with age (the hidden variable), not because they directly influence each other.

4.3 Emergence of Independence at Large n

For $n > 5 \times 10^4$:

- **RDT saturates**: Nearly all integers have depth 5 or 6, with minimal variation
- **Collatz remains chaotic**: Stopping times vary wildly from 12 to 350
- **Shared size-dependence vanishes**: RDT no longer tracks magnitude changes at fine scale

The asymptotic independence reveals their true orthogonality:

1. **RDT (Magnitude Complexity)**: Captures $O(\log \log n)$ growth, measuring "order of magnitude" through logarithmic scaling
2. **Collatz (Path Complexity)**: Captures arithmetic recursion structure through parity patterns, independent of magnitude

4.4 Two-Dimensional Complexity at Asymptotic Scale

For sufficiently large n, integers occupy a two-dimensional complexity space:

- **RDT axis**: Compressed (depth 5–6), reflects magnitude class
- **Collatz axis**: Spread (time 12–350), reflects path tortuosity

This establishes a multi-scale framework: correlation exists as a transient finite-size effect, while independence is the asymptotic ground truth.

4.5 Extreme Cases at Large Scale

Consider integers near $n = 77{,}031$:

Table 6: Extreme Collatz anomalies with typical RDT depth.

n	$R(n)$	$T_C(n)$	Ratio
77,031	5	350	58.3
78,791	5	337	56.2
60,975	5	334	55.7

These numbers are magnitude-typical (RDT = 5 like 78% of integers in this range) yet exhibit extreme Collatz complexity, demonstrating orthogonality.

5 Implications

5.1 Methodological: Importance of Multi-Scale Analysis

This study demonstrates that correlation analysis must account for scale-dependence. A single-range analysis might incorrectly conclude:

- At $n < 10^3$: "Weak correlation exists" ($r = 0.20$)
- At $n > 5 \times 10^4$: "Essentially independent" ($r = 0.03$)

Both statements are correct within their respective domains. The complete picture requires analyzing how correlation evolves across scales.

5.2 For Integer Complexity Theory

The scale-dependent decorrelation suggests:

1. **Finite-size effects dominate at small** n: Shared scaling with size creates spurious correlations
2. **Asymptotic behavior reveals true independence**: For $n \to \infty$, measures capture orthogonal properties
3. **Multiple dimensions necessary**: No single measure captures complete integer structure

5.3 For Collatz Research

RDT provides a magnitude-normalization tool. Define the *residual Collatz complexity*:

$$\Delta(n) = \frac{T_C(n)}{E[T_C \mid R(n)]},$$

where $E[T_C \mid R(n)]$ is the expected Collatz time for a given RDT class. Numbers with $\Delta \gg 1$ exhibit anomalous path complexity independent of magnitude.

6 Open Questions

1. **Theoretical characterization**: Can the correlation decay $r(N)$ for range $[2, N]$ be derived analytically?

2. **Saturation threshold**: At what n^* does RDT saturation begin, marking the transition to independence?

3. **Higher RDT exponents**: Does the pattern persist for $\alpha \neq 1.5$? Do larger α delay saturation?

4. **Comparison with other measures**: Do factorization depth, additive persistence, or other measures exhibit similar scale-dependent decorrelation?

5. **Collatz universality classes**: Can RDT-normalized Collatz complexity identify distinct behavioral classes?

7 Conclusion

We have demonstrated that RDT depth and Collatz stopping time are *fundamentally independent* measures of integer complexity. While raw correlation appears scale-dependent—decaying from $r \approx 0.20$ for $n < 10^3$ to $r \approx 0.03$ for $n > 5 \times 10^4$—partial correlation analysis reveals this apparent relationship is entirely spurious. After controlling for shared size-dependence via $\log(n)$, correlation drops to $r = 0.002$, confirming true independence.

The scale-dependent decay of raw correlation arises from two factors:

1. **Shared size-tracking**: Both RDT (strongly, $r = 0.76$) and Collatz (weakly, $r = 0.22$) correlate with $\log(n)$, creating spurious raw correlation

2. **RDT saturation**: At large n, RDT compresses to depths 5–6, diminishing even its size-tracking, which further reduces raw correlation

The true independence establishes these measures as orthogonal dimensions of integer complexity: RDT captures magnitude through doubly logarithmic scaling ($\sim \log \log n$), while Collatz captures arithmetic path structure through parity-dependent recursion. Neither predicts the other; together they span a two-dimensional complexity space.

This study demonstrates the critical importance of controlling for confounding variables in complexity measure analysis. Conclusions drawn from raw correlations may reflect shared dependencies rather than fundamental relationships. By systematically testing scale-dependence, subsampling stability, and partial correlations, we distinguish spurious from genuine associations, providing methodological guidance for future comparative studies of integer complexity measures.

Data Availability

Complete computational data and source code are available at [https://github.com/RRG314/rdt-collatz-orthogonality].

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