Recursive-Depth Integration: A Depth-Weighted Measure on the Integers

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Abstract

We introduce a new integration theory on the integers based on the Recursive Division Tree (RDT) depth function $R(n)$ introduced by Reid (2025). For a decay parameter $0 < \rho < 1$, we define a depth-weighted measure μ_R on $\mathbb{Z}$ by $\mu_R(A) = \sum_{n\in A} \rho^{R(|n|)}$. The corresponding recursive-depth integral $\int_{\mathbb{Z}} f(n)\, d\mu_R(n) = \sum f(n)\rho^{R(|n|)}$ generalizes the summation of real-valued functions with respect to a structurally induced infinite measure. We prove basic integrability criteria, identify classes of convergent and divergent integrals, and show that the recursive-depth weights behave analogously to a saturated exponential envelope determined by $R(n)$. Using the canonical RDT parameter $\alpha = 1.5$, for which $R(n)$ saturates at $R^* = 3$, we provide numerical experiments that demonstrate rapid convergence for integrable functions. This establishes recursive-depth analysis as a new direction in discrete measure theory, structurally linked to RDT recursion and the recursive-adic framework.

Keywords: discrete measure theory, recursive division tree, depth-weighted integration, infinite measures, number theory
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1 Introduction

The Recursive Division Tree (RDT), introduced in Reid [1], defines a depth function $R(n)$ on positive integers based on optimal recursive splitting. For $\alpha > 1$ the associated depth function satisfies the saturation property

$$\lim_{n\to\infty} R(n) = R^* = \frac{\alpha}{\alpha - 1},$$

and for the canonical value $\alpha = 1.5$, numerical evaluation shows $R(n) = 3$ for all $n \geq 23$ to six decimal places.

The recursive-adic number field introduced in [2] uses $R(n)$ to define valuations and ultrametrics. Here we develop a complementary analytic theory: a depth-weighted measure and integral acting directly on $\mathbb{Z}$.

Given $0 < \rho < 1$, we define

$$\mu_R(A) = \sum_{n\in A} \rho^{R(|n|)}, \qquad \int_{\mathbb{Z}} f\, d\mu_R = \sum_{n\in\mathbb{Z}} f(n)\rho^{R(|n|)}.$$

This creates an infinite measure analogous to counting measure, but weighted by recursive depth. We study integrable functions, divergence criteria, and provide numerical verification of these results.

1.1 Contributions

The main contributions of this paper are:

1. Definition of the recursive-depth measure μ_R and its associated integral (Section 3);
2. Characterization of integrable and non-integrable function classes (Section 4);
3. Numerical verification with reproducible algorithms (Section 6);
4. Discussion of connections to harmonic analysis and future directions (Section 7).

2 Background: Recursive Division Tree Depth

For completeness we recall the RDT depth definition and establish notation.

Definition 2.1 (RDT Depth [1]). *Let $\alpha > 1$. The* RDT depth function $R : \mathbb{Z} \to [0, \infty]$ *is defined by:*

$$R(1) = 0, \tag{1}$$

$$R(n) = 1 + \min_{1 \leq k < n} \frac{R(k) + R(n-k)}{\alpha} \quad \text{for } n \geq 2, \tag{2}$$

with the extensions $R(0) = \infty$ and $R(-n) = R(n)$ for $n > 0$.

Remark 2.2. *The convention $R(0) = \infty$ ensures that zero contributes no mass to the measure μ_R, since $\rho^{R(0)} = \rho^\infty = 0$ for $0 < \rho < 1$. Equivalently, one may define μ_R on $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.*

The fundamental saturation theorem from [2] states:

Theorem 2.3 (Depth Saturation). *If $\alpha > 1$, then $R(n)$ is non-decreasing for $n \geq 1$ and satisfies*

$$\lim_{n \to \infty} R(n) = R^* = \frac{\alpha}{\alpha - 1}.$$

Moreover, $R(n) < R^*$ for all finite n, with $R(n) = R^* - O(\alpha^{-n})$.

Table 1 shows computed values of $R(n)$ for $\alpha = 1.5$.

Table 1: RDT depth $R(n)$ for $\alpha = 1.5$ (saturation value $R^* = 3$).

n	1	2	3	4	5	10	20	23	50
$R(n)$	0.0000	1.0000	1.6667	2.1111	2.4074	2.9220	2.9986	2.9996	3.0000

Throughout the remainder of this paper we use $\alpha = 1.5$, for which $R^* = 3$.

3 Depth-Weighted Measure

Definition 3.1 (Recursive-depth measure)**.** *Let* $0 < \rho < 1$*. For* $A \subseteq \mathbb{Z}$ *define the* recursive-depth measure

$$\mu_R(A) = \sum_{n \in A} \rho^{R(|n|)},$$

where $\rho^{R(0)} = \rho^{\infty} := 0$*.*

Proposition 3.2 (σ-additivity)**.** *The set function* μ_R *is a measure on* $(\mathbb{Z}, 2^{\mathbb{Z}})$*.*

Proof. Clearly $\mu_R(\emptyset) = 0$. For countable additivity, let $\{A_j\}_{j=1}^{\infty}$ be disjoint subsets of $\mathbb{Z}$. Then

$$\mu_R\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{n \in \bigcup_j A_j} \rho^{R(|n|)} = \sum_{j=1}^{\infty} \sum_{n \in A_j} \rho^{R(|n|)} = \sum_{j=1}^{\infty} \mu_R(A_j),$$

where the interchange is justified by absolute convergence (all terms are non-negative). □

Theorem 3.3 (Infinite total mass)**.** $\mu_R(\mathbb{Z}) = \infty$ *for all* $0 < \rho < 1$*.*

Proof. By Theorem 2.3, $R(n) \to R^*$ as $n \to \infty$. Thus for any $\epsilon > 0$, there exists N such that $R(n) < R^* + \epsilon$ for all $n \geq N$. Hence

$$\rho^{R(n)} > \rho^{R^*+\epsilon} > 0 \quad \text{for all } n \geq N.$$

The sum $\sum_{n \in \mathbb{Z}} \rho^{R(|n|)}$ therefore dominates infinitely many terms bounded below by a positive constant, so it diverges. □

Corollary 3.4. *The measure* μ_R *is infinite but* σ*-finite.*

Proof. $\mathbb{Z} = \bigcup_{k=1}^{\infty} [-k, k]$ and each $\mu_R([-k, k]) < \infty$. □

4 Recursive-Depth Integral

Definition 4.1 (Recursive-depth integral)**.** *A function* $f : \mathbb{Z} \to \mathbb{R}$ *is* μ_R-integrable *if*

$$\sum_{n \in \mathbb{Z}} |f(n)| \, \rho^{R(|n|)} < \infty.$$

The recursive-depth integral *is then defined as*

$$\int_{\mathbb{Z}} f(n) \, d\mu_R(n) = \sum_{n \in \mathbb{Z}} f(n) \rho^{R(|n|)}.$$

4.1 Integrability Criteria

Proposition 4.2 (Integrability for ℓ^1 functions)**.** *If* $\sum_{n \in \mathbb{Z}} |f(n)| < \infty$*, then* f *is* μ_R*-integrable and*

$$\left| \int_{\mathbb{Z}} f \, d\mu_R \right| \leq \sum_{n \in \mathbb{Z}} |f(n)|.$$

Proof. Since $0 < \rho < 1$ and $R(|n|) \geq 0$ for $n \neq 0$, we have $0 < \rho^{R(|n|)} \leq 1$. Thus the weighted sum is dominated by $\sum |f(n)|$, which converges by hypothesis. $\square$

Proposition 4.3 (Non-integrability of constants)**.** *The constant function $f(n) \equiv c \neq 0$ is not μ_R-integrable.*

Proof. The integral equals $|c| \cdot \mu_R(\mathbb{Z})$, which diverges by Theorem 3.3. $\square$

Proposition 4.4 (Harmonic-type divergence)**.** *If $|f(n)| \geq c/(1 + |n|)$ for some $c > 0$ and all sufficiently large $|n|$, then f is not μ_R-integrable.*

Proof. Since $\rho^{R(|n|)} \to \rho^{R^*} > 0$, there exists N such that $\rho^{R(|n|)} \geq \frac{1}{2}\rho^{R^*}$ for $|n| \geq N$. Thus

$$\sum_{|n|\geq N} |f(n)|\rho^{R(|n|)} \geq \frac{c\rho^{R^*}}{2} \sum_{|n|\geq N} \frac{1}{1+|n|},$$

which diverges since the harmonic series diverges. $\square$

Theorem 4.5 (Integrability characterization)**.** *Let $f : \mathbb{Z} \to \mathbb{R}$. Then f is μ_R-integrable if and only if*

$$\sum_{n\in\mathbb{Z}} |f(n)| \cdot \rho^{R(|n|)} < \infty.$$

In particular, since $\rho^{R(|n|)} \to \rho^{R^*} > 0$, integrability requires $f \in \ell^1(\mathbb{Z})$.

Proof. The forward direction is the definition. For the converse, if $\sum |f(n)|\rho^{R(|n|)} < \infty$, then since $\rho^{R(|n|)} \geq \rho^{R^*}$ for large $|n|$, we have

$$\rho^{R^*} \sum_{|n|>N} |f(n)| \leq \sum_{|n|>N} |f(n)|\rho^{R(|n|)} < \infty,$$

so $f \in \ell^1(\mathbb{Z})$. $\square$

5 Examples

Example 5.1 (Convergent: $f(n) = 1/(1 + n^2)$)**.** *Let $f_1(n) = 1/(1 + n^2)$. The unweighted sum satisfies*

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi \coth(\pi) \approx 3.1533,$$

so $f_1 \in \ell^1(\mathbb{Z})$ and hence f_1 is μ_R-integrable by Proposition 4.2. Numerical computation (Section 6) gives

$$\int_{\mathbb{Z}} \frac{1}{1+n^2}\, d\mu_R(n) \approx 1.3537 \quad (\rho = 1/2).$$

Example 5.2 (Convergent: $f(n) = e^{-|n|}$)**.** *Let $f_2(n) = e^{-|n|}$. Since $\sum_{n\in\mathbb{Z}} e^{-|n|} = 1+2/(e-1) < \infty$, this function is μ_R-integrable. Numerical computation gives*

$$\int_{\mathbb{Z}} e^{-|n|}\, d\mu_R(n) \approx 0.9147 \quad (\rho = 1/2).$$

Example 5.3 (Divergent: $f(n) = 1/(1+|n|)$)**.** *Let $f_3(n) = 1/(1+|n|)$. Since $|f_3(n)| = 1/(1+|n|)$, this function is not μ_R-integrable by Proposition 4.4.*

6 Numerical Experiments

6.1 Algorithm

We compute the truncated integral

$$I_N = \sum_{n=-N}^{N} f(n)\, \rho^{R(|n|)}$$

using Algorithm 1 for the depth function and Algorithm 2 for the integral.

Algorithm 1 Compute RDT Depth Table

Require: Maximum integer $n_{\max}$, parameter $\alpha > 1$
Ensure: Table $R[0..n_{\max}]$ with $R[n]$ = RDT depth of n

$R[0] \leftarrow \infty$
$R[1] \leftarrow 0$
for $n = 2$ **to** $n_{\max}$ **do**
  $m \leftarrow \infty$
  for $k = 1$ **to** $n - 1$ **do**
    $v \leftarrow 1 + (R[k] + R[n-k])/\alpha$
    if $v < m$ **then**
      $m \leftarrow v$
    end if
  end for
  $R[n] \leftarrow m$
end for
return R

Algorithm 2 Compute Recursive-Depth Integral

Require: Function f, truncation N, decay $\rho \in (0,1)$, depth table R
Ensure: Truncated integral $I_N = \sum_{n=-N}^{N} f(n)\rho^{R(|n|)}$

$S \leftarrow 0$
for $n = -N$ **to** N **do**
  if $n = 0$ **then**
    continue ▷ $R(0) = \infty$ implies weight $= 0$
  end if
  $w \leftarrow \rho^{R[|n|]}$
  $S \leftarrow S + f(n) \cdot w$
end for
return S

6.2 Complexity Analysis

Algorithm 1 has time complexity $O(n_{\max}^2)$ and space complexity $O(n_{\max})$. Algorithm 2 runs in $O(N)$ time given a precomputed depth table.

6.3 Results

Table 2 shows the truncated integral I_N for $f(n) = 1/(1+n^2)$ with $\rho = 1/2$ and $\alpha = 1.5$.

Table 2: Truncated integral I_N for $f(n) = 1/(1+n^2)$, $\rho = 1/2$, $\alpha = 1.5$.

N	10	20	50	100	150	200
I_N	1.3299	1.3416	1.3488	1.3513	1.3521	1.3525

Remark 6.1 (Role of $R(0)$)**.** *Since $R(0) = \infty$, the term at $n = 0$ contributes $\rho^\infty = 0$ to the sum. The dominant early contributions come from $n = \pm 1$, where $R(1) = 0$ gives weight $\rho^0 = 1$.*

6.4 Convergence Analysis

The stabilization after $N \approx 50$ confirms the theoretical prediction: once $R(n) \approx 3$ for $n \geq 23$, the tail behaves like a standard absolutely convergent series with weights bounded by $\rho^3 = 1/8$.

Extended computation shows:

$$\begin{aligned} I_{1000} &= 1.35352136, \\ I_{2000} &= 1.35364626, \\ I_{5000} &= 1.35372123. \end{aligned}$$

The tail contribution $I_{2000} - I_{1000} \approx 1.2 \times 10^{-4}$ demonstrates the expected $O(1/N)$ decay rate for the truncation error.

6.5 Comparison with Unweighted Sum

For reference, the unweighted sum over $\mathbb{Z}$ gives

$$\sum_{n \in \mathbb{Z}} \frac{1}{1+n^2} = \pi \coth(\pi) \approx 3.1533.$$

The ratio of weighted to unweighted sums is approximately $1.354/3.153 \approx 0.429$, reflecting the dampening effect of the depth weights.

7 Discussion

7.1 Structural Interpretation

The key structural insight is that the saturation of $R(n)$ at R^* means the weights $\rho^{R(|n|)}$ converge to a positive constant ρ^{R^*}, rather than decaying to zero. This makes the recursive-depth measure fundamentally different from measures with exponentially decaying weights (such as Poisson-type measures).

In essence, μ_R is "almost" counting measure for large $|n|$, but with a structured dampening near the origin that reflects the recursive complexity of small integers.

7.2 Connections to Harmonic Analysis

The recursive-depth integral suggests natural extensions to harmonic analysis:

1. **Depth-Fourier transform**: For $f : \mathbb{Z} \to \mathbb{C}$, define

$$\hat{f}_R(\theta) = \sum_{n\in\mathbb{Z}} f(n)\rho^{R(|n|)}e^{-2\pi in\theta}.$$

   This is well-defined for $f \in \ell^1(\mathbb{Z})$ and maps to continuous functions on $\mathbb{T}$.

2. **Depth-weighted convolution**: Define

$$(f *_R g)(n) = \sum_{k\in\mathbb{Z}} f(k)g(n-k)\rho^{R(|k|)}.$$

3. **Depth-Laplace transform**: For $f : \mathbb{N} \to \mathbb{R}$, define

$$\mathcal{L}_R[f](s) = \sum_{n=1}^{\infty} f(n)e^{-sn}\rho^{R(n)}.$$

7.3 Future Directions

The recursive-depth integral provides an analytic complement to the recursive-adic number field [2] and opens several directions for future research:

1. **Spectral theory**: Eigenvalue problems for depth-weighted operators;
2. **Probability**: Recursive-depth random walks and their limit distributions;
3. **Analytic number theory**: Depth-weighted Dirichlet series $\sum n^{-s}\rho^{R(n)}$;
4. **Completion theory**: Integration on the recursive-adic completion $\widehat{\mathbb{Z}}_R$.

8 Reproducibility

All numerical results in this paper can be reproduced using the following Python implementation.

```
def compute_R_table(n_max, alpha=1.5):
    """Compute RDT depth for n = 0 to n_max."""
    R = {0: float('inf'), 1: 0.0}
    for n in range(2, n_max + 1):
        R[n] = min(1 + (R[k] + R[n-k])/alpha
                   for k in range(1, n))
    return R

def integral(f, N, rho=0.5, R=None):
    """Compute truncated recursive-depth integral."""
    if R is None:
        R = compute_R_table(N)
    return sum(f(n) * rho**R[abs(n)]
```

```
                for n in range(-N, N+1) if n != 0)

# Example usage:
R = compute_R_table(1000)
f = lambda n: 1/(1 + n**2)
print(f"I_1000 = {integral(f, 1000, R=R):.8f}")
```

Software availability: Reference implementation available at
https://github.com/sreid1118/recursive-depth-integration.

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A Extended Depth Table

Table 3 provides extended values of $R(n)$ for reference.

Table 3: Extended RDT depth values for $\alpha = 1.5$.

n	$R(n)$	n	$R(n)$	n	$R(n)$
1	0.000000	6	2.604938	15	2.989724
2	1.000000	7	2.736626	20	2.998647
3	1.666667	8	2.824417	23	2.999599
4	2.111111	9	2.882945	25	2.999822
5	2.407407	10	2.921963	50	3.000000

B Convergence Data

Table 4 shows detailed convergence data for the integral I_N.

Table 4: Detailed convergence of I_N for $f(n) = 1/(1+n^2)$, $\rho = 1/2$.

N	I_N	ΔI_N
5	1.30472722	—
10	1.32987697	0.02514975
20	1.34158726	0.01171029
50	1.34882154	0.00723428
100	1.35128377	0.00246223
200	1.35252436	0.00124059
500	1.35327173	0.00074737
1000	1.35352136	0.00024963