Derivation and Closure of Euler Potentials in Resistive Magnetohydrodynamics (MHD)

Abstract

This document presents a complete derivation, analysis, and functional implementation of an exact closure method for evolving Euler potentials in the resistive magnetohydrodynamics (MHD) framework. Euler potentials α and β provide a scalar representation of magnetic fields that automatically satisfy the divergence-free condition $\nabla \cdot \mathbf{B} = 0$, via the identity $\mathbf{B} = \nabla\alpha \times \nabla\beta$. In ideal MHD, these scalar potentials are conserved along fluid trajectories, but in resistive MHD, naive scalar diffusion of α and β fails to yield the correct resistive evolution of $\mathbf{B}$. This failure originates from mixed derivative terms that arise in the vector Laplacian of the magnetic field, which are not captured by the Laplacians of α and β individually. To correct this, we introduce scalar source terms S_α and S_β and derive a closure condition that ensures the modified evolution equations for α and β generate a magnetic field consistent with the resistive MHD induction equation. The constraint takes the form of a vector identity involving gradients of the scalar fields and their sources.

We solve this constraint analytically for a class of bilinear Euler potentials where $\alpha = c \cdot u$ and $\beta = c \cdot w$, with $c, u, w \in \{x, y, z\}$ and c being the shared coordinate. For these fields, we derive exact expressions for S_α and S_β that fully recover the missing terms in the magnetic field evolution. The results are validated through symbolic computation for all coordinate permutations, demonstrating exact agreement between the corrected scalar evolution and the full vector induction equation. Additionally, we provide a general computational function capable of detecting valid configurations and producing the required closure sources automatically.

The method is limited to static, structured fields expressible in this bilinear form and does not apply to topologically complex or turbulent configurations. However, within this scope, it offers an exact, reproducible, and computationally efficient means of simulating resistive MHD evolution using scalar potentials. This work corrects previ-

ous erroneous closures and provides a solid mathematical foundation for incorporating resistive effects into Euler potential-based MHD models.

1 Introduction

In resistive magnetohydrodynamics (MHD), the magnetic field evolves according to the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ is the fluid velocity, and η is the magnetic diffusivity. A key physical constraint in MHD is that the magnetic field must remain solenoidal:

$$\nabla \cdot \mathbf{B} = 0$$

An elegant way to guarantee this constraint is to represent $\mathbf{B}$ using Euler potentials α and β such that:

$$\mathbf{B} = \nabla \alpha \times \nabla \beta \tag{2}$$

This representation satisfies $\nabla \cdot \mathbf{B} = 0$ identically due to vector calculus identities.

2 Motivation and Problem

In ideal MHD (when $\eta = 0$), the Euler potentials are materially conserved:

$$\frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0 \tag{3}$$

In resistive MHD, one might attempt to evolve α and β by simple diffusion:

$$\frac{\partial \alpha}{\partial t} = \eta \nabla^2 \alpha, \quad \frac{\partial \beta}{\partial t} = \eta \nabla^2 \beta \tag{4}$$

However, this naive approach fails. To see why, compute the time derivative of $\mathbf{B}$ using this evolution:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \left(\frac{\partial \alpha}{\partial t} \right) \times \nabla \beta + \nabla \alpha \times \nabla \left(\frac{\partial \beta}{\partial t} \right) \tag{5}$$

$$= \nabla(\eta \nabla^2 \alpha) \times \nabla \beta + \nabla \alpha \times \nabla(\eta \nabla^2 \beta) \tag{6}$$

This expression does *not* equal $\eta\nabla^2\mathbf{B}$, because the Laplacian of a cross product is not the cross product of Laplacians:

$$\nabla^2(\nabla\alpha \times \nabla\beta) \neq \nabla(\nabla^2\alpha) \times \nabla\beta + \nabla\alpha \times \nabla(\nabla^2\beta) \tag{7}$$

The correct form includes a remainder term:

$$\eta\nabla^2\mathbf{B} = \nabla(\eta\nabla^2\alpha) \times \nabla\beta + \nabla\alpha \times \nabla(\eta\nabla^2\beta) + \mathbf{R} \tag{8}$$

where $\mathbf{R}$ contains second-order mixed partial derivatives of α and β.

3 Closure Constraint

To compensate for the missing term $\mathbf{R}$, we introduce additional source terms into the scalar evolution equations:

$$\frac{\partial\alpha}{\partial t} = \eta\nabla^2\alpha + S_\alpha \tag{9}$$

$$\frac{\partial\beta}{\partial t} = \eta\nabla^2\beta + S_\beta \tag{10}$$

The magnetic field then evolves as:

$$\frac{\partial\mathbf{B}}{\partial t} = \eta\nabla^2\mathbf{B} + \nabla S_\alpha \times \nabla\beta + \nabla\alpha \times \nabla S_\beta$$

For the closure to be valid, these additional terms must exactly reproduce the missing $\mathbf{R}$:

$$\nabla S_\alpha \times \nabla\beta + \nabla\alpha \times \nabla S_\beta = \mathbf{R} \tag{11}$$

This equation defines a constraint that must be solved to determine the correct forms of S_α and S_β.

4 Solvable Case: Bilinear Shared-Coordinate Fields

Consider a restricted class of Euler potentials where:

$$\alpha = c \cdot u, \quad \beta = c \cdot w$$

with $c, u, w \in \{x, y, z\}$, and $u \neq w$, but both α and β share coordinate c. These are referred to as *bilinear shared-coordinate fields*.

Example

Let $\alpha = xy$, $\beta = xz$. Then:

$$\nabla\alpha = (y, x, 0), \quad \nabla\beta = (z, 0, x)$$

The magnetic field is:

$$\mathbf{B} = \nabla\alpha \times \nabla\beta = (x^2, -xy, -xz)$$

A symbolic calculation shows that:

$$\eta\nabla^2\mathbf{B} = \nabla(\eta\nabla^2\alpha) \times \nabla\beta + \nabla\alpha \times \nabla(\eta\nabla^2\beta) + \mathbf{R}$$

with:

$$\mathbf{R} = (2\eta, 0, 0)$$

We propose source terms:

$$S_\alpha = \eta\frac{y}{x}, \quad S_\beta = \eta\frac{z}{x}$$

Then:

$$\nabla S_\alpha \times \nabla\beta + \nabla\alpha \times \nabla S_\beta = (2\eta, 0, 0)$$

matching $\mathbf{R}$ exactly.

General Closure

For general forms:

$$\alpha = c \cdot u, \quad \beta = c \cdot w$$

the source terms that provide exact closure are:

$$S_\alpha = \eta\frac{u}{c}, \quad S_\beta = \eta\frac{w}{c} \tag{12}$$

This form has been validated symbolically for all permutations of x, y, z.

5 Python Function Implementation

A Python function to compute these closure terms is shown below:

```
def compute_closure_source(alpha, beta, eta, coords):
    x, y, z = coords

    def extract_variables(expr):
        return set(expr.free_symbols).intersection({x, y, z})

    alpha_vars = extract_variables(alpha)
    beta_vars = extract_variables(beta)
    shared = alpha_vars.intersection(beta_vars)

    if len(alpha_vars) != 2 or len(beta_vars) != 2:
        raise ValueError(" and  must each be bilinear in two coordinates.")

    if len(shared) != 1:
        raise ValueError(" and  must share exactly one coordinate.")

    shared_coord = shared.pop()
    u = (alpha_vars - {shared_coord}).pop()
    w = (beta_vars - {shared_coord}).pop()

    S_alpha = eta * u / shared_coord
    S_beta = eta * w / shared_coord

    return S_alpha, S_beta
```

This returns the correct S_α and S_β for any valid bilinear shared-coordinate Euler potential.

6 Limitations

This method is valid only for bilinear Euler potentials with shared coordinates. It is not applicable to general magnetic field configurations, non-bilinear forms, or topologically complex magnetic structures. It also assumes time-independent scalar fields.

7 Conclusion

This work provides a rigorous derivation of closure terms for Euler potentials in resistive MHD. It identifies a restricted class of magnetic fields for which the scalar field evolution can be corrected with exact source terms. The method is validated symbolically and implemented computationally. It offers a precise tool for modeling certain MHD test cases and educational demonstrations while acknowledging its limited scope.