Proof that the Eigenvalues of the Normalized Laplacian Matrix Lie in [0, 2]

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Introduction

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be an undirected graph with n vertices. Let:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the adjacency matrix of $\mathbf{G}$, where $\mathbf{A}_{ij} = 1$ if vertices i and j are connected, and $\mathbf{A}_{ij} = 0$ otherwise.
- $\mathbf{D} \in \mathbb{R}^{n \times n}$ be the diagonal degree matrix, where $\mathbf{D}_{ii} = d_i = \sum_j \mathbf{A}_{ij}$.
- $\mathbf{L} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ be the normalized Laplacian matrix.

Our goal is to prove that all eigenvalues λ of $\mathbf{L}$ satisfy $0 \leq \lambda \leq 2$.

Proof

The proof consists of the following steps:

1. Show that $\mathbf{L}$ is symmetric and positive semidefinite.
2. Prove that the eigenvalues μ of $\mathbf{S} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ satisfy $-1 \leq \mu \leq 1$.
3. Use the relationship between the eigenvalues of $\mathbf{L}$ and $\mathbf{S}$ to conclude that $0 \leq \lambda \leq 2$.

1. Symmetry and Positive Semidefiniteness of L

Symmetry

Since $\mathbf{D}^{-1/2}$ is a diagonal matrix (hence symmetric), and $\mathbf{A}$ is symmetric (because $\mathbf{G}$ is undirected), it follows that:

$$\mathbf{L} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$$

is symmetric, as it is composed of symmetric matrices.

Positive Semidefiniteness

For any vector $\mathbf{x} \in \mathbb{R}^n$, consider the quadratic form:

$$\mathbf{x}^\top \mathbf{L}\mathbf{x} = \mathbf{x}^\top \left(\mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\right)\mathbf{x} = \mathbf{x}^\top\mathbf{x} - \mathbf{x}^\top\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\mathbf{x}$$

Let $\mathbf{y} = \mathbf{D}^{-1/2}\mathbf{x}$. Then:

$$\mathbf{x}^\top \mathbf{L}\mathbf{x} = \mathbf{x}^\top\mathbf{x} - \mathbf{y}^\top\mathbf{A}\mathbf{y} = \|\mathbf{x}\|^2 - \mathbf{y}^\top\mathbf{A}\mathbf{y}$$

Since $\mathbf{A}$ has non-negative entries and $\mathbf{y}^\top\mathbf{A}\mathbf{y} = \sum_{i,j} \mathbf{A}_{ij} y_i y_j$, we can write:

$$\mathbf{x}^\top \mathbf{L}\mathbf{x} = \sum_i x_i^2 - \sum_{i,j} \mathbf{A}_{ij} y_i y_j$$

Noting that $x_i = \sqrt{d_i}\, y_i$ and $x_i^2 = d_i y_i^2$, we have:

$$\mathbf{x}^\top \mathbf{L}\mathbf{x} = \sum_i d_i y_i^2 - \sum_{i,j} \mathbf{A}_{ij} y_i y_j$$

Observe that:

$$\begin{aligned}
\mathbf{x}^\top \mathbf{L}\mathbf{x} &= \sum_i d_i y_i^2 - \sum_{i,j} \mathbf{A}_{ij} y_i y_j \\
&= \frac{1}{2} \sum_{i,j} \mathbf{A}_{ij} \left(d_i y_i^2 + d_j y_j^2 - 2 y_i y_j \right) \\
&= \frac{1}{2} \sum_{i,j} \mathbf{A}_{ij} \left(\sqrt{d_i}\, y_i - \sqrt{d_j}\, y_j \right)^2 \\
&= \frac{1}{2} \sum_{i,j} \mathbf{A}_{ij} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 \\
&\geq 0
\end{aligned}$$

Therefore, $\mathbf{L}$ is positive semidefinite.

2. Eigenvalues of $\mathbf{S} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$

We aim to show that the eigenvalues μ of $\mathbf{S}$ satisfy $-1 \leq \mu \leq 1$.

Symmetry of S

Since $\mathbf{D}^{-1/2}$ and $\mathbf{A}$ are symmetric, $\mathbf{S} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ is symmetric. Therefore, all its eigenvalues are real.

Bounding the Eigenvalues of S

For any non-zero vector $\mathbf{z} \in \mathbb{R}^n$, consider the Rayleigh quotient:

$$\mu = \frac{\mathbf{z}^\top \mathbf{S}\mathbf{z}}{\mathbf{z}^\top \mathbf{z}}$$

Our goal is to show that $|\mu| \leq 1$.

Compute $\mathbf{z}^\top \mathbf{S}\mathbf{z}$:

$$\mathbf{z}^\top \mathbf{S}\mathbf{z} = \mathbf{z}^\top \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\mathbf{z} = \left(\mathbf{D}^{-1/2}\mathbf{z}\right)^\top \mathbf{A} \left(\mathbf{D}^{-1/2}\mathbf{z}\right)$$

Let $\mathbf{w} = \mathbf{D}^{-1/2}\mathbf{z}$. Then:

$$\mathbf{z}^\top \mathbf{S}\mathbf{z} = \mathbf{w}^\top \mathbf{A}\mathbf{w} = \sum_{i,j} \mathbf{A}_{ij} w_i w_j$$

Since $\mathbf{A}_{ij} = 1$ if i and j are connected, and 0 otherwise, we have:

$$\mathbf{z}^\top \mathbf{S}\mathbf{z} = \sum_{(i,j)\in E} w_i w_j$$

Similarly, $\mathbf{z}^\top \mathbf{z} = \left(\mathbf{D}^{1/2}\mathbf{w}\right)^\top \left(\mathbf{D}^{1/2}\mathbf{w}\right) = \sum_i d_i w_i^2$

Now, consider that for any real numbers a and b:

$$2|ab| \leq a^2 + b^2$$

Therefore:

$$|w_i w_j| \leq \frac{1}{2}\left(w_i^2 + w_j^2\right)$$

Thus:

$$\left|\mathbf{z}^\top \mathbf{S}\mathbf{z}\right| = \left| \sum_{(i,j)\in E} w_i w_j \right| \leq \sum_{(i,j)\in E} \frac{1}{2}\left(w_i^2 + w_j^2\right) = \sum_{(i,j)\in E} \frac{1}{2} w_i^2 + \frac{1}{2} w_j^2$$

Since each edge (i,j) contributes to w_i^2 and w_j^2, and the degree d_i counts the number of edges incident to i, we have:

$$\sum_{(i,j)\in E} w_i^2 = \sum_i d_i w_i^2$$

Thus:

$$\left|\mathbf{z}^\top \mathbf{S}\mathbf{z}\right| \leq \sum_i d_i w_i^2 = \mathbf{z}^\top \mathbf{z}$$

Therefore:

$$|\mu| = \left|\frac{\mathbf{z}^\top \mathbf{S}\mathbf{z}}{\mathbf{z}^\top \mathbf{z}}\right| \leq 1$$

Hence, all eigenvalues μ of $\mathbf{S}$ satisfy $-1 \leq \mu \leq 1$.

3. Relationship Between the Eigenvalues of L and S

Since:

$$\mathbf{L} = \mathbf{I} - \mathbf{S}$$

If μ is an eigenvalue of $\mathbf{S}$ with eigenvector $\mathbf{z}$, then:

$$\mathbf{L}\mathbf{z} = (\mathbf{I} - \mathbf{S})\mathbf{z} = \mathbf{z} - \mu\mathbf{z} = (1 - \mu)\mathbf{z}$$

Therefore, the eigenvalues λ of $\mathbf{L}$ are related to those of $\mathbf{S}$ by:

$$\lambda = 1 - \mu$$

Since $-1 \leq \mu \leq 1$, it follows that:

$$0 \leq \lambda = 1 - \mu \leq 2$$

Thus, all eigenvalues of $\mathbf{L}$ lie in the interval $[0, 2]$.

Conclusion

We have shown that the eigenvalues of the normalized Laplacian matrix $\mathbf{L} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ lie within the interval $[0, 2]$, by demonstrating that:

- $\mathbf{L}$ is symmetric and positive semidefinite.
- The eigenvalues μ of $\mathbf{S} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ satisfy $-1 \leq \mu \leq 1$.
- The eigenvalues of $\mathbf{L}$ and $\mathbf{S}$ are related by $\lambda = 1 - \mu$.