# The Art of Linear Algebra

– Graphic Notes on "Linear Algebra for Everyone" –

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with the kindest help of Gilbert Strang †

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**Abstract**

我嘗試爲Gilbert Strang 在書籍"Linear Algebra for Everyone" 中介紹的矩陣的重要概念進形可視化圖釋, 以促進從矩陣分解的角度對向量、矩陣計算和算法的理解. [1] 它們包括矩陣分解(Column-Row, $\boldsymbol{CR}$)、高斯消去法(Gaussian Elimination, $\boldsymbol{LU}$)、格拉姆-施密特正交化(Gram-Schmidt Orthogonalization, $\boldsymbol{QR}$)、特徵值和對角化(Eigenvalues and Diagonalization, $\boldsymbol{Q\Lambda Q^{\mathrm{T}}}$)、和奇異值分解(Singular Value Decomposition, $\boldsymbol{U\Sigma V^{\mathrm{T}}}$).

# 序言

我很高興能看到Kenji Hiranabe 的線性代數中的矩陣運算的圖片! 這樣的圖片是展示代數的絕佳方式. 我們當然可以通過列· 行的點乘來想象矩陣乘法, 但那絕非全部—— 它是"線性組合"與"秩1矩陣"組成的代數與藝術. 我很感激能看到日文翻譯的書籍和Kenji的圖片中的想法.

– Gilbert Strang

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[1] "Linear Algebra for Everyone": http://math.mit.edu/everyone/.

# Contents

# 1 理解矩陣——4個視角

一個矩陣$(m \times n)$ 可以被視爲1個矩陣, $mn$個數, $n$個行和$m$個列.

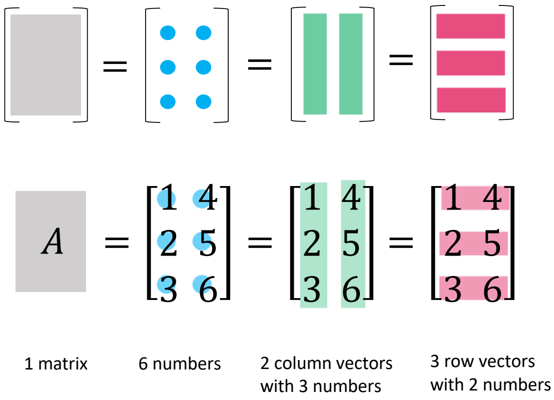



Figure 1: 從四個角度理解矩陣

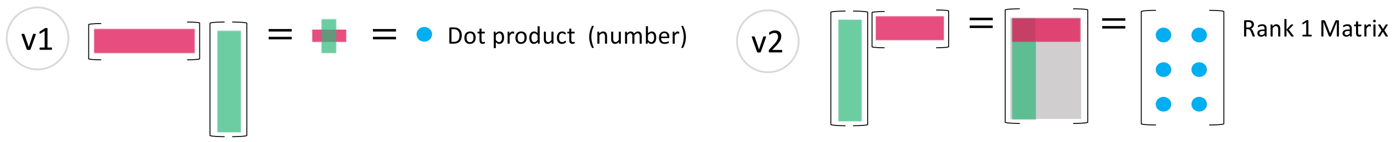
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -\boldsymbol{a}_1^*- \\ -\boldsymbol{a}_2^*- \\ -\boldsymbol{a}_3^*- \end{bmatrix}$$

在這里, 行向量被標記爲粗體$\boldsymbol{a}_1$. 列向量則有一個$*$號, 標記爲$\boldsymbol{a}_1^*$. 轉置向量和矩陣則用T標記爲$\boldsymbol{a}^{\mathrm{T}}$和$A^{\mathrm{T}}$.

# 2 向量乘以向量——2個視角

後文中，我將介紹一些概念，同時列出"Linear Algebra for Everyone"一書中的相應部分(部分編號插入如下)．詳細的內容最好看書，這里我也添加了一個簡短的解釋，以便您可以通過這篇文章盡可能多地理解．此外，每個圖都有一個簡短的名稱，例如v1 (數字1 表示向量的乘積)、Mv1 (數字1 表示矩陣和向量的乘積)，以及如下圖(v1) 所示的彩色圓圈．如你所見，隨著討論的進形，該名稱將被交叉引用．

- 1.1節(p.2) Linear combination and dot products
- 1.3節(p.25) Matrix of Rank One
- 1.4節(p.29) Row way and column way




Dot product ($\boldsymbol{a} \cdot \boldsymbol{b}$) is expressed as $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ in matrix language and yields a number.

$\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$ is a matrix ($\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} = A$). If neither $a, b$ are 0, the result $A$ is a rank 1 matrix.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

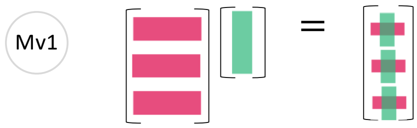
Figure 2: 向量乘以向量- (v1), (v2)

(v1) 是兩個向量之間的基礎運算，而(v2) 將行乘以列並產生一個秩1矩陣．理解(v2) 的結果(秩1) 是接下來章節的關鍵．

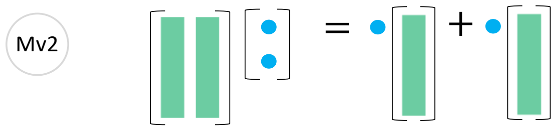
# 3 矩陣乘以向量——2個視角

一個矩陣乘以一個向量將產生三個點積組成的向量(Mv1) 和一種$A$的行向量的線性組合.

- 1.1節(p.3) Linear combinations
- 1.3節(p.21) Matrices and Column Spaces




The row vectors of $A$ are multiplied by a vector $\boldsymbol{x}$ and become the three dot-product elements of $A\boldsymbol{x}$.




The product $A\boldsymbol{x}$ is a linear combination of the column vectors of $A$.

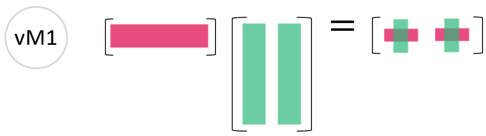
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1+2x_2) \\ (3x_1+4x_2) \\ (5x_1+6x_2) \end{bmatrix}$$

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Figure 3: 矩陣乘以向量- (Mv1), (Mv2)

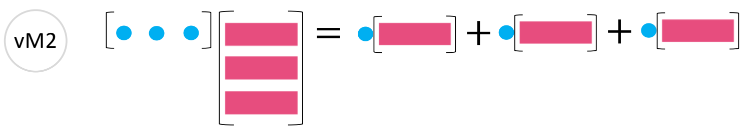
往往你會先學習(Mv1). 但當你習慣了從(Mv2) 的視角看待它, 會理解$A\boldsymbol{x}$是$A$的行的線性組合. 矩陣$A$的行向量的所有線性組合生成的子空間記爲$\mathbf{C}(A)$. $A\boldsymbol{x} = \mathbf{0}$的解空間則是零空間, 記爲$\mathbf{N}(A)$.

同理, 由(vM1) 和(vM2) 可見, 列向量乘以矩陣也是同一種理解方式.




$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = [(y_1+3y_2+5y_3) \quad (2y_1+4y_2+6y_3)]$$

A row vector $\boldsymbol{y}$ is multiplied by the two column vectors of $A$ and become the two dot-product elements of $\boldsymbol{y}A$.




$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = y_1[1 \quad 2] + y_2[3 \quad 4] + y_3[5 \quad 6]$$

The product $\boldsymbol{y}A$ is a linear combination of the row vectors of $A$.

Figure 4: 向量乘以矩陣- (vM1), (vM2)

上圖$A$的列向量的所有線性組合生成的子空間記爲$\mathbf{C}(A^{\mathrm{T}})$. $yA = 0$的解空間是$A$的左零空間, 記爲$\mathbf{N}(A^{\mathrm{T}})$.

本書的一大亮點即爲四個基本子空間: 在$\mathbb{R}^n$ 上的$\mathbf{N}(A)+\mathbf{C}(A^{\mathrm{T}})$ (相互正交) 和在$\mathbb{R}^m$ 上的$\mathbf{N}(A^{\mathrm{T}})+\mathbf{C}(A)$ (相互正交).

- 3.5節(p.124) Dimensions of the Four Subspaces

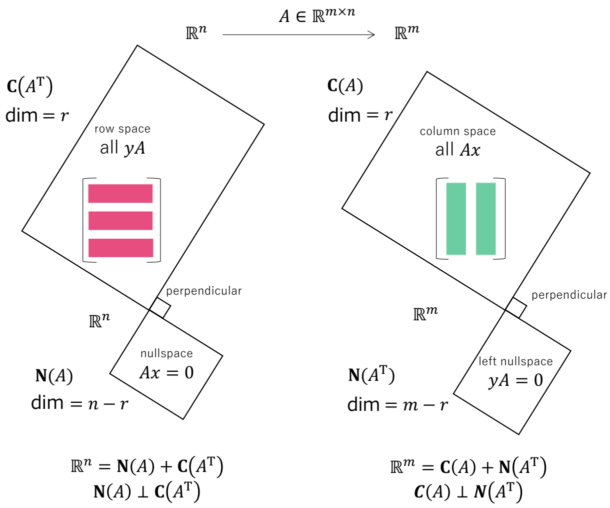


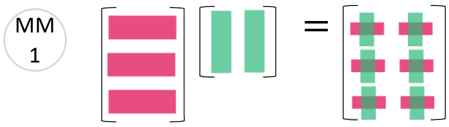

Figure 5: 四個子空間

關於秩$r$, 請見$A = CR$ (6.1節) .

# 4 矩陣乘以矩陣——4個視角

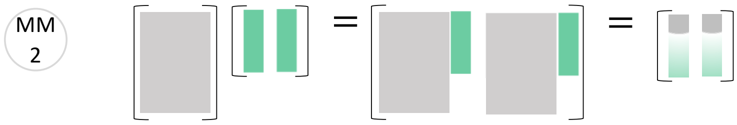
由“矩陣乘以向量”自然延伸到“矩陣乘以矩陣”.

- 1.4節(p.35) Four ways to multiply $\boldsymbol{AB} = \boldsymbol{C}$
- 也可以見書的封底



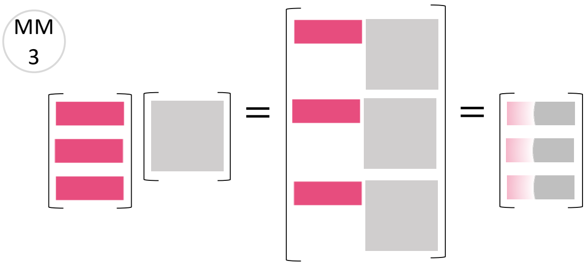

Every element becomes a dot product of row vector and column vector.

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1&y_1\\x_2&y_2\end{bmatrix}=\begin{bmatrix}(x_1+2x_2)&(y_1+2y_2)\\(3x_1+4x_2)&(3y_1+4y_2)\\(5x_1+6x_2)&(5y_1+6y_2)\end{bmatrix}$$



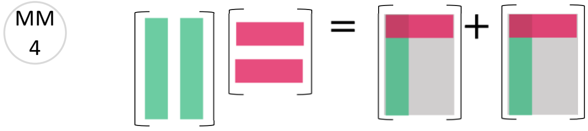

$A\boldsymbol{x}$ and $A\boldsymbol{y}$ are linear combinations of columns of $A$.

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1&y_1\\x_2&y_2\end{bmatrix}=A[\boldsymbol{x}\quad \boldsymbol{y}]=[A\boldsymbol{x}\quad A\boldsymbol{y}]$$




The produced rows are linear combinations of rows.

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1&y_1\\x_2&y_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{a}_1^*\\\boldsymbol{a}_2^*\\\boldsymbol{a}_3^*\end{bmatrix}X=\begin{bmatrix}\boldsymbol{a}_1^*X\\\boldsymbol{a}_2^*X\\\boldsymbol{a}_3^*X\end{bmatrix}$$




Multiplication $AB$ is broken down to a sum of rank 1 matrices.

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}=[\boldsymbol{a}_1\quad \boldsymbol{a}_2]\begin{bmatrix}\boldsymbol{b}_1^*\\\boldsymbol{b}_2^*\end{bmatrix}=\boldsymbol{a}_1\boldsymbol{b}_1^*+\boldsymbol{a}_2\boldsymbol{b}_2^*$$

$$=\begin{bmatrix}1\\3\\5\end{bmatrix}[b_{11}\quad b_{12}]+\begin{bmatrix}2\\4\\6\end{bmatrix}[b_{21}\quad b_{22}]=\begin{bmatrix}b_{11}&b_{12}\\3b_{11}&3b_{12}\\5b_{11}&5b_{12}\end{bmatrix}+\begin{bmatrix}2b_{21}&2b_{22}\\4b_{21}&4b_{22}\\6b_{21}&6b_{22}\end{bmatrix}$$

Figure 6: 矩陣乘以矩陣- (MM1), (MM2), (MM3), (MM4)

# 5　實用模式

在這里, 我展示了一些實用的模式, 可以讓你更直觀地理解接下來的內容。

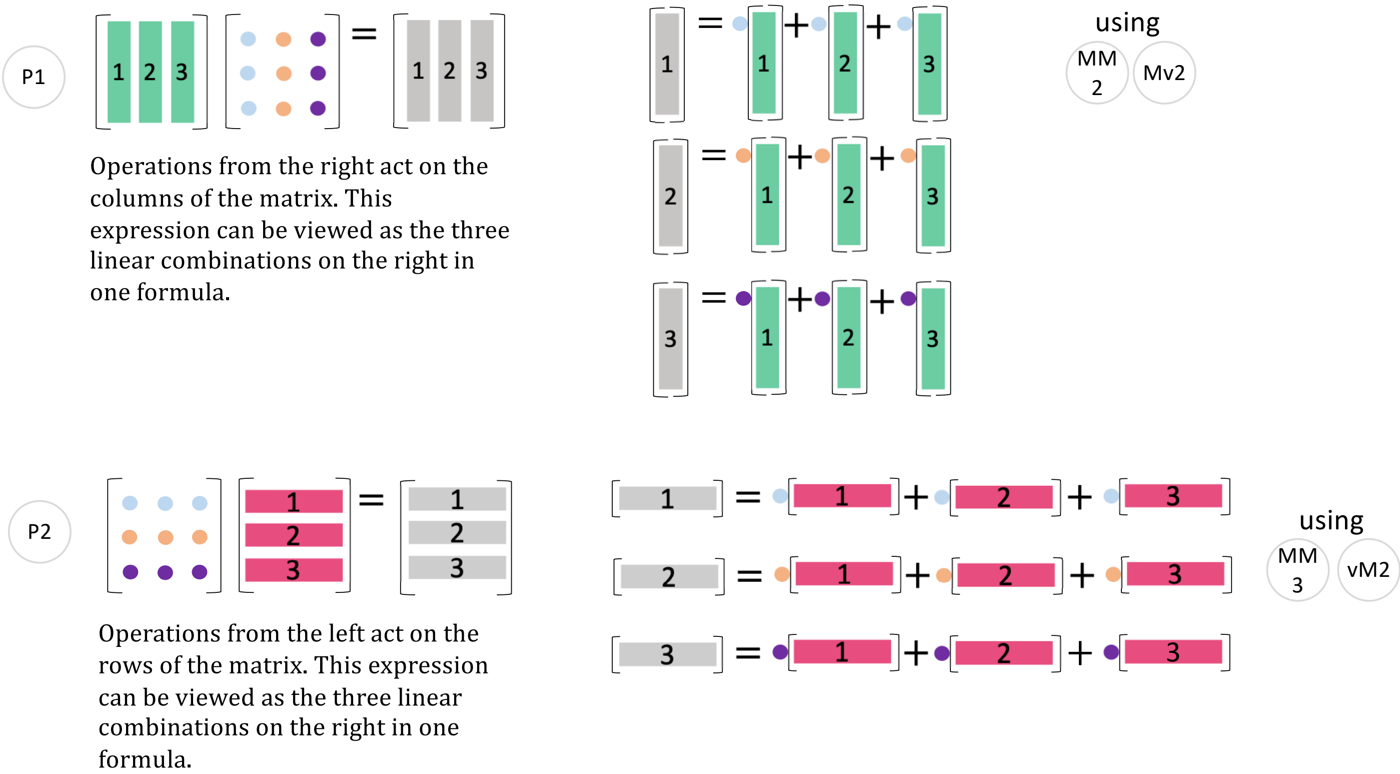
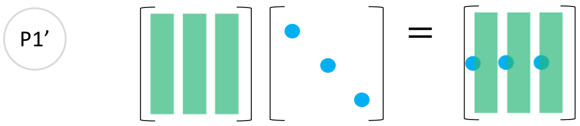
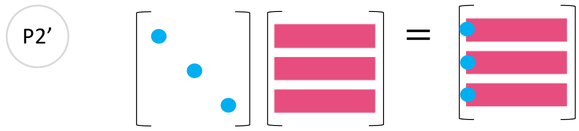



Figure 7: 圖1, 2 - (P1), (P1)

P1 是(MM2) 和(Mv2) 的結合. P2 是(MM3) 和(vM2) 的擴展. 注意, P1 是行運算(右乘一個矩陣), 而P2 是列運算(左乘一個矩陣).




Applying a diagonal matrix from the right scales each column.



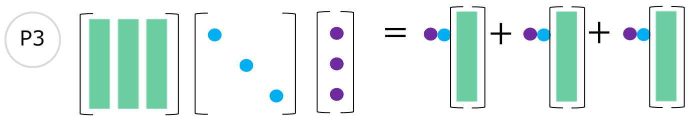

Applying a diagonal matrix from the left scales each row.

$$AD = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \boldsymbol{a_3}]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1\boldsymbol{a_1} \quad d_2\boldsymbol{a_2} \quad d_3\boldsymbol{a_3}]$$

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{b}_1^* \\ \boldsymbol{b}_2^* \\ \boldsymbol{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1\boldsymbol{b}_1^* \\ d_2\boldsymbol{b}_2^* \\ d_3\boldsymbol{b}_3^* \end{bmatrix}$$

Figure 8: 圖1′, 2′ - (P1′), (P2′)

(P1′) 將對角線上的數乘以矩陣的行, 而(P2′) 將對角線上的數乘以矩陣的列. 兩個分別爲(P1) 和(P2) 的變體.




This pattern reveals another combination of columns.
You will encounter this in differential/recurrence equations.

$$X D \boldsymbol{c} = [\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3}]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \boldsymbol{x_1} + c_2 d_2 \boldsymbol{x_2} + c_3 d_3 \boldsymbol{x_3}$$

Figure 9: 圖3 - (P3)

當解決微分方程和遞歸方程時的也會出現這一模式:

- 6節(p.201) Eigenvalues and Eigenvectors
- 6.4節(p.243) Systems of Differential Equations

$$
\frac{d\boldsymbol{u}(t)}{dt} = A\boldsymbol{u}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0
$$

$$
\boldsymbol{u}_{n+1} = A\boldsymbol{u}_n, \quad \boldsymbol{u_0} = \boldsymbol{u}_0
$$

在兩種問題中, 它的解都可以用$A$的特徵值$(\lambda_1, \lambda_2, \lambda_3)$、特徵向量$X = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3\end{bmatrix}$和系數$c = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}^{\mathrm{T}}$表示. 其中$C$是以$X$爲基底的初始值$\boldsymbol{u}(0) = \boldsymbol{u}_0$的坐標.

$$
\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3
$$

$$
\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = X^{-1}\boldsymbol{u}_0
$$

以上兩個問題的通解爲:

$$
\boldsymbol{u}(t) = e^{At}\boldsymbol{u}_0 = Xe^{\Lambda t}X^{-1}\boldsymbol{u_0} \qquad = Xe^{\Lambda t}\boldsymbol{c} = c_1e^{\lambda_1 t}\boldsymbol{x}_1 + c_2e^{\lambda_2 t}\boldsymbol{x}_2 + c_3e^{\lambda_3 t}\boldsymbol{x}_3
$$

$$
\boldsymbol{u}_n = A^n\boldsymbol{u}_0 = X\Lambda^n X^{-1}\boldsymbol{u_0} \qquad = X\Lambda^n\boldsymbol{c} = c_1\lambda_1^n\boldsymbol{x}_1 + c_2\lambda_2^n\boldsymbol{x}_2 + c_3\lambda_3^n\boldsymbol{x}_3
$$

見Figure9: 通過P3可以得到$XDc$.

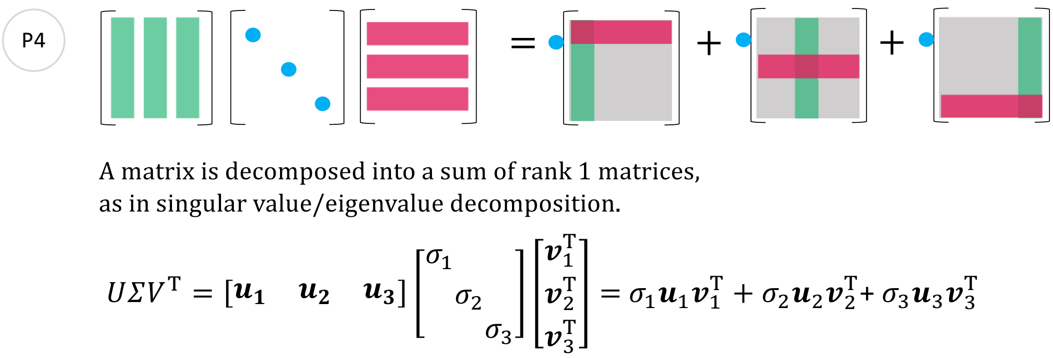



Figure 10: Pattern 4 - (P4)

P4在特徵值分解和特異值分解中都會用到. 兩種分解都可以表示爲三個矩陣之積, 其中中間的矩陣均爲對角矩陣. 且都可以表示爲帶特徵值/特異值系數的秩1矩陣之積.

更多細節將在下一節中討論.

# 6 矩陣的五種分解

- 前言p.vii, The Plan for the Book.

$A=CR, A=LU, A=QR, A=Q\Lambda Q^{\mathrm{T}}, A=U\Sigma V^{\mathrm{T}}$ 將一一說明.

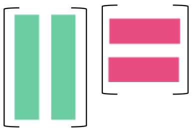
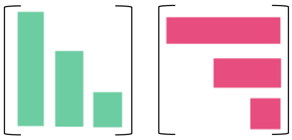
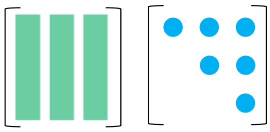
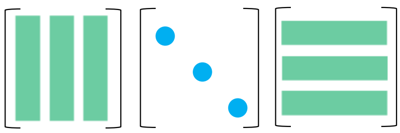
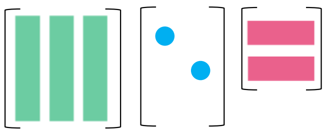
| | | |
|---|---|---|
| $\boldsymbol{A=CR}$ |  | $C$爲$A$的線性無關行<br>$R$爲$A$的列階梯形矩陣<br>可推知行秩$=$ 列秩 |
| $\boldsymbol{A=LU}$ |  | $LU$分解通過<br>高斯消去法<br>(下三角)(上三角) |
| $\boldsymbol{A=QR}$ |  | $QR$分解爲<br>格拉姆-施密特正交化中的<br>正交矩陣$Q$和三角矩陣$R$ |
| $\boldsymbol{S=Q\Lambda Q^{\mathrm{T}}}$ |  | 對稱矩陣$S$可以進形<br>特徵值分解<br>特徵向量組成$Q$, 特徵值組成$\Lambda$ |
| $\boldsymbol{A=U\Sigma V^{\mathrm{T}}}$ |  | 所有矩陣$A$的<br>奇異值分解<br>奇異值組成$\Sigma$ |

Table 1: 五種分解

## 6.1 $\boldsymbol{A = CR}$

- 1.4節Matrix Multiplication and $\boldsymbol{A = CR}$ (p.29)

所有一般的長矩陣$A$都有相同的列秩和行秩. 這個分解是理解這一定理最直觀的方法. $C$由$A$的線性無關行組成, $R$爲$A$的列階梯形矩陣(消除了零列). $A = CR$將$A$化簡爲$r$的線性無關行$C$和線性無關列$R$的乘積.

$$A = CR$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

推導過程: 從左往右看$A$的行. 保留其中線性無關的行, 去掉可以由前者線性表出的行. 則第1、2行被保留, 而第三行因爲可以由前兩行之和表示而被去掉. 而要通過線性無關的1、2兩行重新構造出$A$, 需要右乘一個列階梯矩陣$R$.

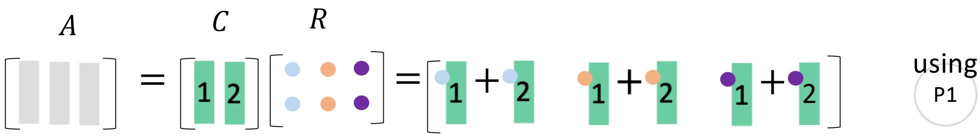



Figure 11: $CR$中行的秩

現在你會發現行的秩爲2, 因爲$C$中只有2個線性無關行. 而$A$中所有的行都可以由$C$中的2行線性表出.

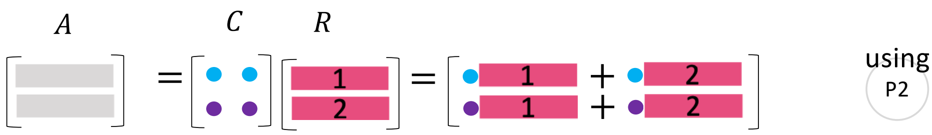



Figure 12: $CR$中列的秩

同樣, 列秩也爲2, 因爲$R$中只有2個線性無關列, 且$A$中所有的列都可以由$R$中的2列線性表出.

## 6.2 $\boldsymbol{A = LU}$

用高斯消除法求解$A\boldsymbol{x} = \boldsymbol{b}$也被稱爲$LU$分解. 通常, 是$A$左乘一個初等列變換矩陣$(E)$來得到一個上三角矩陣$U$.

$$\begin{aligned} EA &= U \\ A &= E^{-1}U \\ \text{let } L = E^{-1}, \quad A &= LU \end{aligned}$$

現在, 求解$A\boldsymbol{x} = \boldsymbol{b}$有2步: (1)求解$L\boldsymbol{c} = \boldsymbol{b}$, (2)代回$U\boldsymbol{x} = \boldsymbol{c}$.

- 2.3節(p.57) Matrix Computations and $\boldsymbol{A = LU}$

在這里, 我們直接通過$A$計算$L$和$U$.

$$A = \begin{bmatrix} | \\ \boldsymbol{l}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_1^* - \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & A_2 \end{bmatrix} = \begin{bmatrix} | \\ \boldsymbol{l}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_1^* - \end{bmatrix} + \begin{bmatrix} | \\ \boldsymbol{l}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{u}_2^* - \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix} = LU$$

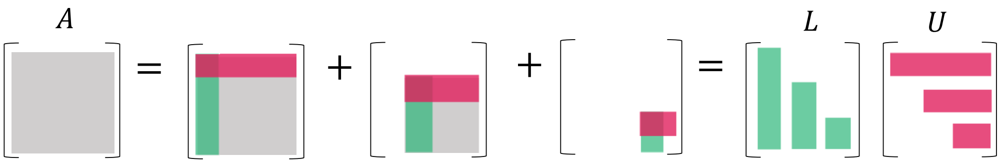



Figure 13: $A$的遞歸秩1矩陣分離

要計算$L$和$U$, 首先分離出由$A$的第一列和第一行組成的外積. 余下的部分爲$A_2$. 遞歸執列此操作, 將$A$分解爲秩1矩陣之和.

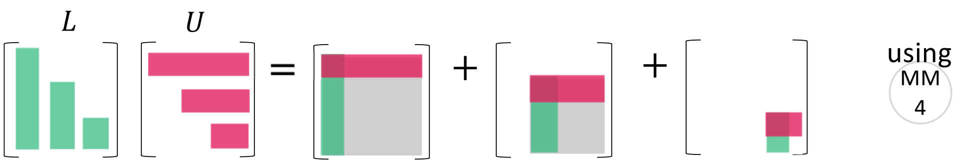



Figure 14: 由$LU$重新構造$A$

由$L$乘以$U$來重新構造$A$則相對簡單.

## 6.3 $A = QR$

$A = QR$是在保持$\boldsymbol{C}(A) = \boldsymbol{C}(Q)$的條件下, 將$A$轉化爲正交矩陣$Q$.

- 4.4節Orthogonal matrices and Gram-Schmidt (p.165)

在格拉姆-施密特正交化中, 首先, 單位化的$\boldsymbol{a}_1$被用作$\boldsymbol{q}_1$, 然後求出$\boldsymbol{a}_2$與$\boldsymbol{q}_1$正交所得到的$\boldsymbol{q}_2$, 以此類推.

$$
\begin{aligned}
&\boldsymbol{q}_1 = \boldsymbol{a}_1/||\boldsymbol{a}_1|| \\
&\boldsymbol{q}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_2)\boldsymbol{q}_1, \quad \boldsymbol{q}_2 = \boldsymbol{q}_2/||\boldsymbol{q}_2|| \\
&\boldsymbol{q}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_2, \quad \boldsymbol{q}_3 = \boldsymbol{q}_3/||\boldsymbol{q}_3||
\end{aligned}
$$

或者你也可以寫作$r_{ij} = \boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{a}_j$:

$$
\begin{aligned}
\boldsymbol{a}_1 &= r_{11}\boldsymbol{q}_1 \\
\boldsymbol{a}_2 &= r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2 \\
\boldsymbol{a}_3 &= r_{13}\boldsymbol{q}_1 + r_{23}\boldsymbol{q}_2 + r_{33}\boldsymbol{q}_3
\end{aligned}
$$

原本的$A$就可以表示爲$QR$: 正交矩陣乘以上三角矩陣.

$$
A = \begin{bmatrix} | & | & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix} = QR
$$

$$
QQ^{\mathrm{T}} = Q^{\mathrm{T}}Q = I
$$

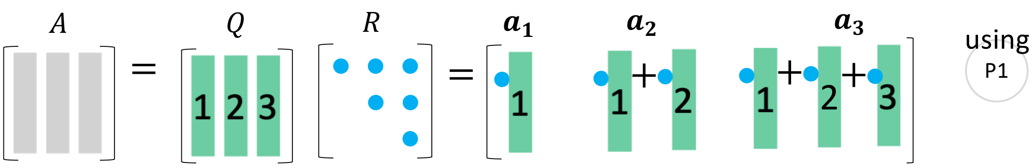



Figure 15: $A = QR$

$A$的行向量就可以轉化爲一個正交集合: $Q$的行向量. $A$的每一個行向量都可以用$Q$和上三角矩陣$R$重新構造出.

圖釋可以回頭看P1.

## 6.4 $\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$

所有對稱矩陣$S$都必須有實特徵值和正交特徵向量. 特徵值是$\Lambda$的對角元素, 特徵向量在$Q$中.

- 6.3節(p.227) Symmetric Positive Definite Matrices

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} | & | & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_1^{\mathrm{T}}- \\ -\boldsymbol{q}_2^{\mathrm{T}}- \\ -\boldsymbol{q}_3^{\mathrm{T}}- \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} | \\ \boldsymbol{q}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_1^{\mathrm{T}}- \end{bmatrix} + \lambda_2 \begin{bmatrix} | \\ \boldsymbol{q}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_2^{\mathrm{T}}- \end{bmatrix} + \lambda_3 \begin{bmatrix} | \\ \boldsymbol{q}_3 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{q}_3^{\mathrm{T}}- \end{bmatrix}$$
$$= \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$

$$P_1 = \boldsymbol{q}_1\boldsymbol{q}_1^{\mathrm{T}}, \quad P_2 = \boldsymbol{q}_2\boldsymbol{q}_2^{\mathrm{T}}, \quad P_3 = \boldsymbol{q}_3\boldsymbol{q}_3^{\mathrm{T}}$$

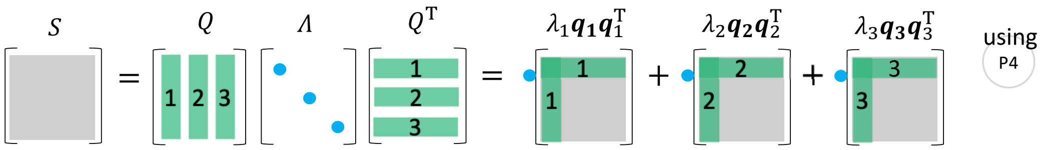



Figure 16: $S = Q\Lambda Q^{\mathrm{T}}$

一個對稱矩陣$S$通過一個正交矩陣$Q$和它的轉置矩陣, 對角化爲$\Lambda$. 然後被分解爲一階投影矩陣$P = qq^{\mathrm{T}}$的組合. 這就是譜定理.

注意, 這里的分解用到了P4.

$$S = S^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$
$$QQ^{\mathrm{T}} = P_1 + P_2 + P_3 = I$$
$$P_1P_2 = P_2P_3 = P_3P_1 = O$$
$$P_1^2 = P_1 = P_1^{\mathrm{T}}, \quad P_2^2 = P_2 = P_2^{\mathrm{T}}, \quad P_3^2 = P_3 = P_3^{\mathrm{T}}$$

## 6.5 $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$

- 7.1節(p.259) Singular Values and Singular Vectors

包括長方陣在內的所有矩陣都具有奇異值分解(SVD). $A = U\Sigma V^{\mathrm{T}}$中, 有$A$的奇異向量$U$和$V$. 奇異值則排列在$\Sigma$的對角線上. 下圖就是"簡化版"的SVD.

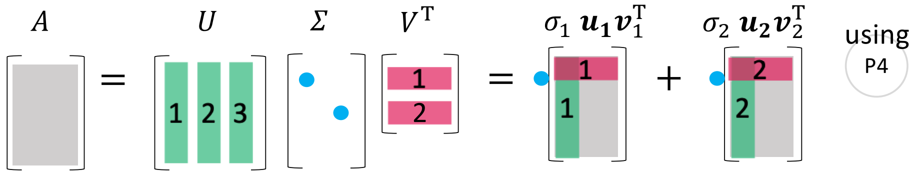



Figure 17: $A = U\Sigma V^{\mathrm{T}}$

你可以發現, $V$是$\mathbb{R}^n$ ($A^{\mathrm{T}}A$的特徵向量) 的標準正交基, 而$U$是$\mathbb{R}^m$ ($AA^{\mathrm{T}}$的特徵向量) 的標準正交基. 它們共同將$A$對角化爲$\Sigma$. 這也可以表示爲秩1矩陣的線性組合.

$$
\begin{aligned}
A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} | & | & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \\ & \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_1^{\mathrm{T}}- \\ -\boldsymbol{v}_2^{\mathrm{T}}- \end{bmatrix} &= \sigma_1 \begin{bmatrix} | \\ \boldsymbol{u}_1 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_1^{\mathrm{T}}- \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \boldsymbol{u}_2 \\ | \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}_2^{\mathrm{T}}- \end{bmatrix} \\
&= \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}
\end{aligned}
$$

注意:

$$
\begin{aligned}
UU^{\mathrm{T}} &= I_m \\
VV^{\mathrm{T}} &= I_n
\end{aligned}
$$

圖釋見P4.

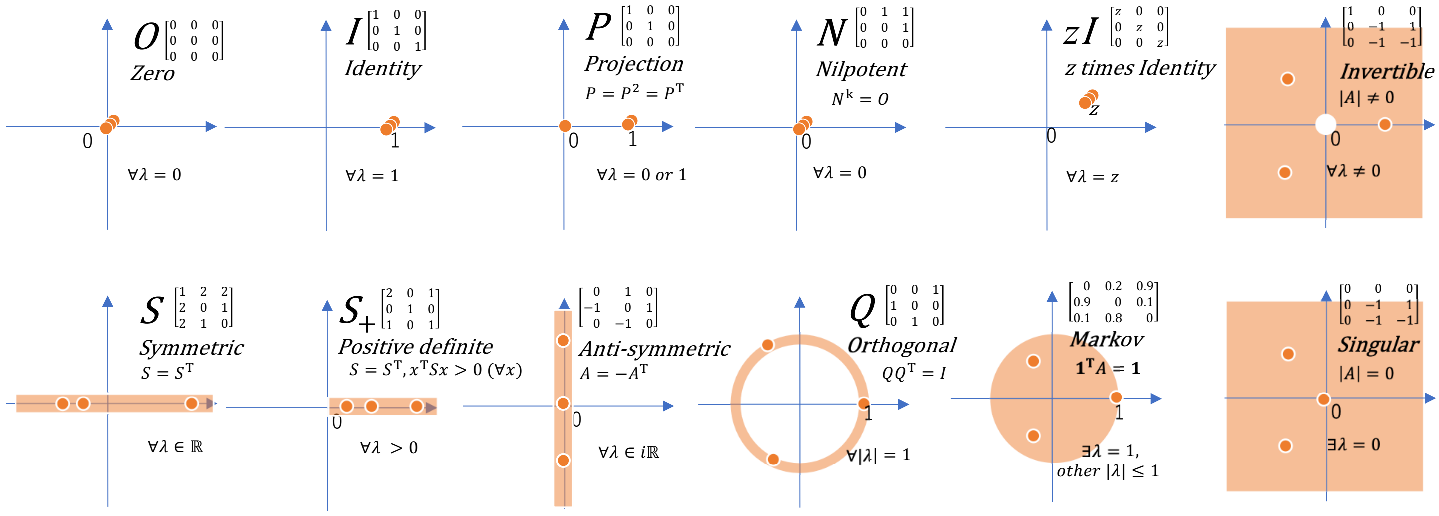



Figure 18: 特徵值圖

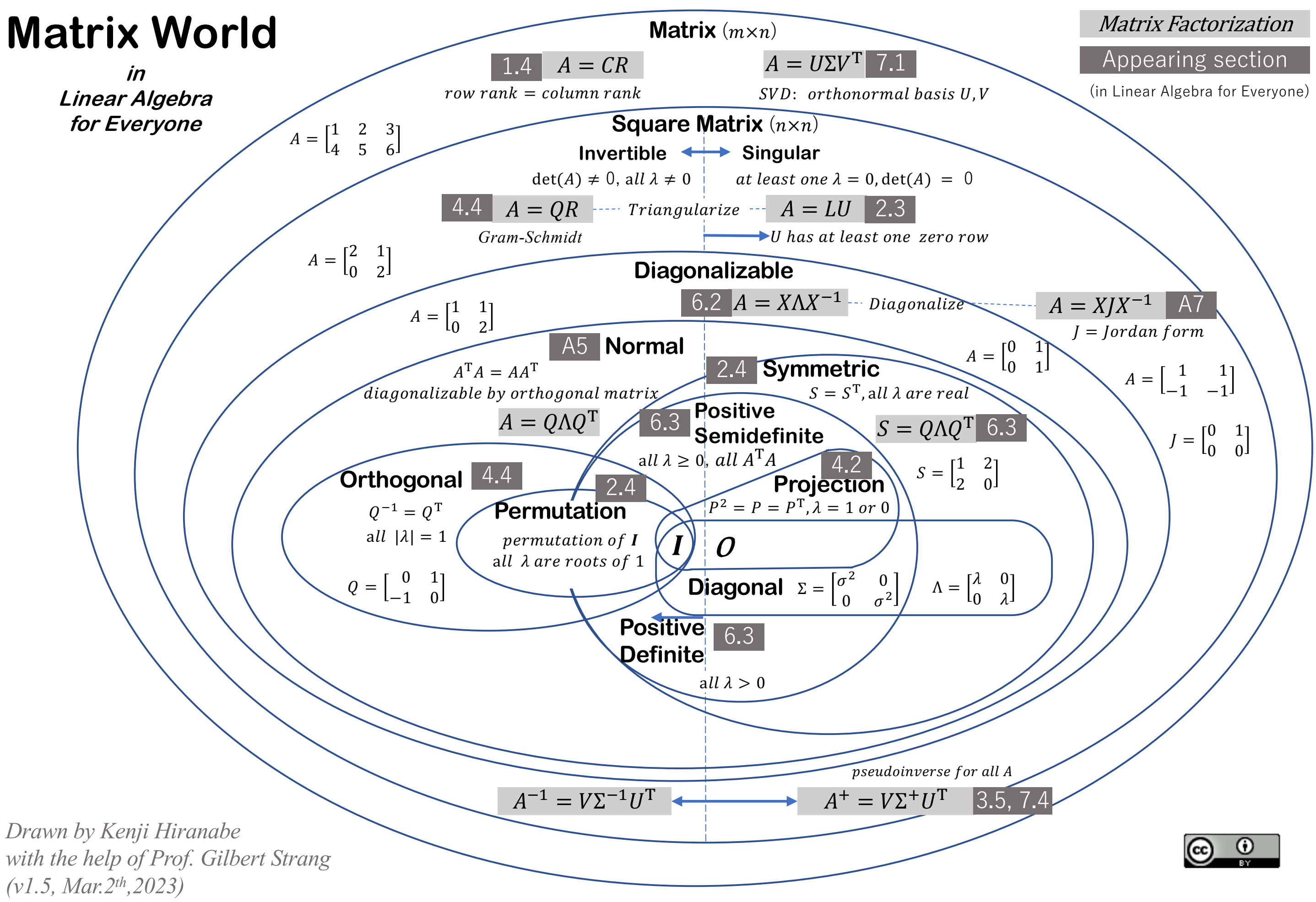



Figure 19: 矩陣世界

# 總結和致謝

我展示了矩陣/向量乘法的系統可視化與它們在五種矩陣分解中的應用. 我希望你能夠喜歡它們、通過它們加深對線性代數的理解.

Ashley Fernandes 在排版時幫我美化了這篇論文, 使它更加一致和專業.

在結束這篇論文之前, 我要感謝Gilbert Strang 教授出版了《Linear Algebra for Everyone》一書. 它引導我們通過新的視角去了解線性代數中這些美麗的風景. 其中介紹了當代和傳統的數據科學和機器學習, 每個人都可以通過實用的方式對它的基本思想進形基本理解. 矩陣世界的重要組成部分.

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