math

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Part I

math languages

Chapter 1

LaTeX

1.1 coloring

1.1.1 single coloring

```
\def\zl{ {\color{blue} z_{\scriptscriptstyle l}} }
```

also can be put into "preamble"

$$0 = \frac{\partial}{\partial {\color{blue} z_{\scriptscriptstyle l}}} \left(\|h(z_{l-1}) \cdot w_l - {\color{blue} z_{\scriptscriptstyle l}}\| + \lambda \|h({\color{blue} z_{\scriptscriptstyle l}}) \cdot w_{l+1} - z_{l+1}\| \right)$$

1.1.2 recolor = coloring with regular expression (= RegEx = re)

LATEX3

https://tex.stackexchange.com/questions/83101/option-clash-for-package-xcolor

Now, the problem was that another package (pgfplots, in this case) had already loaded the xcolor package without options, so loading it after pgfplots with the table option produces the clash. One way to prevent the problem was already presented (using table as class option); another solution is to load xcolor with the table option before pgfplots

```
\usepackage{expl3,xparse}
\usepackage[dvipsnames]{xcolor}

\ExplSyntaxOn
\NewDocumentCommand{\recolor}{m}
{
    \tl_set:Nn \l_tmpa_tl { #1 }
    \regex_replace_all:nnN { 2 } { \c{color}{red}{2} } \l_tmpa_tl
    \tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff
```

$$c^{\color{red}2} = a^{\color{red}2} + b^{\color{red}2}$$

```
\ExplSyntaxOn
\RenewDocumentCommand{\recolor}{m}
{
    \tl_set:Nn \l_tmpa_tl { #1 }

        % e, \rho^2
    \regex_replace_all:nnN { \be\b } { {\c{color}{red}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{rho}\^\{2\} } { {\c{color}{Green}{\0}}
} \l_tmpa_tl

        % rho
        %% \rho_\d
    \regex_replace_all:nnN { \c{rho}_{{\c{scriptscriptstyle} 0}} }
{ {\c{color}{red}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{rho}_{{\c{scriptscriptstyle} 1}} }
{ {\c{color}{blue}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{rho}_{{\c{scriptscriptstyle} 2}} }
{ {\c{color}{Green}{\0}}
} \l_tmpa_tl

        %% \d_\rho
    \regex_replace_all:nnN { 0_{{\c{scriptscriptstyle} \c{rho}}} }
{ {\c{color}{red}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { 1_{{\c{scriptscriptstyle} \c{rho}}} }
{ {\c{color}{blue}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { 2_{{\c{scriptscriptstyle} \c{rho}}} }
```

```
{ {\c{color}{Green}{\0}}
} \l_tmpa_tl

        % pi
        %% \pi_\d
    \regex_replace_all:nnN { \c{pi}_{{\c{scriptscriptstyle} 0}} }
{ {\c{color}{magenta}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{pi}_{{\c{scriptscriptstyle} 1}} }
{ {\c{color}{cyan}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{pi}_{{\c{scriptscriptstyle} 2}} }
{ {\c{color}{orange}{\0}}
} \l_tmpa_tl

        %% \d_\pi
    \regex_replace_all:nnN { 0_{{\c{scriptscriptstyle} \c{pi}}} }
{ {\c{color}{magenta}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { 1_{{\c{scriptscriptstyle} \c{pi}}} }
{ {\c{color}{cyan}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { 2_{{\c{scriptscriptstyle} \c{pi}}} }
{ {\c{color}{orange}{\0}}
} \l_tmpa_tl

        % \d{3}
        %% \[\d{3}\]
    \regex_replace_all:nnN { \c{left}\[(123)\c{right}\] }
{ \c{left}\[{\c{color}{red}{\1}}\c{right}\]
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\[(231)\c{right}\] }
{ \c{left}\[{\c{color}{blue}{\1}}\c{right}\]
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\[(312)\c{right}\] }
{ \c{left}\[{\c{color}{Green}{\1}}\c{right}\]
} \l_tmpa_tl
    \regex_replace_all:nnN { \c{left}\[(213)\c{right}\] }
{ \c{left}\[{\c{color}{magenta}{\1}}\c{right}\]
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\[(132)\c{right}\] }
{ \c{left}\[{\c{color}{cyan}{\1}}\c{right}\]
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\[(321)\c{right}\] }
{ \c{left}\[{\c{color}{orange}{\1}}\c{right}\]
} \l_tmpa_tl

        %% \(\d{3}\)
    \regex_replace_all:nnN { \c{left}\(\c{right}\) }
{ {\c{color}{red}{\0}}
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\((123)\c{right}\) }
{ \c{left}\({\c{color}{blue}{\1}}\c{right}\)
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\((132)\c{right}\) }
{ \c{left}\({\c{color}{Green}{\1}}\c{right}\)
} \l_tmpa_tl
    \regex_replace_all:nnN { \c{left}\((12)\c{right}\) }
{ \c{left}\({\c{color}{magenta}{\1}}\c{right}\)
} \l_tmpa_tl
        \regex_replace_all:nnN { \c{left}\((23)\c{right}\) }
{ \c{left}\({\c{color}{cyan}{\1}}\c{right}\)
} \l_tmpa_tl
```

```
        \regex_replace_all:nnN { \c{left}\((31)\c{right}\) }
{ \c{left}\({\c{color}{orange}{\1}}\c{right}\)
} \l_tmpa_tl

    \tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff
```

$\cdot_{D_3}$	ρ_0	ρ_1	ρ_2	π_0	π_1	π_2	$\cdot_{S_3}$	[123]	[231]	[312]	[213]	[132]	[321]
ρ_0	ρ_0	ρ_1	ρ_2	π_0	π_1	π_2	[123]	[123]	[231]	[312]	[213]	[132]	[321]
ρ_1	ρ_1	ρ_2	ρ_0	π_1	π_2	π_0	[231]	[231]	[312]	[123]	[132]	[321]	[213]
ρ_2	ρ_2	ρ_0	ρ_1	π_2	π_0	π_1	[312]	[312]	[123]	[231]	[321]	[213]	[132]
π_0	π_0	π_2	π_1	ρ_0	ρ_2	ρ_1	[213]	[213]	[321]	[132]	[123]	[312]	[231]
π_1	π_1	π_0	π_2	ρ_1	ρ_0	ρ_2	[132]	[132]	[213]	[321]	[231]	[123]	[312]
π_2	π_2	π_1	π_0	ρ_2	ρ_1	ρ_0	[321]	[321]	[132]	[213]	[312]	[231]	[123]
$\cdot_{S_3}$	e	(123)	(132)	(3) (12)	(1) (23)	(2) (31)	$\cdot_{S_3}$	()	(123)	(132)	(12)	(23)	(31)
e	e	(123)	(132)	(3) (12)	(1) (23)	(2) (31)	()	()	(123)	(132)	(12)	(23)	(31)
(123)	(123)	(132)	e	(1) (23)	(2) (31)	(3) (12)	(123)	(123)	(132)	()	(23)	(31)	(12)
(132)	(132)	e	(123)	(2) (31)	(3) (12)	(1) (23)	(132)	(132)	()	(123)	(31)	(12)	(23)
(3) (12)	(3) (12)	(2) (31)	(1) (23)	e	(132)	(123)	(12)	(12)	(31)	(23)	()	(132)	(123)
(1) (23)	(1) (23)	(3) (12)	(2) (31)	(123)	e	(132)	(23)	(23)	(12)	(31)	(123)	()	(132)
(2) (31)	(2) (31)	(1) (23)	(3) (12)	(132)	(123)	e	(31)	(31)	(23)	(12)	(132)	(123)	()

Chapter 2

LyX

2.1 LyX Chinese environment

https://latexlyx.blogspot.com/2012/06/lyx.html

2014年09月21日 晚上10:58

匿名：Language 那邊改成 Chinese Traditional 之後, Definition 就變成定義, Example 就變成範例, 有沒有辦法維持他們是英文的?

2014年09月22日 上午11:23

Mingyi Wu：這個是 LyX 的特性之一。UI 的語言設定，與編輯區的語言是分開的。 就算 UI 設定為 English, 如果檔案語言設定為 Chinese, 那麼編輯區出現的一些如 Chapter, Section, Definition 等名稱，會自動變成中文。也就是說檔案的語言設定值，會影響 LyX 文字編輯區內呈現的語言。 若使用數學模組或一些數學論文 document class 的時候，甚至連輸出的檔案內容都會根據語言設定而變。(也就是 Definition 變成 定義)

所以您說的狀況，可能有2種情況：

1. Definition 在 LyX 編輯區內變成中文，但輸出檔案時檔案還是出現 Definition 這個只是編輯區呈現的問題，沒辦法只改一部份。如果真的希望檔案設定成中文，但所有介面看起來都要是英文的環境，您可以直接刪掉中文翻譯檔，這樣所有介面都會變成英文的。 以我的環境，繁體中文的翻譯檔路徑在(for Windows): C:\Program Files (x86)\LyX 2.1\Resources\locale\zh_TW\LC_MESSAGES\LyX2.1.mo 把這個檔名改掉，這樣 LyX 就找不到中文翻譯檔，都會以預設的英文呈現。

2. 如果您的問題是輸出的檔案會出現中文的「定義」問題，不管介面顯示。這個問題跟另外一個檔案有關, C:\Program Files (x86)\LyX 2.1\Resources\layouttranslations 您可以用任何文字編輯器開啓此檔，找到 Translation zh_TW 這行以下的設定改成您喜歡的，或是直接把這個檔名改掉或刪掉檔案，這樣輸出檔案也不會自動翻譯了。

https://latexlyx.blogspot.com/2013/06/lyx-2.html

2.2 child document

https://wiki.lyx.org/FAQ/Multidoc[1]
[2]

Bibliography

[1] *LyX wiki — FAQ / Multidoc*. URL: https://wiki.lyx.org/FAQ/Multidoc (visited on 06/13/2024) (cit. on p. 22).

[2] 浩 結城. 數學女孩：伽羅瓦理論. 數學女孩. 世茂, Sept. 2, 2014. URL: https://www.books.com.tw/products/0010647846 (visited on 06/09/2024) (cit. on p. 22).

2.3 mutliple bibliographies

2.3.1 BibLaTeX

https://wiki.lyx.org/BibTeX/Biblatex
https://wiki.lyx.org/BibTeX/Tips#secbib[1]
https://tex.stackexchange.com/questions/606503/how-to-modify-backref-format-in-biblatex

```
\documentclass{scrbook}
\usepackage[natbib=true,backref=true,style=ieee]{biblatex}
\usepackage{hyperref}
\addbibresource{sample.bib}
\DefineBibliographyStrings{english}{
  % backrefpage={see p.},
  backrefpage={},
  % backrefpages={see pp.}
  backrefpages={}
}
```

https://tex.stackexchange.com/questions/36307/formatting-back-references-in-bibliography

```
\usepackage[backref=true]{biblatex}

\usepackage{hyperref}

\DefineBibliographyStrings{english}{%
  backrefpage = {page},% originally "cited on page"
  backrefpages = {pages},% originally "cited on pages"
}
```

Bibliography

[1] *LyX wiki — BibTeX / Tips*. URL: https://wiki.lyx.org/BibTeX/Tips#secbib (visited on 06/13/2024) (cit. on p. 24).

2.4 LyZ: linking Zotero and LyX

https://forums.zotero.org/discussion/78442/connecting-zotero-and-lyx
https://github.com/wshanks/lyz/releases

2.5 list of theorems module

https://tex.stackexchange.com/questions/672794/list-of-theorems-not-working-in-lyx
https://github.com/Udi-Fogiel/LyX-thmtools

2.5.1 list of equations

https://tex.stackexchange.com/questions/173102/table-of-equations-like-list-of-figures
https://stackoverflow.com/questions/61517319/vertical-spacing-adjustment-between-different-chapters-labels-in-the-list-of-eq

2.6 multicolumn

Help ▷ Additional Features

4.1.3 Multiple Columns

This module uses the multicol package and is independent of the option Two-column document in the Document ▷ Settings ▷ Text Layout dialog.

If you want to have two columns in your text, insert a multicolumn inset via the menu Insert ▷ Custom Insets ▷ Multiple Columns where the columns should start. Write all text that should be printed in 2 columns into this inset.

Here is an example:

The Adventure of the Empty House
by Sir Arthur Conan Doyle

It was in the spring of the year 1894 that all London was interested, and the fashionable world dismayed, by the murder of the Honourable Ronald Adair under most unusual and inexplicable circumstances. The public has already learned those particulars of the crime which came out in the police investigation, but a good deal was suppressed upon that occasion, since the case for the prosecution was so overwhelmingly strong that it was not necessary to bring forward all the facts. Only now, at the end of nearly ten years, am I allowed to supply those missing links which make up the whole of that remarkable chain. The crime was of interest in itself, but that interest was as nothing to me compared to the inconceivable sequel, which afforded me the greatest shock and surprise of any event in my adventurous life. Even now, after this long interval, I find myself thrilling as I think of it, and feeling once more that sudden flood of joy, amazement, and incredulity which utterly submerged my mind. Let me say to that public, which has shown some interest in those glimpses which I have occasionally given them of the thoughts and actions of a very remarkable man, that they are not to blame me if I have not shared my knowledge with them, for I should have considered it my first duty to do so, had I not been barred by a positive prohibition from his own lips, which was only withdrawn upon the third of last month.

To get 3 or more columns, set the cursor into the multicolumn inset and use the menu Insert ▷ Number of Columns. The number of the desired columns is written into that inset (for 3 columns write "3").
Here is an example with 3 columns:

It can be imagined that my close intimacy with Sherlock Holmes had interested me deeply in crime, and that after his disappearance I never failed to read with care the various problems which came before the public. And I even attempted, more than once, for my own private satisfaction, to employ his methods in their solution, though with indifferent success. There was none, however, which appealed to me like this tragedy of Ronald Adair. As I read the evidence at the inquest, which led up to a verdict of willful murder against some person or persons unknown, I realized more clearly than I had ever done the loss which the community had sustained by the death of Sherlock Holmes. There were points about this strange business which would, I was sure, have specially appealed to him, and the efforts of the police would have been supplemented, or more probably anticipated, by the trained observation and the alert mind of the first criminal agent in Europe. All day, as I drove upon my round, I turned over the case in my mind and found no explanation which appeared to me to be adequate. At the risk of telling a twice-told tale, I will recapitulate the facts as they were known to the public at the conclusion of the inquest.

You can have up to 10 columns if you want to, but that might not be very pleasant for the readers of your document.

You can also have columns inside columns:

The Honourable Ronald Adair was the second son of the Earl of Maynooth, at that time governor of one of the Australian colonies. Adair's mother had returned from Australia to undergo the operation for cataract, and she, her son Ronald, and her daughter Hilda were living together at 427 Park Lane.

The youth moved in the best society – had, so far as was known, no enemies and no particular vices. He had been engaged to Miss Edith Woodley, of Carstairs, but the engagement had been broken off by mutual consent some months before, and there was no sign that it had left any very profound feeling behind it. For the rest {sic} the man's life moved in a narrow and conventional circle, for his habits were quiet and his nature unemotional. Yet it was upon this easy-going young aristocrat that death came, in most strange and unexpected form, between the hours of ten and eleven-twenty on the night of March 30, 1894.

Ronald Adair was fond of cards – playing continually, but never for such stakes as would hurt him. He was a member of the Baldwin, the Cavendish, and the Bagatelle card clubs. It was shown that, after dinner on the day of his death, he had played a rubber of whist at the latter club. He had also played there in the afternoon. The evidence of those who had played with him – Mr. Murray, Sir John Hardy, and Colonel Moran – showed that the game was whist, and that there was a fairly equal fall of the cards. Adair might have lost five pounds, but not more. His fortune was a considerable one, and such a loss could not in any way affect him. He had played nearly every day at one club or other, but he was a cautious player, and usually rose a winner. It came out in evidence that, in partnership with Colonel Moran, he had actually won as much as four hundred and twenty pounds in a sitting, some weeks before, from Godfrey Milner and Lord Balmoral. So much for his recent history as it came out at the inquest.

2.6.1 Column Breaks

A column break can be forced by inserting the command \columnbreak{} as TEX Code to that position in the text where the column should be broken. Note that this leads in most cases to whitespace in the text.
Here is an example:

"You're surprised to see me, sir," said he, in a strange, croaking voice.

I acknowledged that I was.

"Well, I've a conscience, sir, and when I chanced to see you go into this house, as I came hobbling after you, I thought to myself, I'll just step in and see that kind gentleman, and tell him that if I was a bit gruff in my manner there was not any harm meant, and that I am much obliged to him for picking up my books."

"You make too much of a trifle," said I. "May I ask how you knew who I was?" AFTER THIS SENTENCE THE COLUMN BREAK IS FORCED.

"Well, sir, if it isn't too great a liberty, I am a neighbour of yours, for you'll find my little bookshop at the corner of Church Street, and very happy to see you, I am sure. Maybe you collect yourself, sir. Here's BRITISH BIRDS, and CATULLUS, and THE HOLY WAR – a bargain, every one of them. With five volumes you could just fill that gap on that second shelf. It looks untidy, does it not, sir?"

2.6.2 Column Separation

The width of the columns is automatically calculated, but you can modify the space between the columns. This is done by changing the length \columnsep. Its predefined value is 10 pt. Here is an example where \columnsep is set to 3 cm:

My observations of No. 427 Park Lane did little to clear up the problem in which I was interested. The house was separated from the street by a low wall and railing, the whole not more than five feet high. It was perfectly easy, therefore, for anyone to get into the garden, but the window was entirely inaccessible, since there was no water pipe or anything which could help the most active man to climb it. More puzzled than ever, I retraced my steps to Kensington. I had not been in my study five minutes when the maid entered to say that a person desired to see me. To my astonishment it was none other than my strange old book collector, his sharp, wizened face peering out from a frame of white hair, and his precious volumes, a dozen of them at least, wedged under his right arm.

2.6.3 Vertical Lines

Between the columns a rule with a width of the length \columnseprule is placed. If this rule width is set to 0 pt (this is the default), the rule is suppressed. In the following example the rule is 2 pt wide:

"You're surprised to see me, sir," said he, in a strange, croaking voice.

I acknowledged that I was.

"Well, I've a conscience, sir, and when I chanced to see you go into this house, as I came hobbling after you, I thought to myself, I'll just step in and see that kind gentleman, and tell him that if I was a bit gruff in my manner there was not any harm meant, and that I am much obliged to him for picking up my books."

"You make too much of a trifle," said I. "May I ask how you knew who I was?"

"Well, sir, if it isn't too great a liberty, I am a neighbour of yours, for you'll find my little bookshop at the corner of Church Street, and very happy to see you, I am sure. Maybe you collect yourself, sir. Here's BRITISH BIRDS, and CATULLUS, and THE HOLY WAR – a bargain, every one of them. With five volumes you could just fill that gap on that second shelf. It looks untidy, does it not, sir?"

The rule can be colored by redefining the command \columnseprulecolor. This is done by inserting the command

```
\renewcommand{\columnseprulecolor}{\color{red}}
```

as TEX Code before the multicolumn inset. Replace red in this command by a color of your choice. You can use all pre- and self-defined colors. See the *EmbeddedObjects* manual, section *Colored Tables* for more information about pre- and self-defined colors. To go back to the default color insert the command

```
\renewcommand{\columnseprulecolor}{\normalcolor}
```

Here is the example with a cyan rule and 1 cm column separation:

"You're surprised to see me, sir," said he, in a strange, croaking voice.

I acknowledged that I was.

"Well, I've a conscience, sir, and when I chanced to see you go into this house, as I came hobbling after you, I thought to myself, I'll just step in and see that kind gentleman, and tell him that if I was a bit gruff in my manner there was not any harm meant, and that I am much obliged to him for picking up my books."

"You make too much of a trifle," said I. "May I ask how you knew who I was?"

"Well, sir, if it isn't too great a liberty, I am a neighbour of yours, for you'll find my little bookshop at the corner of Church Street, and very happy to see you, I am sure. Maybe you collect yourself, sir. Here's BRITISH BIRDS, and CATULLUS, and THE HOLY WAR – a bargain, every one of them. With five volumes you could just fill that gap on that second shelf. It looks untidy, does it not, sir?"

Chapter 3

plot

3.1 TikZ

3.1.1 Ti*k*Z-CD = tikz-cd: commutative diagram

```
\usepackage{tikz}
\usepackage{pgfplots}

\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}[column sep=2.75cm, %small,large,huge
                       cells={nodes={draw}}
                      ]
00
\ar[r,"\backslash \text{ar[r]}"]
\ar[d,"\backslash \text{ar[d]}"]
&
01
\ar[r,"\text{[,"swap"']}"']
&
02
\ar[r,"\backslash \text{ar[r]}","\text{[,"swap"']}"']
&
03
\\
10
\ar[d,"\text{[,"swap"']}"']
&
11
\ar[u,"\backslash \text{ar[u]}"]
\ar[l,"\backslash \text{ar[l]}"]
\ar[r,-stealth,"\text{[,-}\text{stealth}\text{]}"]
\ar[d,-{Stealth[reversed]},"\text{[,-}\{\text{Stealth[reversed]}\}\text{]}"]
&
12
\ar[r,-{Stealth[open]},"\text{[,-}\{\text{Stealth[open]}\}\text{]}"]
&
13
\\
20
\ar[r,"\text{[,"r" description]}" description]
\ar[d,"\backslash \text{ar[d]}","\text{[,"swap"']}"']
&
21
\ar[r,-{Stealth[harpoon]},"\text{[,-}\{\text{Stealth[harpoon]}\}\text{]}"]
&
22
\ar[u,shift right=1.75pt,"\text{[,shift right=1.75pt]}"']
\ar[lld,-Stealth,"\backslash \text{ar[lld]}" description]
\ar[r,latex-latex,"\text{[,latex-latex]}"]
\ar[d,shift right=1.75pt,"\text{[,shift right=1.75pt]}"]
&
23
\\
30
\ar[ru,"\backslash \text{ar[ru]}" description]
\ar[r,bend right,-stealth,"\text{bend right}"]
\ar[r,bend right=42,-stealth,"\text{bend right=42}"']
\ar[r,bend right=100,-stealth,"\text{bend right=100}"']
\ar[dd,bend right,-stealth,"\text{[,bend right]}"']
&
31
\ar[r,bend left,stealth-stealth,"\text{bend left}"']
\ar[ddr]
&
32
\ar[l,-{Stealth[harpoon]},"\text{[,-}\{\text{Stealth[harpoon]}\}\text{]}"]
```

```
\ar[r,-{Stealth[harpoon]},shift left=.75pt,"\text{[,shift left=.75pt]}"]
\ar[ddl,crossing over,"\text{[,crossing over]; rounded corneres, to path}"]
\ar[ddr,
    rounded corners,
    to path={--([yshift=-2ex]\tikztostart.south)
             --([yshift=-2ex,xshift=+2ex]\tikztostart.south)
             --([yshift=-2ex,xshift=+8ex]\tikztostart.south)
             --([xshift=-12ex]\tikztotarget.west)
             --(\tikztotarget)
            },
    ]
&
33
\ar[l,-{Stealth[harpoon]},shift left=.75pt,"\text{[,shift left=.75pt]}"]
\\
&
&
&
\\
50
\ar[r,-|,"\text{[,}-|\text{,swap]}",swap]
&
51
\ar[r,-stealth,red,text=black,"|\text{[draw=none]}|" description]
&
|[draw=none]|52
&
53
\end{tikzcd}
```

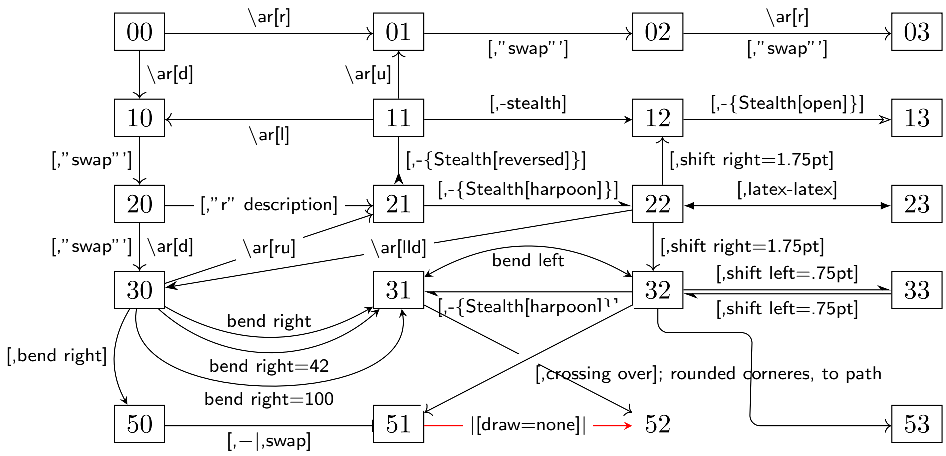

Figure 3.1: learn TikZ-CD = tikz-cd in one picture 2

3.1.2 PGFplots

Chapter 4

LEAN = L∃∀N

4.1 MathLib

4.1.1 MathLib undergraduate

https://leanprover-community.github.io/undergrad.html
2024/06/16

Undergraduate mathematics in mathlib

This gives pointers to undergraduate maths topics that are currently covered in mathlib. The list is gathered from the French curriculum. There is also a page listing undergraduate maths topics that are not yet in mathlib.

To update this list, please submit a PR modifying docs/undergrad.yaml in the mathlib4 repository.

Linear algebra

Fundamentals vector space, product of vector spaces, vector subspace, quotient space, sum of subspaces, direct sum, complementary subspaces, linear independence, generating sets, bases, existence of bases, linear map, range of a linear map, kernel of a linear map, algebra of endomorphisms of a vector space, general linear group.

Duality dual vector space, dual basis, transpose of a linear map.

Finite-dimensional vector spaces finite-dimensionality, isomorphism with K^n, rank of a linear map, rank of a set of vectors, isomorphism with bidual.

Multilinearity multilinear map, determinant of vectors, determinant of endomorphisms, orientation of a $\mathbb{R}$-vector space.

Matrices commutative-ring-valued matrices, field-valued matrices, matrix representation of a linear map, change of basis, rank of a matrix, determinant, invertibility.

Endomorphism polynomials annihilating polynomials, minimal polynomial, characteristic polynomial, Cayley-Hamilton theorem.

Structure theory of endomorphisms eigenvalue, eigenvector, generalized eigenspaces, Jordan-Chevalley-Dunford decomposition.

Linear representations Schur's lemma.

Exponential matrix exponential.

Group Theory

Basic definitions group, group morphism, direct product of groups, subgroup, subgroup generated by a subset, order of an element, normal subgroup, quotient group, group action, stabilizer of a point, orbit, quotient space, class formula, conjugacy classes.

Abelian group cyclic group, finite type abelian groups, complex roots of unity, primitive complex roots of unity.

Permutation group permutation group of a type, decomposition into transpositions, decomposition into cycles with disjoint support, signature, alternating group.

Classical automorphism groups general linear group, special linear group, orthogonal group, unitary group.

Representation theory of finite groups Maschke theorem, orthogonality of irreducible characters, characters of a finite dimensional representation.

Ring Theory

Fundamentals ring, subrings, ring morphisms, ring structure $\mathbb{Z}$, product of rings.

Ideals and Quotients ideal of a commutative ring, quotient rings, prime ideals, maximal ideals, Chinese remainder theorem.

Algebra associative algebra over a commutative ring.

Divisibility in integral domains irreducible elements, invertible elements, coprime elements, unique factorisation domain (UFD), greatest common divisor, least common multiple, $A[X_i]$ is a UFD when A is a UFD, principal ideal domain, Euclidean rings, Euclid's' algorithm, $\mathbb{Z}$ is a euclidean ring, congruence in $\mathbb{Z}$, prime numbers, Bézout's identity, $\mathbb{Z}/n\mathbb{Z}$ and its invertible elements, Euler's totient function (φ).

Polynomial rings $K[X]$ is a euclidean ring when K is a field, irreducible polynomial, cyclotomic polynomials in $\mathbb{Q}[X]$, Eisenstein's criterion, polynomial algebra in one or several indeterminates over a commutative ring, roots of a polynomial, multiplicity, relationship between the coefficients and the roots of a split polynomial, Newton's identities, polynomial derivative, decomposition into sums of homogeneous polynomials, symmetric polynomials.

Field Theory fields, characteristic of a ring, characteristic zero, characteristic p, Subfields, Frobenius morphisms, field $\mathbb{Q}$ of rational numbers, field $\mathbb{R}$ of real numbers, field $\mathbb{C}$ of complex numbers, $\mathbb{C}$ is algebraically closed, field of fractions of an integral domain, algebraic elements, transcendental elements, algebraic extensions, algebraically closed fields, rupture fields, splitting fields, finite fields, rational fraction fields with one indeterminate over a field.

Bilinear and Quadratic Forms Over a Vector Space

Bilinear forms bilinear forms, alternating bilinear forms, symmetric bilinear forms, nondegenerate forms, matrix representation, change of coordinates.

Quadratic forms quadratic form, polar form of a quadratic.

Orthogonality orthogonal elements, adjoint endomorphism, Gram-Schmidt orthogonalisation.

Euclidean and Hermitian spaces Euclidean vector spaces, Hermitian vector spaces, dual isomorphism in the euclidean case, orthogonal complement, Cauchy-Schwarz inequality, norm, orthonormal bases.

Endomorphisms orthogonal group, unitary group, self-adjoint endomorphism, diagonalization of a self-adjoint endomorphism, decomposition of an orthogonal transformation as a product of reflections.

Low dimensions cross product, triple product.

Affine and Euclidean Geometry

General definitions affine space, affine function, affine subspace, barycenter, affine span, affine groups.

Convexity convex subsets, convex hull of a subset of an affine real space, extreme point.

Euclidean affine spaces isometries of a Euclidean affine space, group of isometries of a Euclidean affine space, angles between vectors, cocyclicity.

Single Variable Real Analysis

Real numbers definition of $\mathbb{R}$, field structure, order.

Sequences of real numbers convergence, limit point, recurrent sequences, limit infimum and supremum, Cauchy sequences.

Topology of R metric structure, completeness of R, Bolzano-Weierstrass theorem, compact subsets of $\mathbb{R}$, connected subsets of $\mathbb{R}$, additive subgroups of $\mathbb{R}$.

Numerical series Geometric series, convergence of p-series for $p > 1$, alternating series.

Real-valued functions defined on a subset of $\mathbb{R}$ continuity, limits, intermediate value theorem, image of a segment, continuity of monotone functions, continuity of inverse functions.

Differentiability derivative at a point, differentiable functions, derivative of a composition of functions, derivative of the inverse of a function, Rolle's theorem, mean value theorem, higher order derivatives of functions, C^k functions, Leibniz formula.

Taylor-like theorems Taylor's theorem with Lagrange form for remainder.

Elementary functions (trigonometric, rational, $\exp$, $\log$, etc) polynomial functions, rational functions, logarithms, exponential, power functions, trigonometric functions, hyperbolic trigonometric functions, inverse trigonometric functions, inverse hyperbolic trigonometric functions.

Integration Riemann sums, antiderivative of a continuous function, change of variable, integration by parts.

Sequences and series of functions pointwise convergence, uniform convergence, continuity of the limit of a sequence of functions, continuity of the sum of a series of functions, differentiability of the limit of a sequence of functions, differentiability of the sum of a series of functions, Weierstrass polynomial approximation theorem, Weierstrass trigonometric approximation theorem.

Convexity convex functions of a real variable, characterizations of convexity, convexity inequalities.

Single Variable Complex Analysis

Complex Valued series radius of convergence, continuity, differentiability with respect to the complex variable, complex exponential, extension of trigonometric functions to the complex plane(cos, sin), power series expansion of elementary functions(cos, sin).

Functions on one complex variable holomorphic functions, Cauchy formulas, analyticity of a holomorphic function, principle of isolated zeros, principle of analytic continuation, maximum principle, holomorphic stability under uniform convergence.

Topology

Topology and Metric Spaces topology of a metric space, induced topology, finite product of metric spaces, limits of sequences, cluster points, continuous functions, homeomorphisms, compactness in terms of open covers (Borel-Lebesgue), sequential compactness is equivalent to compactness (Bolzano-Weierstrass), connectedness, connected components, path connectedness, Lipschitz functions, uniformly continuous functions, Heine-Cantor theorem, complete metric spaces, contraction mapping theorem.

Normed vector spaces on $\mathbb{R}$ and $\mathbb{C}$ topology on a normed vector space, Banach open mapping theorem, equivalence of norms in finite dimension, norms $\|\cdot\|_p$ on $\mathbb{R}^n$ and $\mathbb{C}^n$, absolutely convergent series in Banach spaces, continuous linear maps, norm of a continuous linear map, uniform convergence norm (sup-norm), normed space of bounded continuous functions, completeness of the space of bounded continuous functions, Heine-Borel theorem (closed bounded subsets are compact in finite dimension), Riesz' lemma (unit-ball characterization of finite dimension), Arzela-Ascoli theorem.

Hilbert spaces Hilbert projection theorem, orthogonal projection onto closed vector subspaces, dual space, Riesz representation theorem, inner product space l^2, completeness of l^2, inner product space L^2, completeness of L^2, Hilbert bases, example, the Hilbert basis of trigonometric polynomials, Lax-Milgram theorem.

Multivariable calculus

Differential calculus differentiable functions on an open subset of $\mathbb{R}^n$, differentials (linear tangent functions), chain rule, mean value theorem, differentiable functions, k-times continuously differentiable functions, partial derivatives commute, local extrema, convexity of functions on an open convex subset of $\mathbb{R}^n$, diffeomorphisms, inverse function theorem, implicit function theorem.

Differential equations Cauchy-Lipschitz Theorem, Grönwall lemma.

Measures and integral calculus

Measure theory measurable spaces, sigma-algebras, product of sigma-algebras, Borel sigma-algebras, positive measure, counting measure, Lebesgue measure, product measure, measurable functions, approximation by step functions.

Integration integral of positive measurable functions, monotone convergence theorem, Fatou's lemma, integrable functions, dominated convergence theorem, finite dimensional vector-valued integrable functions, continuity of integrals with respect to parameters, L^p spaces where $1 \leq p \leq \infty$, Completeness of L^p spaces, Holder's inequality, Fubini's theorem, change of variables for multiple integrals, change of variables to polar co-ordinates, convolution, approximation by convolution, regularization by convolution.

Fourier analysis Fourier series of locally integrable periodic real-valued functions, Riemann-Lebesgue lemma, Parseval theorem, Fourier transform on $\mathrm{L}^1(\mathbb{R}^d)$, Fourier inversion formula.

Probability Theory

Definitions of a probability space probability measure, events, independent events, sigma-algebra, independent sigma-algebra, 0-1 law, Borel-Cantelli lemma (easy direction), Borel-Cantelli lemma (difficult direction), conditional probability.

Random variables and their laws discrete law, probability density function, independence of random variables, mean of a random variable, variance of a real-valued random variable, moments, Bernoulli law.

Convergence of a sequence of random variables convergence in probability, L^p convergence, almost surely convergence, Markov inequality, Chebychev inequality, strong law of large numbers.

Distribution calculus

Spaces $\mathcal{S}(\mathbb{R}^d)$ Schwartz space of rapidly decreasing functions, stability by derivation.

Numerical Analysis

Approximation of numerical functions Lagrange interpolation, Lagrange polynomial of a function at (n + 1) points.

4.1.2 MathLib undergraduate to-do

https://leanprover-community.github.io/undergrad_todo.html
2024/06/16

Missing undergraduate mathematics in mathlib

This gives pointers to undergraduate maths topics that are currently missing in mathlib. The list is gathered from the French curriculum. There is also a page listing undergraduate maths topics that are already in mathlib.

If you want to work on an item from this list then you should first check the pull requests list to see whether it is already coming, then the issues list to see whether it is discussed there, and finally talk about this idea on Zulip.

To update this list, please submit a PR modifying docs/undergrad.yaml in the mathlib repository.

Linear algebra

Duality: orthogonality.

Finite-dimensional vector spaces: rank of a system of linear equations.

Multilinearity: special linear group.

Matrices: elementary row operations, elementary column operations, Gaussian elimination, row-reduced matrices.

Structure theory of endomorphisms: diagonalization, triangularization, invariant subspaces of an endomorphism, kernels lemma, Jordan normal form.

Linear representations: irreducible representation, examples.

Exponential: endomorphism exponential.

Group Theory

Classical automorphism groups: special orthogonal group, special unitary group.

Representation theory of finite groups: representations of abelian groups, dual groups, Fourier transform for finite abelian groups, convolution, class function over a group, orthonormal basis of irreducible characters, examples of groups with small cardinality.

Ring Theory

Algebra: algebra over a commutative ring.

Field Theory: $\mathbb{R}(X)$-partial fraction decomposition, $\mathbb{C}(X)$-partial fraction decomposition.

Bilinear and Quadratic Forms Over a Vector Space

Bilinear forms: rank of a bilinear form.

Orthogonality: Sylvester's law of inertia, real classification, complex classification.

Endomorphisms: special orthogonal group, special unitary group, normal endomorphism, diagonalization of normal endomorphisms, simultaneous diagonalization of two real quadratic forms, with one positive-definite, polar decompositions in $\mathrm{GL}(n, \mathbb{R})$, polar decompositions in $\mathrm{GL}(n, \mathbb{C})$.

Low dimensions: classification of elements of $\mathrm{O}(2, \mathbb{R})$, classification of elements of $\mathrm{O}(3, \mathbb{R})$.

Affine and Euclidean Geometry

General definitions: equations of affine subspace, affine property, group generated by homotheties and translations, transformations fixing a basis of directions.

Euclidean affine spaces: isometries that do and do not preserve orientation, direct and opposite similarities of the plane, classification of isometries in two and three dimensions, angles between planes, inscribed angle theorem, group of isometries stabilizing a subset of the plane or of space, regular polygons, metric relations in the triangle, using complex numbers in plane geometry.

Application of quadratic forms to study proper conic sections of the affine euclidean plane: focus, eccentricity, quadric surfaces in 3-dimensional Euclidean affine spaces.

Single Variable Real Analysis

Numerical series: Convergence of real valued-series, summation of comparison relations, comparison of a series and an integral, error estimation, absolute convergence, products of series.

Differentiability: piecewise C^k functions.

Taylor-like theorems: Taylor's theorem with little-o remainder, Taylor's theorem with integral form for remainder, Taylor series expansions.

Integration: integral over a segment of piecewise continuous functions, improper integrals, absolute vs conditional convergence of improper integrals, comparison test for improper integrals.

Sequences and series of functions: normal convergence.

Convexity: continuity and differentiability of convex functions.

Single Variable Complex Analysis

Complex Valued series: antiderivative, power series expansion of elementary functions(log).

Functions on one complex variable: Cauchy-Riemann conditions, contour integrals of continuous functions in $\mathbb{C}$, antiderivatives of a holomorphic function, representations of the $\log$ function on $\mathbb{C}$, theorem of holomorphic functions under integral domains, winding number of a closed curve in $\mathbb{C}$ with respect to a point, isolated singularities, Laurent series, meromorphic functions, residue theorem, sequences and series of holomorphic functions.

Topology

Normed vector spaces on $\mathbb{R}$ and $\mathbb{C}$: equivalent norms.

Hilbert spaces: example, classical Hilbert bases of orthogonal polynomials, $H_0^1([0,1])$ and its application to the one-dimensional Dirichlet problem.

Multivariable calculus

Differential calculus: directional derivative, partial derivatives, Jacobian matrix, gradient vector, Hessian matrix, k-th order partial derivatives, Taylor's theorem with little-o remainder, Taylor's theorem with integral form for remainder.

Differential equations: maximal solutions, exit theorem of a compact subspace, autonomous differential equations, phase portraits, qualitative behavior, stability of equilibrium points (linearisation theorem), linear differential systems, method of constant variation (Duhamel’s formula), constant coefficient case, solving systems of differential equations of order > 1.

Submanifolds of $\mathbb{R}^n$: local graphs, local parameterization, local equation, tangent space, position with respect to the tangent plane, gradient, line integral, curve length, Lagrange multipliers.

Measures and integral calculus

Integration: differentiability of integrals with respect to parameters, change of variables to spherical co-ordinates.

Fourier analysis: convolution product of periodic functions, Dirichlet theorem, Fejer theorem, Fourier transform on $\mathrm{L}^2(\mathbb{R}^d)$, Plancherel’s theorem.

Probability Theory

Definitions of a probability space: law of total probability.

Random variables and their laws: absolute continuity of probability laws, law of joint probability, transfer theorem, binomial law, geometric law, Poisson law, uniform law, exponential law, Gaussian law, characteristic function, probability generating functions, applications of probability generating functions to sums of independent random variables.

Convergence of a sequence of random variables: Levy's theorem, weak law of large numbers, central limit theorem.

Distribution calculus

Spaces $\mathcal{D}(\mathbb{R}^d)$: smooth functions with compact support on $\mathbb{R}^d$, stability by derivation, stability by multiplication by a smooth function, partitions of unity, constructing approximations of probability density functions in spaces of common functions (trig, exp, rational, log, etc).

Distributions on $\mathbb{R}^d$: definition of distributions, locally integrable functions as distributions, derivative of a distribution, Dirac measures, derivatives of Dirac measures, derivative of the Heaviside function, Cauchy principal values, multiplication by a smooth function, convergence of sequences of distributions, support of a distribution.

Spaces $\mathcal{S}(\mathbb{R}^d)$: stability by multiplication by a slowly growing smooth function, gaussian functions, Fourier transforms on $\mathcal{S}(\mathbb{R}^d)$, convolution of two functions of $\mathcal{S}(\mathbb{R}^d)$.

Tempered distributions: definition, derivation of tempered distributions, multiplication by a function C^∞ of slow growth, L^2 functions and Riesz representation, L^p functions, periodic functions, Dirac comb, Fourier transforms, inverse Fourier transform, Fourier transform and derivation, Fourier transform and convolution product.

Applications: Poisson's formula, using convolution and Fourier-Laplace transform to solve one dimensional linear differential equations, weak solution of partial derivative equation, fundamental solution of the Laplacian, solving the Laplace equations, heat equations, wave equations.

Numerical Analysis

Solving systems of linear inequalities: conditioning, Gershgorin-Hadamard theorem, Gauss's pivot, LU decomposition.

Iterative methods: Jacobian, Gauss-Seidel, convergence analysis, spectral ray, singular value decomposition, example of discretisation matrix by finite differences of the laplacian in one dimension.

Iterative methods of solving systems of real and vector valued equations: linear systems case, proper element search, brute force method, optimization of convex function in finite dimension, gradient descent square root, nonlinear problems with real and vector values, bisection method, Picard method, Newton's method, rate of convergence and estimation of error.

Numerical integration: Rectangle method, error estimation, Monte-Carlo method, rate of convergence, application to the calculation of multiple integrals.

Approximation of numerical functions: estimation of the error.

Ordinary differential equations: numerical aspects of Cauchy's problem, explicit Euler method, consistency, stability, convergence, order.

Fourier transform: discrete Fourier transform on a finite abelian group, fast Fourier transform.

4.1.3 MathLib overview

https://leanprover-community.github.io/mathlib-overview.html
2024/06/16

A mathlib overview

The goal of this web page is to give a rough list of topics currently covered in mathlib, and provide pointers for exploration. This is not meant to be an exhaustive list, and could be outdated (see the full index for exhaustive and up to date information).

Here topics are listed in the greatest generality we currently have in mathlib, hence some things may be difficult to recognize. We also have a page dedicated to undergraduate mathematics which may be easier to read, as well as a page listing undergraduate maths topics that are not yet in mathlib.

To make updates to this list, please make a pull request to mathlib after editing the yaml source file. This can be done entirely on GitHub, see "Editing files in another user's repository".

General algebra

Category theory category, small category, functor, natural transformation, Yoneda embedding, adjunction, monad, comma category, limits, presheafed space, sheafed space, monoidal category, cartesian closed, abelian category. See also our documentation page about category theory.

Numbers natural number, integer, rational number, continued fraction, real number, extended real number, complex number, p-adic number, p-adic integer, hyper-real number. See also our documentation page about natural numbers.

Group theory group, group morphism, group action, class formula, Burnside lemma, subgroup, subgroup generated by a subset, quotient group, first isomorphism theorem, second isomorphism theorem, third isomorphism theorem, abelianization, free group, presented group, Schreier's lemma, cyclic group, nilpotent group, permutation group of a type, structure of finitely generated abelian groups.

Rings ring, ring morphism, the category of rings, subring, localization, local ring, noetherian ring, ordered ring.

Ideals and quotients ideal of a commutative ring, quotient ring, prime ideal, maximal ideal, Chinese remainder theorem, fractional ideal, first isomorphism theorem for commutative rings.

Divisibility in integral domains irreducible element, coprime element, unique factorisation domain, greatest common divisor, least common multiple, principal ideal domain, Euclidean domain, Euclid's' algorithm, Euler's totient function (φ), Lucas-Lehmer primality test.

Polynomials and power series polynomial in one indeterminate, roots of a polynomial, multiplicity, separable polynomial, $K[X]$ is Euclidean, Hilbert basis theorem, $A[X]$ has gcd and lcm if A does, $A[X_i]$ is a UFD when A is a UFD, irreducible polynomial, Eisenstein's criterion, polynomial in several indeterminates, power series.

Algebras over a ring associative algebra over a commutative ring, the category of algebras over a ring, free algebra of a commutative ring, tensor product of algebras, tensor algebra of a commutative ring, Lie algebra, exterior algebra, Clifford algebra.

Field theory field, characteristic of a ring, characteristic zero, characteristic p, Frobenius morphism, algebraically closed field, existence of algebraic closure of a field, $\mathbb{C}$ is algebraically closed, field of fractions of an integral domain, algebraic extension, rupture field, splitting field, perfect closure, Galois correspondence, Abel-Ruffini theorem (one direction).

Homological algebra chain complex, functorial homology.

Number theory sum of two squares, sum of four squares, quadratic reciprocity, solutions to Pell's equation, Matiyasevič's theorem, arithmetic functions, Bernoulli numbers, Chevalley-Warning theorem, Hensel's lemma (for $\mathbb{Z}_p$), ring of Witt vectors, perfection of a ring.

Transcendental numbers Liouville's theorem on existence of transcendental numbers.

Algebraic Number Theory Finiteness of the class number, Dirichlet unit theorem.

Representation theory representation, category of finite dimensional representations, character, orthogonality of characters.

Linear algebra

Fundamentals module, linear map, the category of modules over a ring, vector space, quotient space, tensor product, noetherian module, basis, multilinear map, alternating map, general linear group.

Duality dual vector space, dual basis.

Finite-dimensional vector spaces finite-dimensionality, isomorphism with K^n, isomorphism with bidual.

Finitely generated modules over a PID structure theorem.

Matrices ring-valued matrix, matrix representation of a linear map, determinant, invertibility.

Endomorphism polynomials minimal polynomial, characteristic polynomial, Cayley-Hamilton theorem.

Structure theory of endomorphisms eigenvalue, eigenvector, existence of an eigenvalue.

Bilinear and quadratic forms bilinear form, alternating bilinear form, symmetric bilinear form, matrix representation, quadratic form, polar form of a quadratic.

Finite-dimensional inner product spaces (see also Hilbert spaces, below) existence of orthonormal basis, diagonalization of self-adjoint endomorphisms.

See also our documentation page about linear algebra.

Topology

General topology filter, limit of a map with respect to filters, topological space, continuous function, the category of topological spaces, induced topology, open map, closed map, closure, cluster point, Hausdorff space, sequential space, extension by continuity, compactness in terms of filters, compactness in terms of open covers (Borel-Lebesgue), connectedness, compact open topology, Stone-Cech compactification, topological fiber bundle, topological vector bundle, Urysohn's lemma, Stone-Weierstrass theorem.

Uniform notions uniform space, uniformly continuous function, uniform convergence, Cauchy filter, Cauchy sequence, completeness, completion, Heine-Cantor theorem.

Topological algebra order topology, intermediate value theorem, extreme value theorem, limit infimum and supremum, topological group, completion of an abelian topological group, infinite sum, topological ring, completion of a topological ring, topological module, continuous linear map, Haar measure on a locally compact Hausdorff group.

Metric spaces metric space, ball, sequential compactness is equivalent to compactness (Bolzano-Weierstrass), Heine-Borel theorem (proper metric space version), Lipschitz continuity, Hölder continuity, contraction mapping theorem, Baire theorem, Arzela-Ascoli theorem, Hausdorff distance, Gromov-Hausdorff space.

See also our documentation page about topology.

Analysis

Topological vector spaces local convexity, Bornology, weak-* topology for dualities.

Normed vector spaces/Banach spaces normed vector space over a normed field, topology on a normed vector space, equivalence of norms in finite dimension, finite dimensional normed spaces over complete normed fields are complete, Heine-Borel theorem (finite dimensional normed spaces are proper), norm of a continuous linear map, Banach-Steinhaus theorem, Banach open mapping theorem, absolutely convergent series in Banach spaces, Hahn-Banach theorem, dual of a normed space, isometric inclusion in double dual, completeness of spaces of bounded continuous functions.

Hilbert spaces Inner product space, over R or C, Cauchy-Schwarz inequality, adjoint operator, self-adjoint operator, orthogonal projection, reflection, orthogonal complement, existence of Hilbert basis, eigenvalues from Rayleigh quotient, Fréchet-Riesz representation of the dual of a Hilbert space, Lax-Milgram theorem.

Differentiability differentiable function between normed vector spaces, derivative of a composition of functions, derivative of the inverse of a function, Rolle's theorem, mean value theorem, Taylor's theorem, C^k function, Leibniz formula, local extrema, inverse function theorem, implicit function theorem, analytic function.

Convexity convex function, characterization of convexity, Jensen's inequality (finite sum version), Jensen's inequality (integral version), convexity inequalities, Carathéodory's theorem.

Special functions logarithm, exponential, trigonometric functions, inverse trigonometric functions, hyperbolic trigonometric functions, inverse hyperbolic trigonometric functions.

Measures and integral calculus sigma-algebra, measurable function, the category of measurable spaces, Borel sigma-algebra, positive measure, Stieltjes measure, Lebesgue measure, Hausdorff measure, Hausdorff dimension, Giry monad, integral of positive measurable functions, monotone convergence theorem, Fatou's lemma, vector-valued integrable function (Bochner integral), uniform integrability, L^p space, Bochner integral, dominated convergence theorem, fundamental theorem of calculus, part 1, fundamental theorem of calculus, part 2, Fubini's theorem, product of finitely many measures, convolution, approximation by convolution, regularization by convolution, change of variables formula, divergence theorem.

Complex analysis Cauchy integral formula, Liouville theorem, maximum modulus principle, principle of isolated zeros, principle of analytic continuation, analyticity of holomorphic functions, Schwarz lemma, removable singularity, Phragmen-Lindelöf principle, fundamental theorem of algebra.

Distribution theory Schwartz space.

Fourier analysis Fourier transform as an integral, inversion formula, Riemann-Lebesgue lemma, Parseval formula for Fourier series.

Probability Theory

Definitions in probability theory probability measure, independent events, independent sigma-algebras, conditional probability, conditional expectation.

Random variables and their laws discrete law, probability density function, variance of a real-valued random variable, independence of random variables, Kolmogorov's 0-1 law, mean of product of independent random variables, moment of a random variable, Bernoulli law.

Convergence of a sequence of random variables convergence in probability, L^p convergence, almost sure convergence, convergence in distribution, Markov inequality, Chebychev inequality, strong law of large numbers.

Stochastic Processes martingale, optional stopping theorem, stopping time, hitting time.

Geometry

Affine and Euclidean geometry affine space, affine function, affine subspace, barycenter, affine span, Euclidean affine space, angle.

Differentiable manifolds smooth manifold (with boundary and corners), smooth map between manifolds, tangent bundle, tangent map, Lie group, sphere.

Algebraic geometry prime spectrum, Zariski topology, locally ringed space, scheme, Nullstellensatz.

Combinatorics

Graph theory simple graph, degree-sum formula, matching, adjacency matrix.

Pigeonhole principles finite, infinite, strong pigeonhole principle.

Transversals Hall's marriage theorem.

See also our documentation page about set-like objects.

Dynamics

Circle dynamics translation number, translation numbers define a group action up to semiconjugacy, translation number defines a homeomorphism up to semiconjugacy.

General theory omega-limit sets, fixed points, periodic points.

Data structures

List-like structures list, array, difference list, lazy list, stream, sequence, weak sequence.

Sets set, finite set, multiset, ordered set.

Maps key-value map, red-black map, hash map, finitely supported function, finite map.

Trees tree, red-black tree, size-balanced binary search tree, W type.

Logic and computation

Computability computable function, Turing machine, halting problem, Rice theorem, combinatorial game.

Set theory ordinal, cardinal, model of ZFC.

Model theory first-order structure, first-order formula, satisfiability, substructure, definable set, elementary embedding, Compactness theorem, Löwenheim-Skolem.

4.2 tutorial

4.2.1 LEAN game server

https://adam.math.hhu.de/#/

4.2.1.1 A LEAN Intro to Logic

https://adam.math.hhu.de/#/g/trequetrum/lean4game-logic

4.2.1.2 Set Theory Game

https://adam.math.hhu.de/#/g/djvelleman/stg4

4.2.1.3 Natural Number Game

https://adam.math.hhu.de/#/g/leanprover-community/NNG4

4.2.2 MathPom

https://www.youtube.com/watch?v=7Ko3QZx4saU

Part II

mathematics

Chapter 5

tensor

5.1 tensor algebra

5.1.1 vector

5.1.1.1 decomposition over bases

$$
\begin{aligned}
&\mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^\infty, \cdots\} && \widetilde{\mathfrak{V}}, \mathfrak{V} \subseteq \mathcal{V} \subseteq \mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\}\\
\mathcal{V} \ni \boldsymbol{v} &= v^j \boldsymbol{v}_j = \sum_j v^j \boldsymbol{v}_j && \boldsymbol{v}_j \in \mathfrak{V} = \{\boldsymbol{v}_j\} = \{\boldsymbol{v}_j\}_j
\end{aligned} \tag{5.1}
$$

$$
= \begin{cases} v^1 \boldsymbol{v}_1 + \cdots + v^n \boldsymbol{v}_n & = \sum_{j=1}^{n} v^j \boldsymbol{v}_j \\ \cdots + v^j \boldsymbol{v}_j + \cdots & = \sum_{j \in J} v^j \boldsymbol{v}_j \end{cases} \qquad = \begin{cases} \{\boldsymbol{v}_j\}_{j=1}^{n} & \\ \{\boldsymbol{v}_j\}_{j \in J} & J \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \cdots\} \end{cases}
$$

$$
= \begin{cases} v^1 \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} + \cdots + v^n \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} & = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\ \cdots + v^j \begin{bmatrix} | \\ \boldsymbol{v}_j \\ | \end{bmatrix} + \cdots & = \begin{bmatrix} & | & \\ \cdots & \boldsymbol{v}_j & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \boldsymbol{v}_j \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} = V\,[\boldsymbol{v}]_V \qquad [\boldsymbol{v}]_V \in \begin{cases} \mathbb{F}^n \\ \mathbb{F}^{|J|} \end{cases} \tag{5.2}
$$

$$
= \widetilde{v}^j \widetilde{\boldsymbol{v}}_j = \sum_j \widetilde{v}^j \widetilde{\boldsymbol{v}}_j \qquad \widetilde{\boldsymbol{v}}_j \in \widetilde{\mathfrak{V}} = \{\widetilde{\boldsymbol{v}}_j\} = \{\widetilde{\boldsymbol{v}}_j\}_j \tag{5.3}
$$

$$
= \begin{cases} \widetilde{v}^1 \widetilde{\boldsymbol{v}}_1 + \cdots + \widetilde{v}^n \widetilde{\boldsymbol{v}}_n & = \sum_{j=1}^{n} \widetilde{v}^j \widetilde{\boldsymbol{v}}_j \\ \cdots + \widetilde{v}^j \widetilde{\boldsymbol{v}}_j + \cdots & = \sum_{j \in J} \widetilde{v}^j \widetilde{\boldsymbol{v}}_j \end{cases} \qquad = \begin{cases} \{\widetilde{\boldsymbol{v}}_j\}_{j=1}^{n} & \\ \{\widetilde{\boldsymbol{v}}_j\}_{j \in J} & J \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \cdots\} \end{cases}
$$

$$
= \begin{cases} \widetilde{v}^1 \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} + \cdots + \widetilde{v}^n \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} & = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} \\ \cdots + \widetilde{v}^j \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_j \\ | \end{bmatrix} + \cdots & = \begin{bmatrix} & | & \\ \cdots & \widetilde{\boldsymbol{v}}_j & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} \vdots \\ \widetilde{v}^j \\ \vdots \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \widetilde{\boldsymbol{v}}_j \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ \widetilde{v}^j \\ \vdots \end{bmatrix} = \widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} \qquad [\boldsymbol{v}]_{\widetilde{V}} \in \begin{cases} \mathbb{F}^n \\ \mathbb{F}^{|J|} \end{cases} \tag{5.4}
$$

$$
\boldsymbol{v} \overset{(5.2)}{=} V\,[\boldsymbol{v}]_V \tag{5.5}
$$

$$
\boldsymbol{v} \overset{(5.4)}{=} \widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} \tag{5.6}
$$

5.1.1.2 change of basis

change of basis and transformation

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
V[\boldsymbol{v}]_V = \boldsymbol{v} \in \mathcal{V} \supseteq \begin{cases} \mathfrak{V} = \{\boldsymbol{v}_j\} \\ \widetilde{\mathfrak{V}} = \{\widetilde{\boldsymbol{v}}_j\} \end{cases} \arrow[r, shift left, "A(\cdot)"'] \arrow[d, shift right, "\overline{V}"'] \arrow[dd, bend right=60, "\overline{\widetilde{V}}"'] & A(\boldsymbol{v}) = \boldsymbol{w} \in \mathcal{W} \supseteq \begin{cases} \mathfrak{W} = \{\boldsymbol{w}_j\} \\ \widetilde{\mathfrak{W}} = \{\widetilde{\boldsymbol{w}}_j\} \end{cases} \arrow[l, shift left, "\overline{A}(\cdot)"'] \arrow[d, shift right, "\overline{W}"'] \arrow[dd, bend left=60, "\overline{\widetilde{W}}"] \\
\overline{V}\boldsymbol{v} = [\boldsymbol{v}]_V \in \mathbb{F}^n \arrow[u, shift right, "V"'] \arrow[r, shift right, "A = {[A(V)]}_W"'] \arrow[d, shift right, "{[V]}_{\widetilde{V}}"'] \arrow[r, bend left=40] & {[A(\boldsymbol{v})]}_W = {[\boldsymbol{w}]}_W \in \mathbb{F}^m \arrow[l, shift right, "\overline{A}"'] \arrow[u, shift right, "W"'] \arrow[d, shift right, "{[W]}_{\widetilde{W}}"'] \\
\overline{\widetilde{V}}\boldsymbol{v} = {[V]}_{\widetilde{V}}\,{[\boldsymbol{v}]}_V = {[\boldsymbol{v}]}_{\widetilde{V}} \in \mathbb{F}^n \arrow[u, shift right, "{[\widetilde{V}]}_V"'] \arrow[uu, bend left=40, "\widetilde{V}"] \arrow[r, shift right, "\widetilde{A} = {[A(\widetilde{V})]}_{\widetilde{W}}"'] \arrow[r, bend left=40] & {[A(\boldsymbol{v})]}_{\widetilde{W}} = {[\boldsymbol{w}]}_{\widetilde{W}} \in \mathbb{F}^m \arrow[l, shift right, "\overline{\widetilde{A}}"'] \arrow[u, shift right, "{[\widetilde{W}]}_W"'] \arrow[uu, bend right=40, "\widetilde{W}"']
\end{tikzcd}
$$

Figure 5.1: change of basis $\widetilde{\ }$ and transformation A

transformation of basis

$$
\begin{aligned}
\begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} &= \boldsymbol{v}_1 \overset{(5.3)}{=} \widetilde{v}_1{}^1 \widetilde{\boldsymbol{v}}_1 + \cdots + \widetilde{v}_1{}^n \widetilde{\boldsymbol{v}}_n = \widetilde{v}_1{}^1 \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} + \cdots + \widetilde{v}_1{}^n \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}_1{}^1 \\ \vdots \\ \widetilde{v}_1{}^n \end{bmatrix} \overset{(5.4)}{=} \widetilde{V} \left[\boldsymbol{v}_1\right]_{\widetilde{V}} \\
&= \widetilde{\boldsymbol{v}}_1 \widetilde{v}^1{}_1 + \cdots + \widetilde{\boldsymbol{v}}_n \widetilde{v}^n{}_1 = \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} \widetilde{v}^1{}_1 + \cdots + \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} \widetilde{v}^n{}_1 = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}^1{}_1 \\ \vdots \\ \widetilde{v}^n{}_1 \end{bmatrix} = \widetilde{V} \left[\boldsymbol{v}_1\right]_{\widetilde{V}} \\
&\vdots \\
\begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} &= \boldsymbol{v}_n \overset{(5.3)}{=} \widetilde{v}_n{}^1 \widetilde{\boldsymbol{v}}_1 + \cdots + \widetilde{v}_n{}^n \widetilde{\boldsymbol{v}}_n = \widetilde{v}_n{}^1 \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} + \cdots + \widetilde{v}_n{}^n \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}_1{}^n \\ \vdots \\ \widetilde{v}_n{}^n \end{bmatrix} \overset{(5.4)}{=} \widetilde{V} \left[\boldsymbol{v}_n\right]_{\widetilde{V}} \\
&= \widetilde{\boldsymbol{v}}_1 \widetilde{v}^1{}_n + \cdots + \widetilde{\boldsymbol{v}}_n \widetilde{v}^n{}_n = \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} \widetilde{v}^n{}_1 + \cdots + \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} \widetilde{v}^n{}_n = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}^1{}_n \\ \vdots \\ \widetilde{v}^n{}_n \end{bmatrix} = \widetilde{V} \left[\boldsymbol{v}_n\right]_{\widetilde{V}}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} &\overset{(5.3)}{=} \begin{matrix} \widetilde{v}_1{}^1 \\ \vdots \\ \widetilde{v}_n{}^1 \end{matrix} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} \begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix} \begin{matrix} \widetilde{v}_1{}^n \\ \vdots \\ \widetilde{v}_n{}^n \end{matrix} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} = \begin{matrix} \widetilde{v}^1{}_1 \\ \vdots \\ \widetilde{v}^1{}_n \end{matrix} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} \begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix} \begin{matrix} \widetilde{v}^n{}_1 \\ \vdots \\ \widetilde{v}^n{}_n \end{matrix} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}_1{}^1 & \cdots & \widetilde{v}_n{}^1 \\ \vdots & \ddots & \vdots \\ \widetilde{v}_1{}^n & \cdots & \widetilde{v}_n{}^n \end{bmatrix} = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ \left[\boldsymbol{v}_1\right]_{\widetilde{V}} & \cdots & \left[\boldsymbol{v}_n\right]_{\widetilde{V}} \\ | & & | \end{bmatrix} \overset{(5.4)}{=} \widetilde{V} \left[V\right]_{\widetilde{V}} \qquad (5.7) \\
&= \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \widetilde{v}^1{}_1 & \cdots & \widetilde{v}^1{}_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n{}_1 & \cdots & \widetilde{v}^n{}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ \left[\boldsymbol{v}_1\right]_{\widetilde{V}} & \cdots & \left[\boldsymbol{v}_n\right]_{\widetilde{V}} \\ | & & | \end{bmatrix} = \widetilde{V} \left[V\right]_{\widetilde{V}} \qquad (5.8)
\end{aligned}
$$

$$V = \widetilde{V} \left[V\right]_{\widetilde{V}} \overset{\text{if } \widetilde{V} \text{ invertible}}{\Longleftrightarrow} \left[V\right]_{\widetilde{V}} = \widetilde{V}^{-1} V = \overline{\widetilde{V}} V \qquad (5.9)$$

$$V = \widetilde{V} \left[V\right]_{\widetilde{V}} \qquad (5.10)$$

$$\left[V\right]_{\widetilde{V}} = \widetilde{V}^{-1} V = \overline{\widetilde{V}} V \qquad (5.11)$$

$$\left[V\right]_{\widetilde{V}} \overset{(5.7)}{=} \begin{bmatrix} \widetilde{v}_1{}^1 & \cdots & \widetilde{v}_n{}^1 \\ \vdots & \ddots & \vdots \\ \widetilde{v}_1{}^n & \cdots & \widetilde{v}_n{}^n \end{bmatrix} \overset{(5.8)}{=} \begin{bmatrix} \widetilde{v}^1{}_1 & \cdots & \widetilde{v}^1{}_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n{}_1 & \cdots & \widetilde{v}^n{}_n \end{bmatrix} = \begin{bmatrix} \widetilde{v}^1_1 & \cdots & \widetilde{v}^1_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n_1 & \cdots & \widetilde{v}^n_n \end{bmatrix} \qquad (5.12)$$

$$\begin{aligned}
\begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_1 \\ | \end{bmatrix} &= \widetilde{\boldsymbol{v}}_1 \overset{(5.1)}{=} v_1{}^1\boldsymbol{v}_1 + \cdots + v_1{}^n\boldsymbol{v}_n = v_1{}^1 \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} + \cdots + v_1{}^n \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v_1{}^1 \\ \vdots \\ v_1{}^n \end{bmatrix} \overset{(5.2)}{=} V\left[\widetilde{\boldsymbol{v}}_1\right]_V \\
&= \boldsymbol{v}_1 v^1{}_1 + \cdots + \boldsymbol{v}_n v^n{}_1 = \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} v^1{}_1 + \cdots + \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} v^n{}_1 = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1{}_1 \\ \vdots \\ v^n{}_1 \end{bmatrix} = V\left[\widetilde{\boldsymbol{v}}_1\right]_V \\
&\vdots \\
\begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}_n \\ | \end{bmatrix} &= \widetilde{\boldsymbol{v}}_n \overset{(5.1)}{=} v_n{}^1\boldsymbol{v}_1 + \cdots + v_n{}^n\boldsymbol{v}_n = v_n{}^1 \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} + \cdots + v_n{}^n \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v_n{}^1 \\ \vdots \\ v_n{}^n \end{bmatrix} \overset{(5.2)}{=} V\left[\widetilde{\boldsymbol{v}}_n\right]_V \\
&= \boldsymbol{v}_1 v^1{}_n + \cdots + \boldsymbol{v}_n v^n{}_n = \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} v^1{}_n + \cdots + \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} v^n{}_n = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1{}_n \\ \vdots \\ v^n{}_n \end{bmatrix} = V\left[\widetilde{\boldsymbol{v}}_n\right]_V
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\begin{bmatrix} | & & | \\ \widetilde{\boldsymbol{v}}_1 & \cdots & \widetilde{\boldsymbol{v}}_n \\ | & & | \end{bmatrix} &\overset{(5.1)}{=} \begin{matrix} v_1{}^1 \\ \vdots \\ v_n{}^1 \end{matrix} \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} \begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix} \begin{matrix} v_1{}^n \\ \vdots \\ v_n{}^n \end{matrix} \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} = \begin{matrix} v^1{}_1 \\ \vdots \\ v^1{}_n \end{matrix} \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} \begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix} \begin{matrix} v^n{}_1 \\ \vdots \\ v^n{}_n \end{matrix} \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v_1{}^1 & \cdots & v_n{}^1 \\ \vdots & \ddots & \vdots \\ v_1{}^n & \cdots & v_n{}^n \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ \left[\widetilde{\boldsymbol{v}}_1\right]_V & \cdots & \left[\widetilde{\boldsymbol{v}}_n\right]_V \\ | & & | \end{bmatrix} \overset{(5.2)}{=} V\left[\widetilde{V}\right]_V
\end{aligned} \tag{5.13}$$

$$= \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1{}_1 & \cdots & v^1{}_n \\ \vdots & \ddots & \vdots \\ v^n{}_1 & \cdots & v^n{}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ \left[\widetilde{\boldsymbol{v}}_1\right]_V & \cdots & \left[\widetilde{\boldsymbol{v}}_n\right]_V \\ | & & | \end{bmatrix} = V\left[\widetilde{V}\right]_V \tag{5.14}$$

$$\widetilde{V} = V\left[\widetilde{V}\right]_V \overset{\text{if } V \text{ invertible}}{\Longleftrightarrow} \left[\widetilde{V}\right]_V = V^{-1}\widetilde{V} = \overline{V}\widetilde{V} \tag{5.15}$$

$$\widetilde{V} = V\left[\widetilde{V}\right]_V \tag{5.16}$$

$$\left[\widetilde{V}\right]_V = V^{-1}\widetilde{V} = \overline{V}\widetilde{V} \tag{5.17}$$

$$\left[\widetilde{V}\right]_V \overset{(5.13)}{=} \begin{bmatrix} v_1{}^1 & \cdots & v_n{}^1 \\ \vdots & \ddots & \vdots \\ v_1{}^n & \cdots & v_n{}^n \end{bmatrix} \overset{(5.14)}{=} \begin{bmatrix} v^1{}_1 & \cdots & v^1{}_n \\ \vdots & \ddots & \vdots \\ v^n{}_1 & \cdots & v^n{}_n \end{bmatrix} = \begin{bmatrix} v^1_1 & \cdots & v^1_n \\ \vdots & \ddots & \vdots \\ v^n_1 & \cdots & v^n_n \end{bmatrix} \tag{5.18}$$

$$[V]_{\widetilde{V}} \overset{(5.9)}{=} \widetilde{V}^{-1}V = \overline{\widetilde{V}}V : \qquad \left[\widetilde{V}\middle|V\right] \xrightarrow{\text{Gauss-Jordan delimination}} \left[I\middle|\widetilde{V}^{-1}V\right] = \left[I\middle|\overline{\widetilde{V}}V\right] = \left[I|[V]_{\widetilde{V}}\right] \tag{5.19}$$

$$\left[\widetilde{V}\right]_V \overset{(5.15)}{=} V^{-1}\widetilde{V} = \overline{V}\widetilde{V} : \qquad \left[V\middle|\widetilde{V}\right] \xrightarrow{\text{Gauss-Jordan delimination}} \left[I\middle|V^{-1}\widetilde{V}\right] = \left[I\middle|\overline{V}\widetilde{V}\right] = \left[I\middle|\left[\widetilde{V}\right]_V\right] \tag{5.20}$$

$$\begin{cases} [V]_{\widetilde{V}} \overset{(5.9)}{=} \widetilde{V}^{-1}V = \overline{\widetilde{V}}V : & \left[\widetilde{V}\middle|V\right] \xrightarrow{\text{Gauss-Jordan delimination}} \left[I\middle|\widetilde{V}^{-1}V\right] = \left[I\middle|\overline{\widetilde{V}}V\right] = \left[I|[V]_{\widetilde{V}}\right] \\ \left[\widetilde{V}\right]_V \overset{(5.15)}{=} V^{-1}\widetilde{V} = \overline{V}\widetilde{V} : & \left[V\middle|\widetilde{V}\right] \xrightarrow{\text{Gauss-Jordan delimination}} \left[I\middle|V^{-1}\widetilde{V}\right] = \left[I\middle|\overline{V}\widetilde{V}\right] = \left[I\middle|\left[\widetilde{V}\right]_V\right] \end{cases} \tag{5.21}$$

$$\begin{aligned}
\boldsymbol{v} &\overset{(5.1)}{=} V[\boldsymbol{v}]_V \overset{(5.2)}{=} \widetilde{V}[\boldsymbol{v}]_{\widetilde{V}} \\
&= V[\boldsymbol{v}]_V \qquad\qquad V \overset{(5.9)}{=} \widetilde{V}[V]_{\widetilde{V}} \\
&= \widetilde{V}[V]_{\widetilde{V}}[\boldsymbol{v}]_V \\
\widetilde{V}[\boldsymbol{v}]_{\widetilde{V}} &= \widetilde{V}[V]_{\widetilde{V}}[\boldsymbol{v}]_V \\
[\boldsymbol{v}]_{\widetilde{V}} &= [V]_{\widetilde{V}}[\boldsymbol{v}]_V
\end{aligned} \tag{5.22}$$

$$[\boldsymbol{v}]_{\widetilde{V}} \overset{(5.22)}{=} [V]_{\widetilde{V}}[\boldsymbol{v}]_V \tag{5.23}$$

symmetrically,

$$[\boldsymbol{v}]_V = \left[\widetilde{V}\right]_V [\boldsymbol{v}]_{\widetilde{V}} \tag{5.24}$$

$$\begin{tikzcd}
V[\boldsymbol{v}]_V = \boldsymbol{v} \in \mathcal{V} \supseteq \begin{cases} \mathfrak{V} = \{\boldsymbol{v}_j\} \\ \widetilde{\mathfrak{V}} = \{\widetilde{\boldsymbol{v}}_j\} \end{cases} \arrow[r, shift right, "A(\cdot)"'] \arrow[d, shift right, "\overline{V}"'] \arrow[dd, bend right=60, "\overline{\widetilde{V}}"'] & A(\boldsymbol{v}) = \boldsymbol{w} \in \mathcal{W} \supseteq \begin{cases} \mathfrak{W} = \{\boldsymbol{w}_j\} \\ \widetilde{\mathfrak{W}} = \{\widetilde{\boldsymbol{w}}_j\} \end{cases} \arrow[l, shift right, "\overline{A}(\cdot)"'] \arrow[d, shift right, "\overline{W}"'] \arrow[dd, bend left=60, "\overline{\widetilde{W}}"] \\
\overline{V}\boldsymbol{v} = [\boldsymbol{v}]_V \in \mathbb{F}^n \arrow[r, shift right, "A = {[A(V)]}_W"'] \arrow[u, shift right, "V"'] \arrow[d, shift right, "{[V]}_{\widetilde{V}}"'] \arrow[r, bend left] & {[A(\boldsymbol{v})]}_W = [\boldsymbol{w}]_W \in \mathbb{F}^m \arrow[l, shift right, "\overline{A}"'] \arrow[u, shift right, "W"'] \arrow[d, shift right, "{[W]}_{\widetilde{W}}"'] \\
\overline{\widetilde{V}}\boldsymbol{v} = [V]_{\widetilde{V}} [\boldsymbol{v}]_V = [\boldsymbol{v}]_{\widetilde{V}} \in \mathbb{F}^n \arrow[r, shift right, "\widetilde{A} = {[A(\widetilde{V})]}_{\widetilde{W}}"'] \arrow[u, shift right, "{[\widetilde{V}]}_V"'] \arrow[uu, bend left=40, "\widetilde{V}"] \arrow[r, bend left] & {[A(\boldsymbol{v})]}_{\widetilde{W}} = [\boldsymbol{w}]_{\widetilde{W}} \in \mathbb{F}^m \arrow[l, shift right, "\overline{\widetilde{A}}"'] \arrow[u, shift right, "{[\widetilde{W}]}_W"'] \arrow[uu, bend right=40, "\widetilde{W}"']
\end{tikzcd}$$

Figure 5.2: change of basis $\widetilde{}$ and transformation A

$[\boldsymbol{v}]_V \rightleftharpoons [\boldsymbol{v}]_{\widetilde{V}}$

$$\begin{aligned}
[\boldsymbol{v}]_V &\overset{[V]_{\widetilde{V}}}{\to} [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.23)}{=} [V]_{\widetilde{V}} [\boldsymbol{v}]_V \\
[\boldsymbol{v}]_V &\overset{[\widetilde{V}]_V}{\leftarrow} [\boldsymbol{v}]_{\widetilde{V}} = \left[\widetilde{V}\right]_V^{-1} [\boldsymbol{v}]_V = \overline{\left[\widetilde{V}\right]_V} [\boldsymbol{v}]_V \\
&\Downarrow \\
[V]_{\widetilde{V}} &= \left[\widetilde{V}\right]_V^{-1} = \overline{\left[\widetilde{V}\right]_V} \Leftrightarrow \left[\widetilde{V}\right]_V = [V]_{\widetilde{V}}^{-1} = \overline{[V]_{\widetilde{V}}}
\end{aligned} \tag{5.25}$$

$$[V]_{\widetilde{V}} \overset{(5.25)}{=} \left[\widetilde{V}\right]_V^{-1} = \overline{\left[\widetilde{V}\right]_V} \tag{5.26}$$

$$\left[\widetilde{V}\right]_V \overset{(5.25)}{=} [V]_{\widetilde{V}}^{-1} = \overline{[V]_{\widetilde{V}}} \tag{5.27}$$

$[\boldsymbol{v}]_V \to [\boldsymbol{v}]_{\widetilde{V}}$

$$\begin{gathered}
[\boldsymbol{v}]_V \overset{[V]_{\widetilde{V}}}{\to} [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.23)}{=} [V]_{\widetilde{V}} [\boldsymbol{v}]_V \\
[\boldsymbol{v}]_V \overset{V}{\to} \boldsymbol{v} \overset{\widetilde{V}^{-1} = \overline{\widetilde{V}}}{\to} [\boldsymbol{v}]_{\widetilde{V}} = \widetilde{V}^{-1} V [\boldsymbol{v}]_V = \overline{\widetilde{V}} V [\boldsymbol{v}]_V \overset{(5.23)}{=} [V]_{\widetilde{V}} [\boldsymbol{v}]_V \Leftrightarrow [\boldsymbol{v}]_V = V^{-1} \widetilde{V} [\boldsymbol{v}]_{\widetilde{V}} = \overline{V} \widetilde{V} [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.24)}{=} [V]_{\widetilde{V}} [\boldsymbol{v}]_V \\
\Downarrow \\
[V]_{\widetilde{V}} = \widetilde{V}^{-1} V = \overline{\widetilde{V}} V \overset{[\widetilde{V}]_V = [V]_{\widetilde{V}}^{-1} = \overline{[V]_{\widetilde{V}}}}{\Longleftrightarrow} \left[\widetilde{V}\right]_V = \left(\widetilde{V}^{-1} V\right)^{-1} = V^{-1} \widetilde{V} = \overline{\overline{\widetilde{V}} V} = \overline{V} \widetilde{V}
\end{gathered} \tag{5.28}$$

transformation of component

$$
\begin{aligned}
\boldsymbol{v} &\overset{(5.1)}{=} V\,[\boldsymbol{v}]_V \overset{(5.2)}{=} \widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} \\
v^i &= V^i{}_j v^j = \widetilde{V}^i{}_j \widetilde{v}^j \\
v^k &= V^k{}_i v^i = \widetilde{V}^k{}_j \widetilde{v}^j \\
[\boldsymbol{v}]_V &= [\boldsymbol{v}]_V \overset{(5.28)}{=} V^{-1}\widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} = \overline{V}\widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} && \overset{(5.24)\text{ or }(5.15)}{=} \left[\widetilde{V}\right]_V [\boldsymbol{v}]_{\widetilde{V}} \\
v^i &= v^i = (V^k{}_i)^{-1}\,\widetilde{V}^k{}_j \widetilde{v}^j = \overline{V^k{}_i}\,\widetilde{V}^k{}_j \widetilde{v}^j && = v^i{}_j \widetilde{v}^j \qquad (5.29)\\
&= (V^{-1})^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j = \overline{V}^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j && = v^i{}_j \widetilde{v}^j \\
&(V^{-1})^i{}_k \widetilde{V}^k{}_j = \overline{V}^i{}_k \widetilde{V}^k{}_j && = v^i{}_j \\
v^i{}_j &= (V^{-1})^i{}_k \widetilde{V}^k{}_j = \overline{V}^i{}_k \widetilde{V}^k{}_j \\
&(V^{-1})^i{}_k = (V^k{}_i)^{-1} && \overline{V}^i{}_k = \overline{V^k{}_i} \\
[\boldsymbol{v}]_{\widetilde{V}} &= [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.28)}{=} \widetilde{V}^{-1} V\,[\boldsymbol{v}]_V = \overline{\widetilde{V}}V\,[\boldsymbol{v}]_V && \overset{(5.23)\text{ or }(5.9)}{=} [V]_{\widetilde{V}}\,[\boldsymbol{v}]_V \\
\widetilde{v}^j &= \widetilde{v}^j = \left(\widetilde{V}^k{}_j\right)^{-1} V^k{}_i v^i = \overline{\widetilde{V}^k{}_j}\,V^k{}_i v^i && = \widetilde{v}^j{}_i v^i \qquad (5.30)\\
&= \left(\widetilde{V}^{-1}\right)^j{}_k V^k{}_i v^i = \overline{\widetilde{V}}^j{}_k V^k{}_i v^i && = \widetilde{v}^j{}_i v^i \\
&\left(\widetilde{V}^{-1}\right)^j{}_k V^k{}_i = \overline{\widetilde{V}}^j{}_k V^k{}_i && = \widetilde{v}^j{}_i \\
\widetilde{v}^j{}_i &= \left(\widetilde{V}^{-1}\right)^j{}_k V^k{}_i = \overline{\widetilde{V}}^j{}_k V^k{}_i \\
&\left(\widetilde{V}^{-1}\right)^j{}_k = \left(\widetilde{V}^k{}_j\right)^{-1} && \overline{\widetilde{V}}^j{}_k = \overline{\widetilde{V}^k{}_j}
\end{aligned}
$$

$$
\begin{aligned}
v^i \boldsymbol{v}_i &\overset{(5.1)(5.3)}{=} \widetilde{v}^j \widetilde{\boldsymbol{v}}_j \overset{(5.30)}{=} \overline{\widetilde{V}^k{}_j}\,V^k{}_i v^i \widetilde{\boldsymbol{v}}_j = \widetilde{v}^j{}_i v^i \widetilde{\boldsymbol{v}}_j && \widetilde{v}^j = \overline{\widetilde{V}^k{}_j}\,V^k{}_i v^i = \widetilde{v}^j{}_i v^i \\
v^i \boldsymbol{v}_i &= \overline{\widetilde{V}^k{}_j}\,V^k{}_i v^i \widetilde{\boldsymbol{v}}_j = \widetilde{v}^j{}_i v^i \widetilde{\boldsymbol{v}}_j && \cancel{v^i} \\
\boldsymbol{v}_i &= \overline{\widetilde{V}^k{}_j}\,V^k{}_i \widetilde{\boldsymbol{v}}_j = \widetilde{v}^j{}_i \widetilde{\boldsymbol{v}}_j && (5.31)\\
\widetilde{v}^j \widetilde{\boldsymbol{v}}_j &\overset{(5.3)(5.1)}{=} v^i \boldsymbol{v}_i \overset{(5.29)}{=} \overline{V^k{}_i}\,\widetilde{V}^k{}_j \widetilde{v}^j \boldsymbol{v}_i = v^i{}_j \widetilde{v}^j \boldsymbol{v}_i && v^i = \overline{V^k{}_i}\,\widetilde{V}^k{}_j \widetilde{v}^j = v^i{}_j \widetilde{v}^j \\
\widetilde{v}^j \widetilde{\boldsymbol{v}}_j &= \overline{V^k{}_i}\,\widetilde{V}^k{}_j \widetilde{v}^j \boldsymbol{v}_i = v^i{}_j \widetilde{v}^j \boldsymbol{v}_i && \cancel{\widetilde{v}^j} \\
\widetilde{\boldsymbol{v}}_j &= \overline{V^k{}_i}\,\widetilde{V}^k{}_j \boldsymbol{v}_i = v^i{}_j \boldsymbol{v}_i && (5.32)
\end{aligned}
$$

$$
v^i \overset{(5.29)}{=} v^i{}_j \widetilde{v}^j = (V^{-1})^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j = \overline{V}^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j \qquad [\boldsymbol{v}]_V \overset{(5.24)}{=} \left[\widetilde{V}\right]_V [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.28)}{=} V^{-1}\widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} = \overline{V}\widetilde{V}\,[\boldsymbol{v}]_{\widetilde{V}} \qquad \text{contravariant} \tag{5.33}
$$

$$
\widetilde{v}^j \overset{(5.30)}{=} \widetilde{v}^j{}_i v^i = \left(\widetilde{V}^{-1}\right)^j{}_k V^k{}_i v^i = \overline{\widetilde{V}}^j{}_k V^k{}_i v^i \qquad [\boldsymbol{v}]_{\widetilde{V}} \overset{(5.23)}{=} [V]_{\widetilde{V}}\,[\boldsymbol{v}]_V \overset{(5.28)}{=} \widetilde{V}^{-1} V\,[\boldsymbol{v}]_V = \overline{\widetilde{V}}V\,[\boldsymbol{v}]_V \qquad \text{contravariant} \tag{5.34}
$$

$$
\boldsymbol{v}_i \overset{(5.31)}{=} \widetilde{v}^j{}_i \widetilde{\boldsymbol{v}}_j = \left(\widetilde{V}^k{}_j\right)^{-1} V^k{}_i \widetilde{\boldsymbol{v}}_j = \overline{\widetilde{V}}^j{}_k V^k{}_i \widetilde{\boldsymbol{v}}_j \qquad \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^{\mathsf T} = \begin{bmatrix}\widetilde{\boldsymbol{v}}_1\\ \vdots \\ \widetilde{\boldsymbol{v}}_n\end{bmatrix}^{\mathsf T} \widetilde{V}^{-1}V = \begin{bmatrix}\widetilde{\boldsymbol{v}}_1\\ \vdots \\ \widetilde{\boldsymbol{v}}_n\end{bmatrix}^{\mathsf T} \overline{\widetilde{V}}V = \begin{bmatrix}\widetilde{\boldsymbol{v}}_1\\ \vdots \\ \widetilde{\boldsymbol{v}}_n\end{bmatrix}^{\mathsf T} [V]_{\widetilde{V}} \qquad \text{covariant} \tag{5.35}
$$

$$
\widetilde{\boldsymbol{v}}_j \overset{(5.32)}{=} v^i{}_j \boldsymbol{v}_i = (V^k{}_i)^{-1}\,\widetilde{V}^k{}_j \boldsymbol{v}_i = \overline{V}^i{}_k \widetilde{V}^k{}_j \boldsymbol{v}_i \qquad \begin{bmatrix}\widetilde{\boldsymbol{v}}_1\\ \vdots \\ \widetilde{\boldsymbol{v}}_n\end{bmatrix}^{\mathsf T} = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^{\mathsf T} V^{-1}\widetilde{V} = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^{\mathsf T} \overline{V}\widetilde{V} = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^{\mathsf T} \left[\widetilde{V}\right]_V \qquad \text{covariant} \tag{5.36}
$$

$$
\left\{
\begin{array}{ll}
\left\{
\begin{array}{l}
v^i \overset{(5.29)}{=} v^i{}_j \widetilde{v}^j = (V^{-1})^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j = \overline{V}^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j \qquad \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \left[\widetilde{V}\right]_V \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = V^{-1}\widetilde{V} \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = \overline{V}\widetilde{V} \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} \\
\widetilde{v}^j \overset{(5.30)}{=} \widetilde{v}^j{}_i v^i = \left(\widetilde{V}^{-1}\right)^j{}_k V^k{}_i v^i = \overline{\widetilde{V}}^j{}_k V^k{}_i v^i \qquad \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = [V]_{\widetilde{V}} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \widetilde{V}^{-1}V \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \overline{\widetilde{V}}V \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}
\end{array}
\right. & \text{contravariant} \\
\left\{
\begin{array}{l}
\boldsymbol{v}_i \overset{(5.31)}{=} \widetilde{\boldsymbol{v}}_j \widetilde{v}^j{}_i = \widetilde{\boldsymbol{v}}_j \left(\widetilde{V}^k{}_j\right)^{-1} V^k{}_i = \widetilde{\boldsymbol{v}}_j \overline{\widetilde{V}}^j{}_k V^k{}_i \qquad \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal [V]_{\widetilde{V}} = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal \widetilde{V}^{-1}V = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal \overline{\widetilde{V}}V \\
\widetilde{\boldsymbol{v}}_j \overset{(5.32)}{=} \boldsymbol{v}_i v^i{}_j = \boldsymbol{v}_i \left(V^k{}_i\right)^{-1} \widetilde{V}^k{}_j = \boldsymbol{v}_i \overline{V}^i{}_k \widetilde{V}^k{}_j \qquad \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal \left[\widetilde{V}\right]_V = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal V^{-1}\widetilde{V} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal \overline{V}\widetilde{V}
\end{array}
\right. & \text{covariant}
\end{array}
\right.
$$

$$
[V]_{\widetilde{V}} \left[\widetilde{V}\right]_V \overset{(5.9)(5.15)}{=} \left(\widetilde{V}^{-1}V\right)\left(V^{-1}\widetilde{V}\right) = \widetilde{V}^{-1}\left(VV^{-1}\right)\widetilde{V} = \widetilde{V}^{-1}1\widetilde{V} = \widetilde{V}^{-1}\widetilde{V} = 1 \qquad \Rightarrow \widetilde{v}^k{}_i v^i{}_j = \delta^k{}_j \tag{5.37}
$$

$$
\left[\widetilde{V}\right]_V [V]_{\widetilde{V}} \overset{(5.15)(5.9)}{=} \left(V^{-1}\widetilde{V}\right)\left(\widetilde{V}^{-1}V\right) = V^{-1}\left(\widetilde{V}\widetilde{V}^{-1}\right)V = V^{-1}1V = V^{-1}V = 1 \qquad \Rightarrow v^i{}_j \widetilde{v}^j{}_k = \delta^i{}_k \tag{5.38}
$$

$$
\left\{
\begin{array}{l}
[V]_{\widetilde{V}} \left[\widetilde{V}\right]_V \overset{(5.9)(5.15)}{=} \left(\overline{\widetilde{V}}V\right)\left(\overline{V}\widetilde{V}\right) = \overline{\widetilde{V}}\left(V\overline{V}\right)\widetilde{V} = \overline{\widetilde{V}}1\widetilde{V} = \overline{\widetilde{V}}\widetilde{V} = 1 \quad \Rightarrow \widetilde{v}^k{}_i v^i{}_j = \delta^k{}_j \\
\left[\widetilde{V}\right]_V [V]_{\widetilde{V}} \overset{(5.15)(5.9)}{=} \left(\overline{V}\widetilde{V}\right)\left(\overline{\widetilde{V}}V\right) = \overline{V}\left(\widetilde{V}\overline{\widetilde{V}}\right)V = \overline{V}1V = \overline{V}V = 1 \quad \Rightarrow v^i{}_j \widetilde{v}^j{}_k = \delta^i{}_k
\end{array}
\right. \tag{5.39}
$$

$$
\begin{aligned}
[V]_{\widetilde{V}} \left[\widetilde{V}\right]_V &\overset{(5.12)(5.18)}{=} \begin{bmatrix} \widetilde{v}_1{}^1 & \cdots & \widetilde{v}_n{}^1 \\ \vdots & \ddots & \vdots \\ \widetilde{v}_1{}^n & \cdots & \widetilde{v}_n{}^n \end{bmatrix} \begin{bmatrix} v_1{}^1 & \cdots & v_n{}^1 \\ \vdots & \ddots & \vdots \\ v_1{}^n & \cdots & v_n{}^n \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{v}^1{}_1 & \cdots & \widetilde{v}^1{}_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n{}_1 & \cdots & \widetilde{v}^n{}_n \end{bmatrix} \begin{bmatrix} v^1{}_1 & \cdots & v^1{}_n \\ \vdots & \ddots & \vdots \\ v^n{}_1 & \cdots & v^n{}_n \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{v}^1_1 & \cdots & \widetilde{v}^1_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n_1 & \cdots & \widetilde{v}^n_n \end{bmatrix} \begin{bmatrix} v^1_1 & \cdots & v^1_n \\ \vdots & \ddots & \vdots \\ v^n_1 & \cdots & v^n_n \end{bmatrix} \qquad \overset{(5.39)}{=} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = 1 \\
\left[\widetilde{V}\right]_V [V]_{\widetilde{V}} &\overset{(5.18)(5.12)}{=} \begin{bmatrix} v_1{}^1 & \cdots & v_n{}^1 \\ \vdots & \ddots & \vdots \\ v_1{}^n & \cdots & v_n{}^n \end{bmatrix} \begin{bmatrix} \widetilde{v}_1{}^1 & \cdots & \widetilde{v}_n{}^1 \\ \vdots & \ddots & \vdots \\ \widetilde{v}_1{}^n & \cdots & \widetilde{v}_n{}^n \end{bmatrix} \\
&= \begin{bmatrix} v^1{}_1 & \cdots & v^1{}_n \\ \vdots & \ddots & \vdots \\ v^n{}_1 & \cdots & v^n{}_n \end{bmatrix} \begin{bmatrix} \widetilde{v}^1{}_1 & \cdots & \widetilde{v}^1{}_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n{}_1 & \cdots & \widetilde{v}^n{}_n \end{bmatrix} \\
&= \begin{bmatrix} v^1_1 & \cdots & v^1_n \\ \vdots & \ddots & \vdots \\ v^n_1 & \cdots & v^n_n \end{bmatrix} \begin{bmatrix} \widetilde{v}^1_1 & \cdots & \widetilde{v}^1_n \\ \vdots & \ddots & \vdots \\ \widetilde{v}^n_1 & \cdots & \widetilde{v}^n_n \end{bmatrix} \qquad \overset{(5.39)}{=} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = 1
\end{aligned}
$$

$$
\begin{cases}
\begin{cases}
v^i \overset{(5.29)}{=} v^i{}_j \widetilde{v}^j = \overline{V}^i{}_k \widetilde{V}^k{}_j \widetilde{v}^j & \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & v^i{}_j & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = \left[\widetilde{V}\right]_V \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = \overline{V}\widetilde{V} \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} \\
\widetilde{v}^j \overset{(5.30)}{=} \widetilde{v}^j{}_i v^i = \overline{\widetilde{V}}^j{}_k V^k{}_i v^i & \begin{bmatrix} \widetilde{v}^1 \\ \vdots \\ \widetilde{v}^n \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & \widetilde{v}^j{}_i & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = [V]_{\widetilde{V}} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \overline{\widetilde{V}} V \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}
\end{cases} & \text{contravariant} \\
\begin{cases}
\boldsymbol{v}_i \overset{(5.31)}{=} \widetilde{\boldsymbol{v}}_j \widetilde{v}^j{}_i = \widetilde{\boldsymbol{v}}_j \overline{\widetilde{V}}^j{}_k V^k{}_i & \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ \cdots & \widetilde{v}^j{}_i & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal [V]_{\widetilde{V}} = \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal \overline{\widetilde{V}} V \\
\widetilde{\boldsymbol{v}}_j \overset{(5.32)}{=} \boldsymbol{v}_i v^i{}_j = \boldsymbol{v}_i \overline{V}^i{}_k \widetilde{V}^k{}_j & \begin{bmatrix} \widetilde{\boldsymbol{v}}_1 \\ \vdots \\ \widetilde{\boldsymbol{v}}_n \end{bmatrix}^\intercal = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ \cdots & v^i{}_j & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal \left[\widetilde{V}\right]_V = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix}^\intercal \overline{V}\widetilde{V}
\end{cases} & \text{covariant}
\end{cases}
$$

We do not denote $V\,[\boldsymbol{v}]_V = V\,[\boldsymbol{v}]_{\mathfrak{V}}$, because $\mathfrak{V}$ can have elements or bases in different orders whereas V cannot.

5.1.2 covector

$$\begin{cases}\boldsymbol{v}\in\mathcal{V}\subseteq\mathbb{F}^n\in\{\mathbb{R}^n,\mathbb{C}^n,\cdots\}\\ \exists!\omega\in\mathbb{F}\left[\boldsymbol{\omega}\left(\boldsymbol{v}\right)=\omega\right]\end{cases}\begin{aligned}&\Leftrightarrow\mathcal{V}\overset{\omega}{\to}\mathbb{F}\Leftrightarrow\boldsymbol{\omega}:\mathcal{V}\to\mathbb{F}\\ &\Leftrightarrow\mathbb{F}^{\mathcal{V}}=\{\boldsymbol{\omega}|\boldsymbol{\omega}:\mathcal{V}\to\mathbb{F}\}\\ &\Downarrow\\ &\left|\mathbb{F}^{\mathcal{V}}\right|=|\mathbb{F}|^{|\mathcal{V}|}\end{aligned}$$

$$\begin{array}{ccccc}\boldsymbol{v}^1\left(\boldsymbol{v}_1\right)=1 & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_n\right)\\ \vdots & & \vdots & & \vdots\\ \boldsymbol{v}^i\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}}{=}\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases}=\delta^i_j & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_n\right)\\ \vdots & & \vdots & & \vdots\\ \boldsymbol{v}^n\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_n\right)=1\end{array}\tag{5.40}$$

$$\boldsymbol{v}^i\left(\boldsymbol{v}\right)=\boldsymbol{v}^i\left(v^j\boldsymbol{v}_j\right)=v^j\boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}(5.40)}{=}v^j\delta^i_j=v^i$$

$$\begin{aligned}&\begin{cases}\boldsymbol{\omega}\in\mathcal{V}^*=(\mathcal{V}^*,\mathbb{F},+,\cdot)=(\mathcal{V}^*,\mathbb{F},+_{\mathcal{V}^*,\mathbb{F}},\cdot_{\mathbb{F}\times\mathcal{V}^*,\mathbb{F}})\\ \boldsymbol{v}\in\mathcal{V}=(\mathcal{V},\mathbb{F},+,\cdot)=(\mathcal{V},\mathbb{F},+_{\mathcal{V},\mathbb{F}},\cdot_{\mathbb{F}\times\mathcal{V},\mathbb{F}})\end{cases}\\ \boldsymbol{\omega}\left(\boldsymbol{v}\right)=&\boldsymbol{\omega}\left(v^j\boldsymbol{v}_j\right)=v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\ =&\boldsymbol{\omega}\left(\sum_j v^j\boldsymbol{v}_j\right)=\sum_j\boldsymbol{\omega}\left(v^j\boldsymbol{v}_j\right)=\sum_j v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\ =&\begin{cases}\boldsymbol{\omega}\left(v^1\boldsymbol{v}_1+\cdots+v^n\boldsymbol{v}_n\right) & =\boldsymbol{\omega}\left(\sum_{j=1}^n v^j\boldsymbol{v}_j\right)\\ \boldsymbol{\omega}\left(\cdots+v^j\boldsymbol{v}_j+\cdots\right) & =\boldsymbol{\omega}\left(\sum_{j\in J}v^j\boldsymbol{v}_j\right)\end{cases} && +=+_{\mathcal{V},\mathbb{F}}\\ =&\begin{cases}v^1\boldsymbol{\omega}\left(\boldsymbol{v}_1\right)+\cdots+v^n\boldsymbol{\omega}\left(\boldsymbol{v}_n\right) & =\sum_{j=1}^n v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\ \cdots+v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)+\cdots & =\sum_{j\in J}v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\end{cases} && +=+_{\mathcal{V}^*,\mathbb{F}}\\ =&v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)=\boldsymbol{v}^j\left(\boldsymbol{v}\right)\boldsymbol{\omega}\left(\boldsymbol{v}_j\right) && v^j=\boldsymbol{v}^j\left(\boldsymbol{v}\right)\Leftarrow\boldsymbol{v}^i\left(\boldsymbol{v}\right)=v^i\Leftarrow\boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}(5.40)}{=}\delta^i_j\\ =&\boldsymbol{v}^j\left(\boldsymbol{v}\right)\omega^{\boldsymbol{v}}_j=\omega^{\boldsymbol{v}}_j\boldsymbol{v}^j\left(\boldsymbol{v}\right)=\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\left(\boldsymbol{v}\right) && \omega^{\boldsymbol{v}}_j\overset{\text{def.}}{=}\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\end{aligned}\tag{5.41}$$

$$\begin{aligned}\boldsymbol{\omega}\left(\boldsymbol{v}\right)=&\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\left(\boldsymbol{v}\right)\\ \boldsymbol{\omega}=&\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\end{aligned}\tag{5.42}$$

$$
\begin{aligned}
\mathcal{V}^* \ni \boldsymbol{\omega} &= \omega_i \boldsymbol{\omega}^i = \sum_i \omega_i \boldsymbol{\omega}^i = \begin{cases} \omega_1 \boldsymbol{\omega}^1 + \cdots + \omega_n \boldsymbol{\omega}^n & = \sum_{i=1}^{n} \omega_i \boldsymbol{\omega}^i \\ \cdots + \omega_i \boldsymbol{\omega}^i + \cdots & = \sum_{i\in I} \omega_i \boldsymbol{\omega}^i \end{cases} \\
&\overset{(5.42)}{=} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i = \sum_i \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i = \begin{cases} \omega_1^{\boldsymbol{v}} \boldsymbol{v}^1 + \cdots + \omega_n^{\boldsymbol{v}} \boldsymbol{v}^n & = \sum_{i=1}^{n} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \\ \cdots + \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i + \cdots & = \sum_{i\in I} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \end{cases}
\end{aligned} \tag{5.43}
$$

$$
= \begin{cases} \omega_1^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^1 \\ | \end{bmatrix}^{\intercal} + \cdots + \omega_n^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^n \\ | \end{bmatrix}^{\intercal} & = \begin{bmatrix} \omega_1^{\boldsymbol{v}} \\ \vdots \\ \omega_n^{\boldsymbol{v}} \end{bmatrix}^{\intercal} \begin{bmatrix} — & \boldsymbol{v}^1 & — \\ & \vdots & \\ — & \boldsymbol{v}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^i \\ | \end{bmatrix}^{\intercal} + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} & \vdots & \\ — & \boldsymbol{v}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ \boldsymbol{v}^i \\ \vdots \end{bmatrix} = [\boldsymbol{\omega}]^{V} V^* \tag{5.44}
$$

$$
\overset{(5.42)}{=} \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i = \sum_i \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i = \begin{cases} \omega_1^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^1 + \cdots + \omega_n^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^n & = \sum_{i=1}^{n} \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i \\ \cdots + \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i + \cdots & = \sum_{i\in I} \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i \end{cases} \tag{5.45}
$$

$$
= \begin{cases} \omega_1^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^1 \\ | \end{bmatrix}^{\intercal} + \cdots + \omega_n^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^n \\ | \end{bmatrix}^{\intercal} & = \begin{bmatrix} \omega_1^{\widetilde{\boldsymbol{v}}} \\ \vdots \\ \omega_n^{\widetilde{\boldsymbol{v}}} \end{bmatrix}^{\intercal} \begin{bmatrix} — & \widetilde{\boldsymbol{v}}^1 & — \\ & \vdots & \\ — & \widetilde{\boldsymbol{v}}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^i \\ | \end{bmatrix}^{\intercal} + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} & \vdots & \\ — & \widetilde{\boldsymbol{v}}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ \widetilde{\boldsymbol{v}}^i \\ \vdots \end{bmatrix} = [\boldsymbol{\omega}]^{\widetilde{V}} \widetilde{V}^* \tag{5.46}
$$

$$
\begin{aligned}
\boldsymbol{\omega} &\overset{(5.44)}{=} [\boldsymbol{\omega}]^{V} V^* \overset{(5.46)}{=} [\boldsymbol{\omega}]^{\widetilde{V}} \widetilde{V}^* \\
&= \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} = \omega_j^{\widetilde{\boldsymbol{v}}} {\widetilde{V}^*}{}^{j}{}_{k} \\
\omega_j^{\widetilde{\boldsymbol{v}}} {\widetilde{V}^*}{}^{j}{}_{k} &= \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} \\
\omega_j^{\widetilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} \left({\widetilde{V}^*}{}^{j}{}_{k} \right)^{-1} \qquad = \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} \left(\widetilde{V}^{*-1} \right)^{k}{}_{j} = \omega_i^{\boldsymbol{v}} q^{i}{}_{j} \\
&= \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} \overline{{\widetilde{V}^*}{}^{j}{}_{k}} \qquad = \omega_i^{\boldsymbol{v}} {V^*}^{i}{}_{k} \overline{\widetilde{V}^*}{}^{k}{}_{j} = \omega_i^{\boldsymbol{v}} q^{i}{}_{j} \\
\boldsymbol{\omega}(\widetilde{\boldsymbol{v}}_j) \overset{(5.41)}{=} \omega_j^{\widetilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} q^{i}{}_{j} = \boldsymbol{\omega}(\boldsymbol{v}_i)\, q^{i}{}_{j} \qquad = \boldsymbol{\omega}(\widetilde{\boldsymbol{v}}_k \widetilde{v}^{k}{}_{i})\, q^{i}{}_{j} = \boldsymbol{\omega}(\widetilde{\boldsymbol{v}}_k)\, \widetilde{v}^{k}{}_{i} q^{i}{}_{j} \\
\boldsymbol{\omega}(\widetilde{\boldsymbol{v}}_j) &= \boldsymbol{\omega}(\widetilde{\boldsymbol{v}}_k)\, \widetilde{v}^{k}{}_{i} q^{i}{}_{j} \\
\widetilde{v}^{k}{}_{i} q^{i}{}_{j} &= \delta^{k}{}_{j} \overset{(5.39)}{\Rightarrow} q^{i}{}_{j} = v^{i}{}_{j} \\
\omega_j^{\widetilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} q^{i}{}_{j} = \omega_i^{\boldsymbol{v}} v^{i}{}_{j} \\
\omega_j^{\widetilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} v^{i}{}_{j} \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^{\intercal} = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^{\intercal} \left[\widetilde{V}\right]_{V} && (5.47) \\
\omega_k^{\widetilde{\boldsymbol{v}}} \widetilde{v}^{k}{}_{j} &= \omega_i^{\boldsymbol{v}} v^{i}{}_{k} \widetilde{v}^{k}{}_{j} = \omega_i^{\boldsymbol{v}} \delta^{i}{}_{j} = \omega_j^{\boldsymbol{v}} \\
\omega_j^{\boldsymbol{v}} &= \omega_k^{\widetilde{\boldsymbol{v}}} \widetilde{v}^{k}{}_{j} \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^{\intercal} = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^{\intercal} [V]_{\widetilde{V}} && (5.48)
\end{aligned}
$$

$$
\begin{aligned}
\omega_j^{\widetilde{\boldsymbol{v}}} \widetilde{v}^{j}{}_{i} \boldsymbol{v}^i \overset{(5.47)}{=} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i &= \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i \overset{(5.48)}{=} \omega_j^{\boldsymbol{v}} v^{j}{}_{i} \widetilde{\boldsymbol{v}}^i && (5.43) = (5.45) \\
\omega_j^{\widetilde{\boldsymbol{v}}} \widetilde{v}^{j}{}_{i} \boldsymbol{v}^i &= \omega_j^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^j \Rightarrow \widetilde{v}^{j}{}_{i} \boldsymbol{v}^i = \widetilde{\boldsymbol{v}}^j \Rightarrow \widetilde{\boldsymbol{v}}^j = \widetilde{v}^{k}{}_{i} \boldsymbol{v}^i && (5.49) \\
\omega_j^{\boldsymbol{v}} v^{j}{}_{i} \widetilde{\boldsymbol{v}}^i &= \omega_j^{\boldsymbol{v}} \boldsymbol{v}^j \Rightarrow v^{j}{}_{i} \widetilde{\boldsymbol{v}}^i = \boldsymbol{v}^j \Rightarrow \boldsymbol{v}^j = v^{j}{}_{i} \widetilde{\boldsymbol{v}}^i && (5.50)
\end{aligned}
$$

$$
\left\{\begin{array}{ll}
\left\{\begin{array}{lll}
\omega_j^{\boldsymbol{v}} = \omega_k^{\widetilde{\boldsymbol{v}}}\,\widetilde{v}^k{}_j & \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^{\intercal} = \begin{bmatrix}\vdots\\ \omega_i^{\widetilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^{\intercal} [V]_{\widetilde{V}} & (5.48)\\
\omega_j^{\widetilde{\boldsymbol{v}}} = \omega_i^{\boldsymbol{v}}\, v^i{}_j & \begin{bmatrix}\vdots\\ \omega_i^{\widetilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^{\intercal} = \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^{\intercal} \left[\widetilde{V}\right]_V & (5.47)
\end{array}\right. & \text{covariant}\\
\left\{\begin{array}{lll}
\boldsymbol{v}^j = v^j{}_i\,\widetilde{\boldsymbol{v}}^i & \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} = \left[\widetilde{V}\right]_V \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} & (5.50)\\
\widetilde{\boldsymbol{v}}^j = \widetilde{v}^k{}_i\,\boldsymbol{v}^i & \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} = [V]_{\widetilde{V}} \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} & (5.49)
\end{array}\right. & \text{contravariant}
\end{array}\right.
$$

$$
\left\{\begin{array}{ll}
\left\{\begin{array}{lll}
\boldsymbol{v}^j = v^j{}_i\,\widetilde{\boldsymbol{v}}^i & \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} = \left[\widetilde{V}\right]_V \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} & (5.50)\\
\widetilde{\boldsymbol{v}}^j = \widetilde{v}^k{}_i\,\boldsymbol{v}^i & \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} = [V]_{\widetilde{V}} \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} & (5.49)
\end{array}\right. & \text{contravariant}\\
\left\{\begin{array}{lll}
\omega_j^{\boldsymbol{v}} = \omega_k^{\widetilde{\boldsymbol{v}}}\,\widetilde{v}^k{}_j & \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^{\intercal} = \begin{bmatrix}\vdots\\ \omega_i^{\widetilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^{\intercal} [V]_{\widetilde{V}} & (5.48)\\
\omega_j^{\widetilde{\boldsymbol{v}}} = \omega_i^{\boldsymbol{v}}\, v^i{}_j & \begin{bmatrix}\vdots\\ \omega_i^{\widetilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^{\intercal} = \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^{\intercal} \left[\widetilde{V}\right]_V & (5.47)
\end{array}\right. & \text{covariant}
\end{array}\right.
$$

covariant | contravariant

$$
\left.\begin{array}{l}\widetilde{\mathfrak{V}}\\ \mathfrak{V}\end{array}\right\} \ni \begin{cases} \widetilde{\boldsymbol{v}}_j = \boldsymbol{v}_i v^i{}_j & (5.32)\\ \boldsymbol{v}_j = \widetilde{\boldsymbol{v}}_i \widetilde{v}^i{}_j & (5.31)\end{cases}
\qquad
\mathbb{F} \ni \begin{cases} \widetilde{v}^i = \widetilde{v}^i{}_j v^j & (5.30)\\ v^i = v^i{}_j \widetilde{v}^j & (5.29)\end{cases}
$$

vector space $\mathcal{V} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j$

$$
\mathbb{F} \ni \begin{cases} \omega_j^{\widetilde{\boldsymbol{v}}} = \omega_i^{\boldsymbol{v}} v^i{}_j & (5.47)\\ \omega_j^{\boldsymbol{v}} = \omega_k^{\widetilde{\boldsymbol{v}}} \widetilde{v}^k{}_j & (5.48)\end{cases}
\qquad
\left.\begin{array}{l}\widetilde{\mathfrak{V}}^*\\ \mathfrak{V}^*\end{array}\right\} \ni \begin{cases} \widetilde{\boldsymbol{v}}^i = \widetilde{v}^i{}_j \boldsymbol{v}^j & (5.49)\\ \boldsymbol{v}^i = v^i{}_j \widetilde{\boldsymbol{v}}^j & (5.50)\end{cases}
$$

dual space $\mathcal{V}^* \ni \boldsymbol{\omega} = \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i$

$$
\widetilde{\boldsymbol{v}}_j \widetilde{v}^j = \boldsymbol{v}_i v^i{}_j \widetilde{v}^j{}_k v^k = \boldsymbol{v}_i \delta^i{}_k v^k = \begin{cases} \boldsymbol{v}_k v^k & \boldsymbol{v}_k = \boldsymbol{v}_i \delta^i{}_k\\ \boldsymbol{v}_i v^i & \delta^i{}_k v^k = v^i\end{cases} = v^j \boldsymbol{v}_j
$$

5.2 tensor calculus

Part III

physics