Knowledge of Everything I Need to Know

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January 27, 2023

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0.1 LyX & LaTeX

致謝 0.1.1. TeX by Turing-Awarded Donald Ervin Knuth (自取中文名: 高德納) https://zh.wikipedia.org/zh-tw/高德納, who also writes multi-volume book The Art of Computer Programming 電腦程式設計藝術 / 計算機程式設計藝術 with TeX.

0.1.1 LyX: 文件 > 設定值 > 文件類別

Document Class: Book

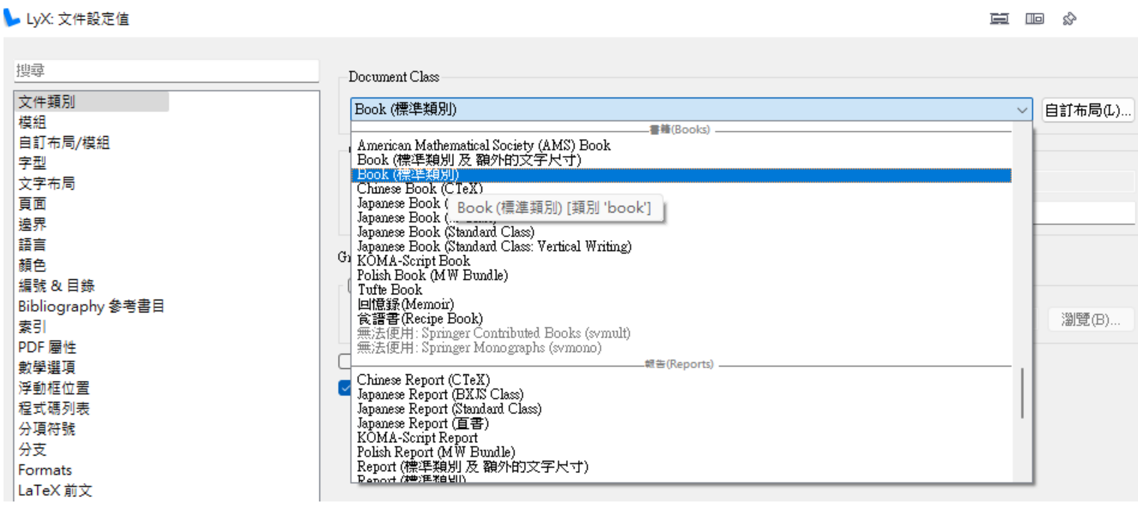


Figure 0.1.1: document class

0.1.2 LyX: 文件 > 設定值 > 模組

Figure 0.1.2: module

0.1.3 LyX: 文件 > 設定值 > 字型

預設字族: 無襯線
    CJK: bsmi

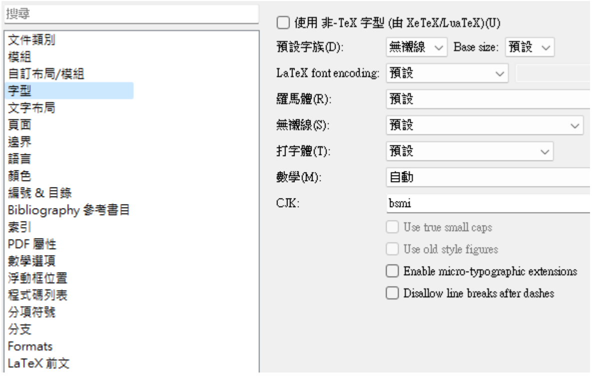


Figure 0.1.3: font

0.1.4 LyX: 文件 > 設定值 > 頁面

格式: A4
    頁面樣式: headings每頁都放
    ✓ 兩面的文件

Figure 0.1.4: page

0.1.5 LyX: 文件 > 設定值 > 邊界

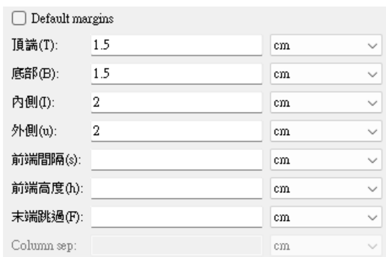


Figure 0.1.5: margin

0.1.6 LyX: 文件 > 設定值 > 語言

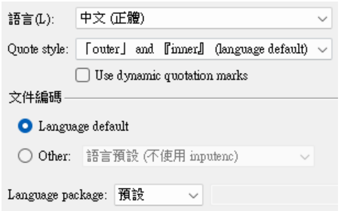


Figure 0.1.6: language

0.1.7 LyX: 文件 > 設定值 > PDF 屬性

超連結

Additional Options: linkcolor=blue[1]

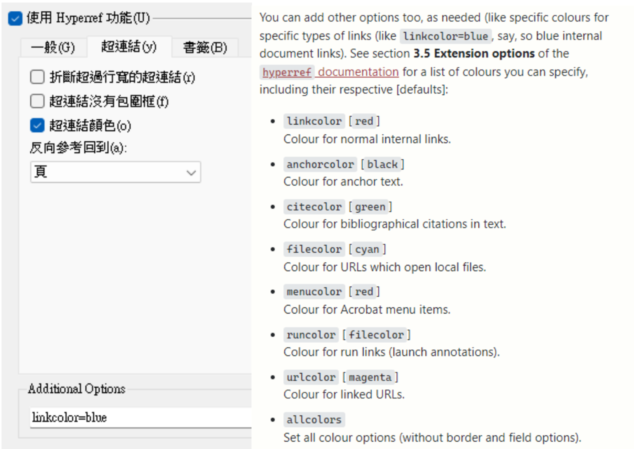


Figure 0.1.7: hyperlink

Figure 0.1.8: before & after

[1] https://tex.stackexchange.com/questions/407340/how-to-highlight-references-instead-of-boxes-in-lyx

書籤

Figure 0.1.9: bookmark

0.1.8 LyX: 文件 > 設定值 > 浮動框位置

肯定在此

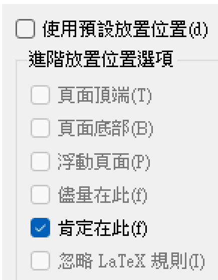


Figure 0.1.10: float

0.2 drawing or plotting

0.2.1 MathCha 數學抹茶 & GeoGebra

I used to use MathJax, a script language for blog HTML embedding; but MathCha[2] is a more powerful web-based .io and easily to use, especially for drawing and plotting.

Geogebra[3], a web-based geometry calculator platform, or other iPad geometry calculators can also be used for drawing or plotting.

MathPix Snip is another useful OCR App to copy MathML or LaTeXetc. forms of equations.

LyX is a WYSIWYG LaTeX platform, way easier than OverLeaf etc. online LaTeXplatforms.

[2] https://www.mathcha.io/editor
[3] https://www.geogebra.org/calculator

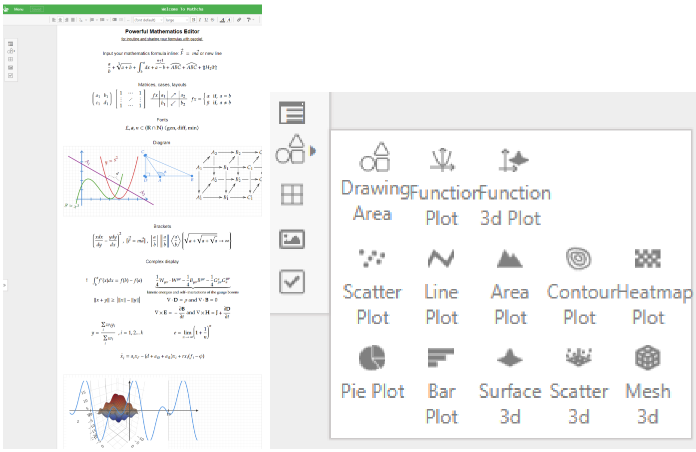


Figure 0.2.1: MathCha

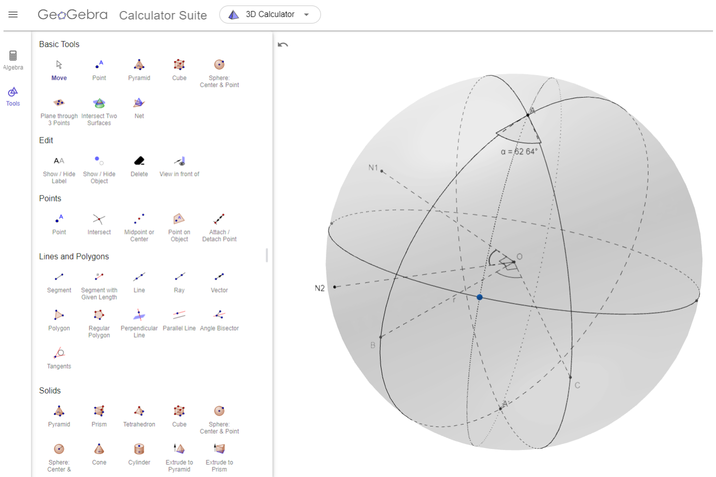


Figure 0.2.2: GeoGebra

0.2.2 TikZ

0.2.2.1 use TikZ package

\usepackage{tikz}

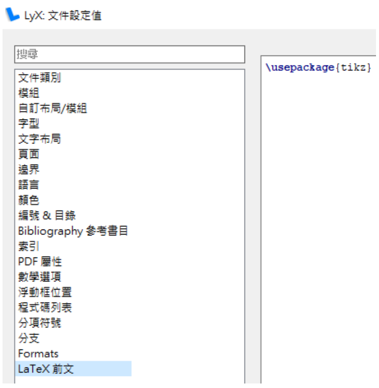


Figure 0.2.3: use TikZ package

0.2.2.2 LyX: 工具 > 偏好設定 > 外觀 & 感覺 > 顯示

Instant preview: 開

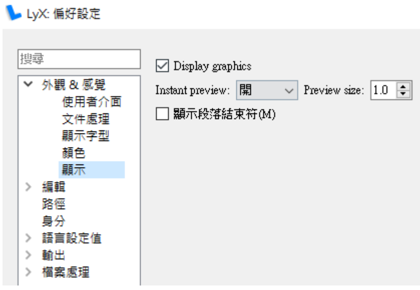
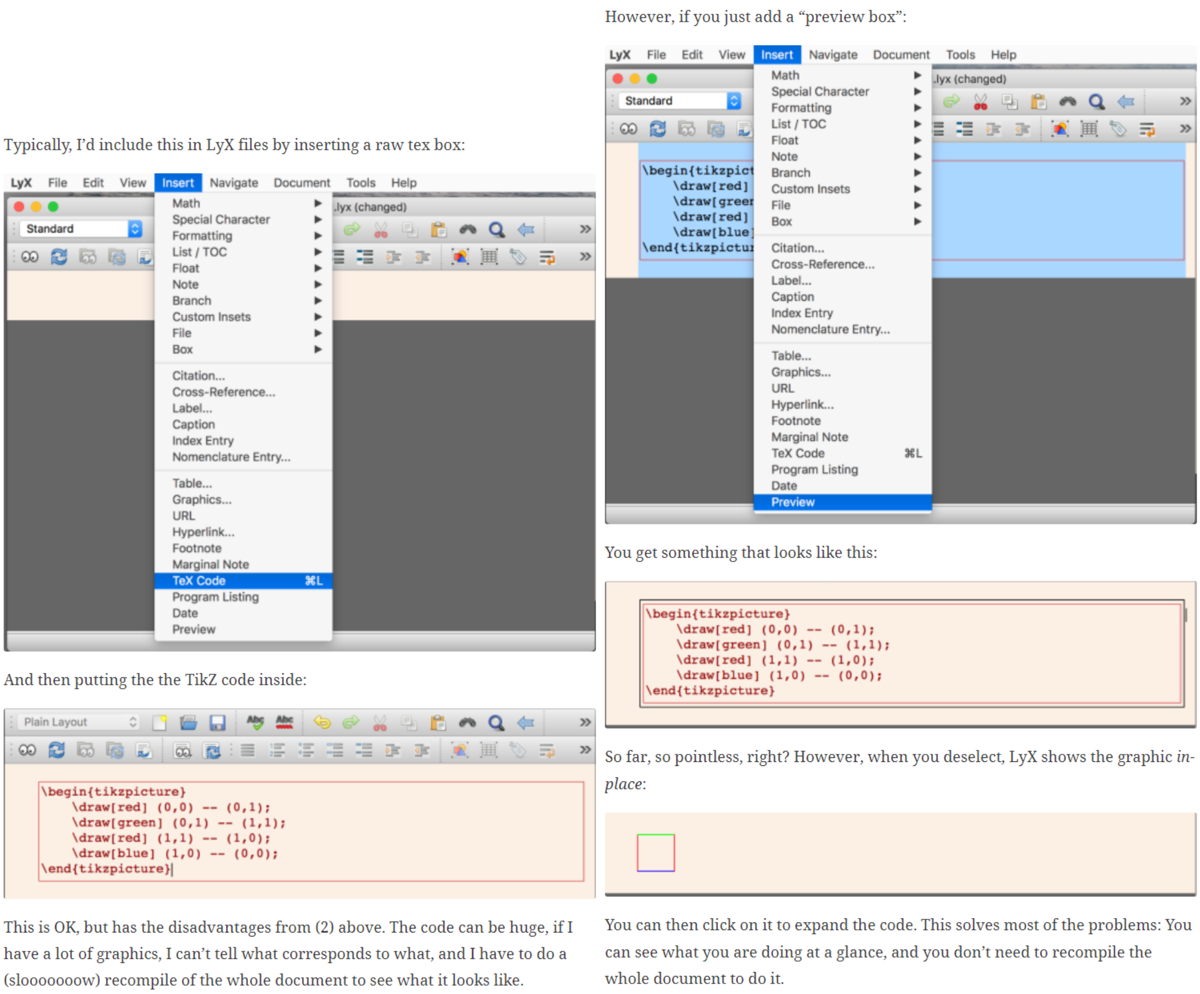


Figure 0.2.4: Instant preview: On

0.2.2.3 preview TikZ in TEXcode box[4]



Typically, I'd include this in LyX files by inserting a raw tex box:

And then putting the the TikZ code inside:

```
\begin{tikzpicture}
    \draw[red] (0,0) -- (0,1);
    \draw[green] (0,1) -- (1,1);
    \draw[red] (1,1) -- (1,0);
    \draw[blue] (1,0) -- (0,0);
\end{tikzpicture}
```

This is OK, but has the disadvantages from (2) above. The code can be huge, if I have a lot of graphics, I can't tell what corresponds to what, and I have to do a (slooooooow) recompile of the whole document to see what it looks like.

However, if you just add a "preview box":

You get something that looks like this:

So far, so pointless, right? However, when you deselect, LyX shows the graphic *in-place*:

You can then click on it to expand the code. This solves most of the problems: You can see what you are doing at a glance, and you don't need to recompile the whole document to do it.

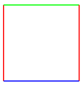
Figure 0.2.5: preview of TikZ code inside a TEXcode box

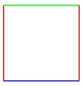
LyX: 插入 > TEX碼, LyX: 插入 > 預覽

put the TEXcode box into a preview box: Insert > TEXCode, then Insert > Preview

\begin{tikzpicture}
\draw[red] (0,0) -- (0,1);
\draw[green] (0,1) -- (1,1);
\draw[red] (1,1) -- (1,0);
\draw[blue] (1,0) -- (0,0);
\end{tikzpicture}

[4] Justin Domke: The second and third best features of LyX you aren't using. https://justindomke.wordpress.com/2017/05/24/the-second-and-third-best-features-of--you-arent-using/





0.2.2.4 copy TikZ code to clipboard from MathCha, then paste with Ctrl+Shift+V into a TEXcode box in LyX to preserve line breaks[5]

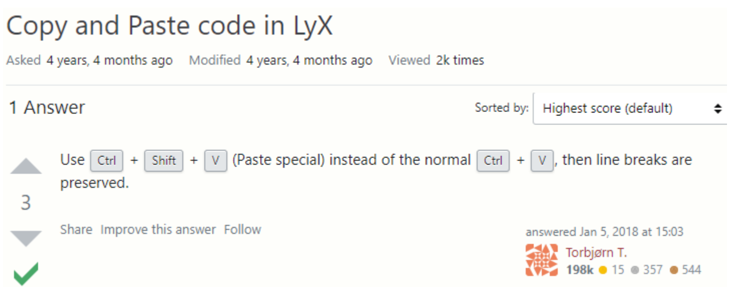


Figure 0.2.6: paste with Ctrl+Shift+V into a TEXcode box in LyX to preserve line breaks

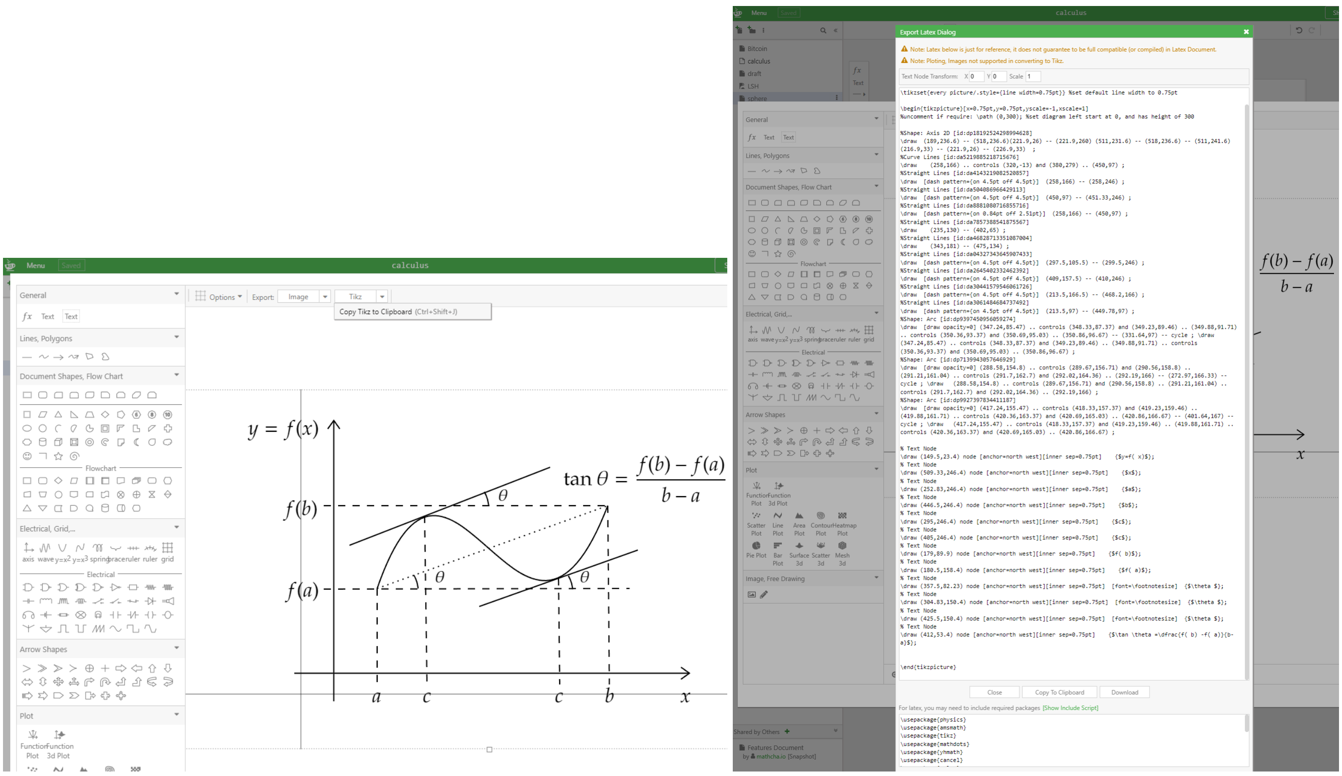


Figure 0.2.7: copy TikZ code to clipboard from MathCha

[5] https://tex.stackexchange.com/questions/408970/copy-and-paste-code-in-

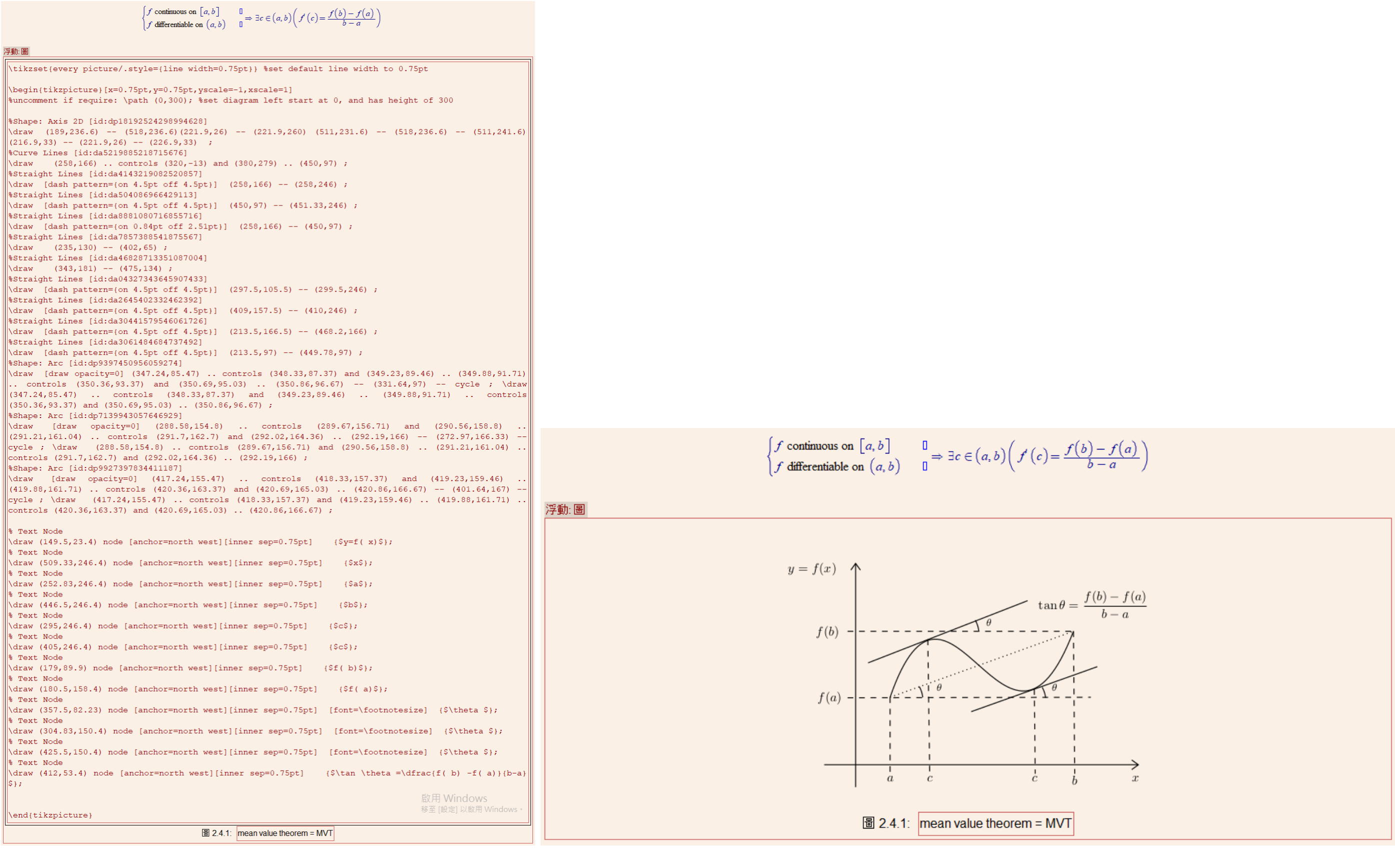


Figure 0.2.8: paste TikZ code with Ctrl+Shift+V into a TEXcode box inside the preview box in LyX to preserve line breaks

0.2.2.5 PGFPlots axis

0.2.2.6 TikZ editor

- TikzEdt http://www.tikzedt.org/
- LaTeXDraw http://latexdraw.sourceforge.net/
- TikZiT https://tikzit.github.io/: for node and edge drawing

0.2.3 xypic in LyX

0.3 reference, citation, and bibliography

0.3.1 Zotero

0.3.1.1 LyZ

0.3.2 LyX

https://www.overleaf.com/learn/latex/Bibtex_bibliography_styles

0.3.2.1 multiple bibliographies

error –> show output anyway

0.3.2.2 NatBib

\usepackage[square]{natbib}

Part I

數學 mathematics

Chapter 1

邏輯 與 集合論 logic & set theory

What is mathematics?

定理 $n \Leftarrow$ 定理 $n-1 \Leftarrow$ 定理 $n-2 \Leftarrow \cdots \Leftarrow$ 定理 $2 \Leftarrow$ 定理 $1 \Leftarrow$ 公理s

定義 $n \leftarrow$ 定義 $n-1 \leftarrow$ 定義 $n-2 \leftarrow \cdots \leftarrow$ 定義 $2 \leftarrow$ 定義 $1 \leftarrow$ 無定義(名詞 或 形容詞 或 關係)s

致謝 1.0.1. Seymour Lipschutz_1964_Schaum's Outline of Theory and Problems of Set Theory_1e

致謝 1.0.2. 黎蒲樹(Seymour Lipschutz)著，劉增福譯_1991_集合論導引_水牛出版

致謝 1.0.3. Seymour Lipschutz_1998_Schaum's Outline of Theory and Problems of Set Theory_2e

1.1 快速基本數學謂詞或述語邏輯 naive predicte logic / naïve logic

定義 1.1.1. 命題 proposition

$$p(x)$$

定義 1.1.2. 非命題 negation of proposition

$$\neg p(x)$$

定義 1.1.3. 真值 / 真假值 truth value

$$\text{either } v(p(x)) = \mathrm{T} \text{ or } v(p(x)) = \mathrm{F}$$

定義 1.1.4. 連詞 conjunctive

- conjunction 且 $\wedge$

$$p \wedge q$$

- disjunction 或 $\vee$

$$p \vee q$$

定義 1.1.5. 量詞 quantifier

- universal quantifier 所有 x

$$\forall x$$

- existential quantifier 存在 x

$$\exists x$$

- uniqueness quantifier 存在唯一 x

$$\exists! x$$

定義 1.1.6. 邏輯等價 logical equivalence

$$p \Leftrightarrow q \text{ means } \forall x\,[v(p(x)) \equiv v(q(x))]$$

備註 1.1.7. 量詞以連詞表命題

- universal quantification 所有 $x \cdots$

$$\forall x\,(p(x)) \Leftrightarrow [p(x_1) \wedge p(x_2) \wedge \cdots] \Leftrightarrow \bigwedge_x p(x)$$

- existential quantification 存在 $x\cdots$

$$\exists x\,(p\,(x)) \Leftrightarrow [p\,(x_1) \vee p\,(x_2) \vee \cdots] \Leftrightarrow \bigvee_x p\,(x)$$

- uniqueness quantification 存在唯一 $x\cdots$[3]
  - equivalent definition 1

$$\begin{aligned}
\exists ! x_1\,(p\,(x_1)) &\Leftrightarrow \exists x_1\,[p\,(x_1) \wedge \neg\exists x_2\,(p\,(x_2) \wedge (x_2 \neq x_1))] \\
&\Leftrightarrow \exists x_1 \left[\begin{cases} p\,(x_1) \\ \neg\exists x_2\,(p\,(x_2) \wedge (x_2 \neq x_1)) \end{cases}\right] \\
&\Leftrightarrow \exists x_1 \left[\begin{cases} p\,(x_1) \\ \neg\exists x_2 \left(\begin{cases} x_2 \neq x_1 \\ p\,(x_2) \end{cases}\right) \end{cases}\right] \\
&\overset{1.1.4}{\Leftrightarrow} \exists x_1 \left[\begin{cases} p\,(x_1) \\ \forall x_2 \left(\neg \begin{cases} x_2 \neq x_1 \\ p\,(x_2) \end{cases}\right) \end{cases}\right] \Leftrightarrow \exists x_1 \left[\begin{cases} p\,(x_1) \\ \forall x_2\,(\neg p\,(x_2) \vee (x_2 = x_1)) \end{cases}\right] \\
&\overset{1.1.5}{\Leftrightarrow} \exists x_1 \left[\begin{cases} p\,(x_1) \\ \forall x_2\,(p\,(x_2) \rightarrow (x_2 = x_1)) \end{cases}\right]
\end{aligned} \tag{1.1.1}$$

  - equivalent definition 2

$$\exists ! x_1\,(p\,(x_1)) \Leftrightarrow \begin{cases} \exists x_1\,(p\,(x_1)) \\ \forall x_2 \forall x_3 \left[\begin{cases} p\,(x_2) \\ p\,(x_3) \end{cases} \rightarrow (x_2 = x_3)\right] \end{cases}$$

  - equivalent definition 3

$$\exists x_1 \forall x_2\,[p\,(x_2) \leftrightarrow (x_2 = x_1)]$$

備註 1.1.8. universal quantification[4]

- universal closure

$$Q : string \rightarrow \{\exists, \forall, \exists !\}$$

$$Q_{\{y_j\}_{j=1}^{\max\{j\}}}\,p\left(\{x_i\}_{i=1}^{\max\{i\}} \cup \{y_j\}_{j=1}^{\max\{j\}}\right)$$

$\{x_i\}_{i=1}^{\max i}$ is the collection of free variables / unbounded variables
$\{y_j\}_{j=1}^{\max j}$ is the collection of bounded variables by the quantifier(s) $Q_{\{y_j\}_{j=1}^{\max\{j\}}}$

$$\begin{aligned}
&Q_{\{y_j\}_{j=1}^{\max\{j\}}}\,p\left(\{x_i\}_{i=1}^{\max\{i\}} \cup \{y_j\}_{j=1}^{\max\{j\}}\right) \\
\Leftrightarrow &\forall_{\{x_i\}_{i=1}^{\max\{i\}}} Q_{\{y_j\}_{j=1}^{\max\{j\}}}\,p\left(\{x_i\}_{i=1}^{\max\{i\}} \cup \{y_j\}_{j=1}^{\max\{j\}}\right)
\end{aligned}$$

e.g.

$$p\,(x) \wedge \exists y\,(q\,(y, z))$$

$$\begin{aligned}
&p\,(x) \wedge \exists y\,(q\,(y, z)) \\
\Leftrightarrow &\forall x \forall z\,[p\,(x) \wedge \exists y\,(q\,(y, z))]
\end{aligned}$$

- rules of inference
  - universal instantiation

$$\forall x \in X\,(p\,(x)) \Rightarrow p\,(x_1)\,, x_1 \in X$$

  - universal generalization

$$\text{arbitrary } x_1 \in X, p\,(x_1) \Rightarrow \forall x \in X\,(p\,(x))$$

定理 1.1.9. *De Morgan laws*

$$\neg\,(p \wedge q) \Leftrightarrow [\neg p \vee \neg q] \tag{1.1.2}$$

$$\neg\,(p \vee q) \Leftrightarrow [\neg p \wedge \neg q] \tag{1.1.3}$$

Proof.

$\neg$	$(p$	$\wedge$	$q)$	v	$[\neg$	p	$\vee$	$\neg$	$q]$
F	T	T	T	=	F	T	F	F	T
T	T	F	F	=	F	T	T	T	F
T	F	F	T	=	T	F	T	F	T
T	F	F	F	=	T	F	T	T	F

□

定理 1.1.10. *negation of quantification*

- negation of universal quantification 並非所有 $x \cdots \Leftrightarrow$ 存在 x 非 $\cdots$

$$\neg\forall x\,(p\,(x)) \Leftrightarrow \exists x\,(\neg p\,(x))$$

Proof.

$$\neg\forall x\,(p\,(x)) \Leftrightarrow \neg\,[p\,(x_1) \wedge p\,(x_2) \wedge \cdots] \Leftrightarrow \neg \bigwedge_x p\,(x) \overset{1.1.9}{\Leftrightarrow} \bigvee_x \neg p\,(x) \Leftrightarrow \exists x\,(\neg p\,(x))$$

□

- negation of existential quantification 並非存在 $x \cdots \Leftrightarrow$ 所有 x 非 $\cdots$

$$\neg\exists x\,(p\,(x)) \Leftrightarrow \forall x\,(\neg p\,(x)) \tag{1.1.4}$$

Proof.

$$\neg\exists x\,(p\,(x)) \Leftrightarrow \neg\,[p\,(x_1) \vee p\,(x_2) \vee \cdots] \Leftrightarrow \neg \bigvee_x p\,(x) \overset{1.1.9}{\Leftrightarrow} \bigwedge_x \neg p\,(x) \Leftrightarrow \forall x\,(\neg p\,(x))$$

□

定理 1.1.11. *conditional statement logical equivalence*

$$[p \to q] \Leftrightarrow [\neg p \vee q] \tag{1.1.5}$$

Proof.

$[p$	$\to$	$q]$	v	$[\neg$	p	$\vee$	$q]$
T	T	T	=	F	T	T	T
T	F	F	=	F	T	F	F
F	T	T	=	T	F	T	T
F	T	F	=	T	F	T	F

□

1.2 樸素集合論 naive set theory / naïve set theory

1.2.1 集合 set

定義 1.2.1. 無定義名詞 集合 set

- roster notation: listing elements

$$A = \{a_1, a_2, \cdots\}$$

註記 1.2.2. duplicate elements in roster notation will be regarded as a unique element, e.g.

$$\{a, a\} = \{a\}$$

etc. and so on,
unlike multiset[1]

- set-builder notation: description by specification

$$A = \{x | p\,(x)\}$$

定義 1.2.3. 無定義關係 屬於 belong

$$A = \{a_1, a_2, \cdots\} \Rightarrow a_1 \in A, a_2 \in A, \cdots$$

$$\begin{cases} A = \{x|p\,(x)\} \\ v\,(p\,(a)) = \mathrm{T} \end{cases} \Rightarrow a \in A$$

i.e.

$$A = \{x|p\,(x)\} \Leftrightarrow \forall x \in A\,(v\,(p\,(x)) = \mathrm{T})$$

定義 1.2.4. 元組 / 多元組 / (多)元組 / 組 / 有序組 tuple

1-tuple = 1-dimensional tuple = 1D tuple

$$\langle x \rangle$$

2-tuple = 2-dimensional tuple = 2D tuple

$$\langle x, y \rangle$$

3-tuple = 3-dimensional tuple = 3D tuple

$$\langle x, y, z \rangle$$

$\vdots$

n-tuple = n-dimensional tuple = nD tuple

$$\langle x_1, x_2, \cdots, x_n \rangle$$

註記 1.2.5. ordering of the elements in tuples matters, e.g.

$$\langle a, b \rangle \neq \langle b, a \rangle \text{ or } \langle a, b \rangle \not\equiv \langle b, a \rangle$$

but ordering of the elements in roster notation of sets doesn't matter, e.g.

$$\{a, b\} = \{b, a\}$$

etc. and so on

定義 1.2.6. 積集合 product set

$$A \times B = \left\{ \langle x, y \rangle \middle| \begin{cases} x \in A \\ y \in B \end{cases} \right\} \Leftrightarrow \forall \langle x, y \rangle \in A \times B \left(\begin{cases} v(x \in A) = \mathrm{T} \\ v(y \in B) = \mathrm{T} \end{cases} \right)$$

$$A^2 = A \times A = \left\{ \langle x_1, x_2 \rangle \middle| \begin{cases} x_1 \in A \\ x_2 \in A \end{cases} \right\} \Leftrightarrow \forall \langle x_1, x_2 \rangle \in A \times A \left(\begin{cases} v(x_1 \in A) = \mathrm{T} \\ v(x_2 \in A) = \mathrm{T} \end{cases} \right)$$

$$\begin{aligned} A_1 \times A_2 \times \cdots \times A_n &= \left\{ \langle x_1, x_2, \cdots, x_n \rangle \middle| \begin{cases} x_1 \in A_1 \\ x_2 \in A_2 \\ \vdots \\ x_n \in A_n \end{cases} \right\} \\ &\Leftrightarrow \forall \langle x_1, x_2, \cdots, x_n \rangle \in A_1 \times A_2 \times \cdots \times A_n \left(\begin{cases} v(x_1 \in A_1) = \mathrm{T} \\ v(x_2 \in A_2) = \mathrm{T} \\ \vdots \\ v(x_n \in A_n) = \mathrm{T} \end{cases} \right) \\ &\Leftrightarrow \forall \langle x_1, x_2, \cdots, x_n \rangle \in A_1 \times A_2 \times \cdots \times A_n \left(\begin{cases} x_1 \in A_1 \\ x_2 \in A_2 \\ \vdots \\ x_n \in A_n \end{cases} \right) \end{aligned}$$

1.2.2 關係 relation

定義 1.2.7. 關係 relation

$$\begin{aligned} & R \text{ is a relation over } A \times B \\ \Leftrightarrow & R = \left\{ \langle x, y \rangle \middle| \begin{cases} x \in A \\ y \in B \\ xRy \end{cases} \right\} \subseteq A \times B \Leftrightarrow \forall \langle a, b \rangle \in R \left(\begin{cases} v(a \in A) = \mathrm{T} \\ v(b \in B) = \mathrm{T} \\ v(R(\langle a, b \rangle)) = v(R(a, b)) = v(aRb) = \mathrm{T} \end{cases} \right) \end{aligned}$$

i.e.

$$\begin{cases} R = \{\langle x, y \rangle | xRy\} \subseteq A \times B \\ v(a \in A) = \mathrm{T} \\ v(b \in B) = \mathrm{T} \\ v(aRb) = \mathrm{T} \end{cases} \Rightarrow \langle a, b \rangle \in R$$

i.e.

$$\begin{cases} R=\{\langle x,y\rangle|xRy\}\subseteq A\times B\\ a\in A\\ b\in B\\ aRb\end{cases}\Rightarrow\langle a,b\rangle\in R$$

1.2.3 等價關係 equivalence relation

致謝 1.2.8. Philip Edward Bertrand Jourdain (16 October 1879 – 1 October 1919) was a British logician and follower of Bertrand Russell.

> "Jourdain was one of the first who suggested a decontextualized term for what we now know as "equivalence relation". However, he was not successful at popularizing the term. Even Russell, one of his closest correspondences, refers to such relations with no name and only qualify them as an "important kind of relation", noting that, "similarity is one of this kind of relations" (Russell 1919, p. 16). By 1919, neither the combination "equivalence relation" nor "equivalence class"was in use. The necessity of naming was first felt with the relation, not with the classes that are formed by that relation... Hasse (1926) wrote:"We call such a decomposition a partition of M,and the subsets thereby determined its classes."
>
> ˜Asghari's Equivalence: an attempt at a history of the idea [1]

定義 1.2.9. 等價關係 equivalence relation

$$\begin{aligned} & R \text{ is an equivalence relation over } A\times B\\ \Leftrightarrow & \begin{cases} R=\sim=\{\langle x,y\rangle|x\sim y\}\subseteq A\times B & \text{(e) equivalence 等價}\\ \vdots & \vdots \end{cases}\\ \Leftrightarrow & \begin{cases} R=\{\langle x,y\rangle|xRy\}\subseteq A\times B & (R)\text{ relation}\\ \forall\langle x,y\rangle\in R\,(xRx) & (r)\text{ reflexive}\\ \forall\langle x,y\rangle\in R\,(xRy\Rightarrow yRx) & (s)\text{ symmetric}\\ \forall\langle x,y\rangle,\langle y,z\rangle\in R\left(\begin{cases}xRy\\ yRz\end{cases}\Rightarrow xRz\right) & (t)\text{ transitive} \end{cases} \Leftrightarrow \begin{cases} R=\{\langle x,y\rangle|xRy\}\subseteq A\times B & \text{關係}\\ \forall\langle x,y\rangle\in R\,(\langle x,x\rangle\in R) & \text{自反}\\ \forall\langle x,y\rangle\in R\,(\langle y,x\rangle\in R) & \text{對稱}\\ \forall\langle x,y\rangle,\langle y,z\rangle\in R\,(\langle x,z\rangle\in R) & \text{遞移} \end{cases} \end{aligned}$$

推論 1.2.10.

$$\sim \textit{ is an equivalence relation over } A\times B \not\Rightarrow \begin{cases}\sim \textit{ is an equivalence relation on } A^2=A\times A\\ B=A\end{cases}$$

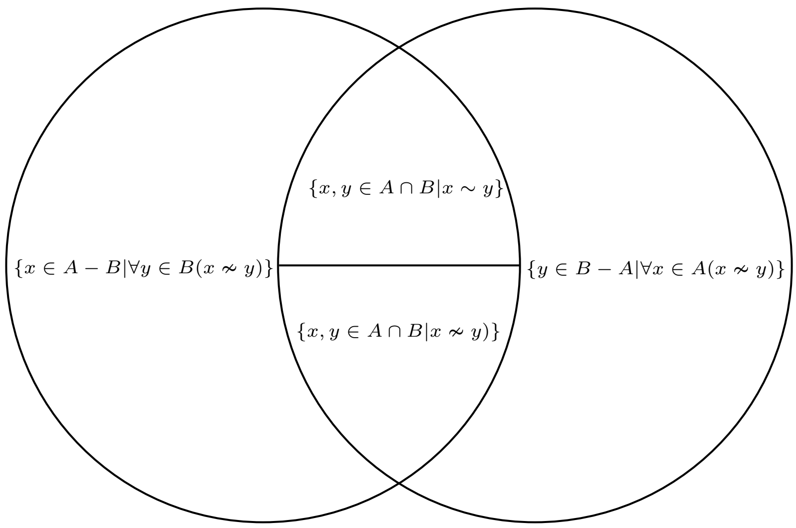


Figure 1.2.1: $\sim$ is an equivalence relation over $A\times B$

Proof. proof by contraposition and proof by contraindication,

$$A\neq B\Rightarrow\exists a\in A\,(a\notin B)$$

[1] https://hsm.stackexchange.com/questions/13450/when-were-equivalence-classes-formalized?rq=1

$$\begin{cases}\sim \text{ is an equivalence relation over } A\times B\\ A\neq B\Rightarrow \exists x=a\in A\,(x\notin B)\Rightarrow a\notin B\end{cases}\Rightarrow \forall y\in B\,(a\nsim y)\Rightarrow \exists x\in A,\forall y\in B\,(x\nsim y)$$

$$\because\begin{cases}\text{if } \exists y=b\in B\,(a\sim b)\\ \sim \text{ is an equivalence relation over } A\times B\end{cases}\Rightarrow \langle a,b\rangle\in\sim\Rightarrow\langle a,a\rangle\in\sim\subseteq A\times B\Rightarrow a\in B\Rightarrow\Leftarrow a\notin B$$

similarly 同理,

$$\exists y\in B,\forall x\in A\,(x\nsim y)$$

however still $\sim$ is an equivalence relation over $A\times B$,

$$\exists x\in A,y\in B\,(x\sim y)$$

$$\exists x=x_1\in A,y=y_1\in B\,(x_1\sim y_1)\Rightarrow\begin{cases}x_1\sim x_1\Rightarrow\langle x_1,x_1\rangle\in\sim\Rightarrow x_1\in B\Rightarrow\exists x\in A\,(x\in B) & \text{自反}\\ y_1\sim x_1\overset{\text{自反}}{\Rightarrow} y_1=y_1\Rightarrow\langle y_1,y_1\rangle\in\sim\Rightarrow y_1\in A\Rightarrow\exists y\in B\,(y\in A) & \text{對稱}\\ \begin{cases}y_1\sim x_1 & \text{對稱}\\ x_1\sim y_1 & \text{存在}\end{cases}\Rightarrow y_1\sim y_1 & \text{遞移}\end{cases}$$

$$\exists x\in A\,(x\in B)$$

$$\exists y\in B\,(y\in A)$$

$$\begin{matrix}\exists x\in A\,(x\in B)\\ \vee\\ \exists y\in B\,(y\in A)\end{matrix}\Rightarrow A\cap B\neq\emptyset$$

□

定義 1.2.11. 等價類 / 等價班 / 班 equivalence class

C is an equivalence class of a on A

$$\Leftrightarrow [a]_\sim=C=\left\{x\,\middle|\,\begin{cases}a\in A\\ x\in A\\ x\sim a\\ \sim \text{ is an equivalence relation over } A\times A=A^2\end{cases}\right\}\subseteq A\neq\emptyset$$

$$\Leftrightarrow [a]=[a]_\sim=\left\{x\,\middle|\,\begin{cases}a\in A\\ x\in A\\ x\sim a\\ \sim \text{ is an equivalence relation on } A\end{cases}\right\}\subseteq A\neq\emptyset$$

$$\Rightarrow [a]_\sim=\{x|x\sim a\}\subseteq A\neq\emptyset$$

i.e.

$$\begin{cases}a\in A\\ x\in A\\ x\sim a\\ \sim \text{ is an equivalence relation on } A\end{cases}\Rightarrow x\in[a]_\sim=[a]$$

i.e.

$$\forall x\in A\,(x\sim a\Rightarrow x\in[a]_\sim=[a])$$

i.e. simply,

$$x\sim a\Leftrightarrow x\in[a]_\sim$$

定義 1.2.12. 劃分 partition

$$\{A_i|i\in I\}\text{ is a partition of } A$$

$$\Leftrightarrow\begin{cases}A \text{ is a set}\\ A=\bigcup_{i\in I}A_i\\ \forall i,j\in I\,(i\neq j\Rightarrow A_i\cap A_j=\emptyset)\end{cases}$$

引理 1.2.13. *if x,y are equivalent to each other, then they representative of the same equivalence class*

$$\begin{cases}\sim \textit{ is an equivalence relation on } A\\ x\sim y\end{cases}\Leftrightarrow\begin{cases}\sim \textit{ is an equivalence relation over } A^2\\ x\sim y\end{cases}$$

$$\Rightarrow [x]_\sim=[y]_\sim\overset{1.2.11}{\Leftrightarrow}[x]=[y]$$

Proof.

$$\begin{cases}\forall \xi \in [x]_\sim (\xi \sim x) & \Leftarrow 1.2.11\\ x \sim y & \text{premise}\end{cases}$$
$$\overset{1.2.9(t)}{\Rightarrow} \forall \xi \in [x]_\sim (\xi \sim y) \Rightarrow \forall \xi \in [x]_\sim (\xi \in [y]_\sim) \Rightarrow [x]_\sim \subseteq [y]_\sim \tag{1.2.1}$$

$$\begin{cases}\forall \eta \in [y]_\sim (\eta \sim y) & \Leftarrow 1.2.11\\ x \sim y \overset{1.2.9(s)}{\Rightarrow} y \sim x & \text{premise}\end{cases}$$
$$\overset{1.2.9(t)}{\Rightarrow} \forall \eta \in [y]_\sim (\eta \sim x) \Rightarrow \forall \eta \in [y]_\sim (\eta \in [x]_\sim) \Rightarrow [y]_\sim \subseteq [x]_\sim \tag{1.2.2}$$

$$\begin{cases}[x]_\sim \subseteq [y]_\sim & 1.2.1\\ [y]_\sim \subseteq [x]_\sim & 1.2.2\end{cases} \Rightarrow [x]_\sim = [y]_\sim$$

□

註記 1.2.14. $x : y : z = \xi : \eta : \zeta =$ (xi) : (eta) : (zeta) = (xi) : ($y$ita) : ($z$eta)

引理 1.2.15. *equivalence classes either equal or disjoint*

$$\sim \text{ is an equivalence relation on } A \Rightarrow \forall x, y \in A \left(\begin{array}{c}[x]_\sim = [y]_\sim\\ \veebar\\ [x]_\sim \cap [y]_\sim = \emptyset\end{array}\right) \overset{1.2.11}{\Leftrightarrow} \forall x, y \in A \left(\begin{array}{c}[x] = [y]\\ \veebar\\ [x] \cap [y] = \emptyset\end{array}\right)$$

Proof. $\forall x, y \in A$,

案例 1. $x \sim y$,

by if x, y are equivalent to each other, then they representative of the same equivalence class 1.2.13,

$$[x]_\sim = [y]_\sim$$

案例 2. $x \nsim y$,

let

$$1.2.9\,(r) \Rightarrow \begin{cases}x \sim x \Rightarrow x \in [x]_\sim\\ y \sim y \Rightarrow y \in [y]_\sim\end{cases}$$

proof by contradiction,

$$\begin{aligned}[x]_\sim \cap [y]_\sim \neq \emptyset &\Rightarrow \exists z \in [x]_\sim \cap [y]_\sim ([x]_\sim \cap [y]_\sim \neq \emptyset)\\ &\Rightarrow \begin{cases}z \sim x \overset{1.2.9(s)}{\Rightarrow} x \sim z\\ z \sim y\end{cases}\\ &\overset{1.2.9(t)}{\Rightarrow} x \sim y \Rightarrow\Leftarrow x \nsim y\\ &\Downarrow\\ [x]_\sim \cap [y]_\sim &= \emptyset\end{aligned}$$

$$\begin{cases}x \sim y \Rightarrow [x]_\sim = [y]_\sim\\ x \nsim y \Rightarrow [x]_\sim \cap [y]_\sim = \emptyset\end{cases} \Rightarrow \begin{array}{c}[x]_\sim = [y]_\sim\\ \veebar\\ [x]_\sim \cap [y]_\sim = \emptyset\end{array}$$

□

定理 1.2.16. 等價關係基本定理 *fundamental theorem on equivalence relation = FTER* [2]

$$\sim \text{ is an equivalence relation on } A \overset{1.2.3}{\Rightarrow} \{[x]_\sim | [x]_\sim \subseteq A\} \text{ is a partition of } A$$

$$\begin{cases}\forall x \in A, \exists i \in I\,(x \in A_i)\\ \forall x, y \in A, \forall x, y \in A_i\,(x \in A_i \wedge y \in A_i)\\ \forall x, y, z \in A, \left(\begin{cases}x, y \in A_i\\ y, z \in A_j\end{cases} \Rightarrow i = j\right)\end{cases} \text{ defines a equivalence relation on } A \overset{1.2.3}{\Leftarrow} \{A_i | i \in I\} \text{ is a partition of } A$$

[2] https://math.libretexts.org/Courses/Monroe_Community_College/MTH_220_Discrete_Math/6:_Relations/6.3:_Equivalence_Relations_and_Partitions

(⇒):

Proof. we can denote different equivalence classes with indices in index set

$$\text{if } \begin{cases} f:\{[x]_\sim | [x]_\sim \subseteq A\} \to I \\ f^{-1}: I \to \{[x]_\sim | [x]_\sim \subseteq A\} \end{cases} \Rightarrow \{[x]_\sim | [x]_\sim \subseteq A\} = \{[x_i]_\sim | i \in I\} \tag{1.2.3}$$

by definition of a partition of A

$$\begin{aligned} &\{A_i | i \in I\} \text{ is a partition of } A \\ \Leftrightarrow &\begin{cases} A \text{ is a set} & (s) \\ A = \bigcup_{i\in I} A_i & (u) \\ \forall i,j \in I\,(i \neq j \Rightarrow A_i \cap A_j = \emptyset) & (d) \end{cases} \end{aligned} \tag{1.2.4}$$

1.2.4(u) consider

$$\bigcup_{i\in I}[x_i]_\sim$$

$$\forall x \in \bigcup_{i\in I}[x_i]_\sim \underset{\forall [x]_\sim \subseteq A([x]_\sim \subseteq A) \Rightarrow \forall i \in I([x_i]_\sim \subseteq A)}{\overset{1.2.11}{\subseteq}} A\,(x \in A) \Rightarrow \bigcup_{i\in I}[x_i]_\sim \subseteq A \tag{1.2.5}$$

$$\begin{aligned} &\forall x \in A\left(x \overset{1.2.9}{\sim} x \overset{1.2.11}{\Rightarrow} x \in [x]_\sim \overset{1.2.3}{\Rightarrow} \exists i \in I\,(x \in [x_i]_\sim)\right) \\ \Rightarrow &\forall x \in A\left(\bigvee_{i\in I} x \in [x_i]_\sim\right) \Rightarrow \forall x \in A\left(x \in \bigcup_{i\in I}[x_i]_\sim\right) \Rightarrow A \subseteq \bigcup_{i\in I}[x_i]_\sim \end{aligned} \tag{1.2.6}$$

$$\begin{cases} \bigcup_{i\in I}[x_i]_\sim \subseteq A & 1.2.5 \\ A \subseteq \bigcup_{i\in I}[x_i]_\sim & 1.2.6 \end{cases} \Rightarrow A = \bigcup_{i\in I}[x_i]_\sim \tag{1.2.7}$$

1.2.4(d) consider
by lemma: equivalence classes either equal or disjoint 1.2.15

$$\sim \text{ is an equivalence relation on } A \Rightarrow \forall x,y \in A \begin{pmatrix} [x]_\sim = [y]_\sim \\ \veebar \\ [x]_\sim \cap [y]_\sim = \emptyset \end{pmatrix}$$

we have

$$\sim \text{ is an equivalence relation on } A \Rightarrow \forall x_i,x_j \in A \begin{pmatrix} [x_i]_\sim = [x_j]_\sim \\ \veebar \\ [x_i]_\sim \cap [x_j]_\sim = \emptyset \end{pmatrix}$$

since we denote different equivalence classes with different indices

$$\begin{cases} \forall x_i,x_j \in A \begin{pmatrix} [x_i]_\sim = [x_j]_\sim \\ \veebar \\ [x_i]_\sim \cap [x_j]_\sim = \emptyset \end{pmatrix} \\ i \neq j \end{cases} \Rightarrow [x_i]_\sim \cap [x_j]_\sim = \emptyset \tag{1.2.8}$$

thus we can let

$$A_i = [x_i]_\sim \tag{1.2.9}$$

$$\begin{aligned} &\begin{cases} A \text{ is a set} & \text{premise} \\ A = \bigcup_{i\in I}[x_i]_\sim & 1.2.7 \\ \forall i,j \in I\left(i \neq j \Rightarrow [x_i]_\sim \cap [x_j]_\sim = \emptyset\right) & \Leftarrow 1.2.8 \end{cases} \\ \overset{1.2.9}{\Rightarrow} &\begin{cases} A \text{ is a set} & \text{premise} \\ A = \bigcup_{i\in I}[x_i]_\sim \overset{1.2.9}{=} \bigcup_{i\in I} A_i & 1.2.7 \\ \forall i,j \in I\left(i \neq j \Rightarrow A_i \cap A_j \overset{1.2.9}{=} [x_i]_\sim \cap [x_j]_\sim = \emptyset\right) & \Leftarrow 1.2.8 \end{cases} \\ \overset{1.2.12}{\Rightarrow} &\{A_i | i \in I\} \text{ is a partition of } A \\ \overset{1.2.9}{\Rightarrow} &\{[x_i]_\sim | i \in I\} \text{ is a partition of } A \\ \overset{1.2.3}{\Rightarrow} &\{[x]_\sim | [x]_\sim \subseteq A\} \text{ is a partition of } A \end{aligned}$$

□

($\Leftarrow$):

Proof. by definition of a partition of A

$$\begin{aligned}&\{A_i | i \in I\} \text{ is a partition of } A\\ \Leftrightarrow &\begin{cases} A \text{ is a set} & (s)\\ A = \bigcup\limits_{i\in I} A_i & (u)\\ \forall i, j \in I\,(i \neq j \Rightarrow A_i \cap A_j = \emptyset) & (d)\end{cases}\end{aligned} \tag{1.2.10}$$

(reflexive):
existence of index

$$1.2.10\,(u) \Rightarrow \forall x \in A\left(x \in \bigcup_{i\in I} A_i\right) \Rightarrow \forall x \in A\left(\bigvee_{i\in I} x \in A_i\right)$$

$$\Rightarrow \forall x \in A, \exists i \in I\,(x \in A_i) \tag{1.2.11}$$

$$\Rightarrow \forall x \in A, \exists i \in I\,(x \in A_i \wedge x \in A_i) \tag{1.2.12}$$

(symmetric):
by 1.2.11,

$$\forall x \in A, \exists i \in I\,(x \in A_i) \tag{1.2.13}$$

by 1.2.11 again,

$$\forall y \in A, \exists j \in I\,(y \in A_j) \tag{1.2.14}$$

$$\begin{cases}\forall x \in A, \exists i \in I\,(x \in A_i) & 1.2.13\\ \forall y \in A, \exists j \in I\,(y \in A_j) & 1.2.14\end{cases}$$

$$\Downarrow$$

$$i = j \Rightarrow (x \in A_i \wedge y \in A_i) \Leftrightarrow (y \in A_i \wedge x \in A_i) \tag{1.2.15}$$

(transitive)
$\forall x, y, z \in A$,

$$\begin{aligned}&\begin{cases} x, y \in A_i\\ y, z \in A_j\end{cases}\\ \Rightarrow &\exists y \in A_i \cap A_j\,(A_i \cap A_j \neq \emptyset)\\ \Rightarrow &\neg\,(A_i \cap A_j = \emptyset) \overset{1.2.10(d)}{\Rightarrow} \neg\,(i \neq j) \Leftrightarrow i = j\end{aligned}$$

i.e.

$$\begin{cases} x \in A_i \wedge y \in A_i\\ y \in A_i \wedge z \in A_i\end{cases} \Rightarrow x \in A_i \wedge z \in A_i \tag{1.2.16}$$

so we can let

$$R = \left\{ \langle x, y\rangle \middle| \begin{cases} x \in A\\ y \in A\\ R\,(x, y) \text{ or } v\,(R\,(x, y)) = \mathrm{T} & \Leftarrow 1.2.7\end{cases}\right\} \subseteq A^2 = A \times A \tag{1.2.17}$$

exists and also in further let 其中 邏輯命題

$$xRy = R\,(x, y) = (x \in A_i \wedge y \in A_i)$$

$$v\,(x \in A_i \wedge x \in A_i) \overset{1.2.12}{=} \mathrm{T} \Rightarrow xRx \tag{1.2.18}$$

$$v\,(x \in A_i \wedge y \in A_i) = v\,(y \in A_i \wedge x \in A_i) \overset{1.2.15}{=} \mathrm{T} \Rightarrow [xRy \Leftrightarrow yRx] \tag{1.2.19}$$

$$v\left(\begin{cases} x \in A_i \wedge y \in A_i\\ y \in A_i \wedge z \in A_i\end{cases} \Rightarrow x \in A_i \wedge z \in A_i\right) \overset{1.2.16}{=} \mathrm{T} \Rightarrow \left[\begin{cases} xRy\\ yRz\end{cases} \Rightarrow xRz\right] \tag{1.2.20}$$

$$\begin{cases} R \subseteq A^2 = A \times A & 1.2.17\,(R) \\ \forall x \in A\,(xRx) & 1.2.18\,(r) \\ \forall x,y \in A\,(xRy \Leftrightarrow yRx) & 1.2.19\,(s) \\ \forall x,y,z \in A\left(\begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz\right) & 1.2.20\,(t) \end{cases}$$

$\overset{1.2.9}{\Rightarrow}$ There defines a equivalence relation R on A according to or induced by partition $\{A_i | i \in I\}$.

□

1.2.4 函 / 函數 function

定義 1.2.17. 函數 function

$$f : X \to Y \Leftrightarrow f \in Y^X \Leftrightarrow \forall x \in X, \exists! y \in Y\,(y = f(x))$$

1.2.5 基 / 基數 cardinality / cardinal number

1.2.6 序 / 序數 ordinality / ordinal number

1.3 公理集合論 axiomatic set theory

公理 1.3.1. *axiom of extension*

公理 1.3.2. *axiom of specification*

公理 1.3.3. *axiom of pairing*

公理 1.3.4. *axiom of unions*

公理 1.3.5. *axiom of powers*

公理 1.3.6. *axiom of infinity*

公理 1.3.7. *axiom of choice*

公理 1.3.8. *axiom of substitution*

1.4 多值邏輯 many-valued logic

- 三值邏輯 three-valued logic
- 直覺邏輯 intuitionistic logic
  - three-valued intuitionistic logic: Heyting logic (HT logic) / Smetanov logic / Gödel G3 logic
- 模糊邏輯 fuzzy logic
- 機率邏輯 probabilistic logic

1.4.1 三值邏輯 three-valued logic 10.1.5

[2]

定義 1.4.1. 真值 / 真假無值 truth value and null
1.1.3

$$\text{either } v(p(x)) = \mathrm{T},\ v(p(x)) = \mathrm{F} \text{ or } v(p(x)) = \mathrm{U}$$

U means unknown or `null`

定義 1.4.2. not $\neg$, and $\wedge$, or $\vee$, conditional statement $\to$ in Kleene algebra or Kleene logic / Priest logic

$\neg$	p
F	T
U	U
T	F

p	$\wedge$	q
T	T	T
T	U	U
U	U	T
T	F	F
F	F	T
U	U	U
U	F	F
F	F	U
F	F	F

$p \wedge q$	T	U	F
T	T	U	F
U	U	U	F
F	F	F	F

$p \vee q$	T	U	F
T	T	T	T
U	T	U	U
F	T	U	F

$\neg$	p	$\vee$q	T	U	F
F	T		T	U	F
U	U		T	U	U
T	F		T	T	T

$p \to q$	T	U	F
T	T	U	F
U	T	U	U
F	T	T	T

定義 1.4.3. conditional statement $\to$ in Łukasiewicz logic

$p \to_{\text{Ł}} q$	T	U	F
T	T	U	F
U	T	T	U
F	T	T	T

定義 1.4.4. Heyting logic / Smetanov logic / Gödel G3 logic

$$\text{either } v\left(p\left(x\right)\right)=\mathrm{T},\ v\left(p\left(x\right)\right)=\mathrm{F} \text{ or } v\left(p\left(x\right)\right)=\mathrm{NF}$$

NF means not false

$\neg$	p
F	T
F	NF
T	F

$p \to_{\mathsf{HT}} q$	T	NF	F
T	T	NF	F
NF	T	T	F
F	T	T	T

Bibliography

[1] Multiset, September 2022. Page Version ID: 1109392985. 19

[2] Three-valued logic, September 2022. Page Version ID: 1112686428. 26

[3] Uniqueness quantification, September 2022. Page Version ID: 1111560253. 18

[4] Universal quantification, June 2022. Page Version ID: 1094078298. 18

Chapter 2

數論 / 整數論 number theory

定理 2.0.1. 良序定理 / 良序原理 *well-ordering theorem / well-ordering principle*

定理 2.0.2. 算術基本定理 / 唯一因數分解定理 / 質因數分解定理 *fundamental theorem of arithmetic = unique factorization theorem = prime factorization theorem*

定義 2.0.3. p進數 p-adic number

Chapter 3

微積分 與 實分析 calculus & real analysis

致謝 3.0.1. 陳金次_2012_高等微積分_advanced calculus http://ocw.aca.ntu.edu.tw/ntu-ocw/ocw/cou/101S130

3.1 實數的建構 construction of the real numbers

3.1.1 有理數 / 有比數 rational number

定義 3.1.1. 正整數 / 自然數 natural number

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

定義 3.1.2. 整數 integer

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

備註 3.1.3. 自然數是正整數

$$\mathbb{N} = \mathbb{Z}^+ = \left\{ n \middle| \begin{cases} n \in \mathbb{Z} \\ n > 0 \end{cases} \right\}$$

$$\{0, 1, 2, 3, \cdots\} = \left\{ m \middle| \begin{cases} m \in \mathbb{Z} \\ m \geq 0 \end{cases} \right\} = \mathbb{N} \cup \{0\} = \mathbb{Z}^+ \cup \{0\}$$

定義 3.1.4. 有理數 / 有比數 rational number

$$\mathbb{Q} = \left\{ \frac{b}{a} \middle| \begin{cases} a, b \in \mathbb{Z} \\ a \neq 0 \end{cases} \right\} = \left\{ \frac{m}{n} \middle| \begin{cases} m \in \mathbb{Z} \\ n \in \mathbb{N} \end{cases} \right\}$$

事實 3.1.5. 有理數 / 有比數 的性質

- 加法
  - 加法封閉性
  $$\forall p, q \in \mathbb{Q} \, (p + q \in \mathbb{Q})$$
  - 加法交換性
  $$\forall p, q \in \mathbb{Q} \, (p + q = q + p)$$
  - 加法結合性
  $$\forall p, q, r \in \mathbb{Q} \, ((p + q) + r = p + (q + r))$$
  - 加法單位元
  $$\exists! 0 \in \mathbb{Q}, \forall p \in \mathbb{Q} \, (p + 0 = p)$$
  - 加法反元素 / 逆元
  $$\forall p \in \mathbb{Q}, \exists! \, (-p) \in \mathbb{Q} \, (p + (-p) = 0)$$
- 乘法 $\forall p, q \in \mathbb{Q} \, (pq = p \cdot q = p \times q)$
  - 乘法封閉性
  $$\forall p, q \in \mathbb{Q} \, (pq \in \mathbb{Q})$$
  - 乘法交換性
  $$\forall p, q \in \mathbb{Q} \, (pq = qp)$$

  - 乘法結合性

$$\forall p, q, r \in \mathbb{Q}\left((pq)\, r = p\,(qr)\right)$$

  - 乘法單位元

$$\exists!1 \in \mathbb{Q}, \forall p \in \mathbb{Q}\,(p \cdot 1 = p)$$

  - 乘法反元素 / 逆元

$$\forall p \in \mathbb{Q}, \exists!\left(p^{-1}\right) \in \mathbb{Q}\left(p \cdot p^{-1} = 1\right)$$

- 分配性

$$\forall p, q, r \in \mathbb{Q}\,(p\,(q + r) = pq + pr)$$

- 減法

$$\forall p, q \in \mathbb{Q}\,(p - q = p + (-q))$$

- 除法

$$\forall p, q \in \mathbb{Q}\left(\frac{p}{q} = p/q = pq^{-1} = p \cdot \frac{1}{q}\right)$$

事實 3.1.6. 有比數 阿基米德性 / 阿基米德性質 / 阿基米德公理 *rational Archimedean property / Archimedean axiom*

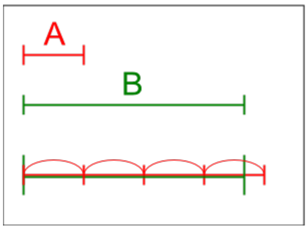


Figure 3.1.1: rational Archimedean property

[1]

$$\forall a, b \left(\begin{cases} a, b \in \mathbb{Q} \\ a, b > 0 \end{cases} \Rightarrow \exists n \in \mathbb{N}\,(na > b)\right)$$

i.e.

$$\forall a, b \in \mathbb{Q}^{+}, \exists n \in \mathbb{N}\,(na > b)$$

3.1.2 無理數 / 無比數 irrational number

定理 3.1.7. $\sqrt{2}$ 不是有比數 或 不存在有比數 p 使得 $p^2 = 2$

$$\left(\sqrt{2}\right)^2 = 2 \Rightarrow \sqrt{2} \notin \mathbb{Q} \Leftrightarrow \forall p \in \mathbb{Q}\left(p^2 \neq 2\right) \Leftrightarrow \nexists p \in \mathbb{Q}\left(p^2 = 2\right)$$

Proof. proof by contradiction[2]

$$\begin{aligned}
&\neg\forall p \in \mathbb{Q}\left(p^2 \neq 2\right) \Leftrightarrow \exists p \in \mathbb{Q}\left(p^2 = 2\right) \\
\Rightarrow &\exists p = \frac{m}{n}, \begin{cases} m \in \mathbb{Z} \\ n \in \mathbb{N} \\ \mathrm{GCD}\,(m, n) = 1 \end{cases} \quad \left(p^2 = 2\right) \Rightarrow \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 2 \\
\Rightarrow &m^2 = 2n^2 \Rightarrow 2|m \Rightarrow \exists k \in \mathbb{Z}\,(m = 2k) \\
\Rightarrow &(2k)^2 = 2n^2 \Leftrightarrow 4k^2 = 2n^2 \Leftrightarrow 2k^2 = n^2 \Rightarrow 2|n \\
\Rightarrow &\begin{cases} 2|m \\ 2|n \end{cases} \Rightarrow \mathrm{GCD}\,(m, n) = 2 \neq 1 \Rightarrow\Leftarrow \exists m \in \mathbb{Z}, n \in \mathbb{N}\,(\mathrm{GCD}\,(m, n) = 1) \\
\Rightarrow &\forall p \in \mathbb{Q}\left(p^2 \neq 2\right) \Leftrightarrow \neg\exists p \in \mathbb{Q}\left(p^2 = 2\right) \Leftrightarrow \nexists p \in \mathbb{Q}\left(p^2 = 2\right)
\end{aligned}$$

□

[1] https://commons.wikimedia.org/wiki/File:Archimedean_property.png
[2] GCD = greatest common divider 最大公因數, $\mathrm{GCD}\,(m, n) = 1$ 意為 m 與 n 互質

定理 3.1.8. $\sqrt{2}$ 的 連分數 型式 *continued fraction*

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}$$

Proof.

$$\left(\sqrt{2}\right)^2=2$$

$$\left(\sqrt{2}\right)^2-1^2=2-1^2=2-1=1$$

$$\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)=\left(\sqrt{2}\right)^2-1^2=1$$

$\sqrt{2}+1>0$

$$\sqrt{2}-1=\frac{1}{\sqrt{2}+1}$$

$$\sqrt{2}=1+\frac{1}{\sqrt{2}+1}$$

$$\sqrt{2}=1+\frac{1}{1+\sqrt{2}}$$

$$\sqrt{2}=1+\frac{1}{1+\sqrt{2}}=1+\cfrac{1}{1+1+\cfrac{1}{1+\sqrt{2}}}=1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}$$

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}=1+\cfrac{1}{2+\cfrac{1}{1+1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}}$$

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}$$

□

定理 3.1.9. $\sqrt{n^2+1}$ 的 連分數 型式 *continued fraction*, $n\in\mathbb{N}$

$$\sqrt{n^2+1}=1+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}}}}=1+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{\ddots}}}}}$$

Proof.

$$\left(\sqrt{n^2+1}\right)^2=n^2+1$$

$$\left(\sqrt{n^2+1}\right)^2-1^2=n^2+1-1^2=n^2+1-1=n^2$$

$$\left(\sqrt{n^2+1}+1\right)\left(\sqrt{n^2+1}-1\right)=\left(\sqrt{n^2+1}\right)^2-1^2=n^2$$

$\sqrt{n^2+1}+1>0$

$$\sqrt{n^2+1}-1=\frac{n^2}{\sqrt{n^2+1}+1}$$

$$\sqrt{n^2+1} = 1 + \frac{n^2}{\sqrt{n^2+1}+1}$$

$$\sqrt{n^2+1} = 1 + \frac{n^2}{1+\sqrt{n^2+1}}$$

$$\sqrt{n^2+1} = 1 + \frac{n^2}{1+\sqrt{n^2+1}} = 1 + \frac{n^2}{1+1+\cfrac{n^2}{1+\sqrt{n^2+1}}} = 1 + \frac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}}$$

$$\sqrt{n^2+1} = 1 + \frac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}} = 1 + \frac{n^2}{2+\cfrac{n^2}{1+1+\cfrac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}}}} = 1 + \frac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}}}}$$

$$\sqrt{n^2+1} = 1 + \frac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{1+\sqrt{n^2+1}}}}} = 1 + \frac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{2+\cfrac{n^2}{\ddots}}}}}$$

□

定理 3.1.10. $\sqrt{n+1} = \sqrt{\left(\sqrt{n}\right)^2+1}$ 的 連分數 型式 *continued fraction*, $n \in \mathbb{N}$

$$\sqrt{n+1} = 1 + \frac{n}{2+\cfrac{n}{2+\cfrac{n}{2+\cfrac{n}{1+\sqrt{n+1}}}}} = 1 + \frac{n}{2+\cfrac{n}{2+\cfrac{n}{2+\cfrac{n}{2+\cfrac{n}{\ddots}}}}}$$

定理 3.1.11. $\left(\sqrt{x}\right)^2 = c \in \mathbb{R}^+$ 的 連分數 型式 *continued fraction*

$$\sqrt{x} = \rho + \frac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\ddots}}}}, \rho > 0$$

Proof.

$$\left(\sqrt{x}\right)^2 = c$$

$$\left(\sqrt{x}\right)^2 - \rho^2 = c - \rho^2$$

$$\left(\sqrt{x}+\rho\right)\left(\sqrt{x}-\rho\right) = \left(\sqrt{x}\right)^2 - \rho^2 = c - \rho^2$$

let $\rho > 0$,

$$\sqrt{x} - \rho = \frac{c-\rho^2}{\sqrt{x}+\rho}$$

$$\sqrt{x} = \rho + \frac{c-\rho^2}{\sqrt{x}+\rho}$$

$$\sqrt{x} = \rho + \frac{c-\rho^2}{\rho+\sqrt{x}}$$

$$\sqrt{x} = \rho + \frac{c-\rho^2}{\rho+\rho+\cfrac{c-\rho^2}{\rho+\sqrt{x}}} = \rho + \frac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{\rho+\sqrt{x}}}$$

$$\sqrt{x} = \rho + \frac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{\rho+\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{\rho+\sqrt{x}}}}} = \rho + \frac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{\rho+\sqrt{x}}}}} = \rho + \frac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\cfrac{c-\rho^2}{2\rho+\ddots}}}}$$

□

定理 3.1.12. $\sqrt{2}$ 不是有比數 $\sqrt{2}$ *is not rational / not a rational number*

- 28 proofs [3]
- 小數討論 [4]
  - 循環小數 [5]
- 連分數討論 [6]

定理 3.1.13. $\sqrt{3}$ 不是有比數 或 不存在有比數 p 使得 $p^2 = 3$

$$\left(\sqrt{3}\right)^2 = 3 \Rightarrow \sqrt{3} \notin \mathbb{Q} \Leftrightarrow \forall p \in \mathbb{Q}\left(p^2 \neq 3\right) \Leftrightarrow \nexists p \in \mathbb{Q}\left(p^2 = 3\right)$$

Proof. proof by contradiction

$$\begin{aligned}
&\neg\forall p \in \mathbb{Q}\left(p^2 \neq 3\right) \Leftrightarrow \exists p \in \mathbb{Q}\left(p^2 = 3\right) \\
\Rightarrow &\exists p = \frac{m}{n}, \begin{cases} m \in \mathbb{Z} \\ n \in \mathbb{N} \\ \operatorname{GCD}(m,n) = 1 \end{cases} \qquad \left(p^2 = 3\right) \Rightarrow \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 3 \\
\Rightarrow &m^2 = 3n^2 \Rightarrow 3|m \Rightarrow \exists k \in \mathbb{Z}\,(m = 3k) \\
\Rightarrow &(3k)^2 = 3n^2 \Leftrightarrow 9k^2 = 3n^2 \Leftrightarrow 3k^2 = n^2 \Rightarrow 3|n \\
\Rightarrow &\begin{cases} 3|m \\ 3|n \end{cases} \Rightarrow \operatorname{GCD}(m,n) = 3 \neq 1 \Rightarrow\Leftarrow \exists m \in \mathbb{Z}, n \in \mathbb{N}\,(\operatorname{GCD}(m,n) = 1) \\
\Rightarrow &\forall p \in \mathbb{Q}\left(p^2 \neq 3\right) \Leftrightarrow \neg\exists p \in \mathbb{Q}\left(p^2 = 3\right) \Leftrightarrow \nexists p \in \mathbb{Q}\left(p^2 = 3\right)
\end{aligned}$$

□

定理 3.1.14. $\log 2 = \log_{10} 2$ 不是有比數

Proof. by 算術基本定理 / 唯一因數分解定理 / 質數分解定理 fundamental theorem of arithmetic = unique factorization theorem = prime factorization theorem, □

定理 3.1.15. $\log 5 = \log_{10} 5$ 不是有比數

Proof. by 算術基本定理 / 唯一因數分解定理 / 質數分解定理 fundamental theorem of arithmetic = unique factorization theorem = prime factorization theorem, □

定理 3.1.16. 無比數 沒有 四則運算 封閉性

Proof.

$$\log 2 + \log 5 = \log_{10} 2 + \log_{10} 5 = \log_{10} 2 \cdot 5 = \log_{10} 10 = 1 \in \mathbb{Q}$$
$$\sqrt{2} - \sqrt{2} = 0 \in \mathbb{Q}$$
$$\sqrt{2}\sqrt{2} = 2 \in \mathbb{Q}$$
$$\frac{\sqrt{2}}{\sqrt{2}} = 1 \in \mathbb{Q}$$

□

定理 3.1.17. $\forall x \notin \mathbb{Q}, y \in \mathbb{Q}\,(x + y \notin \mathbb{Q})$

Proof.

$$\begin{aligned}
&\text{if } x + y \in \mathbb{Q} \Rightarrow \begin{cases} y \in \mathbb{Q} \\ x + y \in \mathbb{Q} \end{cases} \text{premise} \Rightarrow x = (x + y) - y \in \mathbb{Q} \Rightarrow x \in \mathbb{Q} \Rightarrow\Leftarrow x \notin \mathbb{Q} \\
\Rightarrow &x + y \notin \mathbb{Q}
\end{aligned}$$

□

定理 3.1.18. *Hippasus regular pentagon*

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[3] 數學傳播_23_1_蔡聰明_根號 2 為無理數的證明
[4] 數學傳播_30_4_張鎮華_舊題新解一根號 2 是無理數
[5] 數學傳播_25_3_康明昌_循環小數
[6] 數學傳播_32_1_水木耳_再說根號 2 為無理數

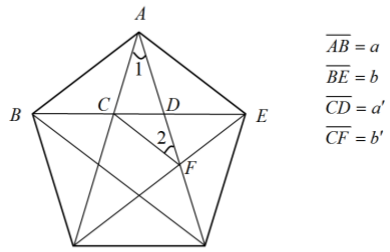


Figure 3.1.2: Hippasus regular pentagon and pentagram (五芒星)

Proof. 外角和 $360°$ 五等分, 則內角

$$\angle BAE = 180° - \frac{360°}{5} = 108°$$

$\triangle ABE$ 以 $\angle BAE$ 為頂角 (vertex angle) 之等腰三角形 (isosceles triangle)

$$\angle ABE = \frac{180° - 108°}{2} = 36°, \wedge \angle ABE = \angle DAE = \angle AED$$

$$\angle BAD = \angle BAE - \angle DAE = 108° - 36° = 72°$$

$$\angle BDA = \angle DAE + \angle AED = 36° + 36° = 72°$$

$$\therefore \angle BAD = \angle BDA \Rightarrow \triangle BDA \text{ isosceles with vertex angle } \angle ABD \Rightarrow \overline{BA} = \overline{BD} \tag{3.1.1}$$

同理

$$\overline{EA} = \overline{EC}$$

且 正五邊形 regular pentagon $\overline{BA} = \overline{EA}$

$$\overline{BD} = \overline{BA} = \overline{EA} = \overline{EC}$$

$$\overline{BD} = \overline{EC} \tag{3.1.2}$$

$$\begin{aligned}\overline{CD} =& \overline{BE} - \overline{BC} - \overline{ED} \\ =& \overline{BE} - \left(\overline{BE} - \overline{EC}\right) - \left(\overline{BE} - \overline{BD}\right) \\ =& \overline{BD} + \overline{EC} + \overline{BE} - 2\overline{BE} = \overline{BD} + \overline{EC} - \overline{BE} \\ \overset{3.1.2}{=}& 2\overline{BD} - \overline{BE} \overset{3.1.1}{=} 2\overline{BA} - \overline{BE}\end{aligned}$$

$$\overline{CD} = 2\overline{BA} - \overline{BE} \tag{3.1.3}$$

$$\angle 1 = \angle CAD = \angle BAE - \angle BAC - \angle DAE \overset{\angle BAC = \angle DAE}{=} \angle BAE - 2\angle DAE = 108° - 2 \cdot 36° = 36°$$

$$\angle 2 = \angle DFC = \angle AED = 36°$$

thus

$$\angle 1 = \angle 2$$

$$\angle 1 = \angle 2 \Rightarrow \overline{CA} = \overline{CF} \tag{3.1.4}$$

$$\overline{CA} = \overline{CB} = \overline{ED} = \overline{BE} - \overline{BD} = \overline{BE} - \overline{BA}$$

$$\triangle_1 = \triangle ABE \sim \triangle_2 = \triangle DEA \sim \triangle_3 = \triangle DCF$$

$$\pentagon_1 \begin{cases} s_1 = \overline{BA} = s > 0 \\ d_1 = \overline{BE} = d > 0 \end{cases} \Rightarrow \pentagon_2 \begin{cases} s_2 = \overline{ED} = \overline{BE} - \overline{BD} \overset{3.1.1}{=} \overline{BE} - \overline{BA} = d - s \\ d_2 = \overline{EA} = \overline{BA} = s \end{cases}$$

$$\Rightarrow \pentagon_3 \begin{cases} s_3 = \overline{CD} = 2\overline{BA} - \overline{BE} = \overline{BA} - \left(\overline{BE} - \overline{BA}\right) = s - (d - s) \\ d_3 = \overline{CF} \overset{3.1.4}{=} \overline{CA} = \overline{ED} = \overline{BE} - \overline{BA} = d - s \end{cases}$$

$$\Rightarrow \cdots$$

$$\vdots$$

$$\frac{\overline{BA}}{\overline{BE}} = \frac{\overline{ED}}{\overline{EA}} = \frac{\overline{CD}}{\overline{CF}} \because \triangle ABE \sim \triangle DEA \sim \triangle DCF$$
$$\frac{s}{d} = \frac{d-s}{s} = \frac{2s-d}{d-s} = \frac{s-(d-s)}{d-s} = \cdots = \frac{\text{以大五邊形對角線減大五邊形邊為小五邊形邊}}{\text{以大五邊形邊為小五邊形對角線}} = \cdots$$

$$\begin{aligned}
\frac{s}{d} &= \frac{d-s}{s} = \frac{s-(d-s)}{d-s} = \frac{2s-d}{d-s} \\
&= \frac{d-s}{s} = \frac{2s-d}{d-s} = \frac{(d-s)-(2s-d)}{2s-d} = \frac{2d-3s}{2s-d} \\
&= \frac{2s-d}{d-s} = \frac{2d-3s}{2s-d} = \frac{(2s-d)-(2d-3s)}{2d-3s} = \frac{5s-3d}{2d-3s} \\
&= \frac{2d-3s}{2s-d} = \frac{5s-3d}{2d-3s} = \frac{(2d-3s)-(5s-3d)}{5s-3d} = \frac{5d-8s}{5s-3d} \\
&= \frac{5s-3d}{2d-3s} = \frac{5d-8s}{5s-3d} = \frac{(5s-3d)-(5d-8s)}{5d-8s} = \frac{13s-8d}{5d-8s} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\frac{F_1 s}{F_1 d} &= \frac{F_1 d - F_2 s}{F_1 s} = \frac{F_1 s - (F_1 d - F_2 s)}{F_1 d - F_2 s} = \frac{(F_1+F_2)\,s - F_1 d}{F_1 d - F_2 s} = \frac{F_3 s - F_2 d}{F_1 d - F_2 s} \\
= \frac{F_1 d - F_2 s}{F_1 s} &= \frac{F_3 s - F_2 d}{F_1 d - F_2 s} = \frac{(F_1 d - F_2 s) - (F_3 s - F_2 d)}{F_3 s - F_2 d} = \frac{(F_1+F_2)\,d - (F_2+F_3)\,s}{F_3 s - F_2 d} = \frac{F_3 d - F_4 s}{F_3 s - F_2 d} \\
= \frac{F_3 s - F_2 d}{F_1 d - F_2 s} &= \frac{F_3 d - F_4 s}{F_3 s - F_2 d} = \frac{(F_3 s - F_2 d) - (F_3 d - F_4 s)}{F_3 d - F_4 s} = \frac{(F_3+F_4)\,s - (F_2+F_3)\,d}{F_3 d - F_4 s} = \frac{F_5 s - F_4 d}{F_3 d - F_4 s} \\
= \frac{F_3 d - F_4 s}{F_3 s - F_2 d} &= \frac{F_5 s - F_4 d}{F_3 d - F_4 s} = \frac{(F_3 d - F_4 s) - (F_5 s - F_4 d)}{F_5 s - F_4 d} = \frac{(F_3+F_4)\,d - (F_4+F_5)\,s}{F_5 s - F_4 d} = \frac{F_5 d - F_6 s}{F_5 s - F_4 d} \\
= \frac{F_5 s - F_4 d}{F_3 d - F_4 s} &= \frac{F_5 d - F_6 s}{F_5 s - F_4 d} = \frac{(F_5 s - F_4 d) - (F_5 d - F_6 s)}{F_5 d - F_6 s} = \frac{(F_5+F_6)\,s - (F_4+F_5)\,d}{F_5 d - F_6 s} = \frac{F_7 s - F_6 d}{F_5 d - F_6 s} \\
&\vdots
\end{aligned}$$

出現 費波那契數列 Fibonacci sequence $\begin{cases} F_1 = F_2 = 1 \\ F_n + F_{n+1} = F_{n+2} \quad \forall n \in \mathbb{N} \end{cases}$

依此類推, 以致無窮 and so on to infinity,

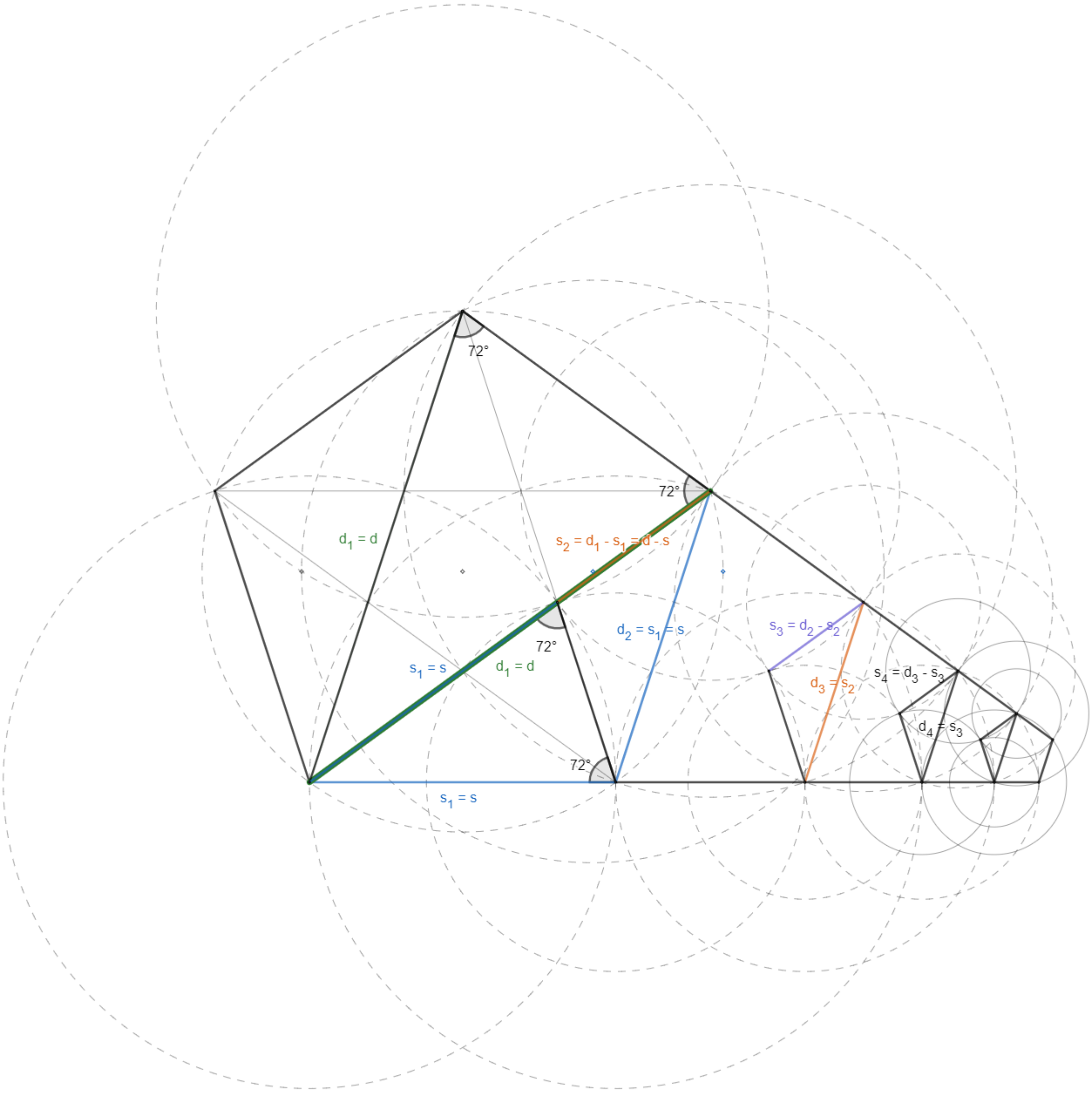


Figure 3.1.3: infinite numbers of Hippasus regular pentagons

就算畫正五邊形到比電子還小還能畫嗎? 還能, 就像用電子排無窮小數把宇宙中電子排完, 還是有更小(更後位)的小數! 也像圍棋的賽局組合數也比可視宇宙物質多, 但圍棋還是存在也可以玩!

$$
\begin{cases}\text{以大五邊形對角線減大五邊形邊為小五邊形邊}\\ \text{以大五邊形邊為小五邊形對角線}\end{cases}
$$

$$
\begin{aligned}
&\Rightarrow \begin{cases}\begin{cases}s_2 = d_1 - s_1 \\ d_2 = s_1\end{cases} & \wedge \dfrac{s_2}{d_2} = \dfrac{s_1}{d_1} \\ \begin{cases}s_3 = d_2 - s_2 \\ d_3 = s_2\end{cases} & \wedge \dfrac{s_3}{d_3} = \dfrac{s_2}{d_2}\end{cases} \\
&\vdots \\
&\Rightarrow \begin{cases}s_{k+1} = d_k - s_k \\ d_{k+1} = s_k\end{cases} \quad \wedge \frac{s_{k+1}}{d_{k+1}} = \frac{s_k}{d_k} \\
&\vdots
\end{aligned} \tag{3.1.5}
$$

by 三角不等式 triangular inequality,

$$
\forall k \in \mathbb{N}, s_{k+1} - s_k = (d_k - s_k) - s_k = d_k - 2s_k \overset{\triangle\text{ineq.}}{<} 0 \Rightarrow s_{k+1} < s_k
$$

$$
\forall k \in \mathbb{N} \cup \{0\}, d_{k+2} - d_{k+1} = s_{k+1} - s_k = (d_k - s_k) - s_k = d_k - 2s_k \overset{\triangle\text{ineq.}}{<} 0 \Rightarrow d_{k+1} < d_k
$$

$$
\forall k \in \mathbb{N}, \begin{cases}s_{k+1} < s_k \\ d_{k+1} < d_k\end{cases} \tag{3.1.6}
$$

其中

$$
\frac{s}{d} = \frac{d-s}{s}
$$

let

$$
\overline{BA} = s = 1
$$

then

$$
\begin{aligned}
\frac{s}{d} &= \frac{d-s}{s} \\
\frac{1}{d} &= \frac{d-1}{1} \\
1 &= d^2 - d \\
0 &= d^2 - d - 1 \\
0 < d &= \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \\
d &= \frac{\sqrt{5}+1}{2}
\end{aligned}
$$

以三角函數觀點計算,

$$
d = \overline{BE} = 2 \cdot \overline{BA} \cdot \sin 54^\circ = 2 \cdot 1 \cdot \sin 54^\circ = 2 \sin 54^\circ = 2 \sin \frac{108^\circ}{2} = 2 \sin \frac{180^\circ - \frac{360^\circ}{5}}{2} \overset{3.1.19}{=} 2 \cdot \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{2}
$$

結果相同.

now we should prove $\dfrac{\overline{BE}}{\overline{BA}} = \dfrac{d}{s} = \dfrac{\frac{\sqrt{5}+1}{2}}{1} = \dfrac{\sqrt{5}+1}{2} = \dfrac{\sqrt{5}}{2} + \dfrac{1}{2} \overset{3.1.17}{\notin} \mathbb{Q}$ or $\dfrac{\overline{BA}}{\overline{BE}} = \dfrac{s}{d} \notin \mathbb{Q}$

by 反證法 / 歸謬證法 proof by contradiction ($\Rightarrow\!\Leftarrow$):
if

$$\frac{\overline{BA}}{\overline{BE}} = \frac{s}{d} \in \mathbb{Q} \Rightarrow \exists m_1, n_1 \in \mathbb{N}, l > 0 \left(\frac{\overline{BA}}{\overline{BE}} = \frac{s}{d} = \frac{s_1}{d_1} = \frac{m_1 l}{n_1 l} = \frac{m_1}{n_1}\right) \tag{3.1.7}$$

$$\Rightarrow \exists \{m_k\}, \{n_k\} \subseteq \mathbb{N}, \forall k \in \mathbb{N} \left(\begin{cases} \begin{cases} m_{k+1} = n_k - m_k \in \mathbb{N} \\ n_{k+1} = m_k \in \mathbb{N} \end{cases} & \Leftarrow 3.1.5 \\ \frac{s_k}{d_k} = \frac{m_k}{n_k} \in \mathbb{Q} & \Leftarrow 3.1.5 \\ \begin{cases} s_{k+1} < s_k \\ d_{k+1} < d_k \end{cases} & \Leftarrow 3.1.6 \end{cases}\right)$$

$$\Rightarrow \exists \{m_k\}, \{n_k\} \subseteq \mathbb{N} \left(\begin{cases} \begin{cases} m_1 \in \{m_k\} \neq \emptyset \\ n_1 \in \{n_k\} \neq \emptyset \end{cases} & \Leftarrow 3.1.7 \\ \frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \cdots = \frac{m_k}{n_k} = \cdots & \Leftarrow 3.1.5 \\ \begin{cases} m_1 > m_2 > m_3 > \cdots > m_k > \cdots \Leftrightarrow \forall k \in \mathbb{N}(m_k > m_{k+1}) \\ n_1 > n_2 > n_3 > \cdots > n_k > \cdots \Leftrightarrow \forall k \in \mathbb{N}(n_k > n_{k+1}) \end{cases} & \Leftarrow 3.1.6 \end{cases}\right)$$

$$\Rightarrow \exists \{m_k\}, \{n_k\} \subseteq \mathbb{N} \left(\begin{cases} \begin{cases} \{m_k\} = \{m_k\}_{k=1}^{\infty} \Leftrightarrow |\{m_k\}| \geq |\mathbb{N}| \Rightarrow |\{m_k\}| = \infty \\ \{n_k\} = \{n_k\}_{k=1}^{\infty} \Leftrightarrow |\{n_k\}| \geq |\mathbb{N}| \Rightarrow |\{n_k\}| = \infty \end{cases} & \Leftarrow 3.1.3 \\ \begin{cases} m_1 > m_2 > m_3 > \cdots > m_k > \cdots \Leftrightarrow \forall k \in \mathbb{N}(m_k > m_{k+1}) \\ n_1 > n_2 > n_3 > \cdots > n_k > \cdots \Leftrightarrow \forall k \in \mathbb{N}(n_k > n_{k+1}) \end{cases} & \Leftarrow 3.1.6 \end{cases}\right)$$

$$\Rightarrow\Leftarrow \forall \{a_k\} \subseteq \mathbb{N}(a_k > a_{k+1} \Rightarrow |\{a_k\}| \in \mathbb{N}) \Leftrightarrow \forall \{a_k\} \subseteq \mathbb{N}\left(a_k > a_{k+1} \Rightarrow \begin{cases} \{a_k\} = \{a_k\}_{k=1}^{N} \\ N \in \mathbb{N} \end{cases}\right) \tag{3.1.8}$$

$$\Leftrightarrow \exists N \in \mathbb{N}, \forall k \in \mathbb{N} \left(\begin{cases} a_N < a_k \\ a_N \in \{a | a = a_k\} \end{cases}\right)$$

有或可作無限多個正五邊形 3.1.5, 但良序原理 (well-ordering principle) 2.0.1 指出嚴格遞減正整數列為有限或有最小元素 3.1.8, 也就是僅有兩組有限正整數列可能符合條件, 但無限又有限兩者矛盾 ($\Rightarrow\Leftarrow$), 也就是不可能有兩組有限嚴格遞減正整數列使無限多正五邊形邊與正五邊形對角線可比

此證法即為 以 無窮遞降法 (proof by infinite descent) 型式的 反證法 / 歸謬證法 (proof by contradiction) thus

$$\frac{\overline{BA}}{\overline{BE}} = \frac{s}{d} \notin \mathbb{Q}$$

□

定理 3.1.19. $\sin 54° = \sin \frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$

$$\sin 54° = \sin(3 \cdot 18°) \overset{3.1.20}{\underset{3.1.13}{=}} \frac{\sqrt{5}+1}{4}$$

定理 3.1.20. $\sin 18° = \sin \frac{\pi}{10}$ *and its related exact values*

$$\sin 18° = \sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$$

θ	$18° = \frac{\pi}{10} = \frac{1}{5}\frac{\pi}{2} = \frac{90°}{5}$	$36° = \frac{2\pi}{10} = \frac{\pi}{5}$	$54° = \frac{3\pi}{10} = \frac{\pi}{2} - \frac{\pi}{5} = 90° - 36°$	$72° = \frac{4\pi}{10} = \frac{2\pi}{5} = 90° - 18°$
$\sin\theta$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
$\cos\theta$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$
$\tan\theta$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$

Table 3.1.1: $\sin 18° = \sin \frac{\pi}{10}$ and its related exact values

Proof.

$$\frac{\pi}{2} = 90^\circ = 5 \cdot 18^\circ$$

let

$$\theta = 18^\circ = \frac{\pi}{10}$$

then

$$5\theta = \frac{\pi}{2}$$

$$2\theta + 3\theta = \frac{\pi}{2}$$

$$2\theta = \frac{\pi}{2} - 3\theta$$

$$\sin 2\theta = \sin\left(\frac{\pi}{2} - 3\theta\right) = \cos 3\theta$$

$$2\sin\theta\cos\theta = \sin 2\theta = \sin\left(\frac{\pi}{2} - 3\theta\right) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$$

$$2\sin\theta = 4\cos^2\theta - 3$$

$$2\sin\theta = 4\cos^2\theta - 3 = 4\left(1 - \sin^2\theta\right) - 3 = 1 - 4\sin^2\theta$$

$$4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\sin\theta = \frac{-2 \pm \sqrt{2^2 - 4\cdot 4\cdot(-1)}}{2\cdot 4} = \frac{-1 \pm \sqrt{1 - 4\cdot(-1)}}{4} = \frac{-1 \pm \sqrt{5}}{4}$$

$$0 < \sin\frac{\pi}{10} = \sin 18^\circ = \sin\theta = \frac{-1+\sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \tag{3.1.9}$$

$$0 < \cos\frac{\pi}{10} = \cos 18^\circ \overset{6.1.7}{\underset{3.1.9}{=}} \sqrt{\frac{4^2 - \left(\sqrt{5}-1\right)^2}{4^2}} = \frac{\sqrt{10+2\sqrt{5}}}{4} \tag{3.1.10}$$

$$= +\sqrt{1 - \sin^2\frac{\pi}{10}} \overset{3.1.9}{=} \sqrt{1 - \frac{5 - 2\sqrt{5} + 1}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\tan\frac{\pi}{10} = \tan 18^\circ \overset{3.1.9}{\underset{3.1.10}{=}} \frac{\frac{\sqrt{5}-1}{4}}{\frac{\sqrt{10+2\sqrt{5}}}{4}} = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}} = \frac{\left(\sqrt{5}-1\right)\sqrt{10-2\sqrt{5}}}{\sqrt{10+2\sqrt{5}}\sqrt{10-2\sqrt{5}}}$$

$$= \frac{\sqrt{\left(\sqrt{5}-1\right)^2\left(10-2\sqrt{5}\right)}}{\sqrt{80}} = \frac{\sqrt{\left(6-2\sqrt{5}\right)\left(10-2\sqrt{5}\right)}}{\sqrt{80}}$$

$$= \frac{\sqrt{60 - 32\sqrt{5} + 20}}{\sqrt{80}} = \sqrt{\frac{80 - 32\sqrt{5}}{80}} = \sqrt{\frac{5 - 2\sqrt{5}}{5}}$$

$$= \sqrt{\frac{25 - 10\sqrt{5}}{25}} = \frac{\sqrt{25 - 10\sqrt{5}}}{5}$$

$$\sin\frac{2\pi}{10} = \sin\frac{\pi}{5} = \sin 36^\circ = \sin\left(2\cdot 18^\circ\right) = 2\sin 18^\circ\cos 18^\circ \overset{3.1.9}{\underset{3.1.10}{=}} 2\cdot\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$= 2\sqrt{\frac{5 - 2\sqrt{5} + 1}{16}}\sqrt{\frac{10+2\sqrt{5}}{16}} = 2\sqrt{\frac{6-2\sqrt{5}}{16}}\sqrt{\frac{10+2\sqrt{5}}{16}}$$

$$= 2\sqrt{\frac{60 - 8\sqrt{5} - 20}{16^2}} = 2\sqrt{\frac{40 - 8\sqrt{5}}{16\cdot 16}} = 2\sqrt{\frac{10-2\sqrt{5}}{4\cdot 16}} = \frac{\sqrt{10-2\sqrt{5}}}{4} \tag{3.1.11}$$

$$
\begin{aligned}
0 < \cos\frac{2\pi}{10} = \cos\frac{\pi}{5} = \cos 36^\circ &\overset{6.1.7}{\underset{3.1.11}{=}} \sqrt{\frac{4^2-\left(\sqrt{10-2\sqrt{5}}\right)^2}{4^2}} = \frac{\sqrt{6+2\sqrt{5}}}{4} = \frac{\sqrt{\left(\sqrt{5}+1\right)^2}}{4} = \frac{\sqrt{5}+1}{4} \\
= \cos\left(2\cdot 18^\circ\right) = 1-2\sin^2 18^\circ &= 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-2\cdot\frac{6-2\sqrt{5}}{16} = \frac{4+4\sqrt{5}}{16} = \frac{\sqrt{5}+1}{4} \qquad (3.1.12)\\
= 2\cos^2 18^\circ - 1 &= 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1 = 2\cdot\frac{10+2\sqrt{5}}{16} - 1 = \frac{4+4\sqrt{5}}{16} = \frac{\sqrt{5}+1}{4}
\end{aligned}
$$

$$
a\pm\sqrt{b} = \sqrt{\left(a\pm\sqrt{b}\right)^2} = \sqrt{a^2\pm 2a\sqrt{b}+b} = \sqrt{\left(a^2+b\right)\pm\left(2a\sqrt{b}\right)}
$$

$$
\begin{aligned}
\sin\frac{3\pi}{10} = \sin 54^\circ &= 3\sin\frac{\pi}{10} - 4\sin^3\frac{\pi}{10} \\
&= 3\cdot\frac{\sqrt{5}-1}{4} - 4\cdot\left(\frac{\sqrt{5}-1}{4}\right)^3 \\
&= \frac{3\sqrt{5}-3}{4} - \frac{\left(\sqrt{5}-1\right)^2\left(\sqrt{5}-1\right)}{4^2} \\
&= \frac{12\sqrt{5}-12-\left(6-2\sqrt{5}\right)\left(\sqrt{5}-1\right)}{16} \\
&= \frac{12\sqrt{5}-12-\left(8\sqrt{5}-16\right)}{16} = \frac{4\sqrt{5}+4}{16} = \frac{\sqrt{5}+1}{4} \qquad (3.1.13)
\end{aligned}
$$

$$
\sin 54^\circ = \sin\left(90^\circ - 36^\circ\right) = \cos 36^\circ \overset{3.1.12}{=} \frac{\sqrt{5}+1}{4}
$$

$$
\begin{aligned}
\sin 54^\circ = \sin\left(36^\circ + 18^\circ\right) &\overset{3.1.11,3.1.10}{\underset{3.1.12,3.1.9}{=}} \sin 36^\circ\cos 18^\circ + \cos 36^\circ \sin 18^\circ \\
&= \frac{\sqrt{10-2\sqrt{5}}}{4}\frac{\sqrt{10+2\sqrt{5}}}{4} + \frac{\sqrt{5}+1}{4}\frac{\sqrt{5}-1}{4} \\
&= \frac{\sqrt{80}+4}{16} = \frac{4\sqrt{5}+4}{16} = \frac{\sqrt{5}+1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\cos\frac{3\pi}{10} = \cos 54^\circ = \sin\left(90^\circ - 36^\circ\right) = \sin 36^\circ &\overset{3.1.11}{=} \frac{\sqrt{10-2\sqrt{5}}}{4} \\
\sin\frac{2\pi}{5} = \sin\left(2\cdot\frac{\pi}{5}\right) = \sin 72^\circ = \sin\left(90^\circ - 18^\circ\right) = \cos 18^\circ &\overset{3.1.10}{=} \frac{\sqrt{10+2\sqrt{5}}}{4} \\
\cos\frac{2\pi}{5} = \cos\left(2\cdot\frac{\pi}{5}\right) = \cos 72^\circ = \cos\left(90^\circ - 18^\circ\right) = \sin 18^\circ &\overset{3.1.9}{=} \frac{\sqrt{5}-1}{4}
\end{aligned}
$$

□

定理 3.1.21. 三角函數哪些是無比數 [7]

3.1.3 遞增有界有比數列 increasing bounded rational sequence = IBQS

因為 戴德金分劃 / 戴德金切割 / 戴德金分割 Dedekind cut 3.1.47 比較抽象, 所以從 小數進位表示 / 十進位數系統 3.1.22 或 遞增有界有比數列 / 單調有界有比數列 3.1.27 先著手

但 其實 戴德金分劃集 與 有界遞增有比數列班 是等價的 (logically equivalent)

而 對 遞增有界有比數列 作分類 其實就是尋找 遞增有界有比數列集 上的等價關係 (equivalence relation) 1.2.16, 3.1.14

定義 3.1.22. p-進制 / p-進位數系統 p-nary number system / arity (Latin) / adicity (Greek) ; multary / multiary / poladic ; 2.0.3

$$
x = \sum_{n=-\infty}^{\infty} a_n p^n
$$

- 0-進制 nullary number system / niladic

$$
\begin{aligned}
x &= a_\infty 0^\infty + \cdots + a_n 0^n + \cdots + a_2 0^2 + a_1 0^1 + a_0 0^0 + a_{-1} 0^{-1} + a_{-2} 0^{-2} + \cdots + a_{-n} 0^{-n} + \cdots + a_{-\infty} 0^{-\infty} \\
&= a_\infty 0 + \cdots + a_n 0 + \cdots + a_2 0 + a_1 0 + a_0 0^0 + a_{-1} 0^{-1} + a_{-2} 0^{-2} + \cdots + a_{-n} 0^{-n} + \cdots + a_{-\infty} 0^{-\infty} \\
&= a_0 0^0 + a_{-1} 0^{-1} + a_{-2} 0^{-2} + \cdots + a_{-n} 0^{-n} + \cdots + a_{-\infty} 0^{-\infty}
\end{aligned}
$$

[7] 數學傳播_44_3_黃越_有理角的三角函數哪些是無理數

  - 0^0
  - 0^{-1} diverges
  - $0^{-\infty}$

- 1-進制 unary number system / monadic [8]

Figure 3.1.4: unary number system

$$
\begin{aligned}
x =& a_\infty 1^\infty + \cdots + a_n 1^n + \cdots + a_2 1^2 + a_1 1^1 + a_0 1^0 + a_{-1} 1^{-1} + a_{-2} 1^{-2} + \cdots + a_{-n} 1^{-n} + \cdots + a_{-\infty} 1^{-\infty} \\
=& a_\infty + \cdots + a_n + \cdots + a_2 + a_1 + a_0 + a_{-1} + a_{-2} + \cdots + a_{-n} + \cdots + a_{-\infty}
\end{aligned}
$$

  - convergent or not
  - $|a_n| < 1$
    * $|a_n| \neq 1$ 否則 or else a_{n+1}, a_n no difference 無區別
    * $|a_n| \not> 1$ 否則 or else 1-進位, 2-進位 no difference 無區別

- 2-進制 binary number system / dyadic

$$
\begin{aligned}
x =& a_\infty 2^\infty + \cdots + a_n 2^n + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + \cdots + a_{-n} 2^{-n} + \cdots + a_{-\infty} 2^{-\infty} \\
=& a_\infty 2^\infty + \cdots + a_n 2^n + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + \cdots + a_{-n} 2^{-n} + \cdots + a_{-\infty} 0 \\
=& a_\infty 2^\infty + \cdots + a_n 2^n + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + \cdots + a_{-n} 2^{-n} + \cdots \\
=& \sum_{k=0}^{\infty} a_k 2^k + \sum_{n=1}^{\infty} a_{-n} 2^{-n} = a_0 + \sum_{n=1}^{\infty} a_n 2^n + a_{-n} 2^{-n}
\end{aligned}
$$

  - $a_\infty 2^\infty, a_{-\infty} 2^{-\infty} = a_{-\infty} 0$ convergent or not
  - 2^∞ diverges, so consider

$$
\begin{aligned}
x =& a_N 2^N + \cdots + a_n 2^n + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + \cdots + a_{-n} 2^{-n} + \cdots, N \in \mathbb{N} \\
=& \sum_{k=0}^{\infty} a_{N-k} 2^{N-k} = \sum_{k=0}^{N} a_k 2^k + \sum_{n=1}^{\infty} a_{-n} 2^{-n}
\end{aligned}
$$

  - $\sum\limits_{n=1}^{\infty} a_{-n} 2^{-n}$ convergent or not
  - $\sum\limits_{k=0}^{N} a_k 2^k$ bounded

$$
-2^N \sum_{k=0}^{N} |a_k| = \sum_{k=0}^{N} - |a_k| \, 2^N \leq \sum_{k=0}^{N} - |a_k| \, 2^k \leq \left(\sum_{k=0}^{N} a_k 2^k \right) \leq \sum_{k=0}^{N} |a_k| \, 2^k \leq \sum_{k=0}^{N} |a_k| \, 2^N = 2^N \sum_{k=0}^{N} |a_k|
$$

  - $|a_n| < 2$
    * $|a_n| \neq 2$ 否則 or else a_{n+1}, a_n no difference 無區別
    * $|a_n| \not> 2$ 否則 or else 2-進位, 3-進位 no difference 無區別

[8] 算籌 https://zh.wikipedia.org/wiki/算籌 → 蘇州碼子

  - * if $x > 0$, let $\forall k \in \mathbb{Z} \left(\begin{cases} a_k \in \mathbb{Z} \\ a_k \geq 0 \end{cases} \right)$

$$a_k \in \{0, 1\}$$

- 3-進制 ternary number system / triadic

$$x = \sum_{k=0}^{\infty} a_{N-k} 3^{N-k} = \sum_{k=0}^{N} a_k 3^k + \sum_{n=1}^{\infty} a_{-n} 3^{-n}$$

  - $\sum\limits_{n=1}^{\infty} a_{-n} 3^{-n}$ convergent or not
  - $\sum\limits_{k=0}^{N} a_k 3^k$ bounded

$$-3^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 3^k \right) \leq 3^N \sum_{k=0}^{N} |a_k|$$

  - $|a_n| < 3$
    - * $|a_n| \neq 3$ 否則 or else a_{n+1}, a_n no difference 無區別
    - * $|a_n| \not> 3$ 否則 or else 3-進位, 4-進位 no difference 無區別

- 4-進制 quaternary number system / tetradic

$$\begin{aligned}
x =& a_\infty 4^\infty + \cdots + a_n 4^n + \cdots + a_2 4^2 + a_1 4^1 + a_0 4^0 + a_{-1} 4^{-1} + a_{-2} 4^{-2} + \cdots + a_{-n} 4^{-n} + \cdots + a_{-\infty} 4^{-\infty} \\
=& a_\infty 4^\infty + \cdots + a_n 4^n + \cdots + a_2 4^2 + a_1 4^1 + a_0 4^0 + a_{-1} 4^{-1} + a_{-2} 4^{-2} + \cdots + a_{-n} 4^{-n} + \cdots + a_{-\infty} 0 \\
=& a_\infty 4^\infty + \cdots + a_n 4^n + \cdots + a_2 4^2 + a_1 4^1 + a_0 4^0 + a_{-1} 4^{-1} + a_{-2} 4^{-2} + \cdots + a_{-n} 4^{-n} + \cdots \\
=& a_\infty \left(2^2\right)^\infty + \cdots + a_n \left(2^2\right)^n + \cdots + a_2 \left(2^2\right)^2 + a_1 \left(2^2\right)^1 + a_0 \left(2^2\right)^0 + \\
& a_{-1} \left(2^2\right)^{-1} + a_{-2} \left(2^2\right)^{-2} + \cdots + a_{-n} \left(2^2\right)^{-n} + \cdots \\
=& a_\infty 2^{2\infty} + \cdots + a_n 2^{2n} + \cdots + a_2 2^{2\cdot 2} + a_1 2^{2\cdot 1} + a_0 2^{2\cdot 0} + a_{-1} 2^{2(-1)} + a_{-2} 2^{2(-2)} + \cdots + a_{-n} 2^{2(-n)} + \cdots \\
=& \sum_{k=0}^{\infty} a_k 2^{2k} + \sum_{n=1}^{\infty} a_{-n} 2^{-2n} = a_0 + \sum_{n=1}^{\infty} a_n 2^{2n} + a_{-n} 2^{-2n}
\end{aligned}$$

  - $a_\infty 4^\infty, a_{-\infty} 4^{-\infty} = a_{-\infty} 0$ convergent or not
  - 4^∞ diverges, so consider

$$\begin{aligned}
x =& a_N 4^N + \cdots + a_n 4^n + \cdots + a_2 4^2 + a_1 4^1 + a_0 4^0 + a_{-1} 4^{-1} + a_{-2} 4^{-2} + \cdots + a_{-n} 4^{-n} + \cdots, N \in \mathbb{N} \\
=& \sum_{k=0}^{\infty} a_{N-k} 4^{N-k} = \sum_{k=0}^{N} a_k 4^k + \sum_{n=1}^{\infty} a_{-n} 4^{-n} \\
=& \sum_{k=0}^{\infty} a_{N-k} 2^{2(N-k)} = \sum_{k=0}^{N} a_k 2^{2k} + \sum_{n=1}^{\infty} a_{-n} 2^{-2n}
\end{aligned}$$

  - $\sum\limits_{n=1}^{\infty} a_{-n} 4^{-n}$ convergent or not
  - $\sum\limits_{k=0}^{N} a_k 4^k$ bounded

$$-4^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 4^k \right) \leq 4^N \sum_{k=0}^{N} |a_k|$$

  - $|a_n| < 4$
    - * $|a_n| \neq 4$ 否則 or else a_{n+1}, a_n no difference 無區別
    - * $|a_n| \not> 4$ 否則 or else 4-進位, 5-進位 no difference 無區別

- 5-進制 quinary number system / pentadic

$$x = \sum_{k=0}^{\infty} a_{N-k} 5^{N-k} = \sum_{k=0}^{N} a_k 5^k + \sum_{n=1}^{\infty} a_{-n} 5^{-n}$$

- $\sum\limits_{n=1}^{\infty} a_{-n}5^{-n}$ convergent or not
- $\sum\limits_{k=0}^{N} a_k 5^k$ bounded

$$-5^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 5^k\right) \leq 5^N \sum_{k=0}^{N} |a_k|$$

- $|a_n| < 5$
  * $|a_n| \neq 5$ 否則 or else a_{n+1}, a_n no difference 無區別
  * $|a_n| \not> 5$ 否則 or else 5-進位, 6-進位 no difference 無區別

• 10-進制 decimal / denary number system / decadic

$$\begin{aligned}
x =& a_\infty 10^\infty + \cdots + a_n 10^n + \cdots + a_2 10^2 + a_1 10^1 + a_0 10^0 + a_{-1}10^{-1} + a_{-2}10^{-2} + \cdots + a_{-n}10^{-n} + \cdots + a_{-\infty}10^{-\infty} \\
=& a_\infty 10^\infty + \cdots + a_n 10^n + \cdots + a_2 10^2 + a_1 10^1 + a_0 10^0 + a_{-1}10^{-1} + a_{-2}10^{-2} + \cdots + a_{-n}10^{-n} + \cdots + a_{-\infty}0 \\
=& a_\infty 10^\infty + \cdots + a_n 10^n + \cdots + a_2 10^2 + a_1 10^1 + a_0 10^0 + a_{-1}10^{-1} + a_{-2}10^{-2} + \cdots + a_{-n}10^{-n} + \cdots \\
=& a_\infty (2\cdot 5)^\infty + \cdots + a_n (2\cdot 5)^n + \cdots + a_2 (2\cdot 5)^2 + a_1 (2\cdot 5)^1 + a_0 (2\cdot 5)^0 + \\
& a_{-1}(2\cdot 5)^{-1} + a_{-2}(2\cdot 5)^{-2} + \cdots + a_{-n}(2\cdot 5)^{-n} + \cdots
\end{aligned}$$

- $a_\infty 10^\infty, a_{-\infty}10^{-\infty} = a_{-\infty}0$ convergent or not
- 10^∞ diverges, so consider

$$\begin{aligned}
x =& a_N 10^N + \cdots + a_n 10^n + \cdots + a_2 10^2 + a_1 10^1 + a_0 10^0 + a_{-1}10^{-1} + a_{-2}10^{-2} + \cdots + a_{-n}10^{-n} + \cdots, N \in \mathbb{N} \\
=& \sum_{k=0}^{\infty} a_{N-k}10^{N-k} = \sum_{k=0}^{N} a_k 10^k + \sum_{n=1}^{\infty} a_{-n}10^{-n}
\end{aligned}$$

- $\sum\limits_{n=1}^{\infty} a_{-n}10^{-n}$ convergent or not
- $\sum\limits_{k=0}^{N} a_k 10^k$ bounded

$$-10^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 10^k\right) \leq 10^N \sum_{k=0}^{N} |a_k|$$

- $|a_n| < 10$
  * $|a_n| \neq 10$ 否則 or else a_{n+1}, a_n no difference 無區別
  * $|a_n| \not> 10$ 否則 or else 10-進位, 11-進位 no difference 無區別

• 16-進制 hexadecimal number system

$$\begin{aligned}
x =& a_\infty 16^\infty + \cdots + a_n 16^n + \cdots + a_2 16^2 + a_1 16^1 + a_0 16^0 + a_{-1}16^{-1} + a_{-2}16^{-2} + \cdots + a_{-n}16^{-n} + \cdots + a_{-\infty}16^{-\infty} \\
=& a_\infty 16^\infty + \cdots + a_n 16^n + \cdots + a_2 16^2 + a_1 16^1 + a_0 16^0 + a_{-1}16^{-1} + a_{-2}16^{-2} + \cdots + a_{-n}16^{-n} + \cdots + a_{-\infty}0 \\
=& a_\infty 16^\infty + \cdots + a_n 16^n + \cdots + a_2 16^2 + a_1 16^1 + a_0 16^0 + a_{-1}16^{-1} + a_{-2}16^{-2} + \cdots + a_{-n}16^{-n} + \cdots \\
=& a_\infty \left(2^4\right)^\infty + \cdots + a_n \left(2^4\right)^n + \cdots + a_2 \left(2^4\right)^2 + a_1 \left(2^4\right)^1 + a_0 \left(2^4\right)^0 + \\
& a_{-1}\left(2^4\right)^{-1} + a_{-2}\left(2^4\right)^{-2} + \cdots + a_{-n}\left(2^4\right)^{-n} + \cdots \\
=& a_\infty 2^{4\infty} + \cdots + a_n 2^{4n} + \cdots + a_2 2^{4\cdot 2} + a_1 2^{4\cdot 1} + a_0 2^{4\cdot 0} + a_{-1}2^{4(-1)} + a_{-2}2^{4(-2)} + \cdots + a_{-n}2^{4(-n)} + \cdots \\
=& \sum_{k=0}^{\infty} a_k 2^{4k} + \sum_{n=1}^{\infty} a_{-n}2^{-4n} = a_0 + \sum_{n=1}^{\infty} a_n 2^{4n} + a_{-n}2^{-4n}
\end{aligned}$$

- $a_\infty 16^\infty, a_{-\infty}16^{-\infty} = a_{-\infty}0$ convergent or not
- 16^∞ diverges, so consider

$$\begin{aligned}
x =& a_N 16^N + \cdots + a_n 16^n + \cdots + a_2 16^2 + a_1 16^1 + a_0 16^0 + a_{-1}16^{-1} + a_{-2}16^{-2} + \cdots + a_{-n}16^{-n} + \cdots, N \in \mathbb{N} \\
=& \sum_{k=0}^{\infty} a_{N-k}16^{N-k} = \sum_{k=0}^{N} a_k 16^k + \sum_{n=1}^{\infty} a_{-n}16^{-n} \\
=& \sum_{k=0}^{\infty} a_{N-k}2^{4(N-k)} = \sum_{k=0}^{N} a_k 2^{4k} + \sum_{n=1}^{\infty} a_{-n}2^{-4n}
\end{aligned}$$

- $\sum_{n=1}^{\infty} a_{-n}16^{-n}$ convergent or not
- $\sum_{k=0}^{N} a_k 16^k$ bounded

$$-16^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 4^k\right) \leq 16^N \sum_{k=0}^{N} |a_k|$$

- $|a_n| < 16$
  * $|a_n| \neq 16$ 否則 or else a_{n+1}, a_n no difference 無區別
  * $|a_n| \not> 16$ 否則 or else 16-進位, 17-進位 no difference 無區別
  * if $x > 0$, let $\forall k \in \mathbb{Z} \left(\begin{cases} a_k \in \mathbb{Z} \\ a_k \geq 0 \end{cases}\right)$

$$a_k \in \left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \overset{10}{\mathrm{A}}, \overset{11}{\mathrm{B}}, \overset{12}{\mathrm{C}}, \overset{13}{\mathrm{D}}, \overset{14}{\mathrm{E}}, \overset{15}{\mathrm{F}}\right\}$$

- $\frac{1}{2}$-進制 half number system

$$\begin{aligned}
x =& a_\infty \left(\frac{1}{2}\right)^\infty + \cdots + a_n \left(\frac{1}{2}\right)^n + \cdots + a_2 \left(\frac{1}{2}\right)^2 + a_1 \left(\frac{1}{2}\right)^1 + a_0 \left(\frac{1}{2}\right)^0 + \\
& a_{-1} \left(\frac{1}{2}\right)^{-1} + a_{-2} \left(\frac{1}{2}\right)^{-2} + \cdots + a_{-n} \left(\frac{1}{2}\right)^{-n} + \cdots + a_{-\infty} \left(\frac{1}{2}\right)^{-\infty} \\
=& a_\infty \left(\frac{1}{2}\right)^\infty + \cdots + a_n \left(\frac{1}{2}\right)^n + \cdots + a_2 \left(\frac{1}{2}\right)^2 + a_1 \left(\frac{1}{2}\right)^1 + a_0 \left(\frac{1}{2}\right)^0 + \\
& a_{-1} \left(\frac{1}{2}\right)^{-1} + a_{-2} \left(\frac{1}{2}\right)^{-2} + \cdots + a_{-n} \left(\frac{1}{2}\right)^{-n} + \cdots + a_{-\infty} 0 \\
=& a_\infty \left(\frac{1}{2}\right)^\infty + \cdots + a_n \left(\frac{1}{2}\right)^n + \cdots + a_2 \left(\frac{1}{2}\right)^2 + a_1 \left(\frac{1}{2}\right)^1 + a_0 \left(\frac{1}{2}\right)^0 + \\
& a_{-1} \left(\frac{1}{2}\right)^{-1} + a_{-2} \left(\frac{1}{2}\right)^{-2} + \cdots + a_{-n} \left(\frac{1}{2}\right)^{-n} + \cdots \\
=& a_\infty \left(2^{-1}\right)^\infty + \cdots + a_n \left(2^{-1}\right)^n + \cdots + a_2 \left(2^{-1}\right)^2 + a_1 \left(2^{-1}\right)^1 + a_0 \left(2^{-1}\right)^0 + \\
& a_{-1} \left(2^{-1}\right)^{-1} + a_{-2} \left(2^{-1}\right)^{-2} + \cdots + a_{-n} \left(2^{-1}\right)^{-n} + \cdots \\
=& a_\infty 2^{-\infty} + \cdots + a_n 2^{-n} + \cdots + a_2 2^{(-1)\cdot 2} + a_1 2^{(-1)\cdot 1} + a_0 2^{(-1)\cdot 0} + \\
& a_{-1} 2^{(-1)(-1)} + a_{-2} 2^{(-1)(-2)} + \cdots + a_{-n} 2^{(-1)(-n)} + \cdots \\
=& a_\infty 2^{-\infty} + \cdots + a_n 2^{-n} + \cdots + a_2 2^{-2} + a_1 2^{-1} + a_0 2^0 + a_{-1} 2^1 + a_{-2} 2^2 + \cdots + a_{-n} 2^n + \cdots \\
=& \sum_{k=0}^{\infty} a_{-k} 2^k + \sum_{n=1}^{\infty} a_{-n} 2^{-n} = a_0 + \sum_{n=1}^{\infty} a_{-n} 2^n + a_n 2^{-n}
\end{aligned}$$

- $|a_n| \neq \frac{1}{2}$ 否則 or else a_{n+1}, a_n no difference 無區別

定理 3.1.23. 十進數轉二進數 *decimal number transformed to binary number*

定義 3.1.24. 有界有比數級數 bounded rational series = BQSe

- e.g. 10-進制 decimal / denary number system / decadic 3.1.3

$$x = \sum_{k=0}^{N} a_k 10^k + \sum_{n=1}^{\infty} a_{-n} 10^{-n}$$

- $\sum_{k=0}^{N} a_k 10^k$ bounded

$$-10^N \sum_{k=0}^{N} |a_k| \leq \left(\sum_{k=0}^{N} a_k 10^k\right) \leq 10^N \sum_{k=0}^{N} |a_k|$$

- consider $\forall k \in \mathbb{Z}$

$$0 \leq |a_k| < 10$$

  i.e.

$$a_k \in \{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- thus

$$a_k 10^k \in \mathbb{Q}$$
$$\sum_{\substack{1\le k\le n\\ k\in\mathbb{N}}} a_k 10^k \in \mathbb{Q}$$
$$\sum_{k=0}^{n} a_{N-k}10^{N-k} \in \mathbb{Q}$$
$$\sum_{k=0}^{\infty} a_{N-k}10^{N-k} \in \mathbb{Q}?$$

- $\sum\limits_{k=0}^{\infty} a_{N-k}10^{N-k} \in \mathbb{Q}?$

$$\sum_{k=0}^{\infty} a_{N-k}10^{N-k} = \begin{cases} \text{有窮小數} \in \mathbb{Q} \\ \text{無窮小數} = \begin{cases} \text{有循環無窮小數} & \in \mathbb{Q} \\ \text{無循環無窮小數} & \notin \mathbb{Q} \end{cases} \end{cases} \tag{3.1.14}$$

對 遞增有界有比數列 作分類 其實就是尋找 (全體)遞增有界有比數列集 上的等價關係 (equivalence relation) 1.2.16

定義 3.1.25. 有比數數列 / 有比數列 rational sequence = QS

$$\boldsymbol{p} = \langle p_1, p_2, \cdots, p_n, \cdots \rangle = \langle p_n \rangle_{n\in\mathbb{N}} = \langle p_n \rangle_{n=1}^{\infty}, \forall n \in \mathbb{N}\,(p_n \in \mathbb{Q})$$

定義 3.1.26. 有界有比數數列 / 有界有比數列 bounded rational sequence = BQS

$$\boldsymbol{p} = \langle p_1, p_2, \cdots, p_n, \cdots \rangle = \langle p_n \rangle_{n\in\mathbb{N}} = \langle p_n \rangle_{n=1}^{\infty}, \begin{cases} \forall n \in \mathbb{N}\,(p_n \in \mathbb{Q}) \\ \exists \beta \in \mathbb{Q}, \forall p_n \in \{p_n\}_{n=1}^{\infty}\,(p_n \le \beta) \end{cases}$$

- e.g. 10-進制 decimal / denary number system / decadic 3.1.3, 3.1.24

$$\begin{aligned}\boldsymbol{x} &= \left\langle \sum_{k=0}^{1} a_{N-k}10^{N-k}, \sum_{k=0}^{2} a_{N-k}10^{N-k}, \cdots, \sum_{k=0}^{n} a_{N-k}10^{N-k}, \cdots \right\rangle \\ &= \langle x_1, x_2, \cdots, x_n, \cdots \rangle = \langle x_n \rangle_{n\in\mathbb{N}} = \langle x_n \rangle_{n=1}^{\infty}\end{aligned}$$

  - $\sum\limits_{k=0}^{N} a_k 10^k$ bounded 3.1.24
  - expansion

$$X = \begin{bmatrix} \sum\limits_{k=0}^{1} a_{1,N-k}10^{N-k} & \sum\limits_{k=0}^{2} a_{1,N-k}10^{N-k} & \cdots & \sum\limits_{k=0}^{n} a_{1,N-k}10^{N-k} & \cdots \\ \sum\limits_{k=0}^{1} a_{2,N-k}10^{N-k} & \sum\limits_{k=0}^{2} a_{2,N-k}10^{N-k} & \cdots & \sum\limits_{k=0}^{n} a_{2,N-k}10^{N-k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum\limits_{k=0}^{1} a_{m,N-k}10^{N-k} & \sum\limits_{k=0}^{2} a_{m,N-k}10^{N-k} & \cdots & \sum\limits_{k=0}^{n} a_{m,N-k}10^{N-k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \langle x_{1,1}, x_{1,2}, \cdots, x_{1,n}, \cdots \rangle \\ \langle x_{2,1}, x_{2,2}, \cdots, x_{2,n}, \cdots \rangle \\ \vdots \\ \langle x_{m,1}, x_{m,2}, \cdots, x_{m,n}, \cdots \rangle \\ \vdots \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle x_{1,n} \rangle_{n\in\mathbb{N}} \\ \langle x_{2,n} \rangle_{n\in\mathbb{N}} \\ \vdots \\ \langle x_{m,n} \rangle_{n\in\mathbb{N}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle x_{1,n} \rangle_{n=1}^{\infty} \\ \langle x_{2,n} \rangle_{n=1}^{\infty} \\ \vdots \\ \langle x_{m,n} \rangle_{n=1}^{\infty} \\ \vdots \end{bmatrix}$$

  - choose $x_{m,n}$ from the above matrix X to form new sequence or infinite dimensional vector $\boldsymbol{y}$

$$\begin{aligned}\boldsymbol{y} = \langle y_n \rangle_{n\in\mathbb{N}} = \langle y_n \rangle_{n=1}^{\infty} &= \langle y_1, y_2, \cdots, y_n, \cdots \rangle \\ &= \langle x_{m_1,n}, x_{m_2,n}, \cdots, x_{m_n,n}, \cdots \rangle \\ &= \langle x_{m(1),n}, x_{m(2),n}, \cdots, x_{m(n),n}, \cdots \rangle\end{aligned}$$

* e.g.

$$\begin{aligned}\boldsymbol{y}=\langle y_n\rangle_{n\in\mathbb{N}}=\langle y_n\rangle_{n=1}^{\infty}&=\langle y_1,y_2,\cdots,y_n,\cdots\rangle\\&=\langle x_{m_1,n},x_{m_2,n},\cdots,x_{m_n,n},\cdots\rangle\\&=\left\langle x_{m(1),n},x_{m(2),n},\cdots,x_{m(n),n},\cdots\right\rangle\\&\overset{\text{if } m_n=m(n)=n}{=}\langle x_{1,1},x_{2,2},\cdots,x_{n,n},\cdots\rangle\\&=\left\langle\sum_{k=0}^{1}a_{1,N-k}10^{N-k},\sum_{k=0}^{2}a_{2,N-k}10^{N-k},\cdots,\sum_{k=0}^{n}a_{n,N-k}10^{N-k},\cdots\right\rangle\end{aligned}$$

However, there exists rational sequence bounded by no rational number, i.e.

$$\exists\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty}\left(\begin{cases}\forall n\in\mathbb{N}\,(p_n\in\mathbb{Q})\\\forall\beta\in\mathbb{Q}^+,\exists p_n\in\{p_n\}_{n=1}^{\infty}\,(|p_n|>\beta)\end{cases}\right)$$

equivalent to rational number system incomplete or no least-upper-bound property 3.3.1.
But we still need to go on further definition of monotone bounded rational sequence 3.1.27.

定義 3.1.27. 遞增有界有比數列 / 單調有界有比數列 increasing bounded rational sequence / monotone bounded rational sequence

monotone including increasing and decreasing

- increasing bounded rational sequence = IBQS

$$\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty},\begin{cases}\forall n\in\mathbb{N}\,(p_n\in\mathbb{Q})\\\exists\beta\in\mathbb{Q},\forall p_n\in\{p_n\}_{n=1}^{\infty}\,(p_n\le\beta)\\\forall n\in\mathbb{N}\,(p_n\le p_{n+1})\end{cases}$$

- decreasing bounded rational sequence = DBQS

$$\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty},\begin{cases}\forall n\in\mathbb{N}\,(p_n\in\mathbb{Q})\\\exists\beta\in\mathbb{Q},\forall p_n\in\{p_n\}_{n=1}^{\infty}\,(p_n\ge\beta)\\\forall n\in\mathbb{N}\,(p_n\ge p_{n+1})\end{cases}$$

定義 3.1.28. set of all increasing bounded rational sequences

$$\boldsymbol{Q}=\left\{\boldsymbol{p}\left|\begin{cases}\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty}\\\forall n\in\mathbb{N}\,(p_n\in\mathbb{Q})\\\exists\beta\in\mathbb{Q},\forall p_n\in\{p_n\}_{n=1}^{\infty}\,(p_n\le\beta)\\\forall n\in\mathbb{N}\,(p_n\le p_{n+1})\end{cases}\right.\right\}$$

according to 等價關係基本定理 fundamental theorem on equivalence relation = FTER 1.2.16, we should find an equivalence relation and its equivalence classes on $\boldsymbol{P}$.

定義 3.1.29. set of all upper bounds of a rational sequence

$$\overline{B}_{\boldsymbol{p}}=\left\{\beta\left|\begin{cases}\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty}\\\beta\in\mathbb{Q},\forall p_n\in\{p_n\}_{n=1}^{\infty}\,(p_n\le\beta)\end{cases}\right.\right\}$$

similarly, set of all lower bounds of a rational sequence

$$\underline{B}_{\boldsymbol{p}}=\left\{\beta\left|\begin{cases}\boldsymbol{p}=\langle p_1,p_2,\cdots,p_n,\cdots\rangle=\langle p_n\rangle_{n\in\mathbb{N}}=\langle p_n\rangle_{n=1}^{\infty}\\\beta\in\mathbb{Q},\forall p_n\in\{p_n\}_{n=1}^{\infty}\,(p_n\ge\beta)\end{cases}\right.\right\}$$

定義 3.1.30. an equivalence relation on $\boldsymbol{P}$, the set of all increasing bounded rational sequences 3.1.28

$$\forall\boldsymbol{p},\boldsymbol{q}\in\boldsymbol{Q}\,(\boldsymbol{p}\sim\boldsymbol{q}\Leftrightarrow\overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{q}})$$

according to set equality, the above equivalence relation is
reflexive

$$\forall\boldsymbol{p}\in\boldsymbol{Q}\,(\boldsymbol{p}\sim\boldsymbol{p}\Leftrightarrow\overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{p}})$$

symmetric

$$\forall\boldsymbol{p},\boldsymbol{q}\in\boldsymbol{Q}\,[(\boldsymbol{p}\sim\boldsymbol{q}\Leftrightarrow\overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{q}})\Rightarrow(\boldsymbol{q}\sim\boldsymbol{p}\Leftrightarrow\overline{B}_{\boldsymbol{q}}=\overline{B}_{\boldsymbol{p}})]$$

transitive

$$\forall \boldsymbol{p},\boldsymbol{q},\boldsymbol{r}\in \boldsymbol{Q}\left[\begin{cases}\boldsymbol{p}\sim\boldsymbol{q}\Leftrightarrow \overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{q}}\\ \boldsymbol{q}\sim\boldsymbol{r}\Leftrightarrow \overline{B}_{\boldsymbol{q}}=\overline{B}_{\boldsymbol{r}}\end{cases}\Rightarrow\left(\boldsymbol{p}\sim\boldsymbol{r}\Leftrightarrow \overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{r}}\right)\right]$$

so according to 等價類 / 等價班 / 班 equivalence class 1.2.11 and 等價關係基本定理 fundamental theorem on equivalence relation = FTER 1.2.16

$$\left(\boldsymbol{p}\sim\boldsymbol{q}\sim\boldsymbol{r}\Leftrightarrow \overline{B}_{\boldsymbol{p}}=\overline{B}_{\boldsymbol{q}}=\overline{B}_{\boldsymbol{r}}\right)\Rightarrow\left([\boldsymbol{p}]_\sim=[\boldsymbol{q}]_\sim=[\boldsymbol{r}]_\sim\Leftrightarrow[\boldsymbol{p}]=[\boldsymbol{q}]=[\boldsymbol{r}]\right)\qquad 1.2.11$$

$$\sim \text{ is an equivalence relation on } \boldsymbol{Q}\overset{1.2.3}{\Rightarrow}\{[\boldsymbol{p}]_\sim|[\boldsymbol{p}]_\sim\subseteq\boldsymbol{Q}\}\text{ is a partition of }\boldsymbol{Q}\qquad 1.2.16$$

定義 3.1.31. 實數系 / 實數系統 / 全體實數集 / (全體)實數集 real number system / set of all real numbers

$$\mathbb{R}=\left\{[\boldsymbol{x}]_\sim\left|\begin{cases}[\boldsymbol{x}]_\sim\subseteq\boldsymbol{Q} & 3.1.28\\ \forall\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{Q}\left(\boldsymbol{x}\sim\boldsymbol{y}\Leftrightarrow\overline{B}_{\boldsymbol{x}}=\overline{B}_{\boldsymbol{y}}\right) & 3.1.30\\ \overline{B}_{\boldsymbol{x}}=\left\{\beta\left|\begin{cases}\boldsymbol{x}=\langle x_n\rangle_{n=1}^\infty\\ \beta\in\mathbb{Q},\forall x_n\in\{x_n\}_{n=1}^\infty(x_n\le\beta)\end{cases}\right.\right\} & 3.1.29\end{cases}\right.\right\}$$

$$=\left\{x\left|\begin{cases}x=[\langle x_n\rangle]_\sim,\langle x_n\rangle=\langle x_n\rangle_{n\in\mathbb{N}}=\boldsymbol{x} & \text{for simplification of symbols}\\ x\subseteq\boldsymbol{Q} & 3.1.28\\ \forall\langle x_n\rangle,\langle y_m\rangle\in\boldsymbol{Q}\left(\langle x_n\rangle\sim\langle y_m\rangle\Leftrightarrow\overline{B}_{\langle x_n\rangle}=\overline{B}_{\langle y_m\rangle}\right) & 3.1.30\\ \overline{B}_{\langle x_n\rangle}=\left\{\beta\middle|\beta\in\mathbb{Q},\forall x_n\in\{x_n\}_{n\in\mathbb{N}}(x_n\le\beta)\right\} & 3.1.29\end{cases}\right.\right\}$$

定義 3.1.32. real order

$$\forall x,y\in\mathbb{R}\left(\begin{cases}x=[\langle x_n\rangle]_\sim,\langle x_n\rangle=\langle x_n\rangle_{n\in\mathbb{N}}\\ y=[\langle y_m\rangle]_\sim,\langle y_m\rangle=\langle y_m\rangle_{m\in\mathbb{N}}\\ \overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\end{cases}\Leftrightarrow x\le y\right)$$

定理 3.1.33. 三一律 *trichotomy law / law of trichotomy*

$$\forall x,y\in\mathbb{R}\begin{pmatrix}x<y\\ \veebar\\ x>y\\ \veebar\\ x=y\end{pmatrix}$$

Proof. by definition of real order 3.1.32,

$$(1)\ x<y\Leftrightarrow\left(\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\right)\wedge\left(\neg\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\ \overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\end{cases}\right)$$

$$(2)\ x>y\Leftrightarrow\left(\overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\right)\wedge\left(\neg\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\ \overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\end{cases}\right)$$

$$(3)\ x=y\Leftrightarrow\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\ \overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\end{cases}$$

$$\forall\langle(i),(j)\rangle\in\{(1),(2),(3)\}^2\left[i\ne j\Rightarrow(i)\wedge(j)\ \text{ includes }\left(\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\ \overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\end{cases}\right)\wedge\left(\neg\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\ \overline{B}_{\langle x_n\rangle}\subseteq\overline{B}_{\langle y_m\rangle}\end{cases}\right)\right]$$
$$\Rightarrow\forall\langle(i),(j)\rangle\in\{(1),(2),(3)\}\times\{(1),(2),(3)\}\left[i\ne j\Rightarrow v((i)\wedge(j))=\mathrm{F}\right]$$

□

引理 3.1.34. *strict real order lemma*

$$\begin{cases}x,y\in\mathbb{R}\\ x<y\\ x=[\langle x_n\rangle]_\sim,\langle x_n\rangle=\langle x_n\rangle_{n\in\mathbb{N}}\\ y=[\langle y_m\rangle]_\sim,\langle y_m\rangle=\langle y_m\rangle_{n\in\mathbb{N}}\end{cases}\Rightarrow\exists m\in\mathbb{N},\forall n\in\mathbb{N}(y_m>x_n)$$

Proof.

$$x<y\Rightarrow\begin{cases}x\le y\\x\ne y\end{cases}\overset{3.1.32}{\Rightarrow}\begin{cases}\overline{B}_{\langle x_n\rangle}\supseteq\overline{B}_{\langle y_m\rangle}\\\overline{B}_{\langle x_n\rangle}\ne\overline{B}_{\langle y_m\rangle}\end{cases}\Rightarrow\overline{B}_{\langle x_n\rangle}\supset\overline{B}_{\langle y_m\rangle}$$

$$\overline{B}_{\langle x_n\rangle}\supset\overline{B}_{\langle y_m\rangle}\Rightarrow\exists\beta_x\in\overline{B}_{\langle x_n\rangle}\left(\beta_x\notin\overline{B}_{\langle y_m\rangle}\right)$$

$$\beta_x\notin\overline{B}_{\langle y_m\rangle}\Rightarrow\exists m\in\mathbb{N},y_m\in\mathbb{Q}\,(y_m>\beta_x)$$

$$\beta_x\in\overline{B}_{\langle x_n\rangle}\Rightarrow\forall n\in\mathbb{N},x_n\in\mathbb{Q}\,(\beta_x\ge x_n)$$

$$\begin{aligned}&\begin{cases}\exists m\in\mathbb{N},y_m\in\mathbb{Q}\,(y_m>\beta_x)\\\forall n\in\mathbb{N},x_n\in\mathbb{Q}\,(\beta_x\ge x_n)\end{cases}\\\Rightarrow&\exists m\in\mathbb{N},y_m\in\mathbb{Q},\forall n\in\mathbb{N},x_n\in\mathbb{Q}\,(y_m>\beta_x\ge x_n)\\\Rightarrow&\exists m\in\mathbb{N},\forall n\in\mathbb{N}\,(y_m>x_n)\end{aligned}$$

□

定義 3.1.35. set of all upper bounds of a real subset

$$\overline{B}_S=\left\{\xi\,\middle|\,\begin{cases}\xi\in\mathbb{R}\\\forall a\in S\subset\mathbb{R}\,(a\le\xi)\end{cases}\right\}$$

similarly, set of all lower bounds of a real subset

$$\underline{B}_S=\left\{\xi\,\middle|\,\begin{cases}\xi\in\mathbb{R}\\\forall a\in S\subset\mathbb{R}\,(a\ge\xi)\end{cases}\right\}$$

定義 3.1.36. 上確界 / 最小上界 與 下確界 / 最大下界 supermum & infimum

$$\exists\alpha\in\mathbb{R}\left[\forall a\in S\left(\begin{cases}a\le\alpha & \text{上界}\\\forall\xi\in\mathbb{R}\,(\forall a\in S\,(a\le\xi)\Rightarrow\alpha\le\xi) & \text{最小}\end{cases}\right)\Leftrightarrow(\alpha=\sup S)\Leftrightarrow(\alpha\text{ 是 }S\text{ 的最小上界})\right]$$

i.e.

$$\sup S\in\overline{B}_S$$

and

$$\forall\xi\in\overline{B}_S\,(\xi\ge\sup S)$$

similarly,

$$\exists\omega\in\mathbb{R}\left[\forall a\in S\left(\begin{cases}a\ge\omega & \text{下界}\\\forall\xi\in\mathbb{R}\,(\forall a\in S\,(a\ge\xi)\Rightarrow\omega\ge\xi) & \text{最大}\end{cases}\right)\Leftrightarrow(\omega=\inf S)\Leftrightarrow(\omega\text{ 是 }S\text{ 的最大下界})\right]$$

i.e.

$$\inf S\in\underline{B}_S$$

and

$$\forall\xi\in\underline{B}_S\,(\xi\le\inf S)$$

引理 3.1.37. 遞增部分實數集上確性 *increasing bounded real subset has a supermum / supermum of IBRS* [9]

$$\forall S=\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}\left[\begin{cases}\forall n\in\mathbb{N}\,(a_n\le a_{n+1}) & \text{遞增}\\\exists\beta\in\mathbb{R},\forall n\in\mathbb{N}\,(a_n\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\,(\alpha=\sup S)\right]$$

Proof.

$$\begin{aligned}\mathbb{R}\supset S=\{a_n\}_{n\in\mathbb{N}}&=\{a_1,a_2,\cdots,a_n,\cdots\}\\&\overset{3.1.31}{=}\{a_1=[\boldsymbol{a}_1]_\sim=\left[\langle a_{1,m}\rangle_{m\in\mathbb{N}}\right]_\sim=[\langle a_{1,1},a_{1,2},\cdots,a_{1,m},\cdots\rangle]_\sim,\\&\qquad a_2=[\boldsymbol{a}_2]_\sim=\left[\langle a_{2,m}\rangle_{m\in\mathbb{N}}\right]_\sim=[\langle a_{2,1},a_{2,2},\cdots,a_{2,m},\cdots\rangle]_\sim,\\&\qquad\vdots\\&\qquad a_n=[\boldsymbol{a}_n]_\sim=\left[\langle a_{n,m}\rangle_{m\in\mathbb{N}}\right]_\sim=[\langle a_{n,1},a_{n,2},\cdots,a_{n,m},\cdots\rangle]_\sim,\\&\qquad\vdots\\&\qquad\cdots\}\end{aligned}$$

[9] 直接規避掉有些 「不可數」【實子集 / 部分實數集】是否 well-ordered 這個問題

$$
\begin{aligned}
a_n \le a_{n+1} \Rightarrow & a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \\
a_n \le a_{n+1} \Rightarrow & a_1 \le a_2 \le \cdots \le a_N \le a_{N+1} \le a_{N+2} \le \cdots
\end{aligned}
$$

case 1: $\exists N \in \mathbb{N}\,(a_N = a_{N+1} = a_{N+2} = \cdots)$

$$
\begin{aligned}
\exists N \in \mathbb{N}\,(a_N = a_{N+1} = a_{N+2} = \cdots) \Leftrightarrow & \exists N \in \mathbb{N}, \forall m \in \mathbb{N}\,(a_N = a_{N+m}) \\
\Rightarrow & \forall n \in \mathbb{N}, \exists N \in \mathbb{N}, \forall m \in \mathbb{N}\,(a_n \le a_N = a_{N+m})
\end{aligned}
$$

$$
\begin{aligned}
a_N = [\boldsymbol{a}_N]_\sim = & \left[\langle a_N \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_N, a_N, \cdots, a_N, \cdots \rangle]_\sim \in \mathbb{R} \\
= & \left[\langle a_{N+m-1} \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_N, a_{N+1}, \cdots, a_{N+m-1}, \cdots \rangle]_\sim \\
= & \left[\langle a_{N,1} \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_{N,1}, a_{N,2}, \cdots, a_{N,m}, \cdots \rangle]_\sim
\end{aligned}
$$

$$\forall a_n \in S\,(a_n \le a_N) \Rightarrow a_N \in \overline{B}_S \neq \emptyset$$

if $\exists \xi \in \overline{B}_S\,(\xi < a_N)$,

$$\xi \in \overline{B}_S \Rightarrow \forall a_n \in S\,(a_n \le \xi)$$

$$a_N \in S \overset{\forall a_n \in S (a_n \le \xi)}{\Rightarrow} a_N \le \xi \Rightarrow\!\!\not\Leftarrow \xi < a_N$$

thus

$$\neg \exists \xi \in \overline{B}_S\,(\xi < a_N) \Leftrightarrow \forall \xi \in \overline{B}_S\,(\xi \ge a_N)$$

i.e.

$$a_N = \sup S$$

case 2: $\exists k \in \mathbb{N}, n_k \in I \subset \mathbb{N}\left(a_{n_k} < a_{n_{k+1}}\right)$, i.e. if there exists strictly increasing subsequence

$$
\begin{aligned}
\mathbb{R} \supset S = \{a_{n_k}\}_{k\in\mathbb{N}} = & \{a_{n_1}, a_{n_2}, \cdots, a_{n_k}, \cdots\} \\
\overset{3.1.31}{=} & \{a_{n_1} = [\boldsymbol{a}_{n_1}]_\sim = \left[\langle a_{n_1,m} \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_{n_1,1}, a_{n_1,2}, \cdots, a_{n_1,m}, \cdots \rangle]_\sim, \\
& a_{n_2} = [\boldsymbol{a}_{n_2}]_\sim = \left[\langle a_{n_2,m} \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_{n_2,1}, a_{n_2,2}, \cdots, a_{n_2,m}, \cdots \rangle]_\sim, \\
& \vdots \\
& a_{n_k} = [\boldsymbol{a}_{n_k}]_\sim = \left[\langle a_{n_k,m} \rangle_{m\in\mathbb{N}}\right]_\sim = [\langle a_{n_k,1}, a_{n_k,2}, \cdots, a_{n_k,m}, \cdots \rangle]_\sim, \\
& \vdots \\
& \cdots\}
\end{aligned}
$$

$$\alpha_1 = a_{n_1,1}$$

$$a_{n_1} < a_{n_2} \Leftrightarrow a_{n_2} > a_{n_1} \overset{3.1.34}{\Rightarrow} \exists m_2 \in \mathbb{N}, \forall m \in \mathbb{N}\,(a_{n_2,m_2} > a_{n_1,m})$$

$$\alpha_2 = a_{n_2,m_2}$$

$$a_{n_2} < a_{n_3} \Leftrightarrow a_{n_3} > a_{n_2} \overset{3.1.34}{\Rightarrow} \exists m_3 \in \mathbb{N}, \forall m \in \mathbb{N}\,(a_{n_3,m_3} > a_{n_2,m})$$

$$\alpha_3 = a_{n_3,m_3}$$

$$\vdots$$

$$a_{n_k} < a_{n_{k+1}} \Leftrightarrow a_{n_{k+1}} > a_{n_k} \overset{3.1.34}{\Rightarrow} \exists m_{k+1} \in \mathbb{N}, \forall m \in \mathbb{N}\left(a_{n_{k+1},m_{k+1}} > a_{n_k,m}\right)$$

$$\alpha_{k+1} = a_{n_{k+1},m_{k+1}}$$

$$\vdots$$

依此類推, 以致無窮 and so on to infinity, we get

$$\langle \alpha_1, \alpha_2, \cdots, \alpha_k, \cdots \rangle = \langle a_{n_1,1}, a_{n_2,m_2}, \cdots, a_{n_k,m_k}, \cdots \rangle$$

$$\alpha_1 = a_{n_1,1} < \alpha_2 = a_{n_2,m_2} < \cdots < \alpha_k = a_{n_k,m_k} < \cdots$$

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k \le \cdots$$

$$\boldsymbol{\alpha} = \langle \alpha_1, \alpha_2, \cdots, \alpha_k, \cdots \rangle \in \boldsymbol{Q}$$

$$\alpha = [\boldsymbol{\alpha}]_\sim = \left[\langle \alpha_k \rangle_{k\in\mathbb{N}}\right]_\sim = [\langle a_{n_1,1}, a_{n_2,m_2}, \cdots, a_{n_k,m_k}, \cdots \rangle]_\sim \in \mathbb{R}$$

compare order

$$\begin{array}{ccccccccc}
\alpha = & [\langle a_{n_1,1} < & a_{n_2,m_2} < & \cdots < & a_{n_k,m_k} < & \cdots\rangle]_\sim \\
& & \vee & & & \\
& \vee| & a_{n_1,m} & \cdots & \vee & \cdots \\
& & \vee| & & & \\
a_{n_1} = & [\langle a_{n_1,1} \le & a_{n_1,2} \le & & a_{n_1,m} \le & \cdots\rangle]_\sim
\end{array}$$

$$\begin{array}{ccccccc}
\alpha = & [\langle a_{n_1,1} < & a_{n_2,m_2} < & a_{n_3,m_3} < & \cdots < & a_{n_k,m_k} < & \cdots\rangle]_\sim \\
& & & \vee & & & \\
& & & a_{n_2,m} & \cdots & \vee & \cdots \\
& & & \vee| & & & \\
a_{n_2} = & [\langle a_{n_2,1} \le & a_{n_2,2} \le & a_{n_2,3} \le & & a_{n_2,m} \le & \cdots\rangle]_\sim
\end{array}$$

$$\vdots$$

$$\forall k \in \mathbb{N}, \exists \alpha_{k+1} = a_{n_{k+1},m_{k+1}} \in \mathbb{Q}\left(\alpha_{k+1} > a_{n_k,k}\right) \Rightarrow \begin{cases} \overline{B}_{\boldsymbol{\alpha}} \subset \overline{B}_{\boldsymbol{a}_{n_1}} \Leftrightarrow \alpha > a_{n_1} \\ \overline{B}_{\boldsymbol{\alpha}} \subset \overline{B}_{\boldsymbol{a}_{n_2}} \Leftrightarrow \alpha > a_{n_2} \\ \vdots \\ \overline{B}_{\boldsymbol{\alpha}} \subset \overline{B}_{\boldsymbol{a}_{n_k}} \Leftrightarrow \alpha > a_{n_k} \\ \vdots \end{cases}$$

$$\forall k \in \mathbb{N}\left(\alpha > a_{n_k}\right) \Rightarrow \forall k \in \mathbb{N}\left(\alpha \ge a_{n_k}\right) \Rightarrow \forall a_{n_k} \in S\left(\alpha \ge a_{n_k}\right) \Rightarrow \alpha \in \overline{B}_S \neq \emptyset$$

if $\exists \xi \in \overline{B}_S\,(\xi < \alpha)$,

$$\xi \in \overline{B}_S \Rightarrow \forall a_{n_k} \in S\left(a_{n_k} \le \xi\right)$$

$$a_{n_k} \le \xi \overset{3.1.34}{\Rightarrow} \exists n \in \mathbb{N}, \forall m \in \mathbb{N}\left(\xi_n \ge a_{n_k,m}\right)$$

$$\alpha_k = a_{n_k,m_k}, m_k \in \mathbb{N}$$

$$\begin{cases} \exists n \in \mathbb{N}, \forall m \in \mathbb{N}\left(\xi_n \ge a_{n_k,m}\right) \\ \alpha_k = a_{n_k,m_k}, m_k \in \mathbb{N} \end{cases} \Rightarrow \exists n \in \mathbb{N}, \forall m_k \in \mathbb{N}\left(\xi_n \ge a_{n_k,m_k} = \alpha_k\right)$$

$$\xi < \alpha \overset{3.1.34}{\Rightarrow} \exists k \in \mathbb{N}, \forall n \in \mathbb{N}\left(\alpha_k > \xi_n\right)$$

$$\begin{aligned} &\begin{cases} \exists n \in \mathbb{N}, \forall m_k \in \mathbb{N}\left(\xi_n \ge a_{n_k,m_k} = \alpha_k\right) \\ \exists k \in \mathbb{N}, \forall n \in \mathbb{N}\left(\alpha_k > \xi_n\right) \end{cases} \\ \Rightarrow &\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, \forall m_k \in \mathbb{N}\left(\alpha_k > \xi_n \ge a_{n_k,m_k} = \alpha_k\right) \\ \Rightarrow &\exists k \in \mathbb{N}\left(\alpha_k > \alpha_k\right), \wedge v\left(\exists k \in \mathbb{N}\left(\alpha_k > \alpha_k\right)\right) = \mathrm{F} \Rightarrow v\left(\exists \xi \in \overline{B}_S\,(\xi < \alpha)\right) = \mathrm{F} \end{aligned}$$

thus

$$\neg \exists \xi \in \overline{B}_S\,(\xi < \alpha) \Leftrightarrow \forall \xi \in \overline{B}_S\,(\xi \ge \alpha)$$

i.e.

$$\alpha = \sup S$$

summary from case 1 and case 2,

$$\begin{aligned} &\forall S = \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}\left[\begin{cases} \forall n \in \mathbb{N}\left(a_n \le a_{n+1}\right) & \text{遞增} \\ \exists \beta \in \overline{B}_S \subset \mathbb{R}, \forall n \in \mathbb{N}\left(a_n \le \beta\right) \Leftrightarrow \overline{B}_S \neq \emptyset & \text{上界} \end{cases}\right] \\ &\Rightarrow \exists \alpha = \begin{cases} a_N = [\langle a_N, a_N, \cdots, a_N, \cdots\rangle]_\sim & \text{case 1} \\ [\langle a_{n_1,1}, a_{n_2,m_2}, \cdots, a_{n_k,m_k}, \cdots\rangle]_\sim & \text{case 2} \end{cases} \in \mathbb{R}\left(\alpha = \sup S\right) \end{aligned}$$

□

定理 3.1.38. 遞增有界實數列 單調收斂性 / 單調收斂定理 *increasing bounded real sequence monotone convergence property / monotone convergence theorem = MCT = MCP / IBRS MCT*

$$\forall \langle a_n \rangle_{n \in \mathbb{N}}, a_n \in \mathbb{R}\left[\begin{cases} \forall n \in \mathbb{N}\left(a_n \le a_{n+1}\right) & \text{遞增} \\ \exists \beta \in \mathbb{R}, \forall n \in \mathbb{N}\left(a_n \le \beta\right) \Leftrightarrow \overline{B}_S \neq \emptyset & \text{上界} \Rightarrow \exists \alpha = \sup S \in \mathbb{R}\left(\textit{let } \lim_{n \to \infty} a_n = \alpha\right) \end{cases}\right]$$

Proof. by lemma 遞增實數集上確性 increasing bounded real set has a supermum / supermum of IBRS 3.1.37

$$\forall S=\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}\left[\begin{cases}\forall n\in\mathbb{N}\,(a_n\le a_{n+1}) & \text{單調}\\ \exists\beta\in\mathbb{R},\forall n\in\mathbb{N}\,(a_n\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\,(\alpha=\sup S)\right]$$

$$\forall\langle a_n\rangle_{n\in\mathbb{N}},a_n\in\mathbb{R}\left[\begin{cases}\forall n\in\mathbb{N}\,(a_n\le a_{n+1}) & \text{遞增}\\ \exists\beta\in\mathbb{R},\forall n\in\mathbb{N}\,(a_n\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha=\sup S\in\mathbb{R}\left(\text{let }\lim_{n\to\infty}a_n=\alpha\right)\right]$$

□

定理 3.1.39. 實子集 / 部分實數集 上確性 / 上確界定理 / 上確界公理 *real subset least-upper-bound property = LUB property = LUBP / previously axiom of least upper bound / Dedekind completeness / RSubS LUBT*

任意上方有界非空實數集 $S\subset\mathbb{R}$, 必有上確界 α (上確界即為最小上界) (若 S 為部分開區間, 該上確界 α 未必屬於 S; 但此處用區間套為閉區間)

$$\forall S\left[\begin{cases}\emptyset\ne S\subset\mathbb{R} & \text{非空}\\ \exists\beta\in\mathbb{R},\forall a\in S\,(a\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R},\forall a\in S\left(\begin{cases}a\le\alpha & \text{上界}\\ \forall\xi\in\mathbb{R}\,(\forall a\in S\,(a\le\xi)\Rightarrow\alpha\le\xi) & \text{最小}\end{cases}\right)\right]$$

$$\forall S\left[\begin{cases}\emptyset\ne S\subset\mathbb{R} & \text{非空}\\ \exists\beta\in\mathbb{R},\forall a\in S\,(a\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\,(\alpha=\sup S)\right]$$

$$\forall S\subset\mathbb{R}\left[\begin{cases}\emptyset\ne S & \text{非空}\\ \exists\beta\in\mathbb{R},\forall a\in S\,(a\le\beta)\Leftrightarrow\overline{B}_S\ne\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\,(\alpha=\sup S)\right]$$

Proof. case 1: $\beta=\sup S$,

$$\exists\alpha=\beta\in\mathbb{R}\,(\alpha=\sup S)$$

case 2: $\beta\ne\sup S$,
construction of 區間套 nested interval = NI 3.3.4,

$$\begin{cases}\forall a\in S\,(a\le\beta)\\ \beta\ne\sup S\end{cases}\Rightarrow\exists a_1\in S\,(a_1<\beta)$$

case 2-1-0:
let

$$\begin{cases}a_1=a<\beta & ,a\in S\\ b_1=\beta\\ m_1=\dfrac{a_1+b_1}{2}\in(a_1,b_1)\\ I_1=[a_1,b_1]\\ \|I_1\|=|b_1-a_1|=b_1-a_1=\beta-a>0\end{cases}$$

case 2-2-1: $m_1\in S$, let

$$\begin{cases}a_2=m_1\\ b_2=b_1\\ m_2=\dfrac{a_2+b_2}{2}\in(a_2,b_2)\\ I_2=[a_2,b_2]=[m_1,b_1]\overset{m_1\in(a_1,b_1)}{\subset}[a_1,b_1]=I_1\Rightarrow I_2\subseteq I_1\\ \|I_2\|=|b_2-a_2|=|b_1-m_1|=b_1-m_1=b_1-\dfrac{a_1+b_1}{2}=\dfrac{b_1-a_1}{2}=\dfrac{\|I_1\|}{2}=\dfrac{\beta-a}{2}\end{cases}$$

case 2-2-2: $m_1\in\overline{B}_S$, let

$$\begin{cases}a_2=a_1\\ b_2=m_1\\ m_2=\dfrac{a_2+b_2}{2}\in(a_2,b_2)\\ I_2=[a_2,b_2]=[a_1,m_1]\overset{m_1\in(a_1,b_1)}{\subset}[a_1,b_1]=I_1\Rightarrow I_2\subseteq I_1\\ \|I_2\|=|b_2-a_2|=|m_1-a_1|=m_1-a_1=\dfrac{a_1+b_1}{2}-a_1=\dfrac{b_1-a_1}{2}=\dfrac{\|I_1\|}{2}=\dfrac{\beta-a}{2}\end{cases}$$

case 2-3-1: $m_2 \in S$, let

$$\begin{cases} a_3 = m_2 \\ b_3 = b_2 \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \\ I_3 = [a_3, b_3] = [m_2, b_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ \|I_3\| = |b_3 - a_3| = |b_2 - m_2| = b_2 - m_2 = b_2 - \dfrac{a_2 + b_2}{2} = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{\beta - a}{2^2} \end{cases}$$

case 2-3-2: $m_2 \in \overline{B}_S$, let

$$\begin{cases} a_3 = a_2 \\ b_3 = m_2 \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \\ I_3 = [a_3, b_3] = [a_2, m_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ \|I_3\| = |b_3 - a_3| = |m_2 - a_2| = m_2 - a_2 = \dfrac{a_2 + b_2}{2} - a_2 = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{\beta - a}{2^2} \end{cases}$$

$\vdots$

case 2-n-1: $m_{n-1} \in S$, let

$$\begin{cases} a_n = m_{n-1} \\ b_n = b_{n-1} \\ m_n = \dfrac{a_n + b_n}{2} \in (a_n, b_n) \\ I_n = [a_n, b_n] = [m_{n-1}, b_{n-1}] \overset{m_{n-1} \in (a_{n-1}, b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |b_{n-1} - m_{n-1}| = b_{n-1} - m_{n-1} = b_{n-1} - \dfrac{a_{n-1} + b_{n-1}}{2} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^{n-1}} = \dfrac{\beta - a}{2^{n-1}} \end{cases}$$

case 2-n-2: $m_{n-1} \in \overline{B}_S$, let

$$\begin{cases} a_n = a_{n-1} \\ b_n = m_{n-1} \\ m_n = \dfrac{a_n + b_n}{2} \in (a_n, b_n) \\ I_n = [a_n, b_n] = [a_{n-1}, m_{n-1}] \overset{m_{n-1} \in (a_{n-1}, b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |m_{n-1} - a_{n-1}| = m_{n-1} - a_{n-1} = \dfrac{a_{n-1} + b_{n-1}}{2} - a_{n-1} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^{n-1}} = \dfrac{\beta - a}{2^{n-1}} \end{cases}$$

$\vdots$

依此類推, 以致無窮 and so on to infinity,

$$\begin{cases} \forall n \in \mathbb{N} \left\{ [a_n, b_n] = I_n, \begin{cases} b_n \in \overline{B}_S \\ a_n \in S \end{cases} \right\} \\ \forall n \in \mathbb{N} \{ I_{n+1} \subseteq I_n \} \\ \forall n \in \mathbb{N} \left\{ \|I_{n+1}\| = \dfrac{\|I_1\|}{2^n} \right\} \end{cases}$$

by 區間套定理 / 區間套原理 nested interval theorem = NIT / nested interval principle = NIP 3.3.4,

$$\begin{cases} \forall n \in \mathbb{N} \left(\begin{cases} I_n = [a_n, b_n] \subset \mathbb{R} \\ I_{n+1} \subseteq I_n \Leftrightarrow I_n \supseteq I_{n+1} \end{cases} \right) \\ \lim\limits_{n\to\infty} (b_n - a_n) = \lim\limits_{n\to\infty} \|I_n\| = \lim\limits_{n\to\infty} \dfrac{\|I_1\|}{2^{n-1}} = \lim\limits_{n\to\infty} \dfrac{\beta - a}{2^{n-1}} = 0 \end{cases} \Rightarrow \exists \alpha \in \mathbb{R} \left(\bigcap_{n=1}^{\infty} I_n = \{\alpha\} \right)$$

$$\alpha = \lim_{n\to\infty} b_n \in \overline{B}_S \cap S \ni \lim_{n\to\infty} a_n = \alpha$$

$$\forall n \in \mathbb{N}, a_n \in S, a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \alpha \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1 = \beta \in \overline{B}_S, \forall n \in \mathbb{N}, b_n \in \overline{B}_S$$

$$\begin{cases} \alpha \in \overline{B}_S \Rightarrow \forall a \in S \, (a \leq \alpha) \\ \alpha \in S \Rightarrow \forall \beta \in \overline{B}_S \, (\alpha \leq \beta) \end{cases} \Rightarrow \alpha = \sup S$$

$\square$

定義 3.1.40. real addition

$$\forall x, y \in \mathbb{R}\left(x+y=z \Leftrightarrow \begin{cases} x=[\boldsymbol{x}]_{\sim}, y=[\boldsymbol{y}]_{\sim} \subseteq \boldsymbol{Q} \\ \Downarrow \langle x_n\rangle \in x, \langle y_n\rangle \in y \\ \boldsymbol{x}=\langle x_n\rangle, \boldsymbol{y}=\langle y_n\rangle \in \boldsymbol{Q} \\ \Downarrow \\ \boldsymbol{x}+\boldsymbol{y}=\langle x_n+y_n\rangle \in \boldsymbol{Q} \\ \Downarrow \langle x_n+y_n\rangle \sim \langle z_n\rangle = \boldsymbol{z} \\ \exists z=[\boldsymbol{z}]_{\sim} \in \mathbb{R} \end{cases}\right)$$

註記 3.1.41. $x:y:z=\xi:\eta:\zeta=$ (xi) : (eta) : (zeta) $=$ (xi) : ($y$ita) : ($z$eta)

$$\forall \xi, \eta \in \mathbb{R}\left(\xi+\eta=\zeta \Leftrightarrow \begin{cases} \xi=[\boldsymbol{x}]_{\sim}, \eta=[\boldsymbol{y}]_{\sim} \subseteq \boldsymbol{Q} \\ \Downarrow \\ \boldsymbol{x}+\boldsymbol{y} \sim \boldsymbol{z} \in \boldsymbol{Q} \\ \Downarrow \\ \exists \zeta=[\boldsymbol{z}]_{\sim} \in \mathbb{R} \end{cases}\right)$$

推論 3.1.42. *well definition of real addition*

$$\forall [\boldsymbol{x}]_{\sim}, [\boldsymbol{y}]_{\sim}, [\boldsymbol{x}']_{\sim}, [\boldsymbol{y}']_{\sim} \in \mathbb{R}\left(\begin{cases} \boldsymbol{x} \sim \boldsymbol{x}' & (1) \\ \boldsymbol{y} \sim \boldsymbol{y}' & (2) \end{cases} \Rightarrow \boldsymbol{x}+\boldsymbol{y} \sim \boldsymbol{x}'+\boldsymbol{y}'\right)$$

Proof. consider target $\boldsymbol{x}+\boldsymbol{y} \sim \boldsymbol{x}'+\boldsymbol{y}'$ for proof idea 證明思路

$$\begin{aligned} & \boldsymbol{x}+\boldsymbol{y} \sim \boldsymbol{x}'+\boldsymbol{y}' \\ \Leftrightarrow & \overline{B}_{\boldsymbol{x}+\boldsymbol{y}} = \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'} \Leftarrow \begin{cases} \overline{B}_{\boldsymbol{x}+\boldsymbol{y}} \subseteq \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'} \\ \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'} \subseteq \overline{B}_{\boldsymbol{x}+\boldsymbol{y}} \end{cases} \Leftrightarrow \begin{cases} \forall \beta \in \overline{B}_{\boldsymbol{x}+\boldsymbol{y}} \left(\beta \in \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'}\right) \\ \forall \beta' \in \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'} \left(\beta' \in \overline{B}_{\boldsymbol{x}+\boldsymbol{y}}\right) \end{cases} \end{aligned}$$

其中

$$\beta = \beta_{\boldsymbol{x}+\boldsymbol{y}} \in \overline{B}_{\boldsymbol{x}+\boldsymbol{y}} \Rightarrow \forall n \in \mathbb{N}\,(x_n+y_n \leq \beta)$$
$$\beta' = \beta_{\boldsymbol{x}'+\boldsymbol{y}'} \in \overline{B}_{\boldsymbol{x}'+\boldsymbol{y}'} \Rightarrow \forall n \in \mathbb{N}\,(x'_n+y'_n \leq \beta')$$

$$\begin{aligned} \boldsymbol{x}+\boldsymbol{y} = \langle x_n+y_n\rangle_{n\in\mathbb{N}} = \langle x_n+y_n\rangle &= \langle x_1+y_1, x_2+y_2, \cdots, x_n+y_n, \cdots\rangle \\ &= \langle x_1+y_1, x_2+y_2, \cdots, x_N+y_N, \cdots\rangle, N \in \mathbb{N} \end{aligned}$$

$$\begin{aligned} \boldsymbol{x}'+\boldsymbol{y}' = \langle x'_n+y'_n\rangle_{n\in\mathbb{N}} = \langle x'_n+y'_n\rangle &= \langle x'_1+y'_1, x'_2+y'_2, \cdots, x'_n+y'_n, \cdots\rangle \\ &= \langle x'_1+y'_1, x'_2+y'_2, \cdots, x'_N+y'_N, \cdots\rangle, N \in \mathbb{N} \end{aligned}$$

$$\begin{aligned} & [\boldsymbol{x}]_{\sim}, [\boldsymbol{y}]_{\sim}, [\boldsymbol{x}']_{\sim}, [\boldsymbol{y}']_{\sim} \in \mathbb{R} \\ \Rightarrow & \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}', \boldsymbol{y}' \in \boldsymbol{Q} = \left\{ \boldsymbol{p} \middle| \begin{cases} \boldsymbol{p} = \langle p_1, p_2, \cdots, p_n, \cdots\rangle = \langle p_n\rangle_{n\in\mathbb{N}} = \langle p_n\rangle_{n=1}^{\infty} \\ \forall n \in \mathbb{N}\,(p_n \in \mathbb{Q}) \\ \exists \beta \in \mathbb{Q}, \forall p_n \in \{p_n\}_{n=1}^{\infty}\,(p_n \leq \beta) \\ \forall n \in \mathbb{N}\,(p_n \leq p_{n+1}) \end{cases} \right\} \\ \Rightarrow & \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}'+\boldsymbol{y}' \in \boldsymbol{Q} \end{aligned}$$

consider $\boldsymbol{x}+\boldsymbol{y}$, $\forall N \in \mathbb{N}$,
if $n \leq N$,

$$x_n+y_n \leq x_n+y_N \leq x_N+y_N \leq \beta$$
$$x_n+y_N \leq \beta$$
$$x_n \leq \beta - y_N$$

if $N < n$,

$$x_N+y_N \leq x_n+y_N \leq x_n+y_n \leq \beta$$
$$x_n+y_N \leq \beta$$
$$x_n \leq \beta - y_N$$

no matter $n \le N$ or $N < n$, $\forall n \in \mathbb{N}$,

$$x_n \le \beta - y_N$$

i.e.

$$\beta - y_N \in \overline{B}_{\boldsymbol{x}}$$

and according to premise(1) $\boldsymbol{x} \sim \boldsymbol{x}'$,

$$\beta - y_N \in \overline{B}_{\boldsymbol{x}} \overset{\boldsymbol{x} \sim \boldsymbol{x}'}{=} \overline{B}_{\boldsymbol{x}'} \Rightarrow x'_n \le \beta - y_N$$

summary from consider $\boldsymbol{x} + \boldsymbol{y}$, $\forall N \in \mathbb{N}$ to above:

$$\forall N \in \mathbb{N}, \forall n \in \mathbb{N} \left(x'_n \le \beta - y_N\right)$$

$$\forall n \in \mathbb{N}, \forall N \in \mathbb{N} \left(x'_n \le \beta - y_N\right)$$
$$\forall n \in \mathbb{N}, \forall N \in \mathbb{N} \left(y_N \le \beta - x'_n\right)$$

consider $y_N \le \beta - x'_n$, $\forall n \in \mathbb{N}$,

$$y_N \le \beta - x'_n \Rightarrow \beta - x'_n \in \overline{B}_{\boldsymbol{y}}$$

and according to premise(2) $\boldsymbol{y} \sim \boldsymbol{y}'$,

$$\beta - x'_n \in \overline{B}_{\boldsymbol{y}} \overset{\boldsymbol{y} \sim \boldsymbol{y}'}{=} \overline{B}_{\boldsymbol{y}'} \Rightarrow y'_n \le \beta - x'_n$$

$$y'_n \le \beta - x'_n \Rightarrow x'_n + y'_n \le \beta$$

summary from consider $y_N \le \beta - x'_n$, $\forall n \in \mathbb{N}$ to above:

$$\forall n \in \mathbb{N} \left(x'_n + y'_n \le \beta\right) \Rightarrow \beta \in \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'}$$

$$\forall \beta \in \overline{B}_{\boldsymbol{x} + \boldsymbol{y}} \left(\beta \in \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'}\right)$$

similarly,

$$\forall \beta' \in \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'} \left(\beta' \in \overline{B}_{\boldsymbol{x} + \boldsymbol{y}}\right)$$

summary:

$$\begin{cases} \forall \beta \in \overline{B}_{\boldsymbol{x} + \boldsymbol{y}} \left(\beta \in \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'}\right) \\ \forall \beta' \in \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'} \left(\beta' \in \overline{B}_{\boldsymbol{x} + \boldsymbol{y}}\right) \end{cases} \Leftrightarrow \begin{cases} \overline{B}_{\boldsymbol{x} + \boldsymbol{y}} \subseteq \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'} \\ \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'} \subseteq \overline{B}_{\boldsymbol{x} + \boldsymbol{y}} \end{cases} \Rightarrow \overline{B}_{\boldsymbol{x} + \boldsymbol{y}} = \overline{B}_{\boldsymbol{x}' + \boldsymbol{y}'} \Rightarrow \boldsymbol{x} + \boldsymbol{y} \sim \boldsymbol{x}' + \boldsymbol{y}'$$

□

推論 3.1.43.

$$\forall x, y \in \mathbb{R} \left(x + y = y + x\right)$$

推論 3.1.44.

$$\forall x, y, z \in \mathbb{R} \left(x + (y + z) = (x + y) + z\right)$$

推論 3.1.45.

$$\exists! 0 = [\langle 0, 0, \cdots, 0, \cdots \rangle]_\sim \in \mathbb{R}, \forall x \in \mathbb{R} \left(x + 0 = x = 0 + x\right)$$

推論 3.1.46.

$$\exists! \, (-x) \in \mathbb{R}, \forall x \in \mathbb{R} \left(x + (-x) = 0 = (-x) + x\right)$$

Proof.

$$x = [\boldsymbol{x}]_\sim = [\langle x_n \rangle]_\sim = \left[\langle x_n \rangle_{n \in \mathbb{N}}\right]_\sim = [\langle x_n \rangle_{n=1}^\infty]_\sim = [\langle x_1, x_2, \cdots, x_n, \cdots \rangle]_\sim$$

證明思路 proof idea

construct $\exists y = [\boldsymbol{y}]_\sim = [\langle y_n \rangle]_\sim = \left[\langle y_n \rangle_{n \in \mathbb{N}}\right]_\sim = [\langle y_n \rangle_{n=1}^\infty]_\sim = [\langle y_1, y_2, \cdots, y_n, \cdots \rangle]_\sim$

使得

$$\lim_{n \to \infty} (x_n + y_n) \to 0$$

$$x \in \mathbb{R} \Rightarrow \exists \beta \in \mathbb{Q}, \forall n \in \mathbb{N} \left(x_n \le \beta\right)$$

construction of 區間套 nested interval = NI 3.3.4,

$$m_0 = \frac{x_1 + \beta}{2} \in (x_1, \beta)$$

case 1-1: $m_0 \in \overline{B}_{\boldsymbol{x}}$, let

$$\begin{cases} a_1 = x_1 \\ b_1 = m_0 \\ m_1 = \dfrac{a_1+b_1}{2} \in (a_1, b_1) \\ I_1 = [a_1, b_1] \\ \|I_1\| = |b_1 - a_1| = b_1 - a_1 = m_0 - x_1 = \dfrac{x_1+\beta}{2} - x_1 = \dfrac{\beta - x_1}{2} > 0 \end{cases}$$

case 1-2: m_0 between $\{x_n\}_{n=1}^{\infty}$, let

$$\begin{cases} a_1 = m_0 \\ b_1 = \beta \\ m_1 = \dfrac{a_1+b_1}{2} \in (a_1, b_1) \\ I_1 = [a_1, b_1] \\ \|I_1\| = |b_1 - a_1| = b_1 - a_1 = \beta - m_0 = \beta - \dfrac{x_1+\beta}{2} = \dfrac{\beta - x_1}{2} > 0 \end{cases}$$

case 2-1: $m_1 \in \overline{B}_{\boldsymbol{x}}$, let

$$\begin{cases} a_2 = a_1 \\ b_2 = m_1 \\ m_2 = \dfrac{a_2+b_2}{2} \in (a_2, b_2) \\ I_2 = [a_2, b_2] = [a_1, m_1] \overset{m_1 \in (a_1,b_1)}{\subset} [a_1, b_1] = I_1 \Rightarrow I_2 \subseteq I_1 \\ \|I_2\| = |b_2 - a_2| = |m_1 - a_1| = m_1 - a_1 = \dfrac{a_1+b_1}{2} - a_1 = \dfrac{b_1 - a_1}{2} = \dfrac{\|I_1\|}{2} = \dfrac{\beta - x_1}{2^2} \end{cases}$$

case 2-2: m_1 between $\{x_n\}_{n=1}^{\infty}$, let

$$\begin{cases} a_2 = m_1 \\ b_2 = b_1 \\ m_2 = \dfrac{a_2+b_2}{2} \in (a_2, b_2) \\ I_2 = [a_2, b_2] = [m_1, b_1] \overset{m_1 \in (a_1,b_1)}{\subset} [a_1, b_1] = I_1 \Rightarrow I_2 \subseteq I_1 \\ \|I_2\| = |b_2 - a_2| = |b_1 - m_1| = b_1 - m_1 = b_1 - \dfrac{a_1+b_1}{2} = \dfrac{b_1 - a_1}{2} = \dfrac{\|I_1\|}{2} = \dfrac{\beta - x_1}{2^2} \end{cases}$$

$\vdots$

case n-1: $m_{n-1} \in \overline{B}_{\boldsymbol{x}}$, let

$$\begin{cases} a_n = a_{n-1} \\ b_n = m_{n-1} \\ m_n = \dfrac{a_n+b_n}{2} \in (a_n, b_n) \\ I_n = [a_n, b_n] = [a_{n-1}, m_{n-1}] \overset{m_{n-1} \in (a_{n-1},b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |b_{n-1} - m_{n-1}| = b_{n-1} - m_{n-1} = b_{n-1} - \dfrac{a_{n-1}+b_{n-1}}{2} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^n} = \dfrac{\beta - x_1}{2^n} \end{cases}$$

case n-2: m_{n-1} between $\{x_n\}_{n=1}^{\infty}$, let

$$\begin{cases} a_n = m_{n-1} \\ b_n = b_{n-1} \\ m_n = \dfrac{a_n+b_n}{2} \in (a_n, b_n) \\ I_n = [a_n, b_n] = [m_{n-1}, b_{n-1}] \overset{m_{n-1} \in (a_{n-1},b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |m_{n-1} - a_{n-1}| = m_{n-1} - a_{n-1} = \dfrac{a_{n-1}+b_{n-1}}{2} - a_{n-1} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^n} = \dfrac{\beta - x_1}{2^n} \end{cases}$$

$\vdots$

依此類推, 以致無窮 and so on to infinity,

$$\begin{cases}\forall n\in\mathbb{N}\left\{[a_n,b_n]=I_n,\begin{cases}b_n\in\overline{B}_{\boldsymbol{x}}\\ a_n \text{ between } \{x_n\}_{n=1}^{\infty}\end{cases}\right\}\\ \forall n\in\mathbb{N}\{I_{n+1}\subseteq I_n\}\\ \forall n\in\mathbb{N}\left\{\|I_n\|=\dfrac{\|I_1\|}{2^n}\right\}\end{cases}$$

by 區間套定理 / 區間套原理 nested interval theorem = NIT / nested interval principle = NIP 3.3.4,

$$\begin{cases}\forall n\in\mathbb{N}\left(\begin{cases}I_n=[a_n,b_n]\subset\mathbb{R}\\ I_{n+1}\subseteq I_n\Leftrightarrow I_n\supseteq I_{n+1}\end{cases}\right)\\ \lim\limits_{n\to\infty}(b_n-a_n)=\lim\limits_{n\to\infty}\|I_n\|=\lim\limits_{n\to\infty}\dfrac{\|I_1\|}{2^n}=\lim\limits_{n\to\infty}\dfrac{\beta-x_1}{2^n}=0\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\left(\bigcap_{n=1}^{\infty}I_n=\{\alpha\}\right)$$

$$\alpha=\lim_{n\to\infty}b_n\in\overline{B}_{\boldsymbol{x}}\cap\{x_n\}_{n=1}^{\infty}\ni\lim_{n\to\infty}a_n=\alpha$$

$$\forall n\in\mathbb{N},a_n \text{ between } \{x_n\}_{n=1}^{\infty},a_1\le a_2\le\cdots\le a_n\le\cdots\le\alpha\le\cdots\le b_n\le\cdots\le b_2\le b_1=\beta\in\overline{B}_{\boldsymbol{x}},\forall n\in\mathbb{N},b_n\in\overline{B}_{\boldsymbol{x}}$$

$$\begin{cases}\alpha\in\overline{B}_S\Rightarrow\forall a\in\{x_n\}_{n=1}^{\infty}(a\le\alpha)\\ \alpha \text{ between } \{x_n\}_{n=1}^{\infty}\Rightarrow\forall\beta\in\overline{B}_S(\alpha\le\beta)\end{cases}\Rightarrow\alpha=\sup\{x_n\}_{n=1}^{\infty}\in\mathbb{R}$$

$$\exists\langle a_n\rangle_{n\in\mathbb{N}},a_n\in\mathbb{Q}\subset\mathbb{R}\left(\begin{cases}\forall n\in\mathbb{N}(a_n\le a_{n+1}) & \text{遞增}\\ \exists\alpha\in\mathbb{R},\forall a_n\in\{a_n\}_{n\in\mathbb{N}}(a_n\le\alpha) & \text{上界}\end{cases}\right)$$
$$\Rightarrow a=[\boldsymbol{a}]_\sim=[\langle a_n\rangle]_\sim=\left[\langle a_n\rangle_{n\in\mathbb{N}}\right]_\sim=[\langle a_n\rangle_{n=1}^{\infty}]_\sim=[\langle a_1,a_2,\cdots,a_n,\cdots\rangle]_\sim\in\mathbb{R}$$

$$\begin{cases}\forall n\in\mathbb{N},a_n\in\mathbb{Q}(a_n \text{ between } \{x_n\}_{n=1}^{\infty})\\ \alpha \text{ between } \{x_n\}_{n=1}^{\infty}\\ \alpha=\sup\{x_n\}_{n=1}^{\infty}\end{cases}$$
$$\begin{aligned}&\Rightarrow\sup\{a_n\}_{n=1}^{\infty}=\sup\{x_n\}_{n=1}^{\infty}=\alpha\\ &\Rightarrow\overline{B}_{\langle a_n\rangle}=\overline{B}_{\langle x_m\rangle}\Rightarrow\langle a_n\rangle\sim\langle x_m\rangle\Rightarrow\boldsymbol{a}\sim\boldsymbol{x}\\ &\Rightarrow a=[\boldsymbol{a}]_\sim=[\boldsymbol{x}]_\sim=x\\ &\Rightarrow a=x\end{aligned}$$

$$\exists\langle b_n\rangle_{n\in\mathbb{N}},b_n\in\mathbb{Q}\left(\begin{cases}\forall n\in\mathbb{N}(b_n\ge b_{n+1}) & \text{遞減}\\ \exists\alpha\in\mathbb{R},\forall b_n\in\{b_n\}_{n\in\mathbb{N}}(b_n\ge\alpha) & \text{下界}\end{cases}\right)$$

$$\exists\langle -b_n\rangle_{n\in\mathbb{N}},-b_n\in\mathbb{Q}\subset\mathbb{R}\left(\begin{cases}\forall n\in\mathbb{N}(-b_n\le -b_{n+1}) & \text{遞增}\\ \exists-\alpha\in\mathbb{R},\forall b_n\in\{b_n\}_{n\in\mathbb{N}}(-b_n\le-\alpha) & \text{上界}\end{cases}\right)$$
$$\Rightarrow -b=[-\boldsymbol{b}]_\sim=[\langle -b_n\rangle]_\sim=\left[\langle -b_n\rangle_{n\in\mathbb{N}}\right]_\sim=[\langle -b_n\rangle_{n=1}^{\infty}]_\sim=[\langle -b_1,-b_2,\cdots,-b_n,\cdots\rangle]_\sim\in\mathbb{R}$$

let

$$\begin{aligned}y&=[\boldsymbol{y}]_\sim=[\langle y_n\rangle]_\sim=\left[\langle y_n\rangle_{n\in\mathbb{N}}\right]_\sim=[\langle y_n\rangle_{n=1}^{\infty}]_\sim=[\langle y_1,y_2,\cdots,y_n,\cdots\rangle]_\sim\\ &=-b=[-\boldsymbol{b}]_\sim=[\langle -b_n\rangle]_\sim=\left[\langle -b_n\rangle_{n\in\mathbb{N}}\right]_\sim=[\langle -b_n\rangle_{n=1}^{\infty}]_\sim=[\langle -b_1,-b_2,\cdots,-b_n,\cdots\rangle]_\sim\in\mathbb{R}\end{aligned}$$

$$\begin{aligned}x+y=a+(-b)&=[\langle a_n\rangle_{n=1}^{\infty}]+[\langle -b_n\rangle_{n=1}^{\infty}]_\sim\\ &=[\langle a_n-b_n\rangle_{n=1}^{\infty}]_\sim\\ &=\left[\left\langle-\frac{\beta-x_1}{2^n}\right\rangle_{n=1}^{\infty}\right]_\sim,\wedge\lim_{n\to\infty}\|I_n\|=\lim_{n\to\infty}\frac{\|I_1\|}{2^n}=\lim_{n\to\infty}\frac{\beta-x_1}{2^n}=0\\ &=\left[\left\langle\frac{\beta-x_1}{2},\frac{\beta-x_1}{2^2},\cdots,\frac{\beta-x_1}{2^n},\cdots,0,0,\cdots,0,\cdots\right\rangle\right]_\sim=[\langle 0,0,\cdots,0,\cdots\rangle]_\sim\\ &=[\langle 0\rangle_{n=1}^{\infty}]_\sim=0\end{aligned}$$

let

$$\begin{aligned}-x = [-\boldsymbol{x}]_\sim = [\langle -x_n\rangle]_\sim = \left[\langle -x_n\rangle_{n\in\mathbb{N}}\right]_\sim = [\langle -x_n\rangle_{n=1}^{\infty}]_\sim = [\langle -x_1, -x_2, \cdots, -x_n, \cdots\rangle]_\sim \\ = y = [\boldsymbol{y}]_\sim = [\langle y_n\rangle]_\sim = \left[\langle y_n\rangle_{n\in\mathbb{N}}\right]_\sim = [\langle y_n\rangle_{n=1}^{\infty}]_\sim = [\langle y_1, y_2, \cdots, y_n, \cdots\rangle]_\sim \\ = -b = [-\boldsymbol{b}]_\sim = [\langle -b_n\rangle]_\sim = \left[\langle -b_n\rangle_{n\in\mathbb{N}}\right]_\sim = [\langle -b_n\rangle_{n=1}^{\infty}]_\sim = [\langle -b_1, -b_2, \cdots, -b_n, \cdots\rangle]_\sim \in \mathbb{R}\end{aligned}$$

$$\exists\, (-x) \in \mathbb{R}, \forall x \in \mathbb{R}\, (x + (-x) = 0)$$

unique proof: We should talk about it later in abstract algebra togetherly including groups and fields. □

Real number system is a complete ordered Archimedean field.
Rational number system is an incomplete ordered Archimedean field.

3.1.4 戴德金分劃 / 戴德金切割 / 戴德金分割 Dedekind cut

定義 3.1.47. Dedekind cut

$$(A, B) \text{ is a Dedekind cut} \Leftrightarrow \begin{cases} A, B \subseteq \mathbb{Q} \\ A \neq \emptyset \\ A \neq \mathbb{Q} \Leftrightarrow B \neq \emptyset \\ \forall p, q \in \mathbb{Q} \left(\begin{cases} p < q \\ q \in A \end{cases} \Rightarrow p \in A\right) \\ \forall p \in A, \exists q \in A\, (p < q) \end{cases} \Leftrightarrow \begin{cases} A, B \subset \mathbb{Q} \\ \forall p, q \in \mathbb{Q} \left(\begin{cases} p < q \\ q \in A \end{cases} \Rightarrow p \in A\right) \\ \forall p \in A, \exists q \in A\, (p < q) \end{cases}$$

3.2 極限 limit

定義 3.2.1. 數列極限 limit of a sequence
存在正整數 N, 當正整數 n 超過 N 以後, 數列 a_n 都與 a 差距小於任意小正實數 ϵ

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\, (n > N \Rightarrow |a_n - a| < \epsilon)$$

定義 3.2.2. 函數極限 limit of a function
存在正實數 δ, 當定義域 D 中 x 與 a 差距小於 δ 且 $x \neq a$, 函數值 $f(x)$ 都與 l 差距小於任意小正實數 ϵ

$$\begin{aligned}\lim_{x\to a} f(x) = l &\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in D\, (0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon) \\ &\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in D - \{a\}\, (|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon)\end{aligned}$$

註記 3.2.3. $f(x) = x$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a \Leftrightarrow x \to a \Leftrightarrow \begin{cases} x \neq a \\ \forall \epsilon > 0, \exists \delta > 0, \forall x \in D\, (0 < |x - a| < \delta \Rightarrow |x - a| < \epsilon) \end{cases}$$

定義 3.2.4. 函數連續性 function continuity

$$f \text{ continuous at } x \Leftrightarrow \lim_{\xi\to x} f(\xi) = f\left(\lim_{\xi\to x} \xi\right) = f(x)$$

$$\begin{aligned}f \text{ continuous on } [a, b] &\Leftrightarrow \forall x \in [a, b] \left(\lim_{\xi\to x} f(\xi) = f\left(\lim_{\xi\to x} \xi\right) = f(x)\right) \\ &\Leftrightarrow \forall x \in [a, b]\, (f \text{ continuous at } x)\end{aligned}$$

3.3 實數的完備性 completeness of the real numbers

- 實數的完備性 的 等價性質 / 等價定理 equivalent properties or theorems for completeness of the real numbers [10]
  - 遞增有界實數列 單調收斂性 / 單調收斂定理 increasing bounded real sequence monotone convergence property / monotone convergence theorem = MCT = MCP / IBRS MCT

[10] 歐基里德作圖畫圓相交時已不知不覺假設此性質/公理成立了

- 區間套定理 / 區間套原理 nested interval theorem = NIT / nested interval principle = NIP
- 實子集 上確性 / 上確界公理 real subset least-upper-bound property = LUB property = LUBP / previously axiom of least upper bound / Dedekind completeness / RSubS LUBT
- 連續函數 介值定理 / 中間值定理 continuous function intermediate value theorem = CF IVT / intermediate value theorem = IVT [11]
- B-W 定理 / 聚點原理 Bolzano-Weierstrass theorem = B-WT / accumulative point principle 波爾查諾-魏爾史特拉斯定理
- 柯西收斂準則 Cauchy sequence / convergence criteria / Cauchy completeness

定理 3.3.1. 有理數 不具 上確性 / 完備性 *rational number system incomplete*

$$\neg\forall S\left[\begin{cases}\emptyset\neq S\subset\mathbb{Q} & \text{非空}\\ \exists\beta\in\mathbb{Q},\forall a\in S\,(a\leq\beta) & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{Q},\forall a\in S\left(\begin{cases}a\leq\alpha & \text{上界}\\ \forall\xi\in\mathbb{Q}\,(\forall a\in S\,(a\leq\xi)\Rightarrow\alpha\leq\xi) & \text{最小}\end{cases}\right)\right]$$

$$\neg\forall S\left[\begin{cases}\emptyset\neq S\subset\mathbb{Q} & \text{非空}\\ \exists\beta\in\mathbb{Q},\forall a\in S\,(a\leq\beta) & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{Q}\,(\alpha\text{ 是 }S\text{ 的最小上界})\right]$$

Proof. need a counter example to show

$$\exists S\left[\begin{cases}\emptyset\neq S\subset\mathbb{Q} & \text{非空}\\ \exists\beta\in\mathbb{Q},\forall a\in S\,(a\leq\beta) & \text{上界}\end{cases}\Rightarrow\neg\exists\alpha\in\mathbb{Q}\,(\alpha\text{ 是 }S\text{ 的最小上界})\right]$$

$$\exists S\left[\begin{cases}\emptyset\neq S\subset\mathbb{Q} & \text{非空}\\ \exists\beta\in\mathbb{Q},\forall a\in S\,(a\leq\beta) & \text{上界}\end{cases}\Rightarrow\forall\alpha\in\mathbb{Q}\,(\alpha\text{ 不是 }S\text{ 的最小上界})\right]$$

let

$$S=\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ a^2<2\end{cases}\right\}\overset{\because\forall a\in S(a\in\mathbb{Q})}{\subset}\mathbb{Q}\tag{3.3.1}$$

$$\begin{cases}1\in\mathbb{Q}\\ 1^2<2\end{cases}\Rightarrow 1\in S\Rightarrow S\neq\emptyset\tag{3.3.2}$$

$$\begin{aligned}\forall a\in S\,(a^2<2)\Rightarrow&\forall a\in S\,(a^2<2\leq 4)\Rightarrow\forall a\in S\,(a^2\leq 4)\Rightarrow\forall a\in S\,(a\leq 2)\\ \Rightarrow&2\in\mathbb{Q}\,(2\text{ 是 }S\text{ 的上界})\\ \Rightarrow&\exists\beta=2\in\mathbb{Q},\forall a\in S\,(a\leq\beta)\end{aligned}\tag{3.3.3}$$

let

$$\zeta\in\mathbb{Q}\,(\zeta\text{ 是 }S\text{ 的最小上界})$$

$$\zeta\in\mathbb{Q}\,(\zeta\text{ 是 }S\text{ 的最小上界})\Rightarrow\forall a\in S\left(\begin{cases}a\leq\zeta & \text{上界}\\ \forall\xi\in\mathbb{Q}\,(\forall a\in S\,(a\leq\xi)\Rightarrow\zeta\leq\xi) & \text{最小}\end{cases}\right)\tag{3.3.4}$$

$$\zeta\in\mathbb{Q}=S\cup(\mathbb{Q}-S)$$

$$S\cap(\mathbb{Q}-S)=\emptyset$$

$$\zeta\in S\veebar\zeta\in(\mathbb{Q}-S)$$

$$\zeta\in S=\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ a^2<2\end{cases}\right\}\veebar\zeta\in(\mathbb{Q}-S)=\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ \neg\,(a^2<2)\end{cases}\right\}$$

$$\zeta\in S=\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ a^2<2\end{cases}\right\}\veebar\zeta\in(\mathbb{Q}-S)=\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ a^2\geq 2\end{cases}\right\}\overset{3.1.7}{=}\left\{a\,\middle|\,\begin{cases}a\in\mathbb{Q}\\ a^2>2\end{cases}\right\}$$

$$\begin{cases}1\in S,\wedge 1>0\\ \zeta\text{ 是 }S\text{ 的最小上界}\Rightarrow\zeta\text{ 是 }S\text{ 的上界}\Leftrightarrow\forall a\in S\,(a\leq\zeta)\end{cases}\Rightarrow 0<1\leq\zeta\Rightarrow\zeta>0\tag{3.3.5}$$

case 1: $\zeta\in S\Rightarrow 0<\zeta^2<2$,

$$\begin{cases}\zeta^2<2 & \text{case 1 premise}\\ \zeta>0\Rightarrow\zeta^2>0 & 3.3.5\end{cases}\Rightarrow 0<\zeta^2<2$$

lemma for case 1: $\zeta\in S\Rightarrow 0<\zeta^2<2\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)^2<2\right]\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)\in S\right]$

[11] 其他完備性似乎無須連續函數假設, 我不確定與其他連續性質是否為等價性質

Proof.

$$\begin{aligned}\zeta^2+\frac{2\zeta}{n}+\frac{1}{n^2}=\left(\zeta+\frac{1}{n}\right)^2 &<2\\ 2-\zeta^2 &>\frac{2\zeta}{n}+\frac{1}{n^2}\\ n\left(2-\zeta^2\right) &>2\zeta+\frac{1}{n}\\ &\wedge\\ 2\zeta+1 &>2\zeta+\frac{1}{n}\end{aligned}$$

by rational number Archimedean property 3.1.6,

$$\forall a,b\in\mathbb{Q}^+,\exists n\in\mathbb{N}\,(na>b)$$
$$0<\left(2-\zeta^2\right),(2\zeta+1)\in\mathbb{Q}^+,\exists n\in\mathbb{N}\left[n\left(2-\zeta^2\right)>(2\zeta+1)\right]$$
$$\exists n\in\mathbb{N}\left[n\left(2-\zeta^2\right)>2\zeta+1\right]$$

and $2\zeta+1>2\zeta+\frac{1}{n}$

$$n\left(2-\zeta^2\right)>2\zeta+1>2\zeta+\frac{1}{n}$$

thus

$$\begin{aligned}&\exists n\in\mathbb{N}\left[n\left(2-\zeta^2\right)>2\zeta+1>2\zeta+\frac{1}{n}\right]\Rightarrow\exists n\in\mathbb{N}\left[n\left(2-\zeta^2\right)>2\zeta+\frac{1}{n}\right]\\ \Rightarrow&\exists n\in\mathbb{N}\left[2-\zeta^2>\frac{2\zeta}{n}+\frac{1}{n^2}\right]\Rightarrow\exists n\in\mathbb{N}\left[2>\zeta^2+\frac{2\zeta}{n}+\frac{1}{n^2}\right]\\ \Rightarrow&\exists n\in\mathbb{N}\left[2>\zeta^2+\frac{2\zeta}{n}+\frac{1}{n^2}=\left(\zeta+\frac{1}{n}\right)^2\right]\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)^2<2\right]\end{aligned}\tag{3.3.6}$$

$$\begin{cases}\begin{cases}\zeta\in\mathbb{Q}\\ \frac{1}{n}\in\mathbb{Q}\end{cases}\Rightarrow\zeta+\frac{1}{n}\in\mathbb{Q} & 3.1.1\\ \exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)^2<2\right] & 3.3.6\end{cases}\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)\in S\right]\tag{3.3.7}$$

□

$$\begin{cases}\zeta\in S & \text{case 1 premise}\\ \exists n\in\mathbb{N}\left[\left(\zeta+\frac{1}{n}\right)\in S\right] & 3.3.7\\ \zeta\text{ 是 }S\text{ 的上界}\Leftrightarrow\forall a\in S\,(a\le\zeta) & 3.3.4\end{cases}\Rightarrow\left(\zeta+\frac{1}{n}\right)\in S\left[\left(\zeta+\frac{1}{n}\right)\le\zeta\right]\Rightarrow\Leftarrow\left(\zeta+\frac{1}{n}\right)>\zeta$$

case 2: $\zeta\in(\mathbb{Q}-S)\Rightarrow\zeta^2>2$,

lemma for case 2: $\zeta\in S\Rightarrow 0<\zeta^2<2\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta-\frac{1}{n}\right)^2>2\right]\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta-\frac{1}{n}\right)\in(\mathbb{Q}-S)\right]$

Proof.

$$\begin{aligned}\zeta^2-\frac{2\zeta}{n}+\frac{1}{n^2}=\left(\zeta-\frac{1}{n}\right)^2 &>2\\ \zeta^2-2 &>\frac{2\zeta}{n}-\frac{1}{n^2}\\ n\left(\zeta^2-2\right) &>2\zeta-\frac{1}{n}\\ &\wedge\\ 2\zeta &>2\zeta-\frac{1}{n}\end{aligned}$$

by rational number Archimedean property 3.1.6,

$$\forall a,b\in\mathbb{Q}^{+},\exists n\in\mathbb{N}\left(na>b\right)$$

$$0<\left(\zeta^{2}-2\right),2\zeta\in\mathbb{Q}^{+},\exists n\in\mathbb{N}\left[n\left(\zeta^{2}-2\right)>2\zeta\right]$$

$$\exists n\in\mathbb{N}\left[n\left(\zeta^{2}-2\right)>2\zeta\right]$$

and $2\zeta>2\zeta-\dfrac{1}{n}$

$$n\left(\zeta^{2}-2\right)>2\zeta>2\zeta-\frac{1}{n}$$

thus

$$\begin{aligned}&\exists n\in\mathbb{N}\left[n\left(\zeta^{2}-2\right)>2\zeta>2\zeta-\frac{1}{n}\right]\Rightarrow\exists n\in\mathbb{N}\left[n\left(\zeta^{2}-2\right)>2\zeta-\frac{1}{n}\right]\\&\Rightarrow\exists n\in\mathbb{N}\left[\zeta^{2}-2>\frac{2\zeta}{n}-\frac{1}{n^{2}}\right]\Rightarrow\exists n\in\mathbb{N}\left[\zeta^{2}-\frac{2\zeta}{n}+\frac{1}{n^{2}}>2\right]\\&\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta-\frac{1}{n}\right)^{2}=\zeta^{2}-\frac{2\zeta}{n}+\frac{1}{n^{2}}>2\right]\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta-\frac{1}{n}\right)^{2}>2\right]\end{aligned}\tag{3.3.8}$$

$$\begin{cases}\begin{cases}\zeta\in\mathbb{Q} & 3.3.4\\ \dfrac{1}{n}\in\mathbb{Q}\end{cases}\Rightarrow\zeta-\dfrac{1}{n}\in\mathbb{Q} & 3.1.1\\ \exists n\in\mathbb{N}\left[\left(\zeta-\dfrac{1}{n}\right)^{2}>2\right] & 3.3.8\end{cases}\Rightarrow\exists n\in\mathbb{N}\left[\left(\zeta-\frac{1}{n}\right)\in\left(\mathbb{Q}-S\right)\right]\tag{3.3.9}$$

□

$$\begin{cases}\zeta\in\left(\mathbb{Q}-S\right)\subset\mathbb{Q} & \text{case 2 premise}\\ \exists n\in\mathbb{N}\left[\left(\zeta-\dfrac{1}{n}\right)\in\left(\mathbb{Q}-S\right)\subset\mathbb{Q}\right]\Rightarrow\begin{cases}\zeta-\dfrac{1}{n}\in\mathbb{Q}\\ \left(\zeta-\dfrac{1}{n}\right)^{2}>2\end{cases} & 3.3.9\\ \zeta\text{ 是 }S\text{ 的最小上界}\Leftrightarrow\forall\xi\in\mathbb{Q}\left(\forall a\in S\left(a\leq\xi\right)\Rightarrow\zeta\leq\xi\right) & 3.3.4\end{cases}$$

$$\Rightarrow\forall a\in S\left(a^{2}<2<\left(\zeta-\frac{1}{n}\right)^{2}\Rightarrow a^{2}<\left(\zeta-\frac{1}{n}\right)^{2}\overset{\zeta>\sqrt{2}>1>\frac{1}{n}>0\Rightarrow\zeta-\frac{1}{n}>0}{\Rightarrow}a<\zeta-\frac{1}{n}\right)$$

$$\Rightarrow\forall a\in S\left(a\leq\zeta-\frac{1}{n}\right)\Leftrightarrow\zeta-\frac{1}{n}\text{ 是 }S\text{ 的上界}$$

$$\Rightarrow\left(\zeta-\frac{1}{n}\right)\in\mathbb{Q}\left[\forall a\in S\left(a\leq\zeta-\frac{1}{n}\right)\Rightarrow\zeta\leq\left(\zeta-\frac{1}{n}\right)\right]\Rightarrow\Leftarrow\left(\zeta-\frac{1}{n}\right)<\zeta$$

no matter case 1 or case 2 both are contradict ($\Rightarrow\Leftarrow$)

$$\nexists\zeta\in\mathbb{Q}\left(\zeta\text{ 是 }S\text{ 的最小上界}\right)$$

$$\neg\exists\zeta\in\mathbb{Q}\left(\zeta\text{ 是 }S\text{ 的最小上界}\right)$$

$$\forall\zeta\in\mathbb{Q}\neg\left(\zeta\text{ 是 }S\text{ 的最小上界}\right)$$

$$\forall\zeta\in\mathbb{Q}\left(\zeta\text{ 不是 }S\text{ 的最小上界}\right)$$

$$\forall\alpha\in\mathbb{Q}\left(\alpha\text{ 不是 }S\text{ 的最小上界}\right)\Leftrightarrow\neg\exists\alpha\in\mathbb{Q},\forall a\in S\left(\begin{cases}a\leq\alpha & \text{上界}\\ \forall\xi\in\mathbb{Q}\left(\forall a\in S\left(a\leq\xi\right)\Rightarrow\alpha\leq\xi\right) & \text{最小}\end{cases}\right)$$

□

定理 3.3.2. 上確性 蘊含 實數 阿基米德性 / 阿基米德性質 *least-upper-bound property = LUB property = LUBP / Dedekind completeness implies real Archimedean property*

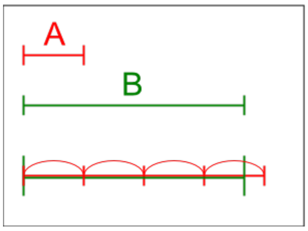


Figure 3.3.1: real Archimedean property

[12]

$$
\begin{aligned}
&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\ \textit{try 5}\ 3.3\Rightarrow\forall x\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(n>x)\ \ 3.3.18\\
\overset{a>0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(n>\frac{b}{a}\right)\ \textit{try 4}\ 3.3\Rightarrow\forall x\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(n>x)\ \ 3.3.18\\
\overset{n>0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(a>\frac{b}{n}\right)\ \textit{try 8}\ 3.3\Rightarrow\forall\epsilon\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(\frac{1}{n}<\epsilon\right)\ \ 3.3.19\\
\overset{n,b>0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(\frac{a}{b}>\frac{1}{n}\right)\ \textit{try 7}\ 3.3\Rightarrow\forall\epsilon\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(\frac{1}{n}<\epsilon\right)\ \ 3.3.19\\
\overset{b>0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(n\frac{a}{b}>1\right)\ \textit{try 6}\ 3.3\\
\overset{n,a>0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\left(1>\frac{1}{n}\frac{b}{a}\right)
\end{aligned}
$$

corollaries

$$
\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\ 3.3.2\Rightarrow\begin{cases}\forall x\geq 1,\exists m,n\in\mathbb{N}\,(m\leq x<n) & 3.3.22\\ \forall x\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(n>x) & 3.3.18\Rightarrow\forall x>0,\exists! n\in\mathbb{N}\,\{n\leq x<n+1\}\ 3.3.29\\ \textit{well-ordering principle} & 2.0.1\end{cases}
$$

Proof. proof by contradiction,

try 1:

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\,(na\leq b)\\
\overset{a>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\left(n\leq\frac{b}{a}\right)
\end{aligned}
$$

$$
\Rightarrow\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\left(\mathbb{N}\ni 2n=n+n\leq\frac{b}{a}+n\Rightarrow 2n\leq\frac{b}{a}\right)
$$

try 2:

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\,(na\leq b)\\
\overset{a>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\left(n\leq\frac{b}{a}\right)
\end{aligned}
$$

$$
\Rightarrow\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\left(0=n-n\leq\frac{b}{a}-n\right)
$$

try 3:

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\,(na\leq b)\\
\overset{n>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^+,\forall n\in\mathbb{N}\left(a\leq\frac{b}{n}\right)
\end{aligned}
$$

[12] https://commons.wikimedia.org/wiki/File:Archimedean_property.png

try 4: from try 1,

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(na\leq b\right)\\
\overset{a>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(n\leq\frac{b}{a}\right)\\
\Rightarrow&\begin{cases}\emptyset\neq S=\left\{n\right\}_{n\in\mathbb{N}}=\mathbb{N}\subseteq\mathbb{N}\subset\mathbb{R}\\ \exists\beta=\dfrac{b}{a}\in\mathbb{R}^{+}\subset\mathbb{R},\forall n\in S\left(n\leq\beta\right)\end{cases}\overset{3.1.39}{\Rightarrow}\exists\alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的最小上界}\right)\\
\Rightarrow&\begin{cases}\forall n\in\mathbb{N}\left(n+1\in\mathbb{N}=S\right)\\ \alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的最小上界}\right)\Rightarrow\alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的上界}\right)\end{cases}\Rightarrow\forall n\in\mathbb{N}\left(n+1\leq\alpha\right)\\
\Rightarrow&\forall n\in\mathbb{N}\left(n\leq\alpha-1\right)\\
\Rightarrow&\forall n\in\mathbb{N}=S\left(n\leq\alpha-1\in\mathbb{R}\right)\Rightarrow\forall n\in S\left(n\leq\alpha-1\in\mathbb{R}\right)\\
\Rightarrow&\exists\alpha-1\in\mathbb{R}\left(\alpha-1\text{ 是 }S\text{ 的上界}\right)\\
\overset{1>0}{\Rightarrow}&\exists\alpha-1<\alpha\left(\alpha-1\text{ 是 }S\text{ 的上界}\right)\Rightarrow\alpha\text{ 不是 }S\text{ 的最小上界}\Rightarrow\Leftarrow\alpha\text{ 是 }S\text{ 的最小上界}\\
&\Downarrow\\
&\neg\left[\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)\right]\Leftrightarrow\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)
\end{aligned}
\tag{3.3.10}
$$

try 5: standard proof on textbooks,

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(na\leq b\right)\\
\Rightarrow&\begin{cases}\emptyset\neq S=\left\{na\right\}_{n\in\mathbb{N}}\subset\mathbb{R}\\ \exists\beta=b\in\mathbb{R}^{+}\subset\mathbb{R},\forall s=na\in S\left(s\leq\beta\right)\end{cases}\overset{3.1.39}{\Rightarrow}\exists\alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的最小上界}\right)\\
\Rightarrow&\begin{cases}\forall n\in\mathbb{N}\left(n+1\in\mathbb{N}\Rightarrow\left(n+1\right)a\in S\right)\\ \alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的最小上界}\right)\Rightarrow\alpha\in\mathbb{R}\left(\alpha\text{ 是 }S\text{ 的上界}\right)\end{cases}\Rightarrow\forall n\in\mathbb{N}\left(\left(n+1\right)a\leq\alpha\right)\\
\Rightarrow&\forall n\in\mathbb{N}\left(na\leq\alpha-a\right)\\
\Rightarrow&\forall s=na\in S\left(s=na\leq\alpha-a\in\mathbb{R}\right)\Rightarrow\forall s\in S\left(s\leq\alpha-a\in\mathbb{R}\right)\\
\Rightarrow&\exists\alpha-a\in\mathbb{R}\left(\alpha-a\text{ 是 }S\text{ 的上界}\right)\\
\overset{a>0}{\Rightarrow}&\exists\alpha-a<\alpha\left(\alpha-a\text{ 是 }S\text{ 的上界}\right)\Rightarrow\alpha\text{ 不是 }S\text{ 的最小上界}\Rightarrow\Leftarrow\alpha\text{ 是 }S\text{ 的最小上界}\\
&\Downarrow\\
&\neg\left[\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)\right]\Leftrightarrow\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)
\end{aligned}
\tag{3.3.11}
$$

try 5.5:

$$
\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(na>b\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(na\leq b\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(na-b\leq 0\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(\begin{vmatrix}n & b\\ 1 & a\end{vmatrix}\leq 0\right)
\end{aligned}
\tag{3.3.12}
$$

try 6:

$$\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\,(na\le b)\\
\overset{b>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(n\frac{a}{b}\le 1\right)\\
\Rightarrow&\begin{cases}\emptyset\ne S=\left\{n\frac{a}{b}\right\}_{n\in\mathbb{N}}\subset\mathbb{R}\\ \exists\beta=1\in\mathbb{R}^{+}\subset\mathbb{R},\forall s=n\frac{a}{b}\in S\,(s\le\beta)\end{cases}\Rightarrow\exists\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\\
\Rightarrow&\begin{cases}\forall n\in\mathbb{N}\left(n+1\in\mathbb{N}\Rightarrow(n+1)\frac{a}{b}\in S\right)\\ \alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\Rightarrow\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的上界})\end{cases}\overset{3.1.39}{\Rightarrow}\forall n\in\mathbb{N}\left((n+1)\frac{a}{b}\le\alpha\right)\\
\Rightarrow&\forall n\in\mathbb{N}\left(n\frac{a}{b}\le\alpha-\frac{a}{b}\right)\\
\Rightarrow&\forall s=n\frac{a}{b}\in S\left(s=n\frac{a}{b}\le\alpha-\frac{a}{b}\in\mathbb{R}\right)\Rightarrow\forall s\in S\left(s\le\alpha-\frac{a}{b}\in\mathbb{R}\right)\\
\Rightarrow&\exists\alpha-\frac{a}{b}\in\mathbb{R}\left(\alpha-\frac{a}{b}\text{ 是 }S\text{ 的上界}\right)\\
\overset{\frac{a}{b}>0}{\Rightarrow}&\exists\alpha-\frac{a}{b}<\alpha\left(\alpha-\frac{a}{b}\text{ 是 }S\text{ 的上界}\right)\Rightarrow\alpha\text{ 不是 }S\text{ 的最小上界}\Rightarrow\Leftarrow\alpha\text{ 是 }S\text{ 的最小上界}\\
&\Downarrow\\
&\neg\left[\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\right]\Leftrightarrow\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)
\end{aligned}\tag{3.3.13}$$

try 7:

$$\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\,(na\le b)\\
\overset{b>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(n\frac{a}{b}\le 1\right)\\
\overset{n>0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(\frac{a}{b}\le\frac{1}{n}\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(-\frac{1}{n}\le-\frac{a}{b}\right)\\
\Rightarrow&\begin{cases}\emptyset\ne S=\left\{-\frac{1}{n}\right\}_{n\in\mathbb{N}}\subset\mathbb{R}\\ \exists\beta=-\frac{a}{b}\in\mathbb{R},\forall s=-\frac{1}{n}\in S\,(s\le\beta)\end{cases}\overset{3.1.39}{\Rightarrow}\exists\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\\
\Rightarrow&\begin{cases}\forall n\in\mathbb{N}\left(n+1\in\mathbb{N}\Rightarrow-\frac{1}{n+1}\in S\right)\\ \alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\Rightarrow\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的上界})\end{cases}\Rightarrow\forall n\in\mathbb{N}\left(-\frac{1}{n+1}\le\alpha\right)\\
\overset{3.3.15}{\Rightarrow}&\forall n\in\mathbb{N}\left(-\frac{1}{n}+\frac{1}{n(n+1)}\le\alpha\right)\Rightarrow\forall n\in\mathbb{N}\left(-\frac{1}{n}\le\alpha-\frac{1}{n(n+1)}\right)\\
\Rightarrow&\forall s=-\frac{1}{n}\in S\left(s=-\frac{1}{n}\le\alpha-\frac{1}{n(n+1)}\in\mathbb{R}\right)\Rightarrow\forall s\in S\left(s\le\alpha-\frac{1}{n(n+1)}\in\mathbb{R}\right)\\
\Rightarrow&\exists\alpha-\frac{1}{n(n+1)}\in\mathbb{R}\left(\alpha-\frac{1}{n(n+1)}\text{ 是 }S\text{ 的上界}\right)\\
\overset{\frac{1}{n(n+1)}>0}{\Rightarrow}&\exists\alpha-\frac{1}{n(n+1)}<\alpha\left(\alpha-\frac{1}{n(n+1)}\text{ 是 }S\text{ 的上界}\right)\Rightarrow\alpha\text{ 不是 }S\text{ 的最小上界}\Rightarrow\Leftarrow\alpha\text{ 是 }S\text{ 的最小上界}\\
&\Downarrow\\
&\neg\left[\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\right]\Leftrightarrow\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)
\end{aligned}\tag{3.3.14}$$

$$-\frac{1}{n+1}-\left(-\frac{1}{n}\right)=-\frac{n}{n(n+1)}-\left(-\frac{n+1}{n(n+1)}\right)=\frac{-n+(n+1)}{n(n+1)}=\frac{1}{n(n+1)}\tag{3.3.15}$$

$$-\frac{1}{n+1}-\left(-\frac{1}{n}\right)=\frac{1}{n(n+1)}$$

$$-\frac{1}{n+1}=-\frac{1}{n}+\frac{1}{n(n+1)}$$

$$-\frac{1}{n+1} \leq \alpha \Rightarrow -\frac{1}{n}+\frac{1}{n(n+1)} \leq \alpha \Rightarrow -\frac{1}{n} \leq \alpha-\frac{1}{n(n+1)}$$

try 8:

$$\begin{aligned}
&\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\,(na\leq b)\\
\overset{n\gtrless 0}{\Leftrightarrow}&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(a\leq\frac{b}{n}\right)\\
\Leftrightarrow&\exists a,b\in\mathbb{R}^{+},\forall n\in\mathbb{N}\left(-\frac{b}{n}\leq -a\right)\\
\Rightarrow&\begin{cases}\emptyset\neq S=\left\{-\frac{b}{n}\right\}_{n\in\mathbb{N}}\subset\mathbb{R}\\ \exists\beta=-a\in\mathbb{R},\forall s=-\frac{b}{n}\in S\,(s\leq\beta)\end{cases}\overset{3.1.39}{\Rightarrow}\exists\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\\
\Rightarrow&\begin{cases}\forall n\in\mathbb{N}\left(n+1\in\mathbb{N}\Rightarrow-\frac{b}{n+1}\in S\right)\\ \alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的最小上界})\Rightarrow\alpha\in\mathbb{R}\,(\alpha\text{ 是 }S\text{ 的上界})\end{cases}\Rightarrow\forall n\in\mathbb{N}\left(-\frac{b}{n+1}\leq\alpha\right)\\
\overset{3.3.17}{\Rightarrow}&\forall n\in\mathbb{N}\left(-\frac{b}{n}+\frac{b}{n(n+1)}\leq\alpha\right)\Rightarrow\forall n\in\mathbb{N}\left(-\frac{b}{n}\leq\alpha-\frac{b}{n(n+1)}\right)\\
\Rightarrow&\forall s=-\frac{b}{n}\in S\left(s=-\frac{b}{n}\leq\alpha-\frac{b}{n(n+1)}\in\mathbb{R}\right)\Rightarrow\forall s\in S\left(s\leq\alpha-\frac{b}{n(n+1)}\in\mathbb{R}\right)\\
\Rightarrow&\exists\alpha-\frac{b}{n(n+1)}\in\mathbb{R}\left(\alpha-\frac{b}{n(n+1)}\text{ 是 }S\text{ 的上界}\right)\\
\overset{\frac{b}{n(n+1)}>0}{\Rightarrow}&\exists\alpha-\frac{b}{n(n+1)}<\alpha\left(\alpha-\frac{b}{n(n+1)}\text{ 是 }S\text{ 的上界}\right)\Rightarrow\alpha\text{ 不是 }S\text{ 的最小上界}\Rightarrow\Leftarrow\alpha\text{ 是 }S\text{ 的最小上界}\\
&\Downarrow\\
&\neg\left[\neg\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\right]\Leftrightarrow\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)
\end{aligned}\tag{3.3.16}$$

$$-\frac{b}{n+1}-\left(-\frac{b}{n}\right)=-\frac{nb}{n(n+1)}-\left(-\frac{(n+1)b}{n(n+1)}\right)=\frac{[-n+(n+1)]b}{n(n+1)}=\frac{b}{n(n+1)}\tag{3.3.17}$$

$$-\frac{b}{n+1}-\left(-\frac{b}{n}\right)=\frac{b}{n(n+1)}$$

$$-\frac{b}{n+1}=-\frac{b}{n}+\frac{b}{n(n+1)}$$

$$-\frac{b}{n+1}\leq\alpha\Rightarrow-\frac{b}{n}+\frac{b}{n(n+1)}\leq\alpha\Rightarrow-\frac{b}{n}\leq\alpha-\frac{b}{n(n+1)}$$

summary from try 4 to try 8:

$$\begin{aligned}
&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\,(na>b)\ \text{try 5 3.3}\\
\overset{a\gtrless 0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(n>\frac{b}{a}\right)\ \text{try 4 3.3}\\
\overset{n\gtrless 0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(a>\frac{b}{n}\right)\ \text{try 8 3.3}\\
\overset{n,b\gtrless 0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(\frac{a}{b}>\frac{1}{n}\right)\ \text{try 7 3.3}\\
\overset{b\gtrless 0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(n\frac{a}{b}>1\right)\ \text{try 6 3.3}\\
\overset{n,a\gtrless 0}{\Leftrightarrow}&\forall a,b\in\mathbb{R}^{+},\exists n\in\mathbb{N}\left(1>\frac{1}{n}\frac{b}{a}\right)
\end{aligned}$$

$$\forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\,(na > b) \overset{b=x>0}{\underset{a=1>0}{\Rightarrow}} \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}\,(n \cdot 1 > x) \Leftrightarrow \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}\,(n > x) \tag{3.3.18}$$

$$\overset{a>0}{\Leftrightarrow} \forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(n > \frac{b}{a}\right) \overset{b=x>0}{\underset{a=1>0}{\Rightarrow}} \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(n > \frac{x}{1}\right) \Leftrightarrow \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}\,(n > x)$$

$$\overset{n>0}{\Leftrightarrow} \forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(a > \frac{b}{n}\right) \overset{b=1>0}{\underset{a=\epsilon>0}{\Rightarrow}} \forall \epsilon \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(\epsilon > \frac{1}{n}\right) \Leftrightarrow \forall \epsilon \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(\frac{1}{n} < \epsilon\right) \tag{3.3.19}$$

$$\begin{aligned}
&\overset{n,b>0}{\Leftrightarrow} \forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(\frac{a}{b} > \frac{1}{n}\right)\\
&\overset{b>0}{\Leftrightarrow} \forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(n\frac{a}{b} > 1\right)\\
&\overset{n,a>0}{\Leftrightarrow} \forall a,b \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(1 > \frac{1}{n}\frac{b}{a}\right)
\end{aligned}$$

try 9:

$$\begin{aligned}
&\forall x \in \mathbb{R}^+, \exists n \in \mathbb{N}\,(n > x) \Leftrightarrow \forall x > 0, \exists n \in \mathbb{N}\,(n > x)\\
\Rightarrow &\forall x + 1 > 1, \exists n \in \mathbb{N}\,(n + 1 > x + 1)
\end{aligned}$$

try 10:

$$\begin{aligned}
&\forall \epsilon \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(\frac{1}{n} < \epsilon\right) \Leftrightarrow \forall \epsilon > 0, \exists n \in \mathbb{N}\left(\frac{1}{n} < \epsilon\right)\\
\Rightarrow &\forall \epsilon + 1 > 1, \exists n \in \mathbb{N}\left(\frac{1}{n} + 1 < \epsilon + 1\right)\\
\Rightarrow &\forall \epsilon + 1 > 1, \exists n \in \mathbb{N}\left(1 < \frac{1}{n} + 1 < \epsilon + 1\right)\\
\Rightarrow &\forall \epsilon + 1 > 1, \exists n \in \mathbb{N}\left(1 \leq \frac{1}{n} + 1 < \epsilon + 1\right)\\
\Rightarrow &\forall \epsilon + 1 > 1, \exists n \in \mathbb{N}\,(1 \leq \epsilon + 1)
\end{aligned}$$

try 11:

$$\begin{aligned}
&\forall \epsilon \in \mathbb{R}^+, \exists n \in \mathbb{N}\left(\frac{1}{n} < \epsilon\right) \Leftrightarrow \forall \epsilon > 0, \exists n \in \mathbb{N}\left(\frac{1}{n} < \epsilon\right)\\
\Rightarrow &\forall \epsilon + n > n, \exists n \in \mathbb{N}\left(\frac{1}{n} + n < \epsilon + n\right)\\
\Rightarrow &\forall \epsilon + n > n \geq 1, \exists n \in \mathbb{N}\left(n < \frac{1}{n} + n < \epsilon + n\right)\\
\Rightarrow &\forall \epsilon + n > n \geq 1, \exists n \in \mathbb{N}\left(n \leq \frac{1}{n} + n < \epsilon + n\right)\\
\Rightarrow &\forall \epsilon + n > 1, \exists n \in \mathbb{N}\,(n \leq \epsilon + n)\\
\Rightarrow &\forall x = \epsilon + n > 1, \exists n \in \mathbb{N}\,(n \leq \epsilon + n = x)\\
\Rightarrow &\forall x > 1, \exists n \in \mathbb{N}\,(n \leq x)
\end{aligned} \tag{3.3.20}$$

summary from try 11 and summary from try 4 to try 8

$$\begin{cases}
\forall x > 1, \exists n \in \mathbb{N}\,(n \leq x) & \Leftrightarrow 3.3.20\\
\forall x > 0, \exists n \in \mathbb{N}\,(n > x) \Rightarrow \forall x > 1 > 0, \exists n \in \mathbb{N}\,(n > x) & \Leftarrow 3.3.18
\end{cases}$$

$$\Rightarrow \begin{cases}
\forall x > 1, \exists m \in \mathbb{N}\,(m \leq x)\\
\forall x > 1 > 0, \exists n \in \mathbb{N}\,(n > x)
\end{cases}$$

$$\Rightarrow \forall x > 1, \exists m, n \in \mathbb{N}\,(m \leq x < n) \tag{3.3.21}$$

$$\begin{cases}
\forall x > 1, \exists m, n \in \mathbb{N}\,(m \leq x < n) & 3.3.21\\
x = 1, \exists m = 1, n = 1 + k \in \mathbb{N}\,(m = 1 \leq x = 1 < 1 + k = n) \Leftrightarrow x = 1, \exists 1, 1 + k \in \mathbb{N}\,(1 \leq x < 1 + k) &
\end{cases}$$

$$\Rightarrow \forall x \geq 1, \exists m, n \in \mathbb{N}\,(m \leq x < n) \tag{3.3.22}$$

$$
\begin{aligned}
&3.3.22\ \forall x \geq 1, \exists m, n \in \mathbb{N}\,(m \leq x < n)\\
&\Rightarrow \forall x \in \mathbb{N} \Rightarrow x \geq 1, \exists m, n \in \mathbb{N}\,(m \leq x < n)\\
&\Rightarrow \forall x \in \mathbb{N}, \exists m, n \in \mathbb{N}\,(m \leq x < n)\\
&\Rightarrow \forall x = m \in \mathbb{N}, \forall k \in \mathbb{N}, \exists m, n = m + k \in \mathbb{N}\,(m \leq x = m < m + k = n)\\
&\Rightarrow \forall m \in \mathbb{N}, 1 \in \mathbb{N}, \exists m, n = m + 1 \in \mathbb{N}\,(m \leq m < m + 1 = n)\\
&\Rightarrow \forall m \in \mathbb{N}\,(m \leq m < m + 1)\\
&\Leftrightarrow \forall n \in \mathbb{N}\,(n \leq n < n + 1)
\end{aligned}
$$

$$
\begin{aligned}
&\forall x \in \left\{ x \middle| \begin{cases} x \in \mathbb{R} \\ x \geq 1 \end{cases} \right\} - \mathbb{N} \Rightarrow x \geq 1, \exists m, n \in \mathbb{N}\,(m \leq x < n)\\
&\Rightarrow \forall x \in \left\{ x \middle| \begin{cases} x \in \mathbb{R} \\ x \geq 1 \end{cases} \right\} - \mathbb{N}, \exists m, n \in \mathbb{N}\,(m \leq x < n)
\end{aligned}
\tag{3.3.23}
$$

aother new proposition

$$
\forall x \in \left\{ x \middle| \begin{cases} x \in \mathbb{R} \\ x \geq 1 \end{cases} \right\} - \mathbb{N}, \exists n \in \mathbb{N}\,(n \leq x < n + 1)?
$$

$$
3.3.23 \Leftrightarrow \forall x \in \left\{ x \middle| \begin{cases} x \in \mathbb{R} \\ x \geq 1 \end{cases} \right\} - \mathbb{N}, \exists m, n \in \mathbb{N}\,(m \leq x < n)
$$

$$
\begin{aligned}
&3.3.18\ \forall x > 0, \exists n \in \mathbb{N}\,(n > x) \Leftrightarrow \forall x > 0, \exists k \in \mathbb{N}\,(k > x)\\
&\Rightarrow \forall x > 0, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}\,(k + m > k > x)\\
&\Rightarrow \forall x > 0, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}\,(k + m > x)\\
&\Rightarrow \forall x > 0, \exists k \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + k > x)\\
&\text{let} K = \left\{ k \middle| \begin{cases} k, m \in \mathbb{N} \\ x \in \mathbb{R}^{+} \\ m + k > x \end{cases} \right\} \subseteq \mathbb{N} \overset{\text{well-ordering principle}}{\Rightarrow} \exists \min K \in K, \forall k \in K\,(\min K \leq k)\\
&\text{let} \kappa = \min K \Rightarrow \begin{cases} \forall x > 0, k = \kappa = \min K \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + \kappa > x) \\ \forall x > 0, \forall \begin{cases} k \in \mathbb{N} \\ k < \kappa \end{cases}, \forall m \in \mathbb{N} \neg\,(m + \kappa > x) \Leftrightarrow \exists \kappa \in \mathbb{N}, \forall x > 0, \forall k \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + k > x \Rightarrow k \geq \kappa) \end{cases}\\
&\Rightarrow \begin{cases} \forall x > 0, k = \kappa = \min K \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + \kappa > x) \\ \forall x > 0, \forall \begin{cases} k \in \mathbb{N} \\ k < \kappa \end{cases}, \forall m \in \mathbb{N}\,(m + \kappa \leq x) \Leftrightarrow \exists \kappa \in \mathbb{N}, \forall x > 0, \forall k \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + k > x \Rightarrow k \geq \kappa) \end{cases}\\
&\Rightarrow \begin{cases} \forall x > 0, \exists \kappa \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + \kappa > x) \\ \forall x > 0, \exists \kappa - 1 \in \mathbb{N} \cup \{0\}, \forall m \in \mathbb{N}\,(m + (\kappa - 1) \leq x) \because \kappa - 1 < \kappa \end{cases}\\
&\Rightarrow \forall x > 0, \exists \kappa \in \mathbb{N}, \forall m \in \mathbb{N}\,(m + (\kappa - 1) \leq x < m + \kappa)
\end{aligned}
\tag{3.3.24}
$$

($\exists$):
case 1: $\kappa = 1$,

$$
\begin{aligned}
3.3.24\ \forall x > 0, \exists \kappa \in \mathbb{N}, \forall m \in \mathbb{N}\{m + (\kappa - 1) \leq& x < m + \kappa\}\\
\forall x > 0, \exists 1 \in \mathbb{N}, \forall m \in \mathbb{N}\{m + (1 - 1) \leq& x < m + 1\}\\
\forall x > 0, \exists 1 + m \in \mathbb{N}\{m + (1 - 1) \leq& x < m + 1\}\\
\forall x > 0, \exists m \in \mathbb{N}\{m + 0 \leq& x < m + 1\}\\
\forall x > 0, \exists m \in \mathbb{N}\{m \leq& x < m + 1\}
\end{aligned}
\tag{3.3.25}
$$

case 2: $\kappa \geq 2$, let $n = m + \kappa - 1 \Rightarrow n + 1 = m + \kappa$,

$$
\begin{aligned}
3.3.24\ \forall x > 0, \exists \kappa \in \mathbb{N}, \forall m \in \mathbb{N}\{m + (\kappa - 1) \leq& x < m + \kappa\}\\
\forall x > 0, \exists \kappa + m - 1 \in \mathbb{N}\{m + \kappa - 1 \leq& x < m + \kappa\}\\
\forall x > 0, \exists n \in \mathbb{N}\{n \leq& x < n + 1\}
\end{aligned}
\tag{3.3.26}
$$

(∃!):

$$\begin{aligned}&\begin{cases}\forall x>0,\exists m\in\mathbb{N}\{m\le x<m+1\} & \Leftrightarrow 3.3.25\\ \forall x>0,\exists n\in\mathbb{N}\{n\le x<n+1\} & \Leftrightarrow 3.3.26\end{cases}\\ \Leftrightarrow&\begin{cases}\forall x>0,\exists n_1\in\mathbb{N}\{n_1\le x<n_1+1\} & (1)\\ \forall x>0,\exists n_2\in\mathbb{N}\{n_2\le x<n_2+1\} & (2)\end{cases}\end{aligned}\tag{3.3.27}$$

if $n_1\neq n_2$, without loss of generality (w/oLoG), let $n_1<n_2$,

$$3.3.27\,(1)\Rightarrow n_1\le x<n_1+1\overset{n_1<n_2}{<}n_2+1\tag{3.3.28}$$

$$\begin{aligned}&\begin{cases}n_1\le x<n_1+1<n_2+1\Rightarrow x<n_1+1<n_2+1 & \Leftarrow 3.3.28\\ n_2\le x<n_2+1 & \Leftarrow 3.3.27\,(2)\end{cases}\\ \Rightarrow& n_2\le x<n_1+1<n_2+1\\ \Rightarrow& n_2\le n_1+1<n_2+1\Rightarrow\Leftarrow\nexists\begin{cases}m\in\mathbb{N}\\ m\neq n\end{cases}\quad\forall n\in\mathbb{N}\,(n\le m<n+1)\\ &\Downarrow\\ &\neg\,(n_1\neq n_2)\Leftrightarrow n_1=n_2\end{aligned}$$

$$\forall x>0,\exists! n\in\mathbb{N}\{n\le x<n+1\}\tag{3.3.29}$$

try 12:

$$\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\overset{?}{\Leftrightarrow}\forall a\in\mathbb{R}^+,\exists n\in\mathbb{N},\forall b\in\mathbb{R}^+\,(na>b)$$

$$\forall a,b\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(na>b)\overset{?}{\Leftrightarrow}\forall a\in\mathbb{R}^+,\exists n\in\mathbb{N}\,(b>0\Rightarrow na>b)$$

□

定理 3.3.3. 遞增有界實數列 單調收斂性 / 單調收斂定理 *increasing bounded real sequence monotone convergence property / monotone convergence theorem = MCT = MCP / IBRS MCT*

$$\forall\langle a_n\rangle_{n\in\mathbb{N}},a_n\in\mathbb{R}\left[\begin{cases}\forall n\in\mathbb{N}\,(a_n\le a_{n+1}) & \text{遞增}\\ \exists\beta\in\mathbb{R},\forall n\in\mathbb{N}\,(a_n\le\beta)\Leftrightarrow\overline{B}_S\neq\emptyset & \text{上界}\end{cases}\Rightarrow\exists\alpha=\sup S\in\mathbb{R}\left(\lim_{n\to\infty}a_n=\alpha\right)\right]$$

Proof. (⇒) :

let

$$\emptyset\neq S=\left\{a\,\middle|\,\begin{cases}n\in\mathbb{N}\\ \exists\,\langle a_n\rangle_{n\in\mathbb{N}}\in\mathbb{R}^{|\mathbb{N}|}\left(\begin{cases}\forall n\in\mathbb{N}\,(a_n\le a_{n+1}) & (1)\text{ 遞增}\\ \exists\beta\in\mathbb{R},\forall n\in\mathbb{N}\,(a_n\le\beta) & (2)\text{ 上界}\end{cases}\right)\\ a=a_n\end{cases}\right\}=\{a_n\}_{n\in\mathbb{N}}=\{a_n\}_{n=1}^{\infty}\subset\mathbb{R}\tag{3.3.30}$$

by 3.1.39,

$$\forall S\left[\begin{cases}\emptyset\neq S\subset\mathbb{R} & \text{非空}\\ \exists\beta\in\mathbb{R},\forall a\in S\,(a\le\beta) & \text{上界}\end{cases}\Rightarrow\exists\alpha\in\mathbb{R},\forall a\in S\left(\begin{cases}a\le\alpha & \text{上界}\\ \forall\xi\in\mathbb{R}\,(\forall a\in S\,(a\le\xi)\Rightarrow\alpha\le\xi) & \text{最小}\end{cases}\right)\right]$$

$$\exists\alpha=\sup S=\sup\{a_n\}_{n\in\mathbb{N}}\in\mathbb{R},\forall a=a_n\in S\left[\begin{cases}a\le\alpha & \text{上界}\\ \forall\xi\in\mathbb{R}\,(\forall a\in S\,(a\le\xi)\Rightarrow\alpha\le\xi) & \text{最小}\end{cases}\right]$$

$$\exists\alpha\in\mathbb{R},\forall a_n\in S\left[\begin{cases}a_n\le\alpha & (3)\text{ 上界}\\ \forall\xi\in\mathbb{R}\,(\forall a_n\in S\,(a_n\le\xi)\Rightarrow\alpha\le\xi) & (4)\text{ 最小}\end{cases}\right]\tag{3.3.31}$$

let

$$\epsilon>0\tag{3.3.32}$$

then

$$\begin{cases}\alpha\text{ is a LUB for }S & 3.3.31\\ \alpha-\epsilon<\alpha & \Leftarrow 3.3.32\end{cases}\overset{3.3.31(4)}{\Rightarrow}\alpha-\epsilon\text{ is not a LUB for }S\Rightarrow\exists a_N\in S,N\in\mathbb{N}\,(a_N>\alpha-\epsilon)\tag{3.3.33}$$

now we already found N and corresponding a_N

$$
\begin{aligned}
3.3.30\,(1) \Rightarrow \forall n \in \mathbb{N}\,(a_n \le a_{n+1}) &\Rightarrow \forall n > N\,(a_N \le a_n) \overset{3.3.33}{\Rightarrow} \forall n > N\,(\alpha - \epsilon < a_N \le a_n) \\
&\overset{3.3.31(3)}{\Rightarrow} \forall n > N\,(\alpha - \epsilon < a_N \le a_n \le \alpha) \\
&\overset{3.3.32}{\Rightarrow} \forall n > N\,(\alpha - \epsilon < a_N \le a_n \le \alpha < \alpha + \epsilon) \\
&\Rightarrow \forall n > N\,(\alpha - \epsilon < a_n < \alpha + \epsilon) \\
&\Rightarrow \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\,(n > N \Rightarrow \alpha - \epsilon < a_n < \alpha + \epsilon) \\
&\Leftrightarrow \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\,(n > N \Rightarrow -\epsilon < a_n - \alpha < \epsilon) \\
&\Leftrightarrow \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\,(n > N \Rightarrow |a_n - \alpha| < \epsilon)
\end{aligned}
$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\,(n > N \Rightarrow |a_n - \alpha| < \epsilon)$$

by definition: 數列極限 limit of a sequence,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\,(n > N \Rightarrow |a_n - \alpha| < \epsilon) \overset{3.2.1}{\Leftrightarrow} \lim_{n\to\infty} a_n = \alpha$$

$$\lim_{n\to\infty} a_n = \alpha$$

$(\Leftarrow)$: apparently,

$$\forall \langle a_n \rangle_{n\in\mathbb{N}} \left[\begin{cases} \forall n \in \mathbb{N}\,(a_n \le a_{n+1}) & \text{遞增} \\ \exists \alpha \in \mathbb{R}\left(\lim_{n\to\infty} a_n = \alpha\right) & \text{收斂} \end{cases} \Rightarrow \exists \beta = \alpha \in \mathbb{R}, \forall n \in \mathbb{N}\,(a_n \le \beta) \right]$$

$\square$

定理 3.3.4. 區間套定理 / 區間套原理 *nested interval theorem = NIT / nested interval principle = NIP*

$$\begin{cases} \forall n \in \mathbb{N} \left(\begin{cases} I_n = [a_n, b_n] \subset \mathbb{R} \\ I_{n+1} \subseteq I_n \Leftrightarrow I_n \supseteq I_{n+1} \end{cases} \right) \\ \lim_{n\to\infty} \|I_n\| = \lim_{n\to\infty} (b_n - a_n) = 0 \end{cases} \Rightarrow \exists x_0 \in \mathbb{R} \left(\bigcap_{n=1}^{\infty} I_n = \{x_0\} \right)$$

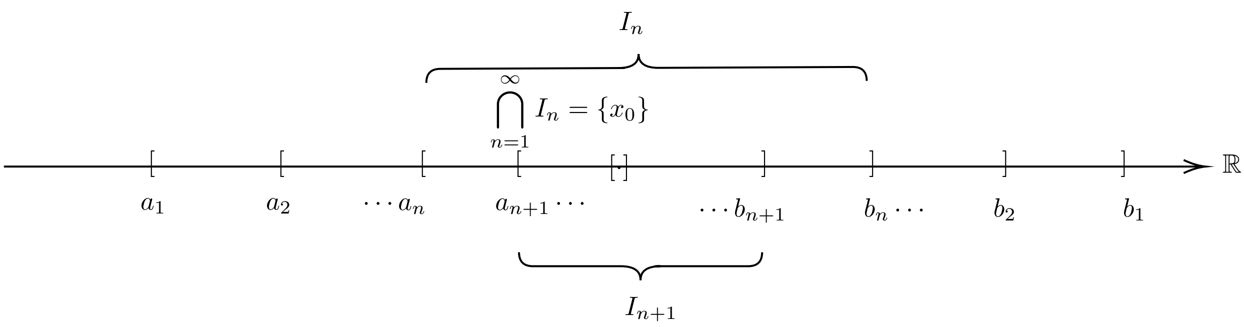


Figure 3.3.2: nested interval theorem = NIT / nested interval principle = NIP

Proof.

$$
\begin{aligned}
& I_n \supseteq I_{n+1} \\
\Leftrightarrow & \{x | a_n \le x \le b_n\} \supseteq \{x | a_{n+1} \le x \le b_{n+1}\} = \{x | a_n \le a_{n+1} \le x \le b_{n+1} \le b_n\} \\
\Rightarrow & \begin{cases} a_n \le a_{n+1} & \Rightarrow \forall n \in \mathbb{N}\,(a_n \le a_{n+1}) \\ b_{n+1} \le b_n \Leftrightarrow b_n \ge b_{n+1} & \Rightarrow -b_n \le -b_{n+1} \Rightarrow \forall n \in \mathbb{N}\,(-b_n \le -b_{n+1}) \end{cases}
\end{aligned} \tag{3.3.34}
$$

$$
\begin{aligned}
& I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots \\
\Leftrightarrow & \{x | a_1 \le x \le b_1\} \supseteq \{x | a_2 \le x \le b_2\} \supseteq \cdots \supseteq \{x | a_n \le x \le b_n\} \\
& \supseteq \{x | a_{n+1} \le x \le b_{n+1}\} = \{x | a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \le x \le \cdots \le b_{n+1} \le b_n \le \cdots \le b_2 \le b_1\} \\
\Rightarrow & a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \le x \le \cdots \le b_{n+1} \le b_n \le \cdots \le b_2 \le b_1 && (3.3.35) \\
\Rightarrow & \forall n \in \mathbb{N}\,(a_n \le b_1) && (3.3.36)
\end{aligned}
$$

$$\begin{aligned}
3.3.35 \Rightarrow & -a_1 \geq -a_2 \geq \cdots \geq -a_n \geq -a_{n+1} \geq \cdots \geq -x \geq \cdots \geq -b_{n+1} \geq -b_n \geq \cdots \geq -b_2 \geq -b_1 \\
\Rightarrow & -b_1 \leq -b_2 \leq \cdots \leq -b_n \leq -b_{n+1} \leq \cdots \leq -x \leq \cdots \leq -a_{n+1} \leq -a_n \leq \cdots \leq -a_2 \leq -a_1 \\
\Rightarrow & \forall n \in \mathbb{N} \left(-b_n \leq -a_1\right) \qquad (3.3.37)
\end{aligned}$$

by 3.1.38,

$$\begin{cases} \forall n \in \mathbb{N} \left(a_n \leq a_{n+1}\right) & 3.3.34 \\ \exists b_1 \in \mathbb{R}, \forall n \in \mathbb{N} \left(a_n \leq b_1\right) & 3.3.36 \end{cases} \overset{\text{monotone convergence theorem}}{\Rightarrow} \exists \alpha \in \mathbb{R} \left(\lim_{n\to\infty} a_n = \alpha \right) \qquad (3.3.38)$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(b_n - a_n + a_n\right) = \lim_{n\to\infty} \left(\left(b_n - a_n\right) + a_n\right) = \lim_{n\to\infty} \left(b_n - a_n\right) + \lim_{n\to\infty} a_n = 0 + \alpha = \alpha$$

$$\begin{aligned}
& \forall n \in \mathbb{N} \left(a_n \leq a_{n+1} \leq x \leq b_{n+1} \leq b_n\right) \\
\Rightarrow & \forall n \in \mathbb{N} \left(a_n \leq a_{n+1} \leq b_{n+1} \leq b_n\right) \\
\Rightarrow & \forall n \in \mathbb{N}, \forall m \in \mathbb{N} \left(n \geq m \Rightarrow a_m \leq a_n \leq b_n \leq b_m\right) \\
\Rightarrow & \forall n \in \mathbb{N}, \forall m \in \mathbb{N} \left(n \geq m \Rightarrow \begin{cases} a_m \leq a_n \leq \lim_{n\to\infty} a_n = \alpha \\ b_m \geq b_n \geq \lim_{n\to\infty} b_n = \alpha \end{cases} \right) \\
\Rightarrow & \forall m \in \mathbb{N} \left(\begin{cases} a_m \leq \alpha \\ b_m \geq \alpha \end{cases} \right) \Leftrightarrow \forall n \in \mathbb{N} \left(\begin{cases} a_n \leq \alpha \\ b_n \geq \alpha \end{cases} \right) \\
\Rightarrow & \forall n \in \mathbb{N} \left(a_n \leq \alpha \leq b_n\right) \Leftrightarrow \forall n \in \mathbb{N} \left(\alpha \in [a_n, b_n]\right) \Leftrightarrow \forall n \in \mathbb{N} \left(\alpha \in I_n\right)
\end{aligned}$$

let $x_0 = \alpha = \beta$

$$\lim_{n\to\infty} I_n = \{\alpha\} = \{\beta\} = \{x_0\}$$

$$\exists x_0 \in \mathbb{R} \left(\lim_{n\to\infty} I_n = \{x_0\} \right)$$

$$I_\infty = \lim_{n\to\infty} I_n = \{x_0\} \qquad (3.3.39)$$

$$\begin{cases} \forall n \in \mathbb{N} \left(a_n \leq a_{n+1}\right) & 3.3.34 \\ \forall n \in \mathbb{N} \left(-b_n \leq -b_{n+1}\right) \Leftrightarrow \forall n \in \mathbb{N} \left(b_{n+1} \leq b_n\right) & 3.3.34 \\ a_\infty = \lim_{n\to\infty} a_n = \alpha \overset{\infty > n}{\geq} a_n \, \forall n \in \mathbb{N} & 3.3.38 \\ b_\infty = \lim_{n\to\infty} b_n = \beta \overset{\infty > n}{\leq} b_n \, \forall n \in \mathbb{N} & \\ x_0 = \alpha = \beta & \end{cases}$$

$$\begin{aligned}
& \Rightarrow a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq \alpha = x_0 = \beta \leq \cdots \leq b_{n+1} \leq b_n \leq \cdots \leq b_2 \leq b_1 \\
& \Rightarrow I_\infty = \{x_0\} = \{x | x = x_0 = \alpha = \beta\} \\
& \qquad = \{x | a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq x = \alpha = x_0 = \beta \leq \cdots \leq b_{n+1} \leq b_n \leq \cdots \leq b_2 \leq b_1\} \\
& \qquad \subseteq \cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1 \Leftrightarrow \forall n \in \mathbb{N} \left(\{x_0\} \subseteq I_n\right) \qquad (3.3.40) \\
& \Rightarrow x_0 \in \cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1 \Leftrightarrow \forall n \in \mathbb{N} \left(x_0 \in I_n\right)
\end{aligned}$$

$$\begin{aligned}
\bigcap_{n=1}^{\infty} I_n = & \lim_{n\to\infty} \bigcap_{k=1}^{n} I_k = \lim_{n\to\infty} \left(I_1 \cap I_2 \cap \cdots \cap I_n\right) \\
= & I_1 \cap I_2 \cap \cdots \cap \lim_{n\to\infty} I_n = I_1 \cap I_2 \cap \cdots \cap I_\infty \\
= & I_1 \cap I_2 \cap \cdots \cap \{x_0\}, \wedge \textbf{??} : \{x_0\} \subseteq \cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1 \\
= & \{x_0\}
\end{aligned}$$

□

定理 3.3.5. 連續函數 介值定理 / 中間值定理 *continuous function intermediate value theorem* = *CF IVT* / *intermediate value theorem* = *IVT*

$$\begin{cases} f : [a, b] \to \mathbb{R} \\ f \text{ continuous on } [a, b] \\ f(a) f(b) < 0 \end{cases} a < b \Rightarrow \exists c \in (a, b) \{f(c) = 0\}$$

or more generalized

$$\begin{cases} g:[a,b]\to\mathbb{R} \\ g \text{ continuous on } [a,b] & a<b \\ \forall k\in\mathbb{R}\left(\begin{cases} k \text{ between } g(a),g(b) \\ k\neq g(a)\wedge k\neq g(b)\end{cases}\right) & g(a)\neq g(b)\end{cases} \Rightarrow \exists c\in(a,b)\{g(c)=k\}$$

Proof. without loss of generality (w/oLoG), let $\begin{cases} f(a)<0 \\ f(b)>0\end{cases}$,

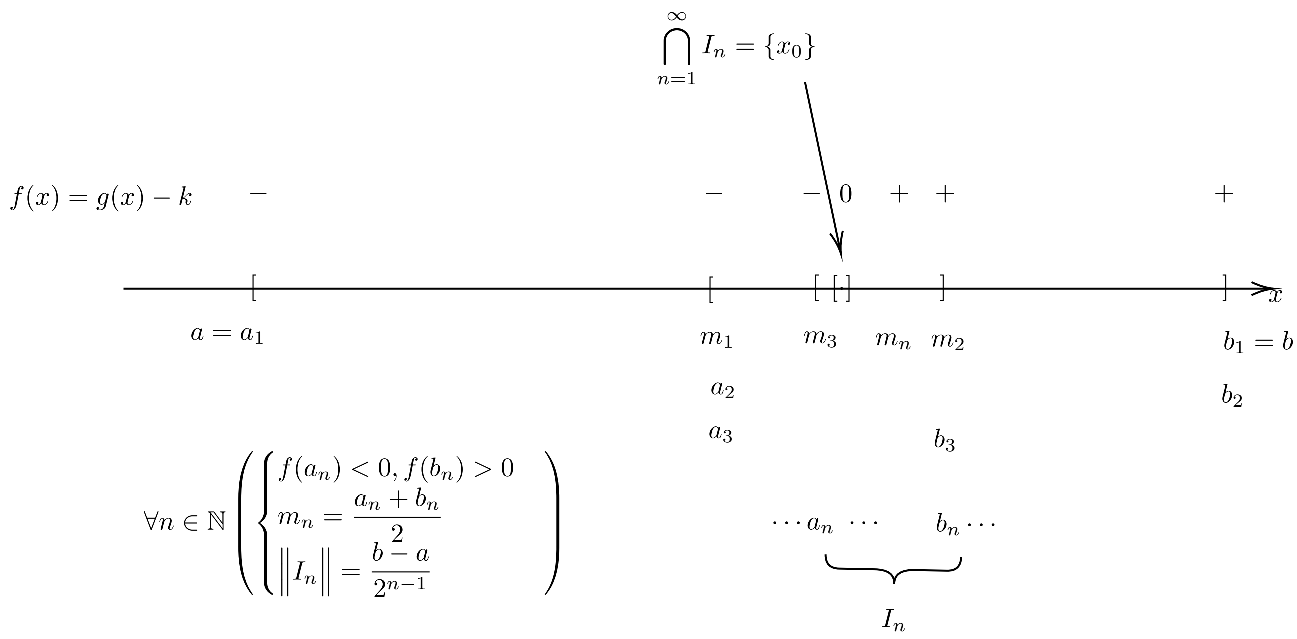


Figure 3.3.3: continuous function intermediate value theorem = CFIVT / intermediate value theorem = IVT

construction of 區間套 nested interval = NI 3.3.4,
case 1-0: let

$$\begin{cases} a_1=a\Rightarrow f(a_1)=f(a)<0\Rightarrow f(a_1)<0 \\ b_1=b\Rightarrow f(b_1)=f(b)>0\Rightarrow f(b_1)>0 \\ I_1=[a_1,b_1]=[a,b] \\ \|I_1\|=|b_1-a_1|=b_1-a_1=b-a \\ m_1=\frac{a_1+b_1}{2}\in(a_1,b_1)\subset I_1=[a,b]\Rightarrow m_1\in(a,b)\end{cases}$$

case 2-1: $f(m_1)=0$,

$$\exists c=m_1\in(a,b)\{f(c)=0\}$$

case 2-2: $f(m_1)<0$, let

$$\begin{cases} a_2=m_1\Rightarrow f(a_2)=f(m_1)<0\Rightarrow f(a_2)<0 \\ b_2=b_1\Rightarrow f(b_2)=f(b_1)>0\Rightarrow f(b_2)>0 \\ I_2=[a_2,b_2]=[m_1,b_1]\overset{m_1\in(a_1,b_1)}{\subset}[a_1,b_1]=I_1\Rightarrow I_2\subseteq I_1 \\ \|I_2\|=|b_2-a_2|=|b_1-m_1|=b_1-m_1=b_1-\frac{a_1+b_1}{2}=\frac{b_1-a_1}{2}=\frac{\|I_1\|}{2}=\frac{b-a}{2} \\ m_2=\frac{a_2+b_2}{2}\in(a_2,b_2)\subset I_2\subseteq I_1=[a,b]\Rightarrow m_2\in(a,b)\end{cases}$$

case 2-3: $f(m_1)>0$, let

$$\begin{cases} a_2=a_1\Rightarrow f(a_2)=f(a_1)<0\Rightarrow f(a_2)<0 \\ b_2=m_1\Rightarrow f(b_2)=f(m_1)>0\Rightarrow f(b_2)>0 \\ I_2=[a_2,b_2]=[a_1,m_1]\overset{m_1\in(a_1,b_1)}{\subset}[a_1,b_1]=I_1\Rightarrow I_2\subseteq I_1 \\ \|I_2\|=|b_2-a_2|=|m_1-a_1|=m_1-a_1=\frac{a_1+b_1}{2}-a_1=\frac{b_1-a_1}{2}=\frac{\|I_1\|}{2}=\frac{b-a}{2} \\ m_2=\frac{a_2+b_2}{2}\in(a_2,b_2)\subset I_2\subseteq I_1=[a,b]\Rightarrow m_2\in(a,b)\end{cases}$$

case 3-1: $f(m_2) = 0$,

$$\exists c = m_2 \in (a, b) \{f(c) = 0\}$$

case 3-2: $f(m_2) < 0$, let

$$\begin{cases} a_3 = m_2 \Rightarrow f(a_3) = f(m_2) < 0 \Rightarrow f(a_3) < 0 \\ b_3 = b_2 \Rightarrow f(b_3) = f(b_2) > 0 \Rightarrow f(b_3) > 0 \\ I_3 = [a_3, b_3] = [m_2, b_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ \|I_3\| = |b_3 - a_3| = |b_2 - m_2| = b_2 - m_2 = b_2 - \dfrac{a_2 + b_2}{2} = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{b-a}{2^2} \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \subset I_3 \subseteq I_2 \subseteq I_1 = [a, b] \Rightarrow m_3 \in (a, b) \end{cases}$$

case 3-3: $f(m_2) > 0$, let

$$\begin{cases} a_3 = a_2 \Rightarrow f(a_3) = f(a_2) < 0 \Rightarrow f(a_3) < 0 \\ b_3 = m_2 \Rightarrow f(b_3) = f(m_2) > 0 \Rightarrow f(b_3) > 0 \\ I_3 = [a_3, b_3] = [a_2, m_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ \|I_3\| = |b_3 - a_3| = |m_2 - a_2| = m_2 - a_2 = \dfrac{a_2 + b_2}{2} - a_2 = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{b-a}{2^2} \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \subset I_3 \subseteq I_2 \subseteq I_1 = [a, b] \Rightarrow m_3 \in (a, b) \end{cases}$$

$\vdots$

case n-1: $f(m_{n-1}) = 0$,

$$\exists c = m_{n-1} \in (a, b) \{f(c) = 0\}$$

case n-2: $f(m_{n-1}) < 0$, let

$$\begin{cases} a_n = m_{n-1} \Rightarrow f(a_n) = f(m_{n-1}) < 0 \Rightarrow f(a_n) < 0 \\ b_n = b_{n-1} \Rightarrow f(b_n) = f(b_{n-1}) > 0 \Rightarrow f(b_n) > 0 \\ I_n = [a_n, b_n] = [m_{n-1}, b_{n-1}] \overset{m_{n-1} \in (a_{n-1}, b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |b_{n-1} - m_{n-1}| = b_{n-1} - m_{n-1} = b_{n-1} - \dfrac{a_{n-1} + b_{n-1}}{2} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^{n-1}} = \dfrac{b-a}{2^{n-1}} \\ m_n = \dfrac{a_n + b_n}{2} \in (a_n, b_n) \subset I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_1 = [a, b] \Rightarrow m_n \in (a, b) \end{cases}$$

case n-3: $f(m_{n-1}) > 0$, let

$$\begin{cases} a_n = a_{n-1} \Rightarrow f(a_n) = f(a_{n-1}) < 0 \Rightarrow f(a_n) < 0 \\ b_n = m_{n-1} \Rightarrow f(b_n) = f(m_{n-1}) > 0 \Rightarrow f(b_n) > 0 \\ I_n = [a_n, b_n] = [a_{n-1}, m_{n-1}] \overset{m_{n-1} \in (a_{n-1}, b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] = I_{n-1} \Rightarrow I_n \subseteq I_{n-1} \\ \|I_n\| = |b_n - a_n| = |b_{n-1} - m_{n-1}| = b_{n-1} - m_{n-1} = b_{n-1} - \dfrac{a_{n-1} + b_{n-1}}{2} = \dfrac{b_{n-1} - a_{n-1}}{2} = \dfrac{\|I_1\|}{2^{n-1}} = \dfrac{b-a}{2^{n-1}} \\ m_n = \dfrac{a_n + b_n}{2} \in (a_n, b_n) \subset I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_1 = [a, b] \Rightarrow m_n \in (a, b) \end{cases}$$

$\vdots$

依此類推, 以致無窮 and so on to infinity,

$$\begin{cases} \forall n \in \mathbb{N} \left\{ I_n = [a_n, b_n], \begin{cases} f(a_n) < 0 \Rightarrow f(a_n) \leq 0 \\ f(b_n) > 0 \Rightarrow f(b_n) \geq 0 \end{cases} \right\} \\ \forall n \in \mathbb{N} \{I_{n+1} \subseteq I_n\} \\ \forall n \in \mathbb{N} \left\{ \|I_n\| = \dfrac{\|I_1\|}{2^{n-1}} \right\} \Leftrightarrow \forall n \in \mathbb{N} \left\{ \|I_{n+1}\| = \dfrac{\|I_1\|}{2^n} \right\} \\ \forall n \in \mathbb{N} \{m_n \in (a, b)\} \end{cases}$$

by 區間套定理 / 區間套原理 nested interval theorem = NIT / nested interval principle = NIP 3.3.4,

$$\begin{cases} \forall n \in \mathbb{N} \left(\begin{cases} I_n = [a_n, b_n] \subset \mathbb{R} \\ I_{n+1} \subseteq I_n \Leftrightarrow I_n \supseteq I_{n+1} \end{cases} \right) \\ \lim\limits_{n\to\infty} \|I_n\| = \lim\limits_{n\to\infty} (b_n - a_n) = \lim\limits_{n\to\infty} \dfrac{\|I_1\|}{2^{n-1}} = \lim\limits_{n\to\infty} \dfrac{b-a}{2^{n-1}} = 0 \end{cases} \Rightarrow \exists x_0 \in \mathbb{R} \left(\bigcap_{n=1}^{\infty} I_n = \{x_0\} \right)$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} [a_n, b_n] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right] = \{x_0\} = [x_0, x_0]$$
$$= \begin{cases} \lim_{n\to\infty} \{m_n\} \subseteq (a,b) & m_n \in (a,b) \\ \lim_{n\to\infty} [a_{n-1}, m_{n-1}] \subseteq (a,b) & [a_n, b_n] = [a_{n-1}, m_{n-1}] \overset{m_{n-1}\in(a_{n-1},b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] \\ \lim_{n\to\infty} [m_{n-1}, b_{n-1}] \subseteq (a,b) & [a_n, b_n] = [m_{n-1}, b_{n-1}] \overset{m_{n-1}\in(a_{n-1},b_{n-1})}{\subset} [a_{n-1}, b_{n-1}] \end{cases}$$
$$\Downarrow$$
$$\{x_0\} \subseteq (a,b) \Leftrightarrow x_0 \in (a,b)$$

by premise and above,

$$\begin{cases} f \text{ continuous on } [a,b] \\ \{x_0\} \subseteq \cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1 = [a,b] \end{cases} \text{premise} \Rightarrow f \text{ continuous at } x_0$$

$$\begin{cases} f(a_n) \le 0 \\ f(b_n) \ge 0 \end{cases} \Rightarrow \begin{cases} \lim_{n\to\infty} f(a_n) \le 0 \\ \lim_{n\to\infty} f(b_n) \ge 0 \end{cases} \underset{f \text{ continuous on } [a,b]}{\overset{[a_n,b_n]\subseteq[a,b]}{\Rightarrow}} \begin{cases} f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) \le 0 \\ f\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} f(b_n) \ge 0 \end{cases}$$
$$\Rightarrow \begin{cases} f(x_0) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) \le 0 \\ f(x_0) = f\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} f(b_n) \ge 0 \end{cases} \Rightarrow \begin{cases} f(x_0) \le 0 \\ f(x_0) \ge 0 \end{cases} \Rightarrow f(x_0) = 0$$
$$\exists c = x_0 \in (a,b) \{f(c) = 0\}$$

□

勘根定理是中間 值定理的特例, 但兩者在邏輯上也是等價的 [13]

定理 3.3.6. 勘根定理

定理 3.3.7. *B-W* 定理 / 聚點原理 *Bolzano-Weierstrass theorem = B-WT / accumulative point principle* 波爾查諾-魏爾史特拉斯定理

任意有界數列必含收斂子數列 *any bounded (infinite) sequence has a convergent subsequence*

$$\begin{cases} \exists \langle a_n\rangle_{n\in\mathbb{N}} = \langle a_n\rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} \\ \exists \beta > 0, \forall n \in \mathbb{N}\,(|a_n| \le \beta) \\ S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \end{cases} \Rightarrow \exists \langle a_{n_k}\rangle_{k\in\mathbb{N}} = \langle a_{n_k}\rangle_{k=1}^{\infty} \text{ is a subsequence of } \langle a_n\rangle_{n\in\mathbb{N}} \left(\lim_{k\to\infty} a_{n_k} = x_0 \in \mathbb{R}\right)$$

Proof. consider (infinite) real sequence 除非特別指明, 不然數列通常是指無限數列

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$
$$a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, a_3 \in \mathbb{R}, \cdots, a_n \in \mathbb{R}, \cdots$$
$$\underbrace{a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, a_3 \in \mathbb{R}, \cdots, a_n \in \mathbb{R}, \cdots}_{|\mathbb{N}|} \in \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots}_{|\mathbb{N}|}$$
$$\langle a_n\rangle_{n\in\mathbb{N}} = \underbrace{\langle a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, a_3 \in \mathbb{R}, \cdots, a_n \in \mathbb{R}, \cdots\rangle}_{|\mathbb{N}|} \in \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots}_{|\mathbb{N}|}$$
$$\langle a_n\rangle_{n\in\mathbb{N}} = \underbrace{\langle a_1, a_2, a_3, \cdots, a_n, \cdots\rangle}_{|\mathbb{N}|} \in \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots}_{|\mathbb{N}|} = \mathbb{R}^{|\mathbb{N}|} = \mathbb{R}^{\aleph_0} = \mathbb{R}^{\infty}$$

$$\langle a_n\rangle_{n\in\mathbb{N}} = \underbrace{\langle a_1, a_2, a_3, \cdots, a_n, \cdots\rangle}_{|\mathbb{N}|} = \underbrace{\langle f(1), f(2), f(3), \cdots, f(n), \cdots\rangle}_{|\mathbb{N}|}$$
$$, f : \mathbb{N} \to \mathbb{R} \Rightarrow f \in \mathbb{R}^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{Q}} \subset \mathbb{R}^{\mathbb{R}}$$

$$\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \mathbb{R}^{\{0,1\}} \ni f : \{0,1\} \to \mathbb{R}$$
$$\langle x, f(x)\rangle \in [a,b] \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2, f : [a,b] \to \mathbb{R} \Rightarrow f \in \mathbb{R}^{[a,b]} \subset \mathbb{R}^{\mathbb{R}}$$

like 3.3.30

[13] 數學傳播_33_2_蔡聰明_均值定理的統合與推廣

let

$$\emptyset \neq S = \left\{ a \middle| \begin{cases} n \in \mathbb{N} \\ \exists \langle a_n \rangle_{n\in\mathbb{N}} \in \mathbb{R}^{|\mathbb{N}|} \left(\exists \beta > 0, \forall n \in \mathbb{N} \left(|a_n| \leq \beta\right)\right) & \text{有界數列} \\ a = a_n \end{cases} \right\} = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$$

i.e.

$$\begin{cases} \exists \langle a_n \rangle_{n\in\mathbb{N}} \in \mathbb{R}^{|\mathbb{N}|} \\ \exists \beta > 0, \forall n \in \mathbb{N} \left(|a_n| \leq \beta\right) \\ \emptyset \neq S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \end{cases}$$

then

$$\begin{cases} \exists \langle a_n \rangle_{n\in\mathbb{N}} = \langle a_n \rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} & \text{數列} \\ \exists \beta > 0, \forall n \in \mathbb{N} \left(|a_n| \leq \beta\right) & \text{上界} \\ \emptyset \neq S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} & \text{集合} \end{cases}$$

$$\Rightarrow \begin{cases} \aleph_0 = |\mathbb{N}| = \left|\langle a_n \rangle_{n\in\mathbb{N}}\right| = |\langle a_n \rangle_{n=1}^{\infty}| = \infty & \langle a_n \rangle \text{ is infinite} \\ |S| = \left|\{a_n\}_{n\in\mathbb{N}}\right| = |\{a_n\}_{n=1}^{\infty}| \begin{array}{cc} \in \mathbb{N} & S \text{ is finite} \\ \veebar & \text{互斥或} \\ \geq |\mathbb{N}| & S \text{ is infinite} \end{array} \end{cases}$$

case 1: S is finite $\Leftrightarrow |S| \in \mathbb{N}$,
舉些例子
exmaple 1-1:

$$\langle c_n \rangle_{n\in\mathbb{N}} = \langle c \rangle_{n\in\mathbb{N}} = \langle c, c, \cdots, c, \cdots \rangle$$

$$S_{\langle c_n \rangle_{n\in\mathbb{N}}} = \left\{ x_n \middle| \begin{cases} n \in \mathbb{N} \\ x_n = c \end{cases} \right\} = \{c, c, \cdots, c, \cdots\} = \{c\}$$

$$\left|S_{\langle c_n \rangle_{n\in\mathbb{N}}}\right| = \left|\left\{ x_n \middle| \begin{cases} n \in \mathbb{N} \\ x_n = c \end{cases} \right\}\right| = |\{c, c, \cdots, c, \cdots\}| = |\{c\}| = 1 \in \mathbb{N}$$

exmaple 1-2:

$$\langle a_n \rangle_{n\in\mathbb{N}} = \langle (-1)^n \rangle_{n\in\mathbb{N}} = \langle -1, 1, -1, 1, -1, 1, \cdots, (-1)^n, \cdots \rangle$$

$$S_{\langle a_n \rangle_{n\in\mathbb{N}}} = \{(-1)^n | n \in \mathbb{N}\} = \{-1, 1, -1, 1, -1, 1, \cdots, (-1)^n, \cdots\} = \{-1, 1\}$$

$$\left|S_{\langle a_n \rangle_{n\in\mathbb{N}}}\right| = |\{(-1)^n | n \in \mathbb{N}\}| = |\{-1, 1, -1, 1, -1, 1, \cdots, (-1)^n, \cdots\}| = |\{-1, 1\}| = 2 \in \mathbb{N}$$

exmaple 1-3:

$$\langle b_n \rangle_{n\in\mathbb{N}} = \left\langle \frac{1}{2} + \left(\frac{-1}{2}\right)^n \right\rangle_{n\in\mathbb{N}} = \left\langle 0, 1, 0, 1, 0, 1, \cdots, \frac{1}{2} + \left(\frac{-1}{2}\right)^n, \cdots \right\rangle$$

$$S_{\langle b_n \rangle_{n\in\mathbb{N}}} = \left\{ \frac{1}{2} + \left(\frac{-1}{2}\right)^n \middle| n \in \mathbb{N} \right\} = \left\{ 0, 1, 0, 1, 0, 1, \cdots, \frac{1}{2} + \left(\frac{-1}{2}\right)^n, \cdots \right\} = \{0, 1\}$$

$$\left|S_{\langle b_n \rangle_{n\in\mathbb{N}}}\right| = \left|\left\{ \frac{1}{2} + \left(\frac{-1}{2}\right)^n \middle| n \in \mathbb{N} \right\}\right| = \left|\left\{ 0, 1, 0, 1, 0, 1, \cdots, \frac{1}{2} + \left(\frac{-1}{2}\right)^n, \cdots \right\}\right| = |\{0, 1\}| = 2 \in \mathbb{N}$$

according to 鴿籠原理 / 鴿巢原理 / 狄利克抽屜原理 / 狄利克箱原理 / pigeonhole principle / Dirichlet drawer principle / Dirichlet box principle / Dirichlet's drawer principle / Dirichlet's box principle ,
e.g.
(infinite) sequence

$$a_1, a_2, a_3, a_4, a_5, a_6, \cdots$$

$$\{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\}$$

e.g. $\{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\} = \{a, b, c\}, \wedge \begin{cases} |\langle a_1, a_2, a_3, a_4, a_5, a_6, \cdots \rangle| = \infty \\ |\{a, b, c\}| = 3 \end{cases}$

e.g. $\{a_1 = a, a_2 = b, a_3 = c, a_4 = a, a_5 = c, a_6 = b, \cdots\}$

$\Rightarrow \exists x \in \{a, b, c\}$ (x appears infinite times in $\langle a_1, a_2, a_3, a_4, a_5, a_6, \cdots \rangle$)

$$|S| \in \mathbb{N} \Leftrightarrow S \text{ is finite} \Rightarrow \exists a \in S \left(a \text{ appears infinite times in } \langle a_n \rangle_{n\in\mathbb{N}}\right)$$

subsequence(s) / subsequences,

$$a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \cdots$$
$$a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \cdots$$
$$a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \cdots$$

e.g.

$$a_{1_1} = a_1 = a, a_{1_2} = a_4 = a, a_{1_3} = a, a_{1_4} = a, \cdots$$
$$a_{2_1} = a_2 = b, a_{2_2} = a_6 = b, a_{2_3} = b, a_{2_4} = b, \cdots$$
$$a_{3_1} = a_3 = c, a_{3_2} = a_5 = c, a_{3_3} = c, a_{3_4} = c, \cdots$$
$$\langle a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \cdots \rangle = \langle a_{1_k} \rangle = \langle a, a, a, \cdots \rangle$$
$$\langle a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \cdots \rangle = \langle a_{2_k} \rangle = \langle b, b, b, \cdots \rangle$$
$$\langle a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \cdots \rangle = \langle a_{3_k} \rangle = \langle c, c, c, \cdots \rangle$$
$$\{a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \cdots\} = \{a_{1_k}\} = \{a, a, a, \cdots\}$$
$$\{a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \cdots\} = \{a_{2_k}\} = \{b, b, b, \cdots\}$$
$$\{a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \cdots\} = \{a_{3_k}\} = \{c, c, c, \cdots\}$$
$$|\{a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \cdots\}| = |\{a_{1_k}\}| = |\{a, a, a, \cdots\}| = 1$$
$$|\{a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \cdots\}| = |\{a_{2_k}\}| = |\{b, b, b, \cdots\}| = 1$$
$$|\{a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \cdots\}| = |\{a_{3_k}\}| = |\{c, c, c, \cdots\}| = 1$$

$$\begin{aligned}
&\text{e.g. } \{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\} = \{a, b, c\}, \wedge \begin{cases} |\{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\}| = \infty \\ |\{a, b, c\}| = 3 \end{cases} \\
&\text{e.g. } \{a_1 = a, a_2 = b, a_3 = c, a_4 = a, a_5 = c, a_6 = b, \cdots\} \\
\Rightarrow &\exists x \in \{a, b, c\} \left(x \text{ appears infinite times in } \langle a_1, a_2, a_3, a_4, a_5, a_6, \cdots \rangle\right) \\
\Leftrightarrow &\exists \langle a_{n_k} \rangle \in \{\langle a_{1_k} \rangle, \langle a_{2_k} \rangle, \langle a_{3_k} \rangle\} \left(\langle a_{n_k} \rangle \text{ is infinite}\right) \\
\Leftrightarrow &\exists \langle a_{n_k} \rangle \in \{\langle a_{1_k} \rangle, \langle a_{2_k} \rangle, \langle a_{3_k} \rangle\} \left(|\langle a_{n_k} \rangle| = \infty\right)
\end{aligned}$$

也就是說至少有一元素在(無限)數列中重複無限次.
重新一般性地說

$$\begin{cases} \exists \langle a_n \rangle_{n\in\mathbb{N}} = \langle a_n \rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} & \text{數列} \Rightarrow \aleph_0 = |\mathbb{N}| = \left|\langle a_n \rangle_{n\in\mathbb{N}}\right| = |\langle a_n \rangle_{n=1}^{\infty}| = \infty \ \langle a_n \rangle \text{ is infinite} \\ \emptyset \neq S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} & \text{集合} \\ |S| = \left|\{a_n\}_{n\in\mathbb{N}}\right| = |\{a_n\}_{n=1}^{\infty}| = N \in \mathbb{N} & S \text{ is finite} \end{cases} \tag{3.3.41}$$

有限實數集其實本身就有界(有上下界)

$$\begin{aligned}
S = \{a_n\}_{n\in\mathbb{N}} \text{ is finite} &\Rightarrow \forall a \in S \left(a \leq \max S = \max \{a_n\}_{n\in\mathbb{N}}\right) \\
&\Rightarrow \begin{cases} \exists \beta = \max S \in \mathbb{R}, \forall a \in S \, (a \leq \beta) \\ \exists \beta = \max \{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}, \forall n \in \mathbb{N} \, (a_n \leq \beta) \end{cases} \\
&\Rightarrow \begin{cases} S \text{ is upper bounded by } \beta = \max S \in \mathbb{R} \\ \langle a_n \rangle_{n\in\mathbb{N}} \text{ is upper bounded by } \beta = \max \{a_n\}_{n\in\mathbb{N}} \in \mathbb{R} \end{cases}
\end{aligned}$$

同理

$$S = \{a_n\}_{n\in\mathbb{N}} \text{ is finite} \Rightarrow \begin{cases} S \text{ is lower bounded by } \min S \in \mathbb{R} \\ \langle a_n \rangle_{n\in\mathbb{N}} \text{ is lower bounded by } \min \{a_n\}_{n\in\mathbb{N}} \in \mathbb{R} \end{cases}$$

故

$$S = \{a_n\}_{n\in\mathbb{N}} \text{ is finite} \Rightarrow \begin{cases} S \text{ is bounded by } \max\{|\min S|, |\max S|\} \in \mathbb{R} \\ \langle a_n \rangle_{n\in\mathbb{N}} \text{ is bounded by } \max\left\{\left|\min \{a_n\}_{n\in\mathbb{N}}\right|, \left|\max \{a_n\}_{n\in\mathbb{N}}\right|\right\} \in \mathbb{R} \end{cases}$$

$$\begin{cases} \exists \langle a_n\rangle_{n\in\mathbb{N}} = \langle a_n\rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} & \text{數列} \\ \emptyset \neq S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} & \text{集合} \\ |S| = \left|\{a_n\}_{n\in\mathbb{N}}\right| = |\{a_n\}_{n=1}^{\infty}| = N \in \mathbb{N} & S \text{ is finite} \end{cases}$$

$$\Rightarrow \begin{cases} \aleph_0 = |\mathbb{N}| = \left|\langle a_n\rangle_{n\in\mathbb{N}}\right| = |\langle a_n\rangle_{n=1}^{\infty}| = \infty & \langle a_n\rangle \text{ is infinite} \\ \exists \beta \in \mathbb{R}, \forall n \in \mathbb{N}\,(a_n \le \beta) & \text{上界} \end{cases}$$

$$\Rightarrow \begin{cases} \aleph_0 = |\mathbb{N}| = \left|\langle a_n\rangle_{n\in\mathbb{N}}\right| = |\langle a_n\rangle_{n=1}^{\infty}| = \infty & \langle a_n\rangle \text{ is infinite} \\ \exists \{x_i\}_{i=1}^{N} = S\left(\{x_i\}_{i=1}^{N} = \{x_1 < x_2 < \cdots < x_N\}\right) & N\text{相異元素} \\ \exists \beta \in \mathbb{R}, \forall n \in \mathbb{N}\,(a_n \le \beta) & \text{上界} \end{cases}$$

$$\begin{cases} \left|\langle a_n\rangle_{n\in\mathbb{N}}\right| = |\langle a_n\rangle_{n=1}^{\infty}| = \infty & \langle a_n\rangle \text{ is infinite} \\ S = \{a_n\}_{n\in\mathbb{N}} & \\ \exists \{x_i\}_{i=1}^{N} = S\left(\{x_i\}_{i=1}^{N} = \{x_1 < x_2 < \cdots < x_N\} \Rightarrow |\{x_1 < x_2 < \cdots < x_N\}| = N \in \mathbb{N}\right) & N\text{相異元素} \end{cases}$$

$$\Rightarrow \exists \langle x_0\rangle \in \{\langle x_1\rangle, \langle x_2\rangle, \cdots, \langle x_N\rangle\}\left(\langle x_0\rangle \text{ is infinite} \Leftrightarrow |\langle x_0\rangle| = \infty \Leftrightarrow \langle x_0\rangle = \langle x_0\rangle_{k\in\mathbb{N}} = \langle x_0\rangle_{k=1}^{\infty}\right)$$

也就是説至少有一元素在(無限)數列中重複無限次.
我們就取重複該元素無限次之數列為收斂子序列

$$\exists \langle a_{n_k}\rangle = \langle x_0\rangle = \langle x_0\rangle_{k\in\mathbb{N}} \text{ is a subsequence of } \langle a_n\rangle_{n\in\mathbb{N}} \left(\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} x_0 = x_0 \in \mathbb{R}\right)$$

case 2: S is infinite,

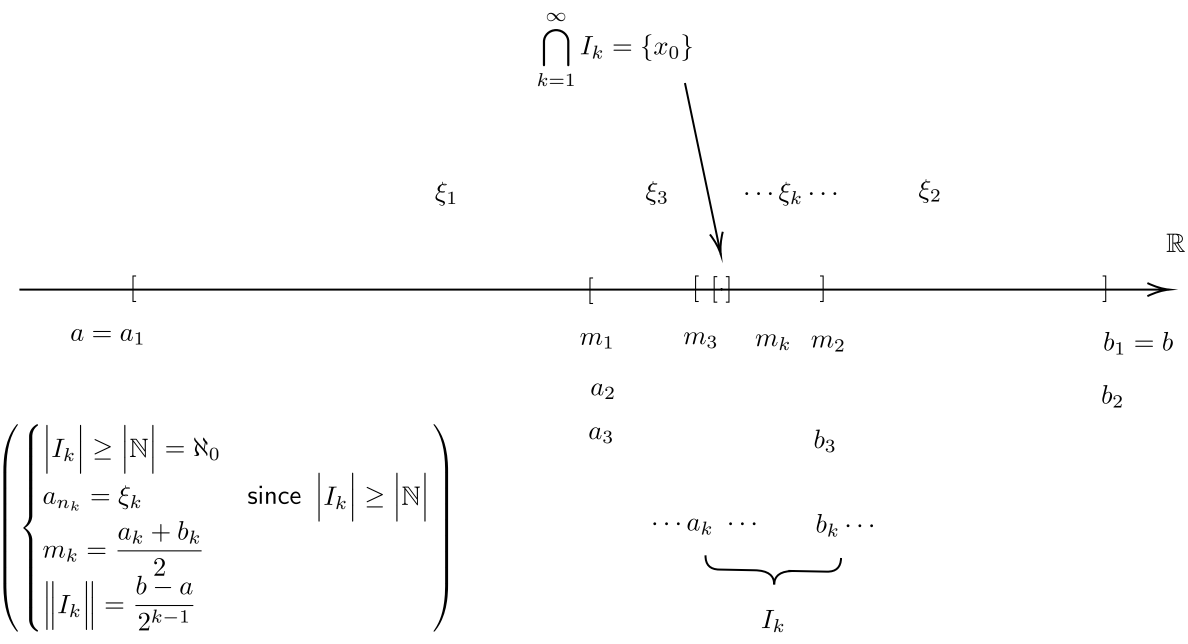
$$\begin{cases} \exists \langle a_n\rangle_{n\in\mathbb{N}} = \langle a_n\rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} & \text{數列} \Rightarrow \aleph_0 = |\mathbb{N}| = \left|\langle a_n\rangle_{n\in\mathbb{N}}\right| = |\langle a_n\rangle_{n=1}^{\infty}| = \infty\ \langle a_n\rangle \text{ is infinite} \\ \emptyset \neq S = \{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} & \text{集合} \\ |S| = \left|\{a_n\}_{n\in\mathbb{N}}\right| = |\{a_n\}_{n=1}^{\infty}| \ge |\mathbb{N}| & S \text{ is infinite} \end{cases} \tag{3.3.42}$$

$$\begin{cases} S \subset \mathbb{R} \\ S \text{ is infinite} \end{cases} \Rightarrow \text{e.g.} \begin{cases} S = \mathbb{N} \\ S = \mathbb{Z} \\ S = \mathbb{Q} \\ S = \mathbb{Q}^+ \\ S = \mathbb{R}^- \\ S = I = \begin{cases} I_i(a_i, b_i) = \begin{cases} (a_i, b_i) \\ (a_i, b_i] \\ [a_i, b_i) \\ [a_i, b_i] \end{cases} \\ I_j(\infty) = \begin{cases} (-\infty, b_j) \\ (-\infty, b_j] \\ (a_j, \infty) \\ [a_j, \infty) \end{cases} \end{cases} \\ S = \bigcup_{i\in\mathcal{I}} I_i(a_i, b_i) \cup \bigcup_{j\in\mathcal{J}} I_j(\infty) \\ \vdots \end{cases}$$

$$\begin{cases} S \subset \mathbb{R} \\ S \text{ is infinite} \end{cases} \Rightarrow \begin{matrix} S \text{ is unbounded, e.g.} & S = \mathbb{Z}, \mathbb{Q}, \cdots \\ S \text{ is only lower bounded, e.g.} & S = \mathbb{N}, \mathbb{Q}^+, \mathbb{R}^+, (a_j, \infty), [a_j, \infty), \cdots \\ S \text{ is only upper bounded, e.g.} & S = \mathbb{Z}^-, \mathbb{Q}^-, \mathbb{R}^-, (-\infty, b_j), (-\infty, b_j], \cdots \\ S \text{ is (upper and lower) bounded, e.g.} & S = I_i(a_i, b_i), \bigcup_{i\in\mathcal{I}} I_i(a_i, b_i), \cdots \end{matrix}$$

$$S \text{ is (upper and lower) bounded} \Leftrightarrow S \text{ is bounded} \Leftrightarrow \exists \beta > 0, \forall n \in \mathbb{N}\,(|a_n| \le \beta)$$

只有 (upper and lower) bounded 實數集 才能構造 區間套 nested interval 吧 like 3.3.4 and 3.3.5



$$\forall k \in \mathbb{N}\left(\begin{cases}\left|I_k\right| \geq \left|\mathbb{N}\right| = \aleph_0 \\ a_{n_k} = \xi_k & \text{since } \left|I_k\right| \geq \left|\mathbb{N}\right| \\ m_k = \dfrac{a_k + b_k}{2} \\ \left\|I_k\right\| = \dfrac{b-a}{2^{k-1}}\end{cases}\right)$$

Figure 3.3.4: Bolzano-Weierstrass theorem = B-WT, case 2: S is infinite
若將基數為無限的集合拆成兩(閉)區間的聯集, 則該兩者其中至少一個區間為基數無限的集合

without loss of generality (w/oLoG), let

$$S = [a,b] \subset \mathbb{R}, a < b \because a = b \Rightarrow |S| = |[a,a]| = |\{a\}| = 1 \in \mathbb{N}$$

$$|S| = |[a,b]| \geq |\mathbb{N}| > 0 = |\emptyset|$$

construction of 區間套 nested interval = NI 3.3.4,

case 2-1-0: let

$$\begin{cases} a_1 = a & , a < b \\ b_1 = b \\ m_1 = \dfrac{a_1 + b_1}{2} \in (a_1, b_1) \\ \quad I_1 \quad = [a_1, b_1] = [a, b] = S \\ \qquad\quad = [a_1, m_1] \cup [m_1, b_1] \\ \quad |I_1| \quad = |[a_1, b_1]| = |[a_1, m_1]| + |[m_1, b_1]| - |[a_1, m_1] \cap [m_1, b_1]| \\ \qquad\qquad\qquad\qquad = |[a_1, m_1]| + |[m_1, b_1]| - |\{m_1\}| \\ \qquad\qquad\qquad\qquad\quad = |[a_1, m_1]| + |[m_1, b_1]| - 1 \\ \|I_1\| = |b_1 - a_1| = b_1 - a_1 = b - a > 0 \\ \text{任取 } a_{n_1} = \xi_1 \in I_1 \text{ since } |I_1| = |[a_1, b_1]| = |S| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

若將基數為無限的集合拆成兩(閉)區間的聯集, 則該兩者其中至少一個區間為基數無限的集合

case 2-2-0: $\begin{cases} |[a_1, m_1]| \in \mathbb{N} \\ |[m_1, b_1]| \in \mathbb{N} \end{cases}$,

$$\begin{cases} |[a_1, m_1]| \in \mathbb{N} \\ |[m_1, b_1]| \in \mathbb{N} \end{cases} \Rightarrow |I_1| \in \mathbb{N} \Rightarrow\!\Leftarrow |I_1| \geq |\mathbb{N}|$$

case 2-2-1: $|[a_1, m_1]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_2 = a_1 \\ b_2 = m_1 \\ m_2 = \dfrac{a_2 + b_2}{2} \in (a_2, b_2) \\ I_2 = [a_2, b_2] = [a_1, m_1] \overset{m_1 \in (a_1, b_1)}{\subset} [a_1, b_1] = I_1 \Rightarrow I_2 \subseteq I_1 \\ |I_2| = |[a_2, b_2]| = |[a_1, m_1]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_2 = [a_2, b_2] = [a_2, m_2] \cup [m_2, b_2] \\ \quad |I_2| \quad = |[a_2, b_2]| = |[a_2, m_2]| + |[m_2, b_2]| - |[a_2, m_2] \cap [m_2, b_2]| \\ \qquad\qquad = |[a_2, m_2]| + |[m_2, b_2]| - |\{m_2\}| \\ \qquad\qquad = |[a_2, m_2]| + |[m_2, b_2]| - 1 \\ \|I_2\| = |b_2 - a_2| = |m_1 - a_1| = m_1 - a_1 = \dfrac{a_1 + b_1}{2} - a_1 = \dfrac{b_1 - a_1}{2} = \dfrac{\|I_1\|}{2} = \dfrac{b - a}{2} \\ \text{任取 } a_{n_2} = \xi_2 \in I_2 \text{ since } |I_2| = |[a_2, b_2]| = |[a_1, m_1]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-2-2: $|[m_1, b_1]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_2 = m_1 \\ b_2 = b_1 \\ m_2 = \dfrac{a_2 + b_2}{2} \in (a_2, b_2) \\ I_2 = [a_2, b_2] = [m_1, b_1] \overset{m_1 \in (a_1, b_1)}{\subset} [a_1, b_1] = I_1 \Rightarrow I_2 \subseteq I_1 \\ |I_2| = |[a_2, b_2]| = |[m_1, b_1]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_2 = [a_2, b_2] = [a_2, m_2] \cup [m_2, b_2] \\ \quad |I_2| \quad = |[a_2, b_2]| = |[a_2, m_2]| + |[m_2, b_2]| - |[a_2, m_2] \cap [m_2, b_2]| \\ \qquad\qquad = |[a_2, m_2]| + |[m_2, b_2]| - |\{m_2\}| \\ \qquad\qquad = |[a_2, m_2]| + |[m_2, b_2]| - 1 \\ \|I_2\| = |b_2 - a_2| = |b_1 - m_1| = b_1 - m_1 = b_1 - \dfrac{a_1 + b_1}{2} = \dfrac{b_1 - a_1}{2} = \dfrac{\|I_1\|}{2} = \dfrac{b - a}{2} \\ \text{任取 } a_{n_2} = \xi_2 \in I_2 \text{ since } |I_2| = |[a_2, b_2]| = |[m_1, b_1]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-2-3: $\begin{cases} |[a_1, m_1]| \geq |\mathbb{N}| \\ |[m_1, b_1]| \geq |\mathbb{N}| \end{cases}$, choose one of case 2-2-1 or case 2-2-2

case 2-3-0: $\begin{cases} |[a_2, m_2]| \in \mathbb{N} \\ |[m_2, b_2]| \in \mathbb{N} \end{cases}$,

$$\begin{cases} |[a_2, m_2]| \in \mathbb{N} \\ |[m_2, b_2]| \in \mathbb{N} \end{cases} \Rightarrow |I_2| \in \mathbb{N} \Rightarrow\!\!\!\!\times\!\!\Leftarrow |I_2| \geq |\mathbb{N}|$$

case 2-3-1: $|[a_2, m_2]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_3 = a_2 \\ b_3 = m_2 \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \\ I_3 = [a_3, b_3] = [a_2, m_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ |I_3| = |[a_3, b_3]| = |[a_2, m_2]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_3 = [a_3, b_3] = [a_3, m_3] \cup [m_3, b_3] \\ \quad |I_3| \quad = |[a_3, b_3]| = |[a_3, m_3]| + |[m_3, b_3]| - |[a_3, m_3] \cap [m_3, b_3]| \\ \qquad\qquad = |[a_3, m_3]| + |[m_3, b_3]| - |\{m_3\}| \\ \qquad\qquad = |[a_3, m_3]| + |[m_3, b_3]| - 1 \\ \|I_3\| = |b_3 - a_3| = |m_2 - a_2| = m_2 - a_2 = \dfrac{a_2 + b_2}{2} - a_2 = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{b - a}{2^2} \\ \text{任取 } a_{n_3} = \xi_3 \in I_3 \text{ since } |I_3| = |[a_3, b_3]| = |[a_2, m_2]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-3-2: $|[m_2, b_2]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_3 = m_2 \\ b_3 = b_2 \\ m_3 = \dfrac{a_3 + b_3}{2} \in (a_3, b_3) \\ I_3 = [a_3, b_3] = [m_2, b_2] \overset{m_2 \in (a_2, b_2)}{\subset} [a_2, b_2] = I_2 \Rightarrow I_3 \subseteq I_2 \\ |I_3| = |[a_3, b_3]| = |[m_2, b_2]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_3 = [a_3, b_3] = [a_3, m_3] \cup [m_3, b_3] \\ \quad |I_3| \quad = |[a_3, b_3]| = |[a_3, m_3]| + |[m_3, b_3]| - |[a_3, m_3] \cap [m_3, b_3]| \\ \qquad = |[a_3, m_3]| + |[m_3, b_3]| - |\{m_3\}| \\ \qquad = |[a_3, m_3]| + |[m_3, b_3]| - 1 \\ \|I_3\| = |b_3 - a_3| = |b_2 - m_2| = b_2 - m_2 = b_2 - \dfrac{a_2 + b_2}{2} = \dfrac{b_2 - a_2}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{b-a}{2^2} \\ \text{任取 } a_{n_3} = \xi_3 \in I_3 \text{ since } |I_3| = |[a_3, b_3]| = |[m_2, b_2]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-3-3: $\begin{cases} |[a_2, m_2]| \geq |\mathbb{N}| \\ |[m_2, b_2]| \geq |\mathbb{N}| \end{cases}$, choose one of case 2-3-1 or case 2-3-2

$\vdots$

$$\begin{cases} m_{k-1} = \dfrac{a_{k-1} + b_{k-1}}{2} \in (a_{k-1}, b_{k-1}) \\ I_{k-1} = [a_{k-1}, b_{k-1}] = [a_{k-1}, m_{k-1}] \cup [m_{k-1}, b_{k-1}] \\ \quad |I_{k-1}| \quad = |[a_{k-1}, b_{k-1}]| = |[a_{k-1}, m_{k-1}]| + |[m_{k-1}, b_{k-1}]| - |[a_{k-1}, m_{k-1}] \cap [m_{k-1}, b_{k-1}]| \\ \qquad = |[a_{k-1}, m_{k-1}]| + |[m_{k-1}, b_{k-1}]| - |\{m_{k-1}\}| \\ \qquad = |[a_{k-1}, m_{k-1}]| + |[m_{k-1}, b_{k-1}]| - 1 \end{cases}$$

case 2-k-0: $\begin{cases} |[a_{k-1}, m_{k-1}]| \in \mathbb{N} \\ |[m_{k-1}, b_{k-1}]| \in \mathbb{N} \end{cases}$,

$$\begin{cases} |[a_{k-1}, m_{k-1}]| \in \mathbb{N} \\ |[m_{k-1}, b_{k-1}]| \in \mathbb{N} \end{cases} \Rightarrow |I_{k-1}| \in \mathbb{N} \Rightarrow\Leftarrow |I_{k-1}| \geq |\mathbb{N}|$$

case 2-k-1: $|[a_{k-1}, m_{k-1}]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_k = a_{k-1} \\ b_k = m_{k-1} \\ m_k = \dfrac{a_k + b_k}{2} \in (a_k, b_k) \\ I_k = [a_k, b_k] = [a_{k-1}, m_{k-1}] \overset{m_{k-1} \in (a_{k-1}, b_{k-1})}{\subset} [a_{k-1}, b_{k-1}] = I_{k-1} \Rightarrow I_k \subseteq I_{k-1} \\ |I_k| = |[a_k, b_k]| = |[a_{k-1}, m_{k-1}]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_k = [a_k, b_k] = [a_k, m_k] \cup [m_k, b_k] \\ \quad |I_k| \quad = |[a_k, b_k]| = |[a_k, m_k]| + |[m_k, b_k]| - |[a_k, m_k] \cap [m_k, b_k]| \\ \qquad = |[a_k, m_k]| + |[m_k, b_k]| - |\{m_k\}| \\ \qquad = |[a_k, m_k]| + |[m_k, b_k]| - 1 \\ \|I_k\| = |b_k - a_k| = |m_{k-1} - a_{k-1}| = m_{k-1} - a_{k-1} = \dfrac{a_{k-1} + b_{k-1}}{2} - a_{k-1} = \dfrac{b_{k-1} - a_{k-1}}{2} = \dfrac{\|I_1\|}{2^{k-1}} = \dfrac{b-a}{2^{k-1}} \\ \text{任取 } a_{n_k} = \xi_k \in I_k \text{ since } |I_k| = |[a_k, b_k]| = |[a_{k-1}, m_{k-1}]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-k-2: $|[m_{k-1}, b_{k-1}]| \geq |\mathbb{N}|$, let

$$\begin{cases} a_k = m_{k-1} \\ b_k = b_{k-1} \\ m_k = \dfrac{a_k + b_k}{2} \in (a_k, b_k) \\ I_k = [a_k, b_k] = [m_{k-1}, b_{k-1}] \overset{m_1 \in (a_1, b_1)}{\subset} [a_{k-1}, b_{k-1}] = I_{k-1} \Rightarrow I_k \subsetneq I_{k-1} \\ |I_k| = |[a_k, b_k]| = |[m_{k-1}, b_{k-1}]| \geq |\mathbb{N}| > 0 = |\emptyset| \\ I_k = [a_k, b_k] = [a_k, m_k] \cup [m_k, b_k] \\ \quad |I_k| \quad = |[a_k, b_k]| = |[a_k, m_k]| + |[m_k, b_k]| - |[a_k, m_k] \cap [m_k, b_k]| \\ \qquad\qquad\qquad = |[a_k, m_k]| + |[m_k, b_k]| - |\{m_k\}| \\ \qquad\qquad\qquad = |[a_k, m_k]| + |[m_k, b_k]| - 1 \\ \|I_k\| = |b_k - a_k| = |b_{k-1} - m_{k-1}| = b_{k-1} - m_{k-1} = b_{k-1} - \dfrac{a_{k-1} + b_{k-1}}{2} = \dfrac{b_{k-1} - a_{k-1}}{2} = \dfrac{\|I_1\|}{2^2} = \dfrac{b-a}{2^2} \\ \text{任取 } a_{n_k} = \xi_k \in I_k \text{ since } |I_k| = |[a_k, b_k]| = |[m_{k-1}, b_{k-1}]| \geq |\mathbb{N}| > 0 = |\emptyset| \end{cases}$$

case 2-k-3: $\begin{cases} |[a_{k-1}, m_{k-1}]| \geq |\mathbb{N}| \\ |[m_{k-1}, b_{k-1}]| \geq |\mathbb{N}| \end{cases}$, choose one of case 2-k-1 or case 2-k-2

⋮

依此類推, 以致無窮 and so on to infinity,
by the above construction,

$$\begin{cases} \forall k \in \mathbb{N} \{I_k = [a_k, b_k], |I_k| \geq |\mathbb{N}| > 0 = |\emptyset|\} \\ \forall k \in \mathbb{N} \{I_{k+1} \subsetneq I_k\} \\ \forall k \in \mathbb{N} \left\{ \|I_k\| = \dfrac{\|I_1\|}{2^{k-1}} \right\} \Leftrightarrow \forall k \in \mathbb{N} \left\{ \|I_{k+1}\| = \dfrac{\|I_1\|}{2^k} \right\} \\ \forall k \in \mathbb{N} \left\{ \text{任取 } a_{n_k} = \xi_k \in I_k \text{ since } |I_k| = |[a_k, b_k]| = \begin{smallmatrix} |[a_{k-1}, m_{k-1}]| \\ \vee \\ |[m_{k-1}, b_{k-1}]| \end{smallmatrix} \geq |\mathbb{N}| > 0 = |\emptyset| \right\} \end{cases}$$

by 區間套定理 / 區間套原理 nested interval theorem = NIT / nested interval principle = NIP 3.3.4,

$$\begin{cases} \forall k \in \mathbb{N} \left(\begin{cases} I_k = [a_k, b_k] \subset \mathbb{R} \\ I_{k+1} \subseteq I_k \Leftrightarrow I_k \supseteq I_{k+1} \end{cases} \right) \\ \lim\limits_{k\to\infty} \|I_k\| = \lim\limits_{k\to\infty} (b_k - a_k) = \lim\limits_{k\to\infty} \dfrac{\|I_1\|}{2^{k-1}} = \lim\limits_{k\to\infty} \dfrac{b-a}{2^{k-1}} = 0 \end{cases} \Rightarrow \exists x_0 \in \mathbb{R} \left(\bigcap_{k=1}^{\infty} I_k = \{x_0\} \right)$$

$$\bigcap_{k=1}^{\infty} I_k = \{x_0\} \Rightarrow \{x_0\} \subseteq \cdots \subseteq I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_2 \subseteq I_1 = [a, b] = S \subset \mathbb{R}$$

$$|a_{n_k} - x_0| = |x_0 - a_{n_k}| \overset{x_0, a_{n_k} \in I_k = [a_k, b_k]}{<} |b_k - a_k| = \|I_k\| = \frac{\|I_1\|}{2^{k-1}} = \frac{b-a}{2^{k-1}}$$

$$|a_{n_k} - x_0| < \frac{b-a}{2^{k-1}}$$

$$\begin{cases} k \to \infty \\ |a_{n_k} - x_0| < \dfrac{b-a}{2^{k-1}} \end{cases} \Rightarrow |a_{n_k} - x_0| \to 0$$

$$\Downarrow$$

$$\lim_{k\to\infty} a_{n_k} = x_0, \wedge x_0 \in \cdots \subseteq I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_{k-1} \subseteq I_1 = [a, b] = S \subset \mathbb{R}$$

from case 1 3.3.41 and case 2 3.3.42,

$$\begin{cases} \exists \langle a_{n_k} \rangle = \langle x_0 \rangle = \langle x_0 \rangle_{k\in\mathbb{N}} \text{ is a subsequence of } \langle a_n \rangle_{n\in\mathbb{N}} \left(\lim\limits_{k\to\infty} a_{n_k} = \lim\limits_{k\to\infty} x_0 = x_0 \in \mathbb{R} \right) & \text{case 1 : } S \text{ is finite} \\ \exists \langle a_{n_k} \rangle = \langle \xi_k \rangle_{k\in\mathbb{N}} \text{ is a subsequence of } \langle a_n \rangle_{n\in\mathbb{N}} \left(\lim\limits_{k\to\infty} a_{n_k} = x_0 \in \mathbb{R} \right) & \text{case 2 : } S \text{ is infinite} \end{cases}$$

$$\Rightarrow \exists \langle a_{n_k} \rangle_{k\in\mathbb{N}} \text{ is a subsequence of } \langle a_n \rangle_{n\in\mathbb{N}} \left(\lim_{k\to\infty} a_{n_k} = x_0 \in \mathbb{R} \right)$$

□

引理 3.3.8. 連續函數 有界定理 *continuous function bounded theorem = CFBT / boundedness theorem*

$$\begin{cases} f:[a,b]\to\mathbb{R} & (1) \\ f \text{ continuous on } [a,b] & (2) \end{cases} \Rightarrow f \text{ is bounded}$$
$$\Leftrightarrow \exists\beta>0,\forall x\in[a,b]\,(|f(x)|\le\beta)$$

Proof. if a function is well-defined 1.2.17,

$$f:X\to Y\Leftrightarrow f\in Y^X\Leftrightarrow\forall x\in X,\exists!y\in Y\,(y=f(x))$$

by premise (1),

$$f:[a,b]\to\mathbb{R}\Leftrightarrow f\in\mathbb{R}^{[a,b]}\Leftrightarrow\forall x\in[a,b],\exists!y\in\mathbb{R}\,(y=f(x))$$

反證法 proof by contradiction,

$$f \text{ is bounded}\Leftrightarrow\exists\beta>0,\forall x\in[a,b]\,(|f(x)|\le\beta)$$

$$\begin{aligned} \neg f \text{ is bounded} &\Leftrightarrow\neg\exists\beta>0,\forall x\in[a,b]\,(|f(x)|\le\beta) && (3.3.43)\\ &\Leftrightarrow\forall\beta>0,\exists x\in[a,b]\,(|f(x)|>\beta)\\ &\overset{\forall n\in\mathbb{N}(n>0)}{\Rightarrow}\forall n\in\mathbb{N},\exists x_n\in[a,b]\,(|f(x_n)|>n)\\ &\Rightarrow\forall n_k\in\mathbb{N},\exists x_{n_k}\in[a,b]\,(|f(x_{n_k})|>n_k) && (3.3.44)\end{aligned}$$

已構造出一個子數列的子數列函數性質.

$$x_n\in[a,b]\Rightarrow S=\{x_n\}_{n\in\mathbb{N}}=\{x_n\}_{n=1}^{\infty}=\left\{x\middle|\begin{cases}n\in\mathbb{N}\\ x=x_n\end{cases}\right\}\subset[a,b]\subset\mathbb{R}$$

by B-W 定理 / 聚點原理 Bolzano-Weierstrass theorem = B-WT / accumulative point principle 波爾查諾-魏爾史特拉斯定理 3.3.7,

$$\begin{cases}\exists\langle a_n\rangle_{n\in\mathbb{N}}=\langle a_n\rangle_{n=1}^{\infty}\in\mathbb{R}^{|\mathbb{N}|}\\ \exists\beta>0,\forall n\in\mathbb{N}\,(|a_n|\le\beta)\\ S=\{a_n\}_{n\in\mathbb{N}}=\{a_n\}_{n=1}^{\infty}\subset\mathbb{R}\end{cases}\Rightarrow\exists\langle a_{n_k}\rangle_{k\in\mathbb{N}}=\langle a_{n_k}\rangle_{k=1}^{\infty}\text{ is a subsequence of }\langle a_n\rangle_{n\in\mathbb{N}}\left(\lim_{k\to\infty}a_{n_k}=x_0\in\mathbb{R}\right)$$

$$\begin{cases}\exists\langle x_n\rangle_{n\in\mathbb{N}}=\langle x_n\rangle_{n=1}^{\infty}\in\mathbb{R}^{|\mathbb{N}|}\\ \exists\beta=\max\{|a|,|b|\}>0,\forall n\in\mathbb{N}\,(|x_n|\le\beta)\\ S=\{x_n\}_{n\in\mathbb{N}}=\{x_n\}_{n=1}^{\infty}\subset\mathbb{R}\end{cases}\Rightarrow\exists\langle x_{n_k}\rangle_{k\in\mathbb{N}}=\langle x_{n_k}\rangle_{k=1}^{\infty}\text{ is a subsequence of }\langle x_n\rangle_{n\in\mathbb{N}}\left(\lim_{k\to\infty}x_{n_k}=x_0\in\mathbb{R}\right)\quad(3.3.45)$$

$$x_0\in\cdots\subseteq I_{k+1}\subseteq I_k\subseteq\cdots\subseteq I_2\subseteq I_1=[a,b]=S\subset\mathbb{R}$$

$$x_0\in[a,b]$$

by premise (2),

$$f \text{ continuous on } [a,b]\overset{x_0\in[a,b]}{\Rightarrow}f\text{ continuous at }x_0$$
$$\Leftrightarrow\lim_{x\to x_0}f(x)=f\left(\lim_{x\to x_0}x\right)=f(x_0)\in\mathbb{R}$$

$$\mathbb{R}\overset{f\text{ is a well-defined function 1.2.17}}{\ni}f(x_0)\overset{\lim_{k\to\infty}x_{n_k}=x_0}{\underset{3.3.45}{=}}f\left(\lim_{k\to\infty}x_{n_k}\right)\overset{(2)}{=}\lim_{k\to\infty}f(x_{n_k})\begin{cases}>\lim_{k\to\infty}n_k=\infty\\ <\lim_{k\to\infty}-n_k=-\infty\end{cases}\quad 3.3.44$$

$$f(x_0)\in\mathbb{R}\Rightarrow\Leftarrow f(x_0)\text{ diverges}\begin{cases}>\lim_{k\to\infty}n_k=\infty\\ <\lim_{k\to\infty}-n_k=-\infty\end{cases}$$

構造出一個子數列的子數列函數性質取極限後與連續性及良好定義函數性質矛盾
thus

$$\neg(3.3.43)\Leftrightarrow\neg(\neg f\text{ is bounded})\Leftrightarrow f\text{ is bounded}$$

□

定理 3.3.9. 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 *continuous function extreme value theorem = CFEVT / extreme value theorem = EVT*

$$\begin{cases} f:[a,b]\to\mathbb{R} & (1) \\ f \text{ continuous on } [a,b] & (2) \end{cases} \Rightarrow \exists x_m, x_M \in [a,b], \forall y = f(x) \in f([a,b]) \{m = f(x_m) \le y = f(x) \le f(x_M) = M\}$$

Proof. 閉區間才必成立

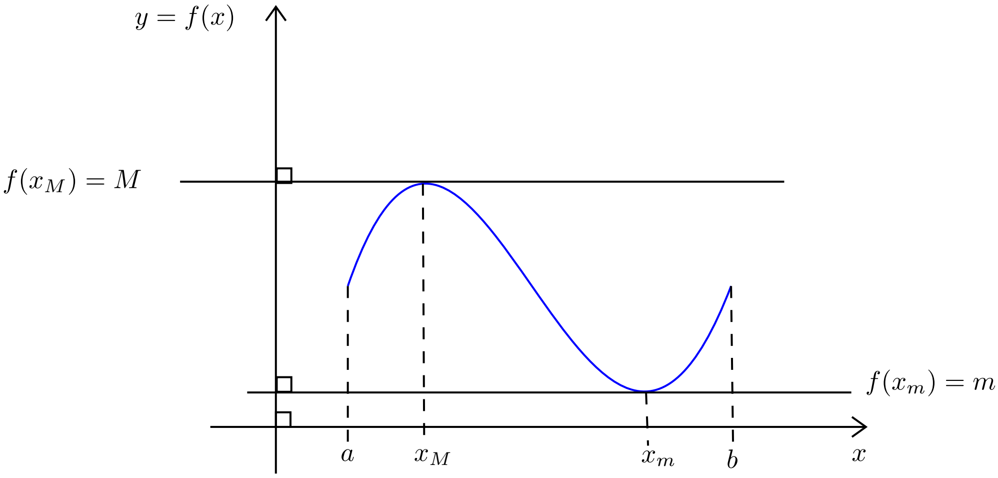


Figure 3.3.5: continuous function extreme value theorem = CFEVT / extreme value theorem = EVT

$$f([a,b]) = \left\{ y \middle| \begin{cases} x \in [a,b] \\ f:[a,b]\to\mathbb{R} \\ y = f(x) \end{cases} \right\} \subseteq \mathbb{R}$$

先證極大值:
by 連續函數 有界定理 continuous function bounded theorem = CFBT / boundedness theorem 3.3.8,

$$\begin{cases} f:[a,b]\to\mathbb{R} & (1) \\ f \text{ continuous on } [a,b] & (2) \end{cases} \overset{3.3.8}{\Rightarrow} f \text{ is bounded}$$
$$\Leftrightarrow \exists \beta > 0, \forall x \in [a,b] (|f(x)| \le \beta)$$
$$\Leftrightarrow \exists \beta > 0, \forall y = f(x) \in f([a,b]) (y \le \beta)$$

by 實數 上確性 / 上確界公理 axiom of least upper bound / least-upper-bound property = LUB property = LUBP / Dedekind completeness 3.1.39,

$$\forall S \left[\begin{cases} \emptyset \ne S \subset \mathbb{R} & \text{非空} \\ \exists \beta \in \mathbb{R}, \forall a \in S (a \le \beta) & \text{上界} \end{cases} \Rightarrow \exists \alpha \in \mathbb{R}, \forall a \in S \left(\begin{cases} a \le \alpha & \text{上界} \\ \forall \xi \in \mathbb{R} (\forall a \in S (a \le \xi) \Rightarrow \alpha \le \xi) & \text{最小} \end{cases} \right) \right]$$

$$\forall S \left[\begin{cases} \emptyset \ne S \subset \mathbb{R} & \text{非空} \\ \exists \beta \in \mathbb{R}, \forall a \in S (a \le \beta) & \text{上界} \end{cases} \Rightarrow \exists \alpha \in \mathbb{R} (\alpha \text{ 是 } S \text{ 的最小上界}) \right]$$

$$\forall S \subset \mathbb{R} \left[\begin{cases} \emptyset \ne S & \text{非空} \\ \exists \beta \in \mathbb{R}, \forall a \in S (a \le \beta) & \text{上界} \end{cases} \Rightarrow \exists \alpha \in \mathbb{R} (\alpha \text{ 是 } S \text{ 的最小上界}) \right]$$

$$\begin{cases}\emptyset = f([a,b]) \subseteq \mathbb{R} \underset{\text{上界}}{\Rightarrow} \emptyset = f([a,b]) \subset \mathbb{R} & \text{非空}\\ \exists \beta > 0, \forall y \in f([a,b])\,(y \le \beta) & \text{上界}\end{cases}$$

$$\overset{3.1.39}{\Rightarrow} \exists M \in \mathbb{R}, \forall y = f(x) \in f([a,b]) \left(\begin{cases} y = f(x) \le M & \text{上界} \\ \forall U \in \mathbb{R}\,(\forall y \in f([a,b])\,(y \le U) \Rightarrow M \le U) & \text{最小}\end{cases}\right) \tag{3.3.46}$$

$$\Leftrightarrow \exists M \in \mathbb{R}\,(M \text{ 是 } f([a,b]) \text{ 的最小上界})$$

$$\Rightarrow \exists M \in \mathbb{R}, \forall y_n = f(x_n) \in f([a,b]) \left(\begin{cases} y_n = f(x_n) \le M & \text{上界} \\ \forall U \in \mathbb{R}\,(\forall y_n \in f([a,b])\,(y_n \le U) \Rightarrow M \le U) & \text{最小}\end{cases}\right), n \in \mathbb{N}$$

$$\Rightarrow \exists M \in \mathbb{R}, \forall n \in \mathbb{N}, \forall y_n = f(x_n) \in f([a,b]) \left(\begin{cases} y_n = f(x_n) \le M & \text{上界} \\ \forall U \in \mathbb{R}\,(\forall y_n \in f([a,b])\,(y_n \le U) \Rightarrow M \le U) & \text{最小}\end{cases}\right) \tag{3.3.47}$$

$$\begin{aligned} M \text{ 是 } f([a,b]) \text{ 的最小上界} &\overset{\forall n \in \mathbb{N}\left(M - \frac{1}{n} < M\right)}{\Rightarrow} M - \frac{1}{n} \text{ 不是 } f([a,b]) \text{ 的最小上界}, n \in \mathbb{N} \\ &\Rightarrow \exists y = f(x) \in f([a,b]) \left(y > M - \frac{1}{n}\right) \\ &\Rightarrow \exists y = f(x) \in f([a,b]) \left(y \ge M - \frac{1}{n}\right) \\ &\Rightarrow \forall n \in \mathbb{N}, \exists y_n = f(x_n) \in f([a,b]) \left(y_n = f(x_n) \ge M - \frac{1}{n}\right) \end{aligned} \tag{3.3.48}$$

$$\begin{cases} \exists M \in \mathbb{R}, \forall n \in \mathbb{N}, \forall y_n = f(x_n) \in f([a,b])\,(y_n = f(x_n) \le M) & 3.3.47 \\ \forall n \in \mathbb{N}, \exists y_n = f(x_n) \in f([a,b]) \left(y_n = f(x_n) \ge M - \frac{1}{n}\right) & 3.3.48 \end{cases}$$

$$\begin{aligned} &\Rightarrow \forall n \in \mathbb{N}, \exists y_n = f(x_n) \in f([a,b]) \left(\begin{cases} y_n = f(x_n) \le M \\ y_n = f(x_n) \ge M - \frac{1}{n} \end{cases}\right) \\ &\Rightarrow \forall n \in \mathbb{N}, \exists y_n = f(x_n) \in f([a,b]) \left(M - \frac{1}{n} \le y_n = f(x_n) \le M\right) \\ &\Rightarrow \forall k \in \mathbb{N}, \forall n_k \in \mathbb{N}, \exists y_{n_k} = f(x_{n_k}) \in f([a,b]) \left(M - \frac{1}{n_k} \le y_{n_k} = f(x_{n_k}) \le M\right) \end{aligned} \tag{3.3.49}$$

by B-W 定理 / 聚點原理 Bolzano-Weierstrass theorem = B-WT / accumulative point principle 波爾查諾-魏爾史特拉斯定理 3.3.7,

任意有界數列必含收斂子數列 any bounded (infinite) sequence has a convergent subsequence

$$\begin{cases} \exists \langle a_n \rangle_{n \in \mathbb{N}} = \langle a_n \rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} \\ \exists \beta > 0, \forall n \in \mathbb{N}\,(|a_n| \le \beta) \\ S = \{a_n\}_{n \in \mathbb{N}} = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \end{cases} \Rightarrow \exists \langle a_{n_k} \rangle_{k \in \mathbb{N}} = \langle a_{n_k} \rangle_{k=1}^{\infty} \text{ is a subsequence of } \langle a_n \rangle_{n \in \mathbb{N}} \left(\lim_{k \to \infty} a_{n_k} = x_0 \in \mathbb{R}\right)$$

$$\begin{cases} \exists \langle x_n \rangle_{n \in \mathbb{N}} = \langle x_n \rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} \\ \exists \beta = \max\{|a|, |b|\} > 0, \forall n \in \mathbb{N}\,(|x_n| \le \beta) \\ X = \{x_n\}_{n \in \mathbb{N}} = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \end{cases} \tag{3.3.50}$$

$$\Rightarrow \exists \langle x_{n_k} \rangle_{k \in \mathbb{N}} = \langle x_{n_k} \rangle_{k=1}^{\infty} \text{ is a subsequence of } \langle x_n \rangle_{n \in \mathbb{N}} \left(\lim_{k \to \infty} x_{n_k} = x_M \in \mathbb{R}\right) \tag{3.3.51}$$

$$\begin{cases} \exists \langle y_n \rangle_{n \in \mathbb{N}} = \langle y_n \rangle_{n=1}^{\infty} = \langle f(x_n) \rangle_{n \in \mathbb{N}} = \langle f(x_n) \rangle_{n=1}^{\infty} \in \mathbb{R}^{|\mathbb{N}|} \\ \exists B = \max\{|m|, |M|\} > 0, \forall n \in \mathbb{N}\,(|y_n| \le B) \\ Y = \{y_n\}_{n \in \mathbb{N}} = \{y_n\}_{n=1}^{\infty} \subset \mathbb{R} \end{cases}$$

$$\Rightarrow \exists \langle y_{n_k} \rangle_{k \in \mathbb{N}} = \langle y_{n_k} \rangle_{k=1}^{\infty} = \langle f(x_{n_k}) \rangle_{k \in \mathbb{N}} = \langle f(x_{n_k}) \rangle_{k=1}^{\infty} \text{ is a subsequence of } \langle y_n \rangle_{n \in \mathbb{N}} \left(\lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} f(x_{n_k}) = y_M \in \mathbb{R}\right) \tag{3.3.52}$$

$$\begin{cases} y_M = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \overset{(1)}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) & 3.3.52 \\ M - \dfrac{1}{n_k} \le y_{n_k} = f(x_{n_k}) \le M & 3.3.49 \end{cases}$$
$$\Rightarrow \begin{cases} y_M = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \overset{(1)}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) \\ M = \lim_{k\to\infty}\left(M - \dfrac{1}{n_k}\right) \le \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \le \lim_{k\to\infty} M = M \end{cases}$$
$$\Rightarrow \begin{cases} y_M = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \overset{(1)}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) \overset{3.3.50}{=} f(x_M) \\ M = \lim_{k\to\infty}\left(M - \dfrac{1}{n_k}\right) \le y_M \overset{3.3.52}{=} \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \le \lim_{k\to\infty} M = M \end{cases}$$
$$\Rightarrow \begin{cases} y_M = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \overset{(1)}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) \overset{3.3.50}{=} f(x_M) \\ M \le y_M \le M \Rightarrow y_M = M \end{cases}$$
$$\Rightarrow \exists x_M \in [a,b]\,(y_M = f(x_M) = M)$$

[14]

$$\exists x_M \in [a,b]\,(y_M = f(x_M) = M, \forall y = f(x) \in f([a,b])\,(y = f(x) \le M)) \tag{3.3.53}$$

再證極小值:
同理, by 連續函數 有界定理 continuous function bounded theorem = CFBT / boundedness theorem 3.3.8,

$$\begin{cases} -f : [a,b] \to \mathbb{R} & (1) \\ -f \text{ continuous on } [a,b] & (2) \end{cases} \overset{3.3.8}{\Rightarrow} -f \text{ is bounded}$$
$$\Leftrightarrow \exists \beta > 0, \forall x \in [a,b]\,(|-f(x)| \le \beta)$$
$$\Leftrightarrow \exists \beta > 0, \forall y = -f(x) \in -f([a,b])\,(y \le \beta)$$

⋮

by 實數 上確性 / 上確界公理 axiom of least upper bound / least-upper-bound property = LUB property = LUBP / Dedekind completeness 3.1.39,

⋮

by B-W 定理 / 聚點原理 Bolzano-Weierstrass theorem = B-WT / accumulative point principle 波爾查諾-魏爾史特拉斯定理 3.3.7,
任意有界數列必含收斂子數列 any bounded (infinite) sequence has a convergent subsequence

⋮

$$\exists x_m \in [a,b]\,(y_m = f(x_m) = m, \forall y = f(x) \in f([a,b])\,(y = f(x) \ge m)) \tag{3.3.54}$$

$$\begin{cases} \exists x_M \in [a,b]\,(y_M = f(x_M) = M, \forall y = f(x) \in f([a,b])\,(y = f(x) \le M)) & 3.3.53 \\ \exists x_m \in [a,b]\,(y_m = f(x_m) = m, \forall y = f(x) \in f([a,b])\,(y = f(x) \ge m)) & 3.3.54 \end{cases}$$
$$\Rightarrow \exists x_m, x_M \in [a,b], \forall y = f(x) \in f([a,b])\,\{m = f(x_m) \le y = f(x) \le f(x_M) = M\}$$

□

Proof. 開區間不成立 or 不必成立?
a counter example

[14]開區間定義域可無此性, 例如極值出現在端點上, 自變數或從定義域出發的 x, e.g. $x = \frac{1}{n} \in (0,1), n \in (1,\infty)$ 可能收斂至定義域外 e.g. $\lim_{n\to\infty} x = \lim_{n\to\infty} \frac{1}{n} = 0 \notin (0,1)$

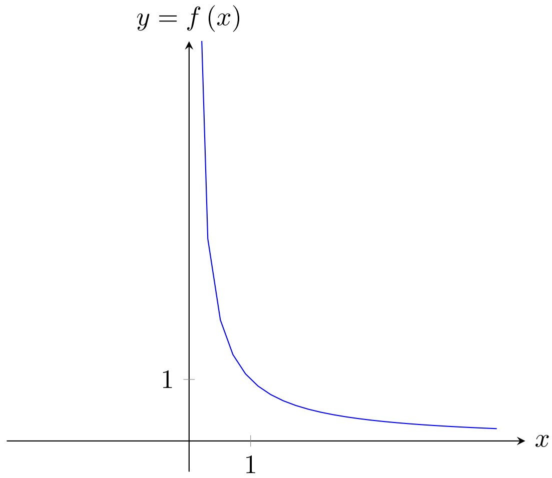


Figure 3.3.6: a counter example for open interval $f(x) = \dfrac{1}{x}$ differentiable on $(0,1)$

e.g.

$$
\begin{aligned}
&\begin{cases} f:(0,1)\to\mathbb{R} \\ f(x)=\dfrac{1}{x} \\ \hookrightarrow\Rightarrow f \text{ continuous on } (0,1) \end{cases} \\
\Rightarrow &\begin{cases} f \text{ is not (upper) bounded} & \text{無(上)界} \\ \forall x\in(a,b), \exists y_<, y_> \in f([a,b]) \left(\begin{cases} y_> = f(x_>) > f(x) \\ y_< = f(x_<) < f(x)\end{cases}\right) & \text{無極值} \end{cases} \\
\Leftrightarrow &\begin{cases} \nexists U\in\mathbb{R}, \forall y\in f([a,b])\{y\le U\} & \text{無界} \\ \neg\exists x_m, x_M\in[a,b], \forall y=f(x)\in f([a,b]) \left(\begin{cases} y=f(x)\le f(x_M)=M \\ m=f(x_m)\le y=f(x)\end{cases}\right) & \text{無極} \end{cases} \\
\Rightarrow \Leftarrow &\exists x_m, x_M\in[a,b], \forall y=f(x)\in f([a,b])\{m=f(x_m)\le y=f(x)\le f(x_M)=M\}
\end{aligned}
$$

就算函數值 $y = f(x)$ 有界或有極值, 也可能剛好落在定義域端點上, 而自變數或從定義域出發的 x, e.g. $x = \dfrac{1}{n} \in (0,1), n\in(1,\infty)$ 可能收斂至定義域外 e.g. $\lim_{n\to\infty} x = \lim_{n\to\infty}\dfrac{1}{n} = 0 \notin (0,1)$ □

定理 3.3.10. *H-B* 定理 *Heine-Borel theorem = H-BT*

定理 3.3.11. $\exists?\sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}} = ?$

定理 3.3.12.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \in \mathbb{R}$$

定理 3.3.13. *Taylor expansion or Maclaurin expansion of* e

定理 3.3.14. e 不是有比數 / e 是無比數 e *is irrational / a irrational number*

3.4 微積分基本定理 fundamental theorems of calculus

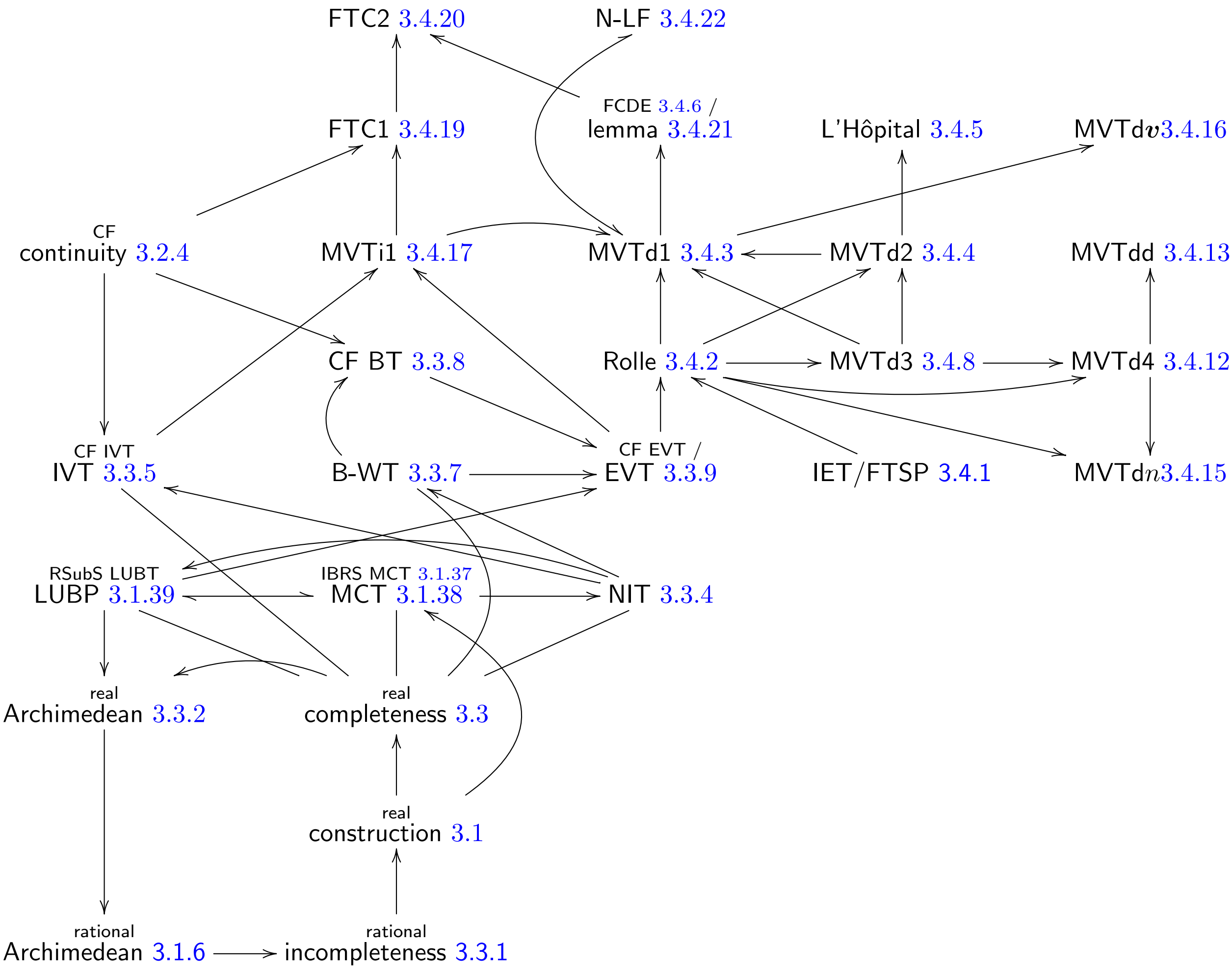


Figure 3.4.1: fundamental theorems of calculus

整理之，Rolle's 定理用到了連續函數的最大最小值定理，以及極值點導數為 0 兩特性。極值點導數為 0，可由導數的定義立即得之，因此，基本上並未再引述新的定理。

故到目前為止，微積分定理的證明推到了連續函數的兩個基本定理上：

$\boxed{\text{微積分基本定理}}$

$\Uparrow$

$\boxed{\text{均值定理}}$

$\Uparrow$

$\boxed{\begin{array}{c}\text{連續函數中間值定理}\\ \text{及最大最小值定理}\end{array}}$

再追問下去，連續函數為什麼具有上述兩個特性，這就要落到實數的完備了。

Figure 3.4.2: fundamental theorems of calculus 任脈

定理 3.4.1. 內點極值定理 / 費馬不動點定理 *Fermat theorem of stationary points / interior extremum theorem* =

IET / FTSP [15]

$$\begin{cases} x_0 \in (a,b) \\ f:[a,b]\to\mathbb{R} \\ f \text{ differentiable at } x_0 \\ \exists\delta>0, \forall x\in[a,b]\left\{0<|x-x_0|<\delta \Rightarrow \begin{matrix} f(x_0)\le f(x) \\ \veebar \\ f(x_0)\ge f(x)\end{matrix}\right\} \text{ local extremum} \end{cases} \Rightarrow f'(x_0)=0$$

[16]

Proof. $(\Rightarrow)$:
反證法 proof by contraposition and proof by contradiction

$$\begin{aligned} f \text{ differentiable at } x_0 &\Rightarrow \exists K\in\mathbb{R}\left(f'(x_0)=K\right) \\ &\Leftrightarrow \exists K\in\mathbb{R}\left(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=K\right)\end{aligned}$$

$$\begin{aligned} &\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=K \\ \overset{3.2.2}{\Leftrightarrow}&\forall\epsilon>0,\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h-0|<\delta\Rightarrow\left|\frac{f(x_0+h)-f(x_0)}{h}-K\right|<\epsilon\right) \\ \Leftrightarrow&\forall\epsilon>0,\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\left|\frac{f(x_0+h)-f(x_0)}{h}-K\right|<\epsilon\right) \\ \Leftrightarrow&\forall\epsilon>0,\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow -\epsilon<\frac{f(x_0+h)-f(x_0)}{h}-K<\epsilon\right) \\ \Leftrightarrow&\forall\epsilon>0,\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\epsilon<\frac{f(x_0+h)-f(x_0)}{h}<K+\epsilon\right) \end{aligned} \tag{3.4.1}$$

$$\begin{aligned} &\forall\epsilon>0,\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\epsilon<\frac{f(x_0+h)-f(x_0)}{h}<K+\epsilon\right) \\ \Leftrightarrow&\forall\epsilon>0,\exists\delta>0,\forall h\in[a-x_0,b-x_0]-\{0\}\left(|h|<\delta\Rightarrow K-\epsilon<\frac{f(x_0+h)-f(x_0)}{h}<K+\epsilon\right)\end{aligned}$$

proof by contraposition, i.e. we want to prove

$$\begin{cases} x_0\in(a,b) \\ f:[a,b]\to\mathbb{R} \\ f'(x_0)\ne 0\end{cases} \overset{?}{\Rightarrow} \neg\exists\delta>0,\forall x\in[a,b]\left\{0<|x-x_0|<\delta\Rightarrow \begin{matrix} f(x_0)\le f(x) \\ \veebar \\ f(x_0)\ge f(x)\end{matrix}\right\}$$

since

$$K=f'(x_0)\ne 0 \tag{3.4.2}$$

let

$$K>0 \tag{3.4.3}$$

let $\epsilon=\dfrac{K}{2}$, since $\epsilon=\dfrac{K}{2}\overset{K>0}{>}0$ and $\forall\epsilon>0$,

$$\begin{aligned} &\overset{3.4.1}{}\ \exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\epsilon<\frac{f(x_0+h)-f(x_0)}{h}<K+\epsilon\right) \\ \overset{\epsilon=\frac{K}{2}}{\Rightarrow}&\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\frac{K}{2}<\frac{f(x_0+h)-f(x_0)}{h}<K+\frac{K}{2}\right) \\ \Leftrightarrow&\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow \frac{K}{2}<\frac{f(x_0+h)-f(x_0)}{h}<\frac{3K}{2}\right) \\ \Rightarrow&\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow \frac{f(x_0+h)-f(x_0)}{h}>\frac{K}{2}\right)\end{aligned} \tag{3.4.4}$$

[15] 數學傳播_33_2_蔡聰明_均值定理的統合與推廣
[16] $\veebar$ 是 互斥或 代表 局部極大或極小只能擇一

case $(K>0)$-1: $h>0$

$$\begin{aligned}
&\overset{3.4.4}{}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{f(x_0+h)-f(x_0)}{h}>\frac{K}{2}\right)\\
&\overset{\Delta x>0}{\Rightarrow}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)>\frac{K}{2}h\underset{h>0}{\overset{K>0}{>}}0\right)\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)>0\}\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)>f(x_0)\}\\
&\Rightarrow\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)>f(x_0)\}
\end{aligned}\tag{3.4.5}$$

case $(K>0)$-2: $h<0$

$$\begin{aligned}
&\overset{3.4.4}{}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{f(x_0+h)-f(x_0)}{h}>\frac{K}{2}\right)\\
&\overset{\Delta x<0}{\Rightarrow}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)<\frac{K}{2}h\underset{h<0}{\overset{K>0}{<}}0\right)\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)<0\}\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)<f(x_0)\}\\
&\Rightarrow\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)<f(x_0)\}
\end{aligned}\tag{3.4.6}$$

$$\begin{aligned}
&\begin{cases}\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)>f(x_0)\} & 3.4.5\\ \exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)<f(x_0)\} & 3.4.6\end{cases}\\
&\Rightarrow\Leftarrow\exists\delta>0,\forall x\in[a,b]\left\{0<|x-x_0|<\delta\Rightarrow\begin{matrix}f(x_0)\le f(x)\\ \veebar\\ f(x_0)\ge f(x)\end{matrix}\right\}
\end{aligned}\tag{3.4.7}$$

thus

$$\begin{cases}\neg K>0 \quad \Leftarrow\neg\text{assumption}\Leftarrow\begin{cases}3.4.3 & \text{assumption}\\ 3.4.7 & \text{contradiction}\end{cases}\\ K\ne 0 \quad 3.4.2\end{cases}\overset{\text{三一律}}{\Rightarrow}K<0$$

let

$$K<0\tag{3.4.8}$$

let $\epsilon=\frac{-K}{2}$, since $\epsilon=\frac{-K}{2}\overset{K<0}{>}0$ and $\forall\epsilon>0$,

$$\begin{aligned}
&\overset{3.4.1}{}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\epsilon<\frac{f(x_0+h)-f(x_0)}{h}<K+\epsilon\right)\\
&\overset{\epsilon=\frac{-K}{2}}{\Rightarrow}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow K-\frac{-K}{2}<\frac{f(x_0+h)-f(x_0)}{h}<K+\frac{-K}{2}\right)\\
&\Leftrightarrow\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{3K}{2}<\frac{f(x_0+h)-f(x_0)}{h}<\frac{K}{2}\right)\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{f(x_0+h)-f(x_0)}{h}<\frac{K}{2}\right)
\end{aligned}\tag{3.4.9}$$

case $(K<0)$-1: $h>0$

$$\begin{aligned}
&\overset{3.4.9}{}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{f(x_0+h)-f(x_0)}{h}<\frac{K}{2}\right)\\
&\overset{\Delta x>0}{\Rightarrow}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)<\frac{K}{2}h\underset{h>0}{\overset{K<0}{<}}0\right)\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)<0\}\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)<f(x_0)\}\\
&\Rightarrow\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)<f(x_0)\}
\end{aligned}\tag{3.4.10}$$

case $(K<0)$-2: $h<0$

$$
\begin{aligned}
&3.4.9\ \exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow\frac{f(x_0+h)-f(x_0)}{h}<\frac{K}{2}\right)\\
&\overset{\Delta x<0}{\Rightarrow}\exists\delta>0,\forall x_0+h\in[a,b]\left(0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)>\frac{K}{2}h\underset{h<0}{\overset{K<0}{>}}0\right)\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)-f(x_0)>0\}\\
&\Rightarrow\exists\delta>0,\forall x_0+h\in[a,b]\{0<|h|<\delta\Rightarrow f(x_0+h)>f(x_0)\}\\
&\Rightarrow\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)>f(x_0)\}
\end{aligned}
\tag{3.4.11}
$$

$$
\begin{aligned}
&\begin{cases}\exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)<f(x_0)\} & 3.4.10\\ \exists\delta>0,\forall x\in[a,b]\{0<|x-x_0|<\delta\Rightarrow f(x)>f(x_0)\} & 3.4.11\end{cases}\\
&\Rightarrow\Leftarrow\exists\delta>0,\forall x\in[a,b]\left\{0<|x-x_0|<\delta\Rightarrow\begin{matrix}f(x_0)\le f(x)\\ \veebar\\ f(x_0)\ge f(x)\end{matrix}\right\}
\end{aligned}
\tag{3.4.12}
$$

thus

$$
\begin{cases}x_0\in(a,b)\\ f:[a,b]\to\mathbb{R}\\ K=f'(x_0)\ne0\end{cases}\overset{三一律}{\Rightarrow}\begin{matrix}K>0\ 3.4.3\\ \vee\\ K<0\ 3.4.8\end{matrix}\Rightarrow\neg\exists\delta>0,\forall x\in[a,b]\left\{0<|x-x_0|<\delta\Rightarrow\begin{matrix}f(x_0)\le f(x)\\ \veebar\\ f(x_0)\ge f(x)\end{matrix}\right\}
$$

or i.e.

$$
\begin{cases}3.4.2 & \text{assumption}\\ \begin{cases}3.4.7\\ 3.4.12\end{cases} & \text{contradiction}\end{cases}\Rightarrow\neg\text{assumption}\Rightarrow\neg(K=f'(x_0)\ne0)\Rightarrow K=f'(x_0)=0
$$

□

Proof. by case discussion

$$
\begin{cases}x_0\in(a,b)\\ f:[a,b]\to\mathbb{R}\\ \exists\delta>0,\forall x\in[a,b]\left\{|x-x_0|<\delta\Rightarrow\begin{matrix}f(x_0)\le f(x)\\ \veebar\\ f(x_0)\ge f(x)\end{matrix}\right\}\\ f\text{ differentiable at }x_0\end{cases}\Rightarrow\begin{cases}f(x_0)\le f(x) & \text{case local minimum}\\ f(x_0)\ge f(x) & \text{case local maximum}\end{cases}
$$

case local minimum: $\exists\delta>0,\forall x\in[a,b]\{|x-x_0|<\delta\Rightarrow f(x_0)\le f(x)\}$

$$f(x_0)\le f(x)$$

$$f(x)-f(x_0)\ge0$$

case local minimum-right limit:

let $\begin{cases}x=x_0+h\\ h\in(0,\delta)\end{cases}\Rightarrow|x-x_0|=|(x_0+h)-x_0|=|h|<\delta,$

$$f(x_0+h)-f(x_0)=f(x)-f(x_0)\ge0$$

$$f(x_0+h)-f(x_0)\ge0$$

since $h>0$,

$$\frac{f(x_0+h)-f(x_0)}{h}\ge0$$

$$\lim_{h\to0^+}\frac{f(x_0+h)-f(x_0)}{h}\ge0\tag{3.4.13}$$

case local minimum-left limit:

let $\begin{cases} x = x_0 + h \\ h \in (-\delta, 0) \end{cases} \Rightarrow |x - x_0| = |(x_0 + h) - x_0| = |h| < \delta,$

$$f(x_0 + h) - f(x_0) = f(x) - f(x_0) \geq 0$$

$$f(x_0 + h) - f(x_0) \geq 0$$

since $h < 0$,

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \tag{3.4.14}$$

$$\begin{aligned}
&\begin{cases} \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0 & 3.4.13 \\ \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 & 3.4.14 \end{cases} \\
\Rightarrow & 0 \leq \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \Rightarrow 0 \leq f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \\
\Rightarrow & f'(x_0) = 0
\end{aligned} \tag{3.4.15}$$

case local maximum: $\exists \delta > 0, \forall x \in [a, b] \{|x - x_0| < \delta \Rightarrow f(x_0) \geq f(x)\}$,
proof similar to case local minimum,

$$\begin{aligned}
&\begin{cases} \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \\ \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \end{cases} \\
\Rightarrow & 0 \leq \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \Rightarrow 0 \leq f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \\
\Rightarrow & f'(x_0) = 0
\end{aligned} \tag{3.4.16}$$

$$\begin{cases} \text{case local minimum: } f'(x_0) = 0 & 3.4.15 \\ \text{case local maximum: } f'(x_0) = 0 & 3.4.16 \end{cases} \Rightarrow f'(x_0) = 0$$

$(\not\Leftarrow)$:

$$\begin{cases} x_0 \in (a, b) \\ f : [a, b] \to \mathbb{R} \\ f \text{ differentiable at } x_0 \\ f'(x_0) = 0 \end{cases} \not\Rightarrow \exists \delta > 0, \forall x \in [a, b] \left\{ 0 < |x - x_0| < \delta \Rightarrow \begin{matrix} f(x_0) \leq f(x) \\ \veebar \\ f(x_0) \geq f(x) \end{matrix} \right\}$$

a counter example

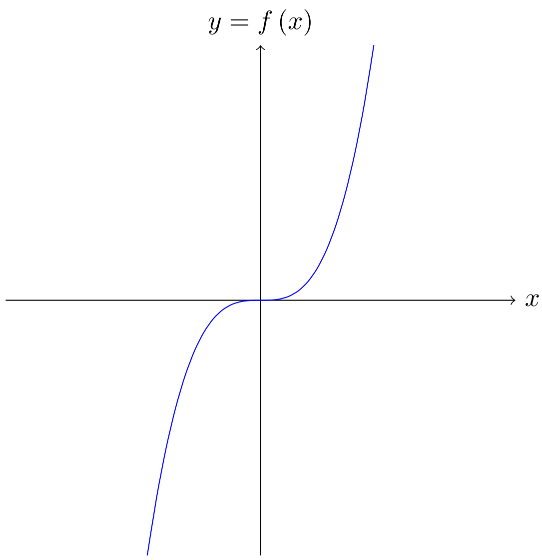


Figure 3.4.3: a counter example $f(x) = x^3$

$$\begin{cases} x_0 = 0 \in (-1,1) \\ f : [-1,1] \to \mathbb{R} \\ f(x) = x^3 \\ f \text{ differentiable at } x_0 = 0 \\ f'(0) = 3x^2|_{x=0} = 0 \end{cases} \Rightarrow \exists \delta > 0, \forall x \in [-1,1] \left\{ 0 < |x - 0| < \delta \Rightarrow \left\{ \begin{array}{c} f(0^-) \le f(x) \\ \wedge \\ f(0^+) \ge f(x) \end{array} \right\} \right.$$

$$\Rightarrow\!\!\not\Leftarrow \exists \delta > 0, \forall x \in [-1,1] \left\{ 0 < |x - x_0| < \delta \Rightarrow \left\{ \begin{array}{c} f(x_0) \le f(x) \\ \veebar \\ f(x_0) \ge f(x) \end{array} \right\} \right.$$

□

定理 3.4.2. *Rolle 定理 Rolle theorem*

$$\begin{cases} f : [a,b] \to \mathbb{R} & (0) \\ f \text{ continuous on } [a,b] & (1) \\ f \text{ differentiable on } (a,b) & (2) \\ f(a) = f(b) & (3) \end{cases} \Rightarrow \exists c \in (a,b) \left(f'(c) = 0\right)$$

$$\begin{cases} f : [a,b] \to \mathbb{R} & (0) \\ f \text{ differentiable on } [a,b] & \Rightarrow (1)\,\&\,(2) \Rightarrow \exists c \in (a,b) \left(f'(c) = 0\right) \\ f(a) = f(b) & (3) \end{cases}$$

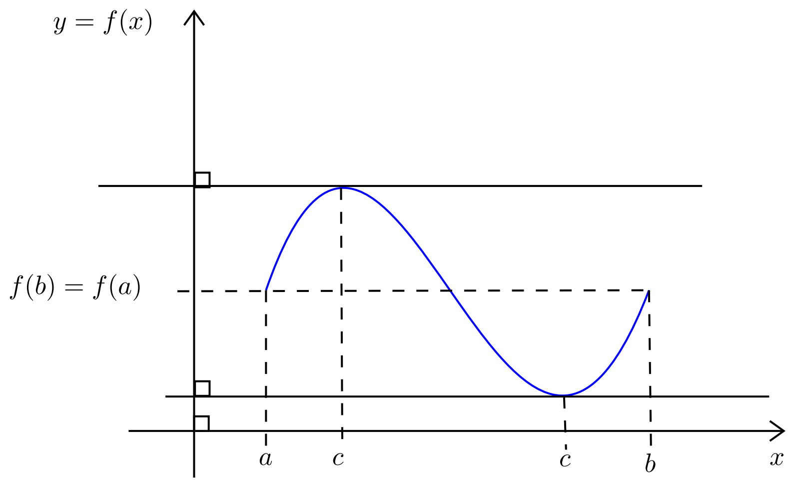


Figure 3.4.4: Rolle theorem

Proof. by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT 3.3.9,

$$f \text{ continuous on } [a,b] \Rightarrow \exists x_m, x_M \in [a,b], \forall x \in [a,b] \{m = f(x_m) \le f(x) \le f(x_M) = M\}$$

case 1: $x_m \in (a,b)$

by 內點極值定理 / 費馬不動點定理 Fermat theorem of stationary points / interior extremum theorem = IET / FTSP 3.4.1,

$$\forall x \in (a,b) \subset [a,b] \left(f(x_m) \le f(x)\right) \overset{3.4.1}{\Rightarrow} f'(x_m) = 0$$

$$\exists c = x_m \in (a,b) \left(f'(c) = 0\right)$$

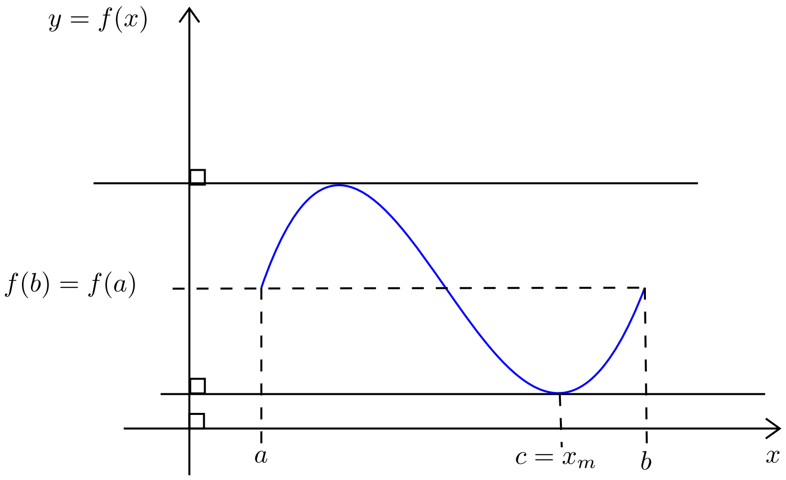


Figure 3.4.5: Rolle theorem case 1: $x_m \in (a, b)$

case 2: $x_M \in (a, b)$
by 內點極值定理 / 費馬不動點定理 Fermat theorem of stationary points / interior extremum theorem = IET / FTSP 3.4.1,

$$\forall x \in (a, b) \subset [a, b] \left(f(x) \leq f(x_M)\right) \overset{3.4.1}{\Rightarrow} f'(x_M) = 0$$

$$\exists c = x_M \in (a, b) \left(f'(c) = 0\right)$$

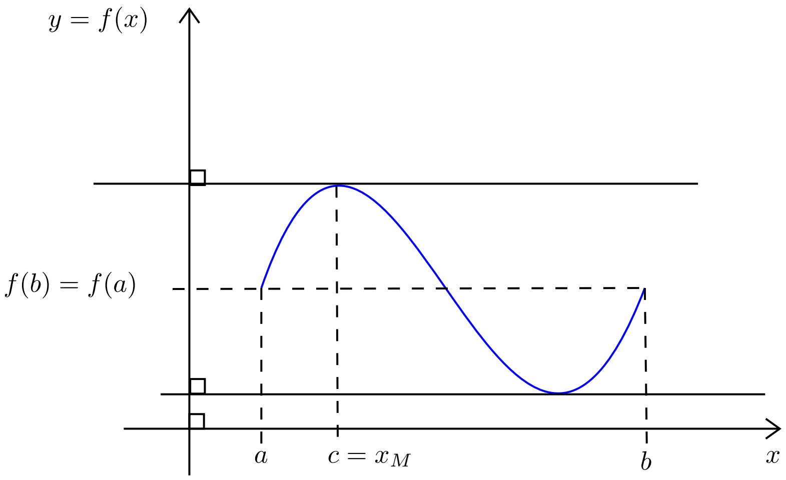


Figure 3.4.6: Rolle theorem case 2: $x_M \in (a, b)$

case 3: $\begin{cases} x_m \notin (a, b) \\ x_M \notin (a, b) \end{cases}$

$$\begin{cases} x_m, x_M \in [a, b] \\ x_m \notin (a, b) \\ x_M \notin (a, b) \end{cases} \Rightarrow \{x_m, x_M\} = \{a, b\} \tag{3.4.17}$$

but

$$f(a) = f(b) \tag{3.4.18}$$

$$\{f(a)\} = \{f(b)\} \overset{3.4.18}{=} \{f(a), f(b)\} = f(\{a, b\}) \overset{3.4.17}{=} f(\{x_m, x_M\}) = \{m, M\}$$

$$\begin{cases} f(a) = m \\ f(a) = M \end{cases} \Rightarrow f(x_m) = m = f(a) = f(b) = M = f(x_M)$$

$$f(x_m) = f(x_M)$$

$$\begin{cases} f(x_m) = f(x_M) \\ f(x_m) \le f(x) \le f(x_M) \end{cases} \begin{aligned} &\Rightarrow f(x) = C = f(x_m) = f(x_M) = \text{constant}\, \forall x \in [a,b] \\ &\Rightarrow f'(x) = 0\, \forall x \in [a,b] \\ &\Rightarrow \forall c \in (a,b)\,(f'(c) = 0) \end{aligned}$$

□

- 均值定理 mean value theorem = MVT
  - 微分均值定理 mean value theorem for derivatives = MVTd
    - 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1
    - 第二微分均值定理 / 柯西均值定理 second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2
    - 推廣三函數微分均值定理 generalized mean value theorem for derivatives of 3 functions = MVTd3
    - 推廣四函數微分均值定理 generalized mean value theorem for derivatives of 4 functions = MVTd4
      - 二階導數的均值公式 mean value theorem for second-order derivatives
      - 三階導數的均值公式 mean value theorem for third-order derivatives
    - 推廣 n 函數微分均值定理 generalized mean value theorem for derivatives of n functions = MVTdn
    - 向量或高維微分均值定理 mean value theorem for vector derivatives = MVTd$\boldsymbol{v}$
  - 積分均值定理 mean value theorem for definite integrals = MVTi
    - 第一積分均值定理 first mean value theorem for definite integrals = MVTi1
    - 第二積分均值定理 second mean value theorem for definite integrals = MVTi2
- 微積分基本定理 fundamental theorem of calculus
  - 第一微積分基本定理 first fundamental theorem of calculus = FTC1
  - 第二微積分基本定理 second fundamental theorem of calculus = FTC2

定理 3.4.3. 第一微分均值定理 / 拉格朗日均值定理 *first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1*

$$\begin{cases} f : [a,b] \to \mathbb{R}, a < b & (0) \\ f \textit{ continuous on } [a,b] & (1) \\ f \textit{ differentiable on } (a,b) & (2) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b) \left\{ f'(c) = \frac{f(b) - f(a)}{b - a} \right\}$$

$$\begin{cases} f : [a,b] \to \mathbb{R}, a < b & (0) \\ f \textit{ differentiable on } [a,b] & \Rightarrow (1) \,\&\, (2) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b) \left\{ f'(c) = \frac{f(b) - f(a)}{b - a} \right\}$$

或是在命題及證明中避免 $b - a$ 出現在分母 即可不必限制 $a < b$ 或 $a \neq b$,

$$\begin{cases} f : [a,b] \to \mathbb{R} & (0) \\ f \textit{ continuous on } [a,b] & (1) \\ f \textit{ differentiable on } (a,b) & (2) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b) \{ f'(c)(b - a) = f(b) - f(a) \} \tag{3.4.19}$$

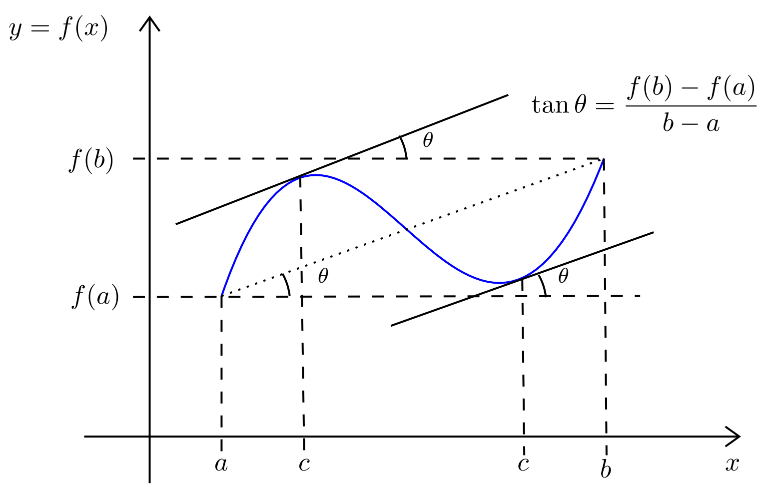


Figure 3.4.7: mean value theorem = MVT

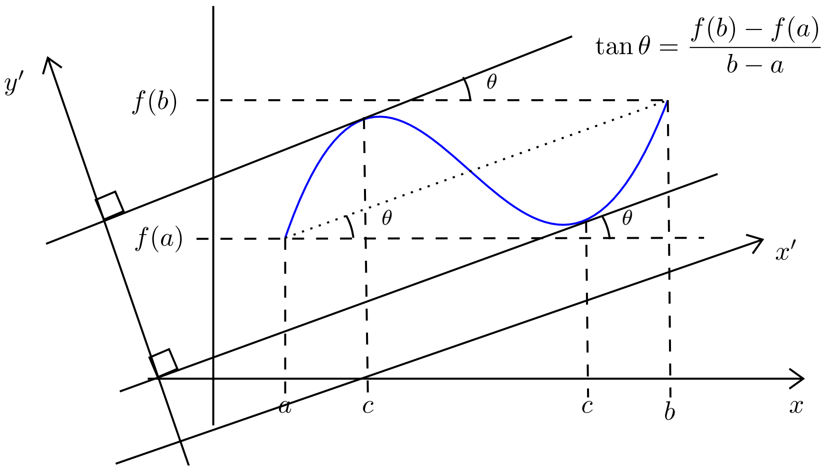


Figure 3.4.8: Rolle theorem directly to mean value theorem = MVT ?

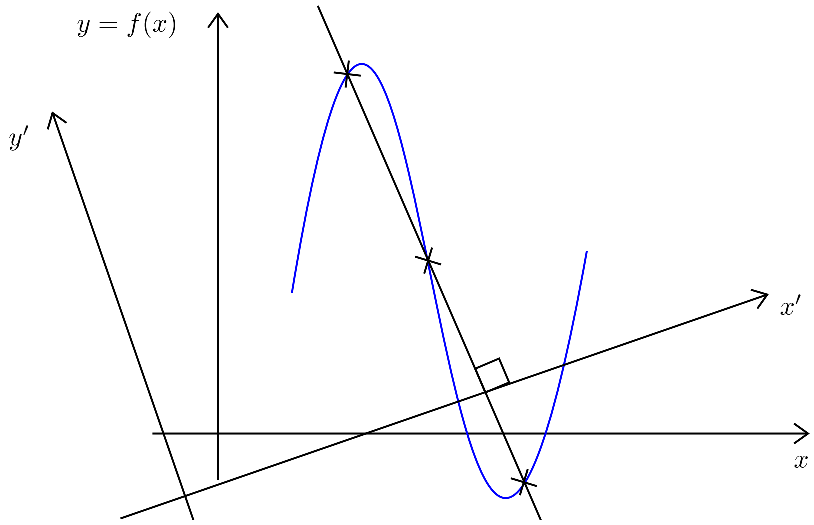


Figure 3.4.9: Rolle theorem directly to mean value theorem = MVT ? No, because of possibly no longer a function after coordinate rotation.

只能規規矩矩地證明

Proof. 構造連續可微且端點值與原函數相同之函數 3.4.21，使兩者形成之差函數 3.4.23 符合 Rolle theorem 3.4.2 條件

let line $l(x)$ through $(a, f(a))$ and $(b, f(b))$

$$l(x) - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$

or

$$[l(x) - f(a)](b-a) = [f(b) - f(a)](x-a) \tag{3.4.20}$$

$$l(x) = \frac{f(b) - f(a)}{b-a}(x-a) + f(a) \tag{3.4.21}$$

$$l'(x) = \frac{f(b) - f(a)}{b-a} \tag{3.4.22}$$

let

$$g(x) = f(x) - l(x) \tag{3.4.23}$$

then

$$\begin{cases} f, l \text{ continuous on } [a,b] \\ f, l \text{ differentiable on } (a,b) \end{cases} \Rightarrow \begin{cases} g \text{ continuous on } [a,b] \\ g \text{ differentiable on } (a,b) \end{cases}$$

and

$$\begin{aligned} g(a) &\overset{3.4.23}{=} f(a) - l(a) = f(a) - \left[\frac{f(b)-f(a)}{b-a}(a-a) + f(a)\right] \\ &= f(a) - \left[\frac{f(b)-f(a)}{b-a} \cdot 0 + f(a)\right] = f(a) - f(a) = 0 \end{aligned}$$

$$\begin{aligned} g(b) &\overset{3.4.23}{=} f(b) - l(b) = f(b) - \left[\frac{f(b)-f(a)}{b-a}(b-a) + f(a)\right] \\ &= f(b) - [(f(b)-f(a)) + f(a)] = f(b) - f(b) = 0 \end{aligned}$$

$$\begin{cases} g \text{ continuous on } [a,b] \\ g \text{ differentiable on } (a,b) \\ g(a) = 0 = g(b) \end{cases}$$

by Rolle 定理 Rolle theorem 3.4.2,

$$\begin{cases} g \text{ continuous on } [a,b] \\ g \text{ differentiable on } (a,b) \\ g(a) = 0 = g(b) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b)\,(g'(c) = 0)$$

$$\exists c \in (a,b)\left(0 = g'(c) = (f-l)'(c) \Rightarrow f'(c) = l'(c) \overset{3.4.22}{=} \frac{f(b)-f(a)}{b-a}\right)$$

$$\exists c \in (a,b)\left(f'(c) = \frac{f(b)-f(a)}{b-a}\right)$$

□

proof by extended case

Proof. by 第二微分均值定理 / 柯西均值定理 second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2 3.4.4,

$$\begin{aligned} &\begin{cases} f,g : [a,b] \to \mathbb{R} \\ f,g \text{ continuous on } [a,b] \\ f,g \text{ differentiable on } (a,b) \end{cases} &&\Rightarrow \exists c \in (a,b)\,\{f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]\} \\ &&&\overset{\text{let } g(x)=x \Rightarrow g'(x)=1}{\Rightarrow} \exists c \in (a,b)\,\{f'(c)[b-a] = 1 \cdot [f(b)-f(a)]\} \\ &&&\Rightarrow \exists c \in (a,b)\,\{f'(c)(b-a) = f(b)-f(a)\} \end{aligned}$$

we get 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3 □

定理 3.4.4. 第二微分均值定理 / 柯西均值定理 *second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2*

在命題及證明中避免 $g(b)-g(a)$ 出現在分母 即可不必限制 $g(b)-g(a) \neq 0$ 或 $g(b) \neq g(a)$,

$$\begin{cases} f,g : [a,b] \to \mathbb{R} \\ f,g \textit{ continuous on } [a,b] \\ f,g \textit{ differentiable on } (a,b) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b)\,\{f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]\}$$

and if $g'(c) \neq 0 \neq g(b)-g(a)$

$$\begin{cases} f,g : [a,b] \to \mathbb{R} \\ f,g \textit{ continuous on } [a,b] \\ f,g \textit{ differentiable on } (a,b) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b)\left\{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}\right\}$$

Proof. 構造連續可微且端點值與原函數相同之函數 3.4.25，使兩者形成之差函數 3.4.27 符合 Rolle theorem 3.4.2 條件

like equation 3.4.20

let line $l(x)$ through $(a, f(a))$ and $(b, f(b))$

$$[l(x) - f(a)][g(b)-g(a)] = [f(b)-f(a)][g(x)-g(a)] \tag{3.4.24}$$

$$l(x)[g(b)-g(a)] \overset{3.4.24}{=} [f(b)-f(a)][g(x)-g(a)]+f(a)[g(b)-g(a)] \tag{3.4.25}$$

$$\begin{aligned}
[l(x)-f(a)][g(b)-g(a)] &= [f(b)-f(a)][g(x)-g(a)] \\
&\Downarrow \text{product rule of derivative} \\
[l(x)-f(a)]'[g(b)-g(a)] &= \ [f(b)-f(a)]'[g(x)-g(a)] \\
+[l(x)-f(a)][g(b)-g(a)]' &\ \ +[f(b)-f(a)][g(x)-g(a)]' \\
&\Downarrow \\
l'(x)[g(b)-g(a)] &= \ 0\cdot[g(x)-g(a)] \\
+[l(x)-f(a)]\cdot 0 &\ \ +[f(b)-f(a)]g'(x) \\
&\Downarrow \\
l'(x)[g(b)-g(a)] &= \ [f(b)-f(a)]g'(x)
\end{aligned} \tag{3.4.26}$$

let

$$h(x)=f(x)-l(x) \tag{3.4.27}$$

then

$$\begin{cases} f,g,l \text{ continuous on } [a,b] \\ f,g,l \text{ differentiable on } (a,b) \end{cases} \Rightarrow \begin{cases} h \text{ continuous on } [a,b] \\ h \text{ differentiable on } (a,b) \end{cases}$$

and

$$\begin{aligned}
h(a)[g(b)-g(a)] &\overset{3.4.27}{=} [f(a)-l(a)][g(b)-g(a)] \\
&= f(a)[g(b)-g(a)]-l(a)[g(b)-g(a)] \\
&\overset{3.4.25}{=} f(a)[g(b)-g(a)]-\{[f(b)-f(a)][g(a)-g(a)]+f(a)[g(b)-g(a)]\} \\
&= f(a)[g(b)-g(a)]-\{[f(b)-f(a)]\cdot 0+f(a)[g(b)-g(a)]\} \\
&= f(a)[g(b)-g(a)]-\{0+f(a)[g(b)-g(a)]\} \\
&= f(a)[g(b)-g(a)]-\{f(a)[g(b)-g(a)]\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
h(b)[g(b)-g(a)] &\overset{3.4.27}{=} [f(b)-l(b)][g(b)-g(a)] \\
&= f(b)[g(b)-g(a)]-l(b)[g(b)-g(a)] \\
&\overset{3.4.25}{=} f(b)[g(b)-g(a)]-\{[f(b)-f(a)][g(b)-g(a)]+f(a)[g(b)-g(a)]\} \\
&= f(b)[g(b)-g(a)]-\{f(b)[g(b)-g(a)]-f(a)[g(b)-g(a)]+f(a)[g(b)-g(a)]\} \\
&= f(b)[g(b)-g(a)]-\{f(b)[g(b)-g(a)]+0\} \\
&= f(b)[g(b)-g(a)]-\{f(b)[g(b)-g(a)]\} \\
&= 0
\end{aligned}$$

$$\begin{cases} h(x)[g(b)-g(a)] \text{ continuous on } [a,b] \\ h(x)[g(b)-g(a)] \text{ differentiable on } (a,b) \\ h(a)[g(b)-g(a)]=0=h(b)[g(b)-g(a)] \end{cases}$$

by Rolle 定理 Rolle theorem 3.4.2,

$$\begin{cases} h(x)[g(b)-g(a)] \text{ continuous on } [a,b] \\ h(x)[g(b)-g(a)] \text{ differentiable on } (a,b) \\ h(a)[g(b)-g(a)]=0=h(b)[g(b)-g(a)] \end{cases} \overset{3.4.2}{\Rightarrow} \exists c\in(a,b)\left(\{h(x)[g(b)-g(a)]\}'(c)=0\right) \tag{3.4.28}$$

$$\begin{aligned}
\{h(x)[g(b)-g(a)]\}' &\overset{\text{product rule of derivative}}{=} h'(x)[g(b)-g(a)]+h(x)[g(x)-g(a)]' \\
&= h'(x)[g(b)-g(a)]+h(x)\cdot 0 \\
&= h'(x)[g(b)-g(a)] \\
&\underset{\text{addition rule of derivative}}{\overset{3.4.27}{=}} [f'(x)-l'(x)][g(b)-g(a)] \\
&= f'(x)[g(b)-g(a)]-l'(x)[g(b)-g(a)] \\
&\overset{3.4.26}{=} f'(x)[g(b)-g(a)]-[f(b)-f(a)]g'(x)
\end{aligned}$$

$$\{h(x)[g(b)-g(a)]\}' = f'(x)[g(b)-g(a)] - [f(b)-f(a)]g'(x) \tag{3.4.29}$$

$$\begin{aligned}
&\overset{3.4.28}{\exists}\ c \in (a,b)\left(\{h(x)[g(b)-g(a)]\}'(c)=0\right)\\
\Leftrightarrow &\exists c \in (a,b)\left(0 = \{h(x)[g(b)-g(a)]\}'(c) \overset{3.4.29}{=} f'(c)[g(b)-g(a)] - [f(b)-f(a)]g'(c)\right)\\
\Leftrightarrow &\exists c \in (a,b)\left(f'(c)[g(b)-g(a)] = [f(b)-f(a)]g'(c)\right)
\end{aligned}$$

$$\exists c \in (a,b)\left(f'(c)[g(b)-g(a)] = [f(b)-f(a)]g'(c)\right)$$

□

- and if $g'(c) \neq 0 \neq g(b)-g(a)$,

Proof.

$$\begin{cases} f,g \text{ continuous on } [a,b] \\ f,g \text{ differentiable on } (a,b) \end{cases} \Rightarrow \exists c \in (a,b)\left\{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}\right\}$$

□

- and if let $g(x) = x \Rightarrow g'(x) = 1$,

Proof.

$$\begin{aligned}
\begin{cases} f,g \text{ continuous on } [a,b] \\ f,g \text{ differentiable on } (a,b) \end{cases} &\Rightarrow \exists c \in (a,b)\{f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]\}\\
&\Rightarrow \exists c \in (a,b)\{f'(c)[b-a] = 1\cdot[f(b)-f(a)]\}\\
&\Rightarrow \exists c \in (a,b)\{f'(c)(b-a) = f(b)-f(a)\}
\end{aligned}$$

we get 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3 □

定理 3.4.5. *L'Hôpital rule = L'Hospital rule = L'Hôpital's rule = L'Hospital's rule*

Proof. by 第二微分均值定理 / 柯西均值定理 second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2 3.4.4, □

推論 3.4.6. 微分方程基本推論 / 微分方程根本補題 *fundamental corollary of differential equation = FCDE* [17]
作為第二微積分基本定理之引理有證明過 *3.4.21*

$$\begin{cases} f:[a,b]\to\mathbb{R} \\ f \text{ continuous on } [a,b] \\ \forall x \in [a,b]\,(f'(x)=0) \end{cases} \Rightarrow \forall x \in [a,b]\{f(x) = C = \textit{constant}\}$$

推論 3.4.7. 不等號形式的均值定理 [18]

$$\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\\ \forall x \in (a,b)\{|f'(x)| \le K\} & (3) \end{cases} \Rightarrow |f(b)-f(a)| \le K\cdot(b-a)$$

$$\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ differentiable on } [a,b] & \Rightarrow (1)\,\&\,(2)\\ \forall x \in (a,b)\{|f'(x)| \le K\} & (3) \end{cases} \Rightarrow |f(b)-f(a)| \le K\cdot(b-a)$$

Proof.

$$[a,b] = \left\{x \,\middle|\, \begin{cases} x \in \mathbb{R} \\ a \le x \le b \end{cases}\right\} \Rightarrow a \le b \Rightarrow b-a \ge 0 \tag{3.4.30}$$

[17]數學傳播_33_2_蔡聰明_均值定理的統合與推廣
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by 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3,

$$
\begin{aligned}
3.4.19 &\Rightarrow \exists c \in (a,b) \left\{ f'(c)(b-a) = f(b) - f(a) \right\} \\
&\Rightarrow \exists c \in (a,b) \left\{ |f'(c)(b-a)| = |f(b) - f(a)| \right\} \\
&\overset{3.4.30: b-a \geq 0}{\Rightarrow} \exists c \in (a,b) \left\{ |f'(c)|(b-a) = |f(b) - f(a)| \right\} \\
&\underset{\text{premise }(3)}{\overset{c\in(a,b)}{\Rightarrow}} K \cdot (b-a) \geq |f'(c)|(b-a) = |f(b) - f(a)| \\
&\Rightarrow |f(b) - f(a)| \leq K \cdot (b-a)
\end{aligned}
$$

□

定理 3.4.8. 推廣三函數微分均值定理 *generalized mean value theorem for derivatives of 3 functions = MVTd3* [19]

$$
\begin{cases}
f,g,h : [a,b] \to \mathbb{R} & (0) \\
f,g,h \text{ continuous on } [a,b] & (1) \\
f,g,h \text{ differentiable on } (a,b) & (2) \\
D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} & (3)
\end{cases}
$$

$$
\overset{3.4.2}{\Rightarrow} \exists c \in (a,b) \left\{ D'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = \begin{vmatrix} f(a) & f(b) & f'(c) \\ g(a) & g(b) & g'(c) \\ h(a) & h(b) & h'(c) \end{vmatrix} = 0 \right\}
$$

以及其幾何意義 *3.4*.

Proof. observe 第二微分均值定理 / 柯西均值定理 second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2 3.4.4

$$
\begin{aligned}
&\begin{cases}
f,g : [a,b] \to \mathbb{R} \\
f,g \text{ continuous on } [a,b] \\
f,g \text{ differentiable on } (a,b)
\end{cases}
\Rightarrow \exists c \in (a,b) \left\{ f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)] \right\} \\
&\Rightarrow f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)] \\
&\Rightarrow 0 = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] \\
&= \begin{vmatrix} f'(c) & g'(c) & 0 \\ f(b) & g(b) & 1 \\ f(a) & g(a) & 1 \end{vmatrix} = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = \left. \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix}' \right|_{x=c} \\
&= D'(x)|_{x=c} = D'(c)
\end{aligned}
$$

其中

$$
\begin{cases}
D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} \\
h(x) = 1
\end{cases}
\Rightarrow
\begin{cases}
h'(x) = 0 & \Rightarrow h'(c) = 0 \\
h(b) = 1 \\
h(a) = 1
\end{cases}
$$

先證明以下兩種行列式表達只差一負號

$$
\begin{bmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{bmatrix} \overset{\text{row}_1 \leftrightarrow \text{row}_3}{\rightarrow} \begin{bmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{bmatrix}
$$

$$
\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{vmatrix} = - \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix}
$$

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let

$$\begin{cases} f,g,h:[a,b]\to\mathbb{R} & (0)\\ f,g,h \text{ continuous on } [a,b] & (1)\\ f,g,h \text{ differentiable on } (a,b) & (2)\\ D(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix} = -\begin{vmatrix} f(a) & f(b) & f(x)\\ g(a) & g(b) & g(x)\\ h(a) & h(b) & h(x)\end{vmatrix} & (3)\end{cases} \tag{3.4.31}$$

$$D(a)=\begin{vmatrix} f(a) & g(a) & h(a)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix} \overset{\text{row}_1\equiv\text{row}_3}{=} 0$$

$$D(b)=\begin{vmatrix} f(b) & g(b) & h(b)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix} \overset{\text{row}_1\equiv\text{row}_2}{=} 0$$

已構造連續可微且端點值為 0 之函數 3.4.31(3)， 符合 Rolle theorem 3.4.2 條件
by Rolle 定理 Rolle theorem 3.4.2,

$$\begin{cases} D \text{ continuous on } [a,b]\\ D \text{ differentiable on } (a,b)\\ D(a)=0=D(b)\end{cases} \overset{3.4.2}{\Rightarrow} \exists c\in(a,b)\,(D'(c)=0)$$

其中

$$D'(c)=\begin{vmatrix} f'(c) & g'(c) & h'(c)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix}$$

□

Proof. 幾何意義
將 3.4.31(3) 中 x 替換成 參數式 常用 t 以表參數, 並調整成不失一般性的順序

$$D_V(t)=\begin{vmatrix} f(t) & g(t) & h(t)\\ f(a) & g(a) & h(a)\\ f(b) & g(b) & h(b)\end{vmatrix} = -\begin{vmatrix} f(t) & g(t) & h(t)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix} = -D(t)$$

考慮 三維空間中曲線 $\Gamma\subset\mathbb{R}^3$

$$\Gamma:\langle x,y,z\rangle=\langle x,y,z\rangle(t)=\langle x(t),y(t),z(t)\rangle=\langle f(t),g(t),h(t)\rangle\ \forall t\in[a,b]$$

其上三點(三個時刻或三個可異定參數)與原點 $O(0,0,0)$ 形成之位置向量

$$P(t),P(a),P(b)$$

$$P(t)=\langle f(t),g(t),h(t)\rangle, P(a)=\langle f(a),g(a),h(a)\rangle, P(b)=\langle f(b),g(b),h(b)\rangle$$

$$\vec{OP}(t),\vec{OP}(a),\vec{OP}(b)$$

$$\vec{OP}(t)=\langle f(t),g(t),h(t)\rangle, \vec{OP}(a)=\langle f(a),g(a),h(a)\rangle, \vec{OP}(b)=\langle f(b),g(b),h(b)\rangle$$

let

$$\boldsymbol{w}=\vec{w}=\vec{OP}(t), \boldsymbol{u}=\vec{u}=\vec{OP}(a), \boldsymbol{v}=\vec{v}=\vec{OP}(b)$$

then

$$\begin{aligned} D_V(t) &= \begin{vmatrix} f(t) & g(t) & h(t)\\ f(a) & g(a) & h(a)\\ f(b) & g(b) & h(b)\end{vmatrix}\\ &= \begin{vmatrix} \vec{w}\\ \vec{u}\\ \vec{v}\end{vmatrix} = \begin{vmatrix} \vec{w} & \vec{u} & \vec{v}\end{vmatrix} = \vec{w}\cdot\left(\vec{u}\times\vec{v}\right)\\ &= \begin{vmatrix} \boldsymbol{w}\\ \boldsymbol{u}\\ \boldsymbol{v}\end{vmatrix} = \begin{vmatrix} \boldsymbol{w} & \boldsymbol{u} & \boldsymbol{v}\end{vmatrix} = \boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{w}(t)\cdot[\boldsymbol{u}(a)\times\boldsymbol{v}(b)] \end{aligned}$$

$\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}$ 所張出平行六面體, 其有號體積就是 $D_V(t)$。 我們欲求 $D_V(t)$ 的極值, 則令 $\pi(\boldsymbol{u}, \boldsymbol{v})$ 表示由 $\boldsymbol{u}, \boldsymbol{v}$ 所張出的平面, 曲線Γ 由 $\langle f(b), g(b), h(b)\rangle$ 點出發到達 $\langle f(a), g(a), h(a)\rangle$, 我們必可在 曲線 Γ 上找得到點 $\langle f(c), g(c), h(c)\rangle$ 或 $t = c$ 使得 $D_V'(c) = 0$ 且使得 點 $\langle f(c), g(c), h(c)\rangle$ 距離由 $\boldsymbol{u}, \boldsymbol{v}$ 所張出的平面 $\pi(\boldsymbol{u}, \boldsymbol{v})$ 為最遠或最近。[20] □

推論 3.4.9. 第二微分均值定理 / 柯西均值定理 *second mean value theorem for derivatives / Cauchy mean value theorem = MVTdC / MVTd2 3.4.4*

$$\begin{cases} f, g, h : [a, b] \to \mathbb{R} & (0) \\ f, g, h \text{ continuous on } [a, b] & (1) \\ f, g, h \text{ differentiable on } (a, b) & (2) \\ D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} & (3) \\ h(x) = 1 \Rightarrow \begin{cases} h'(x) = 0 & \Rightarrow h'(c) = 0 \\ h(b) = 1 \\ h(a) = 1 \end{cases} & (4) \end{cases}$$

$$\begin{aligned} \overset{3.4.8}{\Rightarrow} \exists c \in (a, b) \{0 = D'(c) &= \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} \\ &= \begin{vmatrix} f'(c) & g'(c) & 0 \\ f(b) & g(b) & 1 \\ f(a) & g(a) & 1 \end{vmatrix} \\ &= f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)]\} \end{aligned}$$

推論 3.4.10. 第一微分均值定理 / 拉格朗日均值定理 *first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3*

$$\begin{cases} f, g, h : [a, b] \to \mathbb{R} & (0) \\ f, g, h \text{ continuous on } [a, b] & (1) \\ f, g, h \text{ differentiable on } (a, b) & (2) \\ D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} & (3) \\ h(x) = 1 \Rightarrow \begin{cases} h'(x) = 0 & \Rightarrow h'(c) = 0 \\ h(b) = 1 \\ h(a) = 1 \end{cases} & (4) \\ g(x) = x \Rightarrow \begin{cases} g'(x) = 1 & \Rightarrow g'(c) = 1 \\ g(b) = b \\ g(a) = a \end{cases} & (5) \end{cases}$$

$$\begin{aligned} \overset{3.4.8}{\Rightarrow} \exists c \in (a, b) \{0 = D'(c) &= \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} \\ &= \begin{vmatrix} f'(c) & 1 & 0 \\ f(b) & b & 1 \\ f(a) & a & 1 \end{vmatrix} \\ &= f'(c)[b - a] - [f(b) - f(a)]\} \end{aligned}$$

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推論 3.4.11. *Rolle 定理 Rolle theorem 3.4.2*

$$\begin{cases} f,g,h:[a,b]\to\mathbb{R} & (0)\\ f,g,h \text{ continuous on } [a,b] & (1)\\ f,g,h \text{ differentiable on } (a,b) & (2)\\ D(x)=\begin{vmatrix} f(x) & g(x) & h(x)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix} & (3)\\ h(x)=1\Rightarrow\begin{cases} h'(x)=0 & \Rightarrow h'(c)=0\\ h(b)=1\\ h(a)=1\end{cases} & (4)\\ g(x)=x\Rightarrow\begin{cases} g'(x)=1 & \Rightarrow g'(c)=1\\ g(b)=b\\ g(a)=a\end{cases} & (5)\\ f(a)=0=f(b) & (6)\end{cases}$$

$$\begin{aligned}\overset{3.4.8}{\Rightarrow}\exists c\in(a,b)\{0=D'(c)=&\begin{vmatrix} f'(c) & g'(c) & h'(c)\\ f(b) & g(b) & h(b)\\ f(a) & g(a) & h(a)\end{vmatrix}\\ =&\begin{vmatrix} f'(c) & 1 & 0\\ 0 & b & 1\\ 0 & a & 1\end{vmatrix}\\ =&f'(c)[b-a]\\ \overset{b-a\neq0}{\Rightarrow}&f'(c)=0\}\end{aligned}$$

定理 3.4.12. 推廣四函數微分均值定理 *generalized mean value theorem for derivatives of 4 functions = MVTd4* [21]

$$\begin{cases} f,g,h,k:[a,c]\to\mathbb{R} & (0)\\ f,g,h,k \text{ continuous on } [a,c] & (1)\\ f,g,h,k \text{ differentiable on } (a,c) & (2)\\ D_4(x)=\begin{vmatrix} f(x) & g(x) & h(x) & k(x)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f(x)\\ g(a) & g(b) & g(c) & g(x)\\ h(a) & h(b) & h(c) & h(x)\\ k(a) & k(b) & k(c) & k(x)\end{vmatrix},a<b<c & (3)\end{cases}$$

$$\overset{3.4.2}{\Rightarrow}\exists\begin{cases}\alpha\in(a,b)\\ \beta\in(b,c)\end{cases}\{D_4'(\alpha)=0=D_4'(\beta)\}$$

$$\overset{3.4.2}{\Rightarrow}\exists d\in(\alpha,\beta)\subset(a,c)\{D_4''(d)=0\}$$

Proof. 先證明以下兩種行列式表達相等

$$\begin{bmatrix} f(a) & f(b) & f(c) & f(x)\\ g(a) & g(b) & g(c) & g(x)\\ h(a) & h(b) & h(c) & h(x)\\ k(a) & k(b) & k(c) & k(x)\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix} f(a) & g(a) & h(a) & k(a)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(x) & g(x) & h(x) & k(x)\end{bmatrix}$$

$$\overset{\text{row}_1\leftrightarrow\text{row}_4}{\to}\begin{bmatrix} f(x) & g(x) & h(x) & k(x)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(a) & g(a) & h(a) & k(a)\end{bmatrix}\overset{\text{row}_2\leftrightarrow\text{row}_3}{\to}\begin{bmatrix} f(x) & g(x) & h(x) & k(x)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(a) & g(a) & h(a) & k(a)\end{bmatrix}$$

$$\begin{aligned}\begin{vmatrix} f(a) & f(b) & f(c) & f(x)\\ g(a) & g(b) & g(c) & g(x)\\ h(a) & h(b) & h(c) & h(x)\\ k(a) & k(b) & k(c) & k(x)\end{vmatrix}=&\begin{vmatrix} f(a) & g(a) & h(a) & k(a)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(x) & g(x) & h(x) & k(x)\end{vmatrix}\\ =-&\begin{vmatrix} f(x) & g(x) & h(x) & k(x)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=-\left(-\begin{vmatrix} f(x) & g(x) & h(x) & k(x)\\ f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}\right)\end{aligned}$$

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$$D_4(a)=\begin{vmatrix} f(a) & g(a) & h(a) & k(a)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix} \overset{\text{row}_1=\text{row}_4}{=} 0$$

$$D_4(b)=\begin{vmatrix} f(b) & g(b) & h(b) & k(b)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix} \overset{\text{row}_1=\text{row}_3}{=} 0$$

$$D_4(c)=\begin{vmatrix} f(c) & g(c) & h(c) & k(c)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix} \overset{\text{row}_1=\text{row}_2}{=} 0$$

已構造連續可微且三點值為 0 之函數 D_4， 符合 Rolle theorem 3.4.2 條件
by Rolle 定理 Rolle theorem 3.4.2,

$$\begin{cases} D_4 \text{ continuous on } [a,b],[b,c]\\ D_4 \text{ differentiable on } (a,b),(b,c)\\ D_4(a)=0=D_4(b)=0=D_4(c)\end{cases} \overset{3.4.2}{\Rightarrow} \exists \begin{cases}\alpha\in(a,b)\\ \beta\in(b,c)\end{cases} \{D_4'(\alpha)=0=D_4'(\beta)\}$$

其中

$$D_4'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) & k'(x)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f'(x)\\ g(a) & g(b) & g(c) & g'(x)\\ h(a) & h(b) & h(c) & h'(x)\\ k(a) & k(b) & k(c) & k'(x)\end{vmatrix}$$

$$D_4'(\alpha)=\begin{vmatrix} f'(\alpha) & g'(\alpha) & h'(\alpha) & k'(\alpha)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f'(\alpha)\\ g(a) & g(b) & g(c) & g'(\alpha)\\ h(a) & h(b) & h(c) & h'(\alpha)\\ k(a) & k(b) & k(c) & k'(\alpha)\end{vmatrix}$$

$$D_4'(\beta)=\begin{vmatrix} f'(\beta) & g'(\beta) & h'(\beta) & k'(\beta)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f'(\beta)\\ g(a) & g(b) & g(c) & g'(\beta)\\ h(a) & h(b) & h(c) & h'(\beta)\\ k(a) & k(b) & k(c) & k'(\beta)\end{vmatrix}$$

已構造端點值為 0 之函數 D_4' 若連續且可微， 符合 Rolle theorem 3.4.2 條件
by Rolle 定理 Rolle theorem 3.4.2,

$$\begin{cases} D_4' \text{ continuous on } [\alpha,\beta]\\ D_4' \text{ differentiable on } (\alpha,\beta)\\ D_4'(\alpha)=0=D_4'(\beta)\end{cases} \overset{3.4.2}{\Rightarrow} \exists d\in(\alpha,\beta)\subset(a,c)\{D_4''(d)=0\}$$

其中

$$D_4''(x)=\begin{vmatrix} f''(x) & g''(x) & h''(x) & k''(x)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f''(x)\\ g(a) & g(b) & g(c) & g''(x)\\ h(a) & h(b) & h(c) & h''(x)\\ k(a) & k(b) & k(c) & k''(x)\end{vmatrix}$$

$$D_4''(\alpha)=\begin{vmatrix} f''(\alpha) & g''(\alpha) & h''(\alpha) & k''(\alpha)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f''(\alpha)\\ g(a) & g(b) & g(c) & g''(\alpha)\\ h(a) & h(b) & h(c) & h''(\alpha)\\ k(a) & k(b) & k(c) & k''(\alpha)\end{vmatrix}$$

$$D_4''(\beta)=\begin{vmatrix} f''(\beta) & g''(\beta) & h''(\beta) & k''(\beta)\\ f(c) & g(c) & h(c) & k(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(b) & f(c) & f''(\beta)\\ g(a) & g(b) & g(c) & g''(\beta)\\ h(a) & h(b) & h(c) & h''(\beta)\\ k(a) & k(b) & k(c) & k''(\beta)\end{vmatrix}$$

□

推論 3.4.13. 二階導數的均值公式 *mean value theorem for second-order derivatives*

$$\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ at least } 2^{nd}\text{-order differentiable on } (a,b) & (2)\\ x_0\in(a,b)\Leftrightarrow a<x_0<b & (3)\end{cases} \Rightarrow \exists c\in(a,b)\left\{\frac{f''(c)}{2}=\frac{\frac{f(b)-f(x_0)}{b-x_0}-\frac{f(x_0)-f(a)}{x_0-a}}{b-a}\right\}$$

Proof. by 推廣四函數微分均值定理 generalized mean value theorem for derivatives of 4 functions = MVTd4 3.4.12,

$$\begin{cases} f,g,h,k:[a,b]\to\mathbb{R} & (0)\\ f,g,h,k \text{ continuous on } [a,b] & (1)\\ f,g,h,k \text{ differentiable on } (a,b) & (2)\\ D_4(x)=\begin{vmatrix} f(x) & g(x) & h(x) & k(x)\\ f(b) & g(b) & h(b) & k(b)\\ f(x_0) & g(x_0) & h(x_0) & k(x_0)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}=\begin{vmatrix} f(a) & f(x_0) & f(b) & f(x)\\ g(a) & g(x_0) & g(b) & g(x)\\ h(a) & h(x_0) & h(b) & h(x)\\ k(a) & k(x_0) & k(b) & k(x)\end{vmatrix}, a<x_0<b & (3)\\ k(x)=1\Rightarrow\begin{cases} k(a)=k(x_0)=k(b)=1\\ k'(x)=0\Rightarrow k'(a)=k'(x_0)=k'(b)=0\\ k''(x)=0\Rightarrow k'(a)=k'(x_0)=k'(b)=0\end{cases} & (4)\\ h(x)=x\Rightarrow\begin{cases} h(a)=a;h(x_0)=x_0;h(b)=b\\ h'(x)=1\Rightarrow h'(a)=h'(x_0)=h'(b)=1\\ h''(x)=0\Rightarrow h''(a)=h''(x_0)=h''(b)=0\end{cases} & (5)\\ g(x)=x^2\Rightarrow\begin{cases} g(a)=a^2;g(x_0)=x_0^2;g(b)=b^2\\ g'(x)=2x\Rightarrow g'(a)=2a;g'(x_0)=2x_0;g'(b)=2b\\ g''(x)=2\Rightarrow g''(a)=g''(x_0)=g''(b)=2\end{cases} & (6)\end{cases}$$

$$\overset{3.4.2}{\Rightarrow}\exists\begin{cases}\alpha\in(a,x_0)\\ \beta\in(x_0,b)\end{cases}\quad\{D_4'(\alpha)=0=D_4'(\beta)\}$$

$$\overset{3.4.2}{\Rightarrow}\exists c\in(\alpha,\beta)\subset(a,b)\{0=D_4''(c)=\begin{vmatrix} f''(c) & g''(c) & h''(c) & k''(c)\\ f(b) & g(b) & h(b) & k(b)\\ f(x_0) & g(x_0) & h(x_0) & k(x_0)\\ f(a) & g(a) & h(a) & k(a)\end{vmatrix}$$

$$=\begin{vmatrix} f''(c) & 2 & 0 & 0\\ f(b) & b^2 & b & 1\\ f(x_0) & x_0^2 & x_0 & 1\\ f(a) & a^2 & a & 1\end{vmatrix}$$

$$=f''(c)\begin{vmatrix} b^2 & b & 1\\ x_0^2 & x_0 & 1\\ a^2 & a & 1\end{vmatrix}-2\begin{vmatrix} f(b) & b & 1\\ f(x_0) & x_0 & 1\\ f(a) & a & 1\end{vmatrix}$$

$$\Rightarrow\frac{f''(c)}{2}=\frac{\begin{vmatrix} f(b) & b & 1\\ f(x_0) & x_0 & 1\\ f(a) & a & 1\end{vmatrix}}{\begin{vmatrix} b^2 & b & 1\\ x_0^2 & x_0 & 1\\ a^2 & a & 1\end{vmatrix}}\}$$

$$\frac{f''(c)}{2}=\frac{\begin{vmatrix} f(b) & b & 1\\ f(x_0) & x_0 & 1\\ f(a) & a & 1\end{vmatrix}}{\begin{vmatrix} b^2 & b & 1\\ x_0^2 & x_0 & 1\\ a^2 & a & 1\end{vmatrix}}\underset{4.0.1}{=}\frac{f(b)(x_0-a)-f(x_0)(b-a)+f(a)(b-x_0)}{(x_0-b)(a-x_0)(b-a)}$$

$$=\frac{\frac{f(b)(x_0-a)-f(x_0)(b-a)+f(a)(b-x_0)}{(x_0-b)(a-x_0)}}{b-a}=\frac{\frac{f(b)}{b-x_0}+\frac{-f(x_0)(b-x_0+x_0-a)}{(x_0-b)(a-x_0)}+\frac{f(a)}{x_0-a}}{b-a}$$

$$=\frac{\frac{f(b)}{b-x_0}+\frac{-f(x_0)(b-x_0)-f(x_0)(x_0-a)}{(x_0-b)(a-x_0)}+\frac{f(a)}{x_0-a}}{b-a}$$

$$=\frac{\frac{f(b)}{b-x_0}+\frac{-f(x_0)}{x_0-a}+\frac{-f(x_0)}{b-x_0}+\frac{f(a)}{x_0-a}}{b-a}=\frac{\left[\frac{f(b)}{b-x_0}-\frac{f(x_0)}{b-x_0}\right]-\left[\frac{f(x_0)}{x_0-a}-\frac{f(a)}{x_0-a}\right]}{b-a}$$

$$=\frac{\frac{f(b)-f(x_0)}{b-x_0}-\frac{f(x_0)-f(a)}{x_0-a}}{b-a}$$

□

推論 3.4.14. 三階導數的均值公式 *mean value theorem for third-order derivatives*

定理 3.4.15. 推廣 n 函數微分均值定理 *generalized mean value theorem for derivatives of* n *functions* = *MVTdn*

$$\begin{cases} f_1, f_2, \cdots, f_{n-1}, f_n : [a_1, a_{n-1}] \to \mathbb{R} & (0) \\ f_1, f_2, \cdots, f_{n-1}, f_n \text{ continuous on } [a_1, a_{n-1}] & (1) \\ f_1, f_2, \cdots, f_{n-1}, f_n \text{ at least } (n-2)^{th}\text{-order differentiable on } (a_1, a_{n-1}) & (2) \\ D_n(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) & f_n(x) \\ f_1(a_{n-1}) & f_2(a_{n-1}) & \cdots & f_{n-1}(a_{n-1}) & f_n(a_{n-1}) \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ f_1(a_2) & f_2(a_2) & \cdots & f_{n-1}(a_2) & f_n(a_2) \\ f_1(a_1) & f_2(a_1) & \cdots & f_{n-1}(a_1) & f_n(a_1) \end{vmatrix}, a_1 < a_2 < \cdots < a_{n-1} & (3) \end{cases}$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha_{1,1} \in (a_1, a_2) \\ \alpha_{1,2} \in (a_2, a_3) \\ \vdots \\ \alpha_{1,n-2} \in (a_{n-2}, a_{n-1}) \end{cases} \{D_n'(\alpha_{1,1}) = D_n'(\alpha_{1,2}) = \cdots = D_n'(\alpha_{1,n-2}) = 0\}$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha_{2,1} \in (\alpha_{1,1}, \alpha_{1,2}) \\ \alpha_{2,2} \in (\alpha_{1,2}, \alpha_{1,3}) \\ \vdots \\ \alpha_{2,n-3} \in (\alpha_{1,n-3}, \alpha_{1,n-2}) \end{cases} \{D_n''(\alpha_{2,1}) = D_n''(\alpha_{2,2}) = \cdots = D_n''(\alpha_{2,n-3}) = 0\}$$

$$\vdots$$

$$\overset{3.4.2}{\Rightarrow} \exists \alpha_{n-2,1} \in (\alpha_1, \alpha_{n-2}) \subset (a_1, a_{n-1}) \{D_n^{(n-2)}(\alpha_{n-2,1}) = 0\}$$

Proof. like 推廣四函數微分均值定理 generalized mean value theorem for derivatives of 4 functions = MVTd4 3.4.12

$$\begin{cases} f, g, h, k : [a, c] \to \mathbb{R} & (0) \\ f, g, h, k \text{ continuous on } [a, c] & (1) \\ f, g, h, k \text{ differentiable on } (a, c) & (2) \\ D_4(x) = \begin{vmatrix} f(x) & g(x) & h(x) & k(x) \\ f(c) & g(c) & h(c) & k(c) \\ f(b) & g(b) & h(b) & k(b) \\ f(a) & g(a) & h(a) & k(a) \end{vmatrix} = \begin{vmatrix} f(a) & f(b) & f(c) & f(x) \\ g(a) & g(b) & g(c) & g(x) \\ h(a) & h(b) & h(c) & h(x) \\ k(a) & k(b) & k(c) & k(x) \end{vmatrix}, a < b < c & (3) \end{cases}$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha \in (a, b) \\ \beta \in (b, c) \end{cases} \{D_4'(\alpha) = 0 = D_4'(\beta)\}$$

$$\overset{3.4.2}{\Rightarrow} \exists d \in (\alpha, \beta) \subset (a, c) \{D_4''(d) = 0\}$$

$$\begin{cases} f_1, f_2, f_3, f_4 : [a_1, a_3] \to \mathbb{R} & (0) \\ f_1, f_2, f_3, f_4 \text{ continuous on } [a_1, a_3] & (1) \\ f_1, f_2, f_3, f_4 \text{ differentiable on } (a_1, a_3) & (2) \\ D_4(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & f_4(x) \\ f_1(a_3) & f_2(a_3) & f_3(a_3) & f_4(a_3) \\ f_1(a_2) & f_2(a_2) & f_3(a_2) & f_4(a_2) \\ f_1(a_1) & f_2(a_1) & f_3(a_1) & f_4(a_1) \end{vmatrix} = \begin{vmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) & f_1(x) \\ f_2(a_1) & f_2(a_2) & f_2(a_3) & f_2(x) \\ f_3(a_1) & f_3(a_2) & f_3(a_3) & f_3(x) \\ f_4(a_1) & f_4(a_2) & f_4(a_3) & f_4(x) \end{vmatrix}, a_1 < a_2 < a_3 & (3) \end{cases}$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha_{1,1} \in (a_1, a_2) \\ \alpha_{1,2} \in (a_2, a_3) \end{cases} \{D_4'(\alpha_{1,1}) = D_4'(\alpha_{1,2}) = 0\}$$

if f_1, f_2, f_3, f_4 at least 2^{nd}-order differentiable on (a, b) ⋮

$$\overset{3.4.2}{\Rightarrow} \exists \alpha_{2,1} \in (\alpha_{1,1}, \alpha_{1,2}) \subset (a_1, a_3) \{D_4''(\alpha_{2,1}) = 0\}$$

$$\Leftrightarrow \exists \alpha_{2,1} \in (\alpha_{1,1}, \alpha_{1,2}) \subset (a_1, a_3) \{D_4^{(2)}(\alpha_{2,1}) = 0\}$$

$$\begin{cases} f_1, f_2, \cdots, f_{n-1}, f_n : [a_1, a_{n-1}] \to \mathbb{R} & (0) \\ f_1, f_2, \cdots, f_{n-1}, f_n \text{ continuous on } [a_1, a_{n-1}] & (1) \\ f_1, f_2, \cdots, f_{n-1}, f_n \text{ differentiable on } (a_1, a_{n-1}) & (2) \\ D_n(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) & f_n(x) \\ f_1(a_{n-1}) & f_2(a_{n-1}) & \cdots & f_{n-1}(a_{n-1}) & f_n(a_{n-1}) \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ f_1(a_2) & f_2(a_2) & \cdots & f_{n-1}(a_2) & f_n(a_2) \\ f_1(a_1) & f_2(a_1) & \cdots & f_{n-1}(a_1) & f_n(a_1) \end{vmatrix}, a_1 < a_2 < \cdots < a_{n-1} & (3) \end{cases}$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha_{1,1} \in (a_1, a_2) \\ \alpha_{1,2} \in (a_2, a_3) \\ \vdots \\ \alpha_{1,n-2} \in (a_{n-2}, a_{n-1}) \end{cases} \{D'_n(\alpha_{1,1}) = D'_n(\alpha_{1,2}) = \cdots = D'_n(\alpha_{1,n-2}) = 0\}$$

$$f_1, f_2, \cdots, f_{n-1}, f_n \text{ at least } 2^{\text{nd}}\text{-order differentiable on } (a_1, a_{n-1}) \vdots$$

$$\overset{3.4.2}{\Rightarrow} \exists \begin{cases} \alpha_{2,1} \in (\alpha_{1,1}, \alpha_{1,2}) \\ \alpha_{2,2} \in (\alpha_{1,2}, \alpha_{1,3}) \\ \vdots \\ \alpha_{2,n-3} \in (\alpha_{1,n-3}, \alpha_{1,n-2}) \end{cases} \{D''_n(\alpha_{2,1}) = D''_n(\alpha_{2,2}) = \cdots = D''_n(\alpha_{2,n-3}) = 0\}$$

$$f_1, f_2, \cdots, f_{n-1}, f_n \text{ at least } (n-2)^{\text{th}}\text{-order differentiable on } (a_1, a_{n-1}) \vdots$$

$$\overset{3.4.2}{\Rightarrow} \exists \alpha_{n-2,1} \in (\alpha_1, \alpha_{n-2}) \subset (a_1, a_{n-1}) \{D_n^{(n-2)}(\alpha_{n-2,1}) = 0\}$$

□

從而

$$\frac{\dfrac{f(x_0)-f(a)}{x_0-a}-\dfrac{f(b)-f(a)}{b-a}}{x_0-b}=\frac{1}{2}f''(c).$$

此式可看作是二階導數的均值公式。

6. n 元化的推廣

上述定理 10又可以再推展到 n 個函數的情形, 採用 Rolle 定理就可得證。

定理11: (推廣的 n 元均值定理, $n\geq 3$)
假設 n 個函數 $f_1,f_2,\ldots,f_n$ 在 $[a,b]$ 上連續, $a<x_0<x_1<\cdots<x_{n-3}<b$。令

$$H(t)=\begin{vmatrix} f_1(a) & f_1(x_0) & \cdots & f_1(b) & f_1(t)\\ f_2(a) & f_2(x_0) & \cdots & f_2(b) & f_2(t)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ f_n(a) & f_n(x_0) & \cdots & f_n(b) & f_n(t)\end{vmatrix},\ t\in[a,b].$$

第 1層: 若 $f_1,f_2,\ldots,f_n$ 在 (a,b) 上可微分, 則存在 $c_{1,1}<c_{1,2}<\cdots<c_{1,n-2}$ 使得 $a<c_{1,1}<x_0<c_{1,2}<x_1<c_{1,3}\cdots<x_{n-3}<c_{1,n-2}<b$, 並且

$$H'(c_{1,k})=0,\quad k=1,\ldots,n-2.$$

第 2層: 進一步, 若 $f_1,f_2,\ldots,f_n$ 在 (a,b) 上爲二階可微分, 那麼就存在 $c_{2,1}<c_{2,2}<\cdots<c_{2,n-3}$, 使得 $c_{1,1}<c_{2,1}<c_{1,2}<c_{2,2}<c_{1,3}\cdots<c_{1,n-3}<c_{2,n-3}<c_{1,n-2}$, 並且 $H''(c_{2,k})=0$, $k=1,\ldots,n-3$。
繼續不斷做下去$\cdots\cdots\cdots$
第 $n-2$ 層: 最後, 若 $f_1,f_2,\ldots,f_n$ 在 (a,b) 上爲 $n-2$ 階可微分, 那麼就存在 $c_{n-2,1}$, 使得 $a<c_{n-1,1}<c_{n-2,1}<c_{n-1,2}<b$, 並且 $H^{(n-2)}(c_{n-2,1})=0$。

7. 高維空間的推廣

另一方面, 在高等微積分裡, 定義在高維空間上的實值或向量值的多變數函數 f, 還有各式各樣的均值定理, 我們分成 $f:\mathcal{R}^n\to\mathcal{R}$ 與 $f:\mathcal{R}^n\to\mathcal{R}^m$ 兩種情形來討論。

定理12: (高維空間的均值定理)
(i) 假設函數 $f:U\subset\mathcal{R}^n\to\mathcal{R}$ 在開集 U 上可微分。令 a 與 b 皆屬於 U 且連結它們的線

段 $L(a,b)$ 也都落在 U 裡。那麼存在一點 $c=a+t_0(b-a)$, 其中 $0<t_0<1$, 使得

$$f(b)-f(a)=\nabla f(c)\cdot(b-a). \tag{16}$$

(ii) 假設 $f=(f_1,f_2,\ldots,f_m):U\subset\mathcal{R}^{\,n}\to\mathcal{R}^{\,m}$ 在開集 U 上可微分, 線段 $L(a,b)$ 落在 U 裡, 則存在 $c_1,\ldots,c_m$ 屬於線段 $L(a,b)$, 使得

$$f_k(b)-f_k(a)=\nabla f_k(c_k)\cdot(b-a),\quad k=1,2,\ldots,m.$$

證明: (i) 考慮函數 $g(t)=f[a+t(b-a)]$, 對 g 在 $[0,1]$ 上使用 Lagrange 均值定理與連鎖規則就好了。(ii) 對於每個分量 f_k 皆分別使用 (i) 就得證。

對於 (ii) 的情形, 美中不足的是, 其中落在線段上的諸點 $c_k\in L(a,b)$ 對每個分量函數 f_k 皆不同。因此我們無法期盼如 (16) 式具有共同點之結果, 對所有函數 $f:A\subset\mathcal{R}^n\to\mathcal{R}^m$ 都成立。

反例: 假設 $f_1(x)=\cos x$, $f_2(x)=\sin x$, 並且選取 $a=0$, $b=2\pi$。考慮函數 $f=(f_1,f_2):[0,2\pi]\to\mathcal{R}^{\,2}$, 則 (16) 式之形的均值定理不成立。

修改爲不等式形式的均值定理是可行之道。

定理 13: (高維空間不等式形式的均值定理)
假設 $U\subset\mathcal{R}^{\,n}$ 爲開集, $f:U\to\mathcal{R}^{\,m}$ 爲可微分函數。令 a 與 b 皆屬於 U 且連結它們的線段 $L(a,b)$ 也都落在 U 裡。再假設

$$\|Df(x)\|\le M,\quad \forall\ x\in L(a,b)$$

其中 $\|\ \|$ 表示線性算子的範數 (norm)。那麼我們就有

$$\|f(b)-f(a)\|\le M\cdot\|b-a\|. \tag{17}$$

證明: 考慮函數 $g(t)=f[a+t(b-a)]$, $t\in[0,1]$。利用連鎖規則, 作微分得到 $\dfrac{d}{dt}f[a+t(b-a)]=Df[a+t(b-a)]\cdot(b-a)$。對 t 從 $t=0$ 積分到 $t=1$ 得到

$$f(b)-f(a)=\int_0^1\Big\{Df[a+t(b-a)]\cdot(b-a)\Big\}dt$$

此地的積分是對各成份函數來定義。現在取範數, 因爲積分的範數小於等於範數的積分, 就得到 (17) 式。

將定理13 中的高維歐氏空間改爲更廣泛的賦範向量空間 (normed vector space), 結果仍然成立。

下面我們給出均值定理的一個重要用途:

定理 14: 假設 $U \subset \mathcal{R}^n$ 爲連通的開集, 並且 $f: U \to \mathcal{R}^m$ 爲可微分函數。如果 $Df(x) = 0, \forall\, x \in U$, 則 f 爲常函數。

證明: 取定一點 $a \in U$, 並且令 $V = \{b \in U | f(b) = f(a)\}$, 即 $V = f^{-1}(a)$。因爲 f 爲連續函數, 故 V 爲 U 中的閉集。另一方面, 因爲 U 爲開集, 故對於每個 $x \in V \subset U$, 都存在開的球 B, 半徑大於 0 且被包含於 U 中。這個球 B 是凸集 (convex set), 由定理13 知, $f(y) = f(x) = f(a), \forall\, y \in B$。從而 V 也是 U 中的開集。今因 U 爲連通集, 而連通集中沒有既開且閉的眞子集, 所以 $V = U$。

8. 結語

將數學結果連貫成有機整體, 使得沒有一片知識是孤立的, 這是數學美的要素之一。在微積分裡, 均值定理的知識網是一個著名的例子, 其中 Rolle 定理是核心樞紐!

最後我們整理出本文所涉及的定理之邏輯網絡 (logical net):

$$
\begin{tikzcd}
 & & & & n\text{ 元均值定理} & & \\
 & & & & \vdots & & \\
 & \text{Taylor 定理} \arrow[dl] \arrow[dr] & & & \text{三元均值定理} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\text{N-L 公式} & \Leftrightarrow & \text{Lagrange 均值定理} & \Leftrightarrow & \text{Rolle 定理} & \Leftrightarrow & \text{Cauchy 兩元均值定理} \\
 & \text{積分的均值定理} \arrow[u] & & \text{Fermat 內點極值定理} \arrow[ur] & & \text{連續函數極值定理} \arrow[ul] &
\end{tikzcd}
$$

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定理 3.4.16. 向量或高維微分均值定理 *mean value theorem for vector derivatives = MVTdv*

定理 3.4.17. 第一積分均值定理 *first mean value theorem for definite integrals = MVTi1*

$$\begin{cases} f:[a,b]\to\mathbb{R} & a<b \\ f \text{ continuous on } [a,b] \end{cases} \Rightarrow \exists c\in(a,b)\left(f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right)$$

Proof. closed interval in the proposition conclusion

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT 3.3.9,

$$f \text{ continuous on } [a,b]\Rightarrow \exists x_m,x_M\in[a,b],\forall x\in[a,b]\{m=f(x_m)\le f(x)\le f(x_M)=M\}$$

$$m(b-a)=\int_a^b m\,\mathrm{d}x=\int_a^b f(x_m)\,\mathrm{d}x\le\int_a^b f(x)\,\mathrm{d}x\le\int_a^b f(x_M)\,\mathrm{d}x=\int_a^b M\,\mathrm{d}x=M(b-a)$$

$$m(b-a)\le\int_a^b f(x)\,\mathrm{d}x\le M(b-a)$$

$$f(x_m)=m\le\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\le M=f(x_M)$$

$$\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\in[m,M]=[f(x_m),f(x_M)]$$

by 介值定理 / 中間值定理 intermediate value theorem = IVT 3.3.5,

$$f \text{ continuous on } [a,b]\Rightarrow \exists c \text{ between } x_m,x_M\subseteq[a,b]\left(f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\in[f(x_m),f(x_M)]\right)$$

$$f \text{ continuous on } [a,b]\Rightarrow \exists c\in[a,b]\left(f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right)$$

□

Proof. open interval in the proposition conclusion [22]

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT 3.3.9,

$$\begin{cases} f:[a,b]\to\mathbb{R} \\ f \text{ continuous on } [a,b] \end{cases} \Rightarrow \exists x_m,x_M\in[a,b],\forall x\in[a,b]\{m=f(x_m)\le f(x)\le f(x_M)=M\} \tag{3.4.32}$$

let

$$g:[a,b]\to\mathbb{R}^+\cup\{0\} \tag{3.4.33}$$

non-negative (without loss of generality, w/oLoG), continuous, and integrable on $[a,b]$

$$m=f(x_m)\le f(x)\le f(x_M)=M$$

$$m\cdot g(x)=f(x_m)g(x)\le f(x)g(x)\le f(x_M)g(x)=M\cdot g(x)$$

$$\int_a^b m\cdot g(x)\,\mathrm{d}x=\int_a^b f(x_m)g(x)\,\mathrm{d}x\le\int_a^b f(x)g(x)\,\mathrm{d}x\le\int_a^b f(x_M)g(x)\,\mathrm{d}x=\int_a^b M\cdot g(x)\,\mathrm{d}x$$

$$m\int_a^b g(x)\,\mathrm{d}x=\int_a^b m\cdot g(x)\,\mathrm{d}x\le\int_a^b f(x)g(x)\,\mathrm{d}x\le\int_a^b M\cdot g(x)\,\mathrm{d}x=M\int_a^b g(x)\,\mathrm{d}x$$

let

$$I=\int_a^b g(x)\,\mathrm{d}x$$

then

$$m\cdot I\le\int_a^b f(x)g(x)\,\mathrm{d}x\le M\cdot I$$

case 1: $I=0$,

$$0=m\cdot 0=m\cdot I\le\int_a^b f(x)g(x)\,\mathrm{d}x\le M\cdot I=M\cdot 0=0$$

[22] https://math.stackexchange.com/questions/4120732/integral-mean-value-theorem-open-interval

$$0 \leq \int_a^b f(x) g(x) \, dx \leq 0$$

$$\int_a^b f(x) g(x) \, dx = 0$$

let $g(x) = 1$ for $\int_a^b f(x) g(x) \, dx = \int_a^b f(x) \cdot 1 \, dx = \int_a^b f(x) \, dx$

$$0 = I = \int_a^b g(x) \, dx = \int_a^b 1 \, dx = \int_a^b dx = b - a \Rightarrow b = a$$

is trivial.
case 2 (non-trivial): $I \neq 0$,

$$\begin{cases} I = \int_a^b g(x) \, dx \neq 0 \\ g : [a, b] \to \mathbb{R}^+ \cup \{0\} \\ [a, b] = \{x | a \leq x \leq b\} \end{cases} \Rightarrow \begin{cases} I = \int_a^b g(x) \, dx \neq 0 \\ g : [a, b] \to \mathbb{R}^+ \cup \{0\} \\ a < b & \because a = b \Rightarrow I = 0 \end{cases} \Rightarrow I > 0 \tag{3.4.34}$$

$$m \cdot I \leq \int_a^b f(x) g(x) \, dx \leq M \cdot I$$

$I > 0$

$$m \leq \frac{\int_a^b f(x) g(x) \, dx}{I} \leq M$$

$$f(x_m) = m \leq \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \leq M = f(x_M)$$

let $q = \dfrac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx}$

$$f(x_m) = m \leq q = \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \leq M = f(x_M) \tag{3.4.35}$$

$$f(x_m) = m \leq q \leq M = f(x_M)$$

by easier way of proof, able to omit the following case discussion (case 2-1 to 2-3), we can argue like if

$$\begin{aligned} & \forall x \in (a, b) \, (f(x) > q) \\ \Rightarrow & \forall x \in (a, b) \left(\int_a^b f(x) g(x) \, dx > \int_a^b q \cdot g(x) \, dx = q \int_a^b g(x) \, dx \right) \\ \Leftrightarrow & \forall x \in (a, b) \left(\int_a^b f(x) g(x) \, dx > q \int_a^b g(x) \, dx \right) \\ \Rightarrow\!\Leftarrow & q = \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \Rightarrow \int_a^b f(x) g(x) \, dx = q \int_a^b g(x) \, dx \end{aligned}$$

thus

$$\begin{aligned} & \neg \forall x \in (a, b) \, (f(x) > q) \\ \Leftrightarrow & \exists x \in (a, b) \, \neg (f(x) > q) \\ \Leftrightarrow & \exists x \in (a, b) \, (f(x) \leq q) \end{aligned}$$

with similar argument like the above,

$$\begin{aligned} & \neg \forall x \in (a, b) \, (f(x) < q) \\ \Leftrightarrow & \exists x \in (a, b) \, \neg (f(x) < q) \\ \Leftrightarrow & \exists x \in (a, b) \, (f(x) \geq q) \end{aligned}$$

thus

$$\begin{cases} \exists x \in (a, b) \, (f(x) \leq q) \Rightarrow x_{\leq} \in (a, b) \, (f(x_{\leq}) \leq q) \\ \exists x \in (a, b) \, (f(x) \geq q) \Rightarrow x_{\geq} \in (a, b) \, (f(x_{\geq}) \geq q) \end{cases}$$

by 介值定理 / 中間值定理 intermediate value theorem = IVT 3.3.5,

$$f \text{ continuous on } [a,b] \Rightarrow f \text{ continuous on } (a,b) \Rightarrow \exists c \text{ between } x_{\leq}, x_{\geq} \subseteq (a,b)\,(f(c)=q)$$

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \text{ between } x_{\leq}, x_{\geq} \subseteq (a,b) \left(f(c) = q = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \right)$$

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \text{ between } x_{\leq}, x_{\geq} \subseteq (a,b) \left(f(c) = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \right)$$

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in (a,b) \left(f(c) = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \right)$$

if $g(x) = 1 \geq 0 \Rightarrow I = \int_a^b g(x)\,\mathrm{d}x > 0$ for $\int_a^b f(x)\,g(x)\,\mathrm{d}x = \int_a^b f(x)\cdot 1\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$,

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in (a,b) \left(f(c) = \frac{\int_a^b f(x)\cdot 1\,\mathrm{d}x}{\int_a^b \mathrm{d}x} = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \right)$$

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in (a,b) \left(f(c) = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \right)$$

□

Proof. case 2-1: $m < q < M$,
by properties of the infimum and supremum [23]

$$\exists \begin{cases} \alpha, \beta \in (a,b) \\ a < \alpha < \beta < b \end{cases} \quad [(m < f(\alpha) < q < f(\beta) < M) \vee (m < f(\beta) < q < f(\alpha) < M)]$$

by 介值定理 / 中間值定理 intermediate value theorem = IVT 3.3.5

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in [\alpha,\beta] \subseteq (a,b) \left(f(c) = q = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \in [f(x_m), f(x_M)] \right)$$

if $g(x) = 1 \geq 0 \Rightarrow I = \int_a^b g(x)\,\mathrm{d}x > 0$ for $\int_a^b f(x)\,g(x)\,\mathrm{d}x = \int_a^b f(x)\cdot 1\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$,

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in [\alpha,\beta] \subseteq (a,b) \left(f(c) = \frac{\int_a^b f(x)\cdot 1\,\mathrm{d}x}{\int_a^b \mathrm{d}x} = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \in [f(x_m), f(x_M)] \right)$$

$$f \text{ continuous on } [a,b] \Rightarrow \exists c \in [\alpha,\beta] \subseteq (a,b) \left(f(c) = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \in [f(x_m), f(x_M)] \right)$$

case 2-2: $q = m$,
by inequation 3.4.32,

$$\begin{aligned} m = f(x_m) \leq f(x) \leq f(x_M) = M \\ m \leq f(x) \end{aligned} \tag{3.4.36}$$

$$\begin{aligned} \int_a^b |f(x) - m|\,g(x)\,\mathrm{d}x \overset{f(x)\geq m}{=} & \int_a^b (f(x) - m)\,g(x)\,\mathrm{d}x = \int_a^b f(x)\,g(x)\,\mathrm{d}x - \int_a^b m\cdot g(x)\,\mathrm{d}x \\ & = \int_a^b f(x)\,g(x)\,\mathrm{d}x - m\int_a^b g(x)\,\mathrm{d}x \underset{I>0}{\overset{3.4.34}{=}} \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{I}\cdot I - m\int_a^b g(x)\,\mathrm{d}x \\ & = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}\cdot\int_a^b g(x)\,\mathrm{d}x - m\int_a^b g(x)\,\mathrm{d}x = q\int_a^b g(x)\,\mathrm{d}x - m\int_a^b g(x)\,\mathrm{d}x \\ & = (q - m)\int_a^b g(x)\,\mathrm{d}x \overset{q=m}{=} 0\cdot\int_a^b g(x)\,\mathrm{d}x = 0 \end{aligned}$$

$$\int_a^b (f(x) - m)\,g(x)\,\mathrm{d}x = 0$$

[23] real number property?

and $\begin{cases} I=\int_a^b g(x)\,\mathrm{d}x>0 & 3.4.34 \\ g(x)\geq 0 & 3.4.33 \\ b>a & 3.4.34 \\ f(x)\geq m & 3.4.36 \end{cases}$, thus almost everywhere,

$$(f(x)-m)\,g(x)=0$$

$\begin{cases} g(x)\geq 0 & 3.4.33 \\ I=\int_a^b g(x)\,\mathrm{d}x>0 & 3.4.34 \end{cases} \Rightarrow g(x)>0$ almost everywhere

$$\begin{cases} (f(x)-m)\,g(x)=0 \text{ almost everywhere} \\ g(x)>0 \text{ almost everywhere} \end{cases} \Rightarrow \exists c\in(a,b)\,\{f(c)=m\}$$

$$\exists c\in(a,b)\left\{f(c)=m\overset{q=m}{=}q\overset{3.4.35}{=}\frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}\right\}$$

$$\exists c\in(a,b)\left\{f(c)=\frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}\right\}$$

if $g(x)=1\geq 0\Rightarrow I=\int_a^b g(x)\,\mathrm{d}x>0$ for $\int_a^b f(x)\,g(x)\,\mathrm{d}x=\int_a^b f(x)\cdot 1\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x$,

$$\exists c\in(a,b)\left(f(c)=\frac{\int_a^b f(x)\cdot 1\,\mathrm{d}x}{\int_a^b \mathrm{d}x}=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right)$$

$$\exists c\in(a,b)\left\{f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right\}$$

case 2-3: $q=M$, and the proof is similar to case 2-2. □

定理 3.4.18. 第二積分均值定理 *second mean value theorem for definite integrals = MVTi2*

定理 3.4.19. 第一微積分基本定理 *first fundamental theorem of calculus = FTC1*

$$\begin{cases} f:[a,b]\to\mathbb{R} \\ f \textit{ continuous on } [a,b] \\ F(x)=\int_a^x f(t)\,\mathrm{d}t \end{cases} \Rightarrow \forall x\in[a,b]\,\{F'(x)=f(x)\}$$

Proof.

$$F(x)=\int_a^x f(t)\,\mathrm{d}t$$

$$\begin{aligned} F'(x)&=\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)\,\mathrm{d}t-\int_a^x f(t)\,\mathrm{d}t}{\Delta x} \\ &=\lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{\Delta x} \end{aligned}$$

by 積分均值定理 mean value theorem for definite integrals = MVTi 3.4.17,

$$f \text{ continuous on } [a,b]\Rightarrow\exists c\in(a,b)\left(f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right)$$

$$f \text{ continuous between } x,x+\Delta x\subseteq[a,b]\Rightarrow\exists\xi \text{ between } x,x+\Delta x\left(f(\xi)=\frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{(x+\Delta x)-x}\right)$$

$$\Rightarrow f(\xi)=\frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{(x+\Delta x)-x}=\frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{\Delta x}$$

$$\begin{aligned} F'(x)&=\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)\,\mathrm{d}t-\int_a^x f(t)\,\mathrm{d}t}{\Delta x} \\ &=\lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{\Delta x}=\lim_{\Delta x\to 0} f(\xi) \end{aligned}$$

$$\lim_{\Delta x \to 0} \xi \text{ between } x, x+\Delta x \Rightarrow \lim_{\Delta x \to 0} \xi = \lim_{\xi \to x} \xi = x$$

$$f \text{ continuous on } [a,b] \Rightarrow \lim_{\Delta x \to 0} f(\xi) = \lim_{\xi \to x} f(\xi) = f\left(\lim_{\xi \to x} \xi\right) = f(x)$$

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)\,\mathrm{d}t}{\Delta x} = \lim_{\Delta x \to 0} f(\xi) = \lim_{\xi \to x} f(\xi) = f\left(\lim_{\xi \to x} \xi\right) = f(x) \end{aligned}$$

$$F'(x) = f(x)$$

□

定理 3.4.20. 第二微積分基本定理 *second fundamental theorem of calculus = FTC2*

$$\begin{cases} f : [a,b] \to \mathbb{R} \\ f \text{ continuous on } [a,b] \\ F'(x) = f(x) \end{cases} \Rightarrow \forall x \in [a,b] \left\{ \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \right\}$$

Proof. let

$$G(x) = \int_a^x f(t)\,\mathrm{d}t$$

by 第一微積分基本定理 first fundamental theorem of calculus = FTC1 3.4.19,

$$G'(x) = f(x)$$

by assumption,

$$F'(x) = f(x)$$

thus

$$(F-G)'(x) \overset{\text{addition rule or linearity of derivative}}{=} F'(x) - G'(x) = f(x) - f(x) = 0\,\forall x \in [a,b]$$

by lemma 3.4.21,

$$(F-G)'(x) = 0 \Rightarrow (F-G)(x) = C = \text{constant}\,\forall x \in [a,b]$$

$$F(x) = G(x) + C$$

$$\begin{aligned} F(b) - F(a) &= (G(b)+C) - (G(a)+C) = G(b) - G(a) \\ &= \int_a^b f(t)\,\mathrm{d}t - \int_a^a f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t - 0 \\ &= \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x \end{aligned}$$

□

問題

1. f continuous on $[a,b] \overset{?}{\Rightarrow} f$ Riemann integrable on $[a,b]$
   大哉問, answered by 均勻連續 uniformly continuity, — 督脈 暫且打住
2. proof of 第一微積分基本定理 first fundamental theorem of calculus = FTC1 3.4.19
   第一微積分基本定理 的 溯源 — 任脈
3. $\forall x \in [a,b] \left\{(F-G)'(x) = 0\right\} \overset{?}{\Rightarrow} \forall x \in [a,b] \{(F-G)(x) = C = \text{constant}\}$

   引理 3.4.21. 應該就是 微分方程基本推論 / 微分方程根本補題 *fundamental corollary of differential equation = FCDE 3.4.6*

$$\begin{cases} f : [a,b] \to \mathbb{R} \\ f \text{ continuous on } [a,b] \\ \forall x \in [a,b]\,(f'(x) = 0) \end{cases} \Rightarrow \forall x \in [a,b] \{f(x) = C = \textit{constant}\}$$

Proof. by 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3

$$\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.19}{\Rightarrow} \exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}$$

$$\begin{aligned}&\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,x)\ \forall x\in(a,b] & (2)\end{cases} \Rightarrow \exists c\in(a,x)\left\{f'(c)(x-a)=f(x)-f(a)\right\}\\ &\Rightarrow f(x)-f(a)=f'(c)(x-a)\\ &\underset{c\in(a,x)\subset[a,b]}{\overset{\forall x\in[a,b](f'(x)=0)}{=}} 0\cdot(x-a)=0\\ &\Rightarrow f(x)-f(a)=0\\ &\Rightarrow f(x)=f(a)\end{aligned}$$

$$\begin{cases}\forall x\in(a,b]\left\{f(x)=f(a)\right\}\\ f(a)=f(a)\end{cases} \Rightarrow \forall x\in[a,b]\left\{f(x)=f(a)=\text{constant}\right\}$$

□

定理 3.4.22. 牛頓-萊布尼茨公式 *Newton-Leibniz formula = N-LF, equivalent to* 第一微分均值定理 / 拉格朗日均值定理 *first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3* [24]

$$\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a) \tag{3.4.37}$$

$$\begin{aligned}&\begin{cases}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \textit{ continuous on } [a,b] & (1)\\ f \textit{ differentiable on } (a,b) & (2)\end{cases}\\ \begin{cases} f':[a,b]\to\mathbb{R}\ (4) & a<b\ (3)\\ f' \textit{ continuous on } [a,b] & (5)\end{cases}\end{cases} \Rightarrow \exists c\in(a,b)\left(f'(c)=\frac{\int_a^b f'(x)\,\mathrm{d}x}{b-a}\right) \quad \textit{MVTi1 } 3.4.17\\ &\Rightarrow \begin{array}{c}\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}\ 3.4.39\\ \Updownarrow\\ \int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\ 3.4.38\end{array}\end{aligned}$$

($\Downarrow$):

$$\begin{cases}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \textit{ continuous on } [a,b] & (1)\\ f \textit{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.2}{\Rightarrow} \exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\} \quad \textit{MVTd1 } 3.4.3\\ [a,b]=[x_0,x_n]=[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n] \quad 3.4.40\end{cases}$$

$$\overset{3.4.2}{\underset{3.4.3}{\Rightarrow}} \int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a) \tag{3.4.38}$$

($\Uparrow$):

$$\begin{cases}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \textit{ continuous on } [a,b] & (1)\\ f \textit{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.2}{\underset{3.4.3}{\Rightarrow}} \int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a) \quad \textit{N-LF } 3.4.37\\ \begin{cases} f':[a,b]\to\mathbb{R}\ (4) & a<b\ (3)\\ f' \textit{ continuous on } [a,b] & (5)\end{cases} \Rightarrow \exists c\in(a,b)\left(f'(c)=\frac{\int_a^b f'(x)\,\mathrm{d}x}{b-a}\right) \quad \textit{MVTi1 } 3.4.17\end{cases}$$

$$\Rightarrow \exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\} \tag{3.4.39}$$

[24] 數學傳播_33_2_蔡聰明_均值定理的統合與推廣

Proof. ($\Downarrow$):

重疊端點分割 $[a,b]$ 成 n 部分聯集 $(n \in \mathbb{N})$

$$[a,b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \tag{3.4.40}$$

$$\begin{cases} \begin{cases} f : [a,b] \to \mathbb{R} & (0) \\ f \text{ continuous on } [a,b] & (1) \\ f \text{ differentiable on } (a,b) & (2) \end{cases} \overset{3.4.2}{\Rightarrow} \exists c \in (a,b) \left\{ f'(c)(b-a) = f(b) - f(a) \right\} & \text{MVTd1 } 3.4.3 \\ [a,b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] & 3.4.40 \end{cases}$$

$$\Rightarrow \begin{cases} (f : [x_0, x_n] \to \mathbb{R}) = (f : [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \to \mathbb{R}) & \Leftarrow (0) \\ f \text{ continuous on } [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] & \Leftarrow (1) \\ f \text{ differentiable on } (x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n) & \Leftarrow (2) \end{cases}$$

$$\Rightarrow \begin{cases} f : [x_0, x_1] \to \mathbb{R}, f : [x_0, x_2] \to \mathbb{R}, \cdots, f : [x_{n-1}, x_n] \to \mathbb{R} & \Leftarrow (0) \\ f \text{ continuous on } [x_0, x_1], [x_0, x_2], \cdots, [x_{n-1}, x_n] & \Leftarrow (1) \\ f \text{ differentiable on } (x_0, x_1], [x_0, x_2], \cdots, [x_{n-1}, x_n) \Rightarrow f \text{ differentiable on } (x_0, x_1), (x_0, x_2), \cdots, (x_{n-1}, x_n) & \Leftarrow (2) \end{cases}$$

$$\overset{3.4.2}{\underset{3.4.3}{\Rightarrow}} \begin{cases} \exists c_1 \in (x_0, x_1) \left\{ f'(c_1)(x_1 - x_0) = f(x_1) - f(x_0) \right\} \\ \exists c_2 \in (x_1, x_2) \left\{ f'(c_2)(x_2 - x_1) = f(x_2) - f(x_1) \right\} \\ \vdots \\ \exists c_k \in (x_{k-1}, x_k) \left\{ f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1}) \right\} \qquad \forall k \in \mathbb{N} \cap [1, n] \\ \vdots \\ \exists c_n \in (x_{n-1}, x_n) \left\{ f'(c_n)(x_n - x_{n-1}) = f(x_n) - f(x_{n-1}) \right\} \end{cases}$$

$$\Rightarrow f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$$

$$\Rightarrow \sum_k f'(c_k)(x_k - x_{k-1}) = \sum_k f(x_k) - f(x_{k-1})$$

$$\Rightarrow \sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \cdots + [f(x_n) - f(x_{n-1})]$$

$$= f(x_n) - f(x_0) = f(b) - f(a) \Rightarrow \sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1}) = f(b) - f(a) \overset{\Delta x_k = x_k - x_{k-1}}{\Leftrightarrow} \sum_{k=1}^{n} f'(c_k)\Delta x_k = f(b) - f(a)$$

$$\Rightarrow \lim_{n \to \infty} \lim_{\Delta x_k = \frac{x_n - x_0}{n}} \sum_{k=1}^{n} f'(c_k)\Delta x_k = \lim_{n \to \infty} \lim_{\Delta x_k = \frac{b-a}{n} \to 0} f(b) - f(a) = f(b) - f(a)$$

$$\Rightarrow \int_a^b f'(x)\,\mathrm{d}x = \int_{x_0}^{x_n} f'(x)\,\mathrm{d}x = f(b) - f(a)$$

$$\Rightarrow \int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

得到 牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF 3.4.37 □

Proof. (⇑):

$$\left\{\begin{array}{l}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.2}{\underset{}{\overset{3.4.3}{\Rightarrow}}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \quad \text{N-LF } 3.4.37\\ \begin{cases} f:[a,b]\to\mathbb{R} & a<b\ (3)\\ f \text{ continuous on } [a,b] & (1)\end{cases} \Rightarrow \exists c\in(a,b)\left(f(c)=\frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right) \quad \text{MVTi1 } 3.4.17\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.2}{\underset{}{\overset{3.4.3}{\Rightarrow}}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \quad \text{N-LF } 3.4.37\\ \begin{cases} f:[a,b]\to\mathbb{R} & a<b\ (3)\\ f \text{ continuous on } [a,b] & (1)\end{cases} \Rightarrow \exists c\in(a,b)\left\{f(c)(b-a)=\int_a^b f(x)\,\mathrm{d}x\right\} \quad \text{MVTi1 } 3.4.17\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l}\begin{cases} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{cases} \overset{3.4.2}{\underset{}{\overset{3.4.3}{\Rightarrow}}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \quad \text{N-LF } 3.4.37\\ \begin{cases} \text{if } f':[a,b]\to\mathbb{R} & (4)\\ \text{if } f' \text{ continuous on } [a,b] & (5)\end{cases} \Rightarrow \exists c\in(a,b)\left\{f'(c)(b-a)=\int_a^b f'(x)\,\mathrm{d}x\right\} \quad \text{MVTi1 } 3.4.17\end{array}\right.$$

$$\Rightarrow\exists c\in(a,b)\left\{f'(c)(b-a)=\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\right\}$$

$$\Rightarrow\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}$$

$$\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}$$

得到 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 3.4.3 □

定理 3.4.23. *univariable product rule*

$$\mathrm{d}(uv)=(\mathrm{d}u)v+u\mathrm{d}v=v\mathrm{d}u+u\mathrm{d}v$$

$$\frac{\mathrm{d}(uv)}{\mathrm{d}t}=v\frac{\mathrm{d}u}{\mathrm{d}t}+u\frac{\mathrm{d}v}{\mathrm{d}t}$$

定理 3.4.24. *integration by parts*

$$\begin{aligned}\int\frac{\mathrm{d}(uv)}{\mathrm{d}t}\mathrm{d}t &= \int v\frac{\mathrm{d}u}{\mathrm{d}t}+u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t\\ \int\mathrm{d}(uv)= &= \int v\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t+\int u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t\\ uv= &= \int v\mathrm{d}u+\int u\mathrm{d}v\\ \int u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t &= uv-\int v\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \text{ "switching derivatives"} \quad (3.4.41)\\ \int u\mathrm{d}v &= uv-\int v\mathrm{d}u \text{ "switching differentials"} \quad (3.4.42)\end{aligned}$$

定理 3.4.25. *univariable Taylor theorem*

Proof.

$$
\begin{aligned}
& f(x)-f(a) \\
\overset{3.4.22}{=}\ & \int_a^x f'(t)\,\mathrm{d}t=\int_a^x f'(t)\frac{\mathrm{d}(t-x)}{\mathrm{d}t}\mathrm{d}t \overset{3.4.41}{=} \left[f'(t)(t-x)\right]_{t=a}^{x}-\int_a^x (t-x)\frac{\mathrm{d}f'(t)}{\mathrm{d}t}\mathrm{d}t \\
=\ & \left[-f'(a)(a-x)\right]-\int_a^x f''(t)(t-x)\,\mathrm{d}t=f'(t)(x-a)-\int_a^x f''(t)\frac{\mathrm{d}\frac{(t-x)^2}{2}}{\mathrm{d}t}\mathrm{d}t \\
=\ & f'(a)(x-a)-\left(\left[f''(t)\frac{(t-x)^2}{2}\right]_{t=a}^{x}-\int_a^x \frac{(t-x)^2}{2}\frac{\mathrm{d}f''(t)}{\mathrm{d}t}\mathrm{d}t\right) \\
=\ & f'(a)(x-a)-\left(\left[-f''(a)\frac{(x-a)^2}{2}\right]-\int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t\right) \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\int_a^x f'''(t)\frac{\mathrm{d}\frac{(t-x)^3}{2\cdot 3}}{\mathrm{d}t}\mathrm{d}t \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\left(\left[f'''(t)\frac{(t-x)^3}{2\cdot 3}\right]_{t=a}^{x}-\int_a^x \frac{(t-x)^3}{2\cdot 3}\frac{\mathrm{d}f'''(t)}{\mathrm{d}t}\mathrm{d}t\right) \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\left(\left[f'''(a)\frac{(x-a)^3}{2\cdot 3}\right]-\int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t\right) \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+f'''(t)\frac{(x-a)^3}{2\cdot 3}-\int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+f'''(t)\frac{(x-a)^3}{2\cdot 3}-\int_a^x f^{(4)}(t)\frac{\mathrm{d}\frac{(t-x)^4}{2\cdot 3\cdot 4}}{\mathrm{d}t}\mathrm{d}t \\
=\ & f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+f'''(t)\frac{(x-a)^3}{2\cdot 3} \\
& -\left(\left[f^{(4)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\right]_{t=a}^{x}-\int_a^x \frac{(t-x)^4}{2\cdot 3\cdot 4}\frac{\mathrm{d}f^{(4)}(t)}{\mathrm{d}t}\mathrm{d}t\right) \\
=\ & f'(t)(x-a)+f''(a)\frac{(x-a)^2}{2}+f'''(t)\frac{(x-a)^3}{2\cdot 3} \\
& -\left(\left[-f^{(4)}(a)\frac{(x-a)^4}{2\cdot 3\cdot 4}\right]-\int_a^x f^{(5)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\mathrm{d}t\right) \\
=\ & f'(t)(x-a)+f''(a)\frac{(x-a)^2}{2}+f'''(a)\frac{(x-a)^3}{2\cdot 3}+f^{(4)}(a)\frac{(x-a)^4}{2\cdot 3\cdot 4} \\
\vdots\ & +\int_a^x f^{(5)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\mathrm{d}t \\
=\ & \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\int_a^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\mathrm{d}t \text{ remainder in integral form}
\end{aligned}
$$

□

- remainder in Lagrange form

Proof.

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT 3.3.9 and 連續函數 介值定理 / 中間值定理 continuous function intermediate value theorem = CFIVT / intermediate value theorem = IVT 3.3.5

$$
\begin{array}{rcl}
\text{let } f(x) & = & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + r_n(x) \\
r_n(x) & = & \int_a^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\mathrm{d}t \\
\text{let } n & \in & \{2k-1 | k \in \mathbb{N}\} \\
\text{let } f(t) & \overset{3.3.9}{\in} & [m, M] \subseteq f((a,x)) \text{ when } a < x \\
\int_a^x m\frac{(t-x)^n}{n!}\mathrm{d}t \leq & r_n(x) & \leq \int_a^x M\frac{(t-x)^n}{n!}\mathrm{d}t \\
m\int_a^x \frac{(t-x)^n}{n!}\mathrm{d}t = & & = M\int_a^x \frac{(t-x)^n}{n!}\mathrm{d}t \\
m\left[\frac{(t-x)^{n+1}}{(n+1)!}\right]_{t=a}^{x} = & & = M\left[\frac{(t-x)^{n+1}}{(n+1)!}\right]_{t=a}^{x} \\
m\frac{(x-a)^{n+1}}{(n+1)!} = & & = M\frac{(x-a)^{n+1}}{(n+1)!} \\
 & \Downarrow & 3.3.5 \\
r_n(x) & \overset{\exists\xi\in(a,x)}{=} & f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \\
f(x) & = & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \\
 & & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}
\end{array}
$$

□

- remainder in big O form

Proof.

$$
\begin{array}{rcl}
\text{if } \left|f^{(n+1)}(t)\right| & \leq & K\,\forall t \in (a,x) \\
|r_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}\right| & \leq & \frac{K}{(n+1)!}|x-a|^{n+1} \\
r_n(x) & \in & \mathrm{O}\left((x-a)^{n+1}\right) \\
\text{let } R_n(h) & = & r_n(a+h) \\
R_n(h) & \in & \mathrm{O}\left(h^{n+1}\right) \\
f(a+h) & = & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} \\
 & = & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \mathrm{O}\left(h^{n+1}\right)
\end{array}
$$

□

- multivariable Taylor theorem

$$
\begin{aligned}
\text{let } \boldsymbol{x}(t) &= \boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}) = \boldsymbol{a} + t\boldsymbol{h} \\
\text{let } g(t) &= f(\boldsymbol{x}(t)) \\
g'(t) &= \mathrm{d}_t f(\boldsymbol{x}(t)) = \nabla f(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t) \\
&= \nabla f(\boldsymbol{a} + t\boldsymbol{h}) \cdot \mathrm{d}_t(\boldsymbol{a} + t\boldsymbol{h}) \\
&= \nabla f(\boldsymbol{a} + t\boldsymbol{h}) \cdot \boldsymbol{h} = \boldsymbol{h} \cdot \nabla f(\boldsymbol{a} + t\boldsymbol{h}) \\
g''(t) &= \mathrm{d}_t^2 f(\boldsymbol{x}(t)) \\
&= (\boldsymbol{x}'(t) \cdot \nabla)^2 f(\boldsymbol{x}(t)) + \boldsymbol{x}''(t) \cdot \nabla f(\boldsymbol{x}(t)) \\
&= (\boldsymbol{h} \cdot \nabla)^2 f(\boldsymbol{a} + t\boldsymbol{h}) + \boldsymbol{0} \cdot \nabla f(\boldsymbol{a} + t\boldsymbol{h}) \\
&= (\boldsymbol{h} \cdot \nabla)^2 f(\boldsymbol{a} + t\boldsymbol{h}) \\
&\vdots \\
g^{(k)}(t) &= (\boldsymbol{h} \cdot \nabla)^k f(\boldsymbol{a} + t\boldsymbol{h})
\end{aligned}
$$

$$
\begin{aligned}
g(t) &\overset{3.4.25}{=} \sum_{k=0}^{n} \frac{g^{(k)}(b)}{k!}(t-b)^k + \int_b^t g^{(n+1)}(w) \frac{(w-t)^n}{n!} \mathrm{d}w \\
g(1) &\overset{\text{let} \begin{cases} t = 1 \\ b = 0 \end{cases}}{=} \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!}(1-0)^k + \int_0^1 g^{(n+1)}(w) \frac{(w-1)^n}{n!} \mathrm{d}w \\
f(\boldsymbol{x}) = f(\boldsymbol{a} + \boldsymbol{h}) = &= \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} + \int_0^1 g^{(n+1)}(w) \frac{(w-1)^n}{n!} \mathrm{d}w \\
&= \sum_{k=0}^{n} \frac{(\boldsymbol{h} \cdot \nabla)^k f(\boldsymbol{a} + 0\boldsymbol{h})}{k!} + \int_0^1 (\boldsymbol{h} \cdot \nabla)^k f(\boldsymbol{a} + w\boldsymbol{h}) \frac{(w-1)^n}{n!} \mathrm{d}w \\
&= \sum_{k=0}^{n} \frac{(\boldsymbol{h} \cdot \nabla)^k f(\boldsymbol{a})}{k!} + \int_0^1 (\boldsymbol{h} \cdot \nabla)^k f(\boldsymbol{a} + w\boldsymbol{h}) \frac{(w-1)^n}{n!} \mathrm{d}w \\
&= \sum_{k=0}^{n} \frac{((\boldsymbol{x} - \boldsymbol{a}) \cdot \nabla)^k f(\boldsymbol{a})}{k!} \\
&\quad + \int_0^1 ((\boldsymbol{x} - \boldsymbol{a}) \cdot \nabla)^k f(\boldsymbol{a} + w(\boldsymbol{x} - \boldsymbol{a})) \frac{(w-1)^n}{n!} \mathrm{d}w \\
&= \sum_{k=0}^{n} \frac{((\boldsymbol{x} - \boldsymbol{a}) \cdot \nabla)^k f(\boldsymbol{a})}{k!} + \frac{((\boldsymbol{x} - \boldsymbol{a}) \cdot \nabla)^{n+1} f(\boldsymbol{\xi})}{(n+1)!} \\
f(\boldsymbol{x}) &= \sum_{k=0}^{n} \frac{((\boldsymbol{x} - \boldsymbol{a}) \cdot \nabla)^k f(\boldsymbol{a})}{k!} + r_n(\boldsymbol{x}) \qquad (3.4.43)
\end{aligned}
$$

- second derivative test

$$
\begin{aligned}
f(x+h) - f(x) &\overset{3.4.25}{=} h \cdot f'(x) + \frac{h^2}{2} f''(x) + R_2(h) \\
&\approx \frac{h^2}{2} f''(x) \ \text{ if } \begin{cases} f'(x) = 0 \\ R_2(h) \in \mathrm{O}\left(h^3\right) \end{cases} \\
f(x+h) & \begin{cases} > f(x) & f''(x) > 0 \\ \gtreqqless f(x) & f''(x) = 0 \\ > f(x) & f''(x) < 0 \end{cases}
\end{aligned}
$$

- multivariable second derivative test

$$
\begin{aligned}
f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x}) &\overset{3.4.25}{=} (\boldsymbol{h} \cdot \nabla) f(\boldsymbol{x}) + \frac{(\boldsymbol{h} \cdot \nabla)^2}{2} f(\boldsymbol{x}) + R_2(\boldsymbol{h}) \\
&\approx \frac{(\boldsymbol{h} \cdot \nabla)^2}{2} f(\boldsymbol{x}) \ \text{ if } \begin{cases} \nabla f(\boldsymbol{x}) = \boldsymbol{0} \\ R_2(\boldsymbol{h}) \in \mathrm{O}\left(\boldsymbol{h}^3\right) \end{cases} \\
2(f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})) &\approx (\boldsymbol{h} \cdot \nabla)^2 f(\boldsymbol{x}) \ \text{ if } \begin{cases} \nabla f(\boldsymbol{x}) = \boldsymbol{0} \\ R_2(\boldsymbol{h}) \in \mathrm{O}\left(\boldsymbol{h}^3\right) \end{cases}
\end{aligned}
$$

$$\begin{aligned}
\text{let } \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{h} \end{bmatrix} &= \begin{bmatrix} (x,y) \\ (h,k) \end{bmatrix} \\
2\left(f\left(x+h,y+k\right)-f\left(x,y\right)\right) &\approx f_{xx}h^2+2f_{xy}hk+f_{yy}k^2 \\
&= \begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{f_{xx}f_{yy}-(f_{xy})^2}{f_{xx}}k^2 & f_{xx}\neq 0 \\ f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{f_{xx}f_{yy}-(f_{xy})^2}{f_{yy}}h^2 & f_{yy}\neq 0 \\ 2f_{xy}hk & f_{xx}=f_{yy}=0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{Hessian matrix } H\left(x,y\right) &= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\
\text{Hessian discriminant } \left|H\left(x,y\right)\right| = \det H\left(x,y\right) &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-f_{xy}f_{yx}=f_{xx}f_{yy}-\left(f_{xy}\right)^2
\end{aligned}$$

$$\begin{aligned}
&2\left(f\left(x+h,y+k\right)-f\left(x,y\right)\right) \\
\approx &\begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{f_{xx}f_{yy}-(f_{xy})^2}{f_{xx}}k^2 & f_{xx}\neq 0 \\ f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{f_{xx}f_{yy}-(f_{xy})^2}{f_{yy}}h^2 & f_{yy}\neq 0 \\ 2f_{xy}hk & f_{xx}=f_{yy}=0 \end{cases} \\
= &\begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{|H(x,y)|}{f_{xx}}k^2 & f_{xx}\neq 0 \\ f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{|H(x,y)|}{f_{yy}}h^2 & f_{yy}\neq 0 \\ \begin{cases} 2f_{xy}hk & f_{xy}\neq 0 \\ 0 & f_{xy}=0 \end{cases} & f_{xx}=f_{yy}=0 \end{cases} \\
= &\begin{cases} \begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{|H(x,y)|}{f_{xx}}k^2 & f_{yy}\neq 0 \\ f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2-\frac{(f_{xy})^2}{f_{xx}}k^2 & f_{yy}=0 \end{cases} & f_{xx}\neq 0 \\ \begin{cases} f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{|H(x,y)|}{f_{yy}}h^2 & f_{xx}\neq 0 \\ f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2-\frac{(f_{xy})^2}{f_{yy}}h^2 & f_{xx}=0 \end{cases} & f_{yy}\neq 0 \\ \begin{cases} 2f_{xy}hk & f_{xy}\neq 0 \\ 0 & f_{xy}=0 \end{cases} & f_{xx}=f_{yy}=0 \end{cases} \\
= &\begin{cases} \begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{|H(x,y)|}{f_{xx}}k^2 & f_{yy}\neq 0 \\ f_{xx}h^2+2f_{xy}hk & f_{yy}=0 \end{cases} & f_{xx}\neq 0 \\ \begin{cases} f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{|H(x,y)|}{f_{yy}}h^2 & f_{xx}\neq 0 \\ f_{yy}k^2+2f_{xy}hk & f_{xx}=0 \end{cases} & f_{yy}\neq 0 \\ \begin{cases} 2f_{xy}hk & f_{xy}\neq 0 \\ 0 & f_{xy}=0 \end{cases} & f_{xx}=f_{yy}=0 \end{cases} \\
= &\begin{cases} \begin{cases} f_{xx}\left(h+\frac{f_{xy}}{f_{xx}}k\right)^2+\frac{|H(x,y)|}{f_{xx}}k^2 & f_{yy}\neq 0 \\ \begin{cases} h\left(f_{xx}h+2f_{xy}k\right) & f_{xy}\neq 0 \\ f_{xx}h^2 & f_{xy}=0 \end{cases} & f_{yy}=0 \end{cases} & f_{xx}\neq 0 \\ \begin{cases} f_{yy}\left(k+\frac{f_{xy}}{f_{yy}}h\right)^2+\frac{|H(x,y)|}{f_{yy}}h^2 & f_{xx}\neq 0 \\ \begin{cases} k\left(f_{yy}k+2f_{xy}h\right) & f_{xy}\neq 0 \\ f_{yy}k^2 & f_{xy}=0 \end{cases} & f_{xx}=0 \end{cases} & f_{yy}\neq 0 \\ \begin{cases} 2f_{xy}hk & f_{xy}\neq 0 \\ 0 & f_{xy}=0 \end{cases} & f_{xx}=f_{yy}=0 \end{cases}
\end{aligned}$$

$$f\left(x+h,y+k\right)\begin{cases}\begin{cases}>f\left(x,y\right) & \left(f_{xx}>0\right)\vee\left(f_{yy}>0\right)\\ <f\left(x,y\right) & \left(f_{xx}<0\right)\vee\left(f_{yy}<0\right)\end{cases} & & \left|H\left(x,y\right)\right|>0\\ \begin{cases}\begin{cases}>f\left(x,y\right) & f_{xx}>0\\ <f\left(x,y\right) & f_{xx}<0\end{cases} & f_{xx}\neq0\\ \begin{cases}>f\left(x,y\right) & f_{yy}>0\\ <f\left(x,y\right) & f_{yy}<0\end{cases} & f_{yy}\neq0\\ \text{saddle point} & \\ \gtreqless f\left(x,y\right) & f_{xx}=f_{yy}=f_{xy}=0\end{cases} & & \left|H\left(x,y\right)\right|=0\\ \text{saddle point} & & \left|H\left(x,y\right)\right|<0\end{cases}$$

Chapter 4

線性代數 linear algebra

定理 4.0.1. *Vandermonde determinant* [1]

Proof.

$$\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$\begin{aligned}
\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} &= \begin{vmatrix} 1 & x_1 & x_1^2 - x_1x_1 \\ 1 & x_2 & x_2^2 - x_2x_1 \\ 1 & x_3 & x_3^2 - x_3x_1 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & x_2\left(x_2 - x_1\right) \\ 1 & x_3 & x_3\left(x_3 - x_1\right) \end{vmatrix} \\
&= \begin{vmatrix} 1 & x_1 - x_1 & 0 \\ 1 & x_2 - x_1 & x_2\left(x_2 - x_1\right) \\ 1 & x_3 - x_1 & x_3\left(x_3 - x_1\right) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & x_2\left(x_2 - x_1\right) \\ 1 & x_3 - x_1 & x_3\left(x_3 - x_1\right) \end{vmatrix} \\
&= \begin{vmatrix} x_2 - x_1 & x_2\left(x_2 - x_1\right) \\ x_3 - x_1 & x_3\left(x_3 - x_1\right) \end{vmatrix} = \left(x_2 - x_1\right) \begin{vmatrix} 1 & x_2 \\ x_3 - x_1 & x_3\left(x_3 - x_1\right) \end{vmatrix} \\
&= \left(x_2 - x_1\right)\left(x_3 - x_1\right) \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = \left(x_2 - x_1\right)\left(x_3 - x_1\right)\left(x_3 - x_2\right) \\
&= \left(x_2 - x_1\right)\left(x_3 - x_2\right)\left(x_3 - x_1\right)
\end{aligned}$$

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = -\left(x_2 - x_1\right)\left(x_3 - x_2\right)\left(x_3 - x_1\right) = \left(x_2 - x_1\right)\left(x_3 - x_2\right)\left(x_1 - x_3\right)$$

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (b-a)(c-b)(a-c) = (b-a)(a-c)(c-b) = (a-c)(c-b)(b-a)$$

□

[1] https://ccjou.wordpress.com/2009/12/22/特殊矩陣-八：vandermonde-矩陣/

Chapter 5

代數 / 抽象代數 algebra / abstract algebra

Chapter 6

幾何 geometry

6.1 歐氏幾何 Euclidean geometry

6.1.1 $\mathbb{R}^2$ 平面幾何 plane geometry

事實 6.1.1. 無定義名詞 點 *point*

$$P$$

事實 6.1.2. 無定義名詞 直線 *line*

$$L$$

事實 6.1.3. 無定義關係 線上一點

$$P \in L \text{ or } L \ni P$$

公理 6.1.4. 歐基里德平面幾何公理 / 平面幾何公設 *Euclidean axioms for plane geometry*

1. 兩點可連一(直)線段

2. (直)線段可任意延長

3. 給定一點 O 及 一線段長 r, 可以 O 為圓心, r 為半徑作一圓

4. 凡直角皆相等:

   相異兩對直線形成兩直角, 將該相異兩對直線之各交點及各一直線重合, 則剩餘兩直線重合

5. 平行公理 / 平行公設 parallel axiom:

   - 版本1: 給定一直線之兩相異點各交於另兩相異直線, 形成之同位角兩角和小於 $180°$, 則該另兩相異直線在該給定一直線之該同位角之同側相交
   - 版本2: 過給定一直線外給定一點, 可作另唯一直線與該給定直線平行

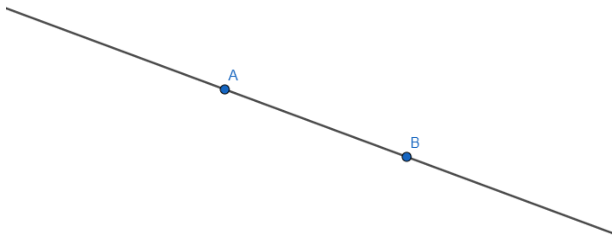
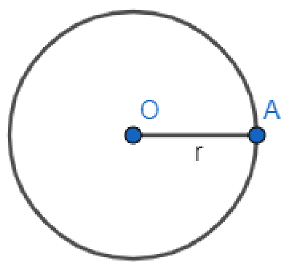
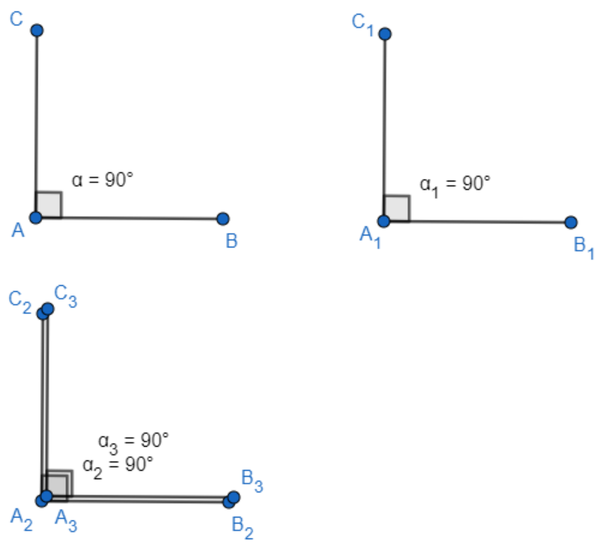
axiom	illustration
1. 兩點可連一(直)線段	 two points A and B have a segment AB or $\overline{AB}$
2. (直)線段可任意延長	 two points A and B actually have a line $\overleftrightarrow{AB}$
3. 可以一點為圓心, 給定線段長為半徑作一圓	 creating the circle through one point A with center a point O, $\overline{OA} = r$
4. 凡直角皆相等	 all right angles are equal to one another

Table 6.1.2: Euclidean axioms for plane geometry

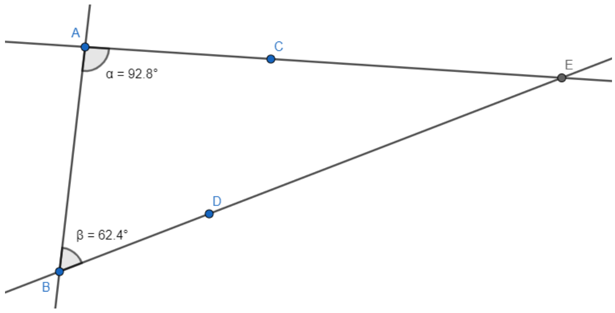
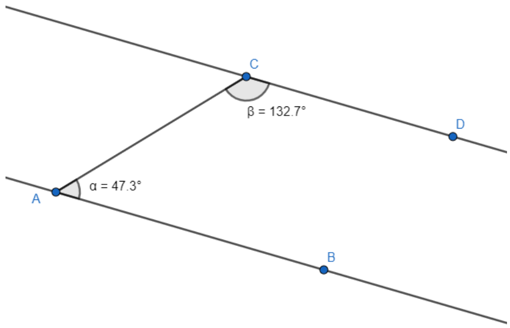
axiom	illustration
5. 平行公設 parallel axiom	 $\angle CAB + \angle DBA < 180° \Rightarrow \exists E \in \overleftrightarrow{AC} \cap \overleftrightarrow{BD}$  $\exists! \overleftrightarrow{CD} \ni C \, (\angle DCA + \angle BAC = 180°)$

Table 6.1.3: The fifth Euclidean axiom for plane geometry

公理 6.1.5. *Hilbert axioms for plane geometry*

Euclidean axioms are not complete.

定義 6.1.6. 平面角 angle

angle defined by ratio of arc length on a circle C

$$\frac{l}{L} = \frac{l_{\text{arc}}}{L_C} = \frac{\theta \cdot r}{2\pi \cdot r}$$

定理 6.1.7. 畢達哥拉斯定理 / 畢氏定理 / 百牛定理 / 勾股定理 / 商高定理 / 新娘座椅定理 *Pythagorean theorem / Pythagoras' theorem*

6.1.1.1 圓函數 / 三角函數 / 角函數 circular function / trigonometric function / angular function / goniometric function

定理 6.1.8.

$$0 \leq \alpha \leq \frac{\pi}{2} \Rightarrow 0 \leq \sin \alpha \leq \alpha \leq \tan \alpha \leq \infty$$

[1]

[1]數學傳播_23_2_徐正梅_談地球上兩點間的球面距離

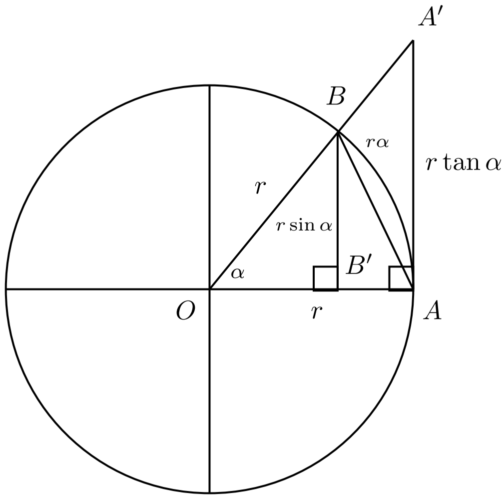


Figure 6.1.1: $0 \leq \alpha \leq \frac{\pi}{2} \Rightarrow 0 \leq \sin\alpha \leq \alpha \leq \tan\alpha \leq \infty$

Proof. according to the figure 6.1.1,

$O\overset{\frown}{AB}$ is a circular sector (扇形),

$$0 \leq a\triangle OAB \leq aO\overset{\frown}{AB} \leq a\triangle OAA'$$

$$0 \leq \frac{1}{2} \cdot \overline{OA} \cdot \overline{OB} \cdot \sin\alpha \leq \pi r^2 \cdot \frac{\alpha}{2\pi} \leq \frac{1}{2} \cdot \overline{OA} \cdot \overline{AA'}$$

$$0 \leq \frac{1}{2} \cdot r \cdot r \cdot \sin\alpha \leq r^2 \cdot \frac{\alpha}{2} \leq \frac{1}{2} \cdot r \cdot r\tan\alpha$$

$$0 \leq \frac{r^2}{2}\sin\alpha \leq \frac{r^2}{2}\alpha \leq \frac{r^2}{2}\tan\alpha$$

$\frac{r^2}{2} > 0$,

$$0 \leq \sin\alpha \leq \alpha \leq \tan\alpha$$

□

定理 6.1.9.

$$0 < \alpha < \beta \leq \frac{\pi}{2}, \frac{\sin\beta}{\sin\alpha} < \frac{\beta}{\alpha} \Leftrightarrow \frac{\sin\alpha}{\alpha} > \frac{\sin\beta}{\beta} \Leftrightarrow \beta\sin\alpha - \alpha\sin\beta > 0$$

[2]

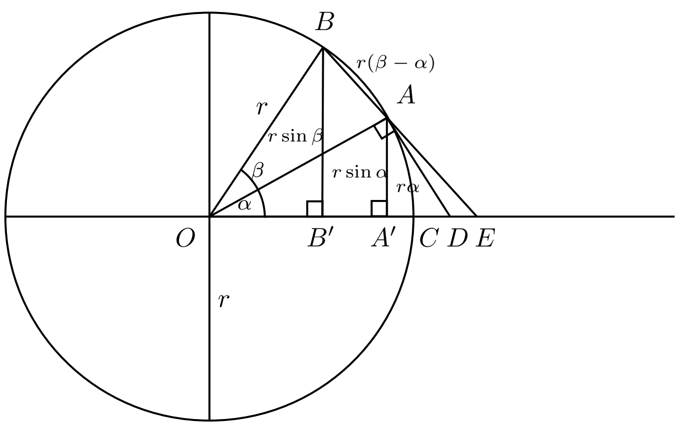


Figure 6.1.2: $0 < \alpha < \beta \leq \frac{\pi}{2}, \frac{\sin\beta}{\sin\alpha} < \frac{\beta}{\alpha} \Leftrightarrow \frac{\sin\alpha}{\alpha} > \frac{\sin\beta}{\beta}$

[2] 數學傳播_23_2_徐正梅_談地球上兩點間的球面距離

Proof. according to the figure 6.1.2,

$$\frac{\sin\beta}{\sin\alpha}=\frac{r\sin\beta}{r\sin\alpha}=\frac{\overline{BB'}}{\overline{AA'}}\overset{\triangle EAA'\sim\triangle EBB'}{=}\frac{\overline{EB}}{\overline{EA}}=\frac{\overline{EA}+\overline{AB}}{\overline{EA}}=1+\frac{\overline{AB}}{\overline{EA}}\overset{\overline{AB}<\widehat{AB}}{<}1+\frac{\widehat{AB}}{\overline{EA}}$$
$$\overset{\widehat{CA}<\overline{DA}<\overline{EA}}{<}1+\frac{\widehat{AB}}{\widehat{CA}}$$
$$=1+\frac{r(\beta-\alpha)}{r\alpha}$$
$$=1+\frac{\beta-\alpha}{\alpha}=\frac{\beta}{\alpha}$$

$$\frac{\sin\beta}{\sin\alpha}<\frac{\beta}{\alpha}$$
$$\frac{\sin\beta}{\beta}<\frac{\sin\alpha}{\alpha}$$
$$\frac{\sin\alpha}{\alpha}>\frac{\sin\beta}{\beta}$$
$$\beta\sin\alpha-\alpha\sin\beta>0$$

□

定理 6.1.10. 直線方程式 *equation of a line in $\mathbb{R}^2$ / line equation*

- 斜截式 / 斜距式 slope-intercept form / gradient-intercept form
$$y=mx+b$$
- 點斜式 point-slope form
$$y-y_0=m(x-x_0)$$
$$y=m(x-x_0)+y_0$$
- 截距式 intercept form
$$\frac{x}{a}+\frac{y}{b}=1$$
- 兩點式 two-point form
$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$
$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\Leftrightarrow\frac{x-x_0}{a}+\frac{y-y_0}{b}=0$$
$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
- 行列式 determinant form
  - determinant form 2 by 2
  - determinant form 3 by 3

定理 6.1.11. 圓方程式 *equation of a circle*

- 標準式 standard form
$$\left(\frac{x-x_0}{r}\right)^2+\left(\frac{y-y_0}{r}\right)^2=1$$
$$(x-x_0)^2+(y-y_0)^2=r^2$$
- 參數式 parametric form
- 直徑式 diameter form
- 一般式 general form

定理 6.1.12. 橢圓方程式 *equation of an ellipse*

- 標準式 standard form
$$\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{b}\right)^2=1$$
- 參數式 parametric form

6.1.1.2 雙曲函數 hyperbolic function

6.1.1.3 橢圓函數 elliptic function

6.1.2 $\mathbb{R}^3$ 立體幾何 solid geometry

定理 6.1.13. 直線方程式 *equation of a line in* $\mathbb{R}^3$

- 向量式 / 參數式

定理 6.1.14. 平面方程式 *equation of a plane*

定理 6.1.15. 球面方程式 *equation of a sphere*

- 標準式 standard form
$$\left(\frac{x-x_0}{r}\right)^2+\left(\frac{y-y_0}{r}\right)^2+\left(\frac{z-z_0}{r}\right)^2=1$$
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

- 參數式 parametric form
- 直徑式 diameter form
- 一般式 general form

定理 6.1.16. *? 圓方程式 equation of a circle in* $\mathbb{R}^3$*?*
https://math.stackexchange.com/questions/1184038/what-is-the-equation-of-a-general-circle-in-3-d-space/1184089

定理 6.1.17. 橢球面方程式 *equation of a spheroid / ellipsoid*

- 標準式 standard form
$$\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{b}\right)^2+\left(\frac{z-z_0}{c}\right)^2=1$$

- 參數式 parametric form

定理 6.1.18. 球面積 *area of a sphere*
$$A_S=4\pi r^2$$

Proof. 1. by tile projection[3]

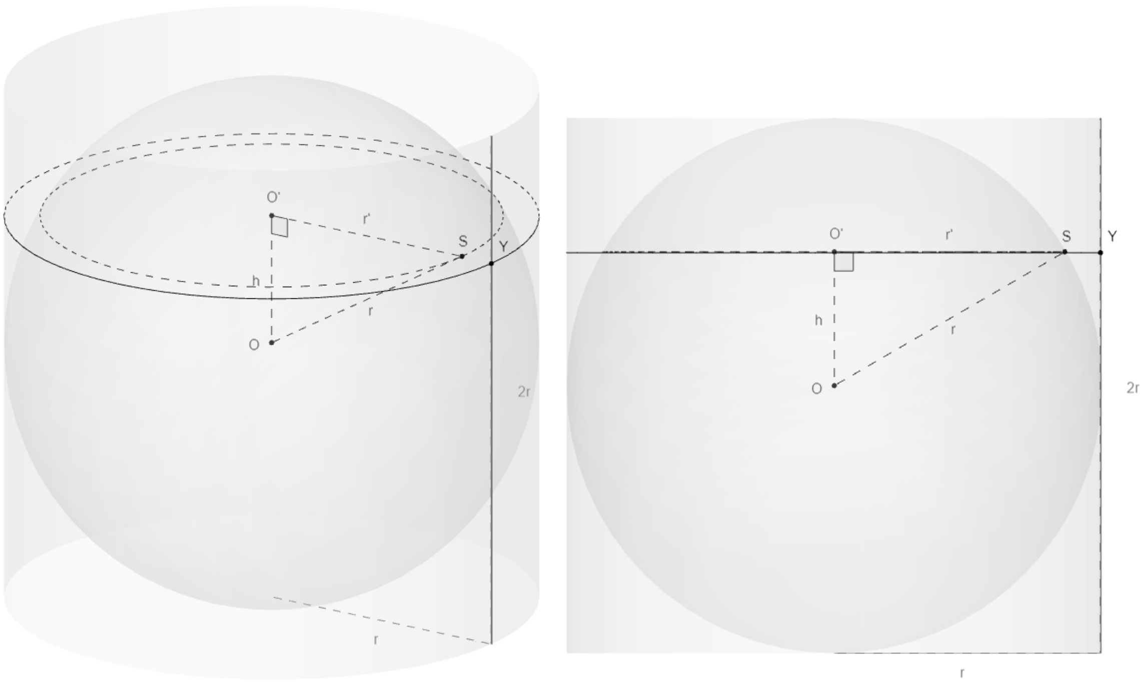


Figure 6.1.3: a sphere with its externally tangent cylinder

[3]3Blue1Brown: But why is a sphere's surface area four times its shadow? https://www.youtube.com/watch?v=GNcFjFmqEc8

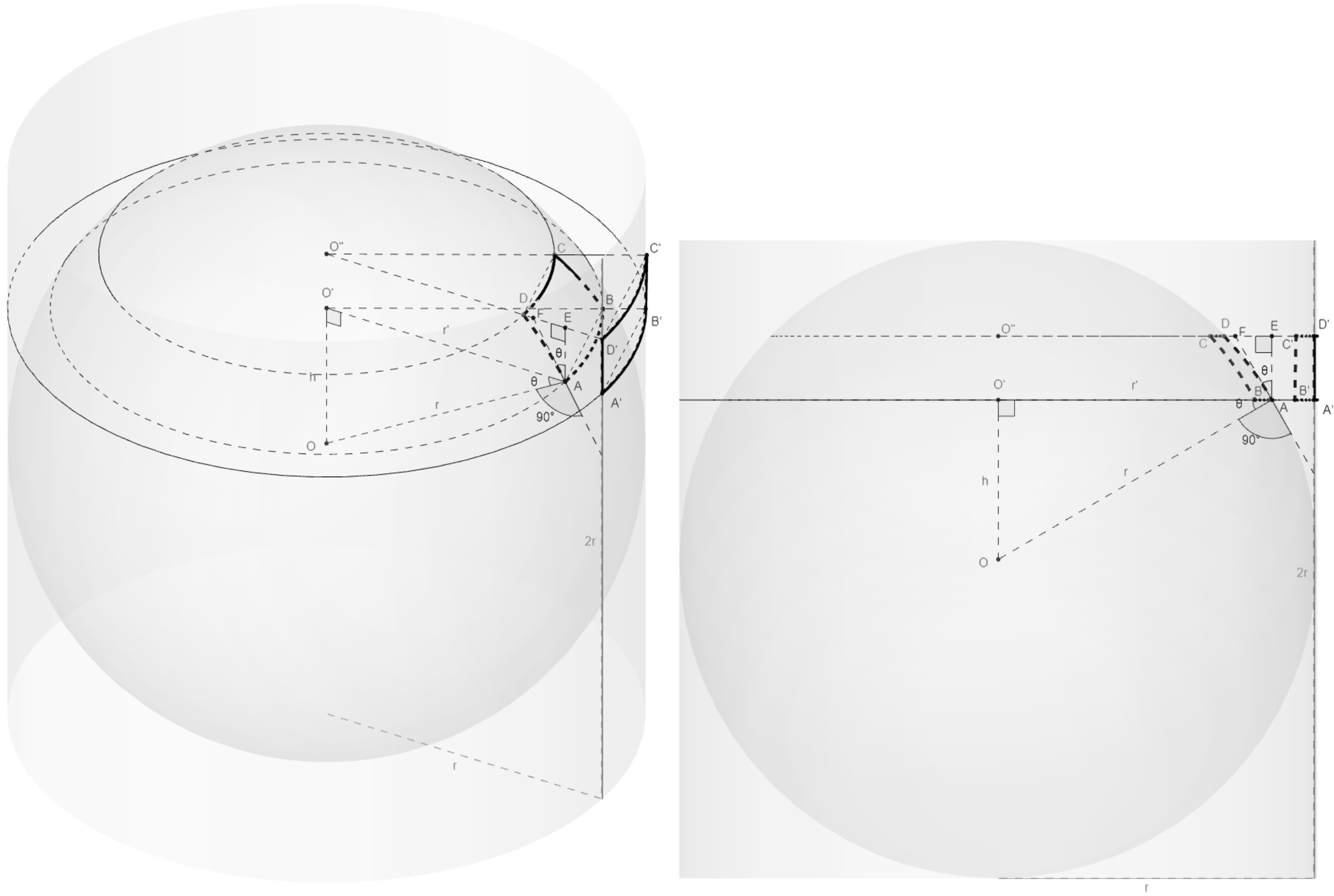


Figure 6.1.4: a sphere S with its externally tangent cylinder Y

$$
\begin{aligned}
aABCD &\approx \overline{AF}\cdot\overline{AB}\\
aA'B'C'D' &\approx \overline{A'D'}\cdot\overline{A'B'}\\
\overline{AE} &= \overline{A'D'}\\
\frac{\overline{AE}}{\overline{AF}} = \sin\theta = \frac{r'}{r} &\Rightarrow \overline{AF} = \frac{r}{r'}\overline{AE} = \frac{r}{r'}\overline{A'D'}\\
\triangle O'AB \sim \triangle O'A'B' \Rightarrow \frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{O'B}}{\overline{O'B'}} &= \frac{r'}{r} = \sin\theta \Rightarrow \overline{AB} = \frac{r'}{r}\overline{A'B'}
\end{aligned}
$$

$$
\begin{aligned}
aABCD \approx \overline{AF}\cdot\overline{AB} &= \frac{r}{r'}\overline{AE}\cdot\overline{AB} = \frac{r}{r'}\overline{A'D'}\cdot\overline{AB}\\
&= \frac{r}{r'}\overline{A'D'}\cdot\frac{r'}{r}\overline{A'B'} = \overline{A'D'}\cdot\overline{A'B'} \approx aA'B'C'D'
\end{aligned}
$$

$$
aABCD \overset{\Downarrow}{\approx} aA'B'C'D'
$$

$$
\lim_{aABCD\to 0} aABCD \overset{\Downarrow}{=} \lim_{aA'B'C'D'\to 0} aA'B'C'D'
$$

$$
\begin{aligned}
A_S &= \sum_{ABCD\in S}\lim_{aABCD\to 0} aABCD = \lim_{aABCD\to 0}\sum_{ABCD\in S} aABCD\\
&= \sum_{A'B'C'D'\in Y}\lim_{aA'B'C'D'\to 0} aA'B'C'D' = \lim_{aA'B'C'D'\to 0}\sum_{A'B'C'D'\in Y} aA'B'C'D'
\end{aligned}
$$

$$
\lim_{aA'B'C'D'\to 0}\sum_{A'B'C'D'\in Y} aA'B'C'D' = 2\pi r\cdot 2r = 4\pi r^2
$$

故一球之表面積與該球外切圓柱之側面積同「因相當於把球面上一極小 tile 轉 $90°$伸縮 $\frac{r'}{r}$ 倍(長伸成 $\frac{r}{r'}$ 倍, 寬縮成 $\frac{r'}{r}$ 倍, 剛好抵銷)投影至圓柱側面」

$$
A_S = 2\pi r\cdot 2r = 4\pi r^2
$$

□

Proof. 2. by calculus □

定理 6.1.19. 錐體體積為柱體體積之 1/3

- 四面體與其所張之平行六面體
- by calculus

定理 6.1.20. 劉-祖[4]原理 / 等冪等積定理 *Cavalieri principle*[5]
specialization of Fubini theorem

定理 6.1.21. 球體積 *valume of a ball*

$$V_B = \frac{4}{3}\pi r^3$$

Proof. 1. by Cavalieri principle

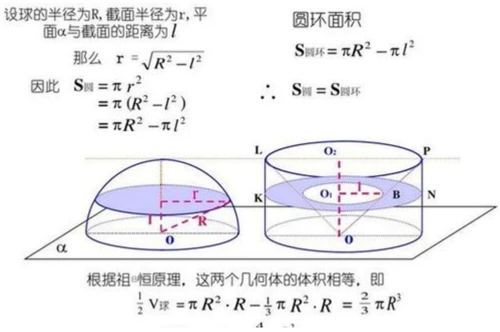


Figure 6.1.5: ball volume by Cavalieri principle_other's illustration

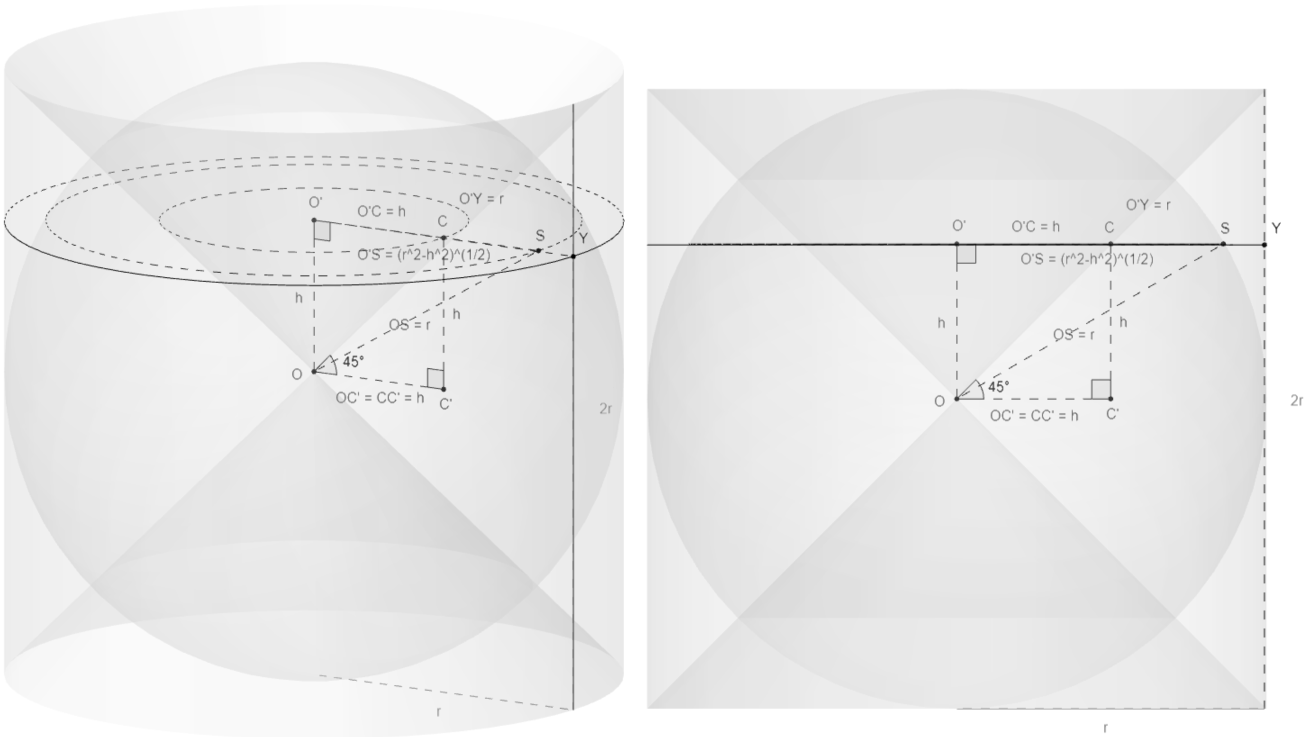


Figure 6.1.6: ball volume by Cavalieri principle

$C_{\overrightarrow{O'S}}$: 以 O'為圓心, $\overline{O'S}$ 為半徑之圓
$A_{C_{\overrightarrow{O'S}}}$: 以 O'為圓心, $\overline{O'S}$ 為半徑之圓之面積

[4]劉徽-祖(日恒)ㄒㄩㄢ
[5]一一對應相同面積鬆餅疊起相同高度之兩鬆餅塔具相同體積

$$\begin{aligned}\overline{O'S} = \sqrt{\overline{OS}^2 - \overline{OO'}^2} = \sqrt{r^2 - h^2} \Rightarrow A_{C_{\overrightarrow{O'S}}} =& \pi\overline{O'S}^2 = \pi\left(r^2 - h^2\right)\\ =& \pi r^2 - \pi h^2 = \pi\overline{O'Y}^2 - \pi\overline{O'C}^2\\ =& A_{C_{\overrightarrow{O'Y}}} - A_{C_{\overrightarrow{O'C}}}\end{aligned}$$

一半球「於某高度 h 截面圓面積 $A_{C_{\overrightarrow{O'S}}} = \pi\left(r^2 - h^2\right)$」恰好等於「該半球外切半圓柱於該高度 h 截面圓面積 $A_{C_{\overrightarrow{O'Y}}} = \pi r^2$ 扣掉該半圓柱同底等高 r 之圓錐於自圓椎頂至該高度 h 截面圓面積 $A_{C_{\overrightarrow{O'C}}} = \pi h^2$」或「$\overrightarrow{CY}$ 所張圓環面積」。又半球與半球外切半圓柱及該半圓柱同底等高圓椎同高 r，或該球與該球外切圓柱及該圓柱同底半高之兩圓椎對頂而置同高 $2r$，根據等冪等積定理 6.1.20，該球體積 $V_{B_{\overrightarrow{O'S}}}$ 與該圓柱扣掉該對頂兩圓椎之體積同。且圓錐體積為圓柱體積之 1/3 6.1.19。

$$\begin{aligned}V_{B_{\overrightarrow{O'S}}} =& \left(\pi r^2\right) \cdot 2r - 2 \cdot \left(\frac{1}{3} \cdot \left(\pi r^2\right) \cdot r\right)\\ =& 2\pi r^3 - \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3\end{aligned}$$

□

Proof. 2. by calculus □

6.2 非歐幾何 non-Euclidean geometry

6.2.1 球面幾何 spherical geometry

定義 6.2.1. 大圓 great circle[6]
a circle, of and on a sphere, with the same center as the sphere's center

- 赤道 the equator
- 經線 / 子午線 circle of longitude / line of longitude / meridian

定義 6.2.2. 小圓 small circle
a circle, of and on a sphere, that is not a great circle

定理 6.2.3. 大圓劣弧為球面上兩點最短路徑 *great-circle distance is the shortest distance between two points on the surface of a sphere*

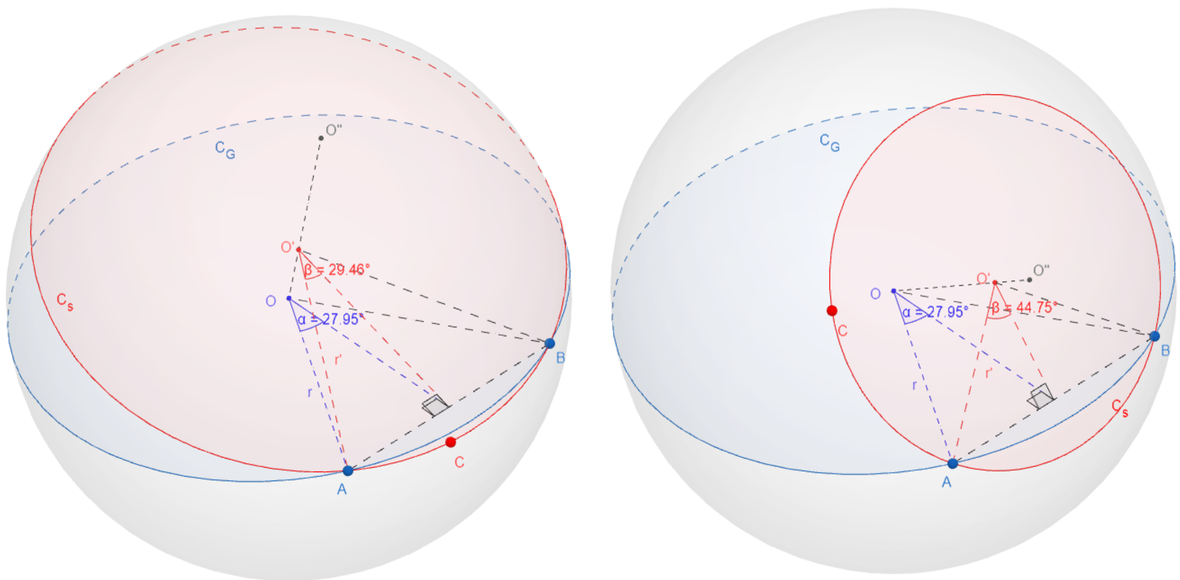


Figure 6.2.1: inferior arc of a great circle is the shortest path on the sphere for the two points on the sphere

[6] 活在2D的螞蟻走在球面上大圓就像走直線一樣為球面上兩點最短距離而渾然不知

Proof. by geometry[7],

according to the figure 6.2.1,

sphere $S \ni A, B, C, O''$ (the center of S is O and the radius of S is r in length),

C_G is the great circle of the sphere S, $C_G \ni A, B$ (the center of C_G is O and the radius of C_G is r in length) ,

C_s is any or small circle of the sphere S, $C_s \ni A, B, C$ (the center of C_s is O' and the radius of C_s is r') ,

$\overline{OO''} \ni O'$,

$\alpha = \frac{1}{2}\angle AOB$ corresponding to the inferior arc $\widehat{AB}_{2\alpha}$ on C_G , $0 < \angle AOB \le \pi \Rightarrow 0 < \alpha = \frac{1}{2}\angle AOB \le \frac{\pi}{2}$,

$\beta = \frac{1}{2}\angle AO'B$ corresponding to the inferior arc $\widehat{AB}_{2\beta}$ on C_s , $0 < \angle AO'B \le \pi \Rightarrow 0 < \beta = \frac{1}{2}\angle AO'B \le \frac{\pi}{2}$,

$$r\sin\alpha = \frac{1}{2}\overline{AB} = r'\sin\beta$$

$0 < r' \le r$ (小圓或任一球面上圓之半徑小於或等於大圓之), $0 < \alpha, \beta \le \frac{\pi}{2} \Rightarrow 0 < \sin\alpha, \sin\beta \le 1$,

$$1 \le \frac{r}{r'} = \frac{\sin\beta}{\sin\alpha} \Rightarrow \sin\beta \ge \sin\alpha \Rightarrow \beta \ge \alpha$$

so $0 < \alpha \le \beta \le \frac{\pi}{2}$,

$$1 \le \frac{r}{r'} = \frac{\sin\beta}{\sin\alpha} \underset{=\text{ when }\beta=\alpha}{\overset{6.1.9}{\le}} \frac{\beta}{\alpha}$$

$$\frac{r}{r'} \le \frac{\beta}{\alpha} \overset{r'>0}{\Rightarrow} r \le \frac{r'\beta}{\alpha} \overset{\alpha>0}{\Rightarrow} r\alpha \le r'\beta$$

$$r\alpha \le r'\beta$$

$$r\cdot(2\alpha) \le r'\cdot(2\beta)$$

$$\widehat{AB}_{2\alpha} = r\cdot(2\alpha) \le r'\cdot(2\beta) = \widehat{AB}_{2\beta}$$

$$\widehat{AB}_{2\alpha} \le \widehat{AB}_{2\beta}$$

意即 包含該兩點之大圓的劣弧短於其餘包含該兩點之小圓的劣弧

任意球面上圓之劣弧短於優弧 (superior arc)
任意球面上連接兩點之曲線或折線長於該兩點之劣弧 (球面「三角不等式」)
又包含該兩點之大圓的劣弧短於其餘包含該兩點之小圓的劣弧
故大圓劣弧為球面上兩點最短路徑 □

Proof. by calculus of variation,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t$$

$$\begin{aligned}\mathrm{d}s &= \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}\\ &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}\end{aligned}$$

under spherical coordinate system,

$$\begin{cases} x = x(r,\phi,\theta) = r\sin\phi\cos\theta \\ y = y(r,\phi,\theta) = r\sin\phi\sin\theta & ,0 \le \phi \le \pi, 0 \le \theta \le 2\pi \\ z = z(r,\phi,\theta) = r\cos\phi \end{cases}$$

[7] 數學傳播_23_2_徐正梅_談地球上兩點間的球面距離

on the surface of a sphere with fixed radius $r > 0$

$$\mathrm{d}r = 0$$

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial x}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial x}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \overset{\mathrm{d}r=0}{=} \frac{\partial x}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial x}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial y}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial y}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \overset{\mathrm{d}r=0}{=} \frac{\partial y}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial y}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial z}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial z}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \overset{\mathrm{d}r=0}{=} \frac{\partial z}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\partial z}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \end{cases}$$

$$\begin{cases} \mathrm{d}x = \frac{\partial x}{\partial \phi}\mathrm{d}\phi + \frac{\partial x}{\partial \theta}\mathrm{d}\theta \\ \mathrm{d}y = \frac{\partial y}{\partial \phi}\mathrm{d}\phi + \frac{\partial y}{\partial \theta}\mathrm{d}\theta \\ \mathrm{d}z = \frac{\partial z}{\partial \phi}\mathrm{d}\phi + \frac{\partial z}{\partial \theta}\mathrm{d}\theta \end{cases} \quad \overset{\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases}}{\Rightarrow} \quad \begin{cases} \mathrm{d}x = r\cos\phi\cos\theta\mathrm{d}\phi - r\sin\phi\sin\theta\mathrm{d}\theta \\ \mathrm{d}y = r\cos\phi\sin\theta\mathrm{d}\phi + r\sin\phi\cos\theta\mathrm{d}\theta \\ \mathrm{d}z = -r\sin\phi\mathrm{d}\phi + 0\cdot\mathrm{d}\theta = -r\sin\phi\mathrm{d}\phi \end{cases}$$

$$\begin{aligned} \mathrm{d}s =& \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \\ =& \sqrt{(r\cos\phi\cos\theta\mathrm{d}\phi - r\sin\phi\sin\theta\mathrm{d}\theta)^2 + (r\cos\phi\sin\theta\mathrm{d}\phi + r\sin\phi\cos\theta\mathrm{d}\theta)^2 + (-r\sin\phi\mathrm{d}\phi)^2} \\ \overset{\cos^2\theta+\sin^2\theta=1}{=}& \sqrt{(r\cos\phi)^2\mathrm{d}\phi^2 + (r\sin\phi)^2\mathrm{d}\theta^2 + (r\sin\phi)^2\mathrm{d}\phi^2} \overset{\cos^2\phi+\sin^2\phi=1}{=} \sqrt{r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2} \\ =& \sqrt{r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2} \overset{r\geq 0}{=} r\sqrt{\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\theta^2} \end{aligned}$$

$$S[\Gamma] = \int_\Gamma \mathrm{d}s = \int_\Gamma r\sqrt{\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\theta^2}$$

let [8]

$$\begin{cases} \phi = \phi(t) \\ \theta = \theta(t) \end{cases} \quad a \leq t \leq b \tag{6.2.1}$$

$$\begin{aligned} S[\Gamma] = \int_\Gamma \mathrm{d}s = \int_\Gamma r\sqrt{\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\theta^2} \\ =& \int_a^b r\sqrt{\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + \sin^2\phi\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}\mathrm{d}t \\ =& \int_a^b r\sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}\mathrm{d}t = S[t] \\ =& r\int_a^b \sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}\mathrm{d}t = r\int_a^b \mathcal{L}\left(t;\phi,\theta,\dot\phi,\dot\theta\right)\mathrm{d}t \end{aligned}$$

$$\mathcal{L}\left(t;\phi,\theta,\dot\phi,\dot\theta\right) = \sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2} \tag{6.2.2}$$

according to action funcitonal

$$\mathcal{A}(\boldsymbol{q}) = \int_a^b \mathcal{L}(t;\boldsymbol{q},\dot{\boldsymbol{q}})\,\mathrm{d}t = \int_a^b \mathcal{L}(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))\,\mathrm{d}t$$

has a smooth path $\boldsymbol{q} = \langle q_i(t)\rangle_{i\in\mathbb{N}} \in \mathcal{P}(\boldsymbol{q}(a),\boldsymbol{q}(b))$ that is stationary for $\mathcal{A}$ if and only if Euler–Lagrange equation(s)

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot q_i} = 0, i\in\mathbb{N} \tag{6.2.3}$$

let 6.2.1

$$\begin{cases} \phi = \phi(t) = q_1(t) \\ \theta = \theta(t) = q_2(t) \end{cases} \quad a \leq t \leq b \tag{6.2.4}$$

$$6.2.3 \Rightarrow \begin{cases} 0 = \frac{\partial\mathcal{L}}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot q_1} \overset{6.2.2}{\underset{6.2.4}{=}} \frac{\partial\sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}}{\partial\dot\phi} \\ 0 = \frac{\partial\mathcal{L}}{\partial q_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot q_2} \overset{6.2.2}{\underset{6.2.4}{=}} \frac{\partial\sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\sqrt{\dot\phi^2 + \sin^2\phi\dot\theta^2}}{\partial\dot\theta} \end{cases}$$

[8] https://en.wikipedia.org/wiki/Great_circle

$$\begin{cases} 0 = \dfrac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\phi} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\phi}} & (1) \\ 0 = \dfrac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\theta} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\theta}} = 0 - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\theta}} & (2) \end{cases}$$

$$\begin{aligned} 0 \overset{(1)}{=} & \frac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\phi}} \\ = & \frac{\partial\left(\dot{\phi}^2+\sin^2\phi\dot{\theta}^2\right)}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\left(\dot{\phi}^2+\sin^2\phi\dot{\theta}^2\right)}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}\partial\dot{\phi}} \\ = & \frac{0+2\sin\phi\cos\phi\dot{\theta}^2}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{2\dot{\phi}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} = & \frac{\sin\phi\cos\phi\dot{\theta}^2}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} \end{aligned}$$

太複雜了, 看下一條似乎比較簡單

$$\begin{aligned} 0 \overset{(2)}{=} & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\theta}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\left(\dot{\phi}^2+\sin^2\phi\dot{\theta}^2\right)}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}\partial\dot{\theta}} \\ = & \frac{\mathrm{d}}{\mathrm{d}t}\frac{0+2\sin^2\phi\dot{\theta}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} \\ 0 = & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\sin^2\phi\dot{\theta}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} \\ \frac{\sin^2\phi\dot{\theta}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} = & C_1, C_1 \text{ is time-independent constant} \end{aligned}$$

$$\begin{aligned} \frac{\sin^2\phi\dot{\theta}}{\sqrt{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2}} = & C_1 \\ \frac{\sin^4\phi\dot{\theta}^2}{\dot{\phi}^2+\sin^2\phi\dot{\theta}^2} = & C_1^2 \\ \sin^4\phi\dot{\theta}^2 = & \left(C_1\dot{\phi}\right)^2 + \left(C_1\sin\phi\right)^2\dot{\theta}^2 \\ \sin^2\phi\left(\sin^2\phi - C_1^2\right)\dot{\theta}^2 = & \left(C_1\dot{\phi}\right)^2 \\ \dot{\theta} = & \frac{C_1\dot{\phi}}{\sin\phi\sqrt{\sin^2\phi-C_1^2}} \end{aligned}$$

$$\begin{aligned} \dot{\theta} = & \frac{C_1\dot{\phi}}{\sin\phi\sqrt{\sin^2\phi-C_1^2}} \\ \mathrm{d}\theta = & \frac{C_1\mathrm{d}\phi}{\sin\phi\sqrt{\sin^2\phi-C_1^2}} \\ \theta = \int\mathrm{d}\theta = & \int\frac{C_1\mathrm{d}\phi}{\sin\phi\sqrt{\sin^2\phi-C_1^2}} = C_1\int\frac{\mathrm{d}\phi}{\sin\phi\sqrt{\sin^2\phi-C_1^2}} \end{aligned}$$

但其實有更簡單的 Lagrangian $\mathcal{L}$,

$$S[\Gamma] = \int_\Gamma \mathrm{d}s = \int_\Gamma r\sqrt{\mathrm{d}\phi^2+\sin^2\phi\mathrm{d}\theta^2}$$

令

$$\theta = \theta(\phi) \tag{6.2.5}$$

其中

$$\begin{cases} 0 \le \phi_A \le \phi \le \phi_B \le \pi \\ 0 \le \theta(\phi) \le 2\pi \end{cases}$$

$$\begin{aligned} S[\Gamma] = \int_\Gamma \mathrm{d}s = \int_\Gamma r\sqrt{\mathrm{d}\phi^2 + \sin^2\phi \mathrm{d}\theta^2} \\ &= \int_{\phi_A}^{\phi_B} r\sqrt{\left(\frac{\mathrm{d}\phi}{\mathrm{d}\phi}\right)^2 + \sin^2\phi\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2}\mathrm{d}\phi \\ &= \int_{\phi_A}^{\phi_B} r\sqrt{1 + \sin^2\phi\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2}\mathrm{d}\phi \\ &= r\int_{\phi_A}^{\phi_B}\sqrt{1+\sin^2\phi\dot{\theta}^2}\mathrm{d}\phi = r\int_{\phi_A}^{\phi_B}\mathcal{L}\left(\phi,\theta,\dot{\theta}\right)\mathrm{d}\phi, \dot{\theta} = \theta'(\phi) = \frac{\mathrm{d}\theta}{\mathrm{d}\phi} \end{aligned}$$

according to Euler–Lagrange equation

$$S[t] = \int_a^b \mathcal{L}(t, q(t), q'(t))\,\mathrm{d}t = \int_a^b \mathcal{L}(t; q, \dot{q})\,\mathrm{d}t$$

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} = 0$$

$$S[\phi] = \int_{\phi_A}^{\phi_B}\mathcal{L}\left(\phi,\theta,\dot{\theta}\right)\mathrm{d}\phi = \int_{\phi_A}^{\phi_B}\sqrt{1+\sin^2\phi\dot{\theta}^2}\mathrm{d}\phi = \int_{\phi_A}^{\phi_B}\mathcal{L}\left(\phi,\dot{\theta}\right)\mathrm{d}\phi, \dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}\phi}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0$$

$$\begin{aligned} 0 =& \frac{\partial\mathcal{L}}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}\left(\phi,\dot{\theta}\right)}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial\mathcal{L}\left(\phi,\dot{\theta}\right)}{\partial\dot{\theta}} \\ =& 0 - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial\sqrt{1+\sin^2\phi\dot{\theta}^2}}{\partial\dot{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial\left(1+\sin^2\phi\dot{\theta}^2\right)}{\sqrt{1+\sin^2\phi\dot{\theta}^2}\partial\dot{\theta}} \\ =& \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{2\sin^2\phi\dot{\theta}}{\sqrt{1+\sin^2\phi\dot{\theta}^2}} \\ \frac{\sin^2\phi\dot{\theta}}{\sqrt{1+\sin^2\phi\dot{\theta}^2}} =& C_1, C_1 \text{ is constant independent of } \phi \end{aligned}$$

$$\begin{aligned} \frac{\sin^2\phi\dot{\theta}}{\sqrt{1+\sin^2\phi\dot{\theta}^2}} =& C_1 \\ \frac{\sin^4\phi\dot{\theta}^2}{1+\sin^2\phi\dot{\theta}^2} =& C_1^2 \\ \sin^4\phi\dot{\theta}^2 =& C_1^2 + (C_1\sin\phi)^2\dot{\theta}^2 \\ \sin^2\phi\left(\sin^2\phi - C_1^2\right)\dot{\theta}^2 =& C_1^2, \text{ let } \sin\phi > 0 \\ 0 \le \dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}\phi} =& \frac{C_1}{\sin\phi\sqrt{\sin^2\phi - C_1^2}} \end{aligned}$$

$$\mathrm{d}\theta = \frac{C_1\mathrm{d}\phi}{\sin\phi\sqrt{\sin^2\phi - C_1^2}}$$

$$\mathrm{d}\theta \in \mathbb{R} \Rightarrow \sin^2\phi - C_1^2 > 0 \Rightarrow C_1^2 < \sin^2\phi \le 1 \Rightarrow |C_1| < 1$$

$$\mathrm{d}\theta=\frac{\dfrac{1}{\sin^2\phi}C_1\mathrm{d}\phi}{\dfrac{\sin\phi}{\sin^2\phi}\sqrt{\sin^2\phi-C_1^2}}=\frac{C_1\csc^2\phi\mathrm{d}\phi}{\sqrt{1-\left(C_1\csc\phi\right)^2}}$$

$$\theta=\int\mathrm{d}\theta=\int\frac{C_1\csc^2\phi\mathrm{d}\phi}{\sqrt{1-\left(C_1\csc\phi\right)^2}}$$

$$-\mathrm{d}u=-C_1\csc^2\phi\mathrm{d}\phi$$

$$\frac{\mathrm{d}\cot\phi}{\mathrm{d}\phi}=\frac{\mathrm{d}\dfrac{\cos\phi}{\sin\phi}}{\mathrm{d}\phi}=\frac{-\sin\phi\sin\phi-\cos\phi\cos\phi}{\sin^2\phi}=-\csc^2\phi$$

$$u=C_1\cot\phi\Rightarrow\cot\phi=\frac{u}{C_1}\Rightarrow\sin\phi=\frac{C_1}{\sqrt{C_1^2+u^2}}\Rightarrow\csc\phi=\frac{\sqrt{C_1^2+u^2}}{C_1}$$

$$\theta=\int\frac{C_1\csc^2\phi\mathrm{d}\phi}{\sqrt{1-\left(C_1\csc\phi\right)^2}}=\int\frac{-\mathrm{d}u}{\sqrt{1-\left(C_1\cdot\dfrac{\sqrt{C_1^2+u^2}}{C_1}\right)^2}}$$

$$=\int\frac{-\mathrm{d}u}{\sqrt{1-C_1^2-u^2}}\overset{|C_1|<1\Rightarrow a^2=1-C_1^2>0}{=}\int\frac{-\mathrm{d}u}{\sqrt{a^2-u^2}}$$

$$\overset{u=a\cos\psi,a>0}{=}\int\frac{-\mathrm{d}a\cos\psi}{\sqrt{a^2-a^2\cos^2\psi}}=\int\frac{a\sin\psi\mathrm{d}\psi}{a\sin\psi}=\psi+C_2$$

$$\psi=\arccos\frac{u}{a}=\arccos\frac{C_1\cot\phi}{\sqrt{1-C_1^2}}$$

$$\theta=\arccos\frac{C_1\cot\phi}{\sqrt{1-C_1^2}}+C_2=\arccos\left(K_1\cot\phi\right)+C_2,K_1=\frac{C_1}{\sqrt{1-C_1^2}}$$

$$\cos\left(\theta-C_2\right)=K_1\cot\phi$$

$$\sin\phi\cos\left(\theta-C_2\right)=K_1\cos\phi$$

$$\sin\phi\left(\cos\theta\cos C_2+\sin\theta\sin C_2\right)=K_1\cos\phi$$

$$\cos C_2\sin\phi\cos\theta+\sin C_2\sin\phi\sin\theta-K_1\cos\phi=0$$

$$\cos C_2\cdot r\sin\phi\cos\theta+\sin C_2\cdot r\sin\phi\sin\theta-K_1\cdot r\cos\phi=0$$

$$\cos C_2\cdot x+\sin C_2\cdot y-K_1\cdot z=0$$

$$n_xx+n_yy+n_zz=0,\begin{cases}n_x=\cos C_2\\n_y=\sin C_2\\n_z=-K_1\end{cases}$$

$$n_x\left(x-0\right)+n_y\left(y-0\right)+n_z\left(z-0\right)=0$$

$$\langle n_x,n_y,n_z\rangle\cdot\langle\left(x-0\right),\left(y-0\right),\left(z-0\right)\rangle=0$$

$$\langle n_x,n_y,n_z\rangle\cdot\left(\langle x,y,z\rangle-\langle 0,0,0\rangle\right)=0$$

雖然是過原點之平面方程的一個三維法向量 $\langle n_x,n_y,n_z\rangle$ 但該法向量只有兩個自由度 C_2,K_1, 所以再根據邊界條件 (boundary condition) 兩點 $\langle x_A,y_A,z_A\rangle,\langle x_B,y_B,z_B\rangle$ 即可定下該平面方程

為免混淆, 令該球面 $r=R\Rightarrow\mathrm{d}r=0$, 球面上起點 A 位置 (同理終點 B)

$$\left\langle\begin{array}{c}x_A\\y_A\\z_A\end{array}\right\rangle=\left\langle\begin{array}{c}R\sin\phi_A\cos\theta_A\\R\sin\phi_A\sin\theta_A\\R\cos\phi_A\end{array}\right\rangle$$

而路徑

$$\left\{\left\langle\begin{array}{c}x_i\\y_i\\z_i\end{array}\right\rangle=\left\langle\begin{array}{c}R\sin\phi_i\cos\theta_i\\R\sin\phi_i\sin\theta_i\\R\cos\phi_i\end{array}\right\rangle,i\in\{A,B\}\cup[0,1]\right\}\subset\left\{\left\langle\begin{array}{c}x\\y\\z\end{array}\right\rangle\middle|\langle n_x,n_y,n_z\rangle\cdot\left(\langle x,y,z\rangle-\langle 0,0,0\rangle\right)=0,\begin{cases}n_x=\cos C_2\\n_y=\sin C_2\\n_z=-K_1\end{cases}\right\}$$

$$\left\{\left\langle\begin{array}{c}x_i\\y_i\\z_i\end{array}\right\rangle, i\in\{A,B\}\cup[0,1]\right\}\subset$$

$$\left\{\left\langle\begin{array}{c}x\\y\\z\end{array}\right\rangle\middle| x^2+y^2+z^2=R^2\right\}\cap\left\{\left\langle\begin{array}{c}x\\y\\z\end{array}\right\rangle\middle|\langle n_x,n_y,n_z\rangle\cdot(\langle x,y,z\rangle-\langle 0,0,0\rangle)=0,\begin{cases}n_x=\cos C_2\\n_y=\sin C_2\\n_z=-K_1\end{cases}\right\}$$

也就是說球面上最短路徑包含於球面與兩端點及球心決定之平面相交的曲線上即大圓之劣弧 □

定義 6.2.4. 球面二角形 / 月牙形 spherical lune

兩相異大圓上交點為界, 各一鄰弧所圍成之形

sphere $S\ni A,N_1,N_2$ (the center of S is O),

circle of longitude $C_1\ni A$, and $\overrightarrow{ON_1}\perp C_1$ (the center of C_1 is O),

circle of longitude $C_2\ni A$, and $\overrightarrow{ON_2}\perp C_2$ (the center of C_2 is O),

the antipode(對蹠點) of A is A', which is also another intersecion (point) of C_1, and C_2,

spherical lunes are defined by two different arcs $\overset{\frown}{AA'}_1\subset C_1$ and $\overset{\frown}{AA'}_2\subset C_2$,

$\alpha=\angle N_1ON_2\in\{\angle C_1C_2,\pi-\angle C_1C_2\}?$[9],

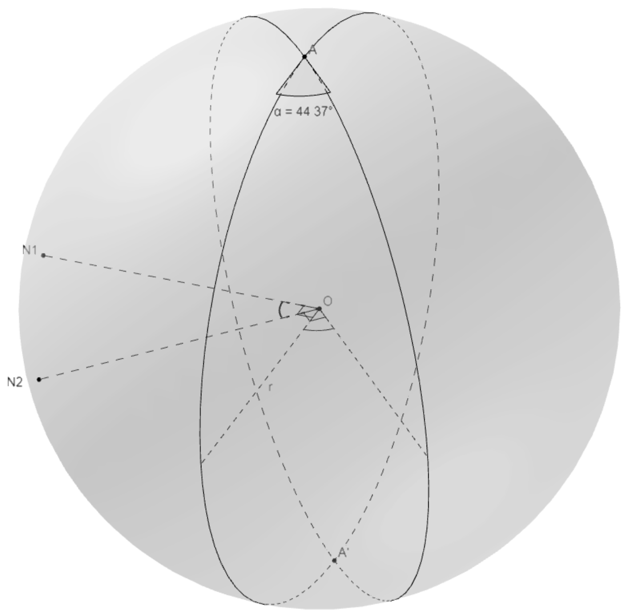


Figure 6.2.2: spherical lune defined by two different circles of longitudes / lines of longitudes / meridians

定理 6.2.5. 球面二角形 / 月牙形 *(表)*面積 *area of a spherical lune*

$$a\overset{\frown}{AA'}_1\overset{\frown}{AA'}_2=2\alpha\cdot r^2$$

Proof.

according to the figure 6.2.2,

the antipode(對蹠點) of N_2 is N_2',

hemisphere $H_{N_2}:\begin{cases}H_{N_2}\subset S\\H_{N_2}\ni N_2\\H_{N_2}\not\ni N_2'\\H_{N_2}\supset C_2\end{cases}$,

[9]要想想如何定義二面角 (dihedral angle) 超過 180° 的狀況, 或是廣義角的狀況, 後續 β,γ 也是

$a\widehat{AA'}_1\widehat{AA'}_2$ is the (surface) area of $\widehat{AA'}_1\widehat{AA'}_2$,

$$a\widehat{AA'}_1\widehat{AA'}_2 = \frac{\alpha}{180°}aH_{N_2} = \frac{\alpha}{\pi}aH_{N_2} = \frac{\alpha}{\pi}\left(\frac{1}{2}A_S\right) = \frac{\alpha}{\pi}\left(\frac{1}{2}\cdot 4\pi r^2\right) = \frac{\alpha}{\pi}\cdot 2\pi r^2 = 2\alpha \cdot r^2$$

□

定義 6.2.6. 球面三角形 spherical triangle

三相異大圓上交點為界, 各一鄰弧所圍成之形, 且其任一弧弧嘗不可為零, 否則可能退化為球面二角形 / 月牙形 spherical lune 6.2.4

sphere $S \ni A, B, C$ (the center of S is O),

circle of a great circle $C_3 \ni B, C, N_1, N_2$ ($\overrightarrow{OA} \perp \overrightarrow{OB}, \overrightarrow{OC}$),

circle of longitude $C_1 \ni A, B$ ($\overrightarrow{ON_1} \perp \overrightarrow{OA}, \overrightarrow{OB}$),

circle of longitude $C_2 \ni C, A$ ($\overrightarrow{ON_2} \perp \overrightarrow{OC}, \overrightarrow{OA}$),

$\triangle_S ABC$ is the one of the spherical triangles defined by three different inferior arcs(劣弧):

$\widehat{AB} \in C_1$, $\widehat{BC} \in C_3$, and $\widehat{CA} \in C_2$,

$\alpha = \angle N_1 O N_2 \in \{\angle C_1C_2, \pi - \angle C_1C_2\}$? 9,

$\beta \in \{\angle C_3C_1, \pi - \angle C_3C_1\}$?,

$\gamma \in \{\angle C_2C_3, \pi - \angle C_2C_3\}$?,

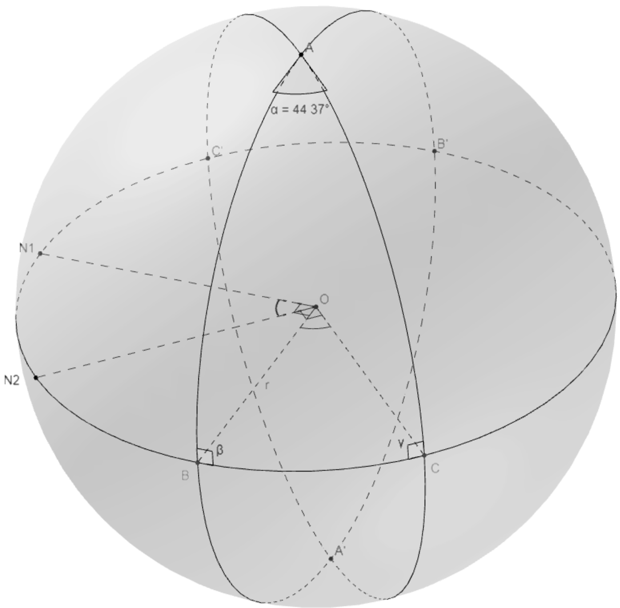


Figure 6.2.3: spherical triangle defined by a transverse great circle and two different circles of longitudes / lines of longitudes / meridians

定理 6.2.7. 半徑為 r 球面上赤道與兩相異經線圍成之球面三角之內角和為

$$\pi + \frac{a\triangle_S ABC}{r^2}$$

同理, 單位球面上赤道與兩相異經線圍成之球面三角之內角和為

$$\pi + a\triangle_{S_{r=1}} ABC$$

其中 $a\triangle_S ABC$ 為該球面三角形之(表)面積

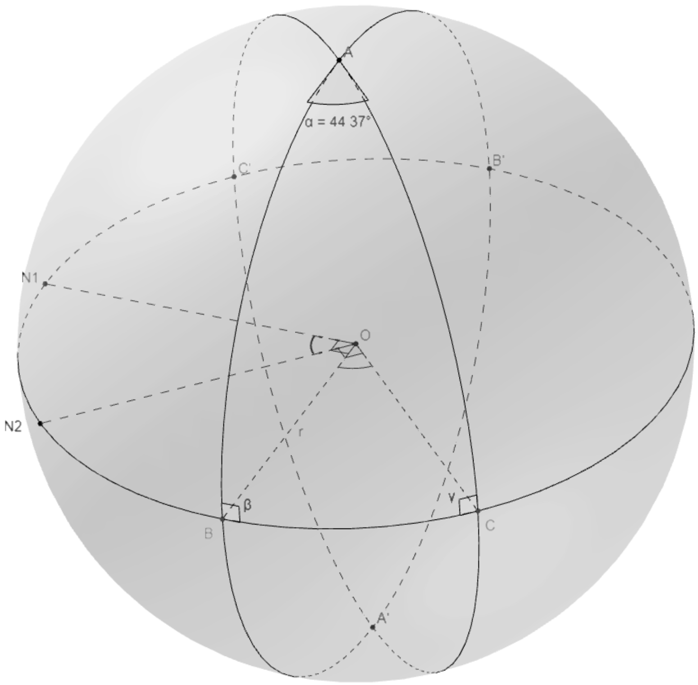


Figure 6.2.4: spherical triangle defined by the equator and two different circles of longitudes / lines of longitudes / meridians

Proof.

sphere $S \ni A, B, C$ (the center of S is O),

circle of the equator $C_{Eq} \ni B, C, N_1, N_2$ ($\overrightarrow{OA} \perp \overrightarrow{OB}, \overrightarrow{OC}$),

circle of longitude $C_1 \ni A, B$ ($\overrightarrow{ON_1} \perp \overrightarrow{OA}, \overrightarrow{OB}$),

circle of longitude $C_2 \ni C, A$ ($\overrightarrow{ON_2} \perp \overrightarrow{OC}, \overrightarrow{OA}$),

the antipode(對蹠點) of A is A',

hemisphere $H_A : \begin{cases} H_A \subset S \\ H_A \ni A \\ H_A \not\ni A' \\ H_A \supset C_{Eq} \end{cases}$,

$\triangle_S$ means spherical triangle,

α, β, γ are interior angles of $\triangle_S ABC$ respectively,

$a\triangle_S ABC$ is the (surface) area of $\triangle_S ABC$,

$\alpha = \angle N_1 O N_2 \in \{\angle C_1 C_2, \pi - \angle C_1 C_2\}$? 9,

$\beta \in \{\angle C_{Eq} C_1, \pi - \angle C_{Eq} C_1\}$?,

$\gamma \in \{\angle C_2 C_{Eq}, \pi - \angle C_2 C_{Eq}\}$?,

$$A_S = 4\pi r^2$$

$$a\triangle_S ABC = \frac{\alpha}{360^\circ} aH_A = \frac{\alpha}{2\pi} aH_A = \frac{\alpha}{2\pi}\left(\frac{1}{2}A_S\right) = \frac{\alpha}{2\pi}\left(\frac{1}{2}\cdot 4\pi r^2\right) = \frac{\alpha}{2\pi}\cdot 2\pi r^2 = \alpha \cdot r^2$$

$$\alpha = \frac{a\triangle_S ABC}{r^2}$$

the sum of interior angles of $\triangle_S ABC$

$$\alpha+\beta+\gamma=\frac{a\triangle_S ABC}{r^2}+90^\circ+90^\circ=\frac{a\triangle_S ABC}{r^2}+\frac{\pi}{2}+\frac{\pi}{2}=\pi+\frac{a\triangle_S ABC}{r^2}$$

and thus unit sphere $r=1$, the sum of interior angles of $\triangle_{S_{r=1}} ABC$

$$\alpha+\beta+\gamma=\pi+\frac{a\triangle_S ABC}{r^2}=\pi+\frac{a\triangle_S ABC}{1^2}=\pi+a\triangle_S ABC=\pi+a\triangle_{S_{r=1}} ABC$$

□

定理 6.2.8. 半徑為 r 球面上之球面三角形之內角和為[10]

$$\pi+\frac{a\triangle_S ABC}{r^2}$$

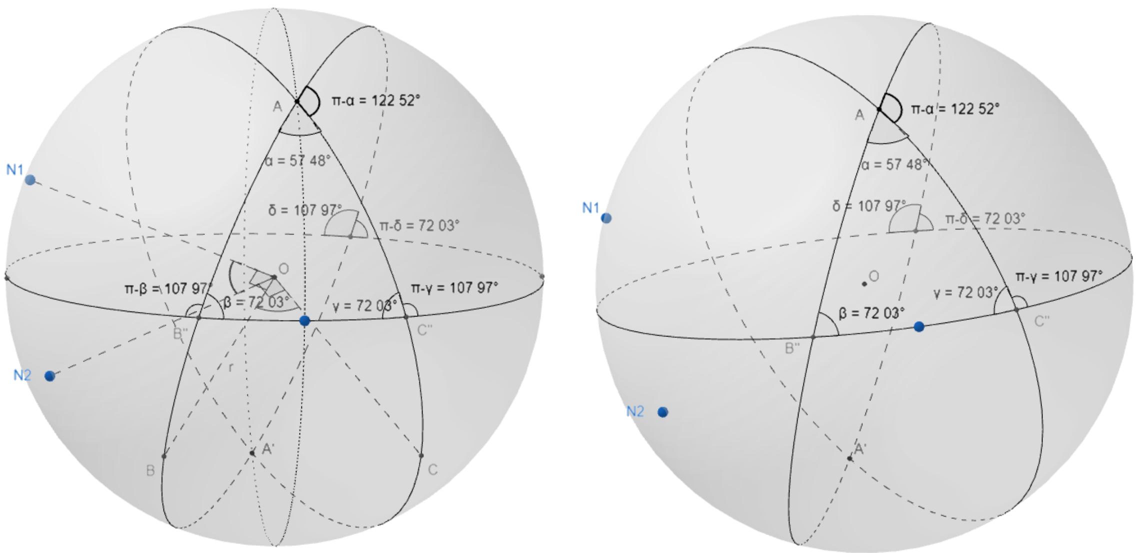


Figure 6.2.5: isosceles(等腰) / symmetric spherical triangle defined by three different great circles

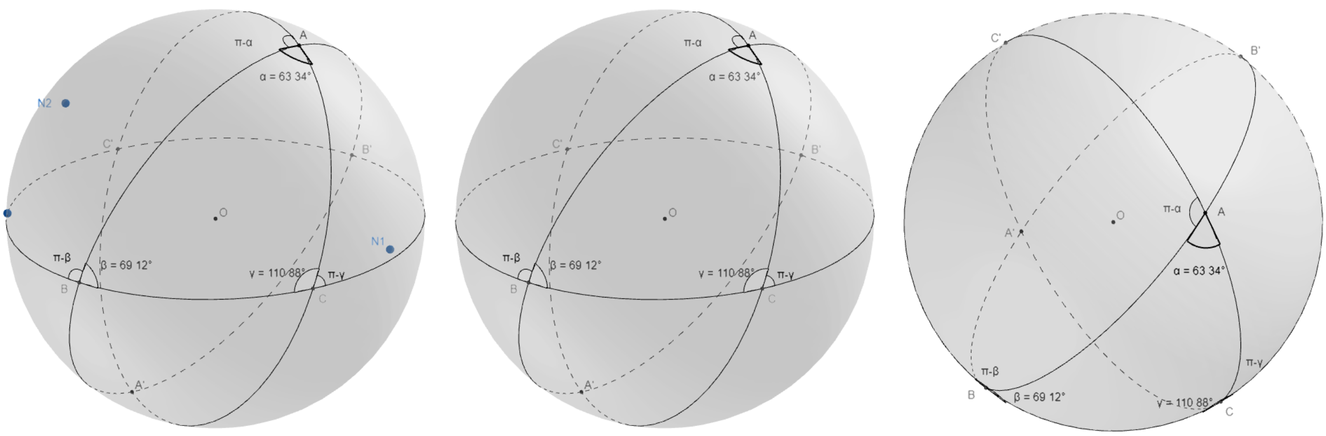


Figure 6.2.6: asymmetric spherical triangle defined by three different great circles

Proof.

according to the figure 6.2.6,

$$a\triangle_S ABC+a\triangle_S BCA'\overset{6.2.5}{=}2\alpha\cdot r^2 \tag{6.2.6}$$

[10]項武義: 七、球面幾何和球面三角學http://episte.math.ntu.edu.tw/articles/ar/ar_wy_geo_07/

$$a\triangle_S BCA + a\triangle_S CAB' \overset{6.2.5}{=} 2\beta \cdot r^2 \tag{6.2.7}$$

$$a\triangle_S CAB + a\triangle_S ABC' \overset{6.2.5}{=} 2\gamma \cdot r^2 \tag{6.2.8}$$

$$\begin{array}{ll} 6.2.6 & a\triangle_S ABC + a\triangle_S A'BC = 2\alpha \cdot r^2 \quad (6.2.9) \\ 6.2.7 \Leftrightarrow & a\triangle_S ABC + a\triangle_S AB'C = 2\beta \cdot r^2 \quad (6.2.10) \\ 6.2.8 & a\triangle_S ABC + a\triangle_S ABC' = 2\gamma \cdot r^2 \quad (6.2.11) \end{array}$$

a spherical triangle with antipodic(對蹠) points to all another one's has the same area as another spherical triangle,

$$a\triangle_S ABC' = a\triangle_S A'B'C \tag{6.2.12}$$

$$2\gamma \cdot r^2 \overset{6.2.11}{=} a\triangle_S ABC + a\triangle_S ABC' \overset{6.2.12}{=} a\triangle_S ABC + a\triangle_S A'B'C$$

observe 6.2.9 to 6.2.11 that, and hemisphere $H_C : \begin{cases} H_C \subset S \\ H_C \ni C \\ H_C \not\ni C' \end{cases}$,

$$\begin{aligned} & a\triangle_S ABC + a\triangle_S A'BC \\ & \quad + a\triangle_S AB'C \\ & \quad + a\triangle_S ABC' \\ \overset{6.2.12}{=} & a\triangle_S ABC + a\triangle_S A'BC \\ & \quad + a\triangle_S AB'C \\ & \quad + a\triangle_S A'B'C = aH_C = \frac{1}{2}A_S = \frac{1}{2} \cdot 4\pi r^2 = 2\pi r^2 \end{aligned} \tag{6.2.13}$$

$$\begin{array}{ll} 6.2.9 & a\triangle_S ABC + a\triangle_S A'BC = 2\alpha \cdot r^2 \\ +6.2.10 \Leftrightarrow & a\triangle_S ABC + a\triangle_S AB'C = 2\beta \cdot r^2 \\ +6.2.11 & a\triangle_S ABC + a\triangle_S A'B'C = 2\gamma \cdot r^2 \\ \| \ 6.2.13 & \\ & 2 \cdot a\triangle_S ABC + 2\pi r^2 = 2(\alpha+\beta+\gamma) \cdot r^2 \\ \Leftrightarrow & a\triangle_S ABC + \pi r^2 = (\alpha+\beta+\gamma) \cdot r^2 \\ \overset{r>0}{\Rightarrow} & \alpha+\beta+\gamma = \pi + \frac{a\triangle_S ABC}{r^2} \end{array}$$

□

推論 6.2.9. 半徑為 r 球面上之球面三角形之內角和上下限

$$\pi < \alpha + \beta + \gamma = \pi + \frac{a\triangle_S ABC}{r^2} < 3\pi$$

且 半徑為 r 球面上之球面三角形之面積上下限

$$0 < a\triangle_S ABC < 2\pi r^2 = \frac{1}{2}A_S$$

Proof.

according to the figure 6.2.5 without loss of generality, w/oLoG (不妨假設等腰球面三角推至極限之狀況),

設 $\triangle_S ABC$ 為嚴格定義的(strictly defined spherical triangle 6.2.6), 考慮與 $\triangle_S ABC$ 形成 球面二角形 / 月牙形 (spherical lune) 的 $\triangle_S A'B''C''$ 其內角和為

$$\angle C''A'B'' + \angle A'B''C'' + \angle B''C''A' = \pi + \frac{a\triangle_S A'B''C''}{r^2}$$

且當 $a\triangle_S A'B''C'' \to 0$ 時, $\triangle_S A'B''C''$ 其內角和為

$$\lim_{a\triangle_S A'B''C'' \to 0} (\angle C''A'B'' + \angle A'B''C'' + \angle B''C''A') = \lim_{a\triangle_S A'B''C'' \to 0} \left(\pi + \frac{a\triangle_S A'B''C''}{r^2}\right) = \pi$$

也就是極小球面幾何逼近平面幾何的狀況三角形內角和又回到 π.

又

$$\angle A'B''C'' = \pi - \angle AB''C'' = \pi - \beta$$

$$\angle B''C''A' = \pi - \angle AC''B'' = \pi - \gamma$$

$$\angle C''A'B'' = \angle B''AC'' = \alpha$$

故當 $a\triangle_S A'B''C'' \to 0$ 時, $\triangle_S A'B''C''$ 其內角和也暗示

$$\begin{aligned}
\alpha + (\pi - \beta) + (\pi - \gamma) =& \pi = \lim_{a\triangle_S A'B''C'' \to 0} (\angle C''A'B'' + \angle A'B''C'' + \angle B''C''A') \\
\alpha + 2\pi - (\beta + \gamma) =& \pi \\
\beta + \gamma =& \pi + \alpha \\
\alpha + \beta + \gamma =& \pi + 2\alpha
\end{aligned}$$

故若承認 α 為狹義二面角 (且若 $\alpha > \pi \Rightarrow \beta + \gamma = \pi + \alpha > 2\pi$, 則對稱球面三角形 $\triangle_S ABC$ 之兩底角會重疊, 可能就已經不是嚴謹定義的球面三角形了吧!)

$$0 < \alpha < \pi$$

則

$$\pi = \pi + 0 < \alpha + \beta + \gamma = \pi + 2\alpha < \pi + 2\pi = 3\pi$$

$$\pi < \alpha + \beta + \gamma < 3\pi$$

啊! 其實也就是說 α, β, γ 是均勻無異的 (isometric), 其實承認 α 為狹義二面角, 同時也承認了 β, γ 為狹義二面角

$$0 < \alpha < \pi$$

$$0 < \beta < \pi$$

$$0 < \gamma < \pi$$

$$\begin{cases} 0 < \alpha < \pi \\ 0 < \beta < \pi \\ 0 < \gamma < \pi \end{cases} \Rightarrow \pi < \alpha + \beta + \gamma < 3\pi$$

又 $\alpha + \beta + \gamma = \pi + \dfrac{a\triangle_S ABC}{r^2}$

$$\pi < \pi + \frac{a\triangle_S ABC}{r^2} < 3\pi$$

$$0 < a\triangle_S ABC < 2\pi r^2 = \frac{1}{2} A_S$$

嚴謹定義的狹義球面三角形其面積大於 0 且小於半球面積 $\dfrac{1}{2}A_S = 2\pi r^2$

□

定義 6.2.10. 立體角 solid angle

平面角 angle defined by ratio of arc length on a circle C

$$\frac{l}{L} = \frac{l_{\text{arc}}}{L_C} = \frac{\theta \cdot r}{2\pi \cdot r}$$

立體角 solid angle defined by ratio surface area on a sphere S

$$\frac{a}{A} = \frac{a_{\text{surface or spherical polygon}}}{A_S} = \frac{\Omega \cdot r^2}{4\pi \cdot r^2}$$

推論 6.2.11. 半徑為 r 球面上之球面三角形之內角 α, β, γ 和為

$$\alpha + \beta + \gamma = \pi + \frac{a\triangle_S ABC}{r^2} = \pi + \Omega_{\triangle_S ABC}$$

其中 $\Omega_{\triangle_S ABC} = \dfrac{a\triangle_S ABC}{r^2}$ 為該球面三角形所張之立體角 *(solid angle) 6.2.10*, 且其與內角和關係為

$$\Omega_{\triangle_S ABC} = \alpha + \beta + \gamma - \pi$$

Part II

統計 statistics

Chapter 7

機率論 probability theory

7.1 樸素機率公理 naive probability axioms / naïve probability axioms

Chapter 8

敘述統計 descriptive statistics

Chapter 9

推論統計 inferential statistics

Part III

資料庫系統論 database system / database theory

Chapter 10

關聯模型 relational model

10.1 component

定義 10.1.1. 關聯基模 / 關聯表基模 relation schema[4, p.43,41]

$$r(R) = r(\{a_1, a_2, \cdots\}) = r\left(\{a_j\}_{j\in\mathbb{N}}\right)$$

R denotes the set of attributes in the schema of relation r, and a_i denotes an atomic attribute, that cannot be further subdivided[3, p.214][4, p.40] or else becoming meaningless

$$R = \{a_1, a_2, \cdots\} = \{a_j\}_{j\in\mathbb{N}}$$

定義 10.1.2. 鍵 key[4, p.43]

$$\begin{cases} r(R) = r\left(\{a_j\}_{j\in\mathbb{N}}\right) \\ K \subseteq R \\ K \neq \emptyset \end{cases} \Leftrightarrow K \text{ is a key for } r \tag{10.1.1}$$

A relation may have many keys

$$\begin{cases} r(R) = r\left(\{a_i\}_{i\in\mathbb{N}}\right) \\ \emptyset \subset K_1, K_2, \cdots, K_n \subseteq R \\ K_1 \cup K_2 \cup \cdots \cup K_n = \bigcup_{\nu=1}^{n} K_\nu = \bigcup_{\nu} K_\nu = R \end{cases}$$

$$\begin{aligned} \Rightarrow r(R) =& r(K_1, K_2, \cdots, K_n) = r\left(\{K_\nu\}_{\nu=1}^{n}\right) = r\left(\bigcup_{\nu=1}^{n} K_\nu\right) \\ =& r\left(\{a_{11}, a_{12}, \cdots\}, \{a_{21}, a_{22}, \cdots\}, \cdots, \{a_{n1}, a_{n2}, \cdots\}\right) = r\left(\left\{\{a_{\nu\eta}\}_{\eta=1}^{n_\nu}\right\}_{\nu=1}^{n}\right) \end{aligned}$$

Actually, keys are either a composite attribute (corresponding to a composite key) or atomic attribute

$$\begin{aligned} r(R) =& r(K_1, K_2, \cdots, K_n) = r\left(\{K_\nu\}_{\nu=1}^{n}\right) = r\left(\bigcup_{\nu=1}^{n} K_\nu\right) \\ =& r\left(\{a_{11}, a_{12}, \cdots\}, \{a_{21}, a_{22}, \cdots\}, \cdots, \{a_{n1}, a_{n2}, \cdots\}\right) = r\left(\left\{\{a_{\nu\eta}\}_{\eta=1}^{n_\nu}\right\}_{\nu=1}^{n}\right) \\ =& r(A_1, A_2, \cdots, A_n) = r\left(\{A_\nu\}_{\nu=1}^{n}\right) = r\left(\bigcup_{\nu=1}^{n} A_\nu\right) \\ =& r(A_\nu)_{\nu=1}^{n} \end{aligned}$$

$$A_\nu = \begin{cases} \{a_i\} & |A_\nu| = 1 \Leftrightarrow A_\nu \text{ is an atomic attribute} \\ \{a_{i_1}, a_{i_2}, \cdots\} & |A_\nu| > 1 \Leftrightarrow A_\nu \text{ is a composite attribute} \end{cases}$$

定義 10.1.3. 域 domain[3, p.115] / attribute domain[3, p.70]

$$D = \mathrm{dom}a = \mathrm{dom}(a)$$

atomic attributes with associated domains, which are also atomic domains

$$D_j = \mathrm{dom}a_j = \mathrm{dom}(a_j)$$

定義 10.1.4. 空 / 空值 null / null value[3, p.74],[4, p.40]

$$\texttt{null}$$

$$\texttt{NULL}$$

$$\texttt{Null}$$

although E.F. Codd also introduced the use of the lowercase Greek omega symbol to represent null in database theory[2]

$$\omega$$

備註 10.1.5. 三值邏輯 1.4.1: Data values or propositions/predicates in database theory are actually three-valued / ternary / trivalent, i.e. either or `true`, `false`, `null`.

定義 10.1.6. 組 / 多元組 / 有序組 tuple 1.2.4,1.2.6

$$\boldsymbol{t} \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\})$$

$$\boldsymbol{t} = \langle v_1, v_2, \cdots, v_j, \cdots, v_N \rangle \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\})$$

A relation may have many tuples

$$\begin{aligned}
\boldsymbol{t}_1 &= \langle v_{11}, v_{12}, \cdots, v_{1j}, \cdots, v_{1N} \rangle \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\}) \\
\boldsymbol{t}_2 &= \langle v_{21}, v_{22}, \cdots, v_{2j}, \cdots, v_{2N} \rangle \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\}) \\
&\vdots \\
\boldsymbol{t}_i &= \langle v_{i1}, v_{i2}, \cdots, v_{ij}, \cdots, v_{iN} \rangle \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\}) \\
&\vdots \\
\boldsymbol{t}_M &= \langle v_{M1}, v_{M2}, \cdots, v_{Mj}, \cdots, v_{MN} \rangle \in (D_1 \cup \{\texttt{null}\}) \times (D_2 \cup \{\texttt{null}\}) \times \cdots \times (D_j \cup \{\texttt{null}\}) \times \cdots \times (D_N \cup \{\texttt{null}\})
\end{aligned}$$

If defining at atomic attribute level

$$\Delta = D \cup \{\texttt{null}\} = \operatorname{dom}(a) \cup \{\texttt{null}\}$$

or

$$\Delta_j = D_j \cup \{\texttt{null}\} = \operatorname{dom}(a_j) \cup \{\texttt{null}\}$$

then

$$\boldsymbol{t} \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N$$

$$\boldsymbol{t} = \langle v_1, v_2, \cdots, v_j, \cdots, v_N \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N$$

or

$$\begin{aligned}
\boldsymbol{t}_1 &= \langle v_{11}, v_{12}, \cdots, v_{1j}, \cdots, v_{1N} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N \\
\boldsymbol{t}_2 &= \langle v_{21}, v_{22}, \cdots, v_{2j}, \cdots, v_{2N} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N \\
&\vdots \\
\boldsymbol{t}_i &= \langle v_{i1}, v_{i2}, \cdots, v_{ij}, \cdots, v_{iN} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N \\
&\vdots \\
\boldsymbol{t}_M &= \langle v_{M1}, v_{M2}, \cdots, v_{Mj}, \cdots, v_{MN} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_j \times \cdots \times \Delta_N
\end{aligned}$$

If defining at key or composite attribute level

$$\Delta = D \cup \{\texttt{null}\} = \operatorname{dom}(A) \cup \{\texttt{null}\}$$

or

$$\Delta_\nu = D_\nu \cup \{\texttt{null}\} = \operatorname{dom}(A_\nu) \cup \{\texttt{null}\} = \operatorname{dom} K_\nu \cup \{\texttt{null}\}$$

then

$$\boldsymbol{t} \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n$$

$$\boldsymbol{t} = \langle v_1, v_2, \cdots, v_\nu, \cdots, v_n \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n$$

or

$$\begin{aligned}
\boldsymbol{t}_1 =& \langle v_{11}, v_{12}, \cdots, v_{1\nu}, \cdots, v_{1n} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n \\
\boldsymbol{t}_2 =& \langle v_{21}, v_{22}, \cdots, v_{2\nu}, \cdots, v_{2n} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n \\
&\vdots \\
\boldsymbol{t}_\mu =& \langle v_{\mu 1}, v_{\mu 2}, \cdots, v_{\mu\nu}, \cdots, v_{\mu n} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n \\
&\vdots \\
\boldsymbol{t}_m =& \langle v_{m1}, v_{m2}, \cdots, v_{m\nu}, \cdots, v_{mn} \rangle \in \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n
\end{aligned}$$

and the relation schema becomes

$$\begin{aligned}
r(R) =& r(K_1, K_2, \cdots, K_n) = r(A_1, A_2, \cdots, A_n) \\
=& r(K_1 : D_1, K_2 : D_2, \cdots, K_n : D_n) = r(A_1 : D_1, A_2 : D_2, \cdots, A_n : D_n) \\
=& r(K_\nu : D_\nu)_{\nu=1}^n = r(A_\nu : D_\nu)_{\nu=1}^n \\
=& r(\langle K_1, D_1 \rangle, \langle K_2, D_2 \rangle, \cdots, \langle K_n, D_n \rangle) = r(\langle A_1, D_1 \rangle, \langle A_2, D_2 \rangle, \cdots, \langle A_n, D_n \rangle) \\
=& r\langle K_\nu, D_\nu \rangle_{\nu=1}^n = r\langle A_\nu, D_\nu \rangle_{\nu=1}^n
\end{aligned}$$

For a tuple $\boldsymbol{t}_\mu$ of the relation schema $r(R)$,

$$\begin{aligned}
\boldsymbol{t}_\mu \in r(R) =& r(K_\nu : D_\nu)_{\nu=1}^n = r(A_\nu : D_\nu)_{\nu=1}^n \\
&= \{\boldsymbol{t}_\mu\}_{\mu=1}^m = \{\langle v_{\mu 1}, v_{\mu 2}, \cdots, v_{\mu\nu}, \cdots, v_{\mu n} \rangle\}_{\mu=1}^m \subseteq \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\nu \times \cdots \times \Delta_n
\end{aligned}$$

and data values are represented by[4, p.43]

$$\boldsymbol{t}_\mu.K_\nu = \boldsymbol{t}_\mu[K_\nu] = \boldsymbol{t}_\mu.A_\nu = \boldsymbol{t}_\mu[A_\nu] = v_{\mu\nu}$$

10.2 representation

定義 10.2.1. 關聯表 relation / table / relational table: relation represented by table

- at key level

$$\begin{aligned}
r(R) =& r(K_1, K_2, \cdots, K_{\nu_K}, \cdots, K_{n_K}) = r(K_1 : D_1, K_2 : D_2, \cdots, K_{\nu_K} : D_{\nu_K}, \cdots, K_{n_K} : D_{n_K}) \\
=& r\langle K_1, K_2, \cdots, K_{\nu_K}, \cdots, K_{n_K} \rangle \\
=& \Big[\langle v_{11}, v_{12}, \cdots, v_{1\nu_K}, \cdots, v_{1n_K} \rangle, \\
& \langle v_{21}, v_{22}, \cdots, v_{2\nu_K}, \cdots, v_{2n_K} \rangle, \\
& \vdots \\
& \langle v_{\mu 1}, v_{\mu 2}, \cdots, v_{\mu\nu_K}, \cdots, v_{\mu n_K} \rangle, \\
& \vdots \\
& \langle v_{m1}, v_{m2}, \cdots, v_{m\nu_K}, \cdots, v_{mn_K} \rangle \Big]
\end{aligned}$$

or more neat,

$$\begin{array}{rcl}
r(R) &=& r\left(\begin{array}{cccccc} K_1 & K_2 & \cdots & K_{\nu_K} & \cdots & K_{n_K} \end{array} \right) \\
&=& r\left\langle \begin{array}{cccccc} K_1 & K_2 & \cdots & K_{\nu_K} & \cdots & K_{n_K} \end{array} \right\rangle \\
&=& r\left\langle \begin{array}{cccccc} K_1 : D_1 & K_2 : D_2 & \cdots & K_{\nu_K} : D_{\nu_K} & \cdots & K_{n_K} : D_{n_K} \end{array} \right\rangle \\
&=& r\left\langle \begin{array}{cccccc} K_1 : \mathrm{dom} K_1 & K_2 : \mathrm{dom} K_2 & \cdots & K_{\nu_K} : \mathrm{dom} K_{\nu_K} & \cdots & K_{n_K} : \mathrm{dom} K_{n_K} \end{array} \right\rangle \\
&=& \left[\begin{array}{cccccc}
v_{11} & v_{12} & \cdots & v_{1\nu_K} & \cdots & v_{1n_K} \\
v_{21} & v_{22} & \cdots & v_{2\nu_K} & \cdots & v_{2n_K} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu 1} & v_{\mu 2} & \cdots & v_{\mu\nu_K} & \cdots & v_{\mu n_K} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{m\nu_K} & \cdots & v_{mn_K}
\end{array} \right]
\end{array}$$

- at attribute level

$$
\begin{aligned}
r(R) =& r(A_1, A_2, \cdots, A_{\nu_A}, \cdots, A_{n_A}) = r(A_1 : D_1, A_2 : D_2, \cdots, A_{\nu_A} : D_{\nu_A}, \cdots, A_{n_A} : D_{n_A}) \\
=& r\langle A_1, A_2, \cdots, A_{\nu_A}, \cdots, A_{n_A}\rangle \\
=& \Big[\langle v_{11}, v_{12}, \cdots, v_{1\nu_A}, \cdots, v_{1n_A}\rangle, \\
& \langle v_{21}, v_{22}, \cdots, v_{2\nu_A}, \cdots, v_{2n_A}\rangle, \\
& \vdots \\
& \langle v_{\mu 1}, v_{\mu 2}, \cdots, v_{\mu\nu_A}, \cdots, v_{\mu n_A}\rangle, \\
& \vdots \\
& \langle v_{m1}, v_{m2}, \cdots, v_{m\nu_A}, \cdots, v_{mn_A}\rangle \Big]
\end{aligned}
$$

or more neat,

$$
\begin{array}{rcl}
r(R) & = & r\begin{pmatrix} A_1 & A_2 & \cdots & A_{\nu_A} & \cdots & A_{n_A} \end{pmatrix} \\
& = & r\left\langle \begin{matrix} A_1 & A_2 & \cdots & A_{\nu_A} & \cdots & A_{n_A} \end{matrix} \right\rangle \\
& = & r\left\langle \begin{matrix} A_1 : D_1 & A_2 : D_2 & \cdots & A_{\nu_A} : D_{\nu_A} & \cdots & A_{n_A} : D_{n_A} \end{matrix} \right\rangle \\
& = & r\left\langle \begin{matrix} A_1 : \mathrm{dom} A_1 & A_2 : \mathrm{dom} A_2 & \cdots & A_{\nu_A} : \mathrm{dom} A_{\nu_A} & \cdots & A_{n_A} : \mathrm{dom} A_{n_A} \end{matrix} \right\rangle \\
& = & \begin{bmatrix}
v_{11} & v_{12} & \cdots & v_{1\nu_A} & \cdots & v_{1n_A} \\
v_{21} & v_{22} & \cdots & v_{2\nu_A} & \cdots & v_{2n_A} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu 1} & v_{\mu 2} & \cdots & v_{\mu\nu_A} & \cdots & v_{\mu n_A} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{m\nu_A} & \cdots & v_{mn_A}
\end{bmatrix}
\end{array}
$$

- at atomic attribute level

$$
\begin{aligned}
r(R) =& r(a_1, a_2, \cdots, a_{\nu_a}, \cdots, a_{n_a}) = r(a_1 : D_1, a_2 : D_2, \cdots, a_{\nu_a} : D_{\nu_a}, \cdots, a_{n_a} : D_{n_a}) \\
=& r\langle a_1, a_2, \cdots, a_{\nu_a}, \cdots, a_{n_a}\rangle \\
=& \Big[\langle v_{11}, v_{12}, \cdots, v_{1\nu_a}, \cdots, v_{1n_a}\rangle, \\
& \langle v_{21}, v_{22}, \cdots, v_{2\nu_a}, \cdots, v_{2n_a}\rangle, \\
& \vdots \\
& \langle v_{\mu 1}, v_{\mu 2}, \cdots, v_{\mu\nu_a}, \cdots, v_{\mu n_a}\rangle, \\
& \vdots \\
& \langle v_{m1}, v_{m2}, \cdots, v_{m\nu_a}, \cdots, v_{mn_a}\rangle \Big]
\end{aligned}
$$

or more neat,

$$\begin{aligned}
r(R) &= r\left(\begin{array}{cccccc} a_1 & a_2 & \cdots & a_{\nu_a} & \cdots & a_{n_a} \end{array}\right) \\
&= r\left\{\begin{array}{cccccc} a_1 & a_2 & \cdots & a_{\nu_a} & \cdots & a_{n_a} \end{array}\right\} \\
&= r\left\langle\begin{array}{cccccc} a_1 : D_1 & a_2 : D_2 & \cdots & a_{\nu_a} : D_{\nu_a} & \cdots & a_{n_a} : D_{n_a} \end{array}\right\rangle \\
&= r\left\langle\begin{array}{cccccc} a_1 : \mathrm{dom}a_1 & a_2 : \mathrm{dom}a_2 & \cdots & a_{\nu_a} : \mathrm{dom}a_{\nu_a} & \cdots & a_{n_a} : \mathrm{dom}a_{n_a} \end{array}\right\rangle \\
&= \left[\begin{array}{cccccc}
v_{11} & v_{12} & \cdots & v_{1\nu_a} & \cdots & v_{1n_a} \\
v_{21} & v_{22} & \cdots & v_{2\nu_a} & \cdots & v_{2n_a} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu 1} & v_{\mu 2} & \cdots & v_{\mu\nu_a} & \cdots & v_{\mu n_a} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{m\nu_a} & \cdots & v_{mn_a}
\end{array}\right]
\end{aligned}$$

備註 10.2.2. Since a relation / table / relational table is a set, its elements with the same data values will be regard as single unique element like 1.2.2, i.e.

if $\langle v_{\mu_1 1}, v_{\mu_1 2}, \cdots, v_{\mu_1 \nu_K}, \cdots, v_{\mu_1 n_K}\rangle = \langle v_{\mu_2 1}, v_{\mu_2 2}, \cdots, v_{\mu_2 \nu_K}, \cdots, v_{\mu_2 n_K}\rangle$,

$$\begin{aligned}
r(R) &= r\left\langle\begin{array}{cccccc} K_1 : D_1 & K_2 : D_2 & \cdots & K_{\nu_K} : D_{\nu_K} & \cdots & K_{n_K} : D_{n_K} \end{array}\right\rangle \\
&= \left[\begin{array}{cccccc}
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu_1 1} & v_{\mu_1 2} & \cdots & v_{\mu_1 \nu_K} & \cdots & v_{\mu_1 n_K} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu_2 1} & v_{\mu_2 2} & \cdots & v_{\mu_2 \nu_K} & \cdots & v_{\mu_2 n_K} \\
\vdots & \vdots & & \vdots & & \vdots
\end{array}\right] \\
&= \left[\begin{array}{cccccc}
\vdots & \vdots & & \vdots & & \vdots \\
v_{\mu_1 1} & v_{\mu_1 2} & \cdots & v_{\mu_1 \nu_K} & \cdots & v_{\mu_1 n_K} \\
\vdots & \vdots & & \vdots & & \vdots
\end{array}\right]
\end{aligned}$$

定義 10.2.3. 命名 naming

$$r(R) \text{ is a relation or relation schema}$$

$$r(R) = r\left(\{a_\nu\}_{\nu=1}^{n}\right) = r\left(\{A_{\nu_A}\}_{\nu_A=1}^{n_A}\right) = r\left(\{K_{\nu_K}\}_{\nu_K=1}^{n_K}\right)$$

$$\begin{aligned}
n(r(R)) = n(r)(n(R)) &= n(r)\left(n\left(\{a_\nu\}_{\nu=1}^{n}\right)\right) = n(r)\left(n\left(\{A_{\nu_A}\}_{\nu_A=1}^{n_A}\right)\right) = n(r)\left(n\left(\{K_{\nu_K}\}_{\nu_K=1}^{n_K}\right)\right) \\
&= n(r)\left(\{n(a_\nu)\}_{\nu=1}^{n}\right) = n(r)\left(\{n(A_{\nu_A})\}_{\nu_A=1}^{n_A}\right) = n(r)\left(\{n(K_{\nu_K})\}_{\nu_K=1}^{n_K}\right)
\end{aligned}$$

- naming of relation

  $$r \to n(r)$$

  e.g.

  $$r \to student$$

- naming of attribute
  - naming of atomic attribute

    $$a_\nu \to n(a_\nu)$$

    e.g. $student$ abbreviated as STU[3, p.115]

    $$a_1 \to STU\_FNAME \tag{10.2.1}$$

  - naming of general attribute

    $$A_{\nu_A} \to n(A_{\nu_A})$$

    e.g.[3, p.116]

    $$A_5 \to STU\_PHONE = \{AREA\_CODE, EXCHAN\_NUMBE, FOUR\_DIGIT\}$$

- naming of key (general attribute or composite attribute)

$$K_{\nu_K} \to n\left(K_{\nu_K}\right)$$

e.g.

$$K_1 \to STU\_ID = \{COLLEGE\_NO, ADMISSION\_YEAR, FOUR\_DIGIT\}$$

- naming or formatting of data value
- naming convention and typography
  - character, case, delimiter, part of speech = word class = grammatical category, typography (typeface (serif vs. sans serif, proportional vs. monospaced), font (roman(upright) vs. italic, bold or not))
  - naming of relation: Latin, lowercase or camelCase, "", noun, (serif, proportional, italic)
    - since naming of a relation to produce a name for the relation, which is still a variable-like object or set object in the relation schema, it should follow the typography (serif, proportional, italic) like the original symbol, i.e. r; this convention is followed by [4, p.41] but not by [3, p.115]
  - naming of attribute: Latin, UPPERCASE or SCREAMING_SNAKE_CASE, "_", relation name abbreviated as prefix followed by delimiter "_" and noun or descriptive of the characteristic (modifier), (serif, proportional, italic)
    - suffixes such as _ID, _NUM, or _CODE for the primary key attribute[3, p.231]
    - since naming of an attribute to produce a name for the attribute, which is still a variable-like object or set object in the relation schema, it should follow the typography (serif, proportional, italic) like the original symbol, i.e. a, A, or K; this convention is followed partially by [4, p.41] (but acceptable since an attribute may be atomic which is lowercase, e.g. stu_fname instead of 10.2.1) but totally not by [3, p.115] `STU_FNAME`

$$\texttt{stu\_fname}$$

$$\texttt{STU\_FNAME}$$

$$STU\_FNAME$$

$$stu\_fname$$

    - however, italic has less readability, so uppercase typewriter (\mathtt) in TEXor LyX may be a compromised option

$$\mathtt{STU\_FNAME}$$

  - data value
    - similar to the cases in statistics[1]:
      - random variable: usually written in upper case roman letters, e.g. X
      - particular realizations of a random variable: written in corresponding lower case letters, e.g. x_1 in $\mathrm{P}\left(X = x_1\right)$
    - the cases in database should be:
      - attribute or key: written in upper case roman letters
      - data value: written in corresponding lower case letters
      - formatted constant value: further upright
      - e.g.

$$\boldsymbol{t}_\mu.K_\nu = v_{\mu\nu} \overset{\text{assign}}{\leftarrow} \mathrm{John}$$

  - tuple
    - a tuple is an n-vector
    - vector: should be bold symbol (\boldsymbol) $\boldsymbol{v}$ in TEXor LyX, not only bold (\mathbf) $\mathbf{v}$ without italic font slope, because vectors are also variables (should follow the typography (serif, proportional, italic))

定義 10.2.4. 資料庫 database
A database may have many relations / tables / relation tables

$$r\left(R\right) = r\left(\{a_\nu\}_{\nu=1}^{n}\right) = r\left(\{A_{\nu_A}\}_{\nu_A=1}^{n_A}\right) = r\left(\{K_{\nu_K}\}_{\nu_K=1}^{n_K}\right)$$

$$r_\lambda\left(R_\lambda\right) = r_\lambda\left(\{a_{\lambda\nu}\}_{\nu=1}^{n_\lambda}\right) = r_\lambda\left(\{A_{\lambda\nu_A}\}_{\nu_A=1}^{n_{A_\lambda}}\right) = r_\lambda\left(\{K_{\lambda\nu_K}\}_{\nu_K=1}^{n_{K_\lambda}}\right)$$

$$\begin{aligned} r_\lambda(R_\lambda) &= r_\lambda \left\langle \begin{array}{cccccc} K_{\lambda 1}:D_{\lambda 1} & K_{\lambda 2}:D_{\lambda 2} & \cdots & K_{\lambda\nu_{K_\lambda}}:D_{\lambda\nu_{K_\lambda}} & \cdots & K_{\lambda n_{K_\lambda}}:D_{n_{K_\lambda}} \end{array} \right\rangle \\ &= \begin{bmatrix} v_{\lambda 11} & v_{\lambda 12} & \cdots & v_{\lambda 1\nu_{K_\lambda}} & \cdots & v_{\lambda 1 n_{K_\lambda}} \\ v_{\lambda 21} & v_{\lambda 22} & \cdots & v_{\lambda 2\nu_{K_\lambda}} & \cdots & v_{\lambda 2 n_{K_\lambda}} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_{\lambda\mu 1} & v_{\lambda\mu 2} & \cdots & v_{\lambda\mu\nu_{K_\lambda}} & \cdots & v_{\lambda\mu n_{K_\lambda}} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_{\lambda m1} & v_{\lambda m2} & \cdots & v_{\lambda m\nu_{K_\lambda}} & \cdots & v_{\lambda m n_{K_\lambda}} \end{bmatrix} \end{aligned}$$

e.g. relation 1: *student* relation

$$\begin{aligned} r_1(R_1) &= r_1 \left\langle \begin{array}{cccccc} K_{11}:D_{11} & K_{12}:D_{12} & \cdots & K_{1\nu_{K_1}}:D_{1\nu_{K_1}} & \cdots & K_{1n_{K_1}}:D_{n_{K_1}} \end{array} \right\rangle \\ &= \begin{bmatrix} v_{111} & v_{112} & \cdots & v_{11\nu_{K_1}} & \cdots & v_{11n_{K_1}} \\ v_{121} & v_{122} & \cdots & v_{12\nu_{K_1}} & \cdots & v_{12n_{K_1}} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_{1\mu 1} & v_{1\mu 2} & \cdots & v_{1\mu\nu_{K_1}} & \cdots & v_{1\mu n_{K_1}} \\ \vdots & \vdots & & \vdots & & \vdots \\ v_{1m1} & v_{1m2} & \cdots & v_{1m\nu_{K_1}} & \cdots & v_{1mn_{K_1}} \end{bmatrix} \\ \overset{\text{assign}}{\leftarrow} \mathit{student}(R_{\mathit{student}}) &= \mathit{student} \left\langle \begin{array}{cccccc} \texttt{STU\_ID} & \texttt{STU\_FNAME} & \cdots & \texttt{STU\_EMAIL} & \cdots & \texttt{STU\_PHONE} \end{array} \right\rangle \\ &= \begin{bmatrix} 120220001 & \text{John} & \cdots & \text{john@gmail.com} & \cdots & 0912-345-678 \\ 120220002 & \text{Jack} & \cdots & \text{jack@gmail.com} & \cdots & 0923-456-781 \\ \vdots & \vdots & & \vdots & & \vdots \\ 220220001 & \text{Mary} & \cdots & \text{mary@gmail.com} & \cdots & 0981-234-567 \\ \vdots & \vdots & & \vdots & & \vdots \\ 920229999 & \text{Xavier} & \cdots & \text{xavier@gmail.com} & \cdots & 0988-888-888 \end{bmatrix} \end{aligned}$$

10.3 database relations

定義 10.3.1. 函數性依賴 / 功能相依 functional dependence
review definition of functions 1.2.17 in set theory,

$$f: X \to Y \Leftrightarrow f \in Y^X \Leftrightarrow \forall x \in X, \exists! y \in Y\,(y = f(x))$$

in relational model or database theory, functional dependence

$$\begin{cases} r(R) \text{ is a relation or relation schema} \\ K_1 \text{ is a key for } r \\ K_2 \text{ is a key for } r \\ f: K_1 \to K_2 \Leftrightarrow K_1 \xrightarrow{f} K_2 \qquad\qquad \Leftrightarrow K_1 \to K_2 \end{cases}$$

$$\Leftrightarrow \forall \boldsymbol{t}.K_1 \in \left(\bigcup_{\boldsymbol{t}\in r(R)} \{\boldsymbol{t}.K_1\}\right), \exists! \boldsymbol{t}.K_2 \in \left(\bigcup_{\boldsymbol{t}\in r(R)} \{\boldsymbol{t}.K_2\}\right) (\boldsymbol{t}.K_2 = f(\boldsymbol{t}.K_1))$$

$$\Leftrightarrow \begin{cases} \text{determinant } K_1 \text{ determines dependent} K_2 & \text{or } K_1 \text{ determines } K_2 \\ \text{dependent } K_2 \text{ is determined by determinant } K_1 & \text{or } K_2 \text{ is functionally dependent on } K_1 \end{cases}$$

定義 10.3.2. 完全函數性依賴 / 功能相依 full functional dependence / fully functional dependence

$$\begin{cases} r(R) \text{ is a relation or relation schema} \\ K_1 \text{ is a key for } r \\ K_2 \text{ is a key for } r \\ K_1 \to K_2 \\ \nexists J \text{ is a key for } r \left(\begin{cases} J \subset K_1 \\ J \to K_2 \end{cases} \right) \overset{?}{\Leftrightarrow} \nexists J \subset K_1 \left(\begin{cases} J \text{ is a key for } r \\ J \to K_2 \end{cases} \right) \end{cases}$$
$$\Leftrightarrow K_2 \text{ is fully functionally dependent on } K_1$$

定義 10.3.3. 超鍵 superkey

$$\begin{cases} r(R) \text{ is a relation or relation schema} \\ \emptyset \subset K \subseteq R \Rightarrow K \text{ is a key for } r & 10.1.1 \\ \forall \boldsymbol{t}_1, \boldsymbol{t}_2 \in r(R)\,(\boldsymbol{t}_1 \neq \boldsymbol{t}_2 \Rightarrow \boldsymbol{t}_1.K \neq \boldsymbol{t}_2.K) \end{cases}$$
$$\Leftrightarrow K \text{ is a superkey for } r\text{, denoted } K = SK_r$$

- 明顯/平凡超鍵 apparent superky / trivial superkey: the key includes all atomic attributes of the relation

$$K = \{a_\nu\}_{\nu \in \mathbb{N}}$$

定義 10.3.4. 候鍵 / 候選鍵 candidate key

$$\begin{cases} r(R) \text{ is a relation or relation schema} \\ K \text{ is a superkey for } r \\ \nexists J \text{ is a superkey for } r\,(J \subset K) \overset{?}{\Leftrightarrow} \nexists J \subset K\,(J \text{ is a superkey for } r) \end{cases}$$
$$\Leftrightarrow K \text{ is a candidate key for } r\text{, denoted } K = CK_r$$

定義 10.3.5. 主鍵 primary key

$$\begin{cases} r(R) \text{ is a relation or relation schema} \\ K \text{ is a candidate key for } r \\ \forall \boldsymbol{t} \in r(R)\,(\boldsymbol{t}.K \neq \texttt{null}) \\ \text{only one } K \text{ is chosen} \end{cases}$$
$$\Leftrightarrow K \text{ is the primary key for } r\text{, denoted } K = PK_r$$

primary-key constraint

$$K \text{ is the primary key for } r$$
$$\Rightarrow \begin{cases} r(R) \text{ is a relation or relation schema} \\ K \text{ is a candidate key for } r \\ \forall \boldsymbol{t} \in r(R)\,(\boldsymbol{t}.K \neq \texttt{null}) \end{cases}$$

primary key represented in relation schema: underlined, or additional boldface
e.g.

$$\begin{aligned} r(R) =& r\left(\underline{K_1}, K_2, \cdots, K_{n_K}\right) = r\left(\underline{A_1, A_2}, \cdots, A_{n_A}\right) \\ =& r\left(\underline{K_1} : D_1, K_2 : D_2, \cdots, K_{n_K} : D_{n_K}\right) \\ =& r\left(\left\langle \underline{K_1}, D_1 \right\rangle, \langle K_2, D_2 \rangle, \cdots, \langle K_{n_K}, D_{n_K} \rangle\right) \end{aligned}$$

$$\begin{aligned} r(R) =& r\left(\underline{\boldsymbol{K_1}}, K_2, \cdots, K_{n_K}\right) = r\left(\underline{\boldsymbol{A_1}, \boldsymbol{A_2}}, \cdots, A_{n_A}\right) \\ =& r\left(\underline{\boldsymbol{K_1}} : D_1, K_2 : D_2, \cdots, K_{n_K} : D_{n_K}\right) \\ =& r\left(\left\langle \underline{\boldsymbol{K_1}}, D_1 \right\rangle, \langle K_2, D_2 \rangle, \cdots, \langle K_{n_K}, D_{n_K} \rangle\right) \end{aligned}$$

K_1 is the primary key for r, also a composite key or composite attribute including attributes A_1 and A_2
e.g.

$$student\,(R_{student}) = student\,(\underline{\texttt{STU\_ID}}, \texttt{STU\_FNAME}, \texttt{STU\_LNAME}, \texttt{STU\_EMAIL}, \texttt{STU\_PHONE})$$

STU_ID is the primary key for the relation *student*

定義 10.3.6. 獨鍵 / 唯鍵 unique key

$$\begin{cases} r\left(R\right) \text{ is a relation or relation schema} \\ K \text{ is a candidate key for } r \\ K \text{ is not the primary key for } r \end{cases}$$
$$\Leftrightarrow K \text{ is a unique key for } r\text{, denoted } K = UK_r$$

- unique key vs. primary key
  - unique key can be assigned `null` value vs. primary key cannot be assigned `null` value
  - a relation may have many unique keys vs. a relation have only one primary key
  - candidate keys not primary key are unique keys

定義 10.3.7. 參照 reference

$$\begin{cases} r_1\left(R_1\right), r_2\left(R_2\right) \text{ are relations or relation schemas} \\ K_1 \text{ is a key for } r_1 \\ K_2 \text{ is a key for } r_2 \\ \forall \boldsymbol{t}_1 \in r_1\left(R_1\right), \exists \boldsymbol{t}_2 \in r_2\left(R_2\right)\left(\boldsymbol{t}_1.K_1 = \boldsymbol{t}_2.K_2\right) \end{cases}$$
$$\Leftrightarrow \begin{cases} K_1 \text{ in } r_1 \text{ references } K_2 \text{ in } r_2 & \text{, denoted } K_1 = RK_{r_1 \to r_2} \\ K_2 \text{ in } r_2 \text{ is referenced by } K_1 \text{ in } r_1 & \text{, denoted } K_2 = RK_{r_2 \leftarrow r_1} \\ r_1 \text{ is the referencing relation} \\ r_2 \text{ is the referenced relation} \end{cases}$$

referential constraint

$$K_1 \text{ is a referencing key for } r_1\text{, referencing the key } K_2 \text{ in } r_2$$
$$\Rightarrow \forall \boldsymbol{t}_1 \in r_1\left(R_1\right)\left(\boldsymbol{t}_1.K_1 \in \left(\bigcup_{\boldsymbol{t}_2 \in r_2\left(R_2\right)} \left\{\boldsymbol{t}_2.K_2\right\}\right) \cup \left\{\texttt{null}\right\}\right)$$

定義 10.3.8. 外鍵 / 外部鍵 / 外來鍵 foreign key

$$\begin{cases} r_1\left(R_1\right), r_2\left(R_2\right) \text{ are relations or relation schemas} \\ K_1 \text{ is a key for } r_1 \\ K_2 \text{ is the primary key for } r_2 \\ \forall \boldsymbol{t}_1 \in r_1\left(R_1\right), \exists \boldsymbol{t}_2 \in r_2\left(R_2\right)\left(\boldsymbol{t}_1.K_1 = \boldsymbol{t}_2.K_2\right) \end{cases}$$
$$\Leftrightarrow \begin{cases} K_1 \text{ is a foreign key for } r_1\text{, referencing the primary key } K_2 \text{ in } r_2 & \text{, denoted } K_1 = FK_{r_1 \to r_2} \\ \text{the primary key } K_2 \text{ in } r_2 \text{ is referenced by a foreign key } K_1 \text{ in } r_1 & \text{, denoted } K_2 = PK_{r_2 \leftarrow r_1} \end{cases}$$

foreign-key constraint

$$K_1 \text{ is a foreign key for } r_1\text{, referencing the primary key } K_2 \text{ in } r_2$$
$$\Rightarrow \begin{cases} \forall \boldsymbol{t}_1 \in r_1\left(R_1\right)\left(\boldsymbol{t}_1.K_1 \in \left(\bigcup_{\boldsymbol{t}_2 \in r_2\left(R_2\right)} \left\{\boldsymbol{t}_2.K_2\right\}\right) \cup \left\{\texttt{null}\right\}\right) \\ K_2 \text{ is the primary key for } r_2 \end{cases}$$

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Chapter 11

實體關係模型 entity-relationship model = ER model

11.1 definitions

定義 **11.1.1.** entity set and entity occurence / entity instance
An entity occurence / entity instance is an element of some entity set

$$E = \{e_1, e_2, \cdots, e_\mu, \cdots, e_m\}$$

定義 **11.1.2.** attribute[1, p.12,13]

$$A_{\nu_A} : E \to D_{\nu_A}$$

$$A_{\nu_A}(e_\mu) = v_{\mu\nu_A}$$

定義 **11.1.3.** entity schema

$$\begin{aligned} r(R) &= r\left(\{a_\nu\}_{\nu=1}^{n}\right) = r\left(\{A_{\nu_A}\}_{\nu_A=1}^{n_A}\right) = r\left(\{K_{\nu_K}\}_{\nu_K=1}^{n_K}\right) \\ \leftrightarrow E\left(\{a_\nu\}_{\nu=1}^{n}\right) &= E\left(\{A_{\nu_A}\}_{\nu_A=1}^{n_A}\right) = E\left(\{K_{\nu_K}\}_{\nu_K=1}^{n_K}\right) \end{aligned} \tag{11.1.1}$$

定義 **11.1.4.** relationship

$$E_1 = \{e_{11}, e_{12}, \cdots, e_{1\mu_1}, \cdots, e_{1m_2}\}$$

$$E_2 = \{e_{21}, e_{22}, \cdots, e_{2\mu_2}, \cdots, e_{2m_2}\}$$

$$E_3 = \{e_{31}, e_{32}, \cdots, e_{3\mu_3}, \cdots, e_{3m_3}\}$$

$$e_{\lambda,\mu_\lambda} = e_{\lambda\mu_\lambda}$$

- relationship degree[2, p.131]
  - binary relationship

$$\begin{aligned} R_{1\times 2} \subseteq E_1 \times E_2 &= \left\{ \langle e_{1\mu_1}, e_{2\mu_2} \rangle \middle| \begin{cases} e_{1\mu_1} \in E_1 \\ e_{2\mu_2} \in E_2 \end{cases} \right\} \\ &= \{\langle e_{1\mu_1}, e_{2\mu_2} \rangle | e_{1\mu_1}, e_{2\mu_2} \in E_1, E_2\} \end{aligned}$$

$$\langle e_{1\mu_1}, e_{2\mu_2} \rangle \in R_{1\times 2} \Leftrightarrow e_{1\mu_1} R_{1\times 2} e_{2\mu_2}$$

    * relationship cardinality[3, p.252 253] / relationship connectivity[2, p.121]

$$E_2^{R_{1\times 2}}(e_{1\mu_1}) = \{e_{2\mu_2} | \langle e_{1\mu_1}, e_{2\mu_2} \rangle \in R_{1\times 2} \subseteq E_1 \times E_2\}$$

$$E_1^{R_{1\times 2}}(e_{2\mu_2}) = \{e_{1\mu_1} | \langle e_{1\mu_1}, e_{2\mu_2} \rangle \in R_{1\times 2} \subseteq E_1 \times E_2\}$$

      · one-to-one relationship

$$\begin{aligned} &\begin{cases} \forall e_{1\mu_1} \in E_1 \left(\left| E_2^{R_{1\times 2}}(e_{1\mu_1}) \right| \in \{0,1\} \right) \\ \forall e_{2\mu_2} \in E_2 \left(\left| E_1^{R_{1\times 2}}(e_{2\mu_2}) \right| \in \{0,1\} \right) \end{cases} \\ \Leftrightarrow &R_{1\times 2} \text{ is a one-to-one relationship} \end{aligned}$$

· one-to-many relationship

$$\begin{cases}\forall e_{1\mu_1} \in E_1 \left(\left|E_2^{R_{1\times 2}}\left(e_{1\mu_1}\right)\right| \in \{0\} \cup \mathbb{N}\right) \\ \forall e_{2\mu_2} \in E_2 \left(\left|E_1^{R_{1\times 2}}\left(e_{2\mu_2}\right)\right| \in \{0,1\}\right)\end{cases}$$
$$\Leftrightarrow R_{1\times 2} \text{ is a one-to-many relationship}$$

· many-to-one relationship

$$\begin{cases}\forall e_{1\mu_1} \in E_1 \left(\left|E_2^{R_{1\times 2}}\left(e_{1\mu_1}\right)\right| \in \{0,1\}\right) \\ \forall e_{2\mu_2} \in E_2 \left(\left|E_1^{R_{1\times 2}}\left(e_{2\mu_2}\right)\right| \in \{0\} \cup \mathbb{N}\right)\end{cases}$$
$$\Leftrightarrow R_{1\times 2} \text{ is a many-to-one relationship}$$

· many-to-many relationship

$$\begin{cases}\forall e_{1\mu_1} \in E_1 \left(\left|E_2^{R_{1\times 2}}\left(e_{1\mu_1}\right)\right| \in \{0\} \cup \mathbb{N}\right) \\ \forall e_{2\mu_2} \in E_2 \left(\left|E_1^{R_{1\times 2}}\left(e_{2\mu_2}\right)\right| \in \{0\} \cup \mathbb{N}\right)\end{cases}$$
$$\Leftrightarrow R_{1\times 2} \text{ is a many-to-many relationship}$$

· composite entity / associative entity / bridge entity

$$\begin{cases}E_{1\circ 2} = \{e_{1\circ 2,\mu_{1\circ 2}}\}_{\mu_{1\circ 2}=1}^{m_{1\circ 2}} \text{ is a entity set with entity schema } E_{1\circ 2}\left(\{A_{1\circ 2,\nu_A}\}_{\nu_{1\circ 2}=1}^{n_{1\circ 2}}\right) \\ \exists A_{1\circ 2,1} \in \{A_{1\circ 2,\nu_A}\}_{\nu_A=1}^{n_A}\left(A_{1\circ 2,1} : E_{1\circ 2} \to \mathrm{dom} PK_{E_1}\right) \\ \exists A_{1\circ 2,2} \in \{A_{1\circ 2,\nu_A}\}_{\nu_A=1}^{n_A}\left(A_{1\circ 2,2} : E_{1\circ 2} \to \mathrm{dom} PK_{E_2}\right)\end{cases}$$
$\Leftrightarrow E_{1\circ 2}$ is a composite entity for E_1 and E_2

$$E_{1\circ 2}^{R_{1\times(1\circ 2)}}\left(e_{1\mu_1}\right) = \left\{e_{1\circ 2,\mu_{1\circ 2}} \middle| \langle e_{1\mu_1}, e_{1\circ 2,\mu_{1\circ 2}}\rangle \in R_{1\times(1\circ 2)} \subseteq E_1 \times E_{1\circ 2}\right\}$$
$$E_1^{R_{1\times(1\circ 2)}}\left(e_{1\circ 2,\mu_{1\circ 2}}\right) = \left\{e_{1\mu_1} \middle| \langle e_{1\mu_1}, e_{1\circ 2,\mu_{1\circ 2}}\rangle \in R_{1\times(1\circ 2)} \subseteq E_1 \times E_{1\circ 2}\right\}$$
$$E_2^{R_{(1\circ 2)\times 2}}\left(e_{1\circ 2,\mu_{1\circ 2}}\right) = \left\{e_{2\mu_2} \middle| \langle e_{1\circ 2,\mu_{1\circ 2}}, e_{2\mu_2}\rangle \in R_{(1\circ 2)\times 2} \subseteq E_{1\circ 2} \times E_2\right\}$$
$$E_{1\circ 2}^{R_{(1\circ 2)\times 2}}\left(e_{2\mu_2}\right) = \left\{e_{1\circ 2,\mu_{1\circ 2}} \middle| \langle e_{1\circ 2,\mu_{1\circ 2}}, e_{2\mu_2}\rangle \in R_{(1\circ 2)\times 2} \subseteq E_{1\circ 2} \times E_2\right\}$$

make many-to-many relationship into
a one-to-many relationship

$$\begin{cases}\forall e_{1\mu_1} \in E_1 \left(\left|E_{1\circ 2}^{R_{1\times(1\circ 2)}}\left(e_{1\mu_1}\right)\right| \in \{0\} \cup \mathbb{N}\right) \\ \forall e_{1\circ 2,\mu_{1\circ 2}} \in E_{1\circ 2} \left(\left|E_1^{R_{1\times(1\circ 2)}}\left(e_{1\circ 2,\mu_{1\circ 2}}\right)\right| \in \{0,1\}\right)\end{cases}$$
$$\Leftrightarrow R_{1\times(1\circ 2)} \text{ is a one-to-many relationship}$$

and a many-to-one relationship

$$\begin{cases}\forall e_{1\circ 2,\mu_{1\circ 2}} \in E_{1\circ 2} \left(\left|E_2^{R_{(1\circ 2)\times 2}}\left(e_{1\circ 2,\mu_{1\circ 2}}\right)\right| \in \{0,1\}\right) \\ \forall e_{2\mu_2} \in E_2 \left(\left|E_{1\circ 2}^{R_{(1\circ 2)\times 2}}\left(e_{2\mu_2}\right)\right| \in \{0\} \cup \mathbb{N}\right)\end{cases}$$
$$\Leftrightarrow R_{(1\circ 2)\times 2} \text{ is a many-to-one relationship}$$

* relationship participation[2, p.128,129][3, p.255]
  · optional participation / partial participation: A relationship in which one entity occurrence does not require a corresponding occurrence in another entity.

$$\forall e_{1\mu_1} \in E_1 \left(\left|E_2^{R_{1\times 2}}\left(e_{1\mu_1}\right)\right| \in \{0\} \cup \mathbb{N}\right)$$
$\Leftrightarrow E_1$ has an optional participation in $R_{1\times 2}$

  · mandatory participation / total participation: A relationship in which one entity occurrence requires at least one corresponding occurrence in another entity.

$$\forall e_{1\mu_1} \in E_1 \left(\left|E_2^{R_{1\times 2}}\left(e_{1\mu_1}\right)\right| \in \mathbb{N}\right)$$
$\Leftrightarrow E_1$ has a mandatory participation in $R_{1\times 2}$

- unary relationship

$$R_{1\times 1} \subseteq E_1 \times E_1 = \left\{ \langle e_{1\mu_1}, e_{1\mu_2} \rangle \middle| \begin{cases} e_{1\mu_1} \in E_1 \\ e_{1\mu_2} \in E_1 \end{cases} \right\}$$
$$= \{\langle e_{1\mu_1}, e_{1\mu_2} \rangle | e_{1\mu_1}, e_{1\mu_2} \in E_1\}$$
$$\langle e_{1\mu_1}, e_{1\mu_2} \rangle \in R_{1\times 1} \Leftrightarrow e_{1\mu_1} R_{1\times 1} e_{1\mu_2}$$

- ternary relationship[2, p.131]
  - conceptual

$$R_{1\times 2\times 3} \subseteq E_1 \times E_2 \times E_3 = \left\{ \langle e_{1\mu_1}, e_{2\mu_2}, e_{3\mu_3} \rangle \middle| \begin{cases} e_{1\mu_1} \in E_1 \\ e_{2\mu_2} \in E_2 \\ e_{3\mu_3} \in E_3 \end{cases} \right\}$$
$$= \{\langle e_{1\mu_1}, e_{2\mu_2}, e_{3\mu_3} \rangle | e_{1\mu_1}, e_{2\mu_2}, e_{3\mu_3} \in E_1, E_2, E_3\}$$
$$\langle e_{1\mu_1}, e_{2\mu_2}, e_{3\mu_3} \rangle \in R_{1\times 2\times 3} \Leftrightarrow e_{1\mu_1} \overset{e_{2\mu_2}}{R_{1\times 2\times 3}} e_{3\mu_3} \Leftrightarrow e_{1\mu_1} \underset{e_{2\mu_2}}{R_{1\times 2\times 3}} e_{3\mu_3}$$

  - logical

$$\begin{cases} E_{1\circ 2\circ 3} = \{e_{1\circ 2\circ 3,\mu_{1\circ 2\circ 3}}\}_{\mu_{1\circ 2\circ 3}=1}^{m_{1\circ 2\circ 3}} \text{ is a entity set with entity schema } E_{1\circ 2\circ 3}\left(\{A_{1\circ 2\circ 3,\nu_{1\circ 2\circ 3}}\}_{\nu_{1\circ 2\circ 3}=1}^{n_{1\circ 2\circ 3}}\right) \\ \exists A_{1\circ 2\circ 3,1} \in \{A_{1\circ 2\circ 3,\nu_A}\}_{\nu_A=1}^{n_A} (A_{1\circ 2\circ 3,1} : E_{1\circ 2\circ 3} \to \mathrm{dom} PK_{E_1}) \\ \exists A_{1\circ 2\circ 3,2} \in \{A_{1\circ 2\circ 3,\nu_A}\}_{\nu_A=1}^{n_A} (A_{1\circ 2\circ 3,2} : E_{1\circ 2\circ 3} \to \mathrm{dom} PK_{E_2}) \\ \exists A_{1\circ 2\circ 3,3} \in \{A_{1\circ 2\circ 3,\nu_A}\}_{\nu_A=1}^{n_A} (A_{1\circ 2\circ 3,3} : E_{1\circ 2\circ 3} \to \mathrm{dom} PK_{E_3}) \end{cases}$$

$\Leftrightarrow E_{1\circ 2\circ 3}$ is a composite entity for E_1, E_2, E_3

定義 11.1.5. ER model schema

$$DB_{ER} = Db_{ER} = \{E_1, E_2, E_3, \cdots, R_{1\times 1}, R_{1\times 2}, \cdots, R_{2\times 1}, R_{2\times 2}, \cdots, R_{1\times 1\times 1}, \cdots, R_{1\times 2\times 3}, \cdots\}$$
$$E_\lambda = E_\lambda\left(\{A_{\lambda\nu_\lambda}\}_{\nu_\lambda=1}^{n_\lambda}\right)$$
$$R_{\lambda_1\times\lambda_2} = R_{\lambda_1\times\lambda_2}\left(\{A_{\lambda_1\times\lambda_2,\nu_{\lambda_1\times\lambda_2}}\}_{\nu_{\lambda_1\times\lambda_2}=1}^{n_{\lambda_1\times\lambda_2}}\right)$$
$$R_{\times_i\lambda_i} = R_{\times_i\lambda_i}\left(\left\{A_{\times_i\lambda_i,\nu_{\times_i\lambda_i}}\right\}_{\nu_{\times_i\lambda_i}=1}^{n_{\times_i\lambda_i}}\right)$$

定義 11.1.6. entity set strength

- weak entity set[3, p.259][2, p.123]

$$\begin{cases} Db_{ER} \ni E_\lambda \\ E_\lambda = \{e_{\lambda,\mu_\lambda}\}_{\mu_\lambda=1}^{m_\lambda} \text{ is a entity set with entity schema } E_\lambda\left(\{A_{\lambda,\nu_\lambda}\}_{\nu_\lambda=1}^{n_\lambda}\right) \\ PK_{E_\lambda} = \bigcup_i A_{\lambda,i} \\ \exists E_\kappa \in Db_{ER}\left(\begin{cases} \kappa \neq \lambda \Leftrightarrow E_\kappa \neq E_\lambda \\ PK_{E_\kappa} \subset PK_{E_\lambda} \end{cases}\right) \end{cases}$$

$\Leftrightarrow E_\lambda$ is a weak entity set, dependent on its identifying entity set E_κ, with strong relationship or identifying relationship $R_{\kappa\times\lambda}$

- discriminator attribute: instead of associating a primary key with a weak entity, we use the primary key of the identifying entity, along with extra attributes, called discriminator attributes to uniquely identify a weak entity[3, p.259], e.g. an atomic attribute of E_λ as the discriminator attribute

$$PK_{E_\kappa} = PK_{E_\lambda} \cup \{\delta\} \Rightarrow PK_{E_\kappa} \subset PK_{E_\lambda}$$

- strong entity set: An entity set that is not a weak entity set is termed a strong entity set.[3, p.259]

$$\begin{cases} Db_{ER} \ni E_\lambda \\ E_\lambda = \{e_{\lambda,\mu_\lambda}\}_{\mu_\lambda=1}^{m_\lambda} \text{ is a entity set with entity schema } E_\lambda\left(\{A_{\lambda,\nu_\lambda}\}_{\nu_\lambda=1}^{n_\lambda}\right) \\ PK_{E_\lambda} = \bigcup_i A_{\lambda,i} \\ \neg\exists E_\kappa \in Db_{ER}\left(\begin{cases} \kappa \neq \lambda \Leftrightarrow E_\kappa \neq E_\lambda \\ PK_{E_\kappa} \subset PK_{E_\lambda} \end{cases}\right) \end{cases}$$

$\Leftrightarrow E_\lambda$ is a strong entity set

– because

$$\begin{aligned}&\neg\exists E_\kappa \in Db_{ER}\left(\begin{cases}\kappa \neq \lambda \Leftrightarrow E_\kappa \neq E_\lambda\\ PK_{E_\kappa} \subset PK_{E_\lambda}\end{cases}\right)\\ \Leftrightarrow&\forall E_\kappa \in Db_{ER}\neg\left(\begin{cases}\kappa \neq \lambda \Leftrightarrow E_\kappa \neq E_\lambda\\ PK_{E_\kappa} \subset PK_{E_\lambda}\end{cases}\right)\\ \Leftrightarrow&\forall E_\kappa \in Db_{ER}\left((\kappa = \lambda)\vee\neg\left(PK_{E_\kappa} \subset PK_{E_\lambda}\right)\right)\end{aligned}$$

– i.e.

$$\begin{cases}Db_{ER} \ni E_\lambda\\ E_\lambda = \{e_{\lambda,\mu_\lambda}\}_{\mu_\lambda=1}^{m_\lambda} \text{ is a entity set with entity schema } E_\lambda\left(\{A_{\lambda,\nu_\lambda}\}_{\nu_\lambda=1}^{n_\lambda}\right)\\ PK_{E_\lambda} = \bigcup_i A_{\lambda,i}\\ \neg\exists E_\kappa \in Db_{ER}\left(\begin{cases}\kappa \neq \lambda \Leftrightarrow E_\kappa \neq E_\lambda\\ PK_{E_\kappa} \subset PK_{E_\lambda}\end{cases}\right) \Leftrightarrow \forall E_\kappa \in Db_{ER}\left((\kappa = \lambda)\vee\neg\left(PK_{E_\kappa} \subset PK_{E_\lambda}\right)\right)\end{cases}$$

$\Leftrightarrow E_\lambda$ is a strong entity set

11.2 擴充實體關係模型 extended ER model = EERM

定義 11.2.1. hierarchy[2, p.168 170][3, p.271 275]

- relationship of specialization / generalization

$$\begin{aligned}&E_{\lambda_i} <: E_\lambda\\ \Leftrightarrow&\begin{cases}E_{\lambda_i} \subset E_\lambda\\ PK_{E_{\lambda_i}} = PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_i}}\right\}\end{cases}\end{aligned} \tag{11.2.1}$$

  – generalization: e.g. E_{λ_i} is a E_λ

$$PK_{E_{\lambda_i}} = PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_i}}\right\}$$

  $\delta_{E_{\lambda_i}}$ is an atomic attribute as the subtype discriminator attribute[2, p.172]
  i.e. $E_{\lambda_1}, E_{\lambda_2}, \cdots, E_{\lambda_i}, \cdots$ are subtypes / subclasses of E_λ

$$E_{\lambda_i} <: E_\lambda \Leftrightarrow \begin{cases}E_{\lambda_1} <: E_\lambda\\ E_{\lambda_2} <: E_\lambda\\ \vdots\\ E_{\lambda_i} <: E_\lambda\\ \vdots\end{cases}$$

$$E_{\lambda_i} \sqsubseteq E_\lambda$$

$$E_{\lambda_i} \leq: E_\lambda$$

  generalization relationship schema: $(\max i + 1)$-nary relationship

$$\begin{aligned}<:_\lambda = <:_{\lambda\times_i\lambda_i} = <:_{\lambda\times_i\lambda_i}&\left(PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_i}}\right\}_{i=1}^{\max\{i\}}\right)\\ = <:_{\lambda\times_i\lambda_i}&\left(PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_1}}\right\} \cup \left\{\delta_{E_{\lambda_2}}\right\} \cup \cdots \cup \left\{\delta_{E_{\lambda_{\max\{i\}}}}\right\}\right)\end{aligned}$$

$$(E_{\lambda_i} <: E_\lambda) \Leftrightarrow (E_{\lambda_i} \sqsubseteq E_\lambda) \Leftrightarrow (E_{\lambda_i} \leq: E_\lambda) \Rightarrow \left(PK_{E_{\lambda_i}} = PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_i}}\right\}\right)$$

  – specialization: or conversely, E_κ is the suptype / supclass of $E_{\lambda_1}, E_{\lambda_2}, \cdots, E_{\lambda_i}, \cdots$

$$(E_\lambda :> E_{\lambda_i}) \Leftrightarrow (E_\lambda \sqsupseteq E_{\lambda_i}) \Leftrightarrow (E_\lambda :> E_{\lambda_i}) \Rightarrow \left(PK_{E_\lambda} \cup \left\{\delta_{E_{\lambda_i}}\right\} = PK_{E_{\lambda_i}}\right)$$

  subtype is a proper subset of its parent entity set

$$(E_{\lambda_i} <: E_\lambda) \Leftrightarrow (E_\lambda :> E_{\lambda_i}) \Rightarrow \forall e \in E_{\lambda_i}\,(e \in E_\lambda) \Leftrightarrow E_{\lambda_i} \subset E_\lambda$$

- disjoint / overlapping constraint[2, p.172,173][3, p.272]

$$E_{\lambda_1}, E_{\lambda_2} <: E_\lambda \Leftrightarrow \begin{cases} E_{\lambda_1} <: E_\lambda \\ E_{\lambda_2} <: E_\lambda \end{cases}$$

  - disjoint = nonoverlapping

$$\begin{aligned} & E_{\lambda_1}, E_{\lambda_2} \text{ are disjoint subtypes of } E_\lambda \\ \Leftrightarrow & \begin{cases} E_{\lambda_1}, E_{\lambda_2} <: E_\lambda \\ (\lambda_1 \neq \lambda_2) \to (E_{\lambda_1} \cap E_{\lambda_2} = \emptyset) \end{cases} \\ \Leftrightarrow & \begin{cases} \begin{cases} PK_{E_{\lambda_1}} = PK_{E_\lambda} \cup \{\delta_{E_{\lambda_1}}\} \\ \forall e \in E_{\lambda_1} \left((e \in E_\lambda) \wedge (e \notin E_{\lambda_2})\right) \end{cases} \\ \begin{cases} PK_{E_{\lambda_2}} = PK_{E_\lambda} \cup \{\delta_{E_{\lambda_2}}\} \\ \forall e \in E_{\lambda_2} \left((e \in E_\lambda) \wedge (e \notin E_{\lambda_1})\right) \end{cases} \end{cases} \end{aligned}$$

  - overlapping

$$\begin{aligned} & E_{\lambda_1}, E_{\lambda_2} \text{ are overlaying subtypes of } E_\lambda \\ \Leftrightarrow & \begin{cases} E_{\lambda_1}, E_{\lambda_2} <: E_\lambda \\ (\lambda_1 \neq \lambda_2) \wedge (E_{\lambda_1} \cap E_{\lambda_2} \neq \emptyset) \end{cases} \\ \Leftrightarrow & \begin{cases} \lambda_1 \neq \lambda_2 \\ \begin{cases} PK_{E_{\lambda_1}} = PK_{E_\lambda} \cup \{\delta_{E_{\lambda_1}}\} \\ \forall e \in E_{\lambda_1} \left(e \in E_\lambda\right) \end{cases} \\ \begin{cases} PK_{E_{\lambda_2}} = PK_{E_\lambda} \cup \{\delta_{E_{\lambda_2}}\} \\ \forall e \in E_{\lambda_2} \left(e \in E_\lambda\right) \end{cases} \\ \exists e \in E_{\lambda_1} \left(e \in E_{\lambda_2}\right) \Leftrightarrow \exists e \in E_{\lambda_2} \left(e \in E_{\lambda_1}\right) \quad \Rightarrow \exists e \in E_\lambda \left(\begin{cases} e \in E_{\lambda_1} \\ e \in E_{\lambda_2} \end{cases} \right) \end{cases} \end{aligned}$$

- completeness constraint[2, p.174,175][3, p.275]
  - total completeness / total specialization or generalization

$$\begin{aligned} & E_\lambda \text{ has total completeness} \\ \Leftrightarrow & \forall e \in E_\lambda, \exists E_{\lambda_i} <: E_\lambda \left(e \in E_{\lambda_i}\right) \end{aligned}$$

  - parital completeness / partial specialization or generalization

$$\begin{aligned} & E_\lambda \text{ has partial completeness} \\ \Leftrightarrow & \neg\forall e \in E_\lambda, \exists E_{\lambda_i} <: E_\lambda \left(e \in E_{\lambda_i}\right) \\ \Leftrightarrow & \exists e \in E_\lambda, \neg\exists E_{\lambda_i} <: E_\lambda \left(e \in E_{\lambda_i}\right) \\ \Leftrightarrow & \exists e \in E_\lambda, \forall E_{\lambda_i} <: E_\lambda \neg \left(e \in E_{\lambda_i}\right) \\ \Leftrightarrow & \exists e \in E_\lambda, \forall E_{\lambda_i} <: E_\lambda \left(e \notin E_{\lambda_i}\right) \end{aligned}$$

- 4 combinations of scenarios[2, p.175]

$$E_\lambda = E_\lambda \left(PK_{E_\lambda}, \left\{\delta_{E_{\lambda_i}}\right\}_{i=1}^{\max\{i\}}, \cdots, A_{\lambda\nu_\lambda}, \cdots \right)$$

$$\begin{array}{ccccccccccc} E_\lambda & PK_{E_\lambda} & \delta_{E_{\lambda_1}} & \delta_{E_{\lambda_2}} & \cdots & \delta_{E_{\lambda_i}} & \cdots & \delta_{E_{\lambda_{\max\{i\}}}} & \cdots & A_{\lambda\nu_\lambda} & \cdots \\ e_{\lambda 1} & PK_{\lambda 1} \\ e_{\lambda 2} & PK_{\lambda 2} \\ \vdots & \vdots \\ e_{\lambda\mu_\lambda} & PK_{\lambda\mu_\lambda} & \begin{Bmatrix} \texttt{true} \\ \texttt{false} \end{Bmatrix} & \begin{Bmatrix} \texttt{true} \\ \texttt{false} \end{Bmatrix} & \cdots & \begin{Bmatrix} \texttt{true} \\ \texttt{false} \end{Bmatrix} & \cdots & \begin{Bmatrix} \texttt{true} \\ \texttt{false} \end{Bmatrix} & \cdots & \cdots & \cdots \\ \vdots & \vdots \\ e_{\lambda m} \end{array}$$

$$
\begin{array}{ccccccccccc}
E_\lambda & PK_{E_\lambda} & \delta_{E_{\lambda_1}} & \delta_{E_{\lambda_2}} & \cdots & \delta_{E_{\lambda_i}} & \cdots & \delta_{E_{\lambda_{\max\{i\}}}} & \cdots & A_{\lambda\nu_\lambda} & \cdots \\
e_{\lambda 1} & PK_{\lambda 1} \\
e_{\lambda 2} & PK_{\lambda 2} \\
\vdots & \vdots \\
e_{\lambda\mu_\lambda} & PK_{\lambda\mu_\lambda} & \left\{\begin{array}{c}1\\0\end{array}\right\} & \left\{\begin{array}{c}1\\0\end{array}\right\} & \cdots & \left\{\begin{array}{c}1\\0\end{array}\right\} & \cdots & \left\{\begin{array}{c}1\\0\end{array}\right\} & \cdots & \cdots & \cdots \\
\vdots & \vdots \\
e_{\lambda m}
\end{array}
$$

$$
\mathrm{dom}\delta_{E_{\lambda_i}} = \left\{\begin{array}{c}1\\0\end{array}\right\} = \{0,1\}
$$

$$
\left(\delta_{E_{\lambda_i}}\right)_{\lambda,\mu_\lambda} = v_{\lambda,\mu_\lambda,\delta_{E_{\lambda_i}}} \sim v_{\lambda\mu\nu}
$$

– parital completeness and overlapping constraints

$$
\sum_{i=1}^{\max\{i\}} \left(\delta_{E_{\lambda_i}}\right)_{\lambda,\mu_\lambda} \in \{0\} \cup \mathbb{N}
$$

– parital completeness and disjoint constraints

$$
\sum_{i=1}^{\max\{i\}} \left(\delta_{E_{\lambda_i}}\right)_{\lambda,\mu_\lambda} \in \{0,1\}
$$

– total completeness and overlapping constraints

$$
\sum_{i=1}^{\max\{i\}} \left(\delta_{E_{\lambda_i}}\right)_{\lambda,\mu_\lambda} \in \mathbb{N}
$$

– total completeness and disjoint constraints

$$
\sum_{i=1}^{\max\{i\}} \left(\delta_{E_{\lambda_i}}\right)_{\lambda,\mu_\lambda} = 1
$$

定義 11.2.2. inheritance / attribute inheritance

- total inheritance

$$
\begin{cases}
E_\lambda = E_\lambda\left(K_\lambda\right) & 11.1.1 \\
E_{\lambda_i} = E_{\lambda_i}\left(K_{\lambda_i}\right) & 11.1.1 \\
E_{\lambda_i} <: E_\lambda & 11.2.1 \\
K_\lambda \subset K_{\lambda_i}
\end{cases}
$$

$\Leftrightarrow E_{\lambda_i}$ has total (attribute) inheritance from E_λ

定義 11.2.3. aggregation[3, p.276,277]

One limitation of the E-R model is that it cannot express relationships among relationships.
e.g.

$$
Ag_{1\times 2} = Ag_{1\times 2}\left(E_1, E_2, R_{1\times 2}\right)
$$

$$
R_{3\times Ag_{1\times 2}} = \left\{ \left\langle e_3, e_{Ag_{1\times 2}} \right\rangle \middle| \begin{cases} e_3 \in E_3 \\ e_{Ag_{1\times 2}} \in Ag_{1\times 2} \end{cases} \right\}
$$

11.3 logical design: mapping conceptual model to relational model / reducing ERD to relational schemas

[2, p.468] [3, p.264]

1. Map strong entities.
2. Map supertype/subtype relationships.
3. Map weak entities.
4. Map binary relationships.
5. Map higher-degree relationships.

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Chapter 12

關聯代數 relational algebra

12.1 basic operation

- operations in relational algebra / algebra on relations
  - select
  - project
  - set operations in set theory
    - product = Cartesian product
    - union
    - intersect = intersection
    - complementation
    - difference
    - symmetric difference = disjoint union
    - power set
  - join
  - assign[4, p.55]
  - rename[4, p.56][1]

定義 12.1.1. select

$$\sigma_{p\left(\{K_\nu\}_{\nu\in N}\right)}(r) = \sigma_{p\left(\{K_\nu\}_{\nu\in N}\right)}(r(R)) = \underset{p\left(\{K_\nu\}_{\nu\in N}\right)}{\sigma}(r) = \underset{p\left(\{K_\nu\}_{\nu\in N}\right)}{\sigma}(r(R))$$

e.g.

$$\sigma_{K_\nu=v}(r) = \sigma_{K_\nu=v}(r(R)) = \underset{K_\nu=v}{\sigma}(r) = \underset{K_\nu=v}{\sigma}(r(R))$$

e.g. relation 1: *student* relation

$$
\begin{aligned}
r_1(R_1) &= r_1\left\langle \begin{array}{cccccc} K_{11}:D_{11} & K_{12}:D_{12} & \cdots & K_{1\nu_{K_1}}:D_{1\nu_{K_1}} & \cdots & K_{1n_{K_1}}:D_{n_{K_1}} \end{array}\right\rangle \\
&= \begin{bmatrix}
v_{111} & v_{112} & \cdots & v_{11\nu_{K_1}} & \cdots & v_{11n_{K_1}} \\
v_{121} & v_{122} & \cdots & v_{12\nu_{K_1}} & \cdots & v_{12n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1\mu 1} & v_{1\mu 2} & \cdots & v_{1\mu\nu_{K_1}} & \cdots & v_{1\mu n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1m1} & v_{1m2} & \cdots & v_{1m\nu_{K_1}} & \cdots & v_{1mn_{K_1}}
\end{bmatrix} \\
\overset{\text{assign}}{\leftarrow} student\left(R_{student}\right) &= student\left\langle \begin{array}{cccccc} \texttt{STU\_ID} & \texttt{STU\_FNAME} & \cdots & \texttt{STU\_EMAIL} & \cdots & \texttt{STU\_PHONE} \end{array}\right\rangle \\
&= \begin{bmatrix}
120220001 & \text{John} & \cdots & \text{john@gmail.com} & \cdots & 0912-345-678 \\
120220002 & \text{Jack} & \cdots & \text{jack@gmail.com} & \cdots & 0923-456-781 \\
\vdots & \vdots & & \vdots & & \vdots \\
220220001 & \text{Mary} & \cdots & \text{mary@gmail.com} & \cdots & 0981-234-567 \\
\vdots & \vdots & & \vdots & & \vdots \\
920229999 & \text{Xavier} & \cdots & \text{xavier@gmail.com} & \cdots & 0988-888-888
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_{\texttt{STU\_FNAME}='\text{John}'}(student) \qquad student\left\langle \begin{array}{cccccc} \texttt{STU\_ID} & \texttt{STU\_FNAME} & \cdots & \texttt{STU\_EMAIL} & \cdots & \texttt{STU\_PHONE} \end{array}\right\rangle \\
&= \underset{\texttt{STU\_FNAME}='\text{John}'}{\sigma}(student) = \begin{bmatrix}
120220001 & \text{John} & \cdots & \text{john@gmail.com} & \cdots & 0912-345-678 \\
120220003 & \text{John} & \cdots & \text{john2@gmail.com} & \cdots & 0934-567-812
\end{bmatrix}
\end{aligned}
$$

定義 12.1.2. project

$$\pi_{\{K_\nu\}_{\nu\in N}}(r) = \pi_{\{K_\nu\}_{\nu\in N}}(r(R)) = \underset{\{K_\nu\}_{\nu\in N}}{\pi}(r) = \underset{\{K_\nu\}_{\nu\in N}}{\pi}(r(R))$$

e.g.

$$\pi_{K_1,K_2,\cdots}(r) = \pi_{K_1,K_2,\cdots}(r(R)) = \underset{K_1,K_2,\cdots}{\pi}(r) = \underset{K_1,K_2,\cdots}{\pi}(r(R))$$

e.g. relation 1: *student* relation

$$
\begin{aligned}
r_1(R_1) &= r_1\left\langle \begin{array}{cccccc} K_{11}:D_{11} & K_{12}:D_{12} & \cdots & K_{1\nu_{K_1}}:D_{1\nu_{K_1}} & \cdots & K_{1n_{K_1}}:D_{n_{K_1}} \end{array}\right\rangle \\
&= \begin{bmatrix}
v_{111} & v_{112} & \cdots & v_{11\nu_{K_1}} & \cdots & v_{11n_{K_1}} \\
v_{121} & v_{122} & \cdots & v_{12\nu_{K_1}} & \cdots & v_{12n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1\mu 1} & v_{1\mu 2} & \cdots & v_{1\mu\nu_{K_1}} & \cdots & v_{1\mu n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1m1} & v_{1m2} & \cdots & v_{1m\nu_{K_1}} & \cdots & v_{1mn_{K_1}}
\end{bmatrix} \\
\overset{\text{assign}}{\leftarrow} student\left(R_{student}\right) &= student\left\langle \begin{array}{cccccc} \texttt{STU\_ID} & \texttt{STU\_FNAME} & \cdots & \texttt{STU\_EMAIL} & \cdots & \texttt{STU\_PHONE} \end{array}\right\rangle \\
&= \begin{bmatrix}
120220001 & \text{John} & \cdots & \text{john@gmail.com} & \cdots & 0912-345-678 \\
120220002 & \text{Jack} & \cdots & \text{jack@gmail.com} & \cdots & 0923-456-781 \\
\vdots & \vdots & & \vdots & & \vdots \\
220220001 & \text{Mary} & \cdots & \text{mary@gmail.com} & \cdots & 0981-234-567 \\
\vdots & \vdots & & \vdots & & \vdots \\
920229999 & \text{Xavier} & \cdots & \text{xavier@gmail.com} & \cdots & 0988-888-888
\end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}&\pi_{\texttt{STU\_EMAIL},\texttt{STU\_PHONE}}\left(student\right) & student\left\langle \begin{array}{cc}\texttt{STU\_EMAIL} & \texttt{STU\_PHONE}\end{array}\right\rangle \\ =&\underset{\texttt{STU\_EMAIL},\texttt{STU\_PHONE}}{\pi}\left(student\right) = & \left[\begin{array}{cc} \text{john@gmail.com} & 0912-345-678 \\ \text{jack@gmail.com} & 0923-456-781 \\ \vdots & \vdots \\ \text{mary@gmail.com} & 0981-234-567 \\ \vdots & \vdots \\ \text{xavier@gmail.com} & 0988-888-888 \end{array}\right]\end{aligned}$$

定義 12.1.3. product = Cartesian product

$$\begin{aligned} r_1 \times r_2 &= r_1\left(R_1\right) \times r_2\left(R_2\right) \\ &= \left\{ \left(\boldsymbol{t}_1, \boldsymbol{t}_2\right) \middle| \begin{cases} \boldsymbol{t}_1 \in r_1\left(R_1\right) \\ \boldsymbol{t}_2 \in r_2\left(R_2\right) \end{cases} \right\} \end{aligned}$$

定義 12.1.4. join

$$\begin{aligned} r_1 \bowtie_p r_2 =& r_1\left(R_1\right) \bowtie_{p\left(\left\{K_{1\nu_1}\right\}_{\nu_1\in N_1}, \left\{K_{2\nu_2}\right\}_{\nu_2\in N_2}\right)} r_2\left(R_2\right) \\ = \sigma_p\left(r_1 \times r_2\right) =& \sigma_{p\left(\left\{K_{1\nu_1}\right\}_{\nu_1\in N_1}, \left\{K_{2\nu_2}\right\}_{\nu_2\in N_2}\right)}\left(r_1\left(R_1\right) \times r_2\left(R_2\right)\right) \end{aligned}$$

or relatively saving space,

$$\begin{aligned} r_1 \underset{p}{\bowtie} r_2 = r_1\left(R_1\right) & \underset{p\left(\left\{K_{1\nu_1}\right\}_{\nu_1\in N_1}, \left\{K_{2\nu_2}\right\}_{\nu_2\in N_2}\right)}{\bowtie} r_2\left(R_2\right) \\ = \underset{p}{\sigma}\left(r_1 \times r_2\right) = & \underset{p\left(\left\{K_{1\nu_1}\right\}_{\nu_1\in N_1}, \left\{K_{2\nu_2}\right\}_{\nu_2\in N_2}\right)}{\sigma}\left(r_1\left(R_1\right) \times r_2\left(R_2\right)\right) \end{aligned}$$

or relatively more simply,

$$\begin{aligned} r \bowtie_\theta s =& r\left(R\right) \bowtie_\theta s\left(S\right) = \sigma_\theta\left(r \times s\right) = \sigma_\theta\left(r\left(R\right) \times s\left(S\right)\right) \\ = r \underset{\theta}{\bowtie} s =& r\left(R\right) \underset{\theta}{\bowtie} s\left(S\right) = \underset{\theta}{\sigma}\left(r \times s\right) = \underset{\theta}{\sigma}\left(r\left(R\right) \times s\left(S\right)\right) \end{aligned}$$

e.g.

$$\sigma_{K_{1\nu_1}=K_{2\nu_2}}\left(r_1 \times r_2\right) = \sigma_{K_{1\nu_1}=K_{2\nu_2}}\left(r_1\left(R_1\right) \times r_2\left(R_2\right)\right)$$

or relatively saving space,

$$\underset{K_{1\nu_1}=K_{2\nu_2}}{\sigma}\left(r_1 \times r_2\right) = \underset{K_{1\nu_1}=K_{2\nu_2}}{\sigma}\left(r_1\left(R_1\right) \times r_2\left(R_2\right)\right)$$

定義 12.1.5. 重命名 rename [1]

$$\rho_{n'\left(K_\nu\right)/n\left(K_\nu\right)}\left(r\right) = \rho_{n'\left(K_\nu\right)/n\left(K_\nu\right)}\left(r\left(R\right)\right)$$

or relatively saving space,

$$\underset{n'\left(K_\nu\right)/n\left(K_\nu\right)}{\rho}\left(r\right) = \underset{n'\left(K_\nu\right)/n\left(K_\nu\right)}{\rho}\left(r\left(R\right)\right)$$

e.g.

$$\begin{aligned} student\left(R_{student}\right) =& student\left\langle \texttt{STU\_ID}, \texttt{STU\_FNAME}, \cdots, \texttt{STU\_EMAIL}, \cdots, \texttt{STU\_PHONE} \right\rangle \\ & \Downarrow \rho_{\texttt{STU\_MOBILE}/\texttt{STU\_PHONE}}\left(student\right) = \rho_{\texttt{STU\_MOBILE}/\texttt{STU\_PHONE}}\left(student\left(R_{student}\right)\right) \\ student\left(R_{student}\right) =& student\left\langle \texttt{STU\_ID}, \texttt{STU\_FNAME}, \cdots, \texttt{STU\_EMAIL}, \cdots, \texttt{STU\_MOBILE} \right\rangle \end{aligned}$$

or relatively saving space,

$$\begin{aligned} student\left(R_{student}\right) =& student\left\langle \texttt{STU\_ID}, \texttt{STU\_FNAME}, \cdots, \texttt{STU\_EMAIL}, \cdots, \texttt{STU\_PHONE} \right\rangle \\ & \Downarrow \underset{\texttt{STU\_MOBILE}/\texttt{STU\_PHONE}}{\rho}\left(student\right) = \underset{\texttt{STU\_MOBILE}/\texttt{STU\_PHONE}}{\rho}\left(student\left(R_{student}\right)\right) \\ student\left(R_{student}\right) =& student\left\langle \texttt{STU\_ID}, \texttt{STU\_FNAME}, \cdots, \texttt{STU\_EMAIL}, \cdots, \texttt{STU\_MOBILE} \right\rangle \end{aligned}$$

定理 12.1.6. *Heath 定理 / 海斯定理 Heath theorem / Heath's theorem [3]*
about functional dependence 10.3.1
represented by project / projection

公理 12.1.7. *Armstrong axiom [2]*

12.2 結構化查詢語言 SQL = structured query language

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Chapter 13

NoSQL = not only SQL

- document store
- key-value store
- column store
- graph sotre
- in-memory store

13.1 document store

定義 13.1.1. document
e.g. relation 1: *student* relation

$$
\begin{array}{rcl}
r_1\left(R_1\right) & = & r_1\left\langle \begin{array}{cccccc} K_{11}:D_{11} & K_{12}:D_{12} & \cdots & K_{1\nu_{K_1}}:D_{1\nu_{K_1}} & \cdots & K_{1n_{K_1}}:D_{n_{K_1}} \end{array}\right\rangle \\
 & = & \left[\begin{array}{cccccc}
v_{111} & v_{112} & \cdots & v_{11\nu_{K_1}} & \cdots & v_{11n_{K_1}} \\
v_{121} & v_{122} & \cdots & v_{12\nu_{K_1}} & \cdots & v_{12n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1\mu 1} & v_{1\mu 2} & \cdots & v_{1\mu\nu_{K_1}} & \cdots & v_{1\mu n_{K_1}} \\
\vdots & \vdots & & \vdots & & \vdots \\
v_{1m1} & v_{1m2} & \cdots & v_{1m\nu_{K_1}} & \cdots & v_{1mn_{K_1}}
\end{array}\right] \\
\overset{\text{assign}}{\leftarrow} student\left(R_{student}\right) & = & student\left\langle \begin{array}{cccccc} \texttt{STU\_ID} & \texttt{STU\_FNAME} & \cdots & \texttt{STU\_EMAIL} & \cdots & \texttt{STU\_PHONE} \end{array}\right\rangle \\
 & = & \left[\begin{array}{cccccc}
120220001 & \text{John} & \cdots & \text{john@gmail.com} & \cdots & 0912-345-678 \\
120220002 & \text{Jack} & \cdots & \text{jack@gmail.com} & \cdots & 0923-456-781 \\
\vdots & \vdots & & \vdots & & \vdots \\
220220001 & \text{Mary} & \cdots & \text{mary@gmail.com} & \cdots & 0981-234-567 \\
\vdots & \vdots & & \vdots & & \vdots \\
920229999 & \text{Xavier} & \cdots & \text{xavier@gmail.com} & \cdots & 0988-888-888
\end{array}\right]
\end{array}
$$

can be viewed as
document 1

$$
\begin{array}{cccc}
\{ & \texttt{"STU\_ID"}: & 120220001 & , \\
 & \texttt{"STU\_FNAME"}: & \text{"John"} & , \\
 & \vdots & \vdots & \\
 & \texttt{"STU\_EMAIL"}: & \text{"john@gmail.com"} & , \\
 & \vdots & \vdots & \\
 & \texttt{"STU\_PHONE"} & \text{"}0912-345-678\text{"} & \}
\end{array}
$$

document 2

$$\begin{array}{llll} \{ & \texttt{"STU\_ID"}: & 120220002 & , \\ & \texttt{"STU\_FNAME"}: & \text{"Jack"} & , \\ & \vdots & \vdots & \\ & \texttt{"STU\_EMAIL"}: & \text{"jack@gmail.com"} & , \\ & \vdots & \vdots & \\ & \texttt{"STU\_PHONE"} & \text{"}0923-456-781\text{"} & \} \end{array}$$

and so on.

13.2 key-value store

13.3 column store

13.4 graph store

Part IV

物理 physics

Part V

機器學習 machine learning = ML

Chapter 14

深度學習 deep learning = DL

Part VI

磁振 magnetic resonance = MR