0.0.1 covector

$$\begin{cases} \boldsymbol{v} \in \mathcal{V} \subseteq \mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\} \\ \exists! \omega \in \mathbb{F} \left[\boldsymbol{\omega}\left(\boldsymbol{v}\right) = \omega\right] \end{cases} \quad \begin{aligned} &\Leftrightarrow \mathcal{V} \xrightarrow{\boldsymbol{\omega}} \mathbb{F} \Leftrightarrow \boldsymbol{\omega} : \mathcal{V} \to \mathbb{F} \\ &\Leftrightarrow \mathbb{F}^{\mathcal{V}} = \{\boldsymbol{\omega} | \boldsymbol{\omega} : \mathcal{V} \to \mathbb{F}\} \\ &\quad \Downarrow \\ &\quad \left|\mathbb{F}^{\mathcal{V}}\right| = \left|\mathbb{F}\right|^{|\mathcal{V}|} \end{aligned}$$

$$\begin{array}{ccccc} \boldsymbol{v}^1\left(\boldsymbol{v}_1\right) = 1 & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_n\right) \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{v}^i\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_j\right) \overset{\text{def.}}{=} \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} = \delta^i_j & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_n\right) \\ \vdots & & \vdots & & \vdots \\ \boldsymbol{v}^n\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_n\right) = 1 \end{array}$$

$$\boldsymbol{v}^i\left(\boldsymbol{v}\right) = \boldsymbol{v}^i\left(v^j \boldsymbol{v}_j\right) = v^j \boldsymbol{v}^i\left(\boldsymbol{v}_j\right) \overset{\text{def.}}{=} v^j \delta^i_j = v^i$$

$$\begin{cases} \boldsymbol{\omega} \in \mathcal{V}^* = \left(\mathcal{V}^*, \mathbb{F}, +, \cdot\right) = \left(\mathcal{V}^*, \mathbb{F}, +_{\mathcal{V}^*,\mathbb{F}}, \cdot_{\mathbb{F}\times\mathcal{V}^*,\mathbb{F}}\right) \\ \boldsymbol{v} \in \mathcal{V} = \left(\mathcal{V}, \mathbb{F}, +, \cdot\right) = \left(\mathcal{V}, \mathbb{F}, +_{\mathcal{V},\mathbb{F}}, \cdot_{\mathbb{F}\times\mathcal{V},\mathbb{F}}\right) \end{cases}$$

$$\begin{aligned} \boldsymbol{\omega}\left(\boldsymbol{v}\right) =& \boldsymbol{\omega}\left(v^j \boldsymbol{v}_j\right) = v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) \\ =& \boldsymbol{\omega}\left(\sum_j v^j \boldsymbol{v}_j\right) = \sum_j \boldsymbol{\omega}\left(v^j \boldsymbol{v}_j\right) = \sum_j v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) \\ =& \begin{cases} \boldsymbol{\omega}\left(v^1 \boldsymbol{v}_1 + \cdots + v^n \boldsymbol{v}_n\right) & = \boldsymbol{\omega}\left(\sum\limits_{j=1}^{n} v^j \boldsymbol{v}_j\right) \\ \boldsymbol{\omega}\left(\cdots + v^j \boldsymbol{v}_j + \cdots\right) & = \boldsymbol{\omega}\left(\sum\limits_{j\in J} v^j \boldsymbol{v}_j\right) \end{cases} && + = +_{\mathcal{V},\mathbb{F}} \\ =& \begin{cases} v^1 \boldsymbol{\omega}\left(\boldsymbol{v}_1\right) + \cdots + v^n \boldsymbol{\omega}\left(\boldsymbol{v}_n\right) & = \sum\limits_{j=1}^{n} v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) \\ \cdots + v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) + \cdots & = \sum\limits_{j\in J} v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) \end{cases} && + = +_{\mathcal{V}^*,\mathbb{F}} \\ =& v^j \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) = \boldsymbol{v}^j\left(\boldsymbol{v}\right) \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) && v^j = \boldsymbol{v}^j\left(\boldsymbol{v}\right) \Leftarrow \boldsymbol{v}^i\left(\boldsymbol{v}\right) = v^i \Leftarrow \boldsymbol{v}^i\left(\boldsymbol{v}_j\right) \overset{\text{def.}}{=} \delta^i_j \\ =& \boldsymbol{v}^j\left(\boldsymbol{v}\right) \omega^{\boldsymbol{v}}_j = \omega^{\boldsymbol{v}}_j \boldsymbol{v}^j\left(\boldsymbol{v}\right) = \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i\left(\boldsymbol{v}\right) && \omega^{\boldsymbol{v}}_j \overset{\text{def.}}{=} \boldsymbol{\omega}\left(\boldsymbol{v}_j\right) \\ \boldsymbol{\omega}\left(\boldsymbol{v}\right) =& \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i\left(\boldsymbol{v}\right) \\ \boldsymbol{\omega} =& \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i \end{aligned}$$

$$
\begin{aligned}
\mathcal{V}^* \ni \boldsymbol{\omega} =& \omega_i \boldsymbol{\omega}^i = \sum_i \omega_i \boldsymbol{\omega}^i = \begin{cases} \omega_1 \boldsymbol{\omega}^1 + \cdots + \omega_n \boldsymbol{\omega}^n & = \sum_{i=1}^n \omega_i \boldsymbol{\omega}^i \\ \cdots + \omega_i \boldsymbol{\omega}^i + \cdots & = \sum_{i \in I} \omega_i \boldsymbol{\omega}^i \end{cases} \\
=& \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i = \sum_i \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i = \begin{cases} \omega_1^{\boldsymbol{v}} \boldsymbol{v}^1 + \cdots + \omega_n^{\boldsymbol{v}} \boldsymbol{v}^n & = \sum_{i=1}^n \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \\ \cdots + \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i + \cdots & = \sum_{i \in I} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \end{cases} \\
=& \begin{cases} \omega_1^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^1 \\ | \end{bmatrix}^\intercal + \cdots + \omega_n^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^n \\ | \end{bmatrix}^\intercal & = \begin{bmatrix} \omega_1^{\boldsymbol{v}} \\ \vdots \\ \omega_n^{\boldsymbol{v}} \end{bmatrix}^\intercal \begin{bmatrix} — & \boldsymbol{v}^1 & — \\ & \vdots & \\ — & \boldsymbol{v}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^i \\ | \end{bmatrix}^\intercal + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ — & \boldsymbol{v}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} \vdots \\ \boldsymbol{v}^i \\ \vdots \end{bmatrix} \quad = [\boldsymbol{\omega}]^V \, V^* \\
=& \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i = \sum_i \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i = \begin{cases} \omega_1^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^1 + \cdots + \omega_n^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^n & = \sum_{i=1}^n \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i \\ \cdots + \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i + \cdots & = \sum_{i \in I} \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i \end{cases} \\
=& \begin{cases} \omega_1^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^1 \\ | \end{bmatrix}^\intercal + \cdots + \omega_n^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^n \\ | \end{bmatrix}^\intercal & = \begin{bmatrix} \omega_1^{\widetilde{\boldsymbol{v}}} \\ \vdots \\ \omega_n^{\widetilde{\boldsymbol{v}}} \end{bmatrix}^\intercal \begin{bmatrix} — & \widetilde{\boldsymbol{v}}^1 & — \\ & \vdots & \\ — & \widetilde{\boldsymbol{v}}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\widetilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \widetilde{\boldsymbol{v}}^i \\ | \end{bmatrix}^\intercal + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ — & \widetilde{\boldsymbol{v}}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} \vdots \\ \widetilde{\boldsymbol{v}}^i \\ \vdots \end{bmatrix} \quad = [\boldsymbol{\omega}]^{\widetilde{V}} \, \widetilde{V}^*
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega} =& [\boldsymbol{\omega}]^V \, V^* = [\boldsymbol{\omega}]^{\widetilde{V}} \, \widetilde{V}^* \\
=& \omega_i^{\boldsymbol{v}} {V^*}^i{}_k = \omega_j^{\widetilde{\boldsymbol{v}}} {\widetilde{V}^*}{}^j{}_k \\
\omega_j^{\widetilde{\boldsymbol{v}}} {\widetilde{V}^*}{}^j{}_k =& \omega_i^{\boldsymbol{v}} {V^*}^i{}_k \\
\omega_j^{\widetilde{\boldsymbol{v}}} =& \omega_i^{\boldsymbol{v}} {V^*}^i{}_k \left({\widetilde{V}^*}{}^j{}_k \right)^{-1} = \omega_i^{\boldsymbol{v}} {V^*}^i{}_k \left(\widetilde{V}^{*-1} \right)^k{}_j = \omega_i^{\boldsymbol{v}} Q^i{}_j \\
\boldsymbol{\omega}\left(\widetilde{\boldsymbol{v}}_j\right) = \omega_j^{\widetilde{\boldsymbol{v}}} =& \omega_i^{\boldsymbol{v}} Q^i{}_j = \boldsymbol{\omega}\left(\boldsymbol{v}_i\right) Q^i{}_j = \boldsymbol{\omega}\left(\widetilde{\boldsymbol{v}}_k B^k{}_i\right) Q^i{}_j = \boldsymbol{\omega}\left(\widetilde{\boldsymbol{v}}_k\right) B^k{}_i Q^i{}_j \\
\boldsymbol{\omega}\left(\widetilde{\boldsymbol{v}}_j\right) =& \boldsymbol{\omega}\left(\widetilde{\boldsymbol{v}}_k\right) B^k{}_i Q^i{}_j \\
B^k{}_i Q^i{}_j =& \delta^k{}_j \Rightarrow Q^i{}_j = F^i{}_j \\
\omega_j^{\widetilde{\boldsymbol{v}}} =& \omega_i^{\boldsymbol{v}} Q^i{}_j = \omega_i^{\boldsymbol{v}} F^i{}_j \\
\omega_j^{\widetilde{\boldsymbol{v}}} =& \omega_i^{\boldsymbol{v}} F^i{}_j \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal F \\
\omega_k^{\widetilde{\boldsymbol{v}}} B^k{}_j =& \omega_i^{\boldsymbol{v}} F^i{}_k B^k{}_j = \omega_i^{\boldsymbol{v}} \delta^i{}_j = \omega_j^{\boldsymbol{v}} \\
\omega_j^{\boldsymbol{v}} =& \omega_k^{\widetilde{\boldsymbol{v}}} B^k{}_j \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal = \begin{bmatrix} \vdots \\ \omega_i^{\widetilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal B
\end{aligned}
$$

$$
\begin{aligned}
\omega_j^{\widetilde{\boldsymbol{v}}} B^j{}_i \boldsymbol{v}^i = \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i =& \omega_i^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^i = \omega_j^{\boldsymbol{v}} F^j{}_i \widetilde{\boldsymbol{v}}^i \\
\omega_j^{\widetilde{\boldsymbol{v}}} B^j{}_i \boldsymbol{v}^i =& \omega_j^{\widetilde{\boldsymbol{v}}} \widetilde{\boldsymbol{v}}^j \Rightarrow B^j{}_i \boldsymbol{v}^i = \widetilde{\boldsymbol{v}}^j \Rightarrow \widetilde{\boldsymbol{v}}^j = B^k{}_i \boldsymbol{v}^i \\
\omega_j^{\boldsymbol{v}} F^j{}_i \widetilde{\boldsymbol{v}}^i =& \omega_j^{\boldsymbol{v}} \boldsymbol{v}^i \Rightarrow F^j{}_i \widetilde{\boldsymbol{v}}^i = \boldsymbol{v}^j \Rightarrow \boldsymbol{v}^j = F^j{}_i \widetilde{\boldsymbol{v}}^i
\end{aligned}
$$

$$
\begin{cases}
\begin{cases}
\omega^{\boldsymbol{v}}_j = \omega^{\widetilde{\boldsymbol{v}}}_k B^k{}_j & \begin{bmatrix}\vdots\\ \omega^{\boldsymbol{v}}_i\\ \vdots\end{bmatrix}^\intercal = \begin{bmatrix}\vdots\\ \omega^{\widetilde{\boldsymbol{v}}}_i\\ \vdots\end{bmatrix}^\intercal B \\
\omega^{\widetilde{\boldsymbol{v}}}_j = \omega^{\boldsymbol{v}}_i F^i{}_j & \begin{bmatrix}\vdots\\ \omega^{\widetilde{\boldsymbol{v}}}_i\\ \vdots\end{bmatrix}^\intercal = \begin{bmatrix}\vdots\\ \omega^{\boldsymbol{v}}_i\\ \vdots\end{bmatrix}^\intercal F
\end{cases} & \text{covariant} \\
\begin{cases}
\boldsymbol{v}^j = F^j{}_i \widetilde{\boldsymbol{v}}^i & \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} = F \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} \\
\widetilde{\boldsymbol{v}}^j = B^k{}_i \boldsymbol{v}^i & \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} = B \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}
\end{cases} & \text{contravariant}
\end{cases}
$$

$$
\begin{cases}
\begin{cases}
\boldsymbol{v}^j = F^j{}_i \widetilde{\boldsymbol{v}}^i & \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix} = F \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} \\
\widetilde{\boldsymbol{v}}^j = B^k{}_i \boldsymbol{v}^i & \begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix} = B \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}
\end{cases} & \text{contravariant} \\
\begin{cases}
\omega^{\boldsymbol{v}}_j = \omega^{\widetilde{\boldsymbol{v}}}_k B^k{}_j & \begin{bmatrix}\vdots\\ \omega^{\boldsymbol{v}}_i\\ \vdots\end{bmatrix}^\intercal = \begin{bmatrix}\vdots\\ \omega^{\widetilde{\boldsymbol{v}}}_i\\ \vdots\end{bmatrix}^\intercal B \\
\omega^{\widetilde{\boldsymbol{v}}}_j = \omega^{\boldsymbol{v}}_i F^i{}_j & \begin{bmatrix}\vdots\\ \omega^{\widetilde{\boldsymbol{v}}}_i\\ \vdots\end{bmatrix}^\intercal = \begin{bmatrix}\vdots\\ \omega^{\boldsymbol{v}}_i\\ \vdots\end{bmatrix}^\intercal F
\end{cases} & \text{covariant}
\end{cases}
$$

covariant contravariant

$$
\left.\begin{matrix}\widetilde{\mathfrak{V}}\\ \mathfrak{V}\end{matrix}\right\} \ni \begin{cases}\widetilde{\boldsymbol{v}}_j = \boldsymbol{v}_i F^i{}_j\\ \boldsymbol{v}_j = \widetilde{\boldsymbol{v}}_i B^i{}_j\end{cases} \qquad \mathbb{F} \ni \begin{cases}\widetilde{v}^i = B^i{}_j v^j\\ v^i = F^i{}_j \widetilde{v}^j\end{cases} \qquad \text{vector space } \mathcal{V} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j
$$

$$
\mathbb{F} \ni \begin{cases}\omega^{\widetilde{\boldsymbol{v}}}_j = \omega^{\boldsymbol{v}}_i F^i{}_j\\ \omega^{\boldsymbol{v}}_j = \omega^{\widetilde{\boldsymbol{v}}}_k B^k{}_j\end{cases} \qquad \left.\begin{matrix}\widetilde{\mathfrak{V}}^*\\ \mathfrak{V}^*\end{matrix}\right\} \ni \begin{cases}\widetilde{\boldsymbol{v}}^i = B^i{}_j \boldsymbol{v}^j\\ \boldsymbol{v}^i = F^i{}_j \widetilde{\boldsymbol{v}}^j\end{cases} \qquad \text{dual space } \mathcal{V}^* \ni \boldsymbol{\omega} = \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i
$$

$$
\widetilde{\boldsymbol{v}}_j \widetilde{v}^j = \boldsymbol{v}_i F^i{}_j B^j{}_k v^k = \boldsymbol{v}_i \delta^i{}_k v^k = \begin{cases}\boldsymbol{v}_k v^k & \boldsymbol{v}_k = \boldsymbol{v}_i \delta^i{}_k\\ \boldsymbol{v}_i v^i & \delta^i{}_k v^k = v^i\end{cases} = v^j \boldsymbol{v}_j
$$