real symmetric matrix diagonalizable

$$\begin{cases}\begin{cases}A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix}\\ A^\intercal = A & \text{symmetric matrix}\end{cases} & \text{real symmetric matrix}\\ A\boldsymbol{x} = \lambda\boldsymbol{x} & \begin{cases}\lambda \in \mathbb{C} & \text{complex eigenvalue}\\ \mathbf{0} \neq \boldsymbol{x} \in \mathbb{C}^n & \text{complex eigenvector}\end{cases}\end{cases}$$

$$\Downarrow$$

$$\begin{cases}\lambda \in \mathbb{R} & \text{real eigenvalue (1)}\\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector (2)}\end{cases}$$

$$\begin{aligned}
A\boldsymbol{x} =& \lambda\boldsymbol{x}\\
\overline{A}\overline{\boldsymbol{x}} = \overline{A\boldsymbol{x}} =& \overline{\lambda\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}\\
\overline{\boldsymbol{x}}^\intercal\overline{A}^\intercal = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal = \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^\intercal = \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\\
\overline{\boldsymbol{x}}^\intercal A \overset{\text{symmetric}}{=} \overline{\boldsymbol{x}}^\intercal A^\intercal \overset{\text{real}}{=}\\
\overline{\boldsymbol{x}}^\intercal A =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\\
\lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal(\lambda\boldsymbol{x}) \underset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{\overset{\cdot\boldsymbol{x}}{=}} \overline{\boldsymbol{x}}^\intercal A\boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\left(\lambda - \overline{\lambda}\right)\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} =& 0 \wedge \begin{cases}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} = \sum\limits_{i=1}^n |x_i|^2\\ \boldsymbol{x} \neq \mathbf{0}\end{cases} \Rightarrow \overline{\boldsymbol{x}}^\intercal\boldsymbol{x} \neq 0\\
\lambda - \overline{\lambda} =& 0\\
\lambda =& \overline{\lambda} \Leftrightarrow \lambda \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
\left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal\boldsymbol{x} = \left(\overline{\boldsymbol{x}}^\intercal\overline{A}^\intercal\right)\boldsymbol{x} \overset{\text{symmetric}}{=} \left(\overline{\boldsymbol{x}}^\intercal\overline{A}\right)\boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal\left(\overline{A}\boldsymbol{x}\right)\\
(L) = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal\boldsymbol{x} =& \overline{\boldsymbol{x}}^\intercal\left(\overline{A}\boldsymbol{x}\right) = (R)\\
(L) = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal\boldsymbol{x} \overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=} \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^\intercal\boldsymbol{x} = \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
(R) = \overline{\boldsymbol{x}}^\intercal\left(\overline{A}\boldsymbol{x}\right) \overset{\text{real}}{=} \overline{\boldsymbol{x}}^\intercal(A\boldsymbol{x}) \overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=} \overline{\boldsymbol{x}}^\intercal(\lambda\boldsymbol{x}) = \lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal\boldsymbol{x} =& \overline{\boldsymbol{x}}^\intercal\left(\overline{A}\boldsymbol{x}\right) = \lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} =& \lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}
\end{aligned}$$

$$\begin{cases}\begin{cases}A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix}\\ A^\intercal = A & \text{symmetric matrix}\end{cases} & \text{real symmetric matrix}\\ A\boldsymbol{x} = \lambda\boldsymbol{x} & \begin{cases}\lambda \in \mathbb{R} & \text{real eigenvalue}\\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector}\end{cases}\\ \begin{cases}A\boldsymbol{x}_1 = \lambda_1\boldsymbol{x}_1 & (e_1)\\ A\boldsymbol{x}_2 = \lambda_2\boldsymbol{x}_2 & (e_2)\end{cases} & \lambda_1 \neq \lambda_2\end{cases}$$

$$\Downarrow$$

$$\boldsymbol{x}_1^\intercal\boldsymbol{x}_2 = 0 \Leftrightarrow \boldsymbol{x}_1 \perp \boldsymbol{x}_2$$

$$
\begin{aligned}
A\boldsymbol{x}_2 =& \lambda_2 \boldsymbol{x}_2 \\
\boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \stackrel{\boldsymbol{x}_1^\intercal \cdot}{=}& \boldsymbol{x}_1^\intercal \lambda_2 \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (1) \\
A\boldsymbol{x}_1 =& \lambda_1 \boldsymbol{x}_1 \\
\boldsymbol{x}_1^\intercal A^\intercal = (A\boldsymbol{x}_1)^\intercal =& (\lambda_1 \boldsymbol{x}_1)^\intercal = \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A^\intercal =& \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \stackrel{\text{symmetric}}{=} \boldsymbol{x}_1^\intercal A^\intercal \boldsymbol{x}_2 \stackrel{\cdot \boldsymbol{x}_2}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (2) \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \stackrel{(1)}{=} \boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \stackrel{(2)}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(\lambda_2 - \lambda_1)\, \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0 \wedge \lambda_1 \neq \lambda_2 \\
\boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0
\end{aligned}
$$

$$
\begin{aligned}
(A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = (\boldsymbol{x}_1^\intercal A^\intercal)\, \boldsymbol{x}_2 \stackrel{\text{symmetric}}{=} (\boldsymbol{x}_1^\intercal A)\, \boldsymbol{x}_2 = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 =& \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) = (R) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 \stackrel{(e_1)}{=}& (\lambda_1 \boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(R) = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) \stackrel{(e_2)}{=}& \boldsymbol{x}_1^\intercal (\lambda_2 \boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 =& \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2
\end{aligned}
$$

distance from a point to a line

$$
\begin{gathered}
\begin{cases}
P = P\left(x_0, y_0\right) \\
L = L\left(x, y\right) = Ax + By + C = 0, A^2 + B^2 \neq 0
\end{cases} \\
\Downarrow \\
d\left(P, L\right) = \frac{\left|Ax_0 + By_0 + C\right|}{\sqrt{A^2 + B^2}}
\end{gathered}
$$

by shortest $\overline{PP'}$

$$
\begin{aligned}
P' = P'\left(x, y\right) \in L = Ax + By + C = 0 \\
\Rightarrow& y = \frac{-1}{B}\left(Ax + C\right)
\end{aligned}
$$

$$
\begin{aligned}
\overline{PP'}^2\left(x, y\right) &= \left(x_0 - x\right)^2 + \left(y_0 - y\right)^2 \\
&= \left(x_0 - x\right)^2 + \left(y_0 - \frac{-1}{B}\left(Ax + C\right)\right)^2 \\
&= \left(x - x_0\right)^2 + \left(\frac{A}{B}x + \frac{C}{B} + y_0\right)^2 = \overline{PP'}^2\left(x\right)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\partial}{\partial x}\overline{PP'}^2\left(x\right) =& 2\left(x - x_0\right) + 2\left(\frac{A}{B}x + \frac{C}{B} + y_0\right)\frac{A}{B} \\
=& \frac{2}{B^2}\left(B^2\left(x - x_0\right) + A^2 x + AC + ABy_0\right) \\
=& \frac{2}{B^2}\left[\left(A^2 + B^2\right)x - \left(B^2 x_0 - ABy_0 - AC\right)\right] \\
x =& \frac{B^2 x_0 - ABy_0 - AC}{A^2 + B^2}
\end{aligned}
$$

or by completing the square

$$
\begin{aligned}
&\overline{PP'}^2\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2}-x_0\right)^2+\left(\frac{A}{B}\frac{B^2x_0-ABy_0-AC}{A^2+B^2}+\frac{C}{B}+y_0\right)^2\\
=&\left(\frac{-A^2x_0-ABy_0-AC}{A^2+B^2}\right)^2+\left(\frac{A\left(B^2x_0-ABy_0-AC\right)+C\left(A^2+B^2\right)+B\left(A^2+B^2\right)y_0}{B\left(A^2+B^2\right)}\right)^2\\
=&\left(\frac{-A\left(Ax_0+By_0+C\right)}{A^2+B^2}\right)^2+\left(\frac{AB^2x_0+B^3y_0+B^2C}{B\left(A^2+B^2\right)}\right)^2\\
=&\frac{A^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}+\frac{B^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}\\
=&\frac{\left(Ax_0+By_0+C\right)^2}{A^2+B^2}\\
\overline{PP'}=&\overline{PP'}\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)=\frac{\left|Ax_0+By_0+C\right|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

by perpendicular foot

$$
\begin{gathered}
y=\frac{-A}{B}x-\frac{C}{B}=\frac{-1}{B}\left(Ax+C\right),\ \text{if } B\neq 0\\
L_\perp:\left(y=\frac{B}{A}x+K\right)\perp\left(y=\frac{-A}{B}x-\frac{C}{B}\right):L\\
L_\perp=L_\perp\left(x,y\right)=Bx-Ay+K=0\\
P=P\left(x_0,y_0\right)\in L_\perp=B\left(x-x_0\right)-A\left(y-y_0\right)=0\\
L_\perp=Bx-Ay-\left(Bx_0-Ay_0\right)=0
\end{gathered}
$$

perpendicular foot $=$ foot of the perpendicular P'

$$
\begin{aligned}
P'\in\left(L_\perp\cap L\right)=&\begin{cases}L=Ax+By+C=0\\ L_\perp=Bx-Ay-\left(Bx_0-Ay_0\right)=0\end{cases}\\
=&\begin{cases}Ax+By=-C\\ Bx-Ay=Bx_0-Ay_0\end{cases}\\
P'=P'\left(x,y\right)=&\left(\frac{\begin{vmatrix}-C & B\\ Bx_0-Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}},\frac{\begin{vmatrix}A & -C\\ B & Bx_0-Ay_0\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}}\right)\\
=&\left(\frac{\begin{vmatrix}C & B\\ -Bx_0+Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}},\frac{\begin{vmatrix}A & C\\ B & -Bx_0+Ay_0\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)
\end{aligned}
$$

$$\begin{aligned}
&d(P,L)=\overline{PP'}\\
=&\left\|(x_0,y_0)-\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)\right\|\\
=&\sqrt{\left(x_0-\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)^2+\left(y_0-\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)^2}\\
=&\sqrt{\left(\frac{A^2x_0+ABy_0+AC}{A^2+B^2}\right)^2+\left(\frac{ABx_0+B^2y_0+BC}{A^2+B^2}\right)^2}\\
=&\sqrt{\frac{A^2\left(Ax_0+By_0+C\right)^2+B^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}}=\sqrt{\frac{\left(Ax_0+By_0+C\right)^2}{A^2+B^2}}\\
=&\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

by normal vector

$$\begin{cases}\vec{n}=(A,B)\perp L=Ax+By+C=0\\ \vec{PP'}=P'-P=(x-x_0,y-y_0)\end{cases}$$

$$\begin{aligned}
&\left|\vec{PP'}\cdot\vec{n}\right|\\
&=\left\|\vec{PP'}\right\|\left\|\vec{n}\right\||\cos\theta|\\
\left\|\vec{PP'}\right\||\cos\theta|&=\left|\vec{PP'}\cdot\widehat{\boldsymbol{n}}\right|\\
&=\frac{\left|\vec{PP'}\cdot\vec{n}\right|}{\left\|\vec{n}\right\|}\\
&=\frac{|(x-x_0,y-y_0)\cdot(A,B)|}{\|(A,B)\|}=\frac{|A(x-x_0)+B(y-y_0)|}{\sqrt{A^2+B^2}}\\
&=\frac{|-Ax_0-By_0+Ax+By|}{\sqrt{A^2+B^2}}\overset{Ax+By+C=0}{\underset{Ax+By=-C}{=}}\frac{|-Ax_0-By_0-C|}{\sqrt{A^2+B^2}}\\
&=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

$$\begin{cases}\boldsymbol{n}=(A,B)\perp L=Ax+By+C=0\\ \overrightarrow{PP'}=P'-P=(x-x_0,y-y_0)\end{cases}$$

$$\begin{aligned}
&\left|\overrightarrow{PP'}\cdot\boldsymbol{n}\right|\\
&=\left\|\overrightarrow{PP'}\right\|\|\boldsymbol{n}\||\cos\theta|\\
\left\|\overrightarrow{PP'}\right\||\cos\theta|&=\left|\overrightarrow{PP'}\cdot\widehat{\boldsymbol{n}}\right|\\
&=\frac{\left|\overrightarrow{PP'}\cdot\boldsymbol{n}\right|}{\|\boldsymbol{n}\|}\\
&=\frac{|(x-x_0,y-y_0)\cdot(A,B)|}{\|(A,B)\|}=\frac{|A(x-x_0)+B(y-y_0)|}{\sqrt{A^2+B^2}}\\
&=\frac{|-Ax_0-By_0+Ax+By|}{\sqrt{A^2+B^2}}\overset{Ax+By+C=0}{\underset{Ax+By=-C}{=}}\frac{|-Ax_0-By_0-C|}{\sqrt{A^2+B^2}}\\
&=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

by Cauchy inequality

$$
\begin{aligned}
Ax+By+C&=0\\
Ax+By&=-C\\
(Ax+By)-(Ax_0+By_0)&=-C-(Ax_0+By_0)\\
A(x-x_0)+B(y-y_0)&=-(Ax_0+By_0+C)
\end{aligned}
$$

$$
\begin{aligned}
\overline{PP'}^2&=(x_0-x)^2+(y_0-y)^2\\
\left[A^2+B^2\right]\overline{PP'}^2&=\left[A^2+B^2\right]\left[(x_0-x)^2+(y_0-y)^2\right]\\
&\ge\left[A(x-x_0)+B(y-y_0)\right]^2\\
&=\left[-(Ax_0+By_0+C)\right]^2=(Ax_0+By_0+C)^2\\
\overline{PP'}^2&\ge\frac{(Ax_0+By_0+C)^2}{A^2+B^2}\\
\overline{PP'}&\ge\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

standard form

$$
\left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2=1
$$

$$
\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2=1
$$

$$
(y-k)=4c(x-h)^2
$$

$$
\left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2=1
$$

$$
\begin{aligned}
\left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2&=1\\
\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2&=1\\
(y-k)-4c(x-h)^2&=0\\
\left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2&=1
\end{aligned}
$$

$$
\begin{array}{cccccc}
\text{circle} & \left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & & & b=a\\
\text{ellipse} & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{vertical} & & b>a\\
 & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{horizontal} & & a>b\\
\text{parabola} & (y-k)-4c(x-h)^2 & =0 & \text{vertical} & & \\
 & -4c(y-k)^2+(x-h) & =0 & \text{horizontal} & & \\
\text{hyperbola} & \left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2 & =1 & \text{vertical} & \frac{x-h}{a}=0\Rightarrow\frac{y-k}{b}=\pm1 & \\
 & -\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{horizontal} & \frac{y-k}{b}=0\Rightarrow\frac{x-h}{a}=\pm1 &
\end{array}
$$

parametric equation

$$
\begin{array}{llll}
\text{circle} & \left(\dfrac{y-k}{a}\right)^2+\left(\dfrac{x-h}{a}\right)^2 &=1 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}a&0&h\\0&a&k\\0&0&1\end{pmatrix}\begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix}=\begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix}\begin{pmatrix}a\\a\\1\end{pmatrix}\\
\text{ellipse} & \left(\dfrac{y-k}{b}\right)^2+\left(\dfrac{x-h}{a}\right)^2 &=1 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix}\begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix}=\begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\1\end{pmatrix}\\
\text{parabola} & (y-k)-4c\,(x-h)^2 &=0 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}1&0&h\\0&4c&k\\0&0&1\end{pmatrix}\begin{pmatrix}t\\t^2\\1\end{pmatrix}=\begin{pmatrix}t&0&h\\0&t^2&k\\0&0&1\end{pmatrix}\begin{pmatrix}1\\4c\\1\end{pmatrix}\\
 & -4c\,(y-k)^2+(x-h) &=0 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}4c&0&h\\0&1&k\\0&0&1\end{pmatrix}\begin{pmatrix}t^2\\t\\1\end{pmatrix}=\begin{pmatrix}t^2&0&h\\0&t&k\\0&0&1\end{pmatrix}\begin{pmatrix}4c\\1\\1\end{pmatrix}\\
\text{hyperbola} & \left(\dfrac{y-k}{b}\right)^2-\left(\dfrac{x-h}{a}\right)^2 &=1 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix}\begin{pmatrix}\pm\cosh t\\\sinh t\\1\end{pmatrix}=\begin{pmatrix}\tan t&0&h\\0&\sec t&k\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\1\end{pmatrix}\\
 & -\left(\dfrac{y-k}{b}\right)^2+\left(\dfrac{x-h}{a}\right)^2 &=1 & \begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix}\begin{pmatrix}\sinh t\\\pm\cosh t\\1\end{pmatrix}=\begin{pmatrix}\sec t&0&h\\0&\tan t&k\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\1\end{pmatrix}
\end{array}
$$

eccentricity

$$
\begin{cases}
F=(0,y_F)\\
L=y-y_L=0\\
\epsilon=\dfrac{\overline{PF}}{d\,(P,L)}=\dfrac{\|(x,y)-(0,y_F)\|}{\|y-y_L\|}
\end{cases}
\qquad
\begin{array}{l}
F:\text{focus}\\
L:\text{directrix}\\
\begin{cases}P=(x,y)\\ \epsilon:\text{eccentricity}\end{cases}
\end{array}
$$

$$
0\le\epsilon=\frac{\overline{PF}}{d\,(P,L)}=\frac{\overline{PF}}{\overline{PP'}}=\frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|}=\frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|}=\frac{\sqrt{x^2+(y-y_F)^2}}{\sqrt{(y-y_L)^2}}
$$

$$
\begin{aligned}
\epsilon^2 &=\frac{x^2+(y-y_F)^2}{(y-y_L)^2}=\frac{x^2+y^2-2y_Fy+y_F^2}{y^2-2y_Ly+y_L^2}\\
0 &=x^2+\left(1-\epsilon^2\right)y^2-2\left(y_F-\epsilon^2y_L\right)y+\left(y_F^2-\epsilon^2y_L^2\right)\\
&\overset{\epsilon\neq1}{=}x^2+\left(1-\epsilon^2\right)\left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right]\\
&=x^2+\left(1-\epsilon^2\right)\\
&\quad\left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2-\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right]\\
&=x^2+\left(1-\epsilon^2\right)\left[\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2}{\left(1-\epsilon^2\right)^2}\right]\\
&=x^2+\left(1-\epsilon^2\right)\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2}{1-\epsilon^2}
\end{aligned}
$$

$$
\begin{aligned}
&\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2\\
&=\left(1-\epsilon^2\right)y_F^2-\left(\epsilon^2-\epsilon^4\right)y_L^2-y_F^2+2\epsilon^2y_Fy_L-\epsilon^4y_L^2\\
&=-\epsilon^2y_F^2-\epsilon^2y_L^2+2\epsilon^2y_Fy_L=-\epsilon^2\left(y_F-y_L\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2\\
&=\left(1-\epsilon^2\right)y_F^2-\left(\epsilon^2-\epsilon^4\right)y_L^2-y_F^2+2\epsilon^2y_Fy_L-\epsilon^4y_L^2\\
&=-\epsilon^2y_F^2-\epsilon^2y_L^2+2\epsilon^2y_Fy_L=-\epsilon^2\left(y_F-y_L\right)^2
\end{aligned}
$$

$$\frac{\epsilon^2 \left(y_F - y_L\right)^2}{1-\epsilon^2} \stackrel{\epsilon\neq 1}{=} x^2 + \left(1-\epsilon^2\right)\left(y - \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2}\right)^2$$

$$1 \stackrel{\epsilon\neq 0,1}{=} \begin{cases} \left(\dfrac{x-0}{\dfrac{\epsilon\left(y_F - y_L\right)}{\sqrt{1-\epsilon^2}}}\right)^2 + \left(\dfrac{y - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2 > 0 \stackrel{\epsilon\geq 0}{\Rightarrow} 0<\epsilon<1 \\ -\left(\dfrac{x-0}{\dfrac{\epsilon\left(y_F - y_L\right)}{\sqrt{\epsilon^2-1}}}\right)^2 + \left(\dfrac{y - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2 < 0 \stackrel{\epsilon\geq 0}{\Rightarrow} \epsilon>1 \end{cases}$$

$$y = \begin{cases} \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2} \pm \dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}\sqrt{1 - \left(\dfrac{x}{\dfrac{\epsilon\left(y_F - y_L\right)}{\sqrt{1-\epsilon^2}}}\right)^2} & 0<\epsilon<1 \\ \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2} \pm \dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}\sqrt{1 + \left(\dfrac{x}{\dfrac{\epsilon\left(y_F - y_L\right)}{\sqrt{\epsilon^2-1}}}\right)^2} & \epsilon>1 \end{cases}$$

$x = 0$

$$\begin{aligned} y - \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} &= \pm \frac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2} \\ y &= \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} \pm \frac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2} \\ &= \frac{\left(1\pm\epsilon\right) y_F - \epsilon\left(\epsilon\pm 1\right) y_L}{1-\epsilon^2} \\ &= \begin{cases} \dfrac{\left(1+\epsilon\right) y_F - \epsilon\left(\epsilon+1\right) y_L}{1-\epsilon^2} & y_+ \\ \dfrac{\left(1-\epsilon\right) y_F - \epsilon\left(\epsilon-1\right) y_L}{1-\epsilon^2} & y_- \end{cases} \end{aligned}$$

$$\frac{y_+ + y_-}{2} - y_F = \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} - y_F = \frac{\epsilon^2\left(y_F - y_L\right)}{1-\epsilon^2}$$

$$\begin{aligned} \frac{y_F + y_{F'}}{2} &= \frac{y_+ + y_-}{2} \\ y_{F'} &= y_+ + y_- - y_F = 2\cdot\frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} - y_F \\ &= \frac{y_F - \epsilon^2\left(2y_L - y_F\right)}{1-\epsilon^2} = \frac{\left(1+\epsilon^2\right) y_F - 2\epsilon^2 y_L}{1-\epsilon^2} \end{aligned}$$

$\epsilon = 0$ or $\lim\limits_{|y_L|\to\infty} \epsilon = 0$

$$r = \overline{PF} = \left\|(x,y) - (0,y_F)\right\| = \left\|(x, y-y_F)\right\| = \sqrt{x^2 + \left(y-y_F\right)^2}$$

$$\epsilon = \frac{r}{d\left(P,L\right)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\left\|(x,y)-(0,y_F)\right\|}{\left\|(x,y)-(x,y_L)\right\|} = \frac{\left\|(x,y-y_F)\right\|}{\left\|(0,y-y_L)\right\|} = \frac{\sqrt{x^2+\left(y-y_F\right)^2}}{\left|y-y_L\right|}$$

$$\lim_{|y_L|\to\infty}\epsilon = \lim_{|y_L|\to\infty}\frac{r}{\overline{PL}} = \lim_{|y_L|\to\infty}\frac{\sqrt{x^2+\left(y-y_F\right)^2}}{\left|y-y_L\right|} = 0$$

$\epsilon = 1$

$$\begin{aligned}
0 =& x^2 + \left(1-\epsilon^2\right) y^2 - 2\left(y_F - \epsilon^2 y_L\right) y + \left(y_F^2 - \epsilon^2 y_L^2\right) \\
\overset{\epsilon=1}{=}& x^2 + \left(1-1^2\right) y^2 - 2\left(y_F - 1^2 y_L\right) y + \left(y_F^2 - 1^2 y_L^2\right) \\
=& x^2 - 2\left(y_F - y_L\right) y + \left(y_F^2 - y_L^2\right) \\
=& x^2 - 2\left(y_F - y_L\right) y + \left(y_F + y_L\right)\left(y_F - y_L\right) \\
x^2 =& 2\left(y_F - y_L\right)\left(y - \frac{y_F + y_L}{2}\right)
\end{aligned}$$

$\epsilon \neq 1$

$$\begin{aligned}
1 &\overset{P(x,y)=V(0,0)}{=} 0 + \left(\frac{0 - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}}\right)^2 \\
&\Rightarrow y_F - \epsilon^2 y_L = \pm\epsilon\left(y_F - y_L\right) \\
&\Rightarrow \begin{cases} \left(1-\epsilon\right) y_F = \epsilon\left(\epsilon - 1\right) y_L & + \\ \left(1+\epsilon\right) y_F = \epsilon\left(\epsilon + 1\right) y_L & - \end{cases} \\
&\Rightarrow y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases}
\end{aligned}$$

$\epsilon = 1$

$$\begin{aligned}
x^2 &= 2\left(y_F - y_L\right)\left(y - \frac{y_F + y_L}{2}\right) \\
\overset{P(x,y)=V(0,0)}{\Rightarrow} 0^2 &= 2\left(y_F - y_L\right)\left(0 - \frac{y_F + y_L}{2}\right) \\
\Rightarrow 0 &= \left(y_F - y_L\right)\left(y_F + y_L\right) \\
\Rightarrow y_F &= \mp y_L
\end{aligned}$$

or by definition of eccentricity

$$\begin{aligned}
0 \leq \epsilon = \frac{\overline{PF}}{d\left(P, L\right)} = \frac{\overline{PF}}{\overline{PP'}} &= \frac{\left\|(x,y) - (0,y_F)\right\|}{\left\|(x,y) - (x,y_L)\right\|} = \frac{\left\|(x, y - y_F)\right\|}{\left\|(0, y - y_L)\right\|} = \frac{\sqrt{x^2 + \left(y - y_F\right)^2}}{\sqrt{\left(y - y_L\right)^2}} \\
&\overset{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2 + \left(0 - y_F\right)^2}}{\sqrt{\left(0 - y_L\right)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2} \\
\epsilon^2 &= \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp\epsilon y_L
\end{aligned}$$

actually,

$$y_F = -\epsilon y_L$$

$0 < \epsilon < 1$
$c^2 = a^2 - b^2$ or $c^2 = b^2 - a^2$

$$\begin{aligned}
&\left(\frac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}\right)^2 - \left(\frac{\epsilon\left(y_F - y_L\right)}{\sqrt{1-\epsilon^2}}\right)^2 = \frac{\epsilon^2\left(y_F - y_L\right)^2}{\left(1-\epsilon^2\right)^2} - \frac{\epsilon^2\left(y_F - y_L\right)^2}{1-\epsilon^2} \\
=& \frac{\epsilon^2\left(y_F - y_L\right)^2}{1-\epsilon^2}\left(\frac{1}{1-\epsilon^2} - 1\right) = \frac{\epsilon^4\left(y_F - y_L\right)^2}{\left(1-\epsilon^2\right)^2} = \left(\frac{\epsilon^2\left(y_F - y_L\right)}{1-\epsilon^2}\right)^2 = \left(\frac{y_+ + y_-}{2} - y_F\right)^2
\end{aligned}$$

$\epsilon > 1$

$c^2 = a^2 + b^2$

$$\begin{aligned}\left(\frac{\epsilon\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2 + \left(\frac{\epsilon\left(y_F - y_L\right)}{\sqrt{\epsilon^2 - 1}}\right)^2 &= \frac{\epsilon^2\left(y_F - y_L\right)^2}{\left(\epsilon^2 - 1\right)^2} + \frac{\epsilon^2\left(y_F - y_L\right)^2}{\epsilon^2 - 1}\\ =\frac{\epsilon^2\left(y_F - y_L\right)^2}{\epsilon^2 - 1}\left(\frac{1}{\epsilon^2 - 1} + 1\right) = \frac{\epsilon^4\left(y_F - y_L\right)^2}{\left(\epsilon^2 - 1\right)^2} &= \left(\frac{\epsilon^2\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2 = \left(\frac{y_+ + y_-}{2} - y_F\right)^2\end{aligned}$$

$$\left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2} = \frac{\left(\dfrac{\epsilon^2\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2}{\left(\dfrac{\epsilon\left(y_F - y_L\right)}{1 - \epsilon^2}\right)^2} = \epsilon^2$$

two definition equivalence of ellipse and hyperbola

$$\begin{cases} P = (x, y) \\ F = \left(x_F, y_F\right) = (\alpha, \varphi) \qquad F' = \left(x_{F'}, y_{F'}\right) = (\chi, \psi) \\ L = A'x + B'y + C' = 0 \end{cases}$$

$$0 \leq \epsilon = \frac{\overline{PF}}{d\left(P, L\right)} = \frac{\sqrt{\left(x - x_F\right)^2 + \left(y - y_F\right)^2}}{\dfrac{\left|A'x + B'y + C'\right|}{\sqrt{A'^2 + B'^2}}} = \frac{\sqrt{\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2}}{\left|Ax + By + C\right|}, \begin{cases} A = \dfrac{A'}{\sqrt{A'^2 + B'^2}} \\ B = \dfrac{B'}{\sqrt{A'^2 + B'^2}} \\ C = \dfrac{C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

$$\epsilon^2 = \frac{\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2}{\left(Ax + By + C\right)^2} = \frac{\left(x - x_F\right)^2 + \left(y - y_F\right)^2}{\dfrac{\left(A'x + B'y + C'\right)^2}{A'^2 + B'^2}}$$

$$\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2 = \left[\epsilon\left(Ax + By + C\right)\right]^2$$

$$\begin{aligned}2c = \overline{FF'} &= \left\|\left(x_F, y_F\right) - \left(x_{F'}, y_{F'}\right)\right\| = \left\|(\alpha, \varphi) - (\chi, \psi)\right\| \\ &= \sqrt{\left(\alpha - \chi\right)^2 + \left(\chi - \psi\right)^2}\end{aligned}$$

$$\begin{aligned}D &= \begin{cases} \sqrt{\left(x - x_F\right)^2 + \left(y - y_F\right)^2} + \sqrt{\left(x - x_{F'}\right)^2 + \left(y - y_{F'}\right)^2} & \text{ellipse} \\ \sqrt{\left(x - x_F\right)^2 + \left(y - y_F\right)^2} - \sqrt{\left(x - x_{F'}\right)^2 + \left(y - y_{F'}\right)^2} & \text{hyperbola} \end{cases} \\ &= \sqrt{\left(x - x_F\right)^2 + \left(y - y_F\right)^2} \pm \sqrt{\left(x - x_{F'}\right)^2 + \left(y - y_{F'}\right)^2} \\ &= \sqrt{\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2} \pm \sqrt{\left(x - \chi\right)^2 + \left(y - \psi\right)^2}\end{aligned}$$

$$\begin{aligned}\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2 &= \left(D \mp \sqrt{\left(x - \chi\right)^2 + \left(y - \psi\right)^2}\right)^2 \\ &= D^2 \mp 2D\sqrt{\left(x - \chi\right)^2 + \left(y - \psi\right)^2} \\ &\quad + \left(x - \chi\right)^2 + \left(y - \psi\right)^2\end{aligned}$$

$$
\begin{aligned}
D^2 =& (x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2 \\
&\pm 2\sqrt{\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]} \\
&(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2-D^2 \\
&= \mp 2\sqrt{\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]} \\
&\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right]^2+D^4 \\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right] \\
&=4\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right] \\
&\left[(x-\alpha)^2+(y-\varphi)^2\right]^2+\left[(x-\chi)^2+(y-\psi)^2\right]^2 \\
&+2\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]+D^4 \\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right] \\
&=4\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right] \\
0 =& \left[(x-\alpha)^2+(y-\varphi)^2\right]^2+\left[(x-\chi)^2+(y-\psi)^2\right]^2 \\
&-2\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]+D^4 \\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right] \\
0 =& \left\{\left[(x-\alpha)^2+(y-\varphi)^2\right]-\left[(x-\chi)^2+(y-\psi)^2\right]\right\}^2+D^4 \\
&-2D^2\left\{\left[(x-\alpha)^2+(y-\varphi)^2\right]+\left[(x-\chi)^2+(y-\psi)^2\right]\right\} \\
0 =& \left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]\right\}^2+D^4 \\
&-2D^2\left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]\right\} \\
&-4D^2\left[(x-\alpha)^2+(y-\varphi)^2\right] \\
&(2D)^2\left[(x-\alpha)^2+(y-\varphi)^2\right] \\
&=\left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]-D^2\right\}^2 \\
&=\left\{\left[(x-\chi)^2-(x-\alpha)^2\right]+\left[(y-\psi)^2-(y-\varphi)^2\right]-D^2\right\}^2 \\
&=\left\{(2x-\chi-\alpha)(\alpha-\chi)+(2y-\psi-\varphi)(\varphi-\psi)-D^2\right\}^2 \\
&=\left\{2(\alpha-\chi)x-\left(\alpha^2-\chi^2\right)+2(\varphi-\psi)y-\left(\varphi^2-\psi^2\right)-D^2\right\}^2 \\
&=\left\{2(\alpha-\chi)x+2(\varphi-\psi)y-\left[\left(\alpha^2-\chi^2\right)+\left(\varphi^2-\psi^2\right)+D^2\right]\right\}^2 \\
D \neq& 0 \\
&(x-\alpha)^2+(y-\varphi)^2 \\
&=\left[\frac{\alpha-\chi}{D}x+\frac{\varphi-\psi}{D}y-\left(\frac{\alpha^2-\chi^2}{2D}+\frac{\varphi^2-\psi^2}{2D}+\frac{D}{2}\right)\right]^2
\end{aligned}
$$

$$
\begin{cases}
(x-\alpha)^2+(y-\varphi)^2=\left[\epsilon\left(Ax+By+C\right)\right]^2 \\
(x-\alpha)^2+(y-\varphi)^2=\left[\frac{\alpha-\chi}{D}x+\frac{\varphi-\psi}{D}y-\left(\frac{\alpha^2-\chi^2}{2D}+\frac{\varphi^2-\psi^2}{2D}+\frac{D}{2}\right)\right]^2
\end{cases}
$$

$$(A,B,C) \rightleftarrows (\chi,\psi,D)$$

$$\begin{cases} \epsilon A = \pm \dfrac{\alpha - \chi}{D} & \chi \pm \epsilon A D = \alpha \\ \epsilon B = \pm \dfrac{\varphi - \psi}{D} & \psi \pm \epsilon B D = \varphi \\ \epsilon C = \mp \left(\dfrac{\alpha^2 - \chi^2}{2D} + \dfrac{\varphi^2 - \psi^2}{2D} + \dfrac{D}{2} \right) \end{cases}$$

$$\begin{aligned} 2\epsilon C &= \mp \left(\frac{\alpha - \chi}{D} (\alpha + \chi) + \frac{\varphi - \psi}{D} (\varphi + \psi) + D \right) \\ &= \mp \left(\pm \epsilon A (\alpha + \chi) \pm \epsilon B (\varphi + \psi) + D \right) \\ \mp \epsilon (A\alpha + B\varphi + 2C) &= \pm \epsilon A \chi \pm \epsilon B \psi + D \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A \\ 0 & 1 & \pm\epsilon B \\ \pm\epsilon A & \pm\epsilon B & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ D \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ \mp\epsilon (A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & \pm\epsilon B & 1 \mp \epsilon^2 A^2 & \mp\epsilon (2A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 \mp \epsilon^2 A^2 \mp \epsilon^2 B^2 & \mp\epsilon (2A\alpha + 2B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 & \dfrac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} \end{pmatrix}$$

$$A^2 + B^2 = \left(\frac{A'}{\sqrt{A'^2 + B'^2}} \right)^2 + \left(\frac{B'}{\sqrt{A'^2 + B'^2}} \right)^2 = 1$$

$$\begin{cases} \chi = \alpha \mp \epsilon A D = \alpha \mp \epsilon \dfrac{A'}{\sqrt{A'^2 + B'^2}} D \\ \psi = \varphi \mp \epsilon B D = \varphi \mp \epsilon \dfrac{B'}{\sqrt{A'^2 + B'^2}} D \\ D = \dfrac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} = \dfrac{\mp 2\epsilon}{1 \mp \epsilon^2} \dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \quad A^2 + B^2 = 1 \end{cases}$$

actually only one solution is true

$$\begin{cases} \chi = \alpha - \epsilon A D = \alpha - \epsilon \dfrac{A'}{\sqrt{A'^2 + B'^2}} D = \alpha - \dfrac{2\epsilon^2}{\epsilon^2 - 1} \dfrac{A'^2\alpha + A'B'\varphi + A'C'}{A'^2 + B'^2} \\ \psi = \varphi - \epsilon B D = \varphi - \epsilon \dfrac{B'}{\sqrt{A'^2 + B'^2}} D = \varphi - \dfrac{2\epsilon^2}{\epsilon^2 - 1} \dfrac{A'B'\alpha + B'^2\varphi + B'C'}{A'^2 + B'^2} \\ D = \dfrac{-2\epsilon (A\alpha + B\varphi + C)}{1 - \epsilon^2 (A^2 + B^2)} = \dfrac{-2\epsilon}{1 - \epsilon^2} \dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} = \dfrac{2\epsilon}{\epsilon^2 - 1} \dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

$$\begin{cases} \chi = \dfrac{(\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2 (A'^2\alpha + A'B'\varphi + A'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\ \psi = \dfrac{(\epsilon^2 - 1)(A'^2 + B'^2)\varphi - 2\epsilon^2 (A'B'\alpha + B'^2\varphi + B'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\ \left| \dfrac{D}{d(F, L)} \right| = \left| \dfrac{2\epsilon}{1 - \epsilon^2} \right| \Rightarrow \left(\dfrac{D}{d(F, L)} \right)^2 = \left(\dfrac{2\epsilon}{1 - \epsilon^2} \right)^2 \end{cases}$$

$$\begin{aligned} &(\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2 (A'^2\alpha + A'B'\varphi + A'C') \\ &= \left(-(\epsilon^2 + 1) A'^2 + (\epsilon^2 - 1) B'^2 \right) \alpha - 2\epsilon^2 (A'B'\varphi + A'C') \\ &= \left(-(\epsilon^2 + 1) A'^2 + (\epsilon^2 - 1) B'^2 \right) \alpha - 2\epsilon^2 (A'B'\varphi + A'C') \end{aligned}$$

$$\begin{aligned}\overline{FF'}^2 &= (\alpha-\chi)^2+(\varphi-\psi)^2\\ &= (\alpha-(\alpha-\epsilon AD))^2+(\varphi-(\varphi-\epsilon BD))^2\\ &= (\epsilon D)^2\left(A^2+B^2\right)\\ &= (\epsilon D)^2\end{aligned}$$

$$\left(\frac{c}{a}\right)^2=\left(\frac{\overline{PF}}{d\left(P,L\right)}\right)^2=\epsilon^2=\left(\frac{\overline{FF'}}{D}\right)^2=\left(\frac{2c}{D}\right)^2\Rightarrow D=2a$$

$$\left(\frac{D}{d\left(F,L\right)}\right)^2=\left(\frac{2\epsilon}{1-\epsilon^2}\right)^2$$

polar coordinate

$$(x-\alpha)^2+(y-\varphi)^2=\left[\epsilon\left(Ax+By+C\right)\right]^2$$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

$$(r\cos\theta-\alpha)^2+(r\sin\theta-\varphi)^2=\left[\epsilon\left(Ar\cos\theta+Br\sin\theta+C\right)\right]^2$$

If $\begin{cases}F=(x_F,y_F)=(\alpha,\varphi)=(0,0)\\ L=Ax+By+C=x+p=0\end{cases}$,

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2 &= \left[\epsilon\left(r\cos\theta+p\right)\right]^2\\ r^2 &= \\ r &= \pm\,\epsilon\left(r\cos\theta+p\right)\\ &= \pm\left(r\epsilon\cos\theta+\epsilon p\right)\\ r\left(1\mp\epsilon\cos\theta\right) &= \epsilon p\\ r &= \frac{\epsilon p}{1\mp\epsilon\cos\theta}\end{aligned}$$

$r=\dfrac{\epsilon p}{1-\epsilon\cos\theta}$ will not cross $L=x+p=0$ on graphs, so maybe it is a more correct solution

$$r=\frac{\epsilon p}{1-\epsilon\cos\theta}$$

$$\epsilon=\pm\frac{r}{r\cos\theta+p}$$

partition

$$\{A_i\}_{i\in I}=\{A_i|i\in I\}\text{ is a partition of a set }A$$

$$\Leftrightarrow\begin{cases}\forall i\in I\left(A_i\neq\emptyset\right)\\ A=\bigcup\limits_{i\in I}A_i\\ \forall i,j\in I\left(i\neq j\Rightarrow A_i\cap A_j=\emptyset\right)\end{cases}$$