real symmetric matrix diagonalizable

$$\begin{cases} \begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix} \\ A^\intercal = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\ A\boldsymbol{x} = \lambda \boldsymbol{x} & \begin{cases} \lambda \in \mathbb{C} & \text{complex eigenvalue} \\ \mathbf{0} \neq \boldsymbol{x} \in \mathbb{C}^n & \text{complex eigenvector} \end{cases} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \lambda \in \mathbb{R} & \text{real eigenvalue (1)} \\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector (2)} \end{cases}$$

$$\begin{aligned} A\boldsymbol{x} =& \lambda \boldsymbol{x} \\ \overline{A}\overline{\boldsymbol{x}} = \overline{A\boldsymbol{x}} =& \overline{\lambda \boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}} \\ \overline{\boldsymbol{x}}^\intercal \overline{A}^\intercal = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal = \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^\intercal =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \\ \overline{\boldsymbol{x}}^\intercal A \overset{\text{symmetric}}{=} \overline{\boldsymbol{x}}^\intercal A^\intercal \overset{\text{real}}{=}& \\ \overline{\boldsymbol{x}}^\intercal A =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \\ \lambda \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal (\lambda \boldsymbol{x}) \underset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{\overset{\cdot \boldsymbol{x}}{=}} \overline{\boldsymbol{x}}^\intercal A \boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \\ \lambda \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \\ \left(\lambda - \overline{\lambda}\right) \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} =& 0 \wedge \begin{cases} \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} = \sum_{i=1}^{n} |x_i|^2 \\ \boldsymbol{x} \neq \mathbf{0} \end{cases} \Rightarrow \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \neq 0 \\ \lambda - \overline{\lambda} =& 0 \\ \lambda =& \overline{\lambda} \Leftrightarrow \lambda \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal \boldsymbol{x} = \left(\overline{\boldsymbol{x}}^\intercal \overline{A}^\intercal\right) \boldsymbol{x} \overset{\text{symmetric}}{=} \left(\overline{\boldsymbol{x}}^\intercal \overline{A}\right) \boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal \left(\overline{A}\boldsymbol{x}\right) \\ (L) = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal \boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal \left(\overline{A}\boldsymbol{x}\right) = (R) \\ (L) = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal \boldsymbol{x} \overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=} \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^\intercal \boldsymbol{x} = \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \\ (R) = \overline{\boldsymbol{x}}^\intercal \left(\overline{A}\boldsymbol{x}\right) \overset{\text{real}}{=} \overline{\boldsymbol{x}}^\intercal (A\boldsymbol{x}) \overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=} \overline{\boldsymbol{x}}^\intercal (\lambda \boldsymbol{x}) = \lambda \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \\ \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \boldsymbol{x} = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal \boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal \left(\overline{A}\boldsymbol{x}\right) = \lambda \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \\ \overline{\lambda}\overline{\boldsymbol{x}}^\intercal \boldsymbol{x} = \lambda \overline{\boldsymbol{x}}^\intercal \boldsymbol{x} \end{aligned}$$

$$\begin{cases} \begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix} \\ A^\intercal = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\ A\boldsymbol{x} = \lambda \boldsymbol{x} & \begin{cases} \lambda \in \mathbb{R} & \text{real eigenvalue} \\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector} \end{cases} \\ \begin{cases} A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1 & (e_1) \\ A\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2 & (e_2) \end{cases} & \lambda_1 \neq \lambda_2 \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = 0 \Leftrightarrow \boldsymbol{x}_1 \perp \boldsymbol{x}_2$$

$$
\begin{aligned}
A\boldsymbol{x}_2 =& \lambda_2 \boldsymbol{x}_2 \\
\boldsymbol{x}_1^\intercal A\boldsymbol{x}_2 \overset{\boldsymbol{x}_1^\intercal \cdot}{=}& \boldsymbol{x}_1^\intercal \lambda_2 \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (1) \\
A\boldsymbol{x}_1 =& \lambda_1 \boldsymbol{x}_1 \\
\boldsymbol{x}_1^\intercal A^\intercal = (A\boldsymbol{x}_1)^\intercal =& (\lambda_1 \boldsymbol{x}_1)^\intercal = \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A^\intercal =& \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \overset{\text{symmetric}}{=} \boldsymbol{x}_1^\intercal A^\intercal \boldsymbol{x}_2 \overset{\cdot \boldsymbol{x}_2}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (2) \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \overset{(1)}{=} \boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \overset{(2)}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(\lambda_2 - \lambda_1)\, \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0 \wedge \lambda_1 \neq \lambda_2 \\
\boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0
\end{aligned}
$$

$$
\begin{aligned}
(A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = (\boldsymbol{x}_1^\intercal A^\intercal)\, \boldsymbol{x}_2 \overset{\text{symmetric}}{=} (\boldsymbol{x}_1^\intercal A)\, \boldsymbol{x}_2 = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) = (R) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 \overset{(e_1)}{=} (\lambda_1 \boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(R) = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) \overset{(e_2)}{=} \boldsymbol{x}_1^\intercal (\lambda_2 \boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = \boldsymbol{x}_1^\intercal (A\boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2
\end{aligned}
$$

$$
\begin{cases}
\begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix} \\ A^\intercal = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\
A\boldsymbol{x}_1 = \lambda \boldsymbol{x}_1 & (e) \\
\boldsymbol{x}_2^\intercal \boldsymbol{x}_1 = 0 \Leftrightarrow \boldsymbol{x}_2 \perp \boldsymbol{x}_1 & (o)
\end{cases}
$$

$$\Downarrow$$

$$A\boldsymbol{x}_2 \perp \boldsymbol{x}_1 \Leftrightarrow (A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 = 0$$

$$
\begin{aligned}
(A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 =& (\boldsymbol{x}_2^\intercal A^\intercal)\, \boldsymbol{x}_1 \overset{\text{symmetric}}{=} (\boldsymbol{x}_2^\intercal A)\, \boldsymbol{x}_1 \\
=& \boldsymbol{x}_2^\intercal (A\boldsymbol{x}_1) \overset{(e)}{=} \boldsymbol{x}_2^\intercal (\lambda \boldsymbol{x}_1) \\
=& \lambda \boldsymbol{x}_2^\intercal \boldsymbol{x}_1 \overset{(o)}{=} \lambda \cdot 0 = 0 \\
(A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 =& 0 \Leftrightarrow A\boldsymbol{x}_2 \perp \boldsymbol{x}_1
\end{aligned}
$$

distance from a point to a line

$$
\begin{cases}
P = P(x_0, y_0) \\
L = L(x, y) = Ax + By + C = 0, A^2 + B^2 \neq 0
\end{cases}
$$

$$\Downarrow$$

$$d(P, L) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

y

by shortest $\overline{PP'}$

$$
\begin{aligned}
P' = P'(x, y) \in L = Ax + By + C = 0 \\
\Rightarrow y = \frac{-1}{B}(Ax + C)
\end{aligned}
$$

$$
\begin{aligned}
\overline{PP'}^2 (x, y) =& (x_0 - x)^2 + (y_0 - y)^2 \\
=& (x_0 - x)^2 + \left(y_0 - \frac{-1}{B}(Ax + C)\right)^2 \\
=& (x - x_0)^2 + \left(\frac{A}{B}x + \frac{C}{B} + y_0\right)^2 = \overline{PP'}^2 (x)
\end{aligned}
$$

$$
\begin{aligned}
0=\frac{\partial}{\partial x}\overline{PP'}^2(x)=&2(x-x_0)+2\left(\frac{A}{B}x+\frac{C}{B}+y_0\right)\frac{A}{B}\\
=&\frac{2}{B^2}\left(B^2(x-x_0)+A^2x+AC+ABy_0\right)\\
=&\frac{2}{B^2}\left[\left(A^2+B^2\right)x-\left(B^2x_0-ABy_0-AC\right)\right]\\
x=&\frac{B^2x_0-ABy_0-AC}{A^2+B^2}
\end{aligned}
$$

or by completing the square

$$
\begin{aligned}
&\overline{PP'}^2\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2}-x_0\right)^2+\left(\frac{A}{B}\frac{B^2x_0-ABy_0-AC}{A^2+B^2}+\frac{C}{B}+y_0\right)^2\\
=&\left(\frac{-A^2x_0-ABy_0-AC}{A^2+B^2}\right)^2+\left(\frac{A\left(B^2x_0-ABy_0-AC\right)+C\left(A^2+B^2\right)+B\left(A^2+B^2\right)y_0}{B\left(A^2+B^2\right)}\right)^2\\
=&\left(\frac{-A\left(Ax_0+By_0+C\right)}{A^2+B^2}\right)^2+\left(\frac{AB^2x_0+B^3y_0+B^2C}{B\left(A^2+B^2\right)}\right)^2\\
=&\frac{A^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}+\frac{B^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}\\
=&\frac{\left(Ax_0+By_0+C\right)^2}{A^2+B^2}\\
\overline{PP'}=&\overline{PP'}\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

by perpendicular foot

$$
\begin{gathered}
y=\frac{-A}{B}x-\frac{C}{B}=\frac{-1}{B}(Ax+C)\,,\ \text{if } B\neq 0\\
L_\perp:\left(y=\frac{B}{A}x+K\right)\perp\left(y=\frac{-A}{B}x-\frac{C}{B}\right):L\\
L_\perp=L_\perp(x,y)=Bx-Ay+K=0\\
P=P(x_0,y_0)\in L_\perp=B(x-x_0)-A(y-y_0)=0\\
L_\perp=Bx-Ay-(Bx_0-Ay_0)=0
\end{gathered}
$$

perpendicular foot = foot of the perpendicular P'

$$
\begin{aligned}
P'\in\left(L_\perp\cap L\right)=&\begin{cases}L=Ax+By+C=0\\ L_\perp=Bx-Ay-(Bx_0-Ay_0)=0\end{cases}\\
=&\begin{cases}Ax+By=-C\\ Bx-Ay=Bx_0-Ay_0\end{cases}\\
P'=P'(x,y)=&\left(\frac{\begin{vmatrix}-C & B\\ Bx_0-Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}},\frac{\begin{vmatrix}A & -C\\ B & Bx_0-Ay_0\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}}\right)\\
=&\left(\frac{\begin{vmatrix}C & B\\ -Bx_0+Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}},\frac{\begin{vmatrix}A & C\\ B & -Bx_0+Ay_0\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& d(P,L) = \overline{PP'} \\
=& \left\| (x_0, y_0) - \left(\frac{B^2x_0 - ABy_0 - AC}{A^2+B^2}, \frac{-ABx_0 + A^2y_0 - BC}{A^2+B^2} \right) \right\| \\
=& \sqrt{\left(x_0 - \frac{B^2x_0 - ABy_0 - AC}{A^2+B^2} \right)^2 + \left(y_0 - \frac{-ABx_0 + A^2y_0 - BC}{A^2+B^2} \right)^2} \\
=& \sqrt{\left(\frac{A^2x_0 + ABy_0 + AC}{A^2+B^2} \right)^2 + \left(\frac{ABx_0 + B^2y_0 + BC}{A^2+B^2} \right)^2} \\
=& \sqrt{\frac{A^2\left(Ax_0 + By_0 + C\right)^2 + B^2\left(Ax_0 + By_0 + C\right)^2}{\left(A^2+B^2\right)^2}} = \sqrt{\frac{\left(Ax_0 + By_0 + C\right)^2}{A^2+B^2}} \\
=& \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

by normal vector

$$
\begin{cases}
\vec{n} = (A,B) \perp L = Ax + By + C = 0 \\
\vec{PP'} = P' - P = (x - x_0, y - y_0)
\end{cases}
$$

$$
\begin{aligned}
\vec{PP'} \cdot \vec{n} &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| \cos\theta \\
\left| \vec{PP'} \cdot \vec{n} \right| &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| |\cos\theta| \\
\left\| \vec{PP'} \right\| |\cos\theta| = \left| \vec{PP'} \cdot \hat{\boldsymbol{n}} \right| &= \frac{\left| \vec{PP'} \cdot \vec{n} \right|}{\left\| \vec{n} \right\|} = \frac{|(x - x_0, y - y_0) \cdot (A,B)|}{\|(A,B)\|} \\
&= \frac{|A(x - x_0) + B(y - y_0)|}{\sqrt{A^2+B^2}} = \frac{|-Ax_0 - By_0 + Ax + By|}{\sqrt{A^2+B^2}} \\
&\overset{Ax+By+C=0}{\underset{Ax+By=-C}{=}} \frac{|-Ax_0 - By_0 - C|}{\sqrt{A^2+B^2}} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

by Cauchy inequality

$$
\begin{aligned}
Ax + By + C &= 0 \\
Ax + By &= -C \\
(Ax + By) - (Ax_0 + By_0) &= -C - (Ax_0 + By_0) \\
A(x - x_0) + B(y - y_0) &= -(Ax_0 + By_0 + C)
\end{aligned}
$$

$$
\begin{aligned}
\overline{PP'}^2 &= (x_0 - x)^2 + (y_0 - y)^2 \\
\left[A^2 + B^2\right] \overline{PP'}^2 &= \left[A^2 + B^2\right] \left[(x_0 - x)^2 + (y_0 - y)^2 \right] \\
&\geq \left[A(x - x_0) + B(y - y_0)\right]^2 \\
&= \left[-(Ax_0 + By_0 + C)\right]^2 = (Ax_0 + By_0 + C)^2 \\
\overline{PP'}^2 &\geq \frac{(Ax_0 + By_0 + C)^2}{A^2+B^2} \\
\overline{PP'} &\geq \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

quadratic form

$$
ax^2 + bxy + cy^2 + dx + ey + f = 0
$$

$$
\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + (b/2)y \\ (b/2)x + cy \end{pmatrix} = ax^2 + bxy + cy^2
$$

$$
\begin{aligned}
0 = &\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f \\
=&\boldsymbol{x}^\intercal A \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + f, \begin{cases} A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} & A \text{ real symmetric} \\ \boldsymbol{b} = \begin{pmatrix} d \\ e \end{pmatrix} \\ \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases}
\end{aligned}
$$

homogeneous coordinate

$$
\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & \\ b/2 & c & \\ & & \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
$$

$$
\begin{aligned}
\begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \kappa \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \alpha x + \beta y + \gamma \\ \delta x + \epsilon y + \zeta \\ \eta x + \theta y + \kappa \end{pmatrix}, \begin{cases} \gamma + \eta = d \\ \zeta + \theta = e \end{cases} \\
&= \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \zeta \\ \eta & \theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & d/2 \\ 0 & 0 & e/2 \\ d/2 & e/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
0 =& ax^2 + bxy + cy^2 + dx + ey + f \\
=& \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = \boldsymbol{x}^\intercal A \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + f \\
=& \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^\intercal & 1 \end{pmatrix} M \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
0 = Q =& Ax^2 + Bxy + Cy^2 + Dx + Ey + F \\
=& \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boldsymbol{x}_\mathrm{h}^\intercal A_Q \boldsymbol{x}_\mathrm{h} \\
=& \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = \boldsymbol{x}^\intercal A_{Q,33} \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + F
\end{aligned}
$$

standard form

$$
\left(\frac{y-k}{a}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1
$$

$$
\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1
$$

$$
(y-k) = 4c\,(x-h)^2
$$

$$
\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 = 1
$$

$$
\begin{aligned}
\left(\frac{y-k}{a}\right)^2 + \left(\frac{x-h}{a}\right)^2 =& 1 \\
\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 =& 1 \\
(y-k) - 4c\,(x-h)^2 =& 0 \\
\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 =& 1
\end{aligned}
$$

$$
\begin{array}{lllll}
\text{circle} & \left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & & b=a \\
\text{ellipse} & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{vertical} & b>a \\
 & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{horizontal} & a>b \\
\text{parabola} & (y-k)-4c\,(x-h)^2 & =0 & \text{vertical} & \\
 & -4c\,(y-k)^2+(x-h) & =0 & \text{horizontal} & \\
\text{hyperbola} & \left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2 & =1 & \text{vertical} & \frac{x-h}{a}=0\Rightarrow\frac{y-k}{b}=\pm1 \\
 & -\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \text{horizontal} & \frac{y-k}{b}=0\Rightarrow\frac{x-h}{a}=\pm1
\end{array}
$$

parametric equation

$$
\begin{array}{llllllll}
\text{circle} & \left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&a&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix} & = \begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\a\\1\end{pmatrix} \\
\text{ellipse} & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix} & = \begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix} \\
\text{parabola} & (y-k)-4c\,(x-h)^2 & =0 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}1&0&h\\0&4c&k\\0&0&1\end{pmatrix} & \begin{pmatrix}t\\t^2\\1\end{pmatrix} & = \begin{pmatrix}t&0&h\\0&t^2&k\\0&0&1\end{pmatrix} & \begin{pmatrix}1\\4c\\1\end{pmatrix} \\
 & -4c\,(y-k)^2+(x-h) & =0 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}4c&0&h\\0&1&k\\0&0&1\end{pmatrix} & \begin{pmatrix}t^2\\t\\1\end{pmatrix} & = \begin{pmatrix}t^2&0&h\\0&t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}4c\\1\\1\end{pmatrix} \\
\text{hyperbola} & \left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\pm\cosh t\\\sinh t\\1\end{pmatrix} & = \begin{pmatrix}\tan t&0&h\\0&\sec t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix} \\
 & -\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\sinh t\\\pm\cosh t\\1\end{pmatrix} & = \begin{pmatrix}\sec t&0&h\\0&\tan t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix}
\end{array}
$$

eccentricity

$$
\begin{cases}
F=(0,y_F) \\
L=y-y_L=0 \\
\epsilon=\dfrac{\overline{PF}}{d\,(P,L)}=\dfrac{\|(x,y)-(0,y_F)\|}{\|y-y_L\|}
\end{cases}
\qquad
\begin{array}{l}
F:\text{focus} \\
L:\text{directrix} \\
\begin{cases} P=(x,y) \\ \epsilon:\text{eccentricity}\end{cases}
\end{array}
$$

$$0 \le \epsilon = \frac{\overline{PF}}{d(P,L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|} = \frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|} = \frac{\sqrt{x^2+(y-y_F)^2}}{\sqrt{(y-y_L)^2}}$$

$$\begin{aligned}
\epsilon^2 =& \frac{x^2+(y-y_F)^2}{(y-y_L)^2} = \frac{x^2+y^2-2y_Fy+y_F^2}{y^2-2y_Ly+y_L^2} \\
0 =& x^2+\left(1-\epsilon^2\right)y^2-2\left(y_F-\epsilon^2y_L\right)y+\left(y_F^2-\epsilon^2y_L^2\right) \\
\overset{\epsilon\neq1}{=}& x^2+\left(1-\epsilon^2\right)\left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right] \\
=& x^2+\left(1-\epsilon^2\right) \\
& \left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2-\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right] \\
=& x^2+\left(1-\epsilon^2\right)\left[\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2}{\left(1-\epsilon^2\right)^2}\right] \\
=& x^2+\left(1-\epsilon^2\right)\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)\text{-}\left(y_F-\epsilon^2y_L\right)^2}{1-\epsilon^2}
\end{aligned}$$

$$\begin{aligned}
&\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)\text{-}\left(y_F-\epsilon^2y_L\right)^2 \\
&=\left(1-\epsilon^2\right)y_F^2-\left(\epsilon^2-\epsilon^4\right)y_L^2-y_F^2+2\epsilon^2y_Fy_L-\epsilon^4y_L^2 \\
&=-\epsilon^2y_F^2-\epsilon^2y_L^2+2\epsilon^2y_Fy_L=-\epsilon^2\left(y_F-y_L\right)^2
\end{aligned}$$

$$\begin{aligned}
&\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)\text{-}\left(y_F-\epsilon^2y_L\right)^2 \\
&=\left(1-\epsilon^2\right)y_F^2-\left(\epsilon^2-\epsilon^4\right)y_L^2-y_F^2+2\epsilon^2y_Fy_L-\epsilon^4y_L^2 \\
&=-\epsilon^2y_F^2-\epsilon^2y_L^2+2\epsilon^2y_Fy_L=-\epsilon^2\left(y_F-y_L\right)^2
\end{aligned}$$

$$\frac{\epsilon^2\left(y_F-y_L\right)^2}{1-\epsilon^2} \overset{\epsilon\neq1}{=} x^2+\left(1-\epsilon^2\right)\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2$$

$$1 \overset{\epsilon\neq0,1}{=} \begin{cases} \left(\dfrac{x-0}{\frac{\epsilon\left(y_F-y_L\right)}{\sqrt{1-\epsilon^2}}}\right)^2+\left(\dfrac{y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}}{\frac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2>0 \overset{\epsilon\geq0}{\Rightarrow} 0<\epsilon<1 \\ -\left(\dfrac{x-0}{\frac{\epsilon\left(y_F-y_L\right)}{\sqrt{\epsilon^2-1}}}\right)^2+\left(\dfrac{y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}}{\frac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2<0 \overset{\epsilon\geq0}{\Rightarrow} \epsilon>1 \end{cases}$$

$$y = \begin{cases} \dfrac{y_F-\epsilon^2y_L}{1-\epsilon^2} \pm \dfrac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}\sqrt{1-\left(\dfrac{x}{\frac{\epsilon\left(y_F-y_L\right)}{\sqrt{1-\epsilon^2}}}\right)^2} & 0<\epsilon<1 \\ \dfrac{y_F-\epsilon^2y_L}{1-\epsilon^2} \pm \dfrac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}\sqrt{1+\left(\dfrac{x}{\frac{\epsilon\left(y_F-y_L\right)}{\sqrt{\epsilon^2-1}}}\right)^2} & \epsilon>1 \end{cases}$$

$x = 0$

$$
\begin{aligned}
y - \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} &= \pm \frac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2} \\
y &= \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} \pm \frac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2} \\
&= \frac{\left(1 \pm \epsilon\right) y_F - \epsilon\left(\epsilon \pm 1\right) y_L}{1-\epsilon^2} \\
&= \begin{cases} \dfrac{\left(1+\epsilon\right) y_F - \epsilon\left(\epsilon+1\right) y_L}{1-\epsilon^2} & y_+ \\ \dfrac{\left(1-\epsilon\right) y_F - \epsilon\left(\epsilon-1\right) y_L}{1-\epsilon^2} & y_- \end{cases}
\end{aligned}
$$

$$
\frac{y_+ + y_-}{2} - y_F = \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} - y_F = \frac{\epsilon^2\left(y_F - y_L\right)}{1-\epsilon^2}
$$

$$
\begin{aligned}
\frac{y_F + y_{F'}}{2} &= \frac{y_+ + y_-}{2} \\
y_{F'} &= y_+ + y_- - y_F = 2 \cdot \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2} - y_F \\
&= \frac{y_F - \epsilon^2\left(2 y_L - y_F\right)}{1-\epsilon^2} = \frac{\left(1+\epsilon^2\right) y_F - 2\epsilon^2 y_L}{1-\epsilon^2}
\end{aligned}
$$

$\epsilon = 0$ or $\lim\limits_{|y_L| \to \infty} \epsilon = 0$

$$
r = \overline{PF} = \left\|(x,y) - (0, y_F)\right\| = \left\|(x, y - y_F)\right\| = \sqrt{x^2 + \left(y - y_F\right)^2}
$$

$$
\epsilon = \frac{r}{d\left(P, L\right)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\left\|(x,y) - (0,y_F)\right\|}{\left\|(x,y) - (x,y_L)\right\|} = \frac{\left\|(x, y - y_F)\right\|}{\left\|(0, y - y_L)\right\|} = \frac{\sqrt{x^2 + \left(y - y_F\right)^2}}{\left|y - y_L\right|}
$$

$$
\lim_{|y_L| \to \infty} \epsilon = \lim_{|y_L| \to \infty} \frac{r}{\overline{PL}} = \lim_{|y_L| \to \infty} \frac{\sqrt{x^2 + \left(y - y_F\right)^2}}{\left|y - y_L\right|} = 0
$$

$\epsilon = 1$

$$
\begin{aligned}
0 &= x^2 + \left(1 - \epsilon^2\right) y^2 - 2\left(y_F - \epsilon^2 y_L\right) y + \left(y_F^2 - \epsilon^2 y_L^2\right) \\
&\overset{\epsilon=1}{=} x^2 + \left(1 - 1^2\right) y^2 - 2\left(y_F - 1^2 y_L\right) y + \left(y_F^2 - 1^2 y_L^2\right) \\
&= x^2 - 2\left(y_F - y_L\right) y + \left(y_F^2 - y_L^2\right) \\
&= x^2 - 2\left(y_F - y_L\right) y + \left(y_F + y_L\right)\left(y_F - y_L\right) \\
x^2 &= 2\left(y_F - y_L\right)\left(y - \frac{y_F + y_L}{2}\right)
\end{aligned}
$$

$\epsilon \neq 1$

$$
\begin{aligned}
1 &\overset{P(x,y)=V(0,0)}{=} 0 + \left(\frac{0 - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F - y_L\right)}{1-\epsilon^2}}\right)^2 \\
&\Rightarrow y_F - \epsilon^2 y_L = \pm\epsilon\left(y_F - y_L\right) \\
&\Rightarrow \begin{cases} \left(1-\epsilon\right) y_F = \epsilon\left(\epsilon - 1\right) y_L & + \\ \left(1+\epsilon\right) y_F = \epsilon\left(\epsilon + 1\right) y_L & - \end{cases} \\
&\Rightarrow y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases}
\end{aligned}
$$

$\epsilon = 1$

$$
\begin{aligned}
x^2 &= 2\left(y_F - y_L\right)\left(y - \frac{y_F + y_L}{2}\right) \\
\overset{P(x,y)=V(0,0)}{\Rightarrow} 0^2 &= 2\left(y_F - y_L\right)\left(0 - \frac{y_F + y_L}{2}\right) \\
\Rightarrow 0 &= \left(y_F - y_L\right)\left(y_F + y_L\right) \\
\Rightarrow y_F &= \mp y_L
\end{aligned}
$$

or by definition of eccentricity

$$
\begin{aligned}
0 \le \epsilon = \frac{\overline{PF}}{d\left(P, L\right)} = \frac{\overline{PF}}{\overline{PP'}} &= \frac{\|(x,y) - (0,y_F)\|}{\|(x,y) - (x,y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + \left(y - y_F\right)^2}}{\sqrt{\left(y - y_L\right)^2}} \\
&\overset{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2 + \left(0 - y_F\right)^2}}{\sqrt{\left(0 - y_L\right)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2}
\end{aligned}
$$

$$
\epsilon^2 = \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp \epsilon y_L
$$

actually,

$$
y_F = -\epsilon y_L
$$

$0 < \epsilon < 1$
$c^2 = a^2 - b^2$ or $c^2 = b^2 - a^2$

$$
\begin{aligned}
&\left(\frac{\epsilon\left(y_F - y_L\right)}{1 - \epsilon^2}\right)^2 - \left(\frac{\epsilon\left(y_F - y_L\right)}{\sqrt{1 - \epsilon^2}}\right)^2 = \frac{\epsilon^2\left(y_F - y_L\right)^2}{\left(1 - \epsilon^2\right)^2} - \frac{\epsilon^2\left(y_F - y_L\right)^2}{1 - \epsilon^2} \\
=&\frac{\epsilon^2\left(y_F - y_L\right)^2}{1 - \epsilon^2}\left(\frac{1}{1 - \epsilon^2} - 1\right) = \frac{\epsilon^4\left(y_F - y_L\right)^2}{\left(1 - \epsilon^2\right)^2} = \left(\frac{\epsilon^2\left(y_F - y_L\right)}{1 - \epsilon^2}\right)^2 = \left(\frac{y_+ + y_-}{2} - y_F\right)^2
\end{aligned}
$$

$\epsilon > 1$
$c^2 = a^2 + b^2$

$$
\begin{aligned}
&\left(\frac{\epsilon\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2 + \left(\frac{\epsilon\left(y_F - y_L\right)}{\sqrt{\epsilon^2 - 1}}\right)^2 = \frac{\epsilon^2\left(y_F - y_L\right)^2}{\left(\epsilon^2 - 1\right)^2} + \frac{\epsilon^2\left(y_F - y_L\right)^2}{\epsilon^2 - 1} \\
=&\frac{\epsilon^2\left(y_F - y_L\right)^2}{\epsilon^2 - 1}\left(\frac{1}{\epsilon^2 - 1} + 1\right) = \frac{\epsilon^4\left(y_F - y_L\right)^2}{\left(\epsilon^2 - 1\right)^2} = \left(\frac{\epsilon^2\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2 = \left(\frac{y_+ + y_-}{2} - y_F\right)^2
\end{aligned}
$$

$$
\left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2} = \frac{\left(\dfrac{\epsilon^2\left(y_F - y_L\right)}{\epsilon^2 - 1}\right)^2}{\left(\dfrac{\epsilon\left(y_F - y_L\right)}{1 - \epsilon^2}\right)^2} = \epsilon^2
$$

two definition equivalence of ellipse and hyperbola

$$
\begin{cases}
P = (x, y) \\
F = \left(x_F, y_F\right) = (\alpha, \varphi) \qquad F' = \left(x_{F'}, y_{F'}\right) = (\chi, \psi) \\
L = A'x + B'y + C' = 0
\end{cases}
$$

$$
0 \le \epsilon = \frac{\overline{PF}}{d\left(P, L\right)} = \frac{\sqrt{\left(x - x_F\right)^2 + \left(y - y_F\right)^2}}{\dfrac{|A'x + B'y + C'|}{\sqrt{A'^2 + B'^2}}} = \frac{\sqrt{\left(x - \alpha\right)^2 + \left(y - \varphi\right)^2}}{|Ax + By + C|}, \begin{cases} A = \dfrac{A'}{\sqrt{A'^2 + B'^2}} \\ B = \dfrac{B'}{\sqrt{A'^2 + B'^2}} \\ C = \dfrac{C'}{\sqrt{A'^2 + B'^2}} \end{cases}
$$

$$\epsilon^2 = \frac{(x-\alpha)^2 + (y-\varphi)^2}{(Ax+By+C)^2} = \frac{(x-x_F)^2 + (y-y_F)^2}{\dfrac{(A'x+B'y+C')^2}{A'^2+B'^2}}$$

$$(x-\alpha)^2 + (y-\varphi)^2 = [\epsilon\,(Ax+By+C)]^2$$

$$\begin{aligned} 2c = \overline{FF'} &= \|(x_F, y_F) - (x_{F'}, y_{F'})\| = \|(\alpha,\varphi) - (\chi,\psi)\| \\ &= \sqrt{(\alpha-\chi)^2 + (\chi-\psi)^2} \end{aligned}$$

$$\begin{aligned} D &= \begin{cases} \sqrt{(x-x_F)^2 + (y-y_F)^2} + \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} & \text{ellipse} \\ \sqrt{(x-x_F)^2 + (y-y_F)^2} - \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} & \text{hyperbola} \end{cases} \\ &= \sqrt{(x-x_F)^2 + (y-y_F)^2} \pm \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} \\ &= \sqrt{(x-\alpha)^2 + (y-\varphi)^2} \pm \sqrt{(x-\chi)^2 + (y-\psi)^2} \end{aligned}$$

$$\begin{aligned} (x-\alpha)^2 + (y-\varphi)^2 &= \left(D \mp \sqrt{(x-\chi)^2 + (y-\psi)^2}\right)^2 \\ &= D^2 \mp 2D\sqrt{(x-\chi)^2 + (y-\psi)^2} \\ &\quad + (x-\chi)^2 + (y-\psi)^2 \end{aligned}$$

$$
\begin{aligned}
D^2 =& (x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\\
&\pm 2\sqrt{\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]}\\
&(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2-D^2\\
=&\mp 2\sqrt{\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]}\\
&\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right]^2+D^4\\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right]\\
=&4\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]\\
&\left[(x-\alpha)^2+(y-\varphi)^2\right]^2+\left[(x-\chi)^2+(y-\psi)^2\right]^2\\
&+2\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]+D^4\\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right]\\
=&4\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]\\
0=&\left[(x-\alpha)^2+(y-\varphi)^2\right]^2+\left[(x-\chi)^2+(y-\psi)^2\right]^2\\
&-2\left[(x-\alpha)^2+(y-\varphi)^2\right]\left[(x-\chi)^2+(y-\psi)^2\right]+D^4\\
&-2D^2\left[(x-\alpha)^2+(y-\varphi)^2+(x-\chi)^2+(y-\psi)^2\right]\\
0=&\left\{\left[(x-\alpha)^2+(y-\varphi)^2\right]-\left[(x-\chi)^2+(y-\psi)^2\right]\right\}^2+D^4\\
&-2D^2\left\{\left[(x-\alpha)^2+(y-\varphi)^2\right]+\left[(x-\chi)^2+(y-\psi)^2\right]\right\}\\
0=&\left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]\right\}^2+D^4\\
&-2D^2\left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]\right\}\\
&-4D^2\left[(x-\alpha)^2+(y-\varphi)^2\right]\\
&(2D)^2\left[(x-\alpha)^2+(y-\varphi)^2\right]\\
=&\left\{\left[(x-\chi)^2+(y-\psi)^2\right]-\left[(x-\alpha)^2+(y-\varphi)^2\right]-D^2\right\}^2\\
=&\left\{\left[(x-\chi)^2-(x-\alpha)^2\right]+\left[(y-\psi)^2-(y-\varphi)^2\right]-D^2\right\}^2\\
=&\left\{(2x-\chi-\alpha)(\alpha-\chi)+(2y-\psi-\varphi)(\varphi-\psi)-D^2\right\}^2\\
=&\left\{2(\alpha-\chi)x-\left(\alpha^2-\chi^2\right)+2(\varphi-\psi)y-\left(\varphi^2-\psi^2\right)-D^2\right\}^2\\
=&\left\{2(\alpha-\chi)x+2(\varphi-\psi)y-\left[\left(\alpha^2-\chi^2\right)+\left(\varphi^2-\psi^2\right)+D^2\right]\right\}^2\\
D\neq& 0\\
&(x-\alpha)^2+(y-\varphi)^2\\
=&\left[\frac{\alpha-\chi}{D}x+\frac{\varphi-\psi}{D}y-\left(\frac{\alpha^2-\chi^2}{2D}+\frac{\varphi^2-\psi^2}{2D}+\frac{D}{2}\right)\right]^2
\end{aligned}
$$

$$
\begin{cases}
(x-\alpha)^2+(y-\varphi)^2=\left[\epsilon\left(Ax+By+C\right)\right]^2\\
(x-\alpha)^2+(y-\varphi)^2=\left[\dfrac{\alpha-\chi}{D}x+\dfrac{\varphi-\psi}{D}y-\left(\dfrac{\alpha^2-\chi^2}{2D}+\dfrac{\varphi^2-\psi^2}{2D}+\dfrac{D}{2}\right)\right]^2
\end{cases}
$$

$$
(A,B,C)\rightleftarrows(\chi,\psi,D)
$$

$$\begin{cases} \epsilon A = \pm\dfrac{\alpha-\chi}{D} & \chi \pm \epsilon AD = \alpha \\ \epsilon B = \pm\dfrac{\varphi-\psi}{D} & \psi \pm \epsilon BD = \varphi \\ \epsilon C = \mp\left(\dfrac{\alpha^2-\chi^2}{2D} + \dfrac{\varphi^2-\psi^2}{2D} + \dfrac{D}{2}\right) \end{cases}$$

$$\begin{aligned} 2\epsilon C &= \mp\left(\frac{\alpha-\chi}{D}(\alpha+\chi) + \frac{\varphi-\psi}{D}(\varphi+\psi) + D\right) \\ &= \mp\left(\pm\epsilon A(\alpha+\chi) \pm \epsilon B(\varphi+\psi) + D\right) \\ \mp\epsilon(A\alpha + B\varphi + 2C) &= \pm\,\epsilon A\chi \pm \epsilon B\psi + D \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A \\ 0 & 1 & \pm\epsilon B \\ \pm\epsilon A & \pm\epsilon B & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ D \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ \mp\epsilon(A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & \pm\epsilon B & 1 \mp \epsilon^2 A^2 & \mp\epsilon(2A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 \mp \epsilon^2 A^2 \mp \epsilon^2 B^2 & \mp\epsilon(2A\alpha + 2B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 & \dfrac{\mp 2\epsilon(A\alpha + B\varphi + C)}{1 \mp \epsilon^2(A^2+B^2)} \end{pmatrix}$$

$$A^2 + B^2 = \left(\frac{A'}{\sqrt{A'^2+B'^2}}\right)^2 + \left(\frac{B'}{\sqrt{A'^2+B'^2}}\right)^2 = 1$$

$$\begin{cases} \chi = \alpha \mp \epsilon AD = \alpha \mp \epsilon\dfrac{A'}{\sqrt{A'^2+B'^2}}D \\ \psi = \varphi \mp \epsilon BD = \varphi \mp \epsilon\dfrac{B'}{\sqrt{A'^2+B'^2}}D \\ D = \dfrac{\mp 2\epsilon(A\alpha + B\varphi + C)}{1 \mp \epsilon^2(A^2+B^2)} = \dfrac{\mp 2\epsilon}{1 \mp \epsilon^2}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} \quad A^2 + B^2 = 1 \end{cases}$$

actually only one solution is true

$$\begin{cases} \chi = \alpha - \epsilon AD = \alpha - \epsilon\dfrac{A'}{\sqrt{A'^2+B'^2}}D = \alpha - \dfrac{2\epsilon^2}{\epsilon^2-1}\dfrac{A'^2\alpha + A'B'\varphi + A'C'}{A'^2+B'^2} \\ \psi = \varphi - \epsilon BD = \varphi - \epsilon\dfrac{B'}{\sqrt{A'^2+B'^2}}D = \varphi - \dfrac{2\epsilon^2}{\epsilon^2-1}\dfrac{A'B'\alpha + B'^2\varphi + B'C'}{A'^2+B'^2} \\ D = \dfrac{-2\epsilon(A\alpha + B\varphi + C)}{1 - \epsilon^2(A^2+B^2)} = \dfrac{-2\epsilon}{1-\epsilon^2}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} = \dfrac{2\epsilon}{\epsilon^2-1}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} \end{cases}$$

$$\begin{cases} \chi = \dfrac{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right)}{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)} \\ \psi = \dfrac{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\varphi - 2\epsilon^2\left(A'B'\alpha + B'^2\varphi + B'C'\right)}{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)} \\ \left|\dfrac{D}{d(F,L)}\right| = \left|\dfrac{2\epsilon}{1-\epsilon^2}\right| \Rightarrow \left(\dfrac{D}{d(F,L)}\right)^2 = \left(\dfrac{2\epsilon}{1-\epsilon^2}\right)^2 \end{cases}$$

$$\begin{aligned} &\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right) \\ &= \left(-\left(\epsilon^2+1\right)A'^2 + \left(\epsilon^2-1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \\ &= \left(-\left(\epsilon^2+1\right)A'^2 + \left(\epsilon^2-1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \end{aligned}$$

$$\begin{aligned}\overline{FF'}^2 &= (\alpha-\chi)^2+(\varphi-\psi)^2\\ &= (\alpha-(\alpha-\epsilon AD))^2+(\varphi-(\varphi-\epsilon BD))^2\\ &= (\epsilon D)^2\left(A^2+B^2\right)\\ &= (\epsilon D)^2\end{aligned}$$

$$\left(\frac{c}{a}\right)^2=\left(\frac{\overline{PF}}{d(P,L)}\right)^2=\epsilon^2=\left(\frac{\overline{FF'}}{D}\right)^2=\left(\frac{2c}{D}\right)^2\Rightarrow D=2a$$

$$\left(\frac{D}{d(F,L)}\right)^2=\left(\frac{2\epsilon}{1-\epsilon^2}\right)^2$$

polar coordinate

$$(x-\alpha)^2+(y-\varphi)^2=[\epsilon(Ax+By+C)]^2$$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

$$(r\cos\theta-\alpha)^2+(r\sin\theta-\varphi)^2=[\epsilon(Ar\cos\theta+Br\sin\theta+C)]^2$$

If $\begin{cases}F=(x_F,y_F)=(\alpha,\varphi)=(0,0)\\ L=Ax+By+C=x+p=0\end{cases}$,

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2 &= [\epsilon(r\cos\theta+p)]^2\\ r^2 &=\\ r &= \pm\,\epsilon(r\cos\theta+p)\\ &= \pm(r\epsilon\cos\theta+\epsilon p)\\ r(1\mp\epsilon\cos\theta) &= \epsilon p\\ r &= \frac{\epsilon p}{1\mp\epsilon\cos\theta}\end{aligned}$$

$r=\dfrac{\epsilon p}{1-\epsilon\cos\theta}$ will not cross $L=x+p=0$ on graphs, so maybe it is a more correct solution

$$r=\frac{\epsilon p}{1-\epsilon\cos\theta}$$

$$\epsilon=\pm\frac{r}{r\cos\theta+p}$$

partition

$$\begin{aligned}&\{A_i\}_{i\in I}=\{A_i|i\in I\}\text{ is a partition of a set }A\\ \Leftrightarrow&\begin{cases}\forall i\in I\,(A_i\neq\emptyset)\\ A=\bigcup_{i\in I}A_i\\ \forall i,j\in I\,(i\neq j\Rightarrow A_i\cap A_j=\emptyset)\end{cases}\end{aligned}$$