

inspiration of operator theory

Joey Hsu, MD

June 2, 2024

lecturer: MathematicS, 姚班

editing since 2024/06/02

Contents

1 algebra	1
2 linear algebra	2
3 calculus	4
4 complex analysis	4
5 matrix calculus	4
6 Lie algebra	4

1 algebra

$$2 + 2^2 + 2^3 + \cdots + 2^n$$

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n$$

$$S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = \sum_{\nu=0}^n 2^\nu$$

$$S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n \quad = \sum_{\nu=0}^n 2^\nu$$

$$2S = 2 \cdot 2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + 2 \cdot 2^3 + \cdots + 2 \cdot 2^n \quad = 2 \sum_{\nu=0}^n 2^\nu$$

$$= 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} \quad = \sum_{\nu=0}^n 2^{\nu+1}$$

$$2S = 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} \quad = 2 \sum_{\nu=0}^n 2^\nu$$

$$S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n \quad = \sum_{\nu=0}^n 2^\nu$$

$$2S - S = (0 - 2^0) + (2^1 - 2^1) + (2^2 - 2^2) + (2^3 - 2^3) + \cdots + (2^n - 2^n) + (2^{n+1} - 0) = 2 \sum_{\nu=0}^n 2^\nu - \sum_{\nu=0}^n 2^\nu$$

$$= (-2^0) + (0) + (0) + (0) + \cdots + (0) + (2^{n+1})$$

$$= (-2^0) + (2^{n+1})$$

$$(2 - 1)S = 2^{n+1} - 2^0 \quad = (2 - 1) \sum_{\nu=0}^n 2^\nu$$

$$S = 2^{n+1} - 1 = \sum_{\nu=0}^n 2^\nu \quad \forall n \in \mathbb{N}$$

$$a + ax + ax^2 + ax^3 + \cdots + ax^n$$

$$\sum_{\nu=0}^n ax^\nu = a + ax^1 + ax^2 + ax^3 + \cdots + ax^n$$

$$x \sum_{\nu=0}^n ax^\nu = x \cdot ax^0 + x \cdot ax^1 + x \cdot ax^2 + x \cdot ax^3 + \cdots + x \cdot ax^n$$

$$\sum_{\nu=0}^n ax^{\nu+1} = ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1}$$

$$x \sum_{\nu=0}^n ax^\nu = ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1}$$

$$\sum_{\nu=0}^n ax^\nu = ax^0 + ax^1 + ax^2 + ax^3 + \cdots + ax^n$$

$$\begin{aligned} x \sum_{\nu=0}^n ax^\nu - \sum_{\nu=0}^n ax^\nu &= (0 - ax^0) + (ax^1 - ax^1) + (ax^2 - ax^2) + (ax^3 - ax^3) + \cdots + (ax^n - ax^n) + (ax^{n+1} - 0) \\ &= (0 - ax^0) + (0) + (0) + (0) + \cdots + (0) + (ax^{n+1}) \\ &= (-ax^0) + (ax^{n+1}) \end{aligned}$$

$$(x - 1) \sum_{\nu=0}^n ax^\nu = ax^{n+1} - ax^0$$

$$a \sum_{\nu=0}^n x^\nu = \sum_{\nu=0}^n ax^\nu = \frac{a(x^{n+1} - 1)}{x - 1} = \frac{(-1)a(x^{n+1} - 1)}{(-1)(x - 1)} = \frac{a(1 - x^{n+1})}{1 - x} = \frac{a(1 - x^{n+1})}{1 - x} \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} a \sum_{\nu=0}^{\infty} x^\nu &= \sum_{\nu=0}^{\infty} ax^\nu = \lim_{n \rightarrow \infty} \frac{a(1 - x^{n+1})}{1 - x} = \frac{ax(1 - 0)}{1 - x} = \frac{a}{1 - x} = (1 - x)^{-1}a \quad n \rightarrow \infty, \forall |x| < 1 \\ (1 - x)^{-1}a &= x^0a + x^1a + x^2a + x^3a + \cdots \quad \forall |x| < 1 \end{aligned}$$

2 linear algebra

$$A^0 \mathbf{x} + A^1 \mathbf{x} + A^2 \mathbf{x} + A^3 \mathbf{x} + \cdots$$

$$\begin{aligned} (1 - x)^{-1}a &= x^0a + x^1a + x^2a + x^3a + \cdots & \forall |x| < 1 \\ x^0a + x^1a + x^2a + x^3a + \cdots &= (1 - x)^{-1}a & \forall |x| < 1 \\ A^0 \mathbf{x} + A^1 \mathbf{x} + A^2 \mathbf{x} + A^3 \mathbf{x} + \cdots &= (I - A)^{-1} \mathbf{x} & \forall \|A\| = |\det A| < 1 \end{aligned}$$

$$\left\{ \begin{array}{l} A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} \right. \quad |\det A| = \frac{1}{2} < 1$$

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x & -x \\ y & y \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1-x)^{-1}a = x^0a + x^1a + x^2a + x^3a + \cdots \quad \forall |x| < 1$$

$$x^0a + x^1a + x^2a + x^3a + \cdots = (1-x)^{-1}a \quad \forall |x| < 1$$

$$A^0\mathbf{x} + A^1\mathbf{x} + A^2\mathbf{x} + A^3\mathbf{x} + \cdots = (I - A)^{-1}\mathbf{x} \quad \forall |\det A| < 1$$

$$\Downarrow$$

$$\begin{aligned} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cdots &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^0\mathbf{x} + A^1\mathbf{x} + A^2\mathbf{x} + A^3\mathbf{x} + \cdots = (I - A)^{-1}\mathbf{x}$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A^0 + A^1 + A^2 + A^3 + \cdots) \mathbf{x} =$$

$$\left(\sum_{\nu=0}^{\infty} A^{\nu} \right) \mathbf{x} =$$

3 calculus

$$\exp(x) = e^x$$

$$e^x \stackrel{\text{def.}}{=} D e^x = D_x e^x = d_x e^x = \frac{d}{dx} e^x = \frac{de^x}{dx} \quad \forall x \in \mathbb{R}$$

$$\Downarrow F(x) = \int_a^x D F(t) \quad \text{FToC1}$$

$$e^x = \int D e^x \quad = \text{fundamental theorem of calculus first part}$$

$$\Downarrow e^x \stackrel{\text{def.}}{=} D e^x$$

$$e^x = \int e^x$$

$$e^x - \int e^x = 0$$

$$\left(1 - \int\right) e^x = 0$$

$$e^x = \left(1 - \int\right)^{-1} 0 = \left(\sum_{\nu=0}^{\infty} \int^{\nu}\right) 0 \quad \forall |\det A| < 1 \left[(I - A)^{-1} x \right.$$

$$= \left(\int^0 + \int^1 + \int^2 + \int^3 + \dots\right) 0 \quad \left. = \left(\sum_{\nu=0}^{\infty} A^{\nu}\right) x \right]$$

$$= \int^0 0 + \int^1 0 + \int^2 0 + \int^3 0 + \dots$$

$$= 0 + \int dx 0 + \iint dx^2 0 + \iiint dx^3 0 + \dots$$

$$= C + \int dx C + \iint dx^2 C + \dots$$

$$= C + Cx + \int dx Cx + \dots$$

$$= C + \frac{Cx}{1} + \frac{Cx^2}{2 \cdot 1} + \dots \quad = C \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!}$$

$$1 = e^0 = e^{x=0} = C + \frac{C0}{1} + \frac{C0^2}{2 \cdot 1} + \dots = C + 0 + 0 + \dots = C \quad \Rightarrow C = 1$$

$$e^x = 1 + \frac{1x}{1} + \frac{1x^2}{2 \cdot 1} + \dots \quad = 1 \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!}$$

$$= \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \dots \quad = \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!}$$

4 complex analysis

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \dots = \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \quad \forall x \in \mathbb{C}, |x| < 1$$

5 matrix calculus

$$e^{Ax} = \frac{1}{0!} + \frac{Ax}{1!} + \frac{A^2 x^2}{2!} + \dots = \sum_{\nu=0}^{\infty} \frac{A^{\nu} x^{\nu}}{\nu!} \quad \forall n \in \mathbb{N}, \forall Ax \in \mathcal{M}_{n \times n}, \|Ax\| = |(\det A) x| < 1$$

6 Lie algebra

$$\text{Ad}_{\exp X} Y = \exp(\text{Ad} X) Y = \frac{Y}{0!} + \frac{[X, Y]}{1!} + \frac{[X, [X, Y]]}{2!} + \frac{[X, [X, [X, Y]]]}{3!} + \dots$$

「Mathematics is the art of giving the same name to different things.」

Henri Poincaré (1854~1912)

「Algebra's like sheet music, the important thing isn't can you read music, it's can you hear it. Can you hear the music, Robert?」

Niels Bohr (1885~1962) ~ Oppenheimer movie (2023)