Part I

tensor

1 tensor algebra

1.1 vector space

$$\mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^\infty, \cdots\} \qquad \tilde{\mathfrak{V}}, \mathfrak{V} \subseteq \mathcal{V} \subseteq \mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\}$$

$$\mathcal{V} \ni \boldsymbol{v} = v^j \boldsymbol{v}_j = \sum_j v^j \boldsymbol{v}_j \qquad \boldsymbol{v}_j \in \mathfrak{V} = \{\boldsymbol{v}_j\} = \{\boldsymbol{v}_j\}_j$$

$$= \begin{cases} v^1 \boldsymbol{v}_1 + \cdots + v^n \boldsymbol{v}_n & = \sum\limits_{j=1}^{n} v^j \boldsymbol{v}_j \\ \cdots + v^j \boldsymbol{v}_j + \cdots & = \sum\limits_{j \in J} v^j \boldsymbol{v}_j \end{cases} \qquad = \begin{cases} \{\boldsymbol{v}_j\}_{j=1}^{n} \\ \{\boldsymbol{v}_j\}_{j \in J} & J \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \cdots\} \end{cases}$$

$$= \begin{cases} v^1 \begin{bmatrix} \vert \\ \boldsymbol{v}_1 \\ \vert \end{bmatrix} + \cdots + v^n \begin{bmatrix} \vert \\ \boldsymbol{v}_n \\ \vert \end{bmatrix} & = \begin{bmatrix} \vert & & \vert \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ \vert & & \vert \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\ \cdots + v^j \begin{bmatrix} \vert \\ \boldsymbol{v}_j \\ \vert \end{bmatrix} + \cdots & = \begin{bmatrix} & \vert & \\ \cdots & \boldsymbol{v}_j & \cdots \\ & \vert & \end{bmatrix} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \boldsymbol{v}_j \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} = V \left[\boldsymbol{v}\right]_V \qquad \left[\boldsymbol{v}\right]_V \in \begin{cases} \mathbb{F}^n \\ \mathbb{F}^{|J|} \end{cases}$$

$$= \tilde{v}^j \tilde{\boldsymbol{v}}_j = \sum_j \tilde{v}^j \tilde{\boldsymbol{v}}_j \qquad \tilde{\boldsymbol{v}}_j \in \tilde{\mathfrak{V}} = \{\tilde{\boldsymbol{v}}_j\} = \{\tilde{\boldsymbol{v}}_j\}_j$$

$$= \begin{cases} \tilde{v}^1 \tilde{\boldsymbol{v}}_1 + \cdots + \tilde{v}^n \tilde{\boldsymbol{v}}_n & = \sum\limits_{j=1}^{n} \tilde{v}^j \tilde{\boldsymbol{v}}_j \\ \cdots + \tilde{v}^j \tilde{\boldsymbol{v}}_j + \cdots & = \sum\limits_{j \in J} \tilde{v}^j \tilde{\boldsymbol{v}}_j \end{cases} \qquad = \begin{cases} \{\tilde{\boldsymbol{v}}_j\}_{j=1}^{n} \\ \{\tilde{\boldsymbol{v}}_j\}_{j \in J} & J \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \cdots\} \end{cases}$$

$$= \begin{cases} \tilde{v}^1 \begin{bmatrix} \vert \\ \tilde{\boldsymbol{v}}_1 \\ \vert \end{bmatrix} + \cdots + \tilde{v}^n \begin{bmatrix} \vert \\ \tilde{\boldsymbol{v}}_n \\ \vert \end{bmatrix} & = \begin{bmatrix} \vert & & \vert \\ \tilde{\boldsymbol{v}}_1 & \cdots & \tilde{\boldsymbol{v}}_n \\ \vert & & \vert \end{bmatrix} \begin{bmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{bmatrix} \\ \cdots + \tilde{v}^j \begin{bmatrix} \vert \\ \tilde{\boldsymbol{v}}_j \\ \vert \end{bmatrix} + \cdots & = \begin{bmatrix} & \vert & \\ \cdots & \tilde{\boldsymbol{v}}_j & \cdots \\ & \vert & \end{bmatrix} \begin{bmatrix} \vdots \\ \tilde{v}^j \\ \vdots \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \tilde{\boldsymbol{v}}_j \\ \vdots \end{bmatrix}^{\intercal} \begin{bmatrix} \vdots \\ \tilde{v}^j \\ \vdots \end{bmatrix} = \tilde{V} \left[\boldsymbol{v}\right]_{\tilde{V}} \qquad \left[\boldsymbol{v}\right]_{\tilde{V}} \in \begin{cases} \mathbb{F}^n \\ \mathbb{F}^{|J|} \end{cases}$$

$$\begin{aligned} \boldsymbol{v} = \quad & V \left[\boldsymbol{v}\right]_V = \tilde{V} \left[\boldsymbol{v}\right]_{\tilde{V}} \\ v^i = \quad & V^i{}_j v^j = \tilde{V}^i{}_j \tilde{v}^j \\ v^k = \quad & V^k{}_i v^i = \tilde{V}^k{}_j \tilde{v}^j \end{aligned}$$

$$\begin{aligned} \left[\boldsymbol{v}\right]_V &= V^{-1} \tilde{V} \left[\boldsymbol{v}\right]_{\tilde{V}} && = F \left[\boldsymbol{v}\right]_{\tilde{V}} \\ v^i &= \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \tilde{v}^j && = F^i{}_j \tilde{v}^j \\ &= \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j \tilde{v}^j && = F^i{}_j \tilde{v}^j \\ \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j & && = F^i{}_j \end{aligned}$$

$$\begin{aligned} F^i{}_j = \quad & \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j \\ & \left(V^{-1}\right)^i{}_k = \left(V^k{}_i\right)^{-1} \end{aligned}$$

$$\begin{aligned} \left[\boldsymbol{v}\right]_{\tilde{V}} &= \tilde{V}^{-1} V \left[\boldsymbol{v}\right]_V && = B \left[\boldsymbol{v}\right]_V \\ \tilde{v}^j &= \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i v^i && = B^j{}_i v^i \\ &= \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i v^i && = B^j{}_i v^i \\ \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i & && = B^j{}_i \end{aligned}$$

$$\begin{aligned} B^j{}_i = \quad & \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i \\ & \left(\tilde{V}^{-1}\right)^j{}_k = \left(\tilde{V}^k{}_j\right)^{-1} \end{aligned}$$

$$
\begin{aligned}
v^i \boldsymbol{v}_i &= \tilde{v}^j \tilde{\boldsymbol{v}}_j = \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i v^i \tilde{\boldsymbol{v}}_j = B^j{}_i v^i \tilde{\boldsymbol{v}}_j \quad \tilde{v}^j = \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i v^i = B^j{}_i v^i \\
v^i \boldsymbol{v}_i &= \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i v^i \tilde{\boldsymbol{v}}_j = B^j{}_i v^i \tilde{\boldsymbol{v}}_j & \cancel{v^i} \\
\boldsymbol{v}_i &= \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i \tilde{\boldsymbol{v}}_j = B^j{}_i \tilde{\boldsymbol{v}}_j \\
\tilde{v}^j \tilde{\boldsymbol{v}}_j &= v^i \boldsymbol{v}_i = \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \tilde{v}^j \boldsymbol{v}_i = F^i{}_j \tilde{v}^j \boldsymbol{v}_i \qquad v^i = \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \tilde{v}^j = F^i{}_j \tilde{v}^j \\
\tilde{v}^j \tilde{\boldsymbol{v}}_j &= \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \tilde{v}^j \boldsymbol{v}_i = F^i{}_j \tilde{v}^j \boldsymbol{v}_i & \cancel{\tilde{v}^j} \\
\tilde{\boldsymbol{v}}_j &= \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \boldsymbol{v}_i = F^i{}_j \boldsymbol{v}_i
\end{aligned}
$$

$$
\left\{\begin{array}{l}
\begin{cases}
v^i = F^i{}_j \tilde{v}^j = \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j \tilde{v}^j & [\boldsymbol{v}]_V = F[\boldsymbol{v}]_{\tilde{V}} = V^{-1}\tilde{V}[\boldsymbol{v}]_{\tilde{V}} \\
\tilde{v}^j = B^j{}_i v^i = \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i v^i & [\boldsymbol{v}]_{\tilde{V}} = B[\boldsymbol{v}]_V = \tilde{V}^{-1}V[\boldsymbol{v}]_V
\end{cases} \quad \text{contravariant} \\
\begin{cases}
\boldsymbol{v}_i = B^j{}_i \tilde{\boldsymbol{v}}_j = \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i \tilde{\boldsymbol{v}}_j & \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal B = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal \tilde{V}^{-1}V \\
\tilde{\boldsymbol{v}}_j = F^i{}_j \boldsymbol{v}_i = \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j \boldsymbol{v}_i & \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal F = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal V^{-1}\tilde{V}
\end{cases} \quad \text{covariant}
\end{array}\right.
$$

$$
\left\{\begin{array}{l}
\begin{cases}
v^i = F^i{}_j \tilde{v}^j = \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j \tilde{v}^j & \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix} = F \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} = V^{-1}\tilde{V} \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} \\
\tilde{v}^j = B^j{}_i v^i = \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i v^i & \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} = B \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix} = \tilde{V}^{-1}V \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix}
\end{cases} \quad \text{contravariant} \\
\begin{cases}
\boldsymbol{v}_i = \tilde{\boldsymbol{v}}_j B^j{}_i = \tilde{\boldsymbol{v}}_j \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i & \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal B = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal \tilde{V}^{-1}V \\
\tilde{\boldsymbol{v}}_j = \boldsymbol{v}_i F^i{}_j = \boldsymbol{v}_i \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j & \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal F = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal V^{-1}\tilde{V}
\end{cases} \quad \text{covariant}
\end{array}\right.
$$

$$
\begin{cases}
BF = \left(\tilde{V}^{-1}V\right)\left(V^{-1}\tilde{V}\right) = \tilde{V}^{-1}\left(VV^{-1}\right)\tilde{V} = \tilde{V}^{-1}1\tilde{V} = \tilde{V}^{-1}\tilde{V} = 1 & \Rightarrow B^k{}_i F^i{}_j = \delta^k{}_j \\
FB = \left(V^{-1}\tilde{V}\right)\left(\tilde{V}^{-1}V\right) = V^{-1}\left(\tilde{V}\tilde{V}^{-1}\right)V = V^{-1}1V = V^{-1}V = 1 & \Rightarrow F^i{}_j B^j{}_k = \delta^i{}_k
\end{cases}
$$

$$
\left\{\begin{array}{l}
\begin{cases}
v^i = F^i{}_j \tilde{v}^j = \left(V^{-1}\right)^i{}_k \tilde{V}^k{}_j \tilde{v}^j & \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & F^i{}_j & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} = V^{-1}\tilde{V} \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} \\
\tilde{v}^j = B^j{}_i v^i = \left(\tilde{V}^{-1}\right)^j{}_k V^k{}_i v^i & \begin{bmatrix}\tilde{v}^1\\ \vdots \\ \tilde{v}^n\end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & B^j{}_i & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix} = \tilde{V}^{-1}V \begin{bmatrix}v^1\\ \vdots \\ v^n\end{bmatrix}
\end{cases} \quad \text{contravariant} \\
\begin{cases}
\boldsymbol{v}_i = \tilde{\boldsymbol{v}}_j B^j{}_i = \tilde{\boldsymbol{v}}_j \left(\tilde{V}^k{}_j\right)^{-1} V^k{}_i & \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ \cdots & B^j{}_i & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal \tilde{V}^{-1}V \\
\tilde{\boldsymbol{v}}_j = \boldsymbol{v}_i F^i{}_j = \boldsymbol{v}_i \left(V^k{}_i\right)^{-1} \tilde{V}^k{}_j & \begin{bmatrix}\tilde{\boldsymbol{v}}_1\\ \vdots \\ \tilde{\boldsymbol{v}}_n\end{bmatrix}^\intercal = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ \cdots & F^i{}_j & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_1\\ \vdots \\ \boldsymbol{v}_n\end{bmatrix}^\intercal V^{-1}\tilde{V}
\end{cases} \quad \text{covariant}
\end{array}\right.
$$

We do not denote $V[\boldsymbol{v}]_V = V[\boldsymbol{v}]_{\mathfrak{V}}$, because $\mathfrak{V}$ can have elements or bases in different orders whereas V cannot.

1.2 dual space

$$\begin{cases}\boldsymbol{v}\in\mathcal{V}\subseteq\mathbb{F}^n\in\{\mathbb{R}^n,\mathbb{C}^n,\cdots\}\\ \exists!\omega\in\mathbb{F}\left[\boldsymbol{\omega}\left(\boldsymbol{v}\right)=\omega\right]\end{cases}\quad\begin{aligned}&\Leftrightarrow\mathcal{V}\xrightarrow{\boldsymbol{\omega}}\mathbb{F}\Leftrightarrow\boldsymbol{\omega}:\mathcal{V}\to\mathbb{F}\\&\Leftrightarrow\mathbb{F}^{\mathcal{V}}=\{\boldsymbol{\omega}|\boldsymbol{\omega}:\mathcal{V}\to\mathbb{F}\}\\&\quad\Downarrow\\&\quad\left|\mathbb{F}^{\mathcal{V}}\right|=\left|\mathbb{F}\right|^{\left|\mathcal{V}\right|}\end{aligned}$$

$$\begin{matrix}\boldsymbol{v}^1\left(\boldsymbol{v}_1\right)=1 & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^1\left(\boldsymbol{v}_n\right)\\ \vdots & & \vdots & & \vdots\\ \boldsymbol{v}^i\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}}{=}\begin{cases}1 & j=i\\0 & j\neq i\end{cases}=\delta^i_j & \cdots & \boldsymbol{v}^i\left(\boldsymbol{v}_n\right)\\ \vdots & & \vdots & & \vdots\\ \boldsymbol{v}^n\left(\boldsymbol{v}_1\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_j\right) & \cdots & \boldsymbol{v}^n\left(\boldsymbol{v}_n\right)=1\end{matrix}$$

$$\boldsymbol{v}^i\left(\boldsymbol{v}\right)=\boldsymbol{v}^i\left(v^j\boldsymbol{v}_j\right)=v^j\boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}}{=}v^j\delta^i_j=v^i$$

$$\begin{aligned}&\begin{cases}\boldsymbol{\omega}\in\mathcal{V}^*=\left(\mathcal{V}^*,\mathbb{F},+,\cdot\right)=\left(\mathcal{V}^*,\mathbb{F},+_{\mathcal{V}^*,\mathbb{F}},\cdot_{\mathbb{F}\times\mathcal{V}^*,\mathbb{F}}\right)\\ \boldsymbol{v}\in\mathcal{V}=\left(\mathcal{V},\mathbb{F},+,\cdot\right)=\left(\mathcal{V},\mathbb{F},+_{\mathcal{V},\mathbb{F}},\cdot_{\mathbb{F}\times\mathcal{V},\mathbb{F}}\right)\end{cases}\\
\boldsymbol{\omega}\left(\boldsymbol{v}\right)=&\boldsymbol{\omega}\left(v^j\boldsymbol{v}_j\right)=v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\
=&\boldsymbol{\omega}\left(\sum_j v^j\boldsymbol{v}_j\right)=\sum_j\boldsymbol{\omega}\left(v^j\boldsymbol{v}_j\right)=\sum_j v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\
=&\begin{cases}\boldsymbol{\omega}\left(v^1\boldsymbol{v}_1+\cdots+v^n\boldsymbol{v}_n\right) & =\boldsymbol{\omega}\left(\sum_{j=1}^n v^j\boldsymbol{v}_j\right)\\ \boldsymbol{\omega}\left(\cdots+v^j\boldsymbol{v}_j+\cdots\right) & =\boldsymbol{\omega}\left(\sum_{j\in J} v^j\boldsymbol{v}_j\right)\end{cases} && +=+_{\mathcal{V},\mathbb{F}}\\
=&\begin{cases}v^1\boldsymbol{\omega}\left(\boldsymbol{v}_1\right)+\cdots+v^n\boldsymbol{\omega}\left(\boldsymbol{v}_n\right) & =\sum_{j=1}^n v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\ \cdots+v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)+\cdots & =\sum_{j\in J}v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\end{cases} && +=+_{\mathcal{V}^*,\mathbb{F}}\\
=&v^j\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)=\boldsymbol{v}^j\left(\boldsymbol{v}\right)\boldsymbol{\omega}\left(\boldsymbol{v}_j\right) && v^j=\boldsymbol{v}^j\left(\boldsymbol{v}\right)\Leftarrow\boldsymbol{v}^i\left(\boldsymbol{v}\right)=v^i\Leftarrow\boldsymbol{v}^i\left(\boldsymbol{v}_j\right)\overset{\text{def.}}{=}\delta^i_j\\
=&\boldsymbol{v}^j\left(\boldsymbol{v}\right)\omega^{\boldsymbol{v}}_j=\omega^{\boldsymbol{v}}_j\boldsymbol{v}^j\left(\boldsymbol{v}\right)=\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\left(\boldsymbol{v}\right) && \omega^{\boldsymbol{v}}_j\overset{\text{def.}}{=}\boldsymbol{\omega}\left(\boldsymbol{v}_j\right)\\
\boldsymbol{\omega}\left(\boldsymbol{v}\right)=&\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\left(\boldsymbol{v}\right)\\
\boldsymbol{\omega}=&\omega^{\boldsymbol{v}}_i\boldsymbol{v}^i\end{aligned}$$

$$
\begin{aligned}
\mathcal{V}^* \ni \boldsymbol{\omega} &= \omega_i \boldsymbol{\omega}^i = \sum_i \omega_i \boldsymbol{\omega}^i \\
&= \begin{cases} \omega_1 \boldsymbol{\omega}^1 + \cdots + \omega_n \boldsymbol{\omega}^n & = \sum\limits_{i=1}^{n} \omega_i \boldsymbol{\omega}^i \\ \cdots + \omega_i \boldsymbol{\omega}^i + \cdots & = \sum\limits_{i \in I} \omega_i \boldsymbol{\omega}^i \end{cases} \\
&= \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i = \sum_i \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \\
&= \begin{cases} \omega_1^{\boldsymbol{v}} \boldsymbol{v}^1 + \cdots + \omega_n^{\boldsymbol{v}} \boldsymbol{v}^n & = \sum\limits_{i=1}^{n} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \\ \cdots + \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i + \cdots & = \sum\limits_{i \in I} \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i \end{cases} \\
&= \begin{cases} \omega_1^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^1 \\ | \end{bmatrix}^\intercal + \cdots + \omega_n^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^n \\ | \end{bmatrix}^\intercal & = \begin{bmatrix} \omega_1^{\boldsymbol{v}} \\ \vdots \\ \omega_n^{\boldsymbol{v}} \end{bmatrix}^\intercal \begin{bmatrix} — & \boldsymbol{v}^1 & — \\ & \vdots & \\ — & \boldsymbol{v}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\boldsymbol{v}} \begin{bmatrix} | \\ \boldsymbol{v}^i \\ | \end{bmatrix}^\intercal + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ — & \boldsymbol{v}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} \vdots \\ \boldsymbol{v}^i \\ \vdots \end{bmatrix} = [\boldsymbol{\omega}]^V \, V^* \\
&= \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i = \sum_i \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i \\
&= \begin{cases} \omega_1^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^1 + \cdots + \omega_n^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^n & = \sum\limits_{i=1}^{n} \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i \\ \cdots + \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i + \cdots & = \sum\limits_{i \in I} \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i \end{cases} \\
&= \begin{cases} \omega_1^{\tilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \tilde{\boldsymbol{v}}^1 \\ | \end{bmatrix}^\intercal + \cdots + \omega_n^{\tilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \tilde{\boldsymbol{v}}^n \\ | \end{bmatrix}^\intercal & = \begin{bmatrix} \omega_1^{\tilde{\boldsymbol{v}}} \\ \vdots \\ \omega_n^{\tilde{\boldsymbol{v}}} \end{bmatrix}^\intercal \begin{bmatrix} — & \tilde{\boldsymbol{v}}^1 & — \\ & \vdots & \\ — & \tilde{\boldsymbol{v}}^n & — \end{bmatrix} \\ \cdots + \omega_i^{\tilde{\boldsymbol{v}}} \begin{bmatrix} | \\ \tilde{\boldsymbol{v}}^i \\ | \end{bmatrix}^\intercal + \cdots & = \begin{bmatrix} \vdots \\ \omega_i^{\tilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} & \vdots & \\ — & \tilde{\boldsymbol{v}}^i & — \\ & \vdots & \end{bmatrix} \end{cases} = \begin{bmatrix} \vdots \\ \omega_i^{\tilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal \begin{bmatrix} \vdots \\ \tilde{\boldsymbol{v}}^i \\ \vdots \end{bmatrix} = [\boldsymbol{\omega}]^{\tilde{V}} \, \tilde{V}^*
\end{aligned}
$$

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\begin{aligned}
\boldsymbol{\omega} = [\boldsymbol{\omega}]^V \, V^* = [\boldsymbol{\omega}]^{\tilde{V}} \, \tilde{V}^* & \\
&= \omega_i^{\boldsymbol{v}} V^{*i}{}_k = \omega_j^{\tilde{\boldsymbol{v}}} \tilde{V}^{*j}{}_k \\
\omega_j^{\tilde{\boldsymbol{v}}} \tilde{V}^{*j}{}_k &= \omega_i^{\boldsymbol{v}} V^{*i}{}_k \\
\omega_j^{\tilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} V^{*i}{}_k \left(\tilde{V}^{*j}{}_k \right)^{-1} = \omega_i^{\boldsymbol{v}} V^{*i}{}_k \left(\tilde{V}^{*-1} \right)^k{}_j = \omega_i^{\boldsymbol{v}} Q^i{}_j \\
\boldsymbol{\omega}\left(\tilde{\boldsymbol{v}}_j\right) = \omega_j^{\tilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} Q^i{}_j = \boldsymbol{\omega}\left(\boldsymbol{v}_i\right) Q^i{}_j = \boldsymbol{\omega}\left(\tilde{\boldsymbol{v}}_k B^k{}_i\right) Q^i{}_j = \boldsymbol{\omega}\left(\tilde{\boldsymbol{v}}_k\right) B^k{}_i Q^i{}_j \\
\boldsymbol{\omega}\left(\tilde{\boldsymbol{v}}_j\right) &= \boldsymbol{\omega}\left(\tilde{\boldsymbol{v}}_k\right) B^k{}_i Q^i{}_j \\
B^k{}_i Q^i{}_j &= \delta^k{}_j \Rightarrow Q^i{}_j = F^i{}_j \\
\omega_j^{\tilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} Q^i{}_j = \omega_i^{\boldsymbol{v}} F^i{}_j \\
\omega_j^{\tilde{\boldsymbol{v}}} &= \omega_i^{\boldsymbol{v}} F^i{}_j \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\tilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal = \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal F \\
\omega_k^{\tilde{\boldsymbol{v}}} B^k{}_j &= \omega_i^{\boldsymbol{v}} F^i{}_k B^k{}_j = \omega_i^{\boldsymbol{v}} \delta^i{}_j = \omega_j^{\boldsymbol{v}} \\
\omega_j^{\boldsymbol{v}} &= \omega_k^{\tilde{\boldsymbol{v}}} B^k{}_j \Rightarrow \begin{bmatrix} \vdots \\ \omega_i^{\boldsymbol{v}} \\ \vdots \end{bmatrix}^\intercal = \begin{bmatrix} \vdots \\ \omega_i^{\tilde{\boldsymbol{v}}} \\ \vdots \end{bmatrix}^\intercal B
\end{aligned}
$$

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\begin{aligned}
\omega_j^{\tilde{\boldsymbol{v}}} B^j{}_i \boldsymbol{v}^i = \omega_i^{\boldsymbol{v}} \boldsymbol{v}^i &= \omega_i^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^i = \omega_j^{\boldsymbol{v}} F^j{}_i \tilde{\boldsymbol{v}}^i \\
\omega_j^{\tilde{\boldsymbol{v}}} B^j{}_i \boldsymbol{v}^i &= \omega_j^{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}^j \Rightarrow B^j{}_i \boldsymbol{v}^i = \tilde{\boldsymbol{v}}^j \Rightarrow \tilde{\boldsymbol{v}}^j = B^k{}_i \boldsymbol{v}^i \\
\omega_j^{\boldsymbol{v}} F^j{}_i \tilde{\boldsymbol{v}}^i &= \omega_j^{\boldsymbol{v}} \boldsymbol{v}^j \Rightarrow F^j{}_i \tilde{\boldsymbol{v}}^i = \boldsymbol{v}^j \Rightarrow \boldsymbol{v}^j = F^j{}_i \tilde{\boldsymbol{v}}^i
\end{aligned}
$$

$$\left\{\begin{aligned}&\left\{\begin{aligned}&\omega_j^{\boldsymbol{v}}=\omega_k^{\tilde{\boldsymbol{v}}}B^k{}_j && \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^\intercal=\begin{bmatrix}\vdots\\ \omega_i^{\tilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^\intercal B\\ &\omega_j^{\tilde{\boldsymbol{v}}}=\omega_i^{\boldsymbol{v}}F^i{}_j && \begin{bmatrix}\vdots\\ \omega_i^{\tilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^\intercal=\begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^\intercal F\end{aligned}\right. && \text{covariant}\\ &\left\{\begin{aligned}&\boldsymbol{v}^j=F^j{}_i\tilde{\boldsymbol{v}}^i && \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}=F\begin{bmatrix}\vdots\\ \tilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix}\\ &\tilde{\boldsymbol{v}}^j=B^k{}_i\boldsymbol{v}^i && \begin{bmatrix}\vdots\\ \tilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix}=B\begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}\end{aligned}\right. && \text{contravariant}\end{aligned}\right.$$

$$\left\{\begin{aligned}&\left\{\begin{aligned}&\boldsymbol{v}^j=F^j{}_i\tilde{\boldsymbol{v}}^i && \begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}=F\begin{bmatrix}\vdots\\ \tilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix}\\ &\tilde{\boldsymbol{v}}^j=B^k{}_i\boldsymbol{v}^i && \begin{bmatrix}\vdots\\ \tilde{\boldsymbol{v}}^i\\ \vdots\end{bmatrix}=B\begin{bmatrix}\vdots\\ \boldsymbol{v}^i\\ \vdots\end{bmatrix}\end{aligned}\right. && \text{contravariant}\\ &\left\{\begin{aligned}&\omega_j^{\boldsymbol{v}}=\omega_k^{\tilde{\boldsymbol{v}}}B^k{}_j && \begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^\intercal=\begin{bmatrix}\vdots\\ \omega_i^{\tilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^\intercal B\\ &\omega_j^{\tilde{\boldsymbol{v}}}=\omega_i^{\boldsymbol{v}}F^i{}_j && \begin{bmatrix}\vdots\\ \omega_i^{\tilde{\boldsymbol{v}}}\\ \vdots\end{bmatrix}^\intercal=\begin{bmatrix}\vdots\\ \omega_i^{\boldsymbol{v}}\\ \vdots\end{bmatrix}^\intercal F\end{aligned}\right. && \text{covariant}\end{aligned}\right.$$

$$\begin{array}{cccc} & \text{covariant} & \text{contravariant} & \\ & \left.\begin{matrix}\tilde{\mathfrak{V}}\\ \mathfrak{V}\end{matrix}\right\}\ni\begin{cases}\tilde{\boldsymbol{v}}_j=\boldsymbol{v}_iF^i{}_j\\ \boldsymbol{v}_j=\tilde{\boldsymbol{v}}_iB^i{}_j\end{cases} & \mathbb{F}\ni\begin{cases}\tilde{v}^i=B^i{}_jv^j\\ v^i=F^i{}_j\tilde{v}^j\end{cases} & \text{vector space } \mathcal{V}\ni\boldsymbol{v}=\boldsymbol{v}_jv^j\\ & \mathbb{F}\ni\begin{cases}\omega_j^{\tilde{\boldsymbol{v}}}=\omega_i^{\boldsymbol{v}}F^i{}_j\\ \omega_j^{\boldsymbol{v}}=\omega_k^{\tilde{\boldsymbol{v}}}B^k{}_j\end{cases} & \left.\begin{matrix}\tilde{\mathfrak{V}}^*\\ \mathfrak{V}^*\end{matrix}\right\}\ni\begin{cases}\tilde{\boldsymbol{v}}^i=B^i{}_j\boldsymbol{v}^j\\ \boldsymbol{v}^i=F^i{}_j\tilde{\boldsymbol{v}}^j\end{cases} & \text{dual space } \mathcal{V}^*\ni\boldsymbol{\omega}=\omega_i^{\boldsymbol{v}}\boldsymbol{v}^i\end{array}$$

$$\tilde{\boldsymbol{v}}_j\tilde{v}^j=\boldsymbol{v}_iF^i{}_jB^j{}_kv^k=\boldsymbol{v}_i\delta^i{}_kv^k=\begin{cases}\boldsymbol{v}_kv^k & \boldsymbol{v}_k=\boldsymbol{v}_i\delta^i{}_k\\ \boldsymbol{v}_iv^i & \delta^i{}_kv^k=v^i\end{cases}=v^j\boldsymbol{v}_j$$

1.3 linear map transformation

$$\begin{cases}\boldsymbol{v}\in\mathcal{V}\subseteq\mathbb{F}^n\in\{\mathbb{R}^n,\mathbb{C}^n,\cdots\}\\ \exists!\boldsymbol{w}\in\mathcal{W}\left[L(\boldsymbol{v})=\boldsymbol{w}\right]\end{cases}\quad\begin{aligned}&\Leftrightarrow\mathcal{V}\xrightarrow{L}\mathcal{W}\Leftrightarrow L:\mathcal{V}\to\mathcal{W}\\ &\Leftrightarrow\mathcal{W}^{\mathcal{V}}=\{L|L:\mathcal{V}\to\mathcal{W}\}\\ &\Downarrow\\ &\left|\mathcal{W}^{\mathcal{V}}\right|=|\mathcal{W}|^{|\mathcal{V}|}\end{aligned}$$

$$\boldsymbol{w}=L(\boldsymbol{v})=L(v^j\boldsymbol{v}_j)=v^jL(\boldsymbol{v}_j)$$

$$
\begin{aligned}
L(\boldsymbol{v}_1) =& \boldsymbol{v}_1 L^1{}_1 + \cdots + \boldsymbol{v}_n L^n{}_1 = \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} L^1{}_1 + \cdots + \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} L^n{}_1 = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} L^1{}_1 \\ \vdots \\ L^n{}_1 \end{bmatrix} \\
&\vdots \\
L(\boldsymbol{v}_j) =& \boldsymbol{v}_1 L^1{}_j + \cdots + \boldsymbol{v}_n L^n{}_j = \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} L^1{}_j + \cdots + \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} L^n{}_j = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} L^1{}_j \\ \vdots \\ L^n{}_j \end{bmatrix} \\
&\vdots \\
L(\boldsymbol{v}_n) =& \boldsymbol{v}_1 L^1{}_n + \cdots + \boldsymbol{v}_n L^n{}_n = \begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} L^1{}_n + \cdots + \begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} L^n{}_n = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} L^1{}_n \\ \vdots \\ L^n{}_n \end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix} | & & | \\ L(\boldsymbol{v}_1) & \cdots & L(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} L^1{}_1 & & L^1{}_n \\ \vdots & \dots & \vdots \\ L^n{}_1 & & L^n{}_n \end{bmatrix}
$$

$$
\begin{aligned}
\boldsymbol{w} = L(\boldsymbol{v}) = v^j L(\boldsymbol{v}_j) = \begin{bmatrix} | & & | \\ L(\boldsymbol{v}_1) & \cdots & L(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} &= \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} L^1{}_1 & & L^1{}_n \\ \vdots & \dots & \vdots \\ L^n{}_1 & & L^n{}_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\
= \begin{bmatrix} & | & \\ \cdots & L(\boldsymbol{v}_n) & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} &= \begin{bmatrix} & | & \\ \cdots & \boldsymbol{v}_i & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} & \vdots & \\ \cdots & L^i{}_j & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ v^j \\ \vdots \end{bmatrix} \\
= \boldsymbol{v}_i w^i_{\boldsymbol{v}} &= \boldsymbol{v}_i L^i{}_j v^j \qquad w^i_{\boldsymbol{v}} = L^i{}_j v^j \\
= \tilde{\boldsymbol{v}}_i w^i_{\tilde{\boldsymbol{v}}} &= \tilde{\boldsymbol{v}}_i \tilde{L}^i{}_j \tilde{v}^j \qquad w^i_{\tilde{\boldsymbol{v}}} = \tilde{L}^i{}_j \tilde{v}^j \\
\tilde{\boldsymbol{v}}_h B^h{}_i L^i{}_j F^j{}_k \tilde{v}^k = \boldsymbol{v}_i L^i{}_j v^j &= \tilde{\boldsymbol{v}}_i \tilde{L}^i{}_j \tilde{v}^j = \boldsymbol{v}_h F^h{}_i \tilde{L}^i{}_j B^j{}_k v^k \\
\tilde{\boldsymbol{v}}_h B^h{}_i L^i{}_j F^j{}_k \tilde{v}^k &= \tilde{\boldsymbol{v}}_h \tilde{L}^h{}_k \tilde{v}^k \qquad \tilde{L}^h{}_k = B^h{}_i L^i{}_j F^j{}_k \\
\boldsymbol{v}_h F^h{}_i \tilde{L}^i{}_j B^j{}_k v^k &= \boldsymbol{v}_h L^h{}_k v^k \qquad L^h{}_k = F^h{}_i \tilde{L}^i{}_j B^j{}_k
\end{aligned}
$$

covariant $(0,1)$-tensor

$$
\left.\begin{matrix} \tilde{\mathfrak{V}} \\ \mathfrak{V} \end{matrix}\right\} \ni \begin{cases} \tilde{\boldsymbol{v}}_j = \boldsymbol{v}_i F^i{}_j \\ \boldsymbol{v}_j = \tilde{\boldsymbol{v}}_i B^i{}_j \end{cases}
\qquad
\mathbb{F} \ni \begin{cases} \omega^{\tilde{\boldsymbol{v}}}_j = \omega^{\boldsymbol{v}}_i F^i{}_j \\ \omega^{\boldsymbol{v}}_j = \omega^{\tilde{\boldsymbol{v}}}_k B^k{}_j \end{cases}
$$

contravariant $(1,0)$-tensor

$$
\mathbb{F} \ni \begin{cases} \tilde{v}^i = B^i{}_j v^j \\ v^i = F^i{}_j \tilde{v}^j \end{cases}
\qquad
\left.\begin{matrix} \tilde{\mathfrak{V}}^* \\ \mathfrak{V}^* \end{matrix}\right\} \ni \begin{cases} \tilde{\boldsymbol{v}}^i = B^i{}_j \boldsymbol{v}^j \\ \boldsymbol{v}^i = F^i{}_j \tilde{\boldsymbol{v}}^j \end{cases}
$$

vector space $\mathcal{V} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j$

dual space $\mathcal{V}^* \ni \boldsymbol{\omega} = \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i$

$(1,1)$-tensor $\qquad \mathcal{V} \xrightarrow{L} \mathcal{W}$

$$
\begin{cases} \tilde{L}^h{}_k = B^h{}_i L^i{}_j F^j{}_k \\ L^h{}_k = F^h{}_i \tilde{L}^i{}_j B^j{}_k \end{cases}
$$

vector space $\mathcal{W} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j$

1.4 metric tensor

$$
\begin{aligned}
\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} =& (\boldsymbol{v}_i v^i) \cdot (\boldsymbol{v}_j v^j) = (\boldsymbol{v}_i \cdot \boldsymbol{v}_j)\, v^i v^j = g_{ij} v^i v^j = v^i (\boldsymbol{v}_i \cdot \boldsymbol{v}_j)\, v^j = v^i g_{ij} v^j \\
=& (\tilde{\boldsymbol{v}}_i \tilde{v}^i) \cdot (\tilde{\boldsymbol{v}}_j \tilde{v}^j) = (\tilde{\boldsymbol{v}}_i \cdot \tilde{\boldsymbol{v}}_j)\, \tilde{v}^i \tilde{v}^j = \tilde{g}_{ij} \tilde{v}^i \tilde{v}^j = \tilde{v}^i (\tilde{\boldsymbol{v}}_i \cdot \tilde{\boldsymbol{v}}_j)\, \tilde{v}^j = \tilde{v}^i \tilde{g}_{ij} \tilde{v}^j \\
=& g_{ij} v^i v^j = \tilde{g}_{ij} \tilde{v}^i \tilde{v}^j = \tilde{g}_{hk} B^h{}_i v^i B^k{}_j v^j = \tilde{g}_{hk} B^h{}_i B^k{}_j v^i v^j \qquad g_{ij} = \tilde{g}_{hk} B^h{}_i B^k{}_j \\
=& \tilde{g}_{ij} \tilde{v}^i \tilde{v}^j = g_{ij} v^i v^j = g_{hk} F^h{}_i \tilde{v}^i F^k{}_j \tilde{v}^j = g_{hk} F^h{}_i F^k{}_j \tilde{v}^i \tilde{v}^j \qquad \tilde{g}_{ij} = g_{hk} F^h{}_i F^k{}_j
\end{aligned}
$$

covariant $(0,1)$-tensor

contravariant $(1,0)$-tensor

$$\left.\begin{matrix}\tilde{\mathfrak{V}}\\ \mathfrak{V}\end{matrix}\right\} \ni \begin{cases}\tilde{\boldsymbol{v}}_j = \boldsymbol{v}_i F^i{}_j\\ \boldsymbol{v}_j = \tilde{\boldsymbol{v}}_i B^i{}_j\end{cases} \qquad \mathbb{F} \ni \begin{cases}\tilde{v}^i = B^i{}_j v^j\\ v^i = F^i{}_j \tilde{v}^j\end{cases} \qquad \text{vector space } \mathcal{V} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j$$

$$\mathbb{F} \ni \begin{cases}\omega^{\tilde{\boldsymbol{v}}}_j = \omega^{\boldsymbol{v}}_i F^i{}_j\\ \omega^{\boldsymbol{v}}_j = \omega^{\tilde{\boldsymbol{v}}}_k B^k{}_j\end{cases} \qquad \left.\begin{matrix}\tilde{\mathfrak{V}}^*\\ \mathfrak{V}^*\end{matrix}\right\} \ni \begin{cases}\tilde{\boldsymbol{v}}^i = B^i{}_j \boldsymbol{v}^j\\ \boldsymbol{v}^i = F^i{}_j \tilde{\boldsymbol{v}}^j\end{cases} \qquad \text{dual space } \mathcal{V}^* \ni \boldsymbol{\omega} = \omega^{\boldsymbol{v}}_i \boldsymbol{v}^i$$

$(1,1)$-tensor $\qquad \mathcal{V} \xrightarrow{L} \mathcal{W}$

$$\begin{cases}\tilde{L}^h{}_k = B^h{}_i L^i{}_j F^j{}_k\\ L^h{}_k = F^h{}_i \tilde{L}^i{}_j B^j{}_k\end{cases} \qquad \text{vector space } \mathcal{W} \ni \boldsymbol{v} = \boldsymbol{v}_j v^j$$

$(0,2)$-tensor $\qquad \mathcal{V}^2 \xrightarrow{g} \mathbb{R}_{\geq 0}$

$$\mathbb{R}^{\mathcal{V}^2}_{\geq 0} \ni \begin{cases}\tilde{g}_{ij} = g_{hk} F^h{}_i F^k{}_j\\ g_{ij} = \tilde{g}_{hk} B^h{}_i B^k{}_j\end{cases} \qquad \text{metric space } \mathcal{V} \times \mathcal{V} \xrightarrow{g} \mathbb{F}$$

$$\begin{aligned}\boldsymbol{u} \cdot \boldsymbol{v} &= (\boldsymbol{u}_i u^i) \cdot (\boldsymbol{v}_j v^j) = (\boldsymbol{u}_i \cdot \boldsymbol{v}_j)\, u^i v^j = g_{ij} u^i v^j = u^i\, (\boldsymbol{u}_i \cdot \boldsymbol{v}_j)\, v^j = u^i g_{ij} v^j\\ &= (\tilde{\boldsymbol{u}}_i \tilde{u}^i) \cdot (\tilde{\boldsymbol{v}}_j \tilde{v}^j) = (\tilde{\boldsymbol{u}}_i \cdot \tilde{\boldsymbol{v}}_j)\, \tilde{u}^i \tilde{v}^j = \tilde{g}_{ij} \tilde{u}^i \tilde{v}^j = \tilde{u}^i\, (\tilde{\boldsymbol{u}}_i \cdot \tilde{\boldsymbol{v}}_j)\, \tilde{v}^j = \tilde{u}^i \tilde{g}_{ij} \tilde{v}^j\end{aligned}$$

1.5 bilinear form

2 tensor calculus

Part II
group theory