inspriation of operator theory

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Contents

1 algebra

$2 + 2^2 + 2^3 + \cdots + 2^n$

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n$$

$$S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = \sum_{\nu=0}^{n} 2^\nu$$

$$
\begin{aligned}
S =& 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n && = \sum_{\nu=0}^{n} 2^\nu \\
2S =& \quad 2 \cdot 2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + 2 \cdot 2^3 + \cdots + 2 \cdot 2^n && = 2\sum_{\nu=0}^{n} 2^\nu \\
=& \quad 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} && = \sum_{\nu=0}^{n} 2^{\nu+1} \\
2S =& \quad 2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} && = 2\sum_{\nu=0}^{n} 2^\nu \\
S =& 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n && = \sum_{\nu=0}^{n} 2^\nu \\
2S - S =& \left(0 - 2^0\right) + \left(2^1 - 2^1\right) + \left(2^2 - 2^2\right) + \left(2^3 - 2^3\right) + \cdots + \left(2^n - 2^n\right) + \left(2^{n+1} - 0\right) && = 2\sum_{\nu=0}^{n} 2^\nu - \sum_{\nu=0}^{n} 2^\nu \\
&= \left(-2^0\right) + (0) + (0) + (0) + \cdots + (0) + \left(2^{n+1}\right) \\
&= \left(-2^0\right) + \left(2^{n+1}\right) \\
(2-1)\,S =& 2^{n+1} - 2^0 && = (2-1)\sum_{\nu=0}^{n} 2^\nu \\
S =& 2^{n+1} - 1 = \sum_{\nu=0}^{n} 2^\nu && \forall n \in \mathbb{N}
\end{aligned}
$$

$a + ax + ax^2 + ax^3 + \cdots + ax^n$

$$\begin{aligned}
\sum_{\nu=0}^{n} ax^{\nu} =& a + ax^1 + ax^2 + ax^3 + \cdots + ax^n \\
x \sum_{\nu=0}^{n} ax^{\nu} =& \qquad x \cdot ax^0 + x \cdot ax^1 + x \cdot ax^2 + x \cdot ax^3 + \cdots + x \cdot ax^n \\
\sum_{\nu=0}^{n} ax^{\nu+1} =& \qquad ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1} \\
x \sum_{\nu=0}^{n} ax^{\nu} =& \qquad ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1} \\
\sum_{\nu=0}^{n} ax^{\nu} =& ax^0 + ax^1 + ax^2 + ax^3 + \cdots + ax^n \\
x \sum_{\nu=0}^{n} ax^{\nu} - \sum_{\nu=0}^{n} ax^{\nu} =& \left(0 - ax^0\right) + \left(ax^1 - ax^1\right) + \left(ax^2 - ax^2\right) + \left(ax^3 - ax^3\right) + \cdots + \left(ax^n - ax^n\right) + \left(ax^{n+1} - 0\right) \\
=& \left(0 - ax^0\right) + (0) + (0) + (0) + \cdots + (0) + \left(ax^{n+1}\right) \\
=& \left(-ax^0\right) + \left(ax^{n+1}\right) \\
(x-1) \sum_{\nu=0}^{n} ax^{\nu} =& ax^{n+1} - ax^0 \\
a \sum_{\nu=0}^{n} x^{\nu} = \sum_{\nu=0}^{n} ax^{\nu} =& \frac{a\left(x^{n+1} - 1\right)}{x-1} = \frac{(-1)\, a \left(x^{n+1} - 1\right)}{(-1)\,(x-1)} = \frac{a\left(1 - x^{n+1}\right)}{1-x} = \frac{a\left(1 - x^{n+1}\right)}{1-x} \qquad \forall n \in \mathbb{N} \\
a \sum_{\nu=0}^{\infty} x^{\nu} = \sum_{\nu=0}^{\infty} ax^{\nu} =& \lim_{n\to\infty} \frac{a\left(1 - x^{n+1}\right)}{1-x} = \frac{ax\,(1-0)}{1-x} = \frac{a}{1-x} = (1-x)^{-1}\, a \qquad n \to \infty, \forall\, |x| < 1 \\
(1-x)^{-1}\, a =& x^0 a + x^1 a + x^2 a + x^3 a + \cdots \qquad \forall\, |x| < 1
\end{aligned}$$

2 linear algebra

$A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots$

$$\begin{aligned}
(1-x)^{-1}\, a =& x^0 a + x^1 a + x^2 a + x^3 a + \cdots && \forall\, |x| < 1 \\
x^0 a + x^1 a + x^2 a + x^3 a + \cdots =& (1-x)^{-1}\, a && \forall\, |x| < 1 \\
A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots =& (I - A)^{-1}\, \boldsymbol{x} && \forall\, ||A|| = |\det A| < 1
\end{aligned}$$

$$\begin{cases} A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad |\det A| = \dfrac{1}{2} < 1 \\ \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x & -x \\ y & y \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
(1-x)^{-1}\,a &= x^0 a + x^1 a + x^2 a + x^3 a + \cdots && \forall\,|x|<1 \\
x^0 a + x^1 a + x^2 a + x^3 a + \cdots &= (1-x)^{-1}\,a && \forall\,|x|<1 \\
A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots &= (I-A)^{-1}\,\boldsymbol{x} && \forall\,|\det A|<1
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^0 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^1 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^2 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^3 \begin{bmatrix}1\\0\end{bmatrix} + \cdots &= \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\right)^{-1} \begin{bmatrix}1\\0\end{bmatrix} \\
&= \left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{bmatrix}\right)^{-1} \begin{bmatrix}1\\0\end{bmatrix} \\
&= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix} \\
&= \begin{bmatrix}1\\1\end{bmatrix} \\
\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^0 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^1 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^2 \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^3 \begin{bmatrix}1\\0\end{bmatrix} + \cdots &= \begin{bmatrix}1\\1\end{bmatrix}
\end{aligned}
$$

$$
A = \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}
$$

$$
A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots = (I-A)^{-1}\,\boldsymbol{x}
$$

$$
\boldsymbol{x} = \begin{bmatrix}1\\0\end{bmatrix}
$$

$$
\begin{aligned}
\left(A^0 + A^1 + A^2 + A^3 + \cdots\right)\boldsymbol{x} &= \\
\left(\sum_{\nu=0}^{\infty} A^\nu\right)\boldsymbol{x} &=
\end{aligned}
$$

3 calculus

$\exp(x) = \mathrm{e}^x$

$$\mathrm{e}^x \overset{\text{def.}}{=} \mathrm{D}\mathrm{e}^x = \mathrm{D}_x\mathrm{e}^x = \mathrm{d}_x\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} \qquad \forall x \in \mathbb{R}$$

$$\Downarrow F(x) = \int_a^x \mathrm{D}F(t) \qquad \text{FToC1}$$

$$\mathrm{e}^x = \int \mathrm{D}\mathrm{e}^x \qquad = \text{fundamental theorem of calculus first part}$$

$$\Downarrow \mathrm{e}^x \overset{\text{def.}}{=} \mathrm{D}\mathrm{e}^x$$

$$\mathrm{e}^x = \int \mathrm{e}^x$$

$$\mathrm{e}^x - \int \mathrm{e}^x = 0$$

$$\left(1 - \int\right)\mathrm{e}^x = 0$$

$$\mathrm{e}^x = \left(1 - \int\right)^{-1} 0 = \left(\sum_{\nu=0}^{\infty} \int^{\nu}\right) 0 \qquad \forall\,|\det A| < 1 \left[(I - A)^{-1}\boldsymbol{x}\right.$$

$$= \left(\int^0 + \int^1 + \int^2 + \int^3 + \cdots\right) 0 \qquad \left.= \left(\sum_{\nu=0}^{\infty} A^{\nu}\right)\boldsymbol{x}\right]$$

$$= \int^0 0 + \int^1 0 + \int^2 0 + \int^3 0 + \cdots$$

$$= 0 + \int \mathrm{d}x 0 + \iint \mathrm{d}x^2 0 + \iiint \mathrm{d}x^3 0 + \cdots$$

$$= \quad C + \int \mathrm{d}x C + \iint \mathrm{d}x^2 C + \cdots$$

$$= \quad C + Cx + \int \mathrm{d}x Cx + \cdots$$

$$= \quad C + \frac{Cx}{1} + \frac{Cx^2}{2 \cdot 1} + \cdots \qquad = C\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}$$

$$1 = \mathrm{e}^0 = \mathrm{e}^{x=0} = C + \frac{C0}{1} + \frac{C0^2}{2 \cdot 1} + \cdots = C + 0 + 0 + \cdots = C \qquad \Rightarrow C = 1$$

$$\mathrm{e}^x = 1 + \frac{1x}{1} + \frac{1x^2}{2 \cdot 1} + \cdots \qquad = 1\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}$$

$$= \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \qquad = \sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}$$

4 complex analysis

$$\mathrm{e}^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!} \quad \forall x \in \mathbb{C}, |x| < 1$$

5 matrix calculus

$$\mathrm{e}^{Ax} = \frac{1}{0!} + \frac{Ax}{1!} + \frac{A^2x^2}{2!} + \cdots = \sum_{\nu=0}^{\infty}\frac{A^{\nu}x^{\nu}}{\nu!} \quad \forall n \in \mathbb{N}, \forall Ax \in \mathcal{M}_{n\times n}, ||Ax|| = |(\det A)\,x| < 1$$

6 Lie algebra

$$\mathrm{Ad}_{\exp X}Y = \exp(\mathrm{Ad}X)\,Y = \frac{Y}{0!} + \frac{[X,Y]}{1!} + \frac{[X,[X,Y]]}{2!} + \frac{[X,[X,[X,Y]]]}{3!} + \cdots$$

「Mathematics is the art of giving the same name to different things.」

Henri Poincré (1854~1912)

「Algebra's like sheet music, the important thing isn't can you read music, it's can you hear it. Can you hear the music, Robert?」

Niels Bohr (1885~1962) $\sim$ Oppenheimer movie (2023)