

math

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2024-04-28



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# **index**

math on bookdown started on 2024/01/28



# Part I

## by discipline



# Chapter 1

## mathematics

### 1.1 tool

- formula typesetting
  - TeX
    - \* LaTeX
      - pdfTeX
      - XeTeX
      - editor/tool:
        - LyX
        - OverLeaf
        - MathPix Snip
        - Microsoft Office Word
          - WordTeX <https://tomwildenhain.com/wordtex/>
          - Pandoc dependent
          - <https://superuser.com/questions/1114697/select-a-different-math-font-in-microsoft-word>
          - [https://www.youtube.com/watch?v=jIX\\_pThh7z8](https://www.youtube.com/watch?v=jIX_pThh7z8)
        - Microsoft Office PowerPoint
          - IguanaTeX <https://www.jonathanleroux.org/software/iguanatex/>
      - MathML
      - MathJax: JavaScript
    - plot<sup>[3]</sup>
    - symbolic computing
      - Maple: by MapleSoft
      - Mathematica: by Wolfram
    - numeric computing
      - MatLab: by MathWorks

equivalence relation<sup>[11]</sup>

equivalence class<sup>[10]</sup>

partition<sup>[9]</sup>

### 1.2 discipline



# Chapter 2

## physics

### 2.1 discipline

- relativity
  - special relativity
    - \* Lorentz transformation<sup>[18]</sup>
  - general relativity
- analytic mechanics
  - Lagrangian mechanics
  - Hamiltonian mechanics
- electromagnetism
- quantum mechanics
- field theory



# Chapter 3

## plot

- LaTeX
  - [TikZ](#)<sup>[13]</sup>
    - \* <https://tikz.dev/>
    - \* TikZ-3Dplot
    - \* [PGFplots](#)<sup>[13.4]</sup>
      - <https://tikz.dev/pgfplots/>
      - <https://pgfplots.sourceforge.net/gallery.html>
      - <https://pgfplots.net/>
    - \* editor / export
      - <https://zhuanlan.zhihu.com/p/660371706>
      - offline
        - [TikzEdt](#): WYSIWYG and live preview
        - [TikZiT](#)
      - online
        - OverLeaf
        - MathCha
        - GeoGebra Classic
      - Python
        - [TikZplotLib](#) / [tikzplotlib](#)<sup>[13.5]</sup>
          - matplotlib export to TikZ .tex
          - [PyPI](#)
          - [GitHub](#)
      - R
        - [TikZDevice](#) / [tikzDevice](#)
          - r chunk engine='tikz' knitr out.width=if (knitr:::is\_html\_output()) '100%'
          - [CRAN](#)
            - [reference manual](#)
            - vignette: [TikZDevice - LaTeX Graphics for R](#)
          - [GitHub](#)
    - \* [TikZ library](#)
  - [xypic](#) = [xy-pic](#)<sup>[14]</sup>
- [OverLeaf](#)
- [MathCha](#)
- [GeoGebra](#)
  - [GeoGebra Classic](#): to export TikZ

- GeoGebra Calculator Suite
- Python
  - Matplotlib / matplotlib<sup>[27]</sup>
  - Seaborn / seaborn<sup>[27.1.3]</sup>
  - Plotly
  - Manim
- R
  - Modern Statistical Graphics
  - ggplot2<sup>[30]</sup>
    - \* Modern Statistical Graphics [section 5.1](#)
  - GraphViz .gv
  - Mermaid .mmd
    - \* **about**
    - \* JavaScript based diagramming and charting tool that renders Markdown-inspired text definitions to create and modify diagrams dynamically
  - Shiny
    - \* R Markdown Guide [section 5.1](#)
  - tool
    - \* Jamovi

neural network plot/draw <https://github.com/ashishpatel26/Tools-to-Design-or-Visualize-Architecture-of-Neural-Network>

# Chapter 4

## programming language

### 4.1 discipline

- [Python](#)<sup>[12]</sup>
- JavaScript
- SQL = structured query language
- [R](#)<sup>[19]</sup>
  - RMarkdown
    - \* Bookdown
  - knitr: engine
    - \* [TikZ](#)<sup>[13]</sup>
  - reticulate: Python
  - Jamovi
- C#
  - web
    - \* MVC
    - \* .NET
  - desktop
    - \* UWP = Universal Windows Platform
    - \* WPF = Windows Presentation Foundation
    - \* WinForms = Windows Forms
  - 3D/game
    - \* Unity

### 4.2 learning map

- [W3School](#)
- [SoloLearn](#)
- [Codecademy](#)



# Chapter 5

## machine learning

我妻幸長

### 5.1 Shai Ben-David

<https://www.youtube.com/playlist?list=PLPW2keNyw-usgymR7FTQ3ZRjfLs5jT4BO>



# **Part II**

## **by date**



# Chapter 6

## A Minimal Book Example

### 6.1 About

This is a *sample* book written in **Markdown**. You can use anything that Pandoc’s Markdown supports; for example, a math equation  $a^2 + b^2 = c^2$ .

#### 6.1.1 Usage

Each **bookdown** chapter is an .Rmd file, and each .Rmd file can contain one (and only one) chapter. A chapter *must* start with a first-level heading: # A good chapter, and can contain one (and only one) first-level heading.

Use second-level and higher headings within chapters like: ## A short section or ### An even shorter section.

The `index.Rmd` file is required, and is also your first book chapter. It will be the homepage when you render the book.

#### 6.1.2 Render book

You can render the HTML version of this example book without changing anything:

1. Find the **Build** pane in the RStudio IDE, and
2. Click on **Build Book**, then select your output format, or select “All formats” if you’d like to use multiple formats from the same book source files.

Or build the book from the R console:

```
bookdown::render_book()
```

To render this example to PDF as a `bookdown::pdf_book`, you’ll need to install XeLaTeX. You are recommended to install TinyTeX (which includes XeLaTeX): <https://yihui.org/tinytex/>.

#### 6.1.3 Preview book

As you work, you may start a local server to live preview this HTML book. This preview will update as you edit the book when you save individual .Rmd files. You can start the server in a work session by using the RStudio add-in “Preview book”, or from the R console:

```
bookdown::serve_book()
```

## 6.2 Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (#) per .Rmd file.

### 6.2.1 A section

All chapter sections start with a second-level (##) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

#### An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a {.unnumbered} or the shorter {-} at the end of the heading, like in this section.

## 6.3 Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

### 6.3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: # Hello world {#nice-label}.
  - Leave the label off if you like the automated heading generated based on your heading title: for example, # Hello world = # Hello world {#hello-world}.
  - To label an un-numbered heading, use: # Hello world {-#nice-label} or {# Hello world .unnumbered}.
2. Next, reference the labeled heading anywhere in the text using \@ref(nice-label); for example, please see Chapter 6.3.
  - If you prefer text as the link instead of a numbered reference use: [any text you want can go here](#).

### 6.3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using \@ref(fig:chunk-label) and \@ref(tab:chunk-label), respectively.

See Figure 6.1.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

Don't miss Table 6.1.

```
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

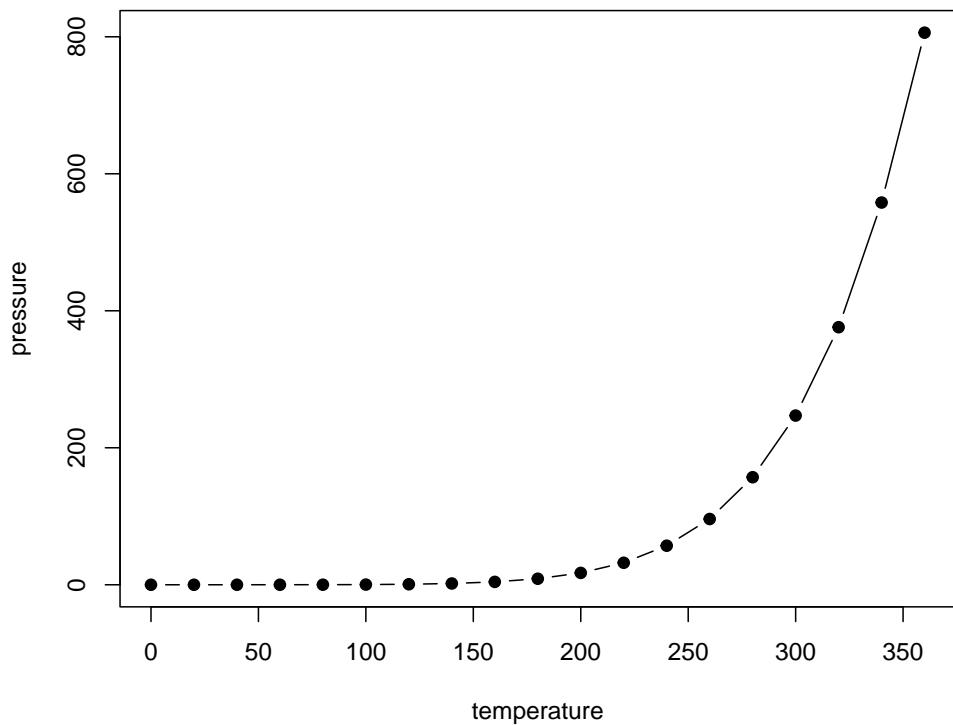


Figure 6.1: Here is a nice figure!

Table 6.1: Here is a nice table!

temperature	pressure
0	0.0002
20	0.0012
40	0.0060
60	0.0300
80	0.0900
100	0.2700
120	0.7500
140	1.8500
160	4.2000
180	8.8000

## 6.4 Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-} (followed by # A chapter)`

Add an unnumbered part: `# (PART\*) Act one {-} (followed by # A chapter)`

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-} (followed by # A chapter)`. Chapters in an appendix are prepended with letters instead of numbers.

## 6.5 Footnotes and citations

### 6.5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one <sup>1</sup>.

### 6.5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the `bookdown` package<sup>1</sup> (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and `knitr`<sup>2</sup> (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: <https://rstudio.github.io/visual-markdown-editing/#/citations>

## 6.6 Blocks

### 6.6.1 Equations

Here is an equation.

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (6.1)$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

### 6.6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

**Theorem 6.1.** *For a right triangle, if  $c$  denotes the length of the hypotenuse and  $a$  and  $b$  denote the lengths of the other two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here <https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html>.

---

<sup>1</sup>This is a footnote.

### 6.6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: <https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html>

## 6.7 Sharing your book

### 6.7.1 Publishing

HTML books can be published online, see: <https://bookdown.org/yihui/bookdown/publishing.html>

### 6.7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

### 6.7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book- all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

<https://pkgs.rstudio.com/bookdown/reference/gitbook.html>

Or use:

```
?bookdown::gitbook
```



# Chapter 7

## test

<https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html>

### 7.1 RStudio

#### 7.1.1 writer options

<https://rstudio.github.io/visual-markdown-editing/markdown.html#writer-options>

##### 7.1.1.1 line wrapping

<https://rstudio.github.io/visual-markdown-editing/markdown.html#line-wrapping>

##### 7.1.1.2 ensuring the same markdown between source / visual mode

<https://stackoverflow.com/questions/71775027/rstudio-switch-markdown-editing-mode-between-source-and-visual-changes-special>

##### canonical mode

<https://rstudio.github.io/visual-markdown-editing/markdown.html#canonical-mode>

```
---
```

```
title: "My Document"
editor_options:
  markdown:
    wrap: 72
    references:
      location: block
    canonical: true
---
```

#### 7.1.2 Rtools

Rtools43 for Windows <https://cran.r-project.org/bin/windows/Rtools/rtools43/rtools.html>

### 7.1.3 addins

```
https://github.com/rstudio/addinexamples

if (!requireNamespace("devtools", quietly = TRUE))
  install.packages("devtools")

devtools::install_github("rstudio/htmltools")
devtools::install_github("rstudio/shiny")
devtools::install_github("rstudio/minUI")
```

### 7.1.4 Git

commit: filename or extension is too long

<https://stackoverflow.com/questions/22575662/filename-too-long-in-git-for-windows>

<https://stackoverflow.com/questions/55327408/how-to-fix-git-for-windows-error-could-not-lock-config-file-c-file-path-to-g>

## 7.2 RMarkdown

R Markdown 指南 <https://cosname.github.io/rmarkdown-guide/index.html>

<https://www.rstudio.com/wp-content/uploads/2015/02/rmarkdown-cheatsheet.pdf>

<https://slides.yihui.org/2020-taipei-saturday-rmarkdown.html#1>

### 7.2.1 Pandoc link

<https://pandoc.org/chunkedhtml-demo/8.16-links-1.html>

<https://stackoverflow.com/questions/39281266/use-internal-links-in-rmarkdown-html-output>

<https://community.rstudio.com/t/how-to-hyperlink-between-different-rmd-files-in-rmarkdown/62289>

### 7.2.2 URL

<https://stackoverflow.com/questions/29787850/how-do-i-add-a-url-to-r-markdown>

[I'm an inline-style link] (<https://www.google.com>)

[I'm an inline-style link with title] (<https://www.google.com> "Google's Homepage")

[I'm a reference-style link] [Arbitrary case-insensitive reference text]

[I'm a relative reference to a repository file] (../blob/master/LICENSE)

[You can use numbers for reference-style link definitions] [1]

Or leave it empty and use the [link text itself]

Some text to show that the reference links can follow later.

[arbitrary case-insensitive reference text]: <https://www.mozilla.org>

```
[1]: http://slashdot.org
[link text itself]: http://www.reddit.com
```

### 7.2.3 arrow

<https://reimbar.org/dev/arrows/>

Up arrow: &uarr;

Down arrow: &darr;

Left arrow: &larr;

Right arrow: &rarr;

Double headed arrow: &harr;

### 7.2.4 superscript and subscript

script<sup>superscript</sup><sub>subscript</sub>

script<sup>superscript</sup><sup>^</sup>

script<sup>superscript</sup>

$\sim$ subscript $\sim$

script<sub>subscript</sub>

#### 7.2.4.1 LaTeX

<https://tex.stackexchange.com/questions/580824/subscript-not-distinguished-enough>

<https://tex.stackexchange.com/questions/262295/make-subscript-size-smaller-always>

### 7.2.5 equation

<https://stackoverflow.com/questions/26049762/erroneous-nesting-of-equation-structures-in-using-beginalign-in-a-multi-l>

#### 7.2.5.1 proof QED

<https://math.meta.stackexchange.com/questions/3582/qed-for-mathjax-here-on-stackexchange>

\tag\*{\$\Box\$}

$$a^2 + b^2 = c^2$$

□

\tag\*{\$\blacksquare\$}

$$a^2 + b^2 = c^2$$

■

### 7.2.6 image

<https://stackoverflow.com/questions/25166624/insert-picture-table-in-r-markdown>

### 7.2.6.1 DiagrammeR / mermaid flowchart

Error: Functions that produce HTML output found in document targeting latex output.  
Please change the output type of this document to HTML.

If you're aiming to have some HTML widgets shown in non-HTML format as a screenshot,  
please install webshot or webshot2 R package for knitr to do the screenshot.

Alternatively, you can allow HTML output in non-HTML formats  
by adding this option to the YAML front-matter of  
your rmarkdown file:

```
always_allow_html: true
```

Note however that the HTML output will not be visible in non-HTML formats.

<https://bookdown.org/yihui/rmarkdown-cookbook/diagrams.html#diagrams>

<https://stackoverflow.com/questions/40803017/how-to-include-diagrammer-mermaid-flowchart-in-a-rmarkdown-file>

```
{r}
library(DiagrammeR)
mermaid("
graph LR
A-->B
",
width = 100
)
```

<https://github.com/rich-iannone/DiagrammeR/issues/364>

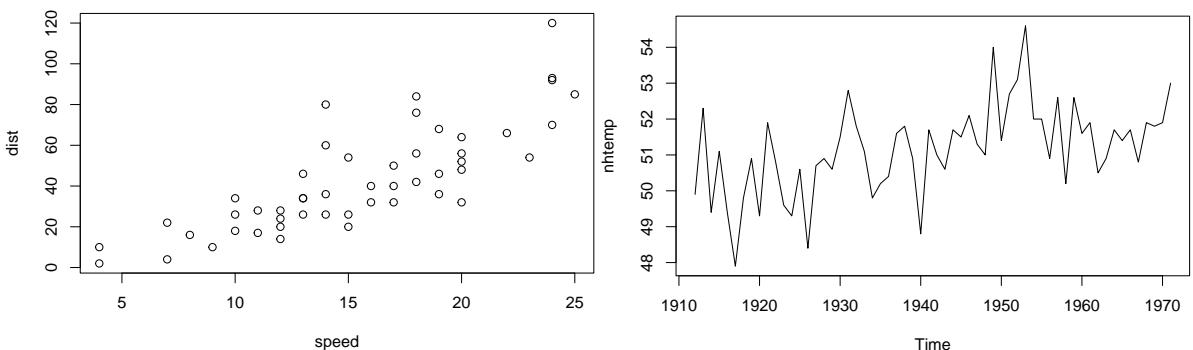
<https://stackoverflow.com/questions/55994210/how-to-solve-diagrammer-waste-of-space-issue-in-rmarkdown>

### 7.2.6.2 multiple images / figures in the same line

<https://cosname.github.io/rmarkdown-guide/rmarkdown-base.html#element-figure>

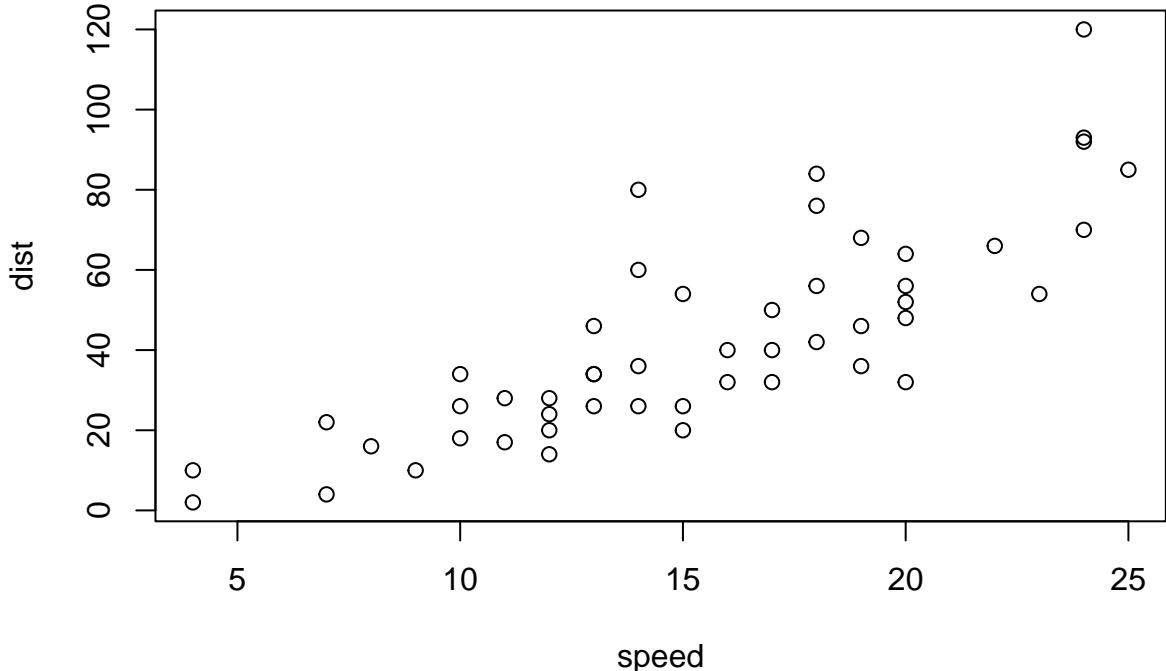
```
{r, fig.show = "hold", out.width = "50%"}
plot(cars)
plot(nhtemp)

plot(cars)
plot(nhtemp)
```

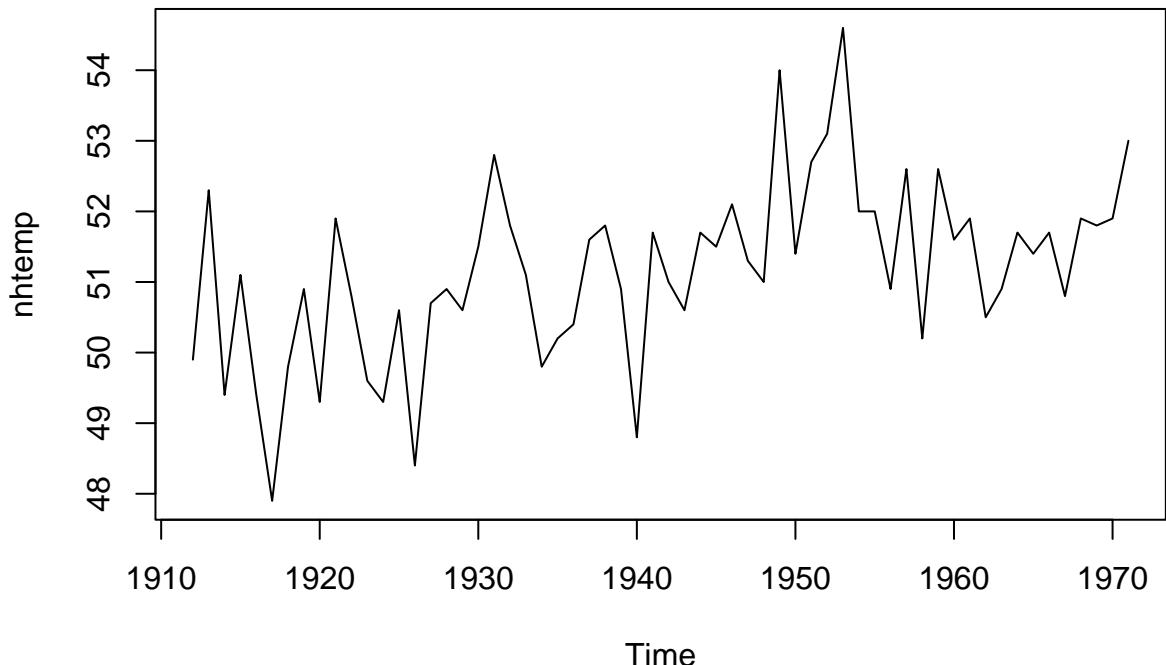


cf.

```
{r}  
plot(cars)  
plot(nhtemp)  
plot(cars)
```



```
plot(nhtemp)
```



### 7.2.6.3 figure size

[https://sebastiansauer.github.io/figure\\_sizing\\_knitr/](https://sebastiansauer.github.io/figure_sizing_knitr/)

YAML in index.Rmd

```
---
title: "My Document"
output: html_document:
fig_width: 6
fig_height: 4
---
```

first R-chunk in your RMD document

```
knitr::opts_chunk$set(fig.width=12, fig.height=8)
```

width, height and options

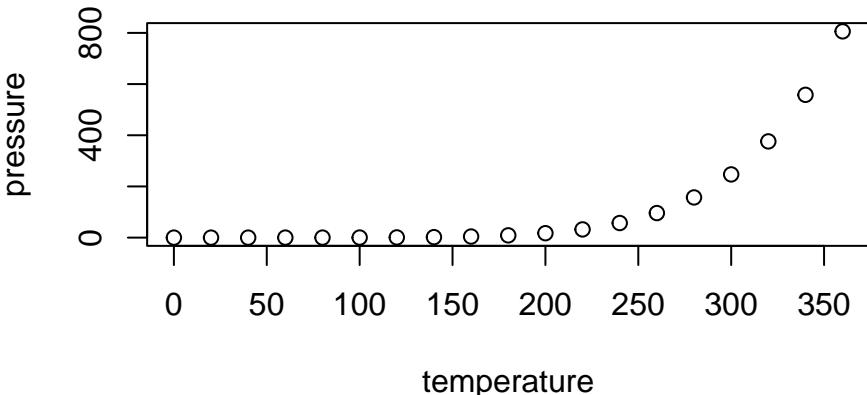
```
```{r fig.height = 3, fig.width = 5}
```

```
plot(pressure)
```

```
```
```

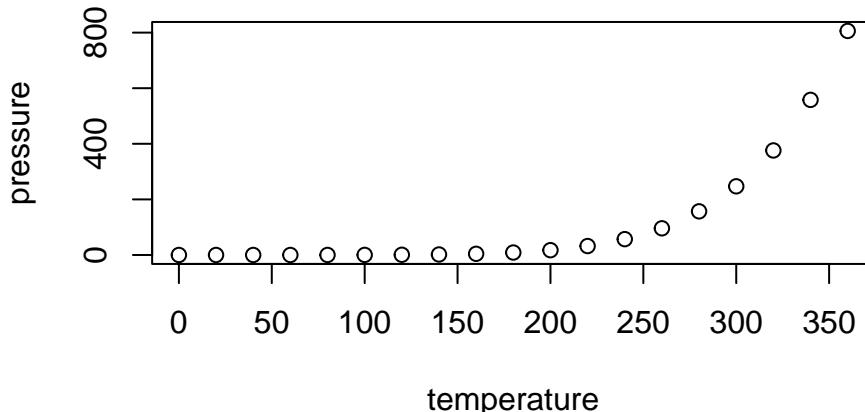
```
{r fig.height = 3, fig.width = 5}
```

```
plot(pressure)
```



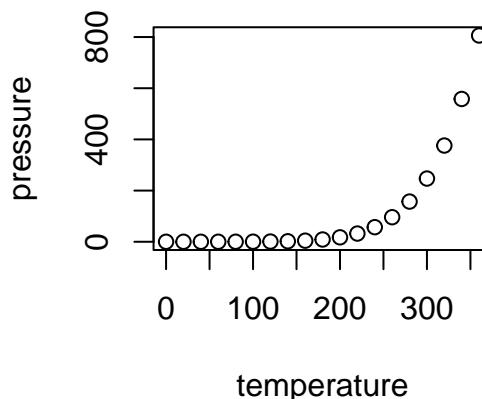
```
{r fig.height = 3, fig.width = 3, fig.align = "center"}
```

```
plot(pressure)
```



```
{r fig.width = 5, fig.asp = .62}
```

```
plot(pressure)
```

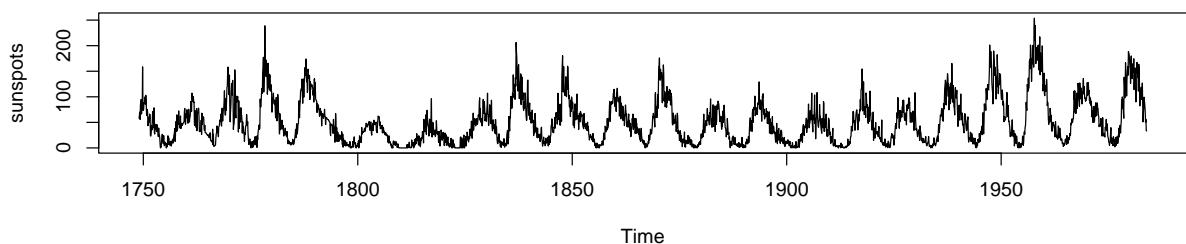


```
<center>
! [] (https://bookdown.org/yihui/rmarkdown-cookbook/images/cover.png) {width=20%}
</center>
```

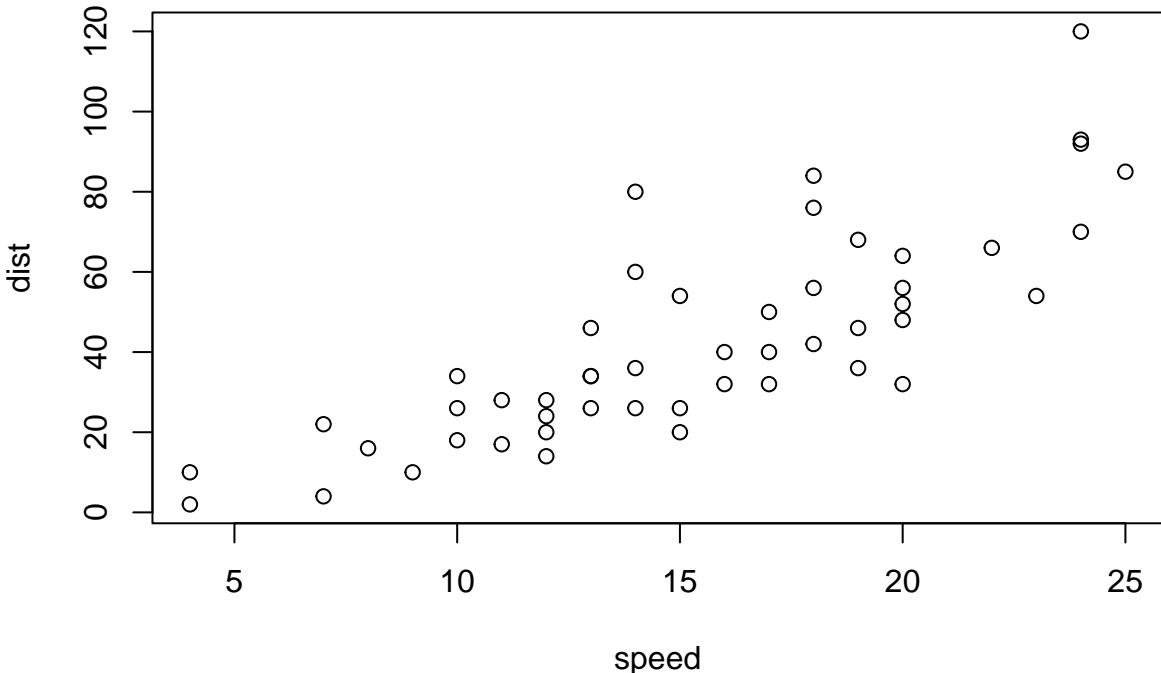
### 7.2.6.3.1 knitr <https://yihui.org/knitr/options/>

<https://bookdown.org/yihui/rmarkdown/tufte-figures.html>

```
par(mar = c(4, 4, .1, .2)); plot(sunspots)
```



```
plot(cars)
```



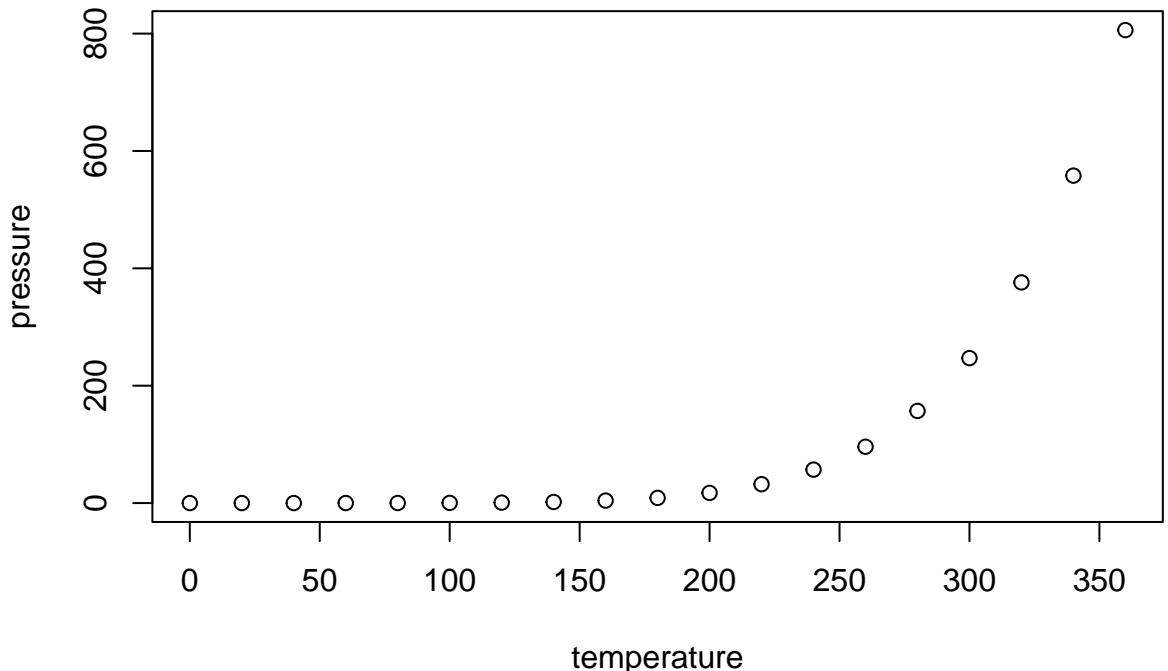
We know from the first fundamental theorem of calculus that  
for  $x$  in  $[a, b]$ :  

$$\frac{d}{dx} \left( \int_a^x f(u) du \right) = f(x).$$

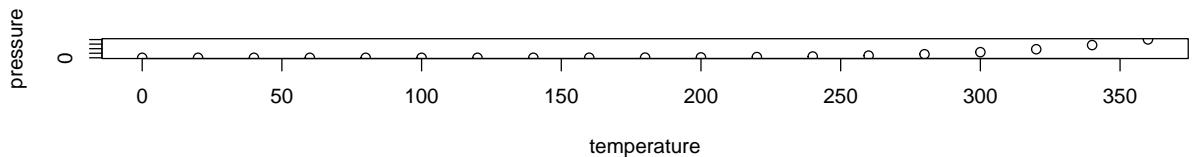
### 7.2.6.3.2 `out.width` vs. `fig.width` <https://stackoverflow.com/questions/29657777/how-to-make-fig-width-and-out-width-consistent-with-knitr>

when chunk option `cache=FALSE` is set, then `out.width` has no effect because no PDF output is created. Hence one has to specify exact measures in inches for `fig.width` and `fig.height` for each chunk

<https://stackoverflow.com/questions/59567235/a-ggmap-too-small-when-rendered-within-a-rmd-file>



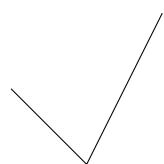
```
plot(pressure)
```



problem: `out.width='100%'` causing LaTeX Error: Not in outer par mode.

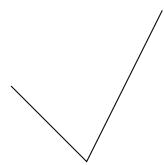
solution: `out.width=if (knitr:::is_html_output()) '100%'`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



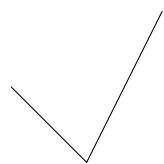
`fig.width=10, fig.height=2`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



`out.width=if (knitr:::is_html_output()) '100%'`

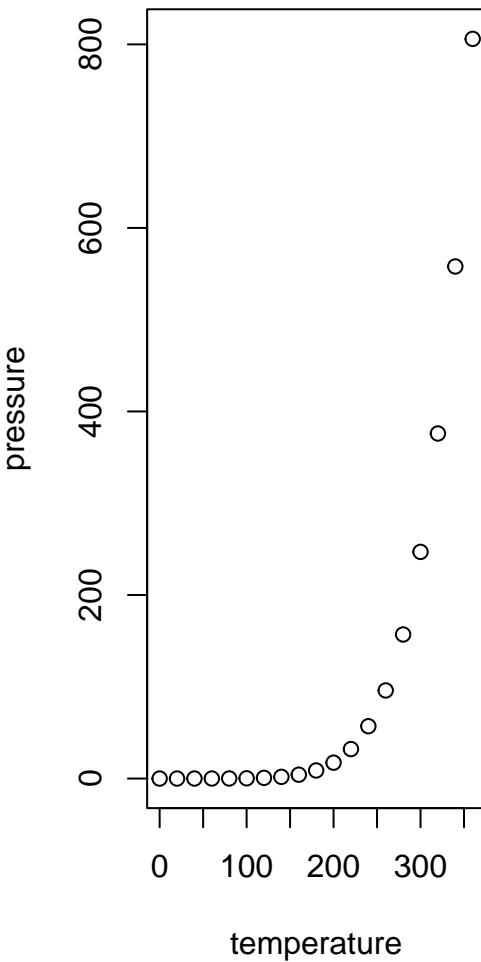
```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



#### 7.2.6.4 dynamic knitr plot width and height

<https://stackoverflow.com/questions/15365829/dynamic-height-and-width-for-knitr-plots>

```
plot(pressure)
```



#### 7.2.6.5 web image in PDF

<https://stackoverflow.com/questions/46331896/how-can-i-insert-an-image-from-internet-to-the-pdf-file-produced-by-r-bookdown-i>

```
cover_url = 'https://bookdown.org/yihui/bookdown/images/cover.jpg'
if (!file.exists(cover_file <- xfun::url_filename(cover_url)))
  xfun::download_file(cover_url)
knitr::include_graphics(if (knitr::pandoc_to('html')) cover_url else cover_file)
```

#### 7.2.6.6 SVG

<https://stackoverflow.com/questions/50165404/how-to-make-a-pdf-using-bookdown-including-svg-images>

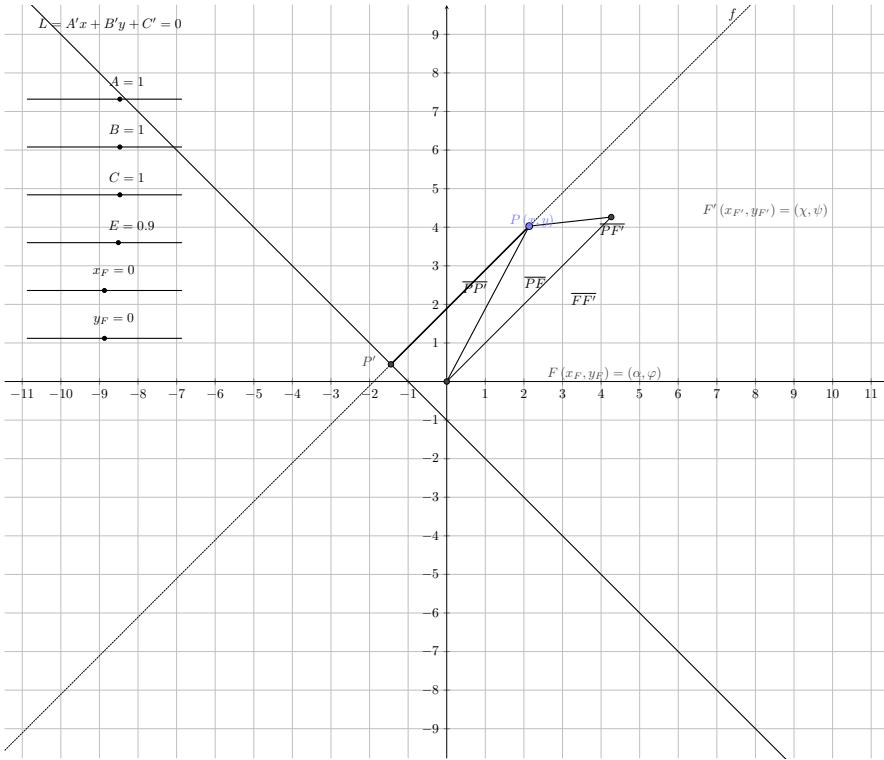
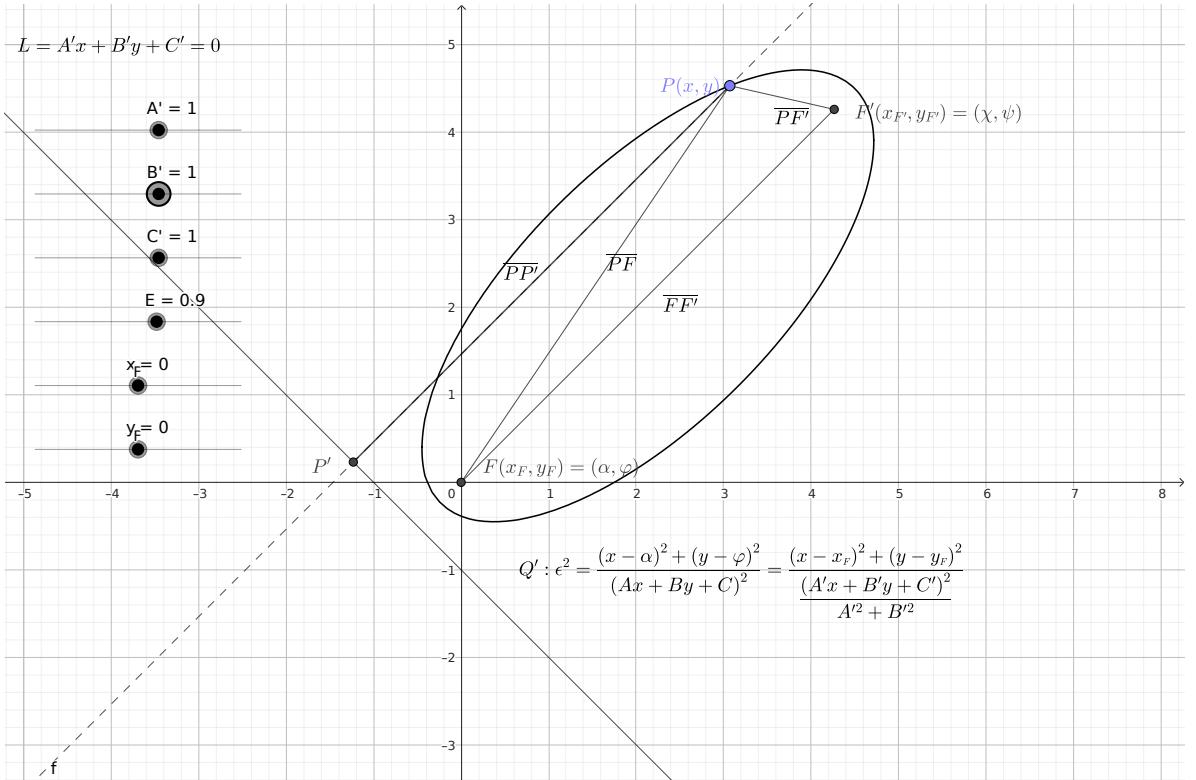


Figure 7.1: conic sections



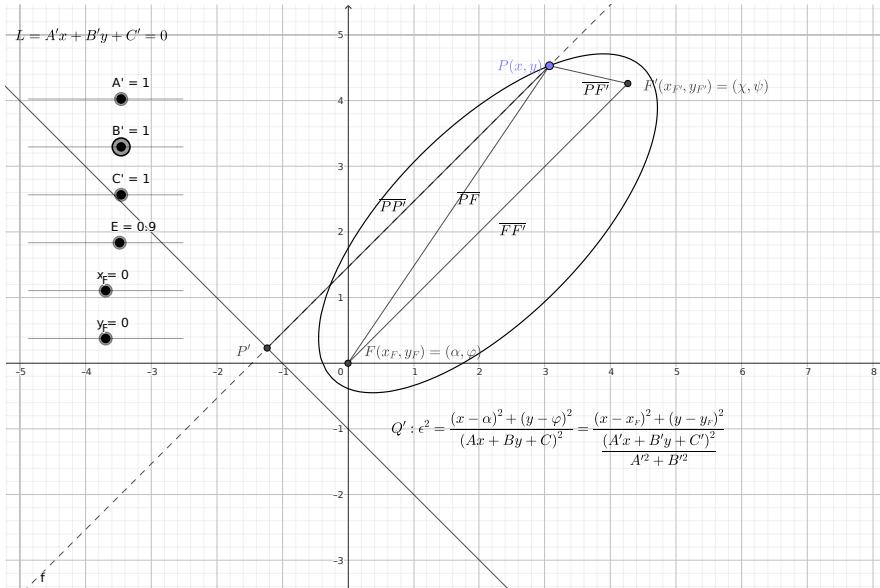


Figure 7.2: conic sections

<https://stackoverflow.com/questions/34064292/is-it-possible-to-include-svg-image-in-pdf-document-rendered-by-rmarkdown>

### 7.2.7 horizontal rule

\*\*\*

horizontal rule (or slide break)

---

```
dim(iris)
```

```
## [1] 150    5
```

### 7.2.8 footnote

### 7.2.9 hyperlink

PDF pandoc internal link will lose focus

equivalence relation [11] equivalence relation<sup>1</sup> equivalence relation<sup>[11]</sup>

equivalence class [10] equivalence class<sup>2</sup> equivalence class<sup>[10]</sup>

partition [9] partition<sup>3</sup> partition<sup>[9]</sup>

- LaTeX
  - TikZ<sup>[13]</sup>
    - \* TikZ-3Dplot

<sup>1</sup>{11} equivalence relation

<sup>2</sup>{10} equivalence class

<sup>3</sup>{9} partition

- \* PGFplots
  - xypic = **xy-pic**<sup>4</sup>
- OverLeaf
- MathCha
- GeoGebra
- Python
  - Matplotlib
  - Seaborn
  - Plotly

## 7.2.10 code chunk

### 7.2.10.1 code folding

<https://cosname.github.io/rmarkdown-guide/rmarkdown-document.html#html-code-folding>

## 7.2.11 xaringan

slide realtime preview with RStudio addin Infinite Moon Reader in RStudio viewer

<https://github.com/yihui/xaringan>

<https://www.youtube.com/watch?v=3n9nASHg9gc>

## 7.3 Bookdown

### 7.3.1 system locale

<https://bookdown.org/tpemartin/ntpu-programming-for-data-science/appendix-d-.html>

`Sys.getlocale()`

Windows

`Sys.setlocale(category = "LC_ALL", locale = "UTF-8")`

MacOS

`Sys.setlocale(category = "LC_ALL", locale = "en_US.UTF-8")`

<https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html>

### 7.3.2 render\_book()

<https://bookdown.org/yihui/bookdown/build-the-book.html>

```
render_book(input = ".", output_format = NULL, ..., clean = TRUE,
  envir = parent.frame(), clean_envir = !interactive(),
  output_dir = NULL, new_session = NA, preview = FALSE,
  config_file = "_bookdown.yml")
```

### 7.3.3 serve\_book()

<https://bookdown.org/yihui/bookdown/serve-the-book.html>

---

<sup>4</sup>{14} **xy-pic**

```
serve_book(dir = ".", output_dir = "_book", preview = TRUE,
  in_session = TRUE, quiet = FALSE, ...)
```

## 7.3.4 LaTeX

### 7.3.4.1 hyperlink, URL, href

<https://www.baeldung.com/cs/latex-hyperref-url-hyperlinks>

<https://www.omidte.com/小技巧讓facebook和line顯示中文網址，網址不再變亂碼/>

### 7.3.4.2 ugly mathptmx $\sum$

PDF LaTeX \usepackage{fdsymbol} to have \overrightharpoon vector; however, there are too many side effects, including ugly mathptmx  $\sum$ , ...

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMylargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMylargesymbols}{50}
```

<https://tex.stackexchange.com/questions/315102/different-sum-signs>

<https://tex.stackexchange.com/questions/275038/how-to-replace-mathptmx-sum-with-cm-sum>

<https://tex.stackexchange.com/questions/391410/calligraphic-symbols-are-too-fancy-with-mathptmx-package>

<https://blog.csdn.net/kongtaoxing/article/details/131005044>

In `preamble.tex`, add

```
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMylargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMylargesymbols}{50}

\DeclareMathAlphabet{\mathcal}{OMS}{cmsy}{m}{n}
\DeclareSymbolFont{largetsymbol}{OMX}{cmex}{m}{n}
```

### 7.3.4.3 LaTeX package in HTML document

<https://github.com/rstudio/rmarkdown/issues/1829>

```
---
title: "assignment"
author: "author"
output: html_document
---

$$
\require{cancel}
\cancel{x}
$$
```

†

<https://stackoverflow.com/questions/18189175/how-to-use-textup-with-mathjax>

\textup is not available in MathJax. You can replace it with \mathrm, but \mathrm does not interpret spaces.

### 7.3.5 depth of table of contents toc\_depth

<https://stackoverflow.com/questions/49009212/how-to-change-toc-depth-in-r-bookdown-gitbook>

```
bookdown::gitbook:
  toc_depth: 2
```

<https://stackoverflow.com/questions/68537309/how-can-i-specific-the-initial-level-to-have-my-table-of-contents-be-expanded-to>

```
toc:
  collapse: section
```

### 7.3.6 multi-column layout / two columns

<https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html>

#### 7.3.6.1 for both HTML and PDF

figure size<sup>[7.2.6.3]</sup>

Below is a Div containing three child Divs side by side. The Div in the middle is empty, just to add more space between the left and right Divs.

```
::::::::::: {.cols data-latex=""}

::: {.col data-latex="{0.55\textwidth}"}
<!-- -->
:::

::: {.col data-latex="{0.05\textwidth}"}
\
<!-- an empty Div (with a white space), serving as
a column separator -->
:::

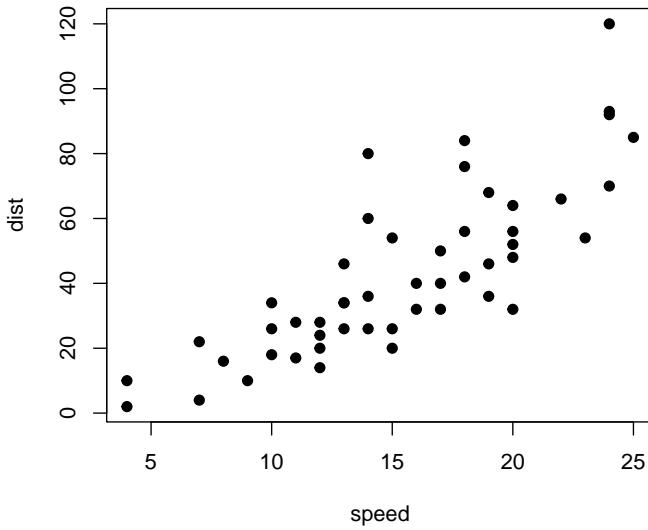
::: {.col data-latex="{0.4\textwidth}"}

The figure on the left-hand side shows the `cars` data.
```

  Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

```
:::
::::::::::
```

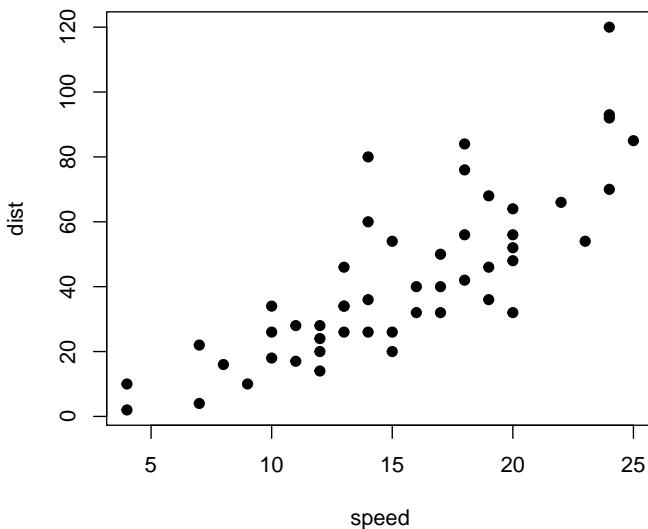
```
{r, echo=FALSE, fig.width=5, fig.height=4}
```



The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

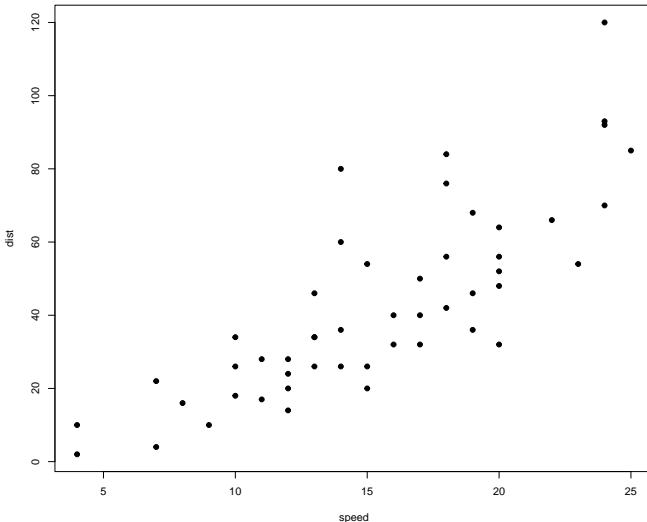
```
{r, echo=FALSE, fig.width=10, fig.height=8, out.width = "100%"}
```



The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

```
{r, echo=FALSE, fig.width=10, fig.height=8}
```



The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

### 7.3.6.2 multi-column `fig.cap` must use `fig.pos="H"`

<https://community.rstudio.com/t/adding-fig-cap-caption-text-to-code-chunk-causes-figure-to-print-at-top-of-page-instead-of-where-it-should-be/30297>

<https://bookdown.org/yihui/rmarkdown-cookbook/figure-placement.html>

to avoid `\LaTeX` Error: Not in outer `par` mode for caption in multi-column `\LaTeX` PDF

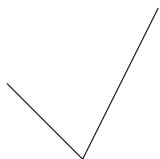
in `output.yml` add `extra_dependencies: ["float"]` under `bookdown::pdf_book`:

include first chunk `knitr::opts_chunk$set(fig.pos = "H", out.extra = "")` in `.Rmd`

add `out.width=if (knitr:::is_html_output()) '50%` for TikZ chunk

thus complete chunk beginning with `{r, echo=FALSE, cache=TRUE, engine='tikz', fig.ext=if (knitr:::is_latex_output()) 'pdf' else 'png', fig.width=10, fig.height=2, out.width=if (knitr:::is_html_output()) '100%', fig.cap='')}`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



`\LaTeX` package `caption`

<https://tex.stackexchange.com/questions/128485/how-to-make-a-caption-via-captionof-and-extra-margins-adhere-to-minipage-marg>

What is the different between using `\captionof{table}{ABC}` and `\caption{ABC}`?

<https://tex.stackexchange.com/questions/514286/what-is-the-different-between-using-captionoftable-abc-and-captionabc>

side-by-side table

<https://stackoverflow.com/questions/73745714/how-to-print-gt-tbl-tables-side-by-side-with-knitr-kable>

`R` ternary operator

<https://stackoverflow.com/questions/8790143/does-the-ternary-operator-exist-in-r>

### 7.3.6.3 caption above figure

<https://stackoverflow.com/questions/56979022/caption-above-figure-in-html-rmarkdown>

```
fig.topcaption=TRUE
```

### 7.3.6.4 for only HTML

#### 7.3.6.4.1 CSS flex

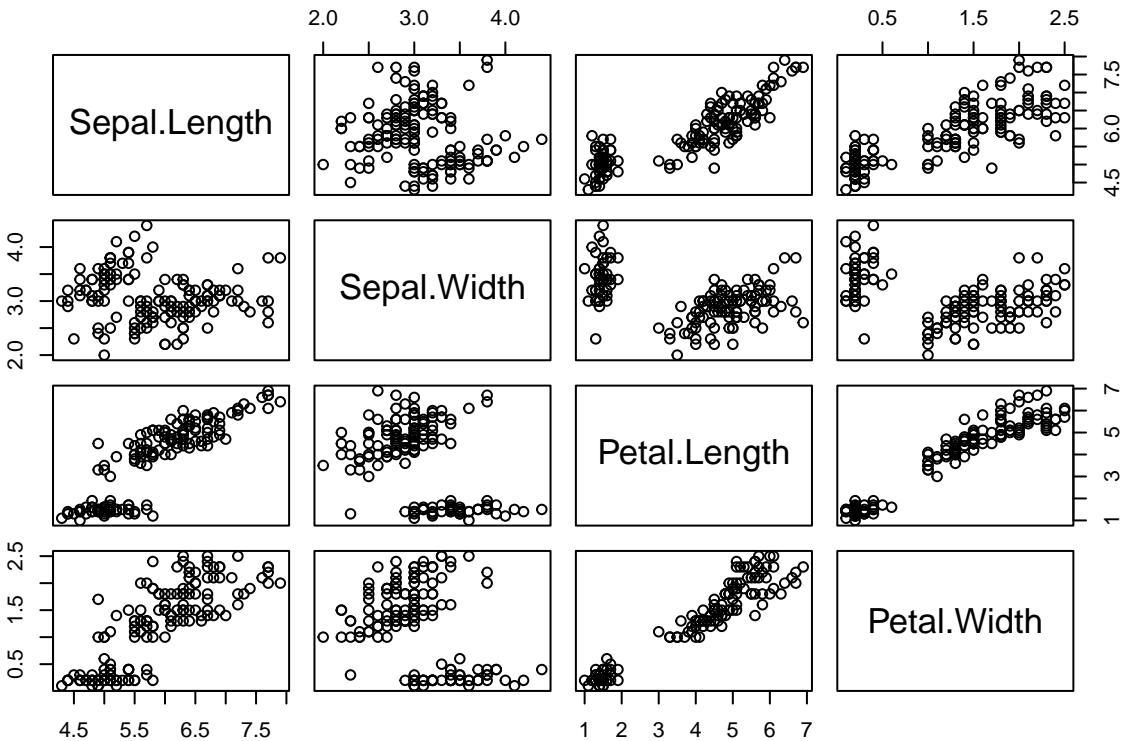
Here is the first Div.

```
str(iris)
```

```
## 'data.frame': 150 obs. of 5 variables:
## $ Sepal.Length: num 5.1 4.9 4.7 4.6 5 5.4 4.6 5 4.4 4.9 ...
## $ Sepal.Width : num 3.5 3 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 ...
## $ Petal.Length: num 1.4 1.4 1.3 1.5 1.4 1.7 1.4 1.5 1.4 1.5 ...
## $ Petal.Width : num 0.2 0.2 0.2 0.2 0.2 0.4 0.3 0.2 0.2 0.1 ...
## $ Species      : Factor w/ 3 levels "setosa","versicolor",...: 1 1 1 1 1 1 1 1 1 1 ...
```

And this block will be put on the right:

```
plot(iris[, -5])
```



#### 7.3.6.4.2 CSS grid

<https://github.com/yihui/knitr/issues/1743>

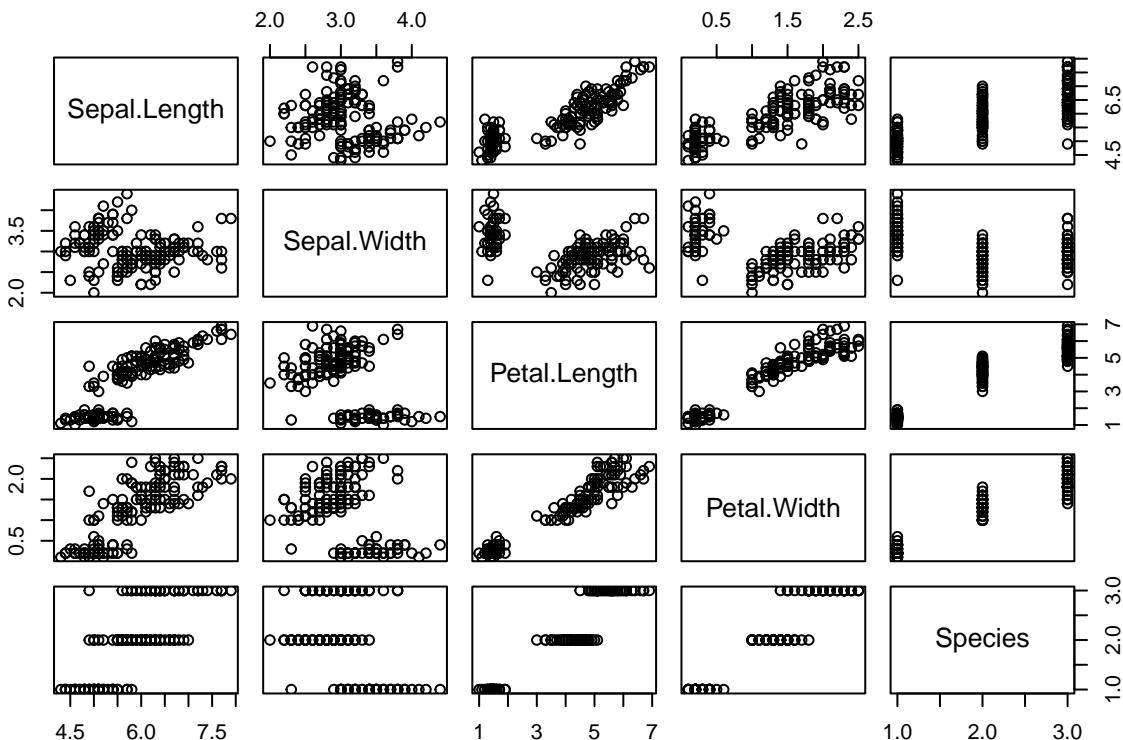
<https://medium.com/enjoy-life-enjoy-coding/css-所以我說那個版能不能好切一點-grid-基本用法-cd763091cf70>

```
head(iris)
```

```
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species
```

```
## 1      5.1     3.5     1.4     0.2   setosa
## 2      4.9     3.0     1.4     0.2   setosa
## 3      4.7     3.2     1.3     0.2   setosa
## 4      4.6     3.1     1.5     0.2   setosa
## 5      5.0     3.6     1.4     0.2   setosa
## 6      5.4     3.9     1.7     0.4   setosa
```

```
plot(iris)
```



## 7.4 conditional block/chunk for either HTML or PDF, and Chinese issue

<https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html>

等價關係 equivalence relation

$R$  is an equivalence relation over  $A \times B$

$$\Leftrightarrow \left\{ \begin{array}{l} R = \sim = \{ \langle x, y \rangle | x \sim y \} \subseteq A \times B \\ \vdots \end{array} \right. \quad (\text{e}) \text{ equivalence 等價}$$

$$\Leftrightarrow \left\{ \begin{array}{ll} R = \{ \langle x, y \rangle | xRy \} \subseteq A \times B & (R) \text{ relation} \\ \forall \langle x, y \rangle \in R (xRx) & (r) \text{ reflexive} \\ \forall \langle x, y \rangle \in R (xRy \Rightarrow yRx) & (s) \text{ symmetric} \\ \forall \langle x, y \rangle, \langle y, z \rangle \in R \left( \begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (t) \text{ transitive} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} R = \{ \langle x, y \rangle | xRy \} \subseteq A \times B \\ \forall \langle x, y \rangle \in R (\langle x, x \rangle \in R) \\ \forall \langle x, y \rangle \in R (\langle y, x \rangle \in R) \\ \forall \langle x, y \rangle, \langle y, z \rangle \in R (\langle x, z \rangle \in R) \end{array} \right. \quad \begin{array}{l} \text{關係} \\ \text{自反} \\ \text{對稱} \\ \text{遞移} \end{array}$$

## 7.5 video embedding

<https://stackoverflow.com/questions/42543206/r-markdown-compile-error>

```
always_allow_html: true
```

```
install.packages("webshot")
webshot::install_phantomjs()
```

however webshot not work

Error: cannot find bilibili.com

<https://cran.r-project.org/web/packages/vembedr/vignettes/embed.html>

```
## embed_youtube("OLFg5dvP0oc")
```

### 7.5.1 timestamp

- YouTube: <https://www.youtube.com/embed/%7BvideoID%7D?start=%7Bsecond%7D>
- BiliBili: <https://player.bilibili.com/player.html?bvid=%7BvideoID%7D&autoplay=0&t=%7Bsecond%7D>

## 7.6 equation term coloring

### 7.6.1 font color

RegEx replacement in RStudio for {\color{(\w+)} in LyX to be replaced with \color{\$1}{ in HTML document, and remain the same for PDF document

In HTML document, if no {} for text range, only the first following term will take effect

```
\color{orange}x=y
```

*x = y*

\color{orange} and \color{cyan} are better color for HTML GitBook White and Night themes and PDF

```
\color{cyan}{x=y}
```

*x = y*

```
\color{cyan}{x=y}
```

*x = y*

```
::: {show-in="html"}
```

```
$$
\frac{\colorbox{#FFD1DC}{$\epsilon^2 \left( y_{\{\{\scriptscriptstyle F\}}}-y_{\{\{\scriptscriptstyle L\}}}\right)^2}}{1-\epsilon^2}
$$
```

```
:::  
::: {show-in="pdf"}  
  
$$  
\frac{\colorbox{red!50}{\text{\ensuremath{\epsilon^2 \left(y_{\{\scriptscriptstyle L\}} - y_{\{\scriptscriptstyle F\}}\right)^2}}}}{1-\epsilon^2}  
$$  
:::
```

## 7.6.2 background color

<https://bookdown.org/yihui/rmarkdown-cookbook/font-color.html>

LaTex color

<https://latexcolor.com/>

[https://www.overleaf.com/learn/latex/Using\\_colors\\_in\\_LaTeX](https://www.overleaf.com/learn/latex/Using_colors_in_LaTeX)

<https://latex-tutorial.com/color-latex/#:~:text=To%20summarize%2C%20pyellow!50efined%20colors%20in,w>

LaTex color methods

color frame

<https://tex.stackexchange.com/questions/582748/highlight-equation-with-boxes-and-arrows>

color box

<https://tex.stackexchange.com/questions/567739/how-to-move-and-size-colorbox>

color box with round corners

<https://tex.stackexchange.com/questions/568880/color-box-with-rounded-corners>

highlighting

<https://tex.stackexchange.com/questions/318991/highlighting-math>

<https://forum.remnote.io/t/highlighting-latex-formulas/149>

LyX

<https://tex.stackexchange.com/questions/250069/create-a-color-box> <https://latexlyx.blogspot.com/2013/12/lyx.html>

<https://tex.stackexchange.com/questions/635486/prevent-lyx-from-escaping-math-in-color-box-title>

Bookdown - conditional display of text and code blocks (LaTeX/PDF vs. HTML) <https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html>

$$F = ma$$

<https://community.rstudio.com/t/highlighting-text-inline-in-rmarkdown-or-bookdown-pdf/35118/4>

$$F = ma$$

$$F = F$$

$$F = ma$$

(7.1)

$$F = ma$$

$$Y = \beta_0 + \beta_1 X_1 + \dots + \beta_n X_n$$

## 7.7 link and reference

<https://stackoverflow.com/questions/57469501/cross-referencing-bookdownhtml-document2-not-working>

$$E = mc^2 \quad (7.2)$$

```
\@ref(nice-label) 7.8
[link to partition] [partition] link to partition
[partition] \@ref(partition)
partition [#partition] (9) @ref(#partition)
[equivalence class] \@ref(equivalence-class)
equivalence class (10)
equivalence class [#equivalence class] (@ref(equivalence class)) @ref(#equivalence class)
[equivalence-class] [#equivalence-class] (10) @ref(#equivalence-class)
X [equivalence-class.html] [equivalence-class.html#equivalence-class] (@ref(equivalence-class.html))
@ref(equivalence-class.html#equivalence-class)
equivalence relation [#equivalence relation] (@ref(equivalence relation)) @ref(#equivalence relation)
[equivalence-relation] [#equivalence-relation] (11) @ref(#equivalence-relation)
X [equivalence-relation.html] [equivalence-relation.html#equivalence-relation] (@ref(equivalence-relation.html))
@ref(equivalence-relation.html#equivalence-relation)
```

## 7.8 number and reference equations

<https://stackoverflow.com/questions/71595882/rstudio-error-in-windows-running-pdflatex-exe-on-file-name-tex-exit-code-10>

[\#eq:emc \@ref\(eq:emc\)](https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#equations)

[\#eq:emc \@ref\(eq:emc\)](https://stackoverflow.com/questions/55923290/consistent-math-equation-numbering-in-bookdown-across-pdf-docx-html-output)

$C$  is an equivalence class of  $a$  on  $A$

$$\Leftrightarrow [a]_\sim = C = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{array} \right. \right\} \subseteq A \neq \emptyset \quad (7.3)$$

$$\Leftrightarrow [a] = [a]_\sim = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{array} \right. \right\} \subseteq A \neq \emptyset$$

$$\Rightarrow [a]_\sim = \{x | x \sim a\} \subseteq A \neq \emptyset$$

<https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#cross-referencing>

This cross reference is the Fig. 7.4

<https://stackoverflow.com/questions/51595939/bookdown-cross-reference-figure-in-another-file>

I ran into the same issue and came up with this solution if you aim at compiling 2 different pdfs. It relies on LaTeX's xr package for cross references: <https://stackoverflow.com/a/52532269/576684>

## 7.9 footnote

```
noun^ [This is a footnote]
noun[^202401260000-test-cross-link-1]
[^202401260000-test-cross-link-1]: This is a footnote.
```

noun<sup>5</sup>

## 7.10 citation

<https://stackoverflow.com/questions/48965247/use-csl-file-for-pdf-output-in-bookdown/49145699#49145699>

citation 1<sup>3</sup> citation 2<sup>3</sup>

citation 3<sup>4</sup> citation 4<sup>4</sup>

---

<sup>5</sup>This is a footnote.

### 7.10.1 citation in fig.cap

<https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-rmarkdown>

```
(ref:rudolph) *nice* cite: [@Lam94] .
(ref:campbell1963) *nice* cite: [@campbell1963] .
(ref:campbell1963) ([@campbell1963]
(ref:campbell1963) \ [ @campbell1963]
```

	Sources of Invalidity								External			
	Internal	External							Interaction of Testing and X	Interaction of Selection and X	Reactive Arrangements	Multiple-X Interference
	History	Maturity	Testing	Instrumentation	Regression	Selection	Mortality	Interaction of Selection and Maturation, etc.	Interaction of Testing and X	Interaction of Selection and X	Reactive Arrangements	Multiple-X Interference
<i>Pre-Experimental Designs:</i>												
1. One-Shot Case Study	X	O	—	—	—	—	—	—	—	—	—	—
2. One-Group Pretest-Posttest Design	O	X	O	—	—	?	+	+	—	—	—	?
3. Static-Group Comparison	X	O	O	+	?	+	+	+	—	—	—	—
<i>True Experimental Designs:</i>												
4. Pretest-Posttest Control Group Design	R	O	X	O	R	O	R	O	+	+	?	?
5. Solomon Four-Group Design	R	O	X	O	R	O	R	X	O	+	?	?
6. Posttest-Only Control Group Design	R	R	X	O	R	R	R	R	+	+	?	?

Figure 7.3: pre- and true experimental designs ( <sup>5</sup> p.8)

### 7.10.2 backreference

<https://community.rstudio.com/t/how-to-create-a-backreference-to-place-of-citation-in-rmarkdown/84866>

<https://blog.csdn.net/RobertChenGuangzhi/article/details/50455429>

<https://latex.org/forum/viewtopic.php?t=3722>

## 7.11 environment for definition, theorem, proof

<https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html>

<https://github.com/rstudio/rstudio/issues/5264>

@howthebodyworks Ideally, previews of such equations should also work inside a theorem, although I could survive without that.

<https://github.com/rstudio/rstudio/issues/8773>

**Theorem 7.1** (Theorem Name). *Here is my theorem.*

*Proof Name.* Here is my proof. □

**Theorem 7.2** (Pythagorean theorem). *For a right triangle, if  $c$  denotes the length of the hypotenuse and  $a$  and  $b$  denote the lengths of the other two sides, we have*

$$a^2 + b^2 \stackrel{7.2}{=} c^2$$

**Definition 7.1** (Definition Name). Here is my definition.

number and reference equations

(7.3)

(7.2)

7.2

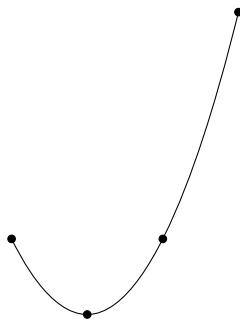


Figure 7.4: parabola arc with points

## 7.12 slide or presentation

### 7.12.1 Xaringan and Infinite Moon Reader

<https://rpubs.com/RW1304/xaringan-zh>

<https://slides.yihui.org/xaringan/#1>

<https://slides.yihui.org/xaringan/zh-CN.html#1>

<https://github.com/yihui/xaringan/tree/master>

<https://bookdown.org/yihui/rmarkdown/xaringan.html>

### 7.12.2 ioslides

<https://www.youtube.com/watch?v=gkyjTcPCTIM>

<https://bookdown.org/yihui/rmarkdown/ioslides-presentation.html>

<https://stackoverflow.com/questions/63749683/how-to-set-up-theorem-environment-in-the-rmarkdown-presentation>

```
---
```

```
title: "Theorem demo"
output:
  ioslides_presentation:
    css: style.css
---
```

```
/* theorem environment _ plain */

/*
.theorem {
  display: block;
  font-style: italic;
  font-size: 24px;
  font-family: "Times New Roman";
  color: black;
}
.theorem::before {
  content: "Theorem. ";
  font-weight: bold;
  font-style: normal;
}
.theorem[text]::before {
  content: "Theorem (" attr(text) ") ";
}
.theorem p {
  display: inline;
}
*/
/* theorem environment _ Copenhagen style */

/*
.theorem {
  display: block;
  font-style: italic;
  font-size: 24px;
  font-family: "Times New Roman";
  color: black;
  border-radius: 10px;
  background-color: rgb(222,222,231);
  box-shadow: 5px 10px 8px #888888;
}
.theorem::before {
  content: "Theorem. ";
  font-weight: bold;
  font-style: normal;
  display: inline-block;
  width: -webkit-fill-available;
  color: white;
```

```
border-radius: 10px 10px 0 0;  
padding: 10px 5px 5px 15px;  
background-color: rgb(38, 38, 134);  
}  
.theorem p {  
    padding: 15px 15px 15px 15px;  
}  
*/
```

### 7.12.3 PowerPoint

<https://bookdown.org/yihui/rmarkdown/powerpoint-presentation.html>



# Chapter 8

## test2

### 8.1 verbatim

<https://community.rstudio.com/t/continued-issues-with-new-verbatim-in-rstudio/139737>

<https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html>

```
```r  
1 + 1  
## [1] 2
```

We can output arbitrary content **\*\*verbatim\*\***.

```
```r  
1 + 1  
## [1] 2
```

The content can contain inline code like  
78.5398163, too.



# Chapter 9

## partition

$$\begin{aligned} \{A_i\}_{i \in I} = \{A_i | i \in I\} \text{ is a partition of a set } A \\ \Leftrightarrow \begin{cases} \forall i \in I (A_i \neq \emptyset) \\ A = \bigcup_{i \in I} A_i \\ \forall i, j \in I (i \neq j \Rightarrow A_i \cap A_j = \emptyset) \end{cases} \end{aligned}$$

[https://proofwiki.org/wiki/Definition:Set\\_Partition](https://proofwiki.org/wiki/Definition:Set_Partition)



# Chapter 10

## equivalence class

$C$  is an equivalence class of  $a$  on  $A$

$$\Leftrightarrow [a]_{\sim} = C = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{array} \right. \right\} \subseteq A \neq \emptyset$$
$$\Leftrightarrow [a] = [a]_{\sim} = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{array} \right. \right\} \subseteq A \neq \emptyset$$
$$\Rightarrow [a]_{\sim} = \{x | x \sim a\} \subseteq A \neq \emptyset$$

where the definition of **equivalence relation** can be found in 11.



# Chapter 11

## equivalence relation

等價關係 equivalence relation

$R$  is an equivalence relation over  $A \times B$

$$\Leftrightarrow \left\{ \begin{array}{l} R = \sim = \{\langle x, y \rangle | x \sim y\} \subseteq A \times B \\ \vdots \end{array} \right. \quad (\text{e}) \text{ equivalence 等價}$$

$$\Leftrightarrow \left\{ \begin{array}{ll} R = \{\langle x, y \rangle | xRy\} \subseteq A \times B & (R) \text{ relation} \\ \forall \langle x, y \rangle \in R (xRx) & (r) \text{ reflexive} \\ \forall \langle x, y \rangle \in R (xRy \Rightarrow yRx) & (s) \text{ symmetric} \\ \forall \langle x, y \rangle, \langle y, z \rangle \in R \left( \begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (t) \text{ transitive} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} R = \{\langle x, y \rangle | xRy\} \subseteq A \times B & \text{關係} \\ \forall \langle x, y \rangle \in R (\langle x, x \rangle \in R) & \text{自反} \\ \forall \langle x, y \rangle \in R (\langle y, x \rangle \in R) & \text{對稱} \\ \forall \langle x, y \rangle, \langle y, z \rangle \in R (\langle x, z \rangle \in R) & \text{遞移} \end{array} \right.$$



# Chapter 12

## Python

### 12.1 using Python in R / RMarkdown

<https://bookdown.org/yihui/rmarkdown/language-engines.html>

```
names(knitr::knit_engines$get())
```

```
## [1] "awk"          "bash"         "coffee"        "gawk"         "groovy"
## [6] "haskell"      "lein"         "mysql"        "node"         "octave"
## [11] "perl"         "php"          "pgsql"        "Rscript"      "ruby"
## [16] "sas"          "scala"        "sed"          "sh"          "stata"
## [21] "zsh"          "asis"         "asy"          "block"        "block2"
## [26] "bslisp"       "c"            "cat"          "cc"          "comment"
## [31] "css"          "dittaa"       "dot"          "embed"       "eviews"
## [36] "exec"         "fortran"      "fortran95"    "go"          "highlight"
## [41] "js"           "julia"        "python"       "R"           "Rcpp"
## [46] "sass"         "scss"         "sql"          "stan"        "targets"
## [51] "tikz"         "verbatim"     "theorem"      "lemma"       "corollary"
## [56] "proposition" "conjecture"   "definition"   "example"     "exercise"
## [61] "hypothesis"  "proof"        "remark"       "solution"
```

[https://rstudio.github.io/reticulate/articles/python\\_packages.html](https://rstudio.github.io/reticulate/articles/python_packages.html)

```
x = 'hello, python world!'
print(x.split(' '))
```

```
## ['hello,', 'python', 'world!']
```

```
library(reticulate)
virtualenv_python()
```

```
library(reticulate)
# conda_list()
```

```
library(reticulate)
virtualenv_list()
```

[https://rstudio.github.io/reticulate/reference/install\\_python.html](https://rstudio.github.io/reticulate/reference/install_python.html)

```
library(reticulate)
version <- "3.9.12"
# install_python(version)

## create a new environment
# virtualenv_create("r-reticulate", version = version)

# use_virtualenv("r-reticulate")

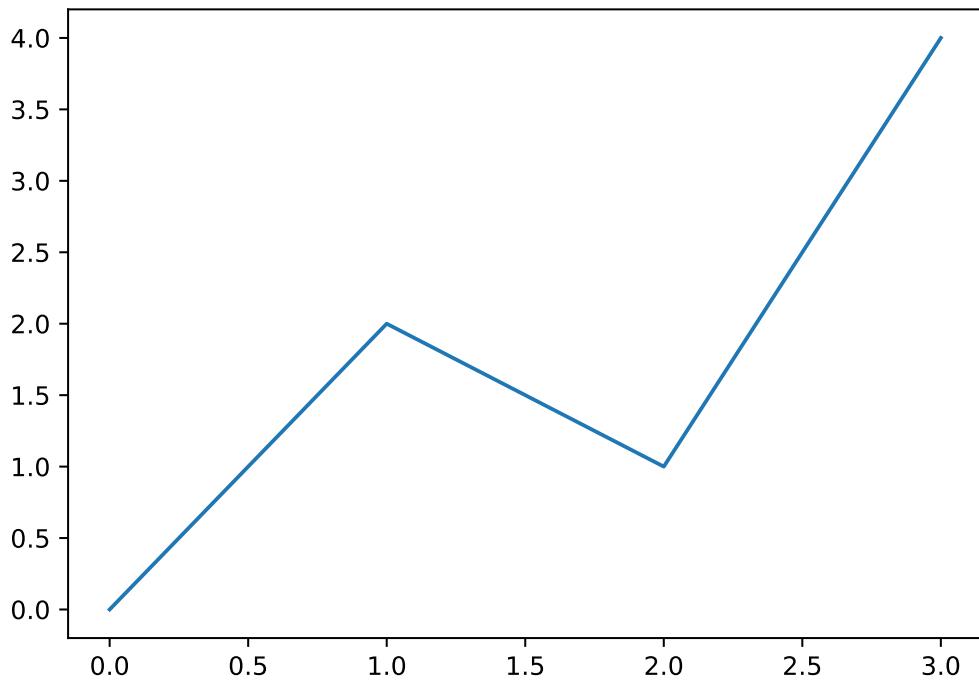
## install Matplotlib
# virtualenv_install("r-reticulate", "matplotlib")

## import Matplotlib (it will be automatically discovered in "r-reticulate")
matplotlib <- import("matplotlib")
```

copy C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl  
and C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl  
two folders to the folder C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\ver

```
# library(reticulate)
# use_virtualenv("r-reticulate")
# # matplotlib <- import("matplotlib")
# matplotlib$use("Agg", force = TRUE)
```

```
import matplotlib.pyplot as plt
plt.plot([0, 2, 1, 4])
plt.show()
```



## 12.2 SoloLearn

<https://www.sololearn.com/>

<https://www.sololearn.com/en/learn/courses/python-intermediate>

## 12.3 list comprehension

<https://www.sololearn.com/en/learn/courses/python-intermediate/lesson/1188906590?p=1>

```
cubes = [i**3 for i in range(5)]  
print(cubes) ## [0, 1, 8, 27, 64]
```

## 12.4 functional programming

- pure function
- lambda
- map
- filter
- generator
- decorator
- recursion
- \*args

- `**kwargs`

## 12.5 object-oriented programming = OOP

- class
- inheritance
- magic method
- operator overloading
- data hiding
- static method
- property

# Chapter 13

## TikZ

- TikZ
  - PGFplots<sup>[13.4]</sup>
  - tikzplotlib<sup>[13.5]</sup>: Python<sup>[12]</sup> matplotlib<sup>[27]</sup> export to TikZ<sup>[13]</sup>.tex

multi-column 7.3.6

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

<https://bookdown.org/yihui/rmarkdown-cookbook/html-scroll.html>

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

How to speed up bookdown generation?

<https://stackoverflow.com/questions/56541371/how-to-speed-up-bookdown-generation>

TikZ and PGFplots

What's the relation between packages PGFplots and TikZ?

<https://tex.stackexchange.com/questions/285925/whats-the-relation-between-packages-pgfplots-and-tikz>

<https://www.youtube.com/watch?v=bQugbYq0BVA>

<https://www.youtube.com/watch?v=ft4Kg9emK1k&list=PLg5nrpKdkk2DWcg3scb75AknF7DJXs81&index=18>

```
\begin{tikzpicture}
\def\aa{1.5} % amplitude
\def\bb{2}    % frequency
\draw[->] (-0.2,0)--(4.2,0)
← node[right, font=\small] {$x$};
\draw[->] (0,-4)--(0,0.5)
← node[above] {$y$};

← \draw[domain=0:4,smooth,variable=\t,blue,thick]
←
plot ({\aa * (\bb*\t -
sin(deg(\bb*\t)))},{-\aa * (1 -
cos(deg(\bb*\t)))});
% \node[above] at (2, 0.5)
← {Brachistochrone Curve};
\node[above, font=\footnotesize] at
← (2, 1) {Brachistochrone Curve};
\node[above, font=\footnotesize] at
← (2, 0) {$\begin{aligned} &x=r(t-\sin t) \\ &y=r(1-\cos t)\end{aligned}$};
\end{aligned}\$};
\end{tikzpicture}
```

Figure 13.1: Brachistochrone Curve

Figure 13.2: Brachistochrone Curve

## 13.1 2D

[https://zhuanlan.zhihu.com/p/127155579?utm\\_psn=1741479950987960320](https://zhuanlan.zhihu.com/p/127155579?utm_psn=1741479950987960320)

```
\begin{tikzpicture}
\draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

---

```
out.width=if (knitr:::is_html_output())
'20%'

\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

---

```
\begin{tikzpicture}
  \draw[rounded corners]
    (-1,1)--(0,0)--(1,2)--(-1,1);
\end{tikzpicture}
```

Figure 13.3: rounded corner pseudo-closed triangle

```
\begin{tikzpicture}
  \draw[rounded corners]
    (-1,1)--(0,0)--(1,2)--cycle;
\end{tikzpicture}
```

Figure 13.4: rounded corner triangle

```
\begin{tikzpicture}
  \draw[rounded corners]
    (-1,1)--(0,0)--(1,2)--cycle;
  \draw[rounded corners]
    (-1,1)--(0,0)--(1,2)--(-1,1);
\end{tikzpicture}
```

Figure 13.5: triangle vs. pseudo-closed triangle

```
\begin{tikzpicture}
  \draw (0,0) rectangle (4,2);
\end{tikzpicture}
```

Figure 13.6: rectangle

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

Figure 13.7: square

---

```
\begin{tikzpicture}
  \draw (0,0) circle (1);
\end{tikzpicture}
```

Figure 13.8: circle

---

```
\begin{tikzpicture}
  \draw (0,0) circle (1);
  \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

Figure 13.9: circle and square

---

```
\begin{tikzpicture}
  \draw (1,1) ellipse (2 and 1);
\end{tikzpicture}
```

Figure 13.10: ellipse

---

```
\begin{tikzpicture}
  \draw (1 ,1) arc (0:270:1);
  \draw (6 ,1) arc (0:270:2 and 1);
\end{tikzpicture}
```

Figure 13.11: circle and ellipse arcs

---

```
\begin{tikzpicture}
\draw (-1,1) parabola bend (0,0)
      .. (2,4);
\end{tikzpicture}
```

Figure 13.12: parabola arc

```
\begin{tikzpicture}
\draw (-1,1) parabola bend (0,0)
      .. (2,4);
\filldraw
(-1,1) circle (.05)
( 0,0) circle (.05)
( 1,1) circle (.05)
( 2,4) circle (.05);
\end{tikzpicture}
```

Figure 13.13: parabola arc with points

```
\begin{tikzpicture}
\draw [step=20pt] (0,0) grid (3,2);
\draw [help lines ,step=20pt] (4,0)
      .. grid (7,2);
\end{tikzpicture}
```

Figure 13.14: grid and help lines

```
\begin{tikzpicture}[scale=0.25]
\draw[->] (0,0)--(9,0);
\draw[<-] (0,1)--(9,1);
\draw[<->] (0,2)--(9,2);
\draw[>->>] (0,3)--(9,3);
\draw[|<->|] (0,4)--(9,4);
\end{tikzpicture}
```

Figure 13.15: arrows

```
\begin{tikzpicture}
\draw[line width =2pt] (0,6)--(9,6);
\draw[dotted] (0,5)--(9,5);
\draw[densely dotted] (0,4)--(9,4);
\draw[loosely dotted] (0,3)--(9,3);
\draw[dashed] (0,2)--(9,2);
\draw[densely dashed] (0,1)--(9,1);
\draw[loosely dashed] (0,0)--(9,0);
\end{tikzpicture}
```

Figure 13.16: lines

```
\begin{tikzpicture}[dline/.style={color=blue, line width=2pt}]
\draw[dline] (0,0)--(9,0);
\end{tikzpicture}
```

Figure 13.17: head styling

```
\begin{tikzpicture}
\draw (0,0) rectangle (2,2);
\draw[shift={( 3, 0)}] (0,0)
\rectangle (2,2);
\draw[shift={( 0, 3)}] (0,0)
\rectangle (2,2);
\draw[shift={( 0,-3)}] (0,0)
\rectangle (2,2);
\draw[shift={(-3, 0)}] (0,0)
\rectangle (2,2);
\draw[shift={( 3, 3)}] (0,0)
\rectangle (2,2);
\draw[shift={(-3, 3)}] (0,0)
\rectangle (2,2);
\draw[shift={( 3,-3)}] (0,0)
\rectangle (2,2);
\draw[shift={(-3,-3)}] (0,0)
\rectangle (2,2);
\end{tikzpicture}
```

Figure 13.18: transform: shift

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt] (0,0) rectangle
    (2,2);
  \draw[xshift=-100pt] (0,0) rectangle
    (2,2);
  \draw[yshift= 100pt] (0,0) rectangle
    (2,2);
  \draw[yshift=-100pt] (0,0) rectangle
    (2,2);
\end{tikzpicture}
```

Figure 13.19: transform: shift x, y

---

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt, xscale=1.5]
    (0,0) rectangle (2,2);
  \draw[yshift= 100pt, xscale=0.5]
    (0,0) rectangle (2,2);
  \draw[xshift=-100pt, yscale=1.5]
    (0,0) rectangle (2,2);
  \draw[yshift=-100pt, yscale=0.5]
    (0,0) rectangle (2,2);
\end{tikzpicture}
```

Figure 13.20: transform: scale x, y

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt, xscale=1.5]
    (0,0) rectangle (2,2);
  \draw[yshift= 100pt, yscale=1.5]
    (0,0) rectangle (2,2);
  \draw[xshift=-100pt, xscale=0.5]
    (0,0) rectangle (2,2);
  \draw[yshift=-100pt, yscale=0.5]
    (0,0) rectangle (2,2);
\end{tikzpicture}
```

Figure 13.21: transform: scale

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift=125pt,rotate=45] (0,0)
    rectangle (2,2);
  \draw[xshift=175pt,rotate
    around={45:(2 ,2)}] (0,0)
    rectangle (2,2);
\end{tikzpicture}
```

Figure 13.22: transform: rotate

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift=70pt,xslant=1] (0,0)
    rectangle (2,2);
  \draw[yshift=70pt,yslant=1] (0,0)
    rectangle (2,2);
\end{tikzpicture}
```

Figure 13.23: transform: slant

```
\tikzset{
  box/.style={
    draw=blue,
    rectangle,
    rounded corners=5pt,
    minimum width=50pt,
    minimum height=20pt,
    inner sep=5pt
  }
}
\begin{tikzpicture}
  \node[box] (1) at (0,0) {1};
  \node[box] (2) at (4,0) {2};
  \node[box] (3) at (8,0) {3};
  \draw[->] (1)--(2);
  \draw[->] (2)--(3);
  \node at (2,1) {a};
  \node at (6,1) {b};
\end{tikzpicture}
```

Figure 13.24: flowchart

```
\tikzset{
  box/.style={
    draw=blue,
    fill=blue!20,
    rectangle,
    rounded corners=5pt,
    minimum height=20pt,
    inner sep=5pt
  }
}
\begin{tikzpicture}
  \node[box] {1}
    child {node[box] {2}}
    child {node[box] {3}
      child {node[box] {4}}
      child {node[box] {5}}
      child {node[box] {6}}
    };
\end{tikzpicture}
```

Figure 13.25: tree

```
\begin{tikzpicture}
  \draw[->] (-0.2,0)--(6,0)
  ↵ node[right] {$x$};
  \draw[->] (0,-0.2)--(0,6)
  ↵ node[above] {$f(x)$};
  \draw[domain=0:4] plot (\x ,{0.1*
  ↵ exp(\x)}) node[right]
  ↵ {$f(x)=\frac{1}{10}e^x$};
\end{tikzpicture}
```

Figure 13.26: function plot

<https://stackoverflow.com/questions/64897575/tikz-libraries-in-bookdown>

It turns out that you can simply put the `\usetikzlibrary{...}` command directly before the `\begin{tikzpicture}` and everything works fine :)

<https://stackoverflow.com/questions/56211210/r-markdown-document-with-html-docx-output-using-latex-package-bbm>

<https://tex.stackexchange.com/questions/171711/how-to-include-latex-package-in-r-markdown>

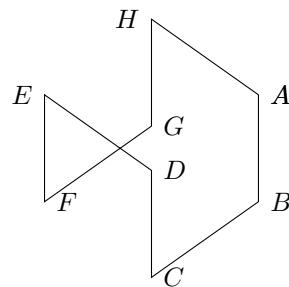
## 13.2 3D

[https://zhuanlan.zhihu.com/p/431732330?utm\\_psn=1741857547550638080](https://zhuanlan.zhihu.com/p/431732330?utm_psn=1741857547550638080)

<https://github.com/RRWWW/Stereometry>

```
\begin{tikzpicture}
\coordinate (A) at ( 1, 1, 1);
\coordinate (B) at ( 1, 1,-1);
\coordinate (C) at ( 1,-1,-1);
\coordinate (D) at ( 1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {$A$}--(B) node[right=1pt] {$B$}--(C) node[right=1pt] {$C$}--(D) node[right=1pt] {$D$}--(E) node[left= 1pt] {$E$}--(F) node[right=1pt] {$F$}--(G) node[right=1pt] {$G$}--(H) node[left= 1pt] {$H$}--(A) node[right=1pt] {$A$};
\end{tikzpicture}
```

Figure 13.27: cube



```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
\tdplotsetmaincoords{45}{45}
\begin{tikzpicture}[tdplot_main_coords]
\coordinate (A) at ( 1, 1, 1);
\coordinate (B) at ( 1, 1,-1);
\coordinate (C) at ( 1,-1,-1);
\coordinate (D) at ( 1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {$A$}--(B) node[right=1pt] {$B$}--(C) node[right=1pt] {$C$}--(D) node[right=1pt] {$D$}--(E) node[left= 1pt] {$E$}--(F) node[right=1pt] {$F$}--(G) node[right=1pt] {$G$}--(H) node[left= 1pt] {$H$}--(A) node[right=1pt] {$A$};
\end{tikzpicture}
```

Figure 13.28: cube rotate

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
%\tdplotsetmaincoords{70}{110}
\begin{tikzpicture}[rotate around
→ y=-15, rotate around z=7]
\coordinate (A) at ( 1, 1, 1);
\coordinate (B) at ( 1, 1,-1);
\coordinate (C) at ( 1,-1,-1);
\coordinate (D) at ( 1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {$A$}--
      (B) node[right=1pt] {$B$}--
      (C) node[right=1pt] {$C$}--
      (D) node[right=1pt] {$D$}--
      (E) node[left= 1pt] {$E$}--
      (F) node[right=1pt] {$F$}--
      (G) node[right=1pt] {$G$}--
      (H) node[left= 1pt] {$H$}--
      (A) node[right=1pt] {$A$};
\end{tikzpicture}
```

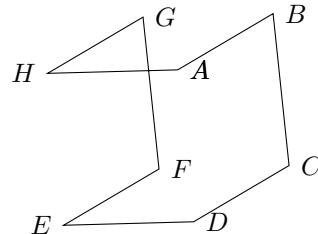


Figure 13.29: cube rotate around

<https://tex.stackexchange.com/questions/388621/optimizing-perspective-tikz-graphic>

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
\begin{tikzpicture}[y={(.5cm,.7cm)}]
\coordinate (A) at ( 1, 1, 1);
\coordinate (B) at ( 1, 1,-1);
\coordinate (C) at ( 1,-1,-1);
\coordinate (D) at ( 1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {$A$}--
      (B) node[right=1pt] {$B$}--
      (C) node[right=1pt] {$C$}--
      (D) node[right=1pt] {$D$}--
      (E) node[left= 1pt] {$E$}--
      (F) node[right=1pt] {$F$}--
      (G) node[right=1pt] {$G$}--
      (H) node[left= 1pt] {$H$}--
      (A) node[right=1pt] {$A$};
\end{tikzpicture}
```

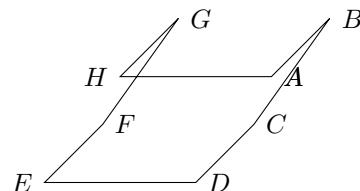


Figure 13.30: cube perspective slant

<https://github.com/XiangyunHuang/bookdown-broken/blob/master/index.Rmd>

```
\smartdiagramset{planet
  ↳ color=gray!40!white,
uniform color list=gray!40!white for
  ↳ 10 items}
\smartdiagram[bubble diagram]{Basic
  ↳ skills,
  Edit~/\` (RStudio),
  Organize~/\` (bookdown),
  Cooperate~/\` (Git),
  Typeset~/\` (LaTeX/Pandoc),
  Compile~/\` (GitHub Action)}
```



Figure 13.31: modern statistics plot skills

### 13.3 plots of functions

<https://tikz.dev/tikz-plots>

A warning before we get started: If you are looking for an easy way to create a normal plot of a function with scientific axes, ignore this section and instead look at the `pgfplots` package or at the `datavisualization` command from Part VI.

<https://tikz.dev/tikz-plots#sec-22.5>

```
\begin{tikzpicture}[domain=0:4]
  \draw[very thin,color=gray]
    (-0.1,-1.1) grid (3.9,3.9);
  \draw[->] (-0.2,0) -- (4.2,0)
    node[right] {$x$};
  \draw[->] (0,-1.2) -- (0,4.2)
    node[above] {$f(x)$};

  \draw[color=red] plot (\x,\x)
    node[right] {$f(x) = x$};
  % \x r means to convert '\x' from
  % degrees to _radians:
  \draw[color=blue] plot (\x,{sin(\x
    r)}) node[right] {$f(x) = \sin x$};
  \draw[color=orange] plot
    (\x,{0.05*exp(\x)}) node[right]
    {$f(x) = \frac{1}{20} e^x$};
\end{tikzpicture}
```

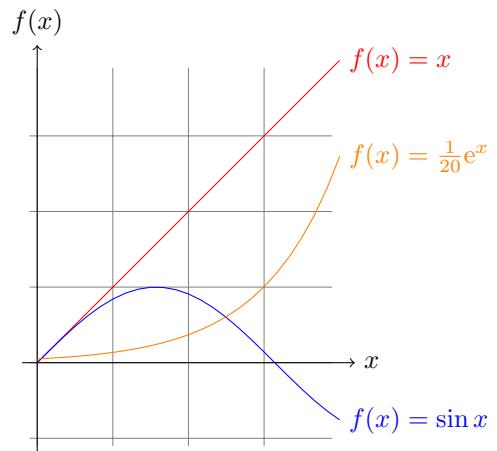


Figure 13.32: plots of functions

```
\tikz
\draw [scale=0.5,domain=-3.141:3.141,smooth,variable=\t]
plot ({\t*sin(\t r)},{\t*cos(\t r)});
```



Figure 13.33: 2D parametric function

```
\tikz \draw [domain=0:360,
            smooth,
            variable=\t]
plot ({sin(\t)},\t/360,{cos(\t)});
```



Figure 13.34: 3D parametric function

## 13.4 PGFplots

axis similar to matplotlib figure anatomy, see Fig: 27.1

<https://tikz.dev/pgfplots/>

<https://tikz.dev/pgfplots/tutorial1>

Not so common is `\pgfplotsset{compat=1.5}`. A statement like this should always be used in order to (a) benefit from a more or less recent feature set and (b) avoid changes to your picture if you recompile it with a later version of pgfplots.

```
\pgfplotsset{width=7cm,compat=1.18}
\begin{tikzpicture}
\begin{axis}[
]
    % density of Normal distribution:
    \addplot [
        red,
        domain=-3e-3:3e-3,
        samples=201,
    ]
        {exp(-x^2 / (2e-3^2)) / (1e-3
        * sqrt(2*pi))};
\end{axis}
\end{tikzpicture}
```

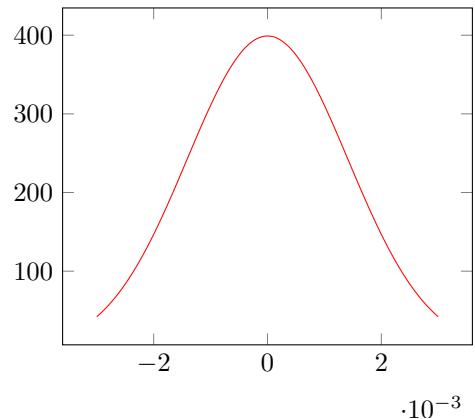


Figure 13.35: PGFplots: normal distribution

### 13.4.1 the axis environments

<https://tikz.dev/pgfplots/reference-axis>

```
\pgfplotsset{every linear axis/.append style={...}}
```

```
\begin{tikzpicture}
  \begin{axis}[
    no markers,
    axis x line = center,
    axis y line = center,
    xlabel = {$x$}, xlabel style
    ← = {right},
    ylabel = {$y$}, ylabel style
    ← = {above},
    xmin = -8, xmax = 8,
    ymin = 0, ymax = 0.45,
    hide obscured x ticks=false, %
    ← for origin x tick label i.e. xtick
    ← = {0}
    xtick={-4, 0, 4},
    xticklabels={$
    ← \mu_{\scriptscriptstyle 1} $,
    $%
    ← \mu_{\scriptscriptstyle 0} $,
    $%
    ← \mu_{\scriptscriptstyle 1} $,
    $},
    %extra x ticks={0},
    ytick = \emptyset,
    x = 1cm, y = 5cm, % x y
    ← scaling
    %grid = major,
    domain = -10:10,
    samples = 1000
  ]
  \end{axis}
\end{tikzpicture}
```

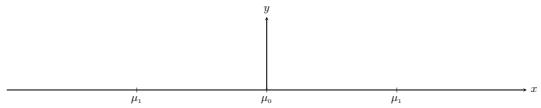


Figure 13.36: beginaxis

<https://tex.stackexchange.com/questions/134959/axis-lines-middle-and-axis-lines-center>

No, there is no difference.

## 13.4.2 axis descriptions

<https://tikz.dev/pgfplots/reference-axisdescription>

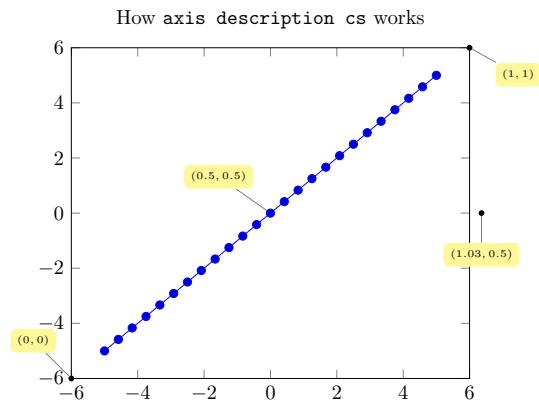
### 13.4.2.1 placement of axis descriptions

13.4.2.1.1 coordinate system axis description cs [https://tikz.dev/pgfplots/reference-axisdescription#pgfp.axis\\_description\\_cs](https://tikz.dev/pgfplots/reference-axisdescription#pgfp.axis_description_cs)

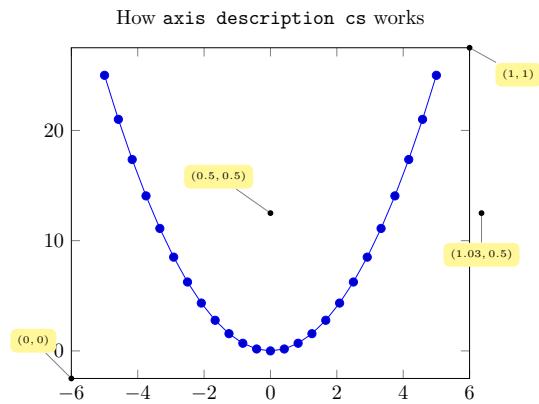
\addplot {x}; can change to \addplot {x^2}; still with auto blue dots

small dot style,pin=angle:LaTeX label at PGFplots axis coordinate system ;

```
\begin{tikzpicture}
  \tikzset{
    every
  → pin/.style={fill=yellow!50!white,rectangle,rounded
  → corners=3pt,font=\tiny},
    small
  → dot/.style={fill=black,circle,scale=0.3},
  }
  \begin{axis}[
    clip=false,
    title=How \texttt{axis
  ← description cs} works,
  ]
    \addplot {x};
    %small dot
    → style,pin=angle:LaTeX
    → label at PGFplots axis
    → coordinate system ;
    \node [small
  ← dot,pin=120:{\$(0,0\$)}
  ← at (axis description
  ← cs:0,0)      {};
    \node [small
  ← dot,pin=-30:{\$(1,1\$)}
  ← at (axis description
  ← cs:1,1)      {};
    \node [small
  ← dot,pin=-90:{\$(1.03,0.5\$)}
  ← at (axis description
  ← cs:1.03,0.5) {};
    \node [small
  ← dot,pin=125:{\$(0.5,0.5\$)}
  ← at (axis description
  ← cs:0.5,0.5) {};
  \end{axis}
\end{tikzpicture}
```

Figure 13.37: PGFplots:  $x$

```
\begin{tikzpicture}
  \tikzset{
    every
    pin/.style={fill=yellow!50!white,rectangle,rounded
    corners=3pt,font=\tiny},
    small
    dot/.style={fill=black,circle,scale=0.3},
  }
  \begin{axis}[
    clip=false,
    title=How \texttt{axis
description cs} works,
]
    \addplot {x^2};
    %small dot
    \style, pin=angle:LaTeX
    \label at PGFplots axis
    coordinate system ;
    \node [small
    dot,pin=120:{\$(0,0)\$}]
    at (axis description
    cs:0,0) {};
    \node [small
    dot,pin=-30:{\$(1,1)\$}]
    at (axis description
    cs:1,1) {};
    \node [small
    dot,pin=-90:{\$(1.03,0.5)\$}]
    at (axis description
    cs:1.03,0.5) {};
    \node [small
    dot,pin=125:{\$(0.5,0.5)\$}]
    at (axis description
    cs:0.5,0.5) {};
  \end{axis}
\end{tikzpicture}
```

Figure 13.38: PGFplots:  $x^2$ 

#### 13.4.2.1.2 legend <https://tikz.dev/pgfplots/reference-axisdescription#sec-4.9.4>

#### 13.4.2.1.3 tick option <https://tikz.dev/pgfplots/reference-tickoptions>

### 13.4.3 declare function

[https://tikz.dev/pgfplots/utility-commands#pgf/declare\\_function](https://tikz.dev/pgfplots/utility-commands#pgf/declare_function)

```
\begin{tikzpicture}
\begin{axis}[
    declare function={
        C=4;
        square(\t)=(\t)^2 + C;
    },
]
\addplot+ [samples=2] {C*x};
\addplot {square(x)};
\end{axis}
\end{tikzpicture}
```

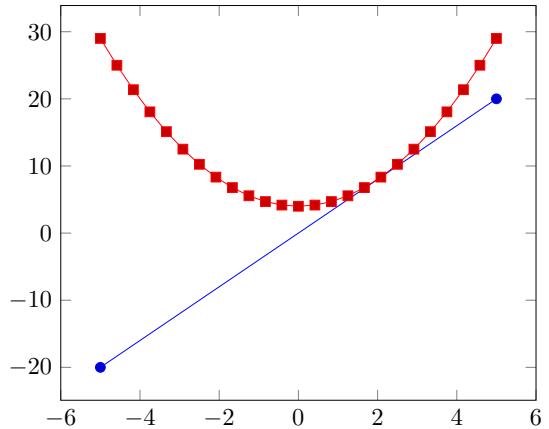


Figure 13.39: declare function

### 13.4.3.1 pgfmathparse

<https://tikz.dev/math-parsing>

<https://tikz.dev/math-parsing#sec-94.1>

This macro parses and returns the result without units in the macro 0.017. Example:  
`\pgfmathparse{2pt+3.5pt}` will set `\pgfmathresult` to the text 5.5.

```
\pgfmathsqrt{x} = \pgfmathparse{sqrt(x)}
```

```
\pgfmathln{x} = \pgfmathparse{ln(x)}
```

...

### 13.4.3.2 pgfmathdeclarefunction

like `pgfplotsinvokeforeach`

replaces any occurrence of #1 inside of (math image)command(math image) once for every element in (math image)list(math image). Thus, it actually assumes that (math image)command(math image) is like a `\newcommand` body.

```
% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
    \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
    \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

\begin{tikzpicture}
\begin{axis}[
    no markers,
```

```

axis x line = center,
axis y line = center,
xlabel = {$x$}, xlabel style = {right},
ylabel = {$y$}, ylabel style = {above},
xmin = -8, xmax = 8,
ymin = 0, ymax = 0.45,
hide obscured x ticks=false, % for origin x tick label i.e. xtick = {0}
xtick={-4, 0, 4},
xticklabels={$\mu_{\scriptscriptstyle 1}$, $\mu_{\scriptscriptstyle 0}$, $\mu_{\scriptscriptstyle 1}$},
%extra x ticks={0},
ytick = \emptyset,
x = 1cm, y = 5cm, % x y scaling
%grid = major,
domain = -10:10,
samples = 1000
]
\addplot [fill=cyan!20, draw=none, domain=-10:-2] {gauss(-4, 1)}
\closedcycle;
\addplot [fill=cyan!20, draw=none, domain= 2:10] {gauss( 4, 1)}
\closedcycle;
\addplot [very thick, cyan!50!black] {gauss(-4, 1)};
\addplot [very thick, cyan!50!black] {gauss( 0, 1)};
\addplot [very thick, cyan!50!black] {gauss( 4, 1)};
%\node [anchor=north] at (axis cs: 0, -0.01) {$\mu$};
%\node at (axis cs: -4, -0.02) {$\mu_1$};
\draw [dashed, thin] (axis cs: -4, 0) -- (axis cs: -4, 1);
\draw [dashed, thin] (axis cs: 4, 0) -- (axis cs: 4, 1);
\end{axis}
\end{tikzpicture}

```

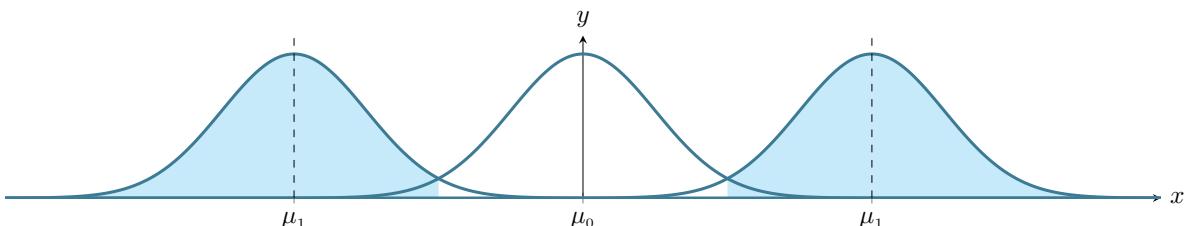


Figure 13.40: PGFmathDeclareFunction: normal distributions hypothesis testing

<https://tex.stackexchange.com/questions/43610/plotting-bell-shaped-curve-in-tikz-pgf>

```

% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
  \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

```

```
\begin{tikzpicture}
\begin{axis}[
    no markers, domain=0:10, samples=100,
    axis lines*=left, xlabel=$x$, ylabel=$y$,
    every axis y label/.style={at=(current axis.above origin), anchor=south},
    every axis x label/.style={at=(current axis.right of origin), anchor=west},
    height=5cm, width=12cm,
    xtick={4,6.5}, ytick=\emptyset,
    enlargelimits=false, clip=false, axis on top,
    grid = major
]
\addplot [fill=cyan!20, draw=none, domain=0:5.96] {gauss(6.5,1)} \closedcycle;
\addplot [very thick,cyan!50!black] {gauss(4,1)};
\addplot [very thick,cyan!50!black] {gauss(6.5,1)};
\draw [yshift=-0.6cm, latex-latex](axis cs:4,0) -- node [fill=white]
    {$1.96\sigma$} (axis cs:5.96,0);
\end{axis}
\end{tikzpicture}
```

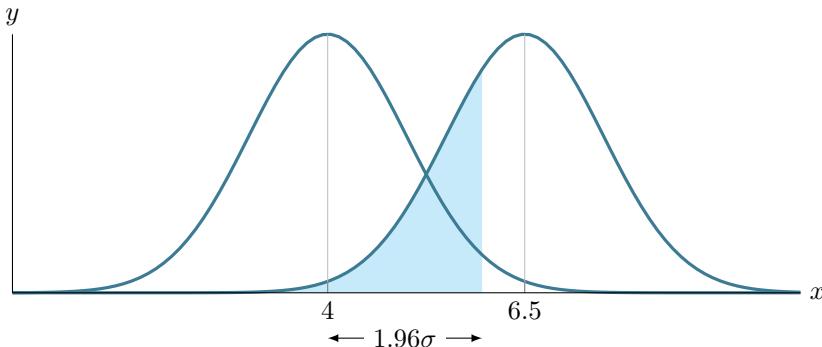


Figure 13.41: PGFmathDeclareFunction: normal distributions

### 13.4.4 |- and -| in TikZ

<https://tex.stackexchange.com/questions/401425/tikz-what-exactly-does-the-the-notation-for-arrows-do>

(a -| b) where a and b are named nodes or coordinates. This means the coordinate that is at the y-coordinate of a, and x-coordinate of b. Similarly, (a |- b) has the x-coordinate of a and y-coordinate of b.

### 13.4.5 pgfplotsinvokeforeach

<https://tikz.dev/pgfplots/pgfplotstable-miscellaneous#/pgfplotsinvokeforeach>

like \foreach in TikZ

A variant of \pgfplotsforeachungrouped (and such also of \foreach) which replaces any occurrence of #1 inside of (math image)command(math image) once for every element in (math image)list(math image). Thus, it actually assumes that (math image)command(math image) is like a \newcommand body.

### 13.4.6 interpolation dashed lines

<https://tex.stackexchange.com/questions/193259/what-is-the-easiest-way-to-accomplish-textual-tick-labels-in-tikz>

```
interpa = (10,10), interpib = (30,30), interp = interpolation

({axis cs:0,0}|-interp#1) = (x of (0,0), y of (interpa)) = (0, 10), ...
```

```
\begin{tikzpicture}
\begin{axis}[
    axis lines=left,
    xmin = 0, xmax = 40,
    ymin = 0, ymax = 40,
    xtick={10,30},
    xticklabels={\$V_i=10\$,\$V_f=30\$},
    ytick={10,30},
    yticklabels={\$P_i=10\$,\$P_f=30\$},
    xlabel={Volume},
    ylabel={Pressure}
]
\addplot [very thick,-latex ]
    coordinates{(10,10) (30,30)}
    % interpa = (10,10), interp =
    % (30,30), interp =
    % interpolation
    coordinate[at
    start](interpa)coordinate[at
    end](interpib);
\pgfplotsinvokeforeach {a,b} {
    \draw[ultra thin, dashed]
        % ({axis cs:0,0}|-interp#1) = (x
        % of (0,0), y of (interpa)) =
        % (0, 10), ...
        ({axis
    cs:0,0}|-interp#1)--(interp#1)--(interp#1|-{axis
    cs:0,0});
}
\end{axis}
\end{tikzpicture}
```

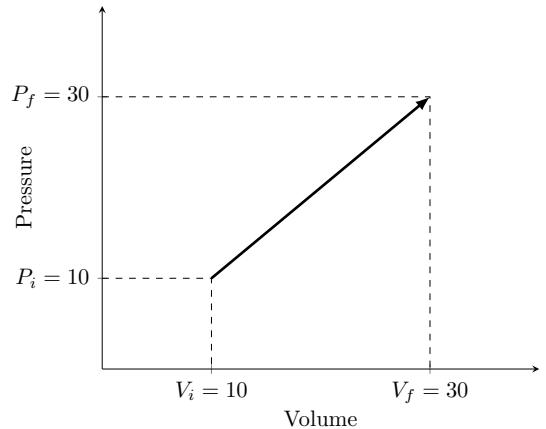


Figure 13.42: tick texts and interpolation dashed lines

### 13.4.7 Zewbie

<https://zhuanlan.zhihu.com/p/551874337>

axis similar to matplotlib figure anatomy, see Fig: 27.1

### 13.4.7.1 coordinate axis/axes fine-tuing

```
\begin{tikzpicture}
  \begin{axis}
    % empty
  \end{axis}
\end{tikzpicture}
```

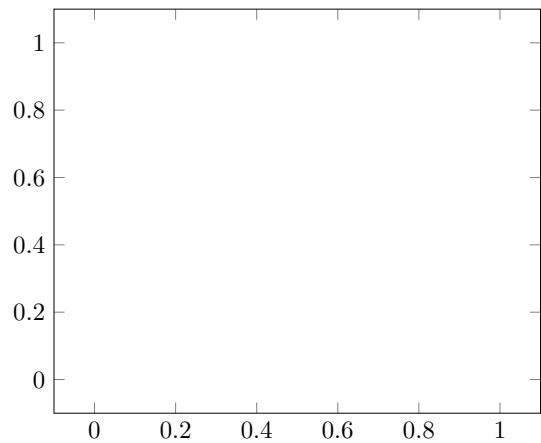


Figure 13.43: PGFplots: 2D axis/axes

#### 13.4.7.1.1 range

---

```
\begin{tikzpicture}
  \begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

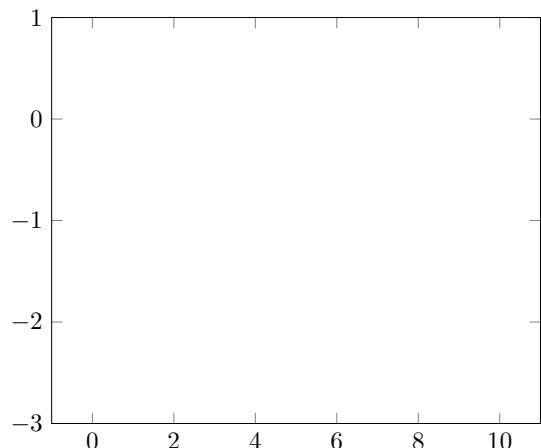


Figure 13.44: PGFplots: axis/axes range

#### 13.4.7.1.2 scaling axis equal image equivalent to unit vector ratio = {1 1 1}

```
\begin{tikzpicture}
  \begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
    axis equal image, % unit
    vector ratio = {1 1 1},
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

`scale only axis`  
`width x axis length, height y axis length`

```
\begin{tikzpicture}
  \begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
    scale only axis,
    width = 5cm, height = 7cm,
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

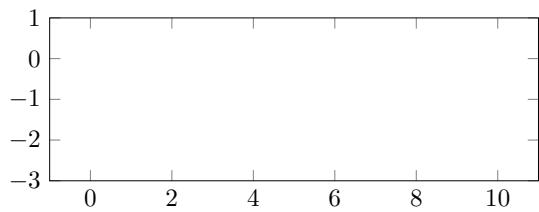


Figure 13.45: PGFplots: axis/axes range

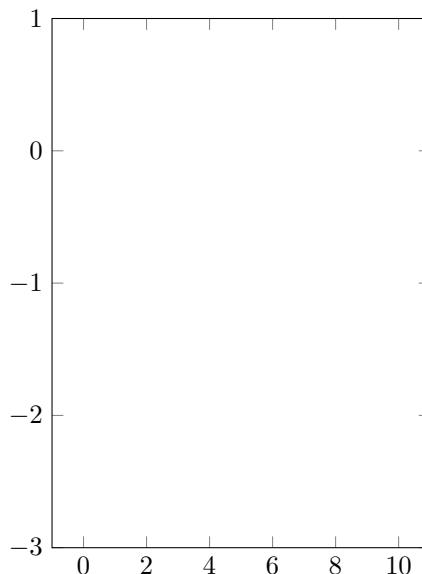


Figure 13.46: PGFplots: axis/axes range

`x x unit vector length, y y unit vector length`

```
\begin{tikzpicture}
  \begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
    x = 1cm, y = 1cm,
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

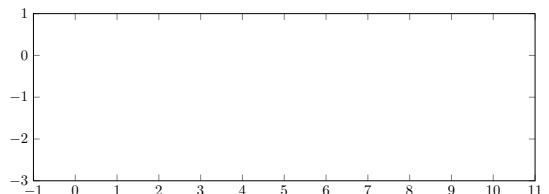


Figure 13.47: PGFplots: axis/axes range

### 13.4.7.1.3 direction vector

```
\begin{tikzpicture}
\begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
    x={(.2cm,-.1cm)},
    y={(-.5cm,.5cm)}, % empty
]
\end{axis}
\end{tikzpicture}
```

Figure 13.48: PGFplots: axis/axes range

```
unit vector ratio = {1 1}
```

```
\begin{tikzpicture}
\begin{axis}[
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
    unit vector ratio = {1 1},
]
% empty
\end{axis}
\end{tikzpicture}
```

Figure 13.49: PGFplots: axis/axes range

**13.4.7.1.4 axis style** axis lines to assign all axes, either `axis lines = box`(default), `axis lines = center`(axis lines with arrows, `center, x: bottom, top, y:` ), or `axis lines = none`(not shown), or even axis lines without arrows `axis lines *= center`

```
axis x line, axis y line to assign the respect axis, e.g. axis x line = center
```

```
axis lines = center:
```

```
\begin{tikzpicture}
\begin{axis}[
    axis lines = center,
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
]
% empty
\end{axis}
\end{tikzpicture}
```

Figure 13.50: PGFplots: axis/axes range

```
axis lines *= center:
```

x axis line without arrow, y axis box

```
\begin{tikzpicture}
\begin{axis}[
    axis x line*=center, % x
    axis line without arrow, y axis
    box
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
]
    % empty
\end{axis}
\end{tikzpicture}
```

Figure 13.51: PGFplots: axis/axes range

---

x axis line with arrow, y axis line without arrow

```
\begin{tikzpicture}
\begin{axis}[
    axis x line = center, % x axis
    line with arrow
    axis y line*=center, % y
    axis line without arrow
    xmin = -1, xmax = 11,
    ymin = -3, ymax = 1,
]
    % empty
\end{axis}
\end{tikzpicture}
```

Figure 13.52: PGFplots: axis/axes range

#### 13.4.7.1.5 axis discontinuity crunch, parallel, none

crunch

```
\begin{tikzpicture}
\begin{axis}[
    axis x line = bottom,
    axis y line = center,
    xmin = -2, xmax = 10,
    ymin = 0, ymax = 12,
    axis y discontinuity = crunch,
]
% empty
\end{axis}
\end{tikzpicture}
```

Figure 13.53: PGFplots: axis/axes range

---

parallel

```
\begin{tikzpicture}
\begin{axis}[
    axis x line = bottom,
    axis y line = center,
    xmin = -2, xmax = 10,
    ymin = 0, ymax = 12,
    axis y discontinuity =
    → parallel,
]
% empty
\end{axis}
\end{tikzpicture}
```

Figure 13.54: PGFplots: axis/axes range

### 13.4.7.1.6 tick tick pos ticklabel pos

```
\begin{tikzpicture}
  \begin{axis}[
    xtick pos = upper,
    yticklabel pos = upper,
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

Figure 13.55: PGFplots: axis/axes range

---

tick distance

tick align: inside, center, outside

```
\begin{tikzpicture}
  \begin{axis}[
    axis lines=center,
    xmin=-1,xmax=3,
    ymin=-3,ymax=3,
    xtick distance=.8,
    ytick distance=1.1,
    tick align=inside,
  ]
    % empty
  \end{axis}
\end{tikzpicture}
```

Figure 13.56: PGFplots: axis/axes range

---

minor tick num

```
\begin{tikzpicture}
\begin{axis}[
    axis y line=none, axis x
    line=center,
    ymin=0, ymax=0, xmin=-3, xmax=3,
    xtick distance=2, tick
    align=inside,
    minor tick num=2,
]
% empty
\end{axis}
\end{tikzpicture}
```

Figure 13.57: PGFplots: axis/axes range

`xtick=\emptyset|data|{<coordinates>}`

```
\begin{tikzpicture}
\begin{axis}[
    axis y line=none, axis x
    line=center,
    ymin=0, ymax=0, xmin=-3, xmax=3,
    xtick={-2.5,0,1},minor
    xtick={1/3,2/3},
    tick align=inside,
]
% empty
\end{axis}
\end{tikzpicture}
```

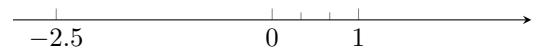


Figure 13.58: PGFplots: axis/axes range

`extra x ticks={<coordinates>}`

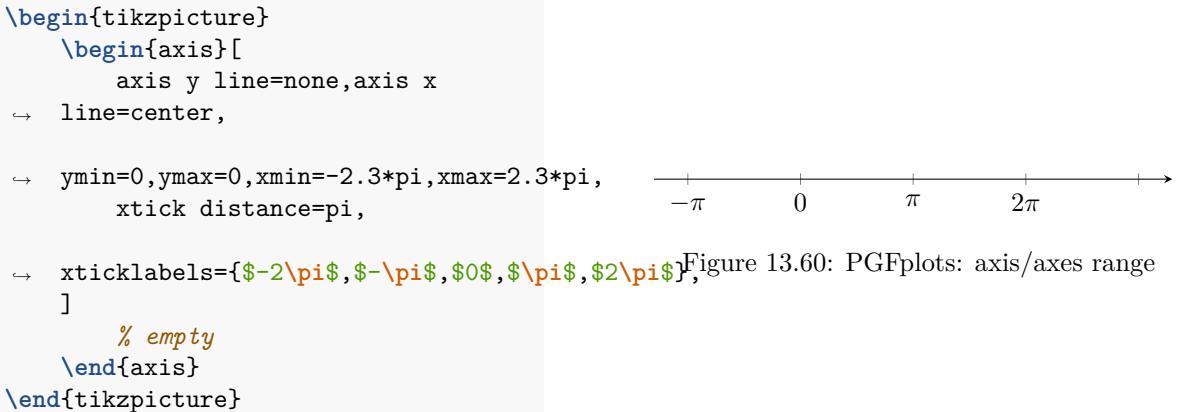
```
\begin{tikzpicture}
\begin{axis}[
    axis y line=none, axis x
    line=center,
    ymin=0, ymax=0, xmin=-2.3, xmax=4.9,
    xtick distance=2,minor tick
    num=1,
    extra x ticks={e,pi},
    extra x tick
    labels={$e$,$\pi$},
    tick align=inside,
]
% empty
\end{axis}
\end{tikzpicture}
```



Figure 13.59: PGFplots: axis/axes range

`ticklabels={<labels>} extra x tick labels={<labels>}`

`hide obscured x ticks = false` for origin x tick label



### 13.4.7.2 addplot+

What does addplot+ do exactly?

<https://tex.stackexchange.com/questions/275959/what-does-addplot-do-exactly>

<https://tikz.dev/pgfplots/reference-addplot#/addplot+>

Every `\addplot` directive receives a pre-defined style (line color, marker style etc) through a pre-defined cycle list that is automatically chosen depending on the index of the current `\addplot` instruction. If you want to add some of your styles manually (like I want red colour instead of blue, say), you can add them through options to `\addplot` like `\addplot[<your options>]`. Now the question is whether you want your own style (your options) to be appended to or replace one of these cycle lists assigned. This is decided by the + sign. If you use `\addplot+ [<your options>]`, your style is appended to and by `\addplot[<your options>]`, your options will replace the assigned cycle list.

### 13.4.7.3 point

`only marks` only points without lines

zero y axis range `ymin=0, ymax=0` and `axis y line=none`, making 1D x axis

```
\begin{tikzpicture}
\begin{axis}[
    xlabel=$x$,
    axis y line=none,
    axis x line=center,
    tick align=inside,
    xmin=-1.5, xmax=4.9, ymin=0,
    ymax=0,
    xtick distance=1,
    x=1cm
]
\addplot+ coordinates {(e,0)}
    node [pin=90:{$e$}] {};
\addplot+ coordinates {(pi,0)}
    node [pin=90:{$\pi$}] {};
\addplot+ coordinates
    {( -1,0)};
\end{axis}
\end{tikzpicture}
```

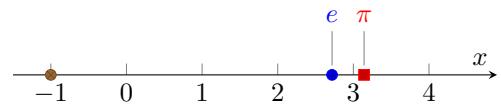


Figure 13.61: PGFplots: 1D points with pins

---

```
\begin{tikzpicture}
\begin{axis}[
    xlabel=$x$, ylabel=$y$,
    axis lines=center,
    tick align=inside,
    xmin=-1.5, xmax=4.9,
    ymin=-3.3, ymax=3.9,
    xtick distance=1, ytick
    distance=1,
    axis equal image
]
\addplot+ [only marks]
    coordinates {
        (-1,-2) (pi,pi/4) (3,2)
        (0,0)};
\addplot+ coordinates {(2,1)};
\addplot+ coordinates
    {(3,-2)};
\end{axis}
\end{tikzpicture}
```

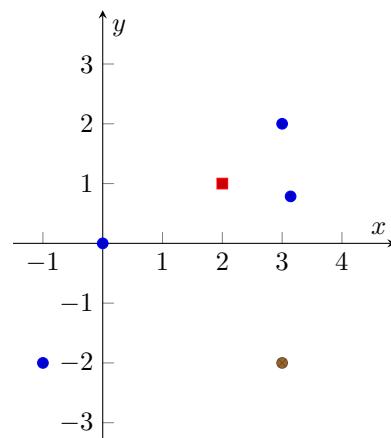


Figure 13.62: PGFplots: 2D points

```
\begin{tikzpicture}
\begin{axis}[axis equal image]
\addplot+ [only marks]
table [x=xdata,y=ydata]
→ {data/func.dat};
\end{axis}
\end{tikzpicture}
```

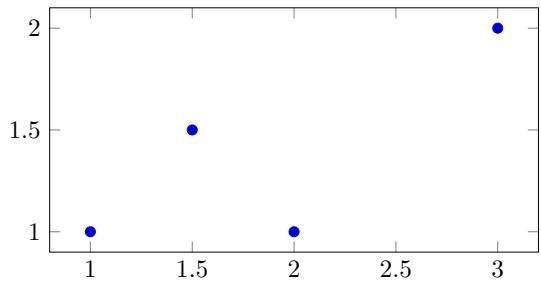


Figure 13.63: PGFplots: data points

`axis equal image` to fix aspect ratio 1:1

---

```
\begin{tikzpicture}
\begin{axis}[title=5 sampling
→ points,
xlabel=$x$,ylabel=$y$,
axis equal image]
\addplot+ [only
→ marks,domain=-2:2,samples=5]
{x^2};
\end{axis}
\end{tikzpicture}
```

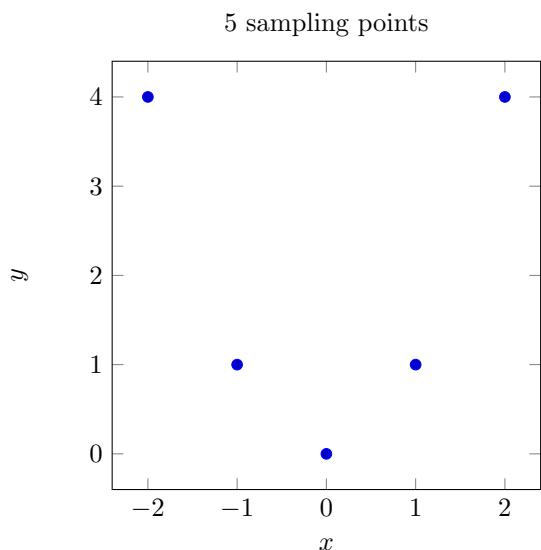


Figure 13.64: PGFplots: function sampling 5 points

```
\begin{tikzpicture}
\begin{axis}[title=55 sampling
points,
xlabel=$x$,ylabel=$y$,
axis equal image]
\addplot+ [only
marks,domain=-2:2,samples=55]
{x^2};
\end{axis}
\end{tikzpicture}
```

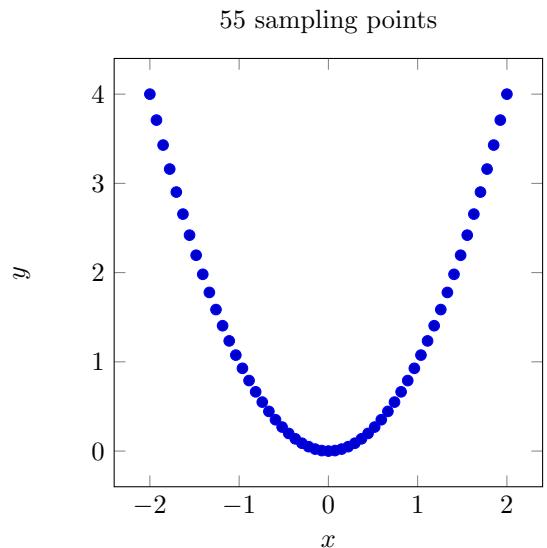


Figure 13.65: PGFplots: function sampling 55 points

```
\begin{tikzpicture}
\begin{axis}[
    trig format plots=rad, %
    angle in radian
    axis equal image]
\addplot+ [only
marks,variable=t,domain=0:pi*3/2,
samples=20]
({cos(t)},{sin(t)});
\end{axis}
\end{tikzpicture}
```

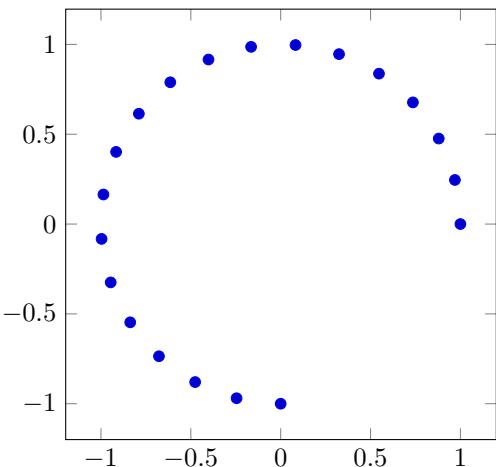


Figure 13.66: PGFplots: parametric function sampling

```
\begin{tikzpicture}
\begin{axis}[
    xlabel=$x$,ylabel=$y$,zlabel=$z$,
    axis lines=center,
    tick align=inside,
    xmin=-1.5,xmax=3.9,
    ymin=-1.5,ymax=3.9,
    zmin=-0.5,zmax=3.9,
    xtick distance=1,
    ytick distance=1,
    ztick distance=1,
    % width=10cm,
    % scale only axis,
    view={120}{30}, % perspective
    angle
    axis equal image,]
\addplot3+
    coordinates{(1,0,0)};
\addplot3+
    coordinates{(0,1,0)};
\addplot3+
    coordinates{(0,0,1)};
\end{axis}
\end{tikzpicture}
```

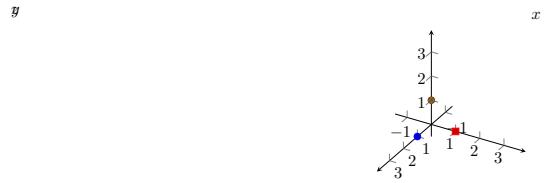


Figure 13.67: PGFplots: 3D points

---

```
\begin{tikzpicture}
\begin{axis}[
    xlabel=$x$,ylabel=$y$,zlabel=$z$,
    grid=major
    ]
\addplot3+ [only marks]
    {x^2+y^2};
\end{axis}
\end{tikzpicture}
```

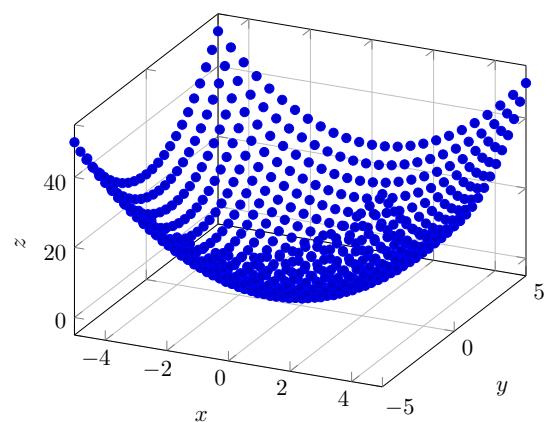


Figure 13.68: PGFplots: 3D function sampling points

#### 13.4.7.4 line

```
\begin{tikzpicture}
\begin{axis}[ytick=data]
\addplot coordinates {
(1,0.15) (2,0.21) (3,0.33)
→ (4,0.4)
(2.5,.1) (3.5,.1)
};
\end{axis}
\end{tikzpicture}
```

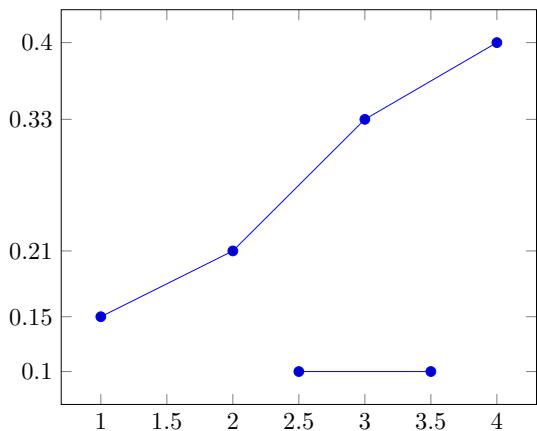


Figure 13.69: lines connecting adjacent points

`smooth` to make smooth curves or lines

```
\begin{tikzpicture}
\begin{axis}[ytick=data]
\addplot+ [smooth] coordinates
→ {
(1,0.15) (2,0.21) (3,0.33)
→ (4,0.4)};
\end{axis}
\end{tikzpicture}
```

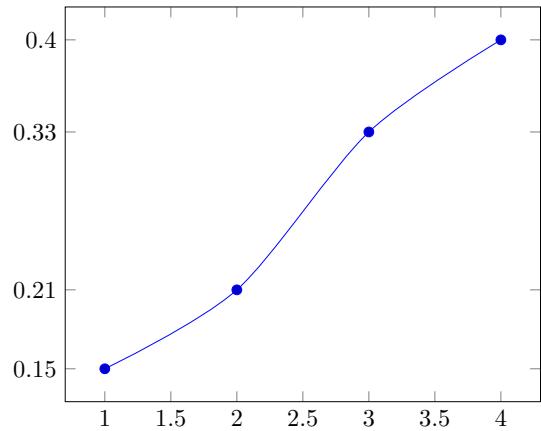
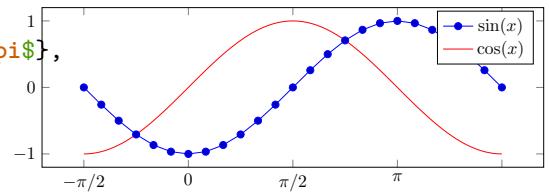


Figure 13.70: smooth lines connecting adjacent points

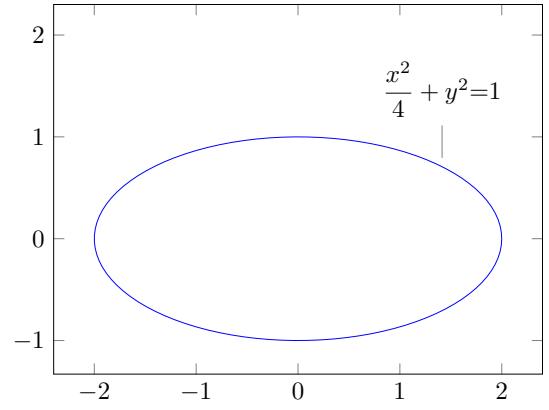
`no markers` to make no markers or points on the curves or lines

`\addlegendentry` to add legends

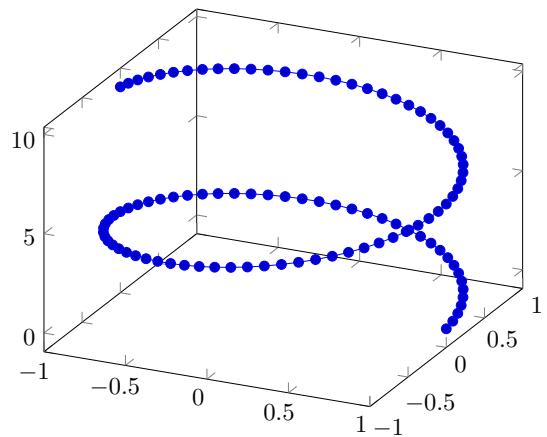
```
\begin{tikzpicture}
\begin{axis}[
    trig format plots=rad,
    xtick distance=pi/2, ytick
    distance=1,
    xticklabels={-$-\pi$,$-\pi/2$,0,$\pi/2$,$\pi$},
    width=10cm, scale only axis,
    axis equal image]
\addplot+ [domain=-pi:pi]
    {sin(x)};
\addlegendentry{$\sin(x)$}
\addplot+ [no
    markers,domain=-pi:pi,samples=100]
    {cos(x)};
\addlegendentry{$\cos(x)$}
\end{axis}
\end{tikzpicture}
```

Figure 13.71:  $\sin(x)$  and  $\cos(x)$ 

```
\begin{tikzpicture}
\begin{axis}[
    trig format plots=rad,
    ymax=2.3,
    axis equal image]
\addplot+ [no markers,
    variable=t,
    domain=0:2*pi,
    samples=100]
    ({2*cos(t)},{sin(t)});
\node
    [pin=90:$\frac{x^2}{4}+y^2=1$]
    at ({2*cos(45)},{sin(45)}) {};
\end{axis}
\end{tikzpicture}
```

Figure 13.72:  $\frac{x^2}{4} + y^2 = 1$

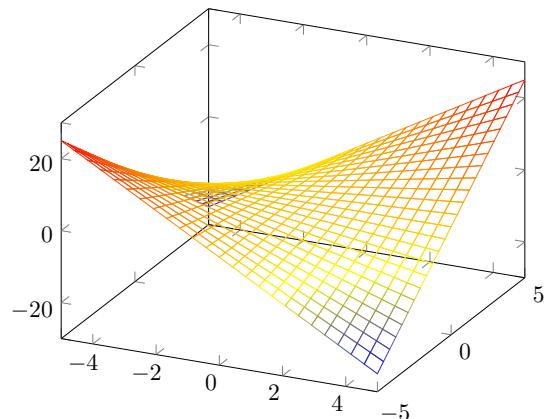
```
\begin{tikzpicture}
\begin{axis}[trig format
    ↵ plots=rad]
\addplot3+ [variable=t,
    domain=0:3*pi,
    samples=100,
    samples y=0]
    ({cos(t)},{sin(t)},{t});
\end{axis}
\end{tikzpicture}
```

Figure 13.73:  $(\cos(t), \sin(t), t)$ 

### 13.4.7.5 plane

[mesh]

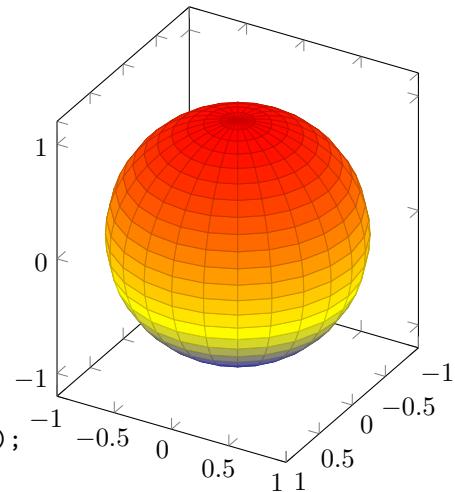
```
\begin{tikzpicture}
\begin{axis}
\addplot3+ [no markers, mesh]
    ↵ {x*y};
\end{axis}
\end{tikzpicture}
```

Figure 13.74:  $f(x, y) = xy$ 


---

[surf] surface

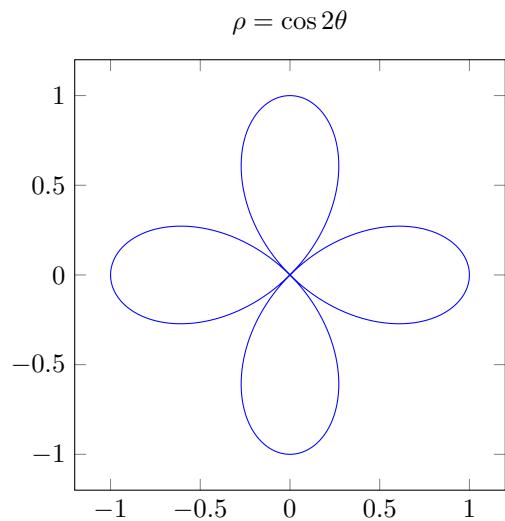
```
\begin{tikzpicture}
\begin{axis}[
    view={120}{30},
    trig format plots=rad,
    width=10cm,
    scale only axis,
    axis equal image
]
\addplot3+ [no markers,
            surf,
            domain=0:2*pi,
            domain y=0:pi]
    ({sin(y)*cos(x)},{sin(y)*sin(x)},{cos(y)});
\end{axis}
\end{tikzpicture}
```

Figure 13.75:  $x^2 + y^2 + z^2 = 1$ 

#### 13.4.7.6 polar coordinate

```
data cs=polar|polarrad
```

```
\begin{tikzpicture}
\begin{axis}[
    title={$\rho=\cos 2\theta$},
    axis equal image
]
\addplot+ [no markers,
            data cs=polar,
            domain=0:360,
            samples=360
            ] (\x,{cos(2*\x)});
\end{axis}
\end{tikzpicture}
```

Figure 13.76: polar coordinate  $\rho = \cos 2\theta$ 


---

```
\usepgfplotslibrary{polar} to use \begin{polaraxis}
```

```
\usepgfplotslibrary{polar}
\begin{tikzpicture}
  \begin{polaraxis}
    \addplot+ coordinates
      {(0,0) (60,1)
      \hookrightarrow (90,{sqrt(3)/2})} -- cycle;
  \end{polaraxis}
\end{tikzpicture}
```

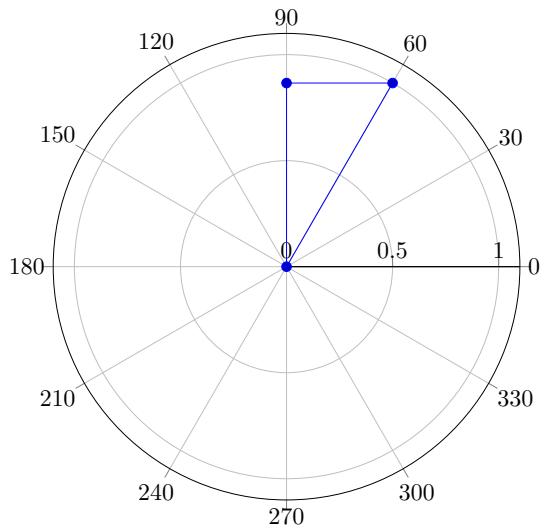


Figure 13.77: polar coordinate axes

<https://zhuanlan.zhihu.com/p/128341873>

### 13.4.8 Arnav Bandekar: Using pgfplots to make economic graphs in LaTeX

<https://towardsdatascience.com/using-pgfplots-to-make-economic-graphs-in-latex-bcdc8e27c0eb>

### 13.4.9 PGFplots gallery

<https://pgfplots.sourceforge.net/gallery.html>

```
\begin{tikzpicture}
\begin{axis}[
    xmin=-3, xmax=3,
    ymin=-3, ymax=3,
    extra x ticks={-1,1},
    extra y ticks={-2,2},
    extra tick style={grid=major},
]
\draw[red] \pgfextra{
    \pgfpathellipse{\pgfplotspointaxisxy{0}{0}}
    {\pgfplotspointaxisdirectionxy{1}{0}}
    {\pgfplotspointaxisdirectionxy{0}{2}}
        % see also the documentation of
        % 'axis direction cs' which
        % allows a simpler way to draw
        % this ellipse
};
\draw[blue] \pgfextra{
    \pgfpathellipse{\pgfplotspointaxisxy{0}{0}}
    {\pgfplotspointaxisdirectionxy{1}{1}}
    {\pgfplotspointaxisdirectionxy{0}{2}}
};
\addplot [only marks,mark=*]
    coordinates { (0,0) };
\end{axis}
\end{tikzpicture}
```

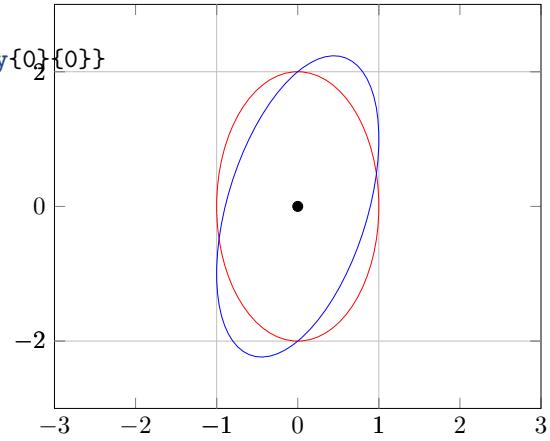


Figure 13.78: declare function

## 13.5 TikZplotLib / tikzplotlib

Python<sup>[12]</sup>

```
library(reticulate)

## Warning: package 'reticulate' was built under R version 4.3.3

# virtualenv_list()
# virtualenv_python()
# use_virtualenv("r-reticulate")

# conda_list()
use_condaenv(condaenv = 'sandbox-3.9')

## install TikZplotLib
# virtualenv_install("r-reticulate", "tikzplotlib")
```

```
## import TikZplotLib (it will be automatically discovered in "r-reticulate")
tikzplotlib <- import("tikzplotlib")
```

Error: ImportError: cannot import name 'common\_texification' from 'matplotlib.backends.bac

<https://github.com/NixOS/nixpkgs/issues/289305>

The “solution” is to use **matplotlib 3.6**, but I guess in nixpkgs a single version is used at a time. The last working upgrade is from 0911608 I guess (I tried using virtualenv + pip + nix-ld + export LD\_LIBRARY\_PATH=“LDLIBRARYPATH :NIX\_LD\_LIBRARY\_PATH”)

<https://stackoverflow.com/questions/60882638/install-a-particular-version-of-python-package-in-a-virtualenv-created-with-reti>

```
#reticulate::virtualenv_install(packages = c("matplotlib==3.6.0"))
```

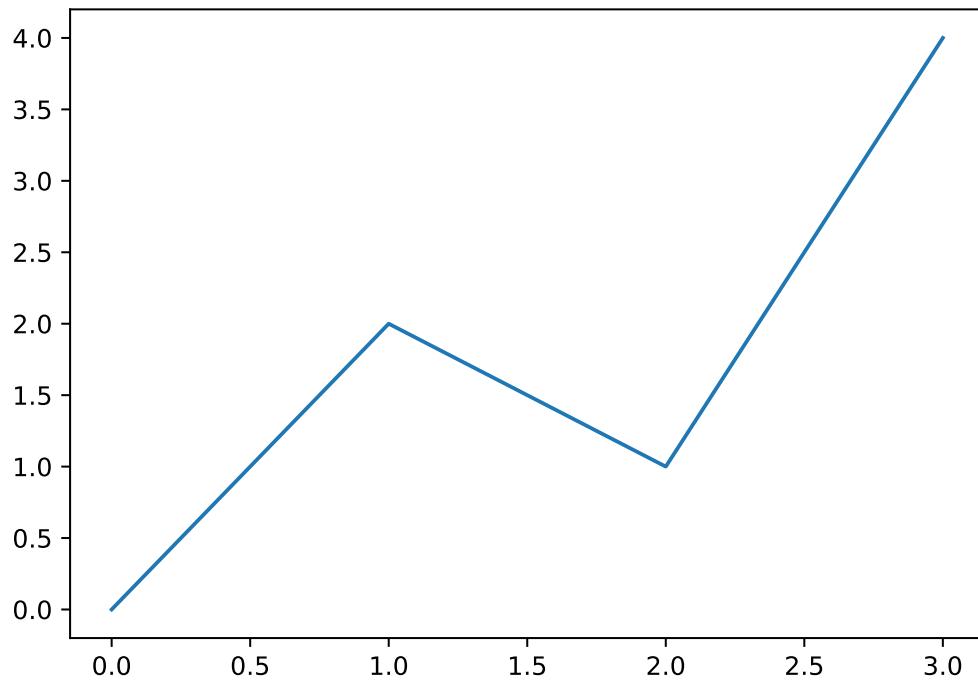
```
reticulate::conda_list()
```

```
##                                     name
## 1                               base
## 2                               mm
## 3                               mmr
## 4                               monai
## 5                               pytorch
## 6   pytorch_1.12.1_cuda_11.6
## 7       r-reticulate
## 8       sandbox
## 9       sandbox-3.9
## 10      sandbox_py_3.10
## 11      v51
##
##                                     python
## 1             C:\\\\Users\\\\RW\\\\anaconda3\\\\python.exe
## 2             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\mm\\\\python.exe
## 3             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\mmr\\\\python.exe
## 4             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\monai\\\\python.exe
## 5             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\pytorch\\\\python.exe
## 6   C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\pytorch_1.12.1_cuda_11.6\\\\python.exe
## 7             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\r-reticulate\\\\python.exe
## 8             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox\\\\python.exe
## 9             C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox-3.9\\\\python.exe
## 10            C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox_py_3.10\\\\python.exe
## 11            C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\v51\\\\python.exe
```

```
reticulate::use_condaenv(condaenv = 'sandbox-3.9')
```

```
import matplotlib.pyplot as plt
```

```
plt.plot([0, 2, 1, 4])
plt.show()
```



```
import tikzplotlib

# tikzplotlib.save("test.tex")
tikzplotlib.get_tikz_code()
```

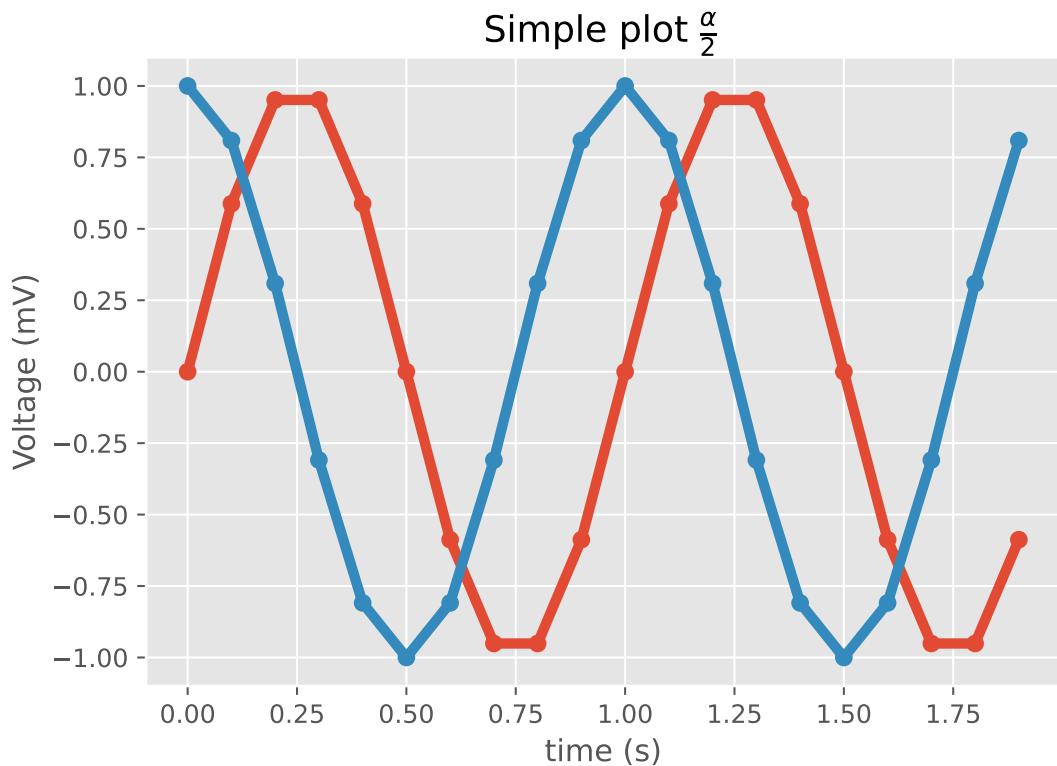
```
## '% This file was created with tikzplotlib v0.10.1.\n\\begin{tikzpicture}\n\n\\end{tikzpicture}
```

```
import matplotlib.pyplot as plt
plt.close()

import matplotlib.pyplot as plt
import numpy as np

plt.style.use("ggplot")

t = np.arange(0.0, 2.0, 0.1)
s = np.sin(2 * np.pi * t)
s2 = np.cos(2 * np.pi * t)
plt.plot(t, s, "o-", lw=4.1)
plt.plot(t, s2, "o-", lw=4.1)
plt.xlabel("time (s)")
plt.ylabel("Voltage (mV)")
plt.title("Simple plot $\frac{\alpha}{2}$")
plt.grid(True)
plt.show()
```



```
import tikzplotlib

# tikzplotlib.save("test.tex")
tikzplotlib.get_tikz_code()
```

## 13.6 animation

<https://zhuanlan.zhihu.com/p/338402487>

# Chapter 14

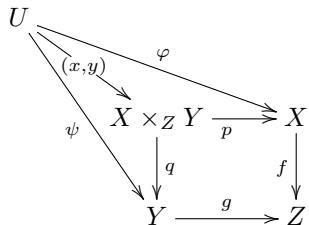
## xy-pic

<https://bookdown.org/yihui/rmarkdown-cookbook/install-latex-pkgs.html>

`tinytex::install_tinytex()`

the following xymatrix from LaTeX package xy for xy-pic is not shown or rendered in HTML:

$\$\\LaTeX\$$  can only be used in HTML, not PDF





# Chapter 15

## statistics

### 15.1 Hung Hung

population

普查 vs. 統計

random variable

$X$

sample has randomness

probability function

$$P_x(E) \in [0, 1]$$

event = subset of sample space

$E$

input event with output probability in 0 to 1

$$P_x : \{E_i\}_{i \in I} \rightarrow [0, 1]$$

target of interest

- probability function

but events are hard to be listed or enumerated

$$X : \{\omega_i\}_{i \in I} \rightarrow \mathbb{R}$$

CDF = cumulative distribution function

$$F_X(x) = P_x((-\infty, x]) = P_x(X \leq x)$$

real function is much easier to be operable, there is differentiation or difference operation target of interest

- CDF = cumulative distribution function
- probability function

target of interest

- $P_x(\cdot)$  PF = probability function
  - $f_x(x) = P_x(X = x)$  PMF = probability mass function
  - $f_x(x) = \frac{d}{dx} P_x(X \leq x)$  PDF = probability density function
- $F_x(x)$  CDF = cumulative distribution function<sup>[15.1.2.1]</sup>
- $M_x(\xi)$  MGF = moment generating function<sup>[15.1.2.7.1]</sup>
- $\varphi_x(\xi)$  CF = characteristic function<sup>[15.1.2.7.2]</sup>

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \xleftrightarrow{\text{FToC}} & f_x(x) & \leftrightarrow & P_x \\ \text{inversion formula : } & \downarrow & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge & & \uparrow \downarrow & \searrow \nearrow & \leftarrow \rho \wedge \text{supp}(f_x) \text{ is bounded} \\ & \varphi_x(\xi) & \xleftrightarrow{} & M_x(\xi) & \rightarrow & & \{\mu_n | n \in \mathbb{N}\} \end{array}$$

In population,

$$X \sim F_x(x)$$

by sampling,

$$X_1, \dots, X_i, \dots, X_n = X_i \sim F_x(x)$$

or

$$X_1, \dots, X_i, \dots, X_n = X_i \stackrel{\text{i.i.d.}}{\sim} F_x(x)$$

i.i.d. = independently identically distributed

and inference back

parametrically

$$\hat{X} \sim \hat{F}_x(x) = \hat{F}_x(x|\theta)$$

or nonparametrically

$$\hat{X} \sim \hat{F}_x(x)$$

inference is function of samples, or called random function, to estimate unknown parameters

$$\hat{\Theta} \leftarrow (X_1, \dots, X_i, \dots, X_n) = T(X_1, \dots, X_n) = T(\dots, X_i, \dots) = T(X_i)$$

corresponding CDF for inference or estimation function of sampling random variables

$$T(X_1, \dots, X_n) = T \sim F_T(t)$$

wish to be unbiased and consistent

$$\begin{cases} E(\hat{\Theta}) = \theta \Leftrightarrow E(\hat{\Theta}) - \theta = 0 & \text{unbiasedness} \\ V(\hat{\Theta}) = 0 & \text{consistency} \end{cases}$$

unbiasedness usually harder than consistency, thus usually first considered consistency.  
modeling or parameterizing with unknown parameter  $\theta$

$$F_X(x) \stackrel{M}{=} F_X(x|\theta) = F_X(x;\theta)$$

parameterization is to reduce unknown parameters from infinite ones to finite ones  
e.g. for normally distributed data

$$f_X(x) \stackrel{M}{=} f_X(x|\theta) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} = f_X(x|\mu, \sigma^2) = f_X(x|\mu, \sigma)$$

the price of parameterization is guess wrong model.

For some non-negative data, instead of normal distribution, consider distributions skewed to the right

- [gamma](#)
- [exponential](#)
- [Weibull](#)
- [log-normal](#)

topics

1.  $P_x$  probability theory
2.  $f_X(x) \stackrel{M}{=} f_X(x|\theta) = f_X(x;\theta)$  various univariable distribution
3.  $f_{\mathbf{X}}(\mathbf{x}) \stackrel{M}{=} f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\theta}) = f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta})$  multivariable distribution
4.  $T(X_1, \dots, X_n)$  inference

- point estimation

$$\hat{\mu} = \begin{cases} \bar{X} & \rightarrow \mu \\ \text{median}(X_1, \dots, X_n) & \rightarrow \mu \\ \vdots \end{cases}$$

- interval estimation = hypothesis testing

$$\begin{cases} H_0 : \theta = \theta_0 & \leftarrow T \in \{0, 1\} \\ H_1 : \theta \neq \theta_0 \end{cases}$$

5. how to find  $T$
6. behavior of random function  $T(X_1, \dots, X_n) = T \sim F_T(t)$ 
  - statistical properties of  $T$
  - asymptotic properties

$$n \rightarrow \infty \begin{cases} \text{CLT} = \text{central limit theorem} \\ \text{LLN} = \text{law of large number} \end{cases}$$

### 15.1.1 probability theory

**Definition 15.1.** sample space: The set  $S$  of all possible outcomes of an experiment is called the sample space

$$S = \{\omega_i\}_{i \in I}$$

**Definition 15.2.** event: An event  $E$  is any collection of possible outcomes of an experiment, i.e. any subset of  $S$

$$E \subseteq S$$

set operation

commutativity, associativity, distributivity

De Morgan law

pairwise disjoint = mutually exclusive

partition

#### 15.1.1.1 probability function

probability function axioms = probability function definition

Kolmogorov axioms of probability<sup>6</sup> p.72

**Definition 15.3.** probability function: Given a sample space  $S$  and its event  $E$ , a probability function is a function  $P$  satisfying

$$\begin{cases} P(S) = 1 \\ \forall E \subseteq S (P(E) \geq 0) \\ E_1, \dots, E_i, \dots \text{ are pairwise disjoint} \Rightarrow P\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} P(E_i) \end{cases}$$

tossing a dice

theorems

$$P(\emptyset) = 0$$

$$P(E) \leq 1$$

$$P(E^C) = P(\overline{E}) = 1 - P(E)$$

$$P(E_2 \cap \overline{E}_1) = P(E_2) - P(E_2 \cap E_1)$$

$$E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$$

addition rule<sup>6</sup> p.75 and extended addition rule<sup>6</sup> p.76

inclusion-exclusion principle = sieve principle

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n \left( (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{i \in \{i_1, \dots, i_k\}} E_i\right) \right)$$

symmetric difference<sup>6</sup> p.75

union probability upper-bounded by sum of individual probability

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \leq P(E_1) + P(E_2)$$

$$E_1 \cap E_2 = \emptyset \Leftrightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Boole inequality

$$P\left(\bigcup_{i \in I} E_i\right) \leq \sum_{i \in I} P(E_i)$$

$$P\left(\widehat{H_0} \mid H_0\right) = P(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha = \text{type 1 error}$$

multiple hypothesis testing

How to control the family-wise error rate?

Ideally,

FWER = family-wise error rate

$$\begin{aligned}
\alpha &= P \left( \overbrace{H_0^1 \cup \dots \cup H_0^m}^{} \middle| H_0^1 \cap \dots \cap H_0^m \right) = P(\text{reject any } H_0^i \mid \text{any } H_0^j \text{ is true}) \\
&= P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) \\
&= 1 - P \left( \bigcap_{i=1}^m \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - P \left( \text{not to reject any } H_0^i \middle| \bigcap_{j=1}^m H_0^j \right) \\
&\stackrel{H_0^j \text{ pairwise independent}}{=} 1 - \prod_{i=1}^m P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - \prod_{i=1}^m \left( 1 - P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) \right) \\
&\stackrel{\forall i, j \left[ P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = \alpha_0 \right]}{=} 1 - \prod_{i=1}^m (1 - \alpha_0) = 1 - (1 - \alpha_0)^m \\
\alpha &= 1 - (1 - \alpha_0)^m \\
\alpha_0 &= 1 - (1 - \alpha)^{\frac{1}{m}} = 1 - \sqrt[m]{1 - \alpha} \\
&\Downarrow \\
\text{set } P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) &= \alpha_0 = 1 - \sqrt[m]{1 - \alpha}
\end{aligned}$$

But condition  $H_0^j$  pairwise independent is too strong.

Practically,

$$\begin{aligned}
\alpha &= P \left( \overbrace{H_0^1 \cup \dots \cup H_0^m}^{} \middle| H_0^1 \cap \dots \cap H_0^m \right) = P(\text{reject any } H_0^i \mid \text{any } H_0^j \text{ is true}) \\
&= P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) \stackrel{P \left( \bigcup_{i \in I} E_i \right) \leq \sum_{i \in I} E_i}{\leq} \stackrel{\text{Boole inequality}}{\leq} \sum_{i=1}^m P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) \stackrel{\uparrow}{=} \sum_{i=1}^m \alpha_0 = m\alpha_0 \stackrel{\Downarrow}{=} \alpha \\
\text{let } \forall i, j \left[ P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = \alpha_0 \right] &\Rightarrow \sum_{i=1}^m P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = \sum_{i=1}^m \alpha_0 = m\alpha_0 \Rightarrow \alpha_0 = \frac{\alpha}{m} \\
&\Downarrow \\
\text{set } P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) &= \alpha_0 = \frac{\alpha}{m}
\end{aligned}$$

Bonferroni correction

$$P \left( \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) = \frac{\alpha}{m} \Rightarrow P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{} \middle| \bigcap_{j=1}^m H_0^j \right) \leq \alpha$$

Bonferroni inequality<sup>6</sup> p.77

Bonferroni inequality and Boole inequality are equivalent inequalities  
birthday problem<sup>6</sup> p.78

### 15.1.1.2 conditional probability

## 15.1.2 univariable distribution

$$\begin{aligned} & \left\{ \begin{array}{ll} P_x(X \in E) & \forall E \subseteq S \\ P_x(X \leq x) & \forall x \in \mathbb{R} \end{array} \right. \\ & \left\{ \begin{array}{ll} P_x(X \in E) & \forall E \subseteq S \\ P_x(X \leq x) = P_x((-\infty, x]) = F_x(x) & \forall x \in \mathbb{R} \end{array} \right. \\ & P_x(X \leq x) = P_x((-\infty, x]) \\ & = P_x\left(\bigcup_{\epsilon>0}(-\infty, x-\epsilon]\right) = \lim_{\epsilon \rightarrow 0} P_x((-\infty, x-\epsilon]) \\ & \leftrightarrow P_x((-\infty, x)) = P_x(E), E = (-\infty, x) \end{aligned}$$

### 15.1.2.1 cumulative distribution function

CDF = cumulative distribution function

$$\begin{aligned} F_x(x) &= P_x((-\infty, x]) = P_x(X \leq x) \\ X \sim P_x &\leftrightarrow F_x(x) \end{aligned}$$

$$X \sim F_x(x) \leftrightarrow P_x$$


---

**Definition 15.4.** CDF = cumulative distribution function: A cumulative distribution function is a function  $F : \mathbb{R} \rightarrow [0, 1]$  satisfying

$$F_x(x) = P_x((-\infty, x]) = P_x(X \leq x)$$

**Theorem 15.1.** CDF = cumulative distribution function:  $F(x)$  is a cumulative distribution function iff

$$\begin{cases} \begin{aligned} & \lim_{x \rightarrow -\infty} F(x) = 0 & \lim_{x \rightarrow +\infty} F(x) = 1 & (01) [0, 1] \\ & \forall x_1 < x_2 [F(x_1) \leq F(x_2)] & & (nd) \text{ non-decreasing} \end{aligned} \\ \begin{aligned} & \lim_{x \rightarrow x_0^+} F(x) = F(x_0) & & (rc) \text{ right-continuous} \end{aligned} \end{cases}$$


---

**Definition 15.5.** RV = r.v. = random variable

$$\begin{cases} X \text{ is a continuous RV} & \lim_{x \rightarrow x_0} F_x(x) = F_x(x_0) \\ X \text{ is a discrete RV} & F_x \text{ is a step function of } x \end{cases}$$

<sup>6</sup> p.103

**Definition 15.6.** RV = r.v. = random variable

<sup>6</sup> p.104

**Definition 15.7.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned}\mathcal{R}_X &= \left\{ x \middle| \left\{ \begin{array}{l} \omega \in S \\ x \in X(\omega) \end{array} \right. \right\} \\ &= \{x | \forall \omega \in S [x \in X(\omega)]\} \\ &= \{x | x \in X(\Omega)\} = X(\Omega)\end{aligned}$$

### 15.1.2.2 probability density function

$$\begin{cases} P_x(X \leq x) = P_x((-\infty, x]) = F_x(x) \\ P_x(X = x) = P_x(x) = ? \end{cases}$$

**Definition 15.8.** PDF = probability density function

PMF = probability mass function

$$\begin{cases} f_x(x) = \frac{d}{dx} F_x(x) & X \text{ continuous RV} \\ f_x(x) = F_x(x) - F_x(x^-) & X \text{ discrete RV} \end{cases}$$

$$\begin{cases} f_x(x) = \text{derivative of } F_x(x) & X \text{ continuous} \\ f_x(x) = \text{difference of } F_x(x) & X \text{ discrete} \end{cases}$$

$$\begin{cases} f_x(x) = \frac{d}{dx} F_x(x) \Leftrightarrow F_x(x) = \int_{-\infty}^x f_x(t) dt \\ f_x(x) = F_x(x) - F_x(x^-) \Leftrightarrow F_x(x) = \sum_{t \leq x} f_x(t) \end{cases}$$

$$\begin{cases} X \sim P_x \Leftrightarrow F_x(x) \Leftrightarrow f_x(x) & \text{e.g. probability theory} \\ X \sim F_x(x) \Leftrightarrow P_x \Rightarrow F_x(x) \stackrel{M}{=} F_x(x|\theta) & \text{e.g. survival analysis} \\ X \sim f_x(x) \Leftrightarrow F_x(x) \Leftrightarrow P_x \Rightarrow f_x(x) \stackrel{M}{=} f_x(x|\theta) & \text{e.g. general statistics} \end{cases}$$


---

**Theorem 15.2.** PDF = probability density function or PMF = probability mass function:  $f(x)$  is a probability density function or probability mass function iff

$$\begin{cases} \forall x \in \mathbb{R} [f(x) \geq 0] \\ \begin{cases} \int_{-\infty}^{+\infty} f(x) dx = 1 & t \text{ continuous} \\ \sum_{x \in X(\Omega)} f(x) = 1 & t \text{ discrete} \end{cases} \end{cases}$$


---

$$\forall E \subseteq S \left[ P_x(X \in E) = \begin{cases} \int_{x \in E} f_x(x) dx & X \text{ continuous} \\ \sum_{x \in E} f_x(x) & X \text{ discrete} \end{cases} \right]$$

$$\begin{aligned}
P_X(X = x) &= \lim_{\epsilon \rightarrow 0} P_X([x - \epsilon, x + \epsilon]) \\
&= \lim_{\epsilon \rightarrow 0} P_X(x - \epsilon \leq X \leq x + \epsilon) \\
&= \lim_{\epsilon \rightarrow 0} [F_X(x + \epsilon) - F_X(x - \epsilon)] \\
&= \begin{cases} F_X(x) - F_X(x^-) = 0 & X \text{ continuous} \\ F_X(x) - F_X(x^-) = f_X(x) & X \text{ discrete} \end{cases}
\end{aligned}$$

$$X \sim F_X(x) \Leftrightarrow P_X$$

$$Y = g(X)$$

$$\begin{cases} Y \sim F_Y(y) \Leftrightarrow f_Y(y) & \Rightarrow F_Y(y) \stackrel{M}{=} F_Y(y|\theta) \\ Y \sim f_Y(y) \Leftrightarrow F_Y(y) \Leftrightarrow P_Y & \Rightarrow f_Y(y) \stackrel{M}{=} f_Y(y|\theta) \end{cases}$$

### 15.1.2.3 range vs. support

**Definition 15.9.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned}
\mathcal{R}_X &= \left\{ x \middle| \left\{ \begin{array}{l} \omega \in S \\ x \in X(\omega) \end{array} \right\} \right\} \\
&= \{x | \forall \omega \in S [x \in X(\omega)]\} \\
&= \{x | x \in X(\Omega)\} = X(\Omega)
\end{aligned}$$

**Definition 15.10.** support

$$\text{supp}(f) = \left\{ x \middle| \left\{ \begin{array}{l} f : D \rightarrow \mathcal{R} \\ x \in D \\ f_x(x) \neq 0 \end{array} \right\} \right\}$$

**Definition 15.11.** support of r.v. = support of RV = the support of a random variable

$$\text{supp}(f_x) = \left\{ x \middle| \left\{ \begin{array}{l} x \in X(\Omega) \\ f_x(x) \neq 0 \end{array} \right\} \right\} \stackrel{f_x(x) \geq 0}{=} \left\{ x \middle| \left\{ \begin{array}{l} x \in X(\Omega) \\ f_x(x) > 0 \end{array} \right\} \right\}$$

### 15.1.2.4 continuous monotone transformation

**Theorem 15.3.** Random variable  $Y$  is monotone transformation of random variable  $X$ ,

i.e.  $\begin{cases} X \sim F_X(x) \Leftrightarrow f_X(x) \\ Y = g(X) \begin{cases} \forall x_1 < x_2 [g(x_1) < g(x_2)] \\ \forall x_1 < x_2 [g(x_1) > g(x_2)] \end{cases} \end{cases} \Rightarrow \exists g^{-1} : Y \rightarrow X, \text{ then}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Proof:

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) \\ &= P(g(X) \leq y) \begin{cases} \forall x_1 < x_2 [g(x_1) < g(x_2)] \Leftrightarrow \forall g(x_1) < g(x_2) [x_1 < x_2] \\ \forall x_1 < x_2 [g(x_1) > g(x_2)] \Leftrightarrow \forall g(x_1) > g(x_2) [x_1 < x_2] \end{cases} \\ &= \begin{cases} P_X(X \leq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ P_X(X \geq g^{-1}(y) = x) & \forall y_1 > y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} P_X(X \leq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ P_X(X \geq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ 1 - F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ F_Y(y) &= \begin{cases} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ 1 - F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \begin{cases} \frac{d}{dy} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{d}{dy} [1 - F_X(g^{-1}(y))] & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} \frac{dF_X(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{-dF_X(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} \frac{dF_X(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{dF_X(g^{-1}(y))}{dg^{-1}(y)} - \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} & \begin{cases} \frac{dg^{-1}(y)}{dy} \geq 0 \\ f_X(g^{-1}(y)) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \\ f_X(g^{-1}(y)) \frac{-dg^{-1}(y)}{dy} & \begin{cases} \frac{-dg^{-1}(y)}{dy} \geq 0 \\ f_X(g^{-1}(y)) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \end{cases} \\ f_Y(y) &= \begin{cases} f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} & \frac{dg^{-1}(y)}{dy} \geq 0 \\ f_X(g^{-1}(y)) \frac{-dg^{-1}(y)}{dy} & \frac{-dg^{-1}(y)}{dy} \geq 0 \end{cases} \\ f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| \end{aligned}$$

□

segment  $g(X)$  into monotone functions

For example,  $\begin{cases} g(x) = x^2 \\ Y = g(X) \end{cases} \Rightarrow Y = g(X) = X^2,$

$$\begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases}$$

$$\begin{aligned} & Y = g(X) = X^2 \\ \Rightarrow & Y = \begin{cases} X^2 = g(X) & X \geq 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2 [X_1^2 < X_2^2] \\ X^2 = g(X) & X < 0 \Leftrightarrow X \in (-\infty, 0) \Rightarrow \forall X_1 < X_2 [X_1^2 > X_2^2] \end{cases} \\ \Rightarrow & X = \begin{cases} \sqrt{Y} = g^{-1}(Y) & X \geq 0 \Rightarrow \forall X_1^2 < X_2^2 [X_1 < X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 < X_2] \\ -\sqrt{Y} = g^{-1}(Y) & X < 0 \Rightarrow \forall X_1^2 < X_2^2 [X_1 > X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 > X_2] \end{cases} \\ \Rightarrow & X = \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \geq 0 \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X < 0 \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases} \end{aligned}$$

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) = P(X^2 \leq y) \\ &= P(\{X^2 \leq y\} \cap (\{X < 0\} \cup \{X \geq 0\})) \\ &= P((\{X^2 \leq y\} \cap \{X < 0\}) \cup (\{X^2 \leq y\} \cap \{X \geq 0\})) \\ &= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\ &\quad - P((\{X^2 \leq y\} \cap \{X < 0\}) \cap (\{X^2 \leq y\} \cap \{X \geq 0\})) \\ &= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) - P(\emptyset) \\ &= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) - 0 \\ &= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\ &= P(\{-X \leq \sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\ &= P(\{X \geq -\sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\ &= P_X(-\sqrt{y} \leq X < 0) + P_X(0 \leq X \leq \sqrt{y}) \\ &= [F_X(0) - F_X(-\sqrt{y})] + [F_X(\sqrt{y}) - F_X(0)] \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

□

Another example,  $\begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases},$

$$\begin{aligned}
Y &= g(X) = X^2 \\
\Rightarrow Y &= \begin{cases} X^2 = g(X) & X \geq 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2 [X_1^2 < X_2^2] \\ X^2 = g(X) & -1 \leq X < 0 \Leftrightarrow X \in [-1, 0) \Rightarrow \forall X_1 < X_2 [X_1^2 > X_2^2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow \forall X_1^2 < X_2^2 [X_1 < X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 < X_2] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in [-1, 0) \Rightarrow \forall X_1^2 < X_2^2 [X_1 > X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 > X_2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \in [0, \infty) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ \sqrt{Y} = g^{-1}(Y) & Y \in [0, 1] \Rightarrow X \in [0, 1] \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases}
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= P_Y(Y \leq y) = P(X^2 \leq y) \begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases} \\
&= P(\{X^2 \leq y\} \cap (\{X < 0\} \cup \{X \geq 0\})) = \dots \text{ as } \begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases} \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\
&= P(\{-X \leq \sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\
&= \begin{cases} P(\{-X \leq \sqrt{y}\} \cap \{X < 0\} \cap \{X > 1\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\} \cap \{X > 1\}) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P(\{-X \leq \sqrt{y}\} \cap \{X < 0\} \cap \{X \geq -1\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\} \cap \{X \leq 1\}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} P(\emptyset) + P_x(1 < X \leq \sqrt{y}) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P(\{X \geq -\sqrt{y}\} \cap \{X < 0\} \cap \{X \geq -1\}) + P_x(0 \leq X \leq \min\{\sqrt{y}, 1\}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} 0 + [F_x(\sqrt{y}) - F_x(1)] & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P_x(\max\{-1, -\sqrt{y}\} \leq X < 0) + P_x(0 \leq X \leq \sqrt{y}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} F_x(\sqrt{y}) - F_x(1) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P_x(-\sqrt{y} \leq X < 0) + [F_x(\sqrt{y}) - F_x(0)] & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} F_x(\sqrt{y}) - F_x(1) & y > 1 \\ F_x(\sqrt{y}) - F_x(-\sqrt{y}) & -1 \leq y \leq 1 \end{cases}
\end{aligned}$$

□

### 15.1.2.5 discrete monotone transformation

$$\begin{cases} Y = g(X) = X^2 \\ X \text{ discrete} \Rightarrow Y \text{ discrete} \end{cases}$$

$$\begin{aligned}
f_Y(y) &= P_Y(Y = y) \\
&= P(X^2 = y) \\
&= P(\{X = \sqrt{y}\} \cup \{X = -\sqrt{y}\}) \\
&= P(\{X = \sqrt{y}\}) + P(\{X = -\sqrt{y}\}) - P(\{X = \sqrt{y}\} \cap \{X = -\sqrt{y}\}) \\
&= P_X(X = \sqrt{y}) + P_X(X = -\sqrt{y}) - P(\emptyset) \\
&= P_X(X = \sqrt{y}) + P_X(X = -\sqrt{y}) - 0 \\
&= P_X(X = \sqrt{y}) + P_X(X = -\sqrt{y}) \\
&= f_X(\sqrt{y}) + f_X(-\sqrt{y})
\end{aligned}$$

□

**Theorem 15.4.** *discrete monotone transformation*

$$\begin{array}{c}
\left\{ \begin{array}{l} Y = g(X) \\ X \text{ discrete} \Rightarrow Y \text{ discrete} \end{array} \right. \\
\Downarrow \\
f_Y(y) = \sum_{\{x|g(x)=y\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x)
\end{array}$$

Proof:

$$\begin{aligned}
f_Y(y) &= P_Y(Y = y) \\
&= P(g(X) = y) = \sum_{t \in \{x|g(x)=y\}} f_X(t) = \sum_{x \in \{x|g(x)=y\}} f_X(x) = \sum_{\{x|g(x)=y\}} f_X(x) \\
&= P_X(X = g^{-1}(y)) = \sum_{t \in \{x|x=g^{-1}(y)\}} f_X(t) = \sum_{x \in \{x|x=g^{-1}(y)\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x) \\
f_Y(y) &= P_Y(Y = y) \\
&= P(g(X) = y) = \sum_{t \in \{x|g(x)=y\}} f_X(t) = \sum_{x \in \{x|g(x)=y\}} f_X(x) = \sum_{\{x|g(x)=y\}} f_X(x) \\
&= P_X(X = g^{-1}(y)) = \sum_{t \in \{x|x=g^{-1}(y)\}} f_X(t) = \sum_{x \in \{x|x=g^{-1}(y)\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x)
\end{aligned}$$

□

**Theorem 15.5.** *probability integral transformation*

$$\begin{array}{c}
\left\{ \begin{array}{l} X \text{ continuous} \quad (c) \\ X \sim F_X(x) \quad (d) \\ Y = F_X(X) \quad (t) \end{array} \right. \\
\Downarrow \\
F_Y(y) = y, \forall y \in [0, 1] \\
\Updownarrow \text{def.} \\
Y \sim U = U(y) \Leftrightarrow Y \sim U(y) \Leftrightarrow Y \text{ is uniformly distributed on } [0, 1]
\end{array}$$

Proof:

$$\begin{aligned}
F_Y(y) &= \Pr_Y(Y \leq y) \stackrel{(t)}{=} \Pr(F_X(X) \leq y), \forall x_1 < x_2 [F_X(x_1) < F_X(x_2)] \Rightarrow \begin{cases} \exists F_X^{-1} : Y \rightarrow X \\ \forall y_1 < y_2 [F_X^{-1}(y_1) < F_X^{-1}(y_2)] \end{cases} \\
&= \Pr_X(X \leq F_X^{-1}(y) = x) = \Pr_X(X \leq x), x = F_X^{-1}(y) \\
&= \Pr_X(X \leq x) = F_X(x) \stackrel{x=F_X^{-1}(y)}{=} F_X(F_X^{-1}(y)) = y \\
F_Y(y) &= y
\end{aligned}$$

□

Note:

According to Theorem 15.5,

$$\begin{aligned}
U &= F_X(X) \stackrel{15.5}{\sim} U(u) \text{ on } [0, 1] \\
\Rightarrow X &= F_X^{-1}(U) \wedge X \sim F_X(x) \Rightarrow F_X^{-1}(U) = X \sim F_X(x) \Rightarrow F_X^{-1}(U) \sim F_X(x) \\
\Rightarrow X &= F_X^{-1}(U) \sim F_X(x) \\
&\text{i.e. uniform random variables substituted into the inverse of } F_X, \\
&\text{we can get random variables follow } F_X(x)
\end{aligned}$$

#### 15.1.2.6 expected value

$$\mathbb{E}(g(X)) = \mathbb{E}[g(X)] = \mathbb{E}g(X) = \mathbb{E}[g(X)] = \mathbb{E}g(X)$$

**Definition 15.12.** expected value: The expected value of a random variable  $g(X)$  is

$$\mathbb{E}(g(X)) = \mathbb{E}[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}$$

<sup>6</sup> p.126

**Definition 15.13.** expected value or expectation function: The expected value of a random variable  $X$  is

$$\mathbb{E}(X) = \mathbb{E}[X] = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx = 1 & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) = 1 & X \text{ discrete} \end{cases}$$

**Theorem 15.6.** *the rule of the lazy statistician*

*the law of the unconscious statistician = the LOTUS*

$$\mathbb{E}(g(X)) = \mathbb{E}[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}$$

Proof:<sup>7</sup> p.162 for p.119

Discrete case:

to be proved

Continuous case:

to be proved

□

By linearity of  $\int$  and  $\sum$ , expected values have the following properties or theorems,

- $E[a_1g_1(X_1) + a_2g_2(X_2) + c] = a_1E[g_1(X_1)] + a_2E[g_2(X_2)] + c$
- $\forall x \in \mathbb{R} [g(x) \geq 0] \Rightarrow E[g(X)] \geq 0$
- $\forall x \in \mathbb{R} [g_1(x) \geq g_2(x)] \Rightarrow E[g_1(X)] \geq E[g_2(X)]$
- $\forall x \in \mathbb{R} [a \leq g(x) \leq b] \Rightarrow a \leq E[g(X)] \leq b$

**Theorem 15.7.**  $E[X]$  minimizes Euclidean distance  $E[(X - b)^2]$  over  $b$ , i.e.

$$E[X] = \arg \min_b E[(X - b)^2]$$

Proof:

$$\begin{aligned} E[(X - b)^2] &= E[(X - E[X] + E[X] - b)^2] \\ &= E[\{(X - E[X]) + (E[X] - b)\}^2] \\ &= E[(X - E[X])^2 + 2(X - E[X])(E[X] - b) + (E[X] - b)^2] \\ &= E[(X - E[X])^2] + 2(E[X] - b)E[(X - E[X])] + E[(E[X] - b)^2] \\ &= E[(X - E[X])^2] + 2(E[X] - b)E[X - E[X]] + (E[X] - b)^2 \\ &= E[(X - E[X])^2] + 2(E[X] - b)(E[X] - E[X]) + (E[X] - b)^2 \\ &= E[(X - E[X])^2] + 2(E[X] - b)0 + (E[X] - b)^2 \\ &= E[(X - E[X])^2] + 0 + (E[X] - b)^2 \\ &= E[(X - E[X])^2] + (E[X] - b)^2 \stackrel{(E[X]-b)^2 \geq 0}{\geq} E[(X - E[X])^2] \\ E[(X - b)^2] &\geq E[(X - E[X])^2] \\ &\Downarrow \\ E[(X - b)^2] &= E[(X - E[X])^2] \text{ holds if } (E[X] - b)^2 = 0 \Rightarrow b = E[X] \Rightarrow E[X] = \arg \min_b E[(X - b)^2] \end{aligned}$$

□

Note:

When  $b = E[X]$ ,  $E[(X - b)^2]$  has minimum loss  $E[(X - E[X])^2] = V[X] = V(X)$ , i.e. defintion of variance appears.

---

**Theorem 15.8.** median  $[X]$  minimizes  $E[|X - b|]$  over  $b$ , i.e.

$$\text{median}[X] = \arg \min_b E[|X - b|]$$

Proof:

to be proved

□

Note:

When  $b = \text{median}[X]$ ,  $E[(X - b)^2]$  has minimum loss  $E[|X - \text{median}[X]|]$ , i.e. defintion of MAD(mean absolute deviation) in robust statistics appears.

---

**Definition 15.14.** indicator function

$$\begin{aligned} 1(E) &= 1(x \in E) = 1(\{x \in E\}) = 1(\{x | x \in E\}) = \begin{cases} 1 & \text{if } E \\ 0 & \text{if } \bar{E} = E^C \end{cases} \\ &= \begin{cases} 1 & \text{if event } E \text{ occurs} \\ 0 & \text{if event } E \text{ does not occur} \end{cases} \end{aligned}$$

Note:

**Theorem 15.9.** probability as expected value

$$P_x(E) = P(x \in E) = E[1(X \in E)]$$

Proof:

$$\begin{aligned} P_x(E) &= P(x \in E) = \int_{x \in E} f_x(x) dx = \int_E f_x(x) dx \\ &= \int 1(x \in E) f_x(x) dx \\ &= \int g(x) f_x(x) dx, g(x) = 1(x \in E) \\ &= E[g(X)], g(X) = 1(X \in E) \\ &= E[1(X \in E)] \end{aligned}$$

$$P_x(E) = P(x \in E) = E[1(X \in E)]$$

□

Iverson bracket [https://en.wikipedia.org/wiki/Iverson\\_bracket](https://en.wikipedia.org/wiki/Iverson_bracket)

$$\begin{cases} v(p(x)) = \text{T} & \Leftrightarrow [p(x)] = 1 \\ v(p(x)) = \text{F} & \Leftrightarrow [p(x)] = 0 \end{cases}$$

$$[p(x)] = \begin{cases} 1 & v(p(x)) = \text{T} \\ 0 & v(\neg p(x)) = \text{T} \end{cases} = \begin{cases} 1 & p(x) \\ 0 & \neg p(x) \end{cases}$$

negation = NOT

$$[\neg p] = 1 - [p]$$

in set theory or domain of events,

$$1(\bar{E}) = 1 - 1(E)$$

conjunction = AND

$$[p \wedge q] = [p][q]$$

in set theory or domain of events,

$$1(E_1 \cap E_2) = 1(E_1)1(E_2)$$

disjunction = OR

$$[p \vee q] = [p] + [q] - [p][q] = [p] + [q] - [p \wedge q]$$

Proof:

in set theory or domain of events,

$$\begin{aligned} 1(E_1 \cup E_2) &\stackrel{\text{de Moivre}}{=} 1(\overline{\bar{E}_1 \cap \bar{E}_2}) \\ &= 1 - 1(\bar{E}_1 \cap \bar{E}_2) = 1 - 1(\bar{E}_1)1(\bar{E}_2) \\ &= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\ &= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\ &= 1 - [1 - 1(E_1) - 1(E_2) + 1(E_1)1(E_2)] \\ &= 1(E_1) + 1(E_2) - 1(E_1)1(E_2) \\ &= 1(E_1) + 1(E_2) - 1(E_1 \cap E_2) \end{aligned}$$

$$1(E_1 \cup E_2) = 1(E_1) + 1(E_2) - 1(E_1)1(E_2) = 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)$$

□

implication = conditional

$$\begin{aligned}[p \rightarrow q] &= [\neg p \vee q] \\ &= [\neg p] + [q] - [\neg p][q] \\ &= 1 - [p] + [q] - (1 - [p])[q] \\ &= 1 - [p] + [p][q]\end{aligned}$$

exclusive disjunction = XOR

$$\begin{aligned}[p \vee q] &= [p \oplus q] = |[p] - [q]| = ([p] - [q])^2 \\ &= [p](1 - [q]) + (1 - [p])[q]\end{aligned}$$

biconditional = XNOR

$$[p \leftrightarrow q] = [p \odot q] = [\neg(p \oplus q)] = [\neg(p \vee q)] = ([p] + (1 - [q]))((1 - [p]) + [q])$$

Kronecker delta

$$\delta_{ij} = [i = j]$$

single-argument notation

$$\delta_i = \delta_{i0} = \begin{cases} 1 & i = j = 0 \\ 0 & i \neq j = 0 \end{cases}$$

sign function

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} = [x > 0] - [x < 0]$$

absolute function

$$\begin{aligned}|x| &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases} = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases} \\ &= \begin{cases} x \cdot 1 & x > 0 \\ x \cdot 0 & x = 0 \\ x \cdot (-1) & x < 0 \end{cases} = \begin{cases} x \cdot \operatorname{sgn}(x) & x > 0 \\ x \cdot \operatorname{sgn}(x) & x = 0 \\ x \cdot \operatorname{sgn}(x) & x < 0 \end{cases} \\ &= x \cdot \operatorname{sgn}(x) = x([x > 0] - [x < 0]) = x[x > 0] - x[x < 0]\end{aligned}$$

binary min and max function

$$\max(x, y) = x [x > y] + y [x \leq y]$$

$$\min(x, y) = x [x \leq y] + y [x > y]$$

binary max function

$$\max(x, y) = \frac{x + y + |x - y|}{2}$$

floor and ceiling functions

floor function

$$\begin{aligned} \lfloor x \rfloor &= n, n \leq x < n + 1 \\ &= \sum_{n \in \mathbb{N}} n [n \leq x < n + 1] \end{aligned}$$

ceiling function

$$\begin{aligned} \lceil x \rceil &= n, n - 1 < x \leq n \\ &= \sum_{n \in \mathbb{N}} n [n - 1 < x \leq n] \end{aligned}$$

Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} = [x > 0] = 1_{(0, \infty)}(x)$$

or conveniently define “unit step function”

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} = [x \geq 0] = 1_{[0, \infty)}(x)$$

ramp function = rectified linear unit activation function = ReLU

$$\text{ReLU}(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} = x [x \geq 0]$$

indicator function

$$A \subseteq X \Rightarrow \begin{cases} 1_A : X \rightarrow \{0, 1\} & \Leftrightarrow x \in X \xrightarrow{1_A} \{0, 1\} \\ 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} & = [x \in A] = \begin{cases} 1 & v(x \in A) = T \\ 0 & v(\neg(x \in A)) = T \end{cases} \end{cases}$$

$A, B \subseteq \Omega$ ,

$$A = B \Leftrightarrow 1_A = 1_B$$

$$A = \Omega \Leftrightarrow 1_A(x) = 1$$

$$A = \emptyset \Leftrightarrow 1_A(x) = 0$$

**Theorem 15.10.** *subset indicator order*

$$A \subset B \Rightarrow 1_A(x) \leq 1_B(x)$$

Proof:

$$\begin{aligned} & \forall x (1_A(x) = 1 \Rightarrow 1_B(x) = 1) \\ \Leftrightarrow & \forall x (\neg 1_A(x) = 1 \vee 1_B(x) = 1) \\ \Leftrightarrow & \forall x (\neg (1_A(x) = 1) \wedge \neg 1_B(x) = 1) \\ \Rightarrow & \neg \exists x (1_A(x) = 1 \wedge 1_B(x) = 0) \\ \Rightarrow & \neg \exists x (1_B(x) = 0 < 1 = 1_A(x)) \\ \Rightarrow & \neg \exists x (1_B(x) < 1_A(x)) \\ \Rightarrow & \forall x (1_B(x) \geq 1_A(x)) \end{aligned}$$

□

in set theory or domain of events,

$$1(E_1 \cap E_2) = 1(E_1) 1(E_2)$$

$$1(\overline{E}) = 1 - 1(E)$$

$$1(E_1 \cup E_2) = 1(E_1) + 1(E_2) - 1(E_1) 1(E_2) = 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)$$

expectation in many perspectives

$$Y = g(X)$$

$$\int_{-\infty}^{+\infty} y f_Y(y) dy = E[Y] = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

$$E_Y[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy = E[Y] = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx = E_X[g(X)]$$

$$\mathbb{E}_Y [Y] = \mathbb{E}_X [g(X)]$$


---

$$\begin{aligned} & \mathbb{E}[a_1 g_1(X_1) + a_2 g_2(X_2) + c], \begin{cases} Y_1 = g_1(X_1) \\ Y_2 = g_2(X_2) \end{cases} \\ &= \mathbb{E}[a_1 Y_1 + a_2 Y_2 + c] \end{aligned}$$

$$\begin{aligned} & \mathbb{E}[a_1 g_1(X_1) + a_2 g_2(X_2) + c] = a_1 \mathbb{E}[g_1(X_1)] + a_2 \mathbb{E}[g_2(X_2)] + c \\ &= \mathbb{E}[a_1 Y_1 + a_2 Y_2 + c] = a_1 \mathbb{E}[Y_1] + a_2 \mathbb{E}[Y_2] + c \end{aligned}$$

$$a_1 \mathbb{E}[g_1(X_1)] + a_2 \mathbb{E}[g_2(X_2)] + c = a_1 \mathbb{E}_{X_1}[g_1(X_1)] + a_2 \mathbb{E}_{X_2}[g_2(X_2)] + c$$

$$a_1 \mathbb{E}[Y_1] + a_2 \mathbb{E}[Y_2] + c = a_1 \mathbb{E}_{Y_1}[Y_1] + a_2 \mathbb{E}_{Y_2}[Y_2] + c$$

$$a_1 \mathbb{E}_{X_1}[g_1(X_1)] + a_2 \mathbb{E}_{X_2}[g_2(X_2)] + c = a_1 \mathbb{E}_{Y_1}[Y_1] + a_2 \mathbb{E}_{Y_2}[Y_2] + c$$

#### 15.1.2.7 moment

**Definition 15.15.**  $n^{\text{th}}$  moment: For each integer  $n$ , the  $n^{\text{th}}$  moment of  $X$  is  $\mathbb{E}[X^n]$ .

The  $n^{\text{th}}$  central moment of  $X$  is  $\mu_n = \mathbb{E}[(X - \mathbb{E}[X])^n]$ .

$$\mathbb{E}[X^n] = \begin{cases} \int_{-\infty}^{+\infty} x^n f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^n f_X(x) & X \text{ discrete} \end{cases}$$

$$\mu = \mathbb{E}[X^1] = \mathbb{E}[X] = \begin{cases} \int_{-\infty}^{+\infty} x^1 f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) & X \text{ discrete} \end{cases}$$

$$\mu_n = \mathbb{E}[(X - \mathbb{E}[X])^n] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^n f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^n f_X(x) & X \text{ discrete} \end{cases}$$


---

1<sup>st</sup> moment of  $X$  = mean

$$\mu = \mathbb{E}[X^1] = \mathbb{E}[X] = \begin{cases} \int_{-\infty}^{+\infty} x^1 f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) & X \text{ discrete} \end{cases}$$

1<sup>st</sup> central moment of  $X$  = 0

$$\begin{aligned}\mu_1 &= \mathbb{E}[(X - \mathbb{E}[X])^1] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^1 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^1 f_x(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu) f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu) f_x(x) & X \text{ discrete} \end{cases} \\ &= \begin{cases} \int_{-\infty}^{+\infty} x f_x(x) dx - \int_{-\infty}^{+\infty} \mu f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_x(x) - \sum_{x \in X(\Omega)} \mu f_x(x) & X \text{ discrete} \end{cases} \\ &= \begin{cases} \mathbb{E}[X] - \mu \int_{-\infty}^{+\infty} f_x(x) dx & X \text{ continuous} \\ \mathbb{E}[X] - \mu \sum_{x \in X(\Omega)} f_x(x) & X \text{ discrete} \end{cases} = \begin{cases} \mathbb{E}[X] - \mu \cdot 1 & X \text{ continuous} \\ \mathbb{E}[X] - \mu \cdot 1 & X \text{ discrete} \end{cases} \\ &= \begin{cases} \mathbb{E}[X] - \mu & X \text{ continuous} \\ \mathbb{E}[X] - \mu & X \text{ discrete} \end{cases} = \begin{cases} \mathbb{E}[X] - \mathbb{E}[X] & X \text{ continuous} \\ \mathbb{E}[X] - \mathbb{E}[X] & X \text{ discrete} \end{cases} \\ &= \begin{cases} 0 & X \text{ continuous} \\ 0 & X \text{ discrete} \end{cases} = 0\end{aligned}$$

$$\mathbb{E}[(X - \mathbb{E}[X])] = 0$$

$$\mathbb{E}[X - \mathbb{E}[X]] = 0$$

$$\forall X (\mathbb{E}[X - \mathbb{E}[X]] = 0)$$

For normal distribution, actually for any distribution,

$$\begin{array}{c} X \sim n(0, 1) = \mathcal{N}(0, 1^2) \\ \Downarrow \\ \mathbb{E}[X - \mathbb{E}[X]] = 0 \end{array}$$


---

$2^{\text{nd}}$  central moment of  $X$  = variance

$$\begin{aligned}\mu_2 &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x) & X \text{ discrete} \end{cases} \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] = V[X] = V(X)\end{aligned}$$

For normal distribution,

$$\begin{array}{c} X \sim n(0, 1) = \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ \Downarrow \\ V[X] = V(X) = 1 \end{array}$$

variance properties

$$V[aX + b] = a^2 V[X]$$

Proof:

to be proved

□

3<sup>rd</sup> central moment of  $X$

$$\mu_3 = E[(X - E[X])^3] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^3 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^3 f_x(x) & X \text{ discrete} \end{cases}$$

skewness

偏度

$$\begin{aligned} \text{skewness}[X] &= \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{\left(E[(X - E[X])^2]\right)^{\frac{3}{2}}} \\ &= \begin{cases} \frac{\int_{-\infty}^{+\infty} (x - \mu)^3 f_x(x) dx}{\left(\int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx\right)^{\frac{3}{2}}} & X \text{ continuous} \\ \frac{\sum_{x \in X(\Omega)} (x - \mu)^3 f_x(x)}{\left(\sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x)\right)^{\frac{3}{2}}} & X \text{ discrete} \end{cases} \end{aligned}$$

For normal distribution,

$$\begin{aligned} X \sim n(0, 1) &= \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ \Downarrow \\ \text{skewness}[X] &= \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{1^{\frac{3}{2}}} = 0 \end{aligned}$$

Proof:

to be proved

□

$4^{\text{th}}$  central moment of  $X$

$$\mu_4 = E[(X - E[X])^4] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^4 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^4 f_x(x) & X \text{ discrete} \end{cases}$$

kurtosis

峰度

$$\begin{aligned} \text{kurtosis}[X] &= \frac{\mu_4}{\mu_2^2} = \frac{E[(X - E[X])^4]}{(V[X])^2} = \frac{E[(X - E[X])^4]}{\left(E[(X - E[X])^2]\right)^2} \\ &= \begin{cases} \frac{\int_{-\infty}^{+\infty} (x - \mu)^4 f_x(x) dx}{\left(\int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx\right)^2} & X \text{ continuous} \\ \frac{\sum_{x \in X(\Omega)} (x - \mu)^4 f_x(x)}{\left(\sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x)\right)^2} & X \text{ discrete} \end{cases} \end{aligned}$$

For normal distribution,

$$\begin{aligned} X \sim n(0, 1) &= \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ \downarrow \\ \text{kurtosis}[X] &= \frac{E[(X - E[X])^4]}{(V[X])^2} = \frac{[(X - E[X])^4]}{1^2} = 3 \end{aligned}$$

Proof:

to be proved

□

For normal distribution,

$$\begin{aligned} X \sim n(0, 1) &= \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ \downarrow \end{aligned}$$

$$\left\{ \begin{array}{ll} \mu = E[X] & = 0 \\ \mu_1 = E[X - E[X]] & = 0 \\ \text{variance}[X] = V[X] = E[(X - E[X])^2] & = 1 \\ \text{skewness}[X] = \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} & = 0 \\ \text{kurtosis}[X] = \frac{E[(X - E[X])^4]}{(V[X])^2} & = 3 \end{array} \right.$$

$$\begin{aligned} \mu &= E[X] = 0 \\ \mu_1 &= E[X - E[X]] = 0 \\ \text{variance}[X] &= V[X] = E[(X - E[X])^2] = 1 \\ \text{skewness}[X] &= \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} = 0 \\ \text{kurtosis}[X] &= \frac{E[(X - E[X])^4]}{(V[X])^2} = 3 \end{aligned}$$


---

$$X \sim F_x(x) \leftrightarrow f_x(x) \rightarrow \{\mu_n | n \in \mathbb{N}\} = \left\{ \mu_n \left| \begin{cases} n \in \mathbb{N} \\ \mu_n = E[(X - E[X])^n] \end{cases} \right. \right\}$$


---

#### 15.1.2.7.1 moment generating function

**Definition 15.16.** MGF = moment generating function: The moment generating function of  $X$  is  $M(\xi) = M_x(\xi) = E[e^{\xi X}]$ , provided that the expression exists for  $t \approx 0$ .

$$M(t) = M_x(t) = E[e^{tX}]$$

$$M(\xi) = M_x(\xi) = E[e^{\xi X}]$$

**Theorem 15.11.** moment generating function(MGF) generating moment

$$M_x^{(n)}(\xi) = E[X^n]$$

where

$$M_x^{(n)}(\xi) = \frac{d^n}{d\xi^n} M_x(\xi)$$

Proof:

to be proved

□

$$X \sim F_x(x) \leftrightarrow f_x(x) \rightarrow \{\mu_n | n \in \mathbb{N}\} = \left\{ \mu_n \middle| \begin{cases} n \in \mathbb{N} \\ \mu_n = \mathbb{E}[(X - \mathbb{E}[X])^n] \end{cases} \right\}$$

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \leftrightarrow & f_x(x) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \\ & & & & \downarrow & \nearrow & \\ & & & & M_x(\xi) & & \end{array}$$

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \leftrightarrow & f_x(x) & & \\ & & & & \downarrow & \searrow & \\ & & & & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \end{array}$$

**Theorem 15.12.** If  $X$  and  $Y$  have bounded support, then  $\forall u [F_x(u) = F_y(u)]$  iff  $\forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n])$ .

$$\forall u [F_x(u) = F_y(u)] \Rightarrow \forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n])$$

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n]) \\ \left\{ \begin{array}{l} \text{supp}(f_x) \text{ is bounded} \\ \text{supp}(f_y) \text{ is bounded} \end{array} \right. \Rightarrow \forall u [F_x(u) = F_y(u)] \end{array} \right.$$

Proof:

to be proved

□

**Theorem 15.13.** If  $M_x(t)$  and  $M_y(t)$  exist, then  $\forall u [F_x(u) = F_y(u)]$  iff  $\forall t \approx 0 [M_x(t) = M_y(t)]$ .

$$\forall u [F_x(u) = F_y(u)] \Rightarrow \forall t \approx 0 [M_x(t) = M_y(t)]$$

$$\left\{ \begin{array}{l} \forall t \approx 0 [M_x(t) = M_y(t)] \\ \left\{ \begin{array}{l} \exists M_x(t) \in \mathbb{R} \\ \exists M_y(t) \in \mathbb{R} \end{array} \right. \Rightarrow \forall u [F_x(u) = F_y(u)] \end{array} \right.$$

Proof:

to be proved

□

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \leftrightarrow & f_x(x) & & \\ & & \uparrow & \downarrow & \searrow \nwarrow & \leftarrow \wedge \text{ supp}(f_x) \text{ is bounded} & \\ & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] & \wedge & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} & \end{array}$$

### 15.1.2.7.2 characteristic function

**Definition 15.17.** CF = characteristic function: The characteristic function of  $X$  is  $\varphi(\xi) = \varphi_x(\xi) = E[e^{i\xi X}]$ , provided that the expression always exists.

$$\varphi(t) = \varphi_x(t) = E[e^{itX}]$$

$$\varphi(\xi) = \varphi_x(\xi) = E[e^{i\xi X}]$$

Note:

1.  $\varphi(\xi) = \varphi_x(\xi) = E[e^{i\xi X}]$  always exists.

$$\forall X (\varphi(\xi) = \varphi_x(\xi) = E[e^{i\xi X}] \in \mathbb{R})$$

2. moment generating function to characteristic function

$$M(\xi) = M_x(\xi) = E[e^{\xi X}] \in \mathbb{R} \Rightarrow M_x(i\xi) = \varphi_x(\xi)$$

3. inversion theorem or inversion formula

For  $a < b$ ,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^{+T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_x(t) dt \\ &= P(a < X < b) + \frac{1}{2} [P(X = a) + P(X = b)] \end{aligned}$$

i.e.  $\varphi_x(\xi)$  determines  $F_x(x)$ .

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \leftrightarrow & f_x(x) & & \\ \uparrow & & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge & \uparrow \downarrow & \searrow \nearrow & \leftrightarrow \rho \wedge \text{supp}(f_x) \text{ is bounded} \\ \varphi_x(\xi) & & \Leftrightarrow & M_x(\xi) & \rightarrow & & \{\mu_n | n \in \mathbb{N}\} \end{array}$$

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \xleftrightarrow{\text{FToC}} & f_x(x) & \leftrightarrow & P_x \\ \text{inversion formula :} & & \downarrow & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge & \uparrow \downarrow & \searrow \nearrow & \leftrightarrow \rho \wedge \text{supp}(f_x) \text{ is bounded} \\ & & \varphi_x(\xi) & \Leftrightarrow & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \end{array}$$


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MGF theorems

<https://www.youtube.com/watch?v=fSbs6im6wqY&t=524s>

**Theorem 15.14.** If

### 15.1.2.8 common families of distributions

mean and variance of discrete probability distributions<sup>[43.5]</sup>

#### 15.1.2.8.1 discrete distribution

##### 15.1.2.8.1.1 discrete uniform distribution

$$\begin{aligned} X &\sim \mathcal{DU}(1, N) \\ &\Updownarrow \\ \begin{cases} f_x(x|N) = \frac{1}{N} \\ x \in X(\Omega) = \{1, 2, \dots, N\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \frac{N+1}{2} \\ V[X] = \frac{(N+1)(N-1)}{12} \end{cases} \end{aligned}$$

$$\begin{aligned} E[X] &= \sum_{x \in X(\Omega)} x f_x(x|N) = \sum_{x=1}^N x \frac{1}{N} \\ &= \frac{1}{N} \sum_{x=1}^N x = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2} \\ E[X] &= \frac{N+1}{2} \end{aligned}$$

$$\begin{aligned} V[X] &= \sum_{x \in X(\Omega)} (x - E[X])^2 f_x(x|N) = \sum_{x=1}^N \left( x - \frac{N+1}{2} \right)^2 \frac{1}{N} \\ &= \frac{1}{N} \sum_{x=1}^N \left[ x^2 - (N+1)x + \left( \frac{N+1}{2} \right)^2 \right] \\ &= \frac{1}{N} \left[ \sum_{x=1}^N x^2 - (N+1) \sum_{x=1}^N x + N \left( \frac{N+1}{2} \right)^2 \right] \\ &= \frac{1}{N} \left[ \frac{N(N+1)(2N+1)}{6} - (N+1) \frac{N(N+1)}{2} + N \left( \frac{N+1}{2} \right)^2 \right] \\ &= (N+1) \left[ \frac{2N+1}{6} - \frac{N+1}{2} + \frac{N+1}{4} \right] = (N+1) \frac{4N+2-(3N+3)}{12} \\ &= \frac{(N+1)(N-1)}{12} \\ V[X] &= \frac{(N+1)(N-1)}{12} \end{aligned}$$


---

$$X \sim \mathcal{DU}(a, b) \Updownarrow \begin{cases} f_x(x|N) = \frac{1}{N} = \frac{1}{b-a+1} \\ x \in X(\Omega) = \{a, a+1, \dots, b\} \end{cases} \quad N = b - a + 1 \quad \Updownarrow F_x(x|a, b) = \frac{\lfloor x \rfloor - a + 1}{b - a + 1}$$

### 15.1.2.8.1.2 hypergeometric distribution

$$\begin{aligned}
 X &\sim \mathcal{HG}(N, M, K) \\
 f_X(x|N, M, K) &= \frac{\binom{N-M}{K-x} \binom{M}{x}}{\binom{N}{K}} \\
 x \in X(\Omega) &= \{\max\{0, K - (N - M)\}, \dots, \min\{K, M\}\} \\
 &\quad \Downarrow K \ll N, M \\
 x \in X(\Omega) &= \{0, 1, \dots, K\} \\
 &\quad \Downarrow \\
 \begin{cases} E[X] = \frac{KM}{N} \\ V[X] = \frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)} \end{cases}
 \end{aligned}$$

### 15.1.2.8.1.3 Bernoulli distribution

$$\begin{aligned}
 X &\sim \mathcal{B}(p), p = P(X = 1) \\
 f_X(x|p) &= (1-p)^{1-x} p^x \\
 x \in X(\Omega) &= \{0, 1\} \\
 &\quad \Downarrow \\
 \begin{cases} E[X] = p \\ V[X] = p(1-p) \\ M_x(\xi) = (1-p) + pe^\xi \end{cases}
 \end{aligned}$$

### 15.1.2.8.1.4 binomial distribution independent and identical Bernoulli trials

$$\begin{aligned}
 X &\sim \mathcal{B}(n, p), p = P(X = 1) \\
 f_X(x|n, p) &= \binom{n}{x} (1-p)^{n-x} p^x \\
 x \in X(\Omega) &= \{0, 1, \dots, n\} \\
 &\quad \Downarrow \\
 \begin{cases} E[X] = np \\ V[X] = np(1-p) \\ M_x(\xi) = [(1-p) + pe^\xi]^n \end{cases}
 \end{aligned}$$

reparameterization technique

$$\begin{aligned}
M_X(\xi) &= \text{E} [e^{\xi X}] = \sum_{x \in X(\Omega)} e^{\xi x} f_X(x|n,p) \\
&= \sum_{x=1}^n e^{\xi x} \binom{n}{x} (1-p)^{n-x} p^x = \sum_{x=1}^n \binom{n}{x} (1-p)^{n-x} (pe^\xi)^x \\
&= \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p)+pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p)+pe^\xi} \right]^x [(1-p)+pe^\xi]^{n-x} [(1-p)+pe^\xi]^x \\
&= \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p)+pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p)+pe^\xi} \right]^x [(1-p)+pe^\xi]^n \\
&= [(1-p)+pe^\xi]^n \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p)+pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p)+pe^\xi} \right]^x \\
&= [(1-p)+pe^\xi]^n \sum_{x=1}^n \binom{n}{x} [p^*]^{n-x} [1-p^*]^x, p^* = \frac{1-p}{(1-p)+pe^\xi} \\
&= [(1-p)+pe^\xi]^n \cdot 1 = [(1-p)+pe^\xi]^n \\
M_X(\xi) &= [(1-p)+pe^\xi]^n
\end{aligned}$$

### 15.1.2.8.1.5 Poisson distribution count = number of events

an unbounded discrete distribution we first see or met

$$\begin{aligned}
X &\sim \mathcal{P}(\lambda) \\
\Updownarrow \\
\begin{cases} f_X(x|\lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \\ x \in X(\Omega) = \{0, 1, \dots\} \end{cases} \\
\Downarrow \\
\begin{cases} \text{E}[X] = \lambda \\ \text{V}[X] = \lambda = \text{E}[X] \\ M_X(\xi) = \exp[\lambda(e^\xi - 1)] = e^{\lambda(e^\xi - 1)} \end{cases}
\end{aligned}$$

the Poisson postulates

$$\begin{cases} N_t \\ N_0 = N_{t=0} = 0 \\ \forall s < t [N_s \perp N_t - N_s] \\ N_s = N_{t+s} - N_t \\ \lim_{t \rightarrow 0} \frac{P(N_t = 1)}{t} = \lambda \\ \lim_{t \rightarrow 0} \frac{P(N_t > 1)}{t} = 0 \end{cases}
\begin{aligned}
&\text{a r.v. denoting the number of events in } [0, t] \\
&\text{reset the count at the initial point} \\
&\text{disjoint intervals independent} \\
&\text{depends on length instead of initial point} \\
&\Rightarrow \forall t \approx 0 [P(N_t = 1) \approx \lambda t] \\
&\text{no coincidence for small } t
\end{aligned}$$

solve the differential equations  $\Downarrow$  with probability axioms

$$f_X(x|\lambda t) = P(N_t = x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}$$

### 15.1.2.8.1.6 negative binomial distribution

also an unbounded discrete distribution

Count the number of independent and identical Bernoulli trials until  $r$  a fixed number of success.

$$\begin{aligned} X &\sim \mathcal{NB}(r, p) \\ &\Updownarrow \\ \begin{cases} f_x(x|r, p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \\ x \in X(\Omega) = \{r, r+1, \dots\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \\ V[X] = \\ M_x(\xi) = \end{cases} \end{aligned}$$

$$\begin{aligned} Y = X - r, X \sim \mathcal{NB}(r, p) \Rightarrow X = Y + r \\ &\Updownarrow \\ \begin{cases} f_Y(y|r, p) = \binom{r+y-1}{r-1} (1-p)^y p^r \\ y \in Y(\Omega) = \{0, 1, \dots\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[Y] = r \frac{1-p}{p} = r \left( \frac{1}{p} - 1 \right) \\ V[Y] = r \frac{1-p}{p^2} = r \left( \frac{1}{p^2} - \frac{1}{p} \right) \\ M_Y(\xi) = \left[ \frac{p}{1 - (1-p)e^\xi} \right]^r \end{cases} \end{aligned}$$

Note:

$$\begin{aligned} Y = X - r \Rightarrow f_Y(y|r, p) &= P(Y = y) \\ &= P(X - r = y) \\ &= P(X = r + y) \end{aligned}$$

reparameterization technique

$$\begin{aligned} M_Y(\xi) &= E[e^{\xi Y}] = \sum_{y \in Y(\Omega)} e^{\xi y} f_Y(y|r, p) \\ &= \sum_{y=0}^{\infty} e^{\xi y} \binom{r+y-1}{r-1} (1-p)^y p^r \\ &= p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} [(1-p)e^\xi]^y \\ &= p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} [1-p^*]^y [p^*]^r \frac{1}{[p^*]^r}, 1-p^* = (1-p)e^\xi \\ &= \left[ \frac{p}{p^*} \right]^r \sum_{y=0}^{\infty} f_Y(y|r, p^*), Y \sim \mathcal{NB}(r, p^*), p^* = 1 - (1-p)e^\xi \\ &= \left[ \frac{p}{p^*} \right]^r \cdot 1 = \left[ \frac{p}{p^*} \right]^r = \left[ \frac{p}{1 - (1-p)e^\xi} \right]^r \end{aligned}$$

Note:

For sufficiently small  $t$  such that

$$0 \leq p^* = (1 - p) e^\xi \leq 1$$

or else  $t \gg 1$

$$p^* = (1 - p) e^\xi > 1$$

#### 15.1.2.8.1.7 geometric distribution

$$\begin{aligned} X &\sim \mathcal{NB}(r, p) \\ &\Downarrow \\ \begin{cases} f_x(x|r, p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \\ x \in X(\Omega) = \{r, r+1, \dots\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \\ V[X] = \\ M_x(\xi) = \end{cases} \end{aligned}$$

$$r = 1,$$

$$\begin{aligned} X &\sim \mathcal{NB}(r=1, p) = \mathcal{G}(p) \\ &\Downarrow \\ \begin{cases} f_x(x|r=1, p) = \left[ \binom{x-1}{r-1} (1-p)^{x-r} p^r \right]_{r=1} = \binom{x-1}{1-1} (1-p)^{x-1} p^1 = (1-p)^{x-1} p \\ x \in X(\Omega) = \{r, r+1, \dots\}_{r=1} = \{1, 2, \dots\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \\ V[X] = \\ M_x(\xi) = \end{cases} \end{aligned}$$

the only “memoryless” discrete distribution, and there is also a “memoryless” continuous distribution.  
memoryless property

$$\forall s > t [P(X > s | X > t) = P(X > s - t)]$$

Survival depends on length instead of initial point; it might be proper assumption for stuff survival, but might not be proper for human survival.

#### 15.1.2.8.2 continuous distribution

### 15.1.2.8.2.1 uniform distribution = continuous uniform distribution

$$\begin{aligned}
 X &\sim \mathcal{U}(a, b) \\
 \Updownarrow \\
 \begin{cases} f_x(x|a, b) = \frac{1(a \leq x \leq b)}{b-a} = \frac{1(x \in [a, b])}{b-a} \\ x \in X(\Omega) = [a, b] \end{cases} \\
 \Downarrow \\
 \begin{cases} E[X] = \frac{a+b}{2} \\ V[X] = \frac{(b-a)^2}{12} \\ M_x(\xi) = \frac{e^{b\xi} - e^{a\xi}}{(b-a)\xi} \end{cases}
 \end{aligned}$$

$$a = 0, b = 1$$

$$\begin{aligned}
 X &\sim \mathcal{U}(a=0, b=1) = \mathcal{U}(0, 1) \\
 \Updownarrow \\
 \begin{cases} f_x(x|a=0, b=1) = \left[ \frac{1(a \leq x \leq b)}{b-a} \right]_{a=0, b=1} = 1(0 \leq x \leq 1) = 1(x \in [0, 1]) \\ x \in X(\Omega) = [a, b]|_{a=0, b=1} = [0, 1] \end{cases} \\
 \Downarrow \\
 \begin{cases} E[X] = \frac{1}{2} \\ V[X] = \frac{1}{12} \\ M_x(\xi) = \frac{e^\xi - 1}{\xi} \end{cases}
 \end{aligned}$$

### 15.1.2.8.2.2 gamma distribution

### 15.1.2.8.2.3 exponential distribution

### 15.1.2.8.2.4 Chi-square distribution

### 15.1.2.8.2.5 Weibull distribution

### 15.1.2.8.2.6 normal distribution

$$\begin{aligned}
 X &\sim \mathcal{N}(\mu, \sigma^2) \\
 \Updownarrow \\
 \begin{cases} f_x(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} = \frac{\exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\
 \Downarrow \\
 \begin{cases} E[X] = \mu \\ V[X] = \sigma^2 \\ M_x(\xi) = e^{(\mu\xi + \frac{\sigma^2}{2}\xi^2)} = e^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases}
 \end{aligned}$$

### 15.1.2.8.2.7 beta distribution random success probability

**15.1.2.8.2.8 Cauchy distribution** <https://www.youtube.com/watch?v=oJZb6nZWdvI&t=33m34s>

[https://en.wikipedia.org/wiki/Cauchy\\_distribution](https://en.wikipedia.org/wiki/Cauchy_distribution)

$$\begin{aligned}
 X &\sim \mathcal{C}(\theta, \sigma) \\
 &\Downarrow \\
 \left\{ \begin{array}{l} f_X(x|\theta, \sigma) = \frac{1}{\pi\sigma} \left[ 1 + \left( \frac{x-\theta}{\sigma} \right)^2 \right]^{-1} = \frac{1}{\pi\sigma \left[ 1 + \left( \frac{x-\theta}{\sigma} \right)^2 \right]} = \frac{1}{\pi} \left[ \frac{\sigma}{(x-\theta)^2 + \sigma^2} \right] \\ x \in X(\Omega) = \mathbb{R} \end{array} \right. \\
 &\Downarrow \\
 \left\{ \begin{array}{ll} E[X] & \text{diverges} \\ V[X] & \text{diverges} \\ M_X(\xi) & \text{diverges} \end{array} \right.
 \end{aligned}$$

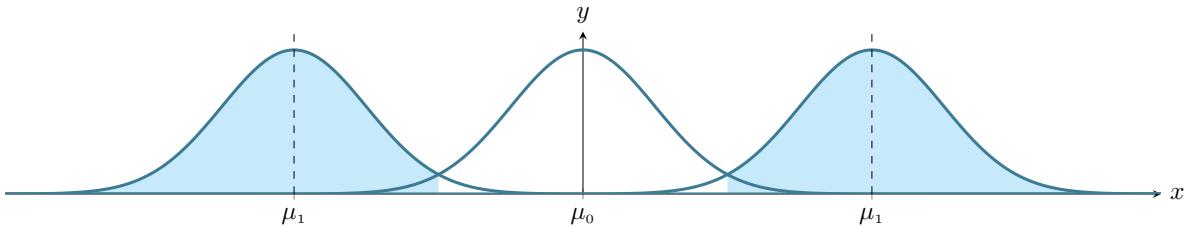


Figure 15.1: not completed: Cauchy distribution vs. normal distribution

heavy tail

t distribution is also heavy tail

box plot

### 15.1.2.8.2.9 log-normal distribution

#### 15.1.2.9 exponential family

**Definition 15.18.** exponential family: A family of PDF/PMF is called exponential family if

$$f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right)$$

with  $\boldsymbol{\theta} = \boldsymbol{\theta}(\theta_1, \dots, \theta_k) = (\theta_1, \dots, \theta_k)$  for some  $h(x), c(\boldsymbol{\theta}), w_j(\boldsymbol{\theta}), t_j(x)$ , where

$$h(x) c(\boldsymbol{\theta}) \geq 0 \Rightarrow f(x|\boldsymbol{\theta}) \geq 0$$

and parameters  $\boldsymbol{\theta}$  and statistic or real number  $x$  can be separated.

$$\mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

- exponential family
  - normal distribution<sup>[15.1.2.8.2.6]</sup>
  - gamma distribution<sup>[15.1.2.8.2.2]</sup>
  - log-normal distribution<sup>[15.1.2.8.2.9]</sup>
- non-exponential family
  - Cauchy distribution<sup>[15.1.2.8.2.8]</sup> <https://www.youtube.com/watch?v=oJZb6nZWdvI&t=1h7m43s>

#### 15.1.2.9.1 binomial distribution with known $n$

$$f(x|p) = \binom{n}{x} (1-p)^{n-x} p^x$$

or

$$f(x|p) = \binom{n}{x} (1-p)^{n-x} p^x = f(x|n=n, p)$$


---

not

$$f(x|n, p) = \binom{n}{x} (1-p)^{n-x} p^x$$


---

$$\begin{aligned} f(x|p) &= \binom{n}{x} (1-p)^{n-x} p^x \\ &= \binom{n}{x} (1-p)^n \left( \frac{p}{1-p} \right)^x \\ &= \binom{n}{x} (1-p)^n e^{(\ln \frac{p}{1-p})x} \end{aligned}$$

$$= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \quad \begin{cases} h(x) = \binom{n}{x} \\ c(\boldsymbol{\theta}) = c(\theta_1) = c(p) = (1-p)^n \\ w_1(\boldsymbol{\theta}) = w(\theta_1) = w(p) = \ln \frac{p}{1-p} \\ t_1(x) = x \\ k = 1 \end{cases}$$

↓

$$f(x|p) = \binom{n}{x} (1-p)^{n-x} p^x \in \mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

why known  $n$ ?

known  $n$

$$\binom{n}{x} = h(x)$$

unknown  $n$

$$\binom{n}{x} = h(x, n) \neq h_1(n) h_2(x)$$

### 15.1.2.9.2 continuous uniform distribution not in exponential family

$$X \sim \mathcal{U}(a, b)$$

$$\begin{aligned} f_x(x|a, b) &= \frac{1(x \in [a, b])}{b - a} \\ &= \frac{1}{b - a}, x \in [a, b] \end{aligned}$$

$$\frac{1}{b - a} = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \begin{cases} h(x) = 1 \\ c(\boldsymbol{\theta}) = c(\theta_1, \theta_2) = c(a, b) = \frac{1}{b - a} \\ w_1(\theta_1) = w_2(\theta_2) = 0 \\ t_1(x) = t_2(x) = x \\ k = 2 \end{cases}$$

however,

$$1(x \in [a, b]) \neq h(x) c(\boldsymbol{\theta}) = h(x) c(a, b)$$

thus

$$f_x(x|a, b) = \frac{1(x \in [a, b])}{b - a} \notin \mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

### 15.1.2.9.3 normal distribution is in exponential family

$$\begin{aligned} X &\sim \mathcal{N}(\mu, \sigma^2) \\ &\Downarrow \\ \begin{cases} f_x(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} = \frac{e^{\frac{-1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}}{\sigma\sqrt{2\pi}} = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} = \frac{\exp \left\{ \frac{-1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \mu \\ V[X] = \sigma^2 \\ M_X(\xi) = e^{(\mu\xi + \frac{\sigma^2}{2}\xi^2)} = e^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases} \end{aligned}$$

$$\begin{aligned}
f_x(x|\mu, \sigma^2) &= f_x(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)} = \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} = \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} e^{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2} \\
&= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \quad \begin{cases} h(x) = & 1 \\ c(\boldsymbol{\theta}) = c(\theta_1, \theta_2) = c(\mu, \sigma) = & \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} \\ w_1(\theta_1, \theta_2) = w_1(\mu, \sigma) = & \frac{\mu}{\sigma^2} \\ w_2(\theta_2) = w_2(\sigma) = & \frac{-1}{2\sigma^2} \\ t_1(x) = & x \\ t_2(x) = & x^2 \\ k = & 2 \end{cases} \\
&\Downarrow \\
f_x(x|\mu, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \in \mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}
\end{aligned}$$

#### 15.1.2.9.4 normal distribution with unknown equal mean and standard deviation in curved exponential family

$$\begin{aligned}
X &\sim \mathcal{N}(\mu, \sigma^2 = \mu^2) = \mathcal{N}(\mu, \mu^2) \\
&\Updownarrow \\
\begin{cases} f_x(x|\mu, \mu^2) = f_x(x|\mu) = \left[ \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \right]_{\sigma=\mu} = \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\mu})^2} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\
&\Downarrow \\
\begin{cases} E[X] = \mu \\ V[X] = \mu^2 \\ M_X(\xi) = e^{\left( \mu\xi + \frac{\mu^2}{2}\xi^2 \right)} = e^{\mu\xi + \frac{\mu^2}{2}\xi^2} \end{cases}
\end{aligned}$$


---

$$\begin{aligned}
f_x(x|\mu, \mu^2) &= f_x(x|\mu) = \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\mu})^2} \\
&= \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2\mu^2}(x^2 - 2x\mu + \mu^2)} = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} e^{\frac{-1}{2\mu^2}x^2 + \frac{\mu}{\mu^2}x} = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} e^{\frac{1}{\mu}x - \frac{1}{2\mu^2}x^2} \\
&= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \quad \begin{cases} h(x) = 1 \\ c(\boldsymbol{\theta}) = c(\theta_1) = c(\mu) = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} \\ w_1(\theta_1) = w_1(\mu) = \frac{1}{\mu} \\ w_2(\theta_1) = w_2(\mu) = \frac{-1}{2\mu^2} \\ t_1(x) = x \\ t_2(x) = x^2 \\ k = 2 > 1 = p \end{cases} \\
&\Downarrow \begin{cases} p = \dim \boldsymbol{\theta} \\ k = \dim \mathbf{w} \end{cases} \\
f_x(x|\mu, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \in \mathcal{C}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^{k>p} w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \\
&\subset \mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}
\end{aligned}$$


---

curved exponential family

$$\mathcal{C}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^{k>p} w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$


---

exponential family

$$\mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$


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<https://tex.stackexchange.com/questions/145969/filling-specified-area-by-random-dots-in-tikz>

### 15.1.2.9.5 properties of exponential family

#### 15.1.2.9.5.1 fundamentals of statistical inference

**Lemma 15.1.** *Leibnitz rule*

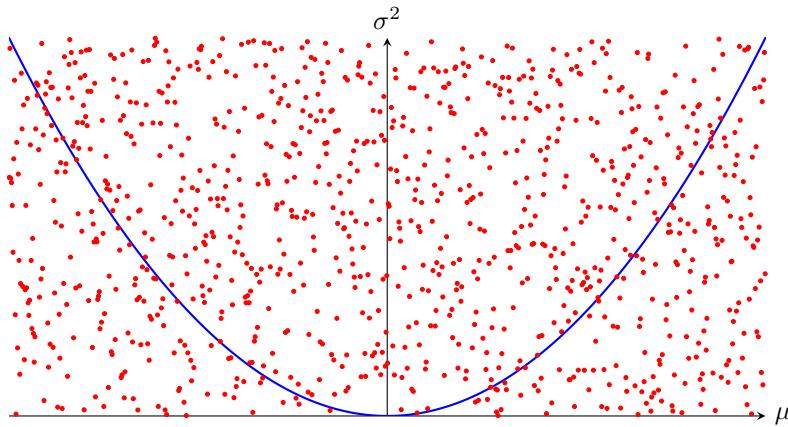


Figure 15.2: curved exponential family vs. exponential family

$$\begin{aligned}
 \frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx &= \frac{d \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx}{d\theta} \\
 &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{d}{d\theta} g(x, \theta) dx \\
 &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx
 \end{aligned}$$


---


$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx = g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx$$

$$\begin{aligned}
 \frac{d}{d\theta} \int_{a(\theta)=a}^{b(\theta)=b} g(x, \theta) dx &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx \\
 &= g(x, b) \frac{db}{d\theta} - g(x, a) \frac{da}{d\theta} + \int_a^b \frac{dg(x, \theta)}{d\theta} dx \\
 &= g(x, b) 0 - g(x, a) 0 + \int_a^b \frac{dg(x, \theta)}{d\theta} dx = 0 + 0 + \int_a^b \frac{dg(x, \theta)}{d\theta} dx \\
 &= \int_a^b \frac{dg(x, \theta)}{d\theta} dx
 \end{aligned}$$


---


$$\frac{d}{d\theta} \int_{X(\Omega) \perp \theta}^{b(\theta)=b} g(x, \theta) dx = \frac{d}{d\theta} \int_{a(\theta)=a}^{b(\theta)=b} g(x, \theta) dx = \int_a^b \frac{dg(x, \theta)}{d\theta} dx = \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx$$

$$\frac{d}{d\theta} \int_{X(\Omega) \perp \theta} g(x, \theta) dx = \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx$$


---


$$\frac{d}{d\theta} \int_{X(\Omega) \perp \theta} g(x, \theta) dx = \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx$$

**Lemma 15.2.** *parameter-independent expectation:*

Assume the domain of  $X$  independent of  $\theta$ , then

$$\mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = 0, \forall i = 1, \dots, p$$

Proof:

$$\begin{aligned} \mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\ &= \int \left[ \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} \right] f(x|\boldsymbol{\theta}) dx \\ &= \int \left[ \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right] f(x|\boldsymbol{\theta}) dx = \int \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\ &= \int \frac{f(x|\boldsymbol{\theta})}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx = \int 1 \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx = \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx \\ &\stackrel{\text{Leibnitz rule}}{=} \frac{\partial}{\partial \theta_i} \int f(x|\boldsymbol{\theta}) dx \end{aligned}$$

□

Note:

$$\begin{aligned} \mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right) f(x|\boldsymbol{\theta}) dx \\ X \sim f_x(x|\boldsymbol{\theta}) \Rightarrow \mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f_x(x|\boldsymbol{\theta})}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f_x(x|\boldsymbol{\theta})}{\partial \theta_i}}{f_x(x|\boldsymbol{\theta})} \right) f_x(x|\boldsymbol{\theta}) dx \\ &= \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx \\ X \sim f_x(x|\boldsymbol{\theta}) \Rightarrow \mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] &= \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i}}{f_x(x|\boldsymbol{\theta}^*)} \right) f(x|\boldsymbol{\theta}) dx \\ &= \int \frac{\partial f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} \left( \frac{f_x(x|\boldsymbol{\theta})}{f_x(x|\boldsymbol{\theta}^*)} \right) dx \end{aligned}$$


---

$$\mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \mathbb{E}_{\boldsymbol{\theta}^*} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}^*) dx$$

$$\mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] \not\equiv 0$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] \underset{\boldsymbol{\theta}^* = \boldsymbol{\theta}}{=} 0$$

as the fundamental to estimate parameters.

interval estimation

**Lemma 15.3.** *parameter-independent variance:*

Assume the domain of  $X$  independent of  $\theta$ , then

$$\mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right], \forall i = 1, \dots, p$$

Proof:

$$\begin{aligned} \mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] &= \int \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta}) dx = \int \frac{\partial}{\partial \theta_i} \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\ &= \int \left\{ \frac{\partial}{\partial \theta_i} \left( \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right) \right\} f(x|\boldsymbol{\theta}) dx = \int \left\{ \frac{\partial}{\partial \theta_i} \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right) \right\} f(x|\boldsymbol{\theta}) dx \\ &= \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta}) - \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{[f(x|\boldsymbol{\theta})]^2} f(x|\boldsymbol{\theta}) dx = \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} [f(x|\boldsymbol{\theta})]^2 - \left[ \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right]^2 [f(x|\boldsymbol{\theta})]}{[f(x|\boldsymbol{\theta})]^2} dx \\ &= \int \frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} dx - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \stackrel{\text{Leibnitz rule}}{=} \frac{\partial^2}{\partial \theta_i^2} \int f(x|\boldsymbol{\theta}) dx - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \\ &= \frac{\partial^2}{\partial \theta_i^2} 1 - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx = 0 - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx = - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \\ &= - \int \left( \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 f(x|\boldsymbol{\theta}) dx = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right] \end{aligned}$$

□

$$\begin{cases} \mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = 0 \\ \mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right], \forall i = 1, \dots, p \end{cases}$$

exponential family expectation and variance

**Theorem 15.15.** *exponential family expectation*

$$X \in \mathcal{E}^f = \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

$\Downarrow \forall i = 1, \dots, p$

$$\mathbb{E} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}$$

Proof:

$$\begin{aligned}
 f(x|\boldsymbol{\theta}) &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \\
 \ln f(x|\boldsymbol{\theta}) &= \ln h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
 &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \ln e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
 &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
 \ln f(x|\boldsymbol{\theta}) &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
 \ln f(X|\boldsymbol{\theta}) &= \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
 \frac{\partial}{\partial \theta_i} \ln f(X|\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_i} \left[ \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \right] \\
 &= \frac{\partial}{\partial \theta_i} \ln h(X) + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
 &= 0 + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k \frac{\partial}{\partial \theta_i} \{w_j(\boldsymbol{\theta}) t_j(X)\} \\
 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) \left( \frac{\partial}{\partial \theta_i} t_j(X) \right) \right\} \\
 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) 0 \right\} \\
 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) \right\} \\
 \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}
 \end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \\
0 &\stackrel{\text{lemma: } E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right]=0}{=} E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = E\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}\right] \\
&= E\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}\right] + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&\stackrel{E[c]=c}{=} \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k E\left[\frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} E[t_j(X)] \\
0 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] &= \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}
\end{aligned}$$

□

**Theorem 15.16.** exponential family variance

$$\begin{aligned}
X \in \mathcal{E}^f &= \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \\
&\Downarrow \forall i = 1, \dots, p \\
V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] &= \frac{-\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X)\right]
\end{aligned}$$

Proof:

same as the above

$$\begin{aligned}
f(x|\boldsymbol{\theta}) &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \\
\ln f(x|\boldsymbol{\theta}) &= \ln h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
&= \ln h(x) + \ln c(\boldsymbol{\theta}) + \ln e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
&= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
\ln f(x|\boldsymbol{\theta}) &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
\ln f(X|\boldsymbol{\theta}) &= \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
\frac{\partial}{\partial \theta_i} \ln f(X|\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_i} \left[ \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \right] \\
&= \frac{\partial}{\partial \theta_i} \ln h(X) + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
&= 0 + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k \frac{\partial}{\partial \theta_i} \{w_j(\boldsymbol{\theta}) t_j(X)\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) \left( \frac{\partial}{\partial \theta_i} t_j(X) \right) \right\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) 0 \right\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) \right\} \\
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}
\end{aligned}$$

$$\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} = \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)$$

$$V\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = V\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right]$$

$$V[aX+b] = a^2 V[X] V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right]$$

$$V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] = V\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right]$$

$$V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] = V\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = E\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} - E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right]\right)^2\right]$$

$$\text{lemma: } E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = 0 \quad E\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} - 0\right)^2\right] = E\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right)^2\right]$$

$$\text{lemma: } E\left[\frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2}\right] = -E\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right)^2\right] - E\left[\frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2}\right] = -E\left[\frac{\partial}{\partial \theta_i} \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right]$$

$$\begin{aligned} \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \\ &= -E\left[\frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}\right] \\ &= -E\left[\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \end{aligned}$$

$$= -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}\right]$$

$$= -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} \right) t_j(X) + \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial}{\partial \theta_i} t_j(X) \right\}\right]$$

$$= -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \left\{ \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) + \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} 0 \right\}\right]$$

$$= -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X)\right]$$

$$V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X)\right]$$

□

---

exponential family expectation and variance

sense of downgrading by differentiation instead of integration

$$X \in \mathcal{E}^f = \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

$$\Rightarrow \begin{cases} E \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = -\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} \\ V \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E \left[ \sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) \right], \forall i = 1, \dots, p \end{cases}$$

### 15.1.3 multivariable distribution

## 15.2 Chen, Lin-An

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUPXtNAX9lro-leGgEK0OSEW>

[https://en.wikipedia.org/wiki/Inverse\\_distance\\_weighting](https://en.wikipedia.org/wiki/Inverse_distance_weighting) Shepard interpolation

# Chapter 16

## covariance matrix

### 16.1 covariance matrix

8

#### 16.1.1 calculation

$$\begin{aligned} C[\mathbf{X}] &= \text{Cov}[\mathbf{X}] = V[\mathbf{X}] = E\left[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^T\right] \\ &= E\left[(\mathbf{X} - E(\mathbf{X}))\left(\mathbf{X}^T - E(\mathbf{X})^T\right)\right] \\ &= E\left[\mathbf{X}\mathbf{X}^T - E(\mathbf{X})\mathbf{X}^T - \mathbf{X}E(\mathbf{X})^T + E(\mathbf{X})E(\mathbf{X})^T\right] \\ &= E\left[\mathbf{X}\mathbf{X}^T\right] - E\left[E(\mathbf{X})\mathbf{X}^T\right] - E\left[\mathbf{X}E(\mathbf{X})^T\right] + E\left[E(\mathbf{X})E(\mathbf{X})^T\right] \\ &= E\left[\mathbf{X}\mathbf{X}^T\right] - E(\mathbf{X})E\left[\mathbf{X}^T\right] - E[\mathbf{X}]E(\mathbf{X})^T + E(\mathbf{X})E(\mathbf{X})^T \\ &= E\left[\mathbf{X}\mathbf{X}^T\right] - E(\mathbf{X})E(\mathbf{X})^T - E(\mathbf{X})E(\mathbf{X})^T + E(\mathbf{X})E(\mathbf{X})^T \\ &= E\left[\mathbf{X}\mathbf{X}^T\right] - E(\mathbf{X})E(\mathbf{X})^T \end{aligned}$$

$$\begin{aligned} \mathbf{X} &= [X]_{1 \times 1} = X \Rightarrow C(X) = C[\mathbf{X}] = E\left[\mathbf{X}\mathbf{X}^T\right] - E(\mathbf{X})E(\mathbf{X})^T \\ &= E[XX] - E(X)E(X) \\ &= E(X^2) - [E(X)]^2 = V(X) \end{aligned}$$

#### 16.1.2 $V[\mathbf{X} + \mathbf{b}] = V[\mathbf{X}]$

$$\begin{aligned} V[\mathbf{X} + \mathbf{b}] &= E\left[((\mathbf{X} + \mathbf{b}) - E(\mathbf{X} + \mathbf{b}))((\mathbf{X} + \mathbf{b}) - E(\mathbf{X} + \mathbf{b}))^T\right] \\ &\stackrel{E(\mathbf{X}+\mathbf{b})=E(\mathbf{X})+\mathbf{b}}{=} E\left[(\mathbf{X} + \mathbf{b} - E(\mathbf{X}) - \mathbf{b})(\mathbf{X} + \mathbf{b} - E(\mathbf{X}) - \mathbf{b})^T\right] \\ &= E\left[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^T\right] = V[\mathbf{X}] \end{aligned}$$

**16.1.3**  $\text{V}[A\mathbf{X}] = A\text{V}[\mathbf{X}]A^T$

$$\begin{aligned}\text{V}[A\mathbf{X}] &= \text{E} \left[ [(A\mathbf{X}) - \text{E}(A\mathbf{X})] [(A\mathbf{X}) - \text{E}(A\mathbf{X})]^T \right] \\ &\stackrel{\text{E}(A\mathbf{X})=A\text{E}(\mathbf{X})}{=} \text{E} \left[ [A\mathbf{X} - A\text{E}(\mathbf{X})] [A\mathbf{X} - A\text{E}(\mathbf{X})]^T \right] \\ &= \text{E} \left[ A [\mathbf{X} - \text{E}(\mathbf{X})] [A [\mathbf{X} - \text{E}(\mathbf{X})]]^T \right] \\ &= \text{E} \left[ A [\mathbf{X} - \text{E}(\mathbf{X})] [\mathbf{X} - \text{E}(\mathbf{X})]^T A^T \right] \\ &= A\text{E} \left[ [\mathbf{X} - \text{E}(\mathbf{X})] [\mathbf{X} - \text{E}(\mathbf{X})]^T \right] A^T = A\text{V}[\mathbf{X}]A^T\end{aligned}$$

**16.1.4**  $\text{V}[A\mathbf{X} + \mathbf{b}] = A\text{V}[\mathbf{X}]A^T$

$$\text{V}[A\mathbf{X} + \mathbf{b}] = \text{V}[A\mathbf{X}] = A\text{V}[\mathbf{X}]A^T$$

## Chapter 17

# Gosper algorithm



# Chapter 18

## Lorentz transformation

### 18.1 Einstein

<https://wap.hillpublisher.com/UpFile/202204/20220414165340.pdf>

### 18.2 Bondi $k$ -calculus

[https://en.wikipedia.org/wiki/Bondi\\_k-calculus](https://en.wikipedia.org/wiki/Bondi_k-calculus)

### 18.3 wordline in Minkowski space

#### 18.3.1 Wick rotation

<https://ncatlab.org/nlab/show/Wick+rotation>

##### 18.3.1.1 Osterwalder-Schrader reconstruction theorem

<https://ncatlab.org/nlab/show/Osterwalder-Schrader+theorem>



# Chapter 19

## R

### 19.1 TonyKuoYJ

郭耀仁 認識 R 的美好

<https://bookdown.org/tonykuoyj/eloquentr/getting-started.html>

<https://bookdown.org/tonykuoyj/eloquentr/easy-installation.html#about-packages>

```
install.packages()
```

```
library()
```

<https://bookdown.org/tonykuoyj/eloquentr/getting-started.html>

#### 19.1.1 quick intro

Ctrl + Alt + I to insert a new code chunk in RStudio

Ctrl + Enter to run the current line

Ctrl + Shift + Enter to run the current chunk

```
R.version
```

```
##           -
## platform      x86_64-w64-mingw32
## arch         x86_64
## os            mingw32
## crt          ucrt
## system       x86_64, mingw32
## status
## major        4
## minor        3.2
## year         2023
## month        10
## day          31
## svn rev     85441
## language     R
## version.string R version 4.3.2 (2023-10-31 ucrt)
```

```
## nickname      Eye Holes
a <- 23 # prime
a

## [1] 23

combine <- c(11, 13) # twin prime
combine

## [1] 11 13

# ?c
# help(c)
```

Ctrl + L to clean R console

path with slash / in R, differing backslash \ in M\$ Windows

### 19.1.1.1 function

```
add <- function(x, y) {
  return(x + y)
}

add(11, 13)
```

```
## [1] 24
```

$$BMI = \frac{BW \text{ [Kg]}}{BH \text{ [m]}^2}$$

```
get_bmi <- function (bw, bh) {
  return (bw/(bh/100)^2)
}

get_bmi(70, 170)
```

```
## [1] 24.22145
```

### 19.1.2 R style

<https://bookdown.org/tonykuoyj/eloquentr/styleguide.html>

snake\_case rather than camelCase

### 19.1.3 data workflow or forward pipe

from *chaining method* in *object-oriented programming* to **functional programming**

#### 19.1.3.1 %>% operator

```
abs(-5:5)
```

```

## [1] 5 4 3 2 1 0 1 2 3 4 5
# install.packages("magrittr")

library(magrittr)

##
## Attaching package: 'magrittr'
## The following object is masked _by_ '.GlobalEnv':
##      add
-5:5 %>% abs()

## [1] 5 4 3 2 1 0 1 2 3 4 5

# with readability but too many lines
sys_date <- Sys.Date()
sys_date_yr <- format(sys_date, format = "%Y")
sys_date_num <- as.numeric(sys_date_yr)
sys_date_num

## [1] 2024

# less line but also less readability
sys_date_num <- as.numeric(format(Sys.Date(), format = "%Y"))
sys_date_num

## [1] 2024

# use %>% operator to demonstrate data workflow or forward pipe
sys_date_num <- Sys.Date() %>%
  format(format = "%Y") %>%
  as.numeric()
sys_date_num

## [1] 2024

```

#### 19.1.4 data processing with dplyr

<https://bookdown.org/tonykuoyj/eloquentr/dplyr.html>

some functions functioning like those in **SQL**

```

library(dplyr)

##
## Attaching package: 'dplyr'
## The following objects are masked from 'package:stats':
##      filter, lag
## The following objects are masked from 'package:base':
##      intersect, setdiff, setequal, union

```

```
# install.packages("gapminder")

library(gapminder)

## Warning: package 'gapminder' was built under R version 4.3.3

head(gapminder)

## # A tibble: 6 x 6
##   country   continent year lifeExp      pop gdpPercap
##   <fct>     <fct>    <int>  <dbl>    <int>    <dbl>
## 1 Afghanistan Asia     1952    28.8  8425333    779.
## 2 Afghanistan Asia     1957    30.3  9240934    821.
## 3 Afghanistan Asia     1962    32.0  10267083   853.
## 4 Afghanistan Asia     1967    34.0  11537966   836.
## 5 Afghanistan Asia     1972    36.1  13079460   740.
## 6 Afghanistan Asia     1977    38.4  14880372   786.

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007)

## # A tibble: 142 x 6
##   country   continent year lifeExp      pop gdpPercap
##   <fct>     <fct>    <int>  <dbl>    <int>    <dbl>
## 1 Afghanistan Asia     2007    43.8  31889923    975.
## 2 Albania     Europe    2007    76.4  3600523    5937.
## 3 Algeria     Africa    2007    72.3  33333216   6223.
## 4 Angola      Africa    2007    42.7  12420476   4797.
## 5 Argentina   Americas   2007    75.3  40301927  12779.
## 6 Australia   Oceania   2007    81.2  20434176  34435.
## 7 Austria     Europe    2007    79.8  8199783   36126.
## 8 Bahrain     Asia      2007    75.6  708573    29796.
## 9 Bangladesh   Asia      2007    64.1  150448339   1391.
## 10 Belgium    Europe    2007    79.4  10392226   33693.
## # i 132 more rows

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  select(country)

## # A tibble: 142 x 1
##   country
##   <fct>
## 1 Afghanistan
## 2 Albania
```

```
## 3 Algeria
## 4 Angola
## 5 Argentina
## 6 Australia
## 7 Austria
## 8 Bahrain
## 9 Bangladesh
## 10 Belgium
## # i 132 more rows
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  mutate(pop_in_thousands = pop / 1000)
```

```
## # A tibble: 1,704 x 7
##   country   continent year lifeExp      pop gdpPercap pop_in_thousands
##   <fct>     <fct>    <int>   <dbl>    <int>     <dbl>          <dbl>
## 1 Afghanistan Asia     1952    28.8  8425333    779.       8425.
## 2 Afghanistan Asia     1957    30.3  9240934   821.       9241.
## 3 Afghanistan Asia     1962    32.0 10267083   853.      10267.
## 4 Afghanistan Asia     1967    34.0 11537966   836.      11538.
## 5 Afghanistan Asia     1972    36.1 13079460   740.      13079.
## 6 Afghanistan Asia     1977    38.4 14880372   786.      14880.
## 7 Afghanistan Asia     1982    39.9 12881816   978.      12882.
## 8 Afghanistan Asia     1987    40.8 13867957   852.      13868.
## 9 Afghanistan Asia     1992    41.7 16317921   649.      16318.
## 10 Afghanistan Asia    1997    41.8 22227415   635.      22227.
## # i 1,694 more rows
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  arrange(year)
```

```
## # A tibble: 1,704 x 6
##   country   continent year lifeExp      pop gdpPercap
##   <fct>     <fct>    <int>   <dbl>    <int>     <dbl>
## 1 Afghanistan Asia     1952    28.8  8425333    779.
## 2 Albania     Europe    1952    55.2 1282697   1601.
## 3 Algeria     Africa    1952    43.1  9279525   2449.
## 4 Angola      Africa    1952    30.0  4232095   3521.
## 5 Argentina   Americas   1952    62.5 17876956   5911.
## 6 Australia   Oceania   1952    69.1  8691212  10040.
## 7 Austria     Europe    1952    66.8  6927772   6137.
## 8 Bahrain     Asia      1952    50.9  120447   9867.
## 9 Bangladesh  Asia      1952    37.5  46886859   684.
## 10 Belgium    Europe   1952     68   8730405   8343.
## # i 1,694 more rows
```

total population in the world in 2007

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  summarise(ttl_pop = sum(as.numeric(pop)))
```

```
## # A tibble: 1 x 1
##       ttl_pop
##   <dbl>
## 1 6251013179
```

total population group by the continents in 2007

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  group_by(continent) %>%
  summarise(ttl_pop = sum(as.numeric(pop)))
```

```
## # A tibble: 5 x 2
##   continent     ttl_pop
##   <fct>        <dbl>
## 1 Africa      929539692
## 2 Americas    898871184
## 3 Asia        3811953827
## 4 Europe      586098529
## 5 Oceania     24549947
```

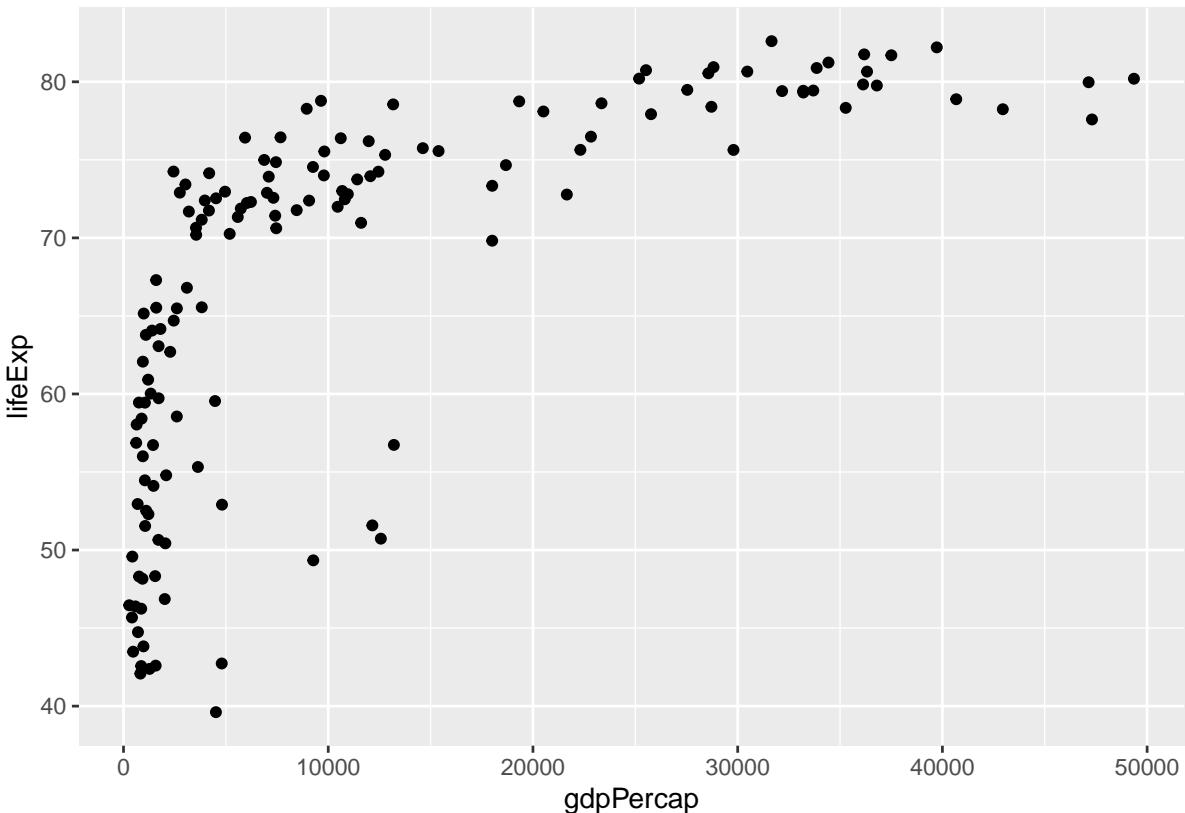
### 19.1.5 visualization statically with `ggplot2`

```
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 4.3.3
```

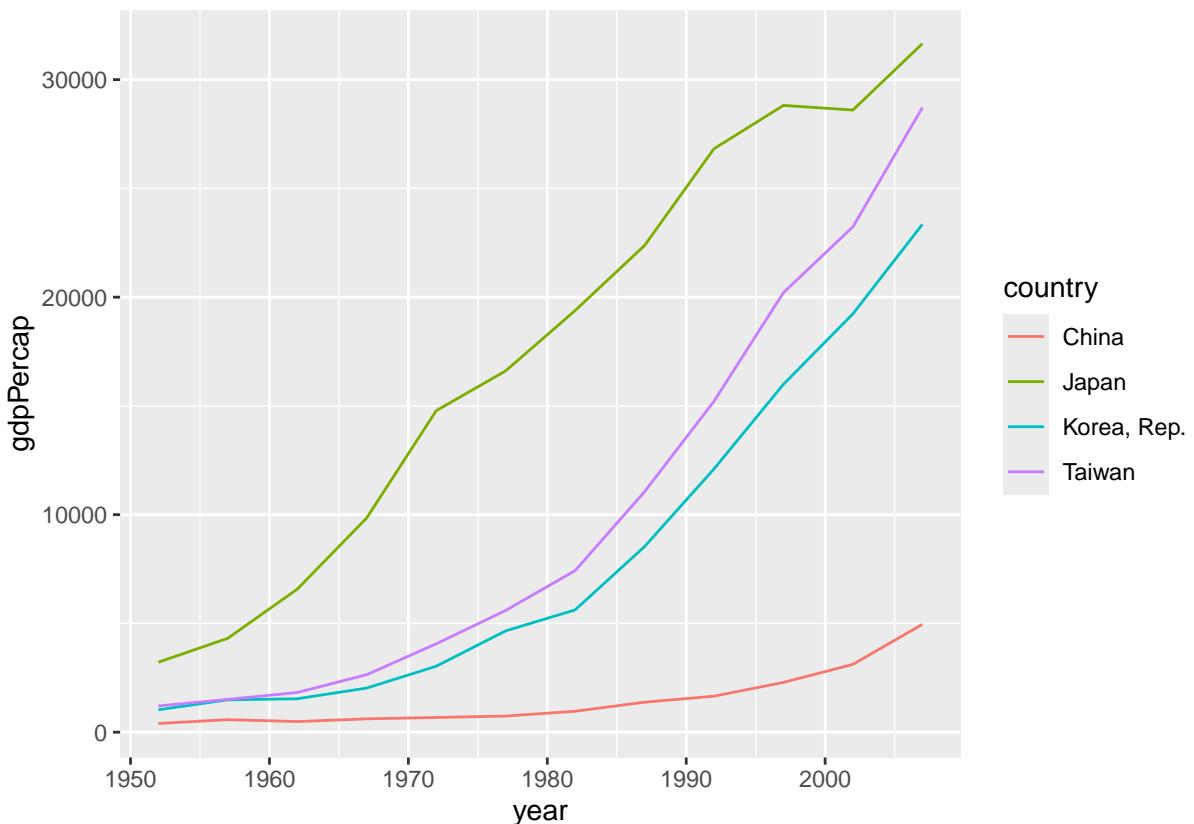
```
library(gapminder)

gapminder_2007 <- gapminder %>%
  filter(year == 2007)
scatter_plot <- ggplot(gapminder_2007, aes(x = gdpPercap, y = lifeExp)) +
  geom_point()
scatter_plot
```



```
library(ggplot2)
library(gapminder)

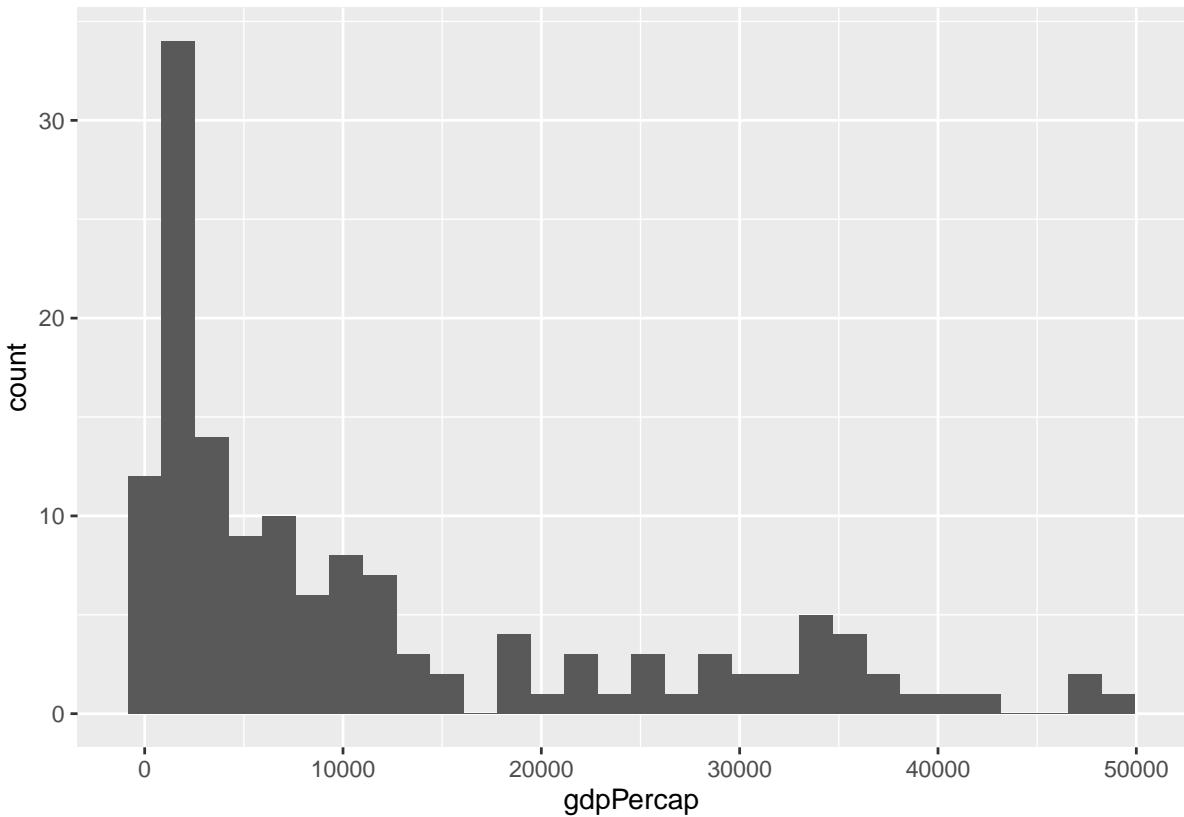
north_asia <- gapminder %>%
  filter(country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
line_plot <- ggplot(north_asia, aes(x = year, y = gdpPercap, colour = country)) +
  geom_line()
line_plot
```



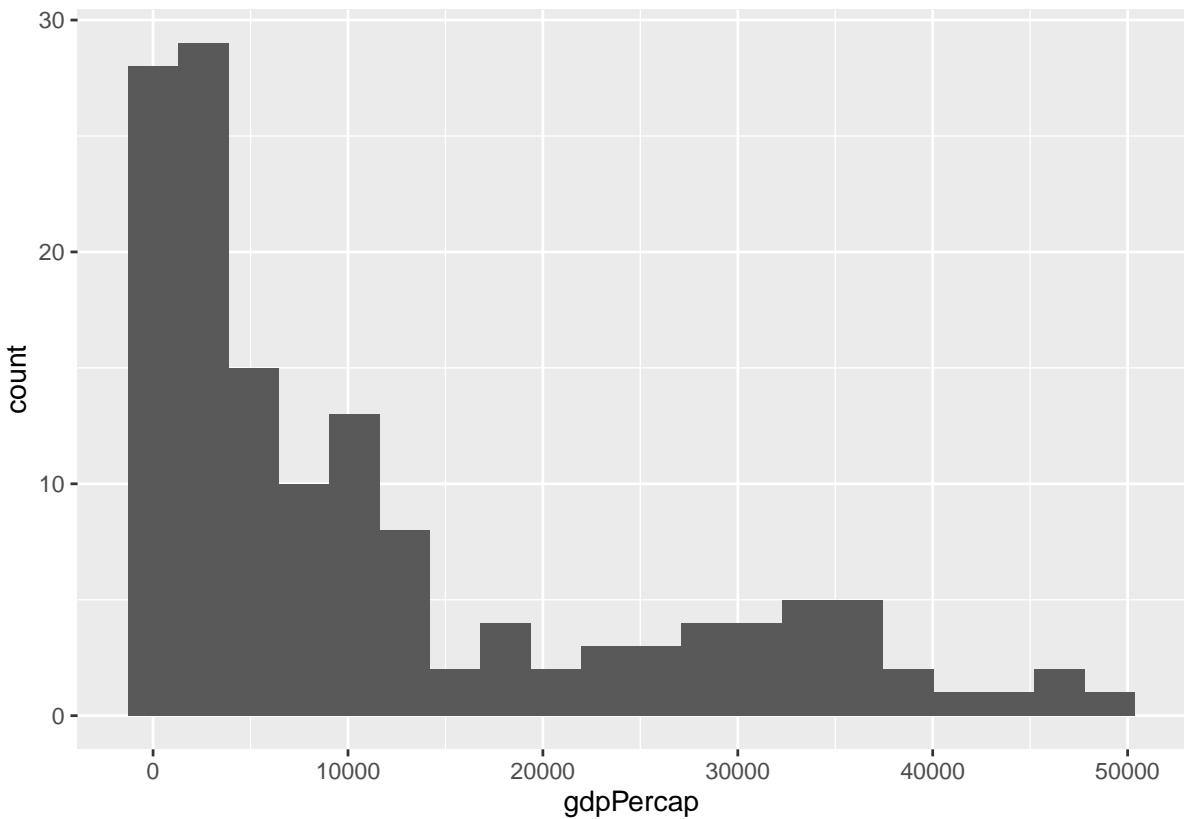
```
library(ggplot2)
library(gapminder)

hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +
  geom_histogram()
hist_plot
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

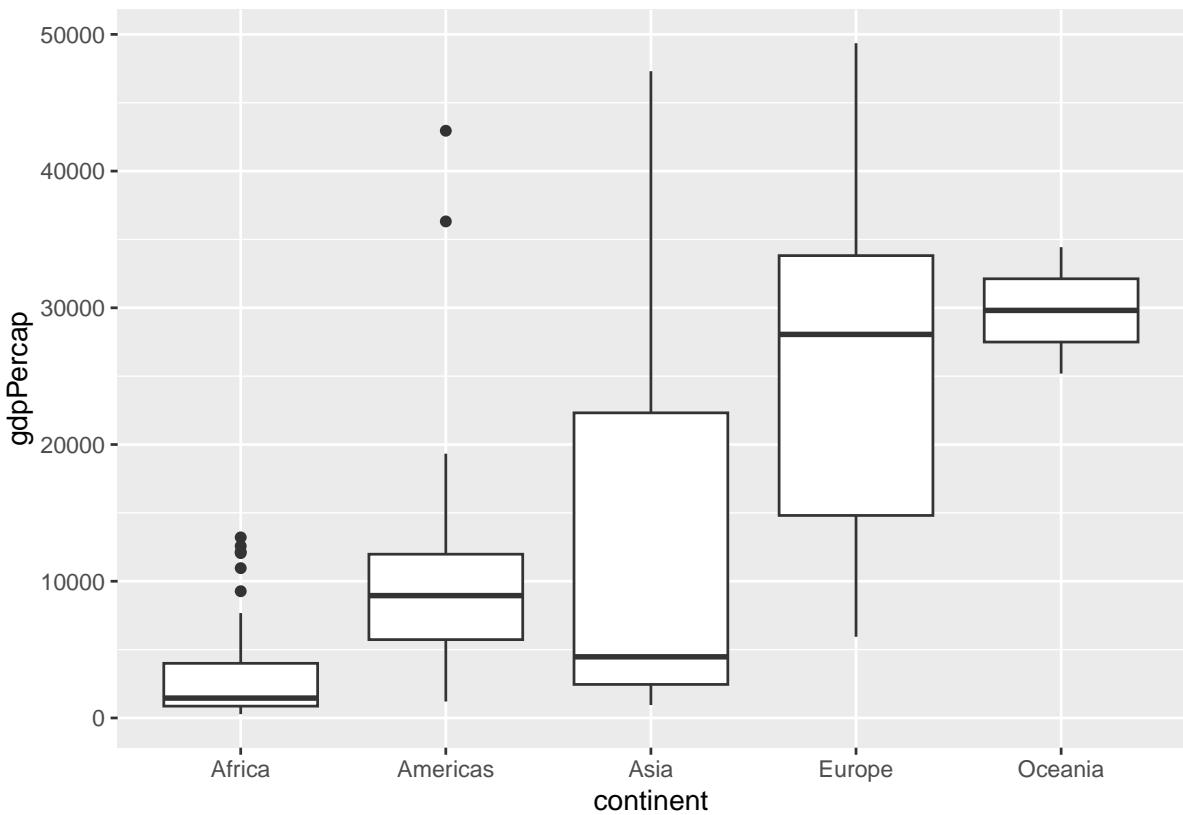


```
hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +  
  geom_histogram(bins = 20)  
hist_plot
```



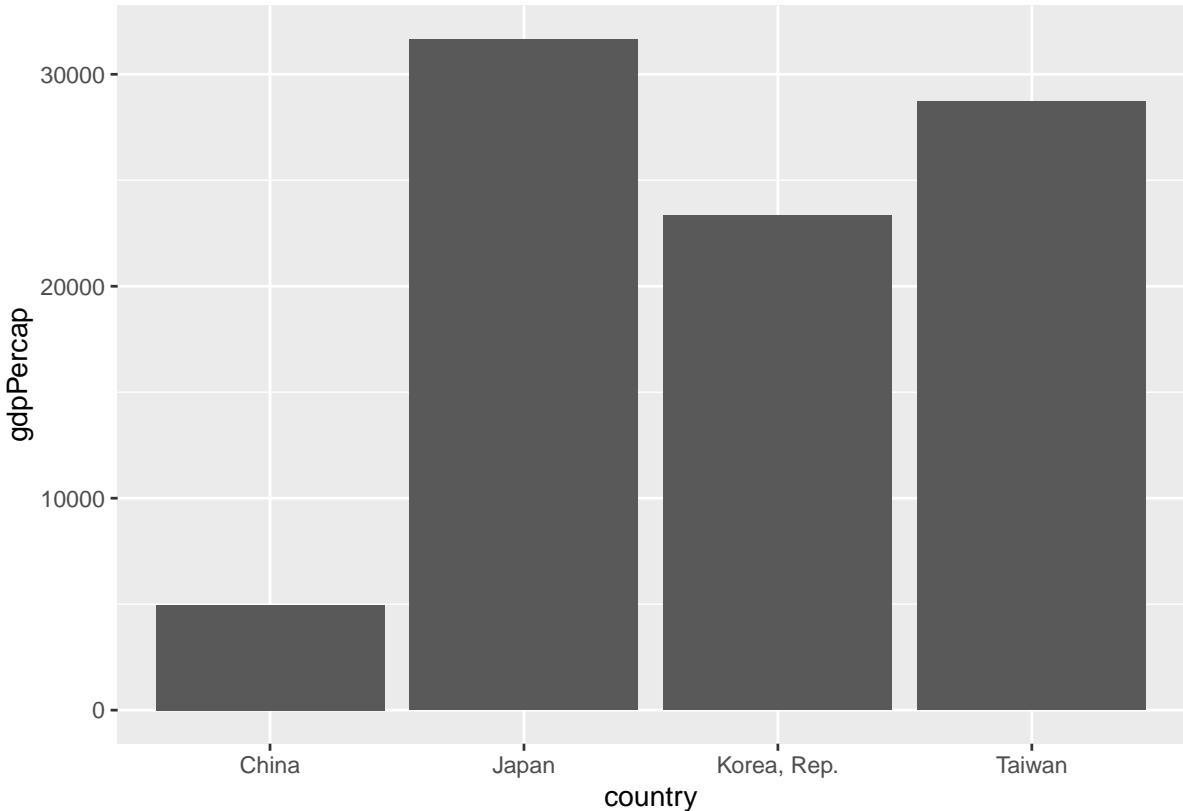
```
library(ggplot2)
library(gapminder)

box_plot <- ggplot(gapminder_2007, aes(x = continent, y = gdpPercap)) +
  geom_boxplot()
box_plot
```



```
library(ggplot2)
library(gapminder)

gdpPercap_2007_na <- gapminder %>%
  filter(year == 2007 & country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
bar_plot <- ggplot(gdpPercap_2007_na, aes(x = country, y = gdpPercap)) +
  geom_bar(stat = "identity")
bar_plot
```



### 19.1.6 loop

<https://bookdown.org/tonykuoyj/eloquentr/for.html>

```
month.name
```

```
## [1] "January"    "February"   "March"      "April"       "May"        "June"  
## [7] "July"        "August"      "September"  "October"     "November"   "December"
```

```
month.name[1]
```

```
## [1] "January"
```

```
for (month in month.name) {  
  print(month)  
}
```

```
## [1] "January"  
## [1] "February"  
## [1] "March"  
## [1] "April"  
## [1] "May"  
## [1] "June"  
## [1] "July"  
## [1] "August"  
## [1] "September"  
## [1] "October"
```

```
## [1] "November"
## [1] "December"
```

### 19.1.7 variable type

<https://bookdown.org/tonykuoyj/eloquentr/variable-types.html>

[https://www.w3schools.com/r/r\\_data\\_types.asp](https://www.w3schools.com/r/r_data_types.asp)

- numeric
- integer
- complex = complex number
- character
- logical = boolean

```
class(2L)
```

```
## [1] "integer"
```

```
class(2.0L)
```

```
## [1] "integer"
```

```
class(2.3L)
```

```
## [1] "numeric"
```

time: POSIXct POSIXt

```
class(Sys.time())
```

```
## [1] "POSIXct" "POSIXt"
```

```
0 %in% -5:5
```

```
## [1] TRUE
```

#### 19.1.7.1 date

1970-01-01 = 0L

```
date_of_origin <- as.Date("1970-01-01")
as.integer(date_of_origin)
```

```
## [1] 0
```

check if type of x is Date

```
inherits(x, what = "Date")
```

convert character to Date

```
as.Date("01-01-1970", format = "%m-%d-%Y")
```

#### 19.1.7.2 time

1970-01-01 00:00:00 GMT = 0L

tz = time zone

```
time_of_origin <- as.POSIXct("1970-01-01 00:00:00", tz = "GMT")
as.integer(time_of_origin)
```

```
## [1] 0
check if type of x is time
inherits(x, what = "POSIXct")
convert character to time
as.POSIXct("1970-01-01 00:00:00", tz = "GMT")
```

### 19.1.7.3 quotient %/% operator

[https://www.w3schools.com/r/r\\_operators.asp](https://www.w3schools.com/r/r_operators.asp)

```
7 %/% 3
```

```
## [1] 2
```

### 19.1.8 data type

<https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html>

- 1D
  - `vector`<sup>[19.1.8.1]</sup>
  - `factor`<sup>[19.1.8.2]</sup>
- 2D
  - `matrix`<sup>[19.1.8.3]</sup>
  - `data frame`<sup>[19.1.8.4]</sup>
- $n$ D
  - `array`<sup>[19.2.6.1]</sup>
  - `list`<sup>[19.2.6.2]</sup>

#### 19.1.8.1 vector

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons
```

```
## [1] "spring" "summer" "autumn" "winter"
```

```
favorite_season <- four_seasons[3]
favorite_season
```

```
## [1] "autumn"
```

```
favorite_seasons <- four_seasons[c(-2, -4)]
favorite_seasons
```

```
## [1] "spring" "autumn"
```

only one variable type for a vector

```
lucky_numbers <- c(7L, 24)
class(lucky_numbers[1])
```

```

## [1] "numeric"
lucky_numbers <- c(7L, FALSE)
lucky_numbers

## [1] 7 0

class(lucky_numbers[2])

## [1] "integer"
mixed_vars <- c(TRUE, 7L, 24, "spring")
mixed_vars

## [1] "TRUE"    "7"       "24"      "spring"
class(mixed_vars[1])

## [1] "character"
class(mixed_vars[2])

## [1] "character"
class(mixed_vars[3])

## [1] "character"

four_seasons <- c("spring", "summer", "autumn", "winter")
my_favorite_seasons <- four_seasons == "spring" | four_seasons == "autumn"
four_seasons[my_favorite_seasons]

```

#### 19.1.8.1.1 logic

```
## [1] "spring" "autumn"
```

```
rep(7L, times = 8)
```

#### 19.1.8.1.2 rep repeat

```
## [1] 7 7 7 7 7 7 7 7
```

```
rep("R", times = 10)
```

```
## [1] "R" "R" "R" "R" "R" "R" "R" "R" "R" "R"
```

```
seq(from = 7, to = 77, by = 7)
```

#### 19.1.8.1.3 seq sequence

```
## [1] 7 14 21 28 35 42 49 56 63 70 77
```

```
11:20
```

```
## [1] 11 12 13 14 15 16 17 18 19 20
```

### 19.1.8.2 factor

<https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html#factor>

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons

## [1] "spring" "summer" "autumn" "winter"

four_seasons_factor <- factor(four_seasons)
four_seasons_factor

## [1] spring summer autumn winter
## Levels: autumn spring summer winter

four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons_factor <- factor(four_seasons, ordered = TRUE, levels = c("summer",
  ↪ "winter", "spring", "autumn"))
four_seasons_factor

## [1] spring summer autumn winter
## Levels: summer < winter < spring < autumn

temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE, levels = c("cold", "warm",
  ↪ "hot"))
temp_factors

## [1] warm hot cold
## Levels: cold < warm < hot

if no levels specified, the levels will be specified alphabetically, sometimes not really true

temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE)
temp_factors

## [1] warm hot cold
## Levels: cold < hot < warm
```

### 19.1.8.3 matrix

<https://bookdown.org/tonykuoyj/eloquentr/matrix-dataframe-more.html>

```
my_mat <- matrix(1:6, nrow = 2)
my_mat

##      [,1] [,2] [,3]
## [1,]    1    3    5
## [2,]    2    4    6

class(my_mat)

## [1] "matrix" "array"

my_mat2 <- matrix(1:6, nrow = 2, byrow = TRUE)
my_mat2
```

```

##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    4    5    6
my_mat2[2, 3]

## [1] 6
my_mat2[2, ]

## [1] 4 5 6
my_mat2[, 3]

## [1] 3 6
filter <- my_mat2 < 6 & my_mat2 > 1
my_mat2[filter]

## [1] 4 2 5 3

```

boolean will become value in a matrix, like vector

```

my_mat3 <- matrix(c(1, 2, TRUE, FALSE, 3, 4), nrow = 2)
my_mat3

##      [,1] [,2] [,3]
## [1,]    1    1    3
## [2,]    2    0    4
class(my_mat3[, 2])

## [1] "numeric"

```

#### 19.1.8.4 data frame

- variable: column
- observation: row
- value: cell

```

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season)
great_nba_teams

##              team_name wins losses is_champion   season
## 1          Chicago Bulls    72     10        TRUE 1995-96
## 2 Golden State Warriors    73      9       FALSE 2015-16
great_nba_teams[1, 1]

## [1] "Chicago Bulls"

```

```

great_nba_teams[1, ]

##      team_name wins losses is_champion season
## 1 Chicago Bulls    72     10      TRUE 1995-96
great_nba_teams[, 1]

## [1] "Chicago Bulls"      "Golden State Warriors"
stringsAsFactors = TRUE

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season,
                                stringsAsFactors = TRUE)
great_nba_teams[, 1]

## [1] Chicago Bulls      Golden State Warriors
## Levels: Chicago Bulls Golden State Warriors

stringsAsFactors = FALSE

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season,
                                stringsAsFactors = FALSE)
great_nba_teams[, 1]

## [1] "Chicago Bulls"      "Golden State Warriors"

great_nba_teams$team_name

```

#### 19.1.8.4.1 selecting variable or column

```

## [1] "Chicago Bulls"      "Golden State Warriors"
great_nba_teams[, "team_name"]

## [1] "Chicago Bulls"      "Golden State Warriors"

filter <- great_nba_teams$is_champion == TRUE
great_nba_teams[filter, ]

```

#### 19.1.8.4.2 filtering observation or row

```
##      team_name wins losses is_champion season
## 1 Chicago Bulls    72     10      TRUE 1995-96

str(great_nba_teams)
```

### 19.1.8.4.3 check mixed data type

```
## 'data.frame':   2 obs. of  5 variables:
## $ team_name : chr "Chicago Bulls" "Golden State Warriors"
## $ wins       : num  72 73
## $ losses     : num  10 9
## $ is_champion: logi TRUE FALSE
## $ season     : chr "1995-96" "2015-16"
```

## 19.2 W3School

<https://www.w3schools.com/r/default.asp>

### 19.2.1 same multiple variable

[https://www.w3schools.com/r/r\\_variables\\_multiple.asp](https://www.w3schools.com/r/r_variables_multiple.asp)

```
# Assign the same value to multiple variables in one line
var1 <- var2 <- var3 <- "Orange"

# Print variable values
var1

## [1] "Orange"

var2

## [1] "Orange"

var3

## [1] "Orange"
```

### 19.2.2 legal variable name

[https://www.w3schools.com/r/r\\_variables\\_name.asp](https://www.w3schools.com/r/r_variables_name.asp)

```
# Legal variable names:
myvar <- "John"
my_var <- "John"
myVar <- "John"
MYVAR <- "John"
myvar2 <- "John"
.myvar <- "John"

## Illegal variable names:
# 2myvar <- "John"
# my-var <- "John"
```

```
# my var <- "John"
# _my_var <- "John"
# my_v@ar <- "John"
# TRUE <- "John"
```

### 19.2.3 complex number

[https://www.w3schools.com/r/r\\_data\\_types.asp](https://www.w3schools.com/r/r_data_types.asp)

[https://www.w3schools.com/r/r\\_numbers.asp](https://www.w3schools.com/r/r_numbers.asp)

### 19.2.4 escape character

[https://www.w3schools.com/r/r\\_strings\\_esc.asp](https://www.w3schools.com/r/r_strings_esc.asp)

### 19.2.5 global assignment <<-

```
my_function <- function() {
  txt <<- "fantastic"
  paste("R is", txt)
}
```

```
my_function()
```

```
## [1] "R is fantastic"
```

```
print(txt)
```

```
## [1] "fantastic"
```

```
txt <- "awesome"
my_function <- function() {
  txt <<- "fantastic"
  paste("R is", txt)
}
```

```
my_function()
```

```
## [1] "R is fantastic"
```

```
paste("R is", txt)
```

```
## [1] "R is fantastic"
```

### 19.2.6 data type

#### 19.2.6.1 array

[https://www.w3schools.com/r/r\\_arrays.asp](https://www.w3schools.com/r/r_arrays.asp)

```
# An array with one dimension with values ranging from 1 to 24
thisarray <- c(1:24)
thisarray
```

```

## [1] 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

# An array with more than one dimension
multiarray <- array(thisarray, dim = c(4, 3, 2))
multiarray

## , , 1
##
## [,1] [,2] [,3]
## [1,]    1    5    9
## [2,]    2    6   10
## [3,]    3    7   11
## [4,]    4    8   12
##
## , , 2
##
## [,1] [,2] [,3]
## [1,] 13 17 21
## [2,] 14 18 22
## [3,] 15 19 23
## [4,] 16 20 24
multiarray[2, 3, 2]

## [1] 22

```

### 19.2.6.2 list

[https://www.w3schools.com/r/r\\_lists.asp](https://www.w3schools.com/r/r_lists.asp)

## 19.3 Apan Liao

R 演習室

<https://www.youtube.com/playlist?list=PL5AC0ADBF65924EAD>

### 19.3.1 data input

[https://www.youtube.com/watch?v=STcIxf\\_vUWY&list=PL5AC0ADBF65924EAD&index=1](https://www.youtube.com/watch?v=STcIxf_vUWY&list=PL5AC0ADBF65924EAD&index=1)

- `scan()`
- `read`
  - `read.table()`
  - `read.csv()`

### 19.3.2 descriptive statistics

[https://www.youtube.com/watch?v=GL3Wv\\_45LaU&list=PL5AC0ADBF65924EAD&index=2](https://www.youtube.com/watch?v=GL3Wv_45LaU&list=PL5AC0ADBF65924EAD&index=2)



## Chapter 20

# Laplace transform



# Chapter 21

## conic section

conic section 圓錐曲線 / 圓錐截痕

[https://en.wikipedia.org/wiki/Conic\\_section](https://en.wikipedia.org/wiki/Conic_section)

<https://tex.stackexchange.com/questions/222882/drawing-minimal-xy-axis>

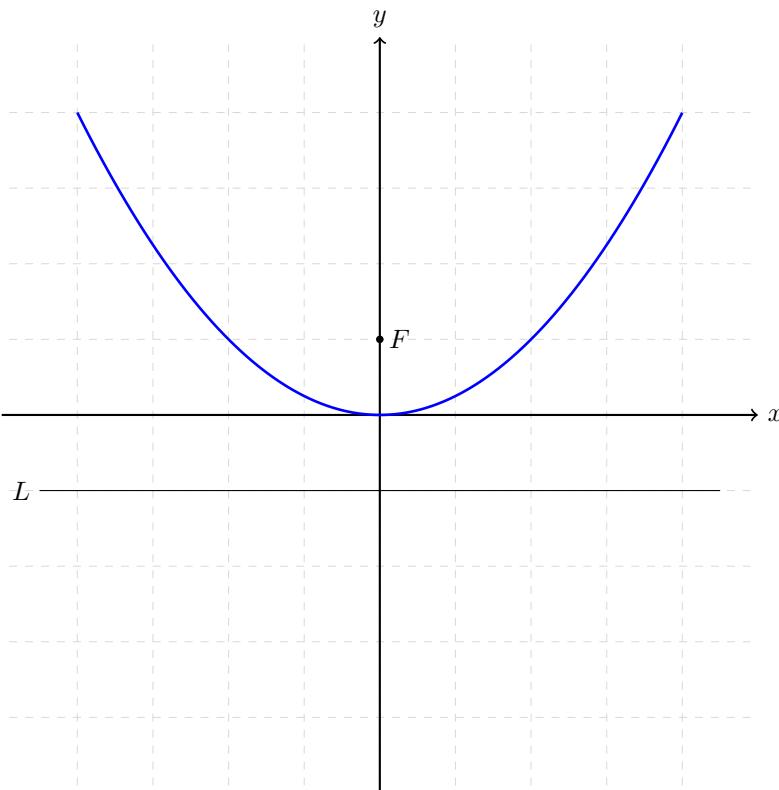


Figure 21.1: parabola defined by focus, directrix, eccentricity

## 21.1 Cartesian coordinate: focus, directrix, eccentricity

focus, directrix, eccentricity 焦點, 準線, 離心率

$$\begin{cases} F = (0, y_F) \\ L = y - y_L = 0 \\ \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\|(x, y) - (0, y_F)\|}{\|y - y_L\|} \end{cases} \quad \begin{array}{l} F : \text{focus} \\ L : \text{directrix} \\ \left\{ \begin{array}{l} P = (x, y) \\ \epsilon : \text{eccentricity} \end{array} \right. \end{array}$$

$$0 \leq \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{\sqrt{(y - y_L)^2}} \quad (21.1)$$

$$\epsilon^2 = \frac{x^2 + (y - y_F)^2}{(y - y_L)^2} = \frac{x^2 + y^2 - 2y_F y + y_F^2}{y^2 - 2y_L y + y_L^2} \quad (21.2)$$

$$0 = x^2 + (1 - \epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \quad (21.3)$$

$$\stackrel{\epsilon \neq 1}{=} x^2 + (1 - \epsilon^2) \left[ y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2} \right] \quad (21.4)$$

$$= x^2 + (1 - \epsilon^2) \quad (21.5)$$

$$\left[ y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \left( \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 - \left( \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2} \right] \quad (21.6)$$

$$= x^2 + (1 - \epsilon^2) \left[ \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2}{(1 - \epsilon^2)^2} \right] \quad (21.7)$$

$$= x^2 + (1 - \epsilon^2) \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2}{1 - \epsilon^2} \quad (21.8)$$

$$\begin{aligned} & (y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2 \\ &= (1 - \epsilon^2) y_F^2 - (\epsilon^2 - \epsilon^4) y_L^2 - y_F^2 + 2\epsilon^2 y_F y_L - \epsilon^4 y_L^2 \\ &= -\epsilon^2 y_F^2 - \epsilon^2 y_L^2 + 2\epsilon^2 y_F y_L = -\epsilon^2 (y_F - y_L)^2 \end{aligned}$$

$$\begin{aligned} & \frac{\epsilon^2 (y_F - y_L)^2}{1 - \epsilon^2} \stackrel{\epsilon \neq 1}{=} x^2 + (1 - \epsilon^2) \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 \\ & 1 \stackrel{\epsilon \neq 0, 1}{=} \begin{cases} \left( \frac{x - 0}{\frac{\epsilon(y_F - y_L)}{\sqrt{1 - \epsilon^2}}} \right)^2 + \left( \frac{y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\frac{\epsilon(y_F - y_L)}{1 - \epsilon^2}} \right)^2 & 1 - \epsilon^2 > 0 \stackrel{\epsilon \geq 0}{\Rightarrow} 0 < \epsilon < 1 \\ -\left( \frac{x - 0}{\frac{\epsilon(y_F - y_L)}{\sqrt{\epsilon^2 - 1}}} \right)^2 + \left( \frac{y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\frac{\epsilon(y_F - y_L)}{1 - \epsilon^2}} \right)^2 & 1 - \epsilon^2 < 0 \stackrel{\epsilon \geq 0}{\Rightarrow} \epsilon > 1 \end{cases} \end{aligned}$$

$$\epsilon = 0 \text{ or } \lim_{|y_L| \rightarrow \infty} \epsilon = 0$$

$$r = \overline{PF} = \|(x, y) - (0, y_F)\| = \|(x, y - y_F)\| = \sqrt{x^2 + (y - y_F)^2}$$

$$\epsilon = \frac{r}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{|y - y_L|}$$

$$\lim_{|y_L| \rightarrow \infty} \epsilon = \lim_{|y_L| \rightarrow \infty} \frac{r}{d(P, L)} = \lim_{|y_L| \rightarrow \infty} \frac{\sqrt{x^2 + (y - y_F)^2}}{|y - y_L|} = 0$$

$$\epsilon = 1$$

$$\begin{aligned} 0 &= x^2 + (1 - \epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \\ &\stackrel{\epsilon=1}{=} x^2 + (1 - 1^2) y^2 - 2(y_F - 1^2 y_L) y + (y_F^2 - 1^2 y_L^2) \\ &= x^2 - 2(y_F - y_L) y + (y_F^2 - y_L^2) \\ &= x^2 - 2(y_F - y_L) y + (y_F + y_L)(y_F - y_L) \\ x^2 &= 2(y_F - y_L) \left( y - \frac{y_F + y_L}{2} \right) \end{aligned}$$

Let one curve vertex  $P = V = (0, 0)$  on the curve, and fix the directrix  $L$  or  $y_L$ ,

$$\epsilon \neq 1$$

$$\begin{aligned} 1 &\stackrel{P(x,y)=V(0,0)}{=} 0 + \left( \frac{0 - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\frac{\epsilon(y_F - y_L)}{1 - \epsilon^2}} \right)^2 \\ &\Rightarrow y_F - \epsilon^2 y_L = \pm \epsilon(y_F - y_L) \\ &\Rightarrow \begin{cases} (1 - \epsilon)y_F = \epsilon(\epsilon - 1)y_L & + \\ (1 + \epsilon)y_F = \epsilon(\epsilon + 1)y_L & - \end{cases} \\ &\Rightarrow y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases} \end{aligned}$$

$$\epsilon = 1$$

$$\begin{aligned} x^2 &= 2(y_F - y_L) \left( y - \frac{y_F + y_L}{2} \right) \\ &\stackrel{P(x,y)=V(0,0)}{\Rightarrow} 0^2 = 2(y_F - y_L) \left( 0 - \frac{y_F + y_L}{2} \right) \\ &\Rightarrow 0 = (y_F - y_L)(y_F + y_L) \\ &\Rightarrow y_F = \mp y_L \end{aligned}$$

or by definition of eccentricity (21.1)

$$0 \leq \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{\sqrt{(y - y_L)^2}}$$

$$\stackrel{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2 + (0 - y_F)^2}}{\sqrt{(0 - y_L)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2}$$

$$\epsilon^2 = \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp \epsilon y_L$$

actually,

$$y_F = -\epsilon y_L$$

## 21.2 two-definition equivalence for ellipse and hyperbola

<https://math.stackexchange.com/questions/1833973/prove-that-the-directrix-focus-and-focus-definitions-are-equivalent>

<https://www.geogebra.org/calculator/zkppuxwp>

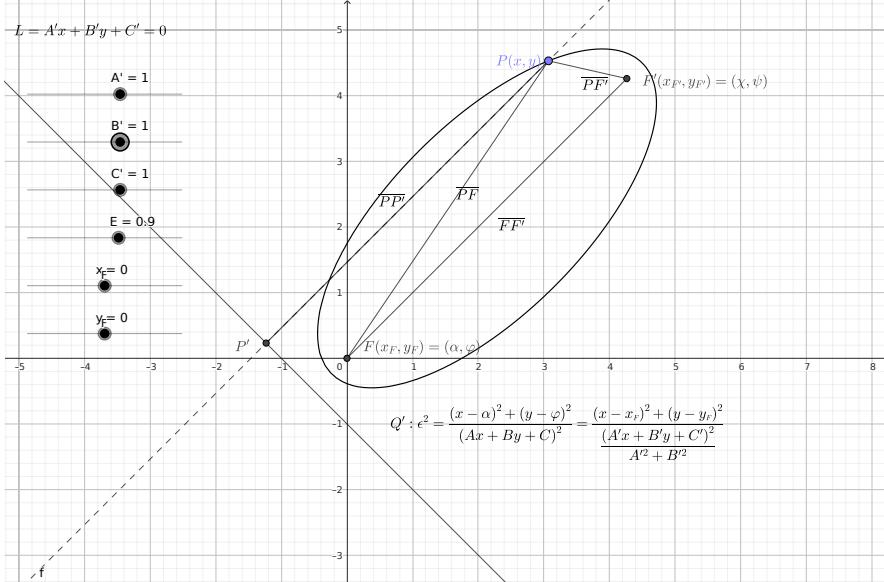


Figure 21.2: conic sections

$$\begin{cases} P = (x, y) \\ F = (x_F, y_F) = (\alpha, \varphi) & F' = (x_{F'}, y_{F'}) = (\chi, \psi) \\ L = A'x + B'y + C' = 0 \end{cases}$$

### 21.2.1 first definition for conic sections including ellipses and hyperbolas

distance from a point to a line<sup>[^22^]</sup>

$$0 \leq \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\sqrt{(x - x_F)^2 + (y - y_F)^2}}{\frac{|A'x + B'y + C'|}{\sqrt{A'^2 + B'^2}}} = \frac{\sqrt{(x - \alpha)^2 + (y - \varphi)^2}}{|Ax + By + C|}, \begin{cases} A = \frac{A'}{\sqrt{A'^2 + B'^2}} \\ B = \frac{B'}{\sqrt{A'^2 + B'^2}} \\ C = \frac{C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

$$A^2 + B^2 = \left( \frac{A'}{\sqrt{A'^2 + B'^2}} \right)^2 + \left( \frac{B'}{\sqrt{A'^2 + B'^2}} \right)^2 = 1$$

or allowing  $\epsilon < 0$  by squaring the definition

$$\epsilon^2 = \frac{(x - \alpha)^2 + (y - \varphi)^2}{(Ax + By + C)^2} = \frac{(x - x_F)^2 + (y - y_F)^2}{\frac{(A'x + B'y + C')^2}{A'^2 + B'^2}}$$

$$(x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2$$

### 21.2.2 second definition for ellipses and hyperbolas

$$2c = \overline{FF'} = \|(x_F, y_F) - (x_{F'}, y_{F'})\| = \|(\alpha, \varphi) - (\chi, \psi)\| \\ = \sqrt{(\alpha - \chi)^2 + (\chi - \psi)^2}$$

$$D = \begin{cases} \sqrt{(x - x_F)^2 + (y - y_F)^2} + \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} & \text{ellipse} \\ \sqrt{(x - x_F)^2 + (y - y_F)^2} - \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} & \text{hyperbola} \end{cases} \\ = \sqrt{(x - x_F)^2 + (y - y_F)^2} \pm \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} \\ = \sqrt{(x - \alpha)^2 + (y - \varphi)^2} \pm \sqrt{(x - \chi)^2 + (y - \psi)^2}$$

$$(x - \alpha)^2 + (y - \varphi)^2 = \left( D \mp \sqrt{(x - \chi)^2 + (y - \psi)^2} \right)^2 \\ = D^2 \mp 2D\sqrt{(x - \chi)^2 + (y - \psi)^2} \\ + (x - \chi)^2 + (y - \psi)^2$$

$$\begin{aligned}
D^2 &= (x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2 \\
&\pm 2\sqrt{\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right]} \\
&\frac{(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2 - D^2}{=} \\
&= \mp 2\sqrt{\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right]} \\
&\quad \left[\left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right]^2 + D^4\right. \\
&\quad \left.- 2D^2\left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right]\right] \\
&= 4\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right] \\
&\quad \left[\left[(x - \alpha)^2 + (y - \varphi)^2\right]^2 + \left[(x - \chi)^2 + (y - \psi)^2\right]^2\right. \\
&\quad \left.+ 2\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right] + D^4\right. \\
&\quad \left.- 2D^2\left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right]\right] \\
&= 4\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right] \\
0 &= \left[(x - \alpha)^2 + (y - \varphi)^2\right]^2 + \left[(x - \chi)^2 + (y - \psi)^2\right]^2 \\
&\quad - 2\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right] + D^4 \\
&\quad - 2D^2\left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right] \\
0 &= \left\{\left[(x - \alpha)^2 + (y - \varphi)^2\right] - \left[(x - \chi)^2 + (y - \psi)^2\right]\right\}^2 + D^4 \\
&\quad - 2D^2\left\{\left[(x - \alpha)^2 + (y - \varphi)^2\right] + \left[(x - \chi)^2 + (y - \psi)^2\right]\right\} \\
0 &= \left\{\left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right]\right\}^2 + D^4 \\
&\quad - 2D^2\left\{\left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right]\right\} \\
&\quad - 4D^2\left[(x - \alpha)^2 + (y - \varphi)^2\right] \\
&\quad (2D)^2\left[(x - \alpha)^2 + (y - \varphi)^2\right] \\
&= \left\{\left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right] - D^2\right\}^2 \\
&= \left\{\left[(x - \chi)^2 - (x - \alpha)^2\right] + \left[(y - \psi)^2 - (y - \varphi)^2\right] - D^2\right\}^2 \\
&= \{(2x - \chi - \alpha)(\alpha - \chi) + (2y - \psi - \varphi)(\varphi - \psi) - D^2\}^2 \\
&= \{2(\alpha - \chi)x - (\alpha^2 - \chi^2) + 2(\varphi - \psi)y - (\varphi^2 - \psi^2) - D^2\}^2 \\
&= \{2(\alpha - \chi)x + 2(\varphi - \psi)y - [(\alpha^2 - \chi^2) + (\varphi^2 - \psi^2) + D^2]\}^2
\end{aligned}$$

$D \neq 0$

$$\begin{aligned}
&(x - \alpha)^2 + (y - \varphi)^2 \\
&= \left[\frac{\alpha - \chi}{D}x + \frac{\varphi - \psi}{D}y - \left(\frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2}\right)\right]^2
\end{aligned}$$

$$\begin{cases} (x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2 \\ (x - \alpha)^2 + (y - \varphi)^2 = \left[ \frac{\alpha - \chi}{D}x + \frac{\varphi - \psi}{D}y - \left( \frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2} \right) \right]^2 \end{cases}$$

$$(A, B, C) \rightleftarrows (\chi, \psi, D)$$

$$\begin{cases} \epsilon A = \pm \frac{\alpha - \chi}{D} & \chi \pm \epsilon AD = \alpha \\ \epsilon B = \pm \frac{\varphi - \psi}{D} & \psi \pm \epsilon BD = \varphi \\ \epsilon C = \mp \left( \frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2} \right) & \end{cases}$$

$$\begin{aligned} 2\epsilon C &= \mp \left( \frac{\alpha - \chi}{D}(\alpha + \chi) + \frac{\varphi - \psi}{D}(\varphi + \psi) + D \right) \\ &= \mp (\pm \epsilon A(\alpha + \chi) \pm \epsilon B(\varphi + \psi) + D) \\ \mp \epsilon (A\alpha + B\varphi + 2C) &= \pm \epsilon A\chi \pm \epsilon B\psi + D \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A \\ 0 & 1 & \pm \epsilon B \\ \pm \epsilon A & \pm \epsilon B & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ D \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ \mp \epsilon (A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & \pm \epsilon B & 1 \mp \epsilon^2 A^2 & \mp \epsilon (2A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & 0 & 1 \mp \epsilon^2 A^2 \mp \epsilon^2 B^2 & \mp \epsilon (2A\alpha + 2B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & 0 & 1 & \frac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} \end{pmatrix}$$

$$A^2 + B^2 = \left( \frac{A'}{\sqrt{A'^2 + B'^2}} \right)^2 + \left( \frac{B'}{\sqrt{A'^2 + B'^2}} \right)^2 = 1$$

$$\begin{cases} \chi = \alpha \mp \epsilon AD = \alpha \mp \epsilon \frac{A'}{\sqrt{A'^2 + B'^2}} D \\ \psi = \varphi \mp \epsilon BD = \varphi \mp \epsilon \frac{B'}{\sqrt{A'^2 + B'^2}} D \\ D = \frac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} = \frac{\mp 2\epsilon}{1 \mp \epsilon^2} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \quad A^2 + B^2 = 1 \end{cases}$$

actually, only one of two solutions is true

$$\begin{cases} \chi = \alpha - \epsilon AD = \alpha - \epsilon \frac{A'}{\sqrt{A'^2 + B'^2}} D = \alpha - \frac{2\epsilon^2}{\epsilon^2 - 1} \frac{A'^2 \alpha + A'B'\varphi + A'C'}{A'^2 + B'^2} \\ \psi = \varphi - \epsilon BD = \varphi - \epsilon \frac{B'}{\sqrt{A'^2 + B'^2}} D = \varphi - \frac{2\epsilon^2}{\epsilon^2 - 1} \frac{A'B'\alpha + B'^2\varphi + B'C'}{A'^2 + B'^2} \\ D = \frac{-2\epsilon(A\alpha + B\varphi + C)}{1 - \epsilon^2(A^2 + B^2)} = \frac{-2\epsilon}{1 - \epsilon^2} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} = \frac{2\epsilon}{\epsilon^2 - 1} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

$$\begin{cases} \chi = \frac{(\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2(A'^2\alpha + A'B'\varphi + A'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\ \psi = \frac{(\epsilon^2 - 1)(A'^2 + B'^2)\varphi - 2\epsilon^2(A'B'\alpha + B'^2\varphi + B'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\ \left| \frac{D}{d(F, L)} \right| = \left| \frac{2\epsilon}{1 - \epsilon^2} \right| \Rightarrow \left( \frac{D}{d(F, L)} \right)^2 = \left( \frac{2\epsilon}{1 - \epsilon^2} \right)^2 \end{cases}$$

$$\begin{aligned} & (\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2(A'^2\alpha + A'B'\varphi + A'C') \\ &= (-(\epsilon^2 + 1)A'^2 + (\epsilon^2 - 1)B'^2)\alpha - 2\epsilon^2(A'B'\varphi + A'C') \\ &= (-(\epsilon^2 + 1)A'^2 + (\epsilon^2 - 1)B'^2)\alpha - 2\epsilon^2(A'B'\varphi + A'C') \end{aligned}$$

Can the above be more simplified?

$$\begin{aligned} \overline{FF'}^2 &= (\alpha - \chi)^2 + (\varphi - \psi)^2 \\ &= (\alpha - (\alpha - \epsilon AD))^2 + (\varphi - (\varphi - \epsilon BD))^2 \\ &= (\epsilon D)^2 (A^2 + B^2) \\ &= (\epsilon D)^2 \end{aligned}$$

### 21.2.3 eccentricity and its equivalent representation

$$\left( \frac{c}{a} \right)^2 = \left( \frac{\overline{PF}}{d(P, L)} \right)^2 = \epsilon^2 = \left( \frac{\overline{FF'}}{D} \right)^2 = \left( \frac{2c}{D} \right)^2 \Rightarrow D = 2a$$

$$\left( \frac{D}{d(F, L)} \right)^2 = \left( \frac{2\epsilon}{1 - \epsilon^2} \right)^2$$

## 21.3 Cartesian coordinate: standard form / standard equation

circle	$\left( \frac{y - k}{a} \right)^2 + \left( \frac{x - h}{a} \right)^2 = 1$	$b = a$
ellipse	$\left( \frac{y - k}{b} \right)^2 + \left( \frac{x - h}{a} \right)^2 = 1$	vertical $b > a$
	$\left( \frac{y - k}{b} \right)^2 + \left( \frac{x - h}{a} \right)^2 = 1$	horizontal $a > b$
parabola	$(y - k) - 4c(x - h)^2 = 0$	vertical
	$-4c(y - k)^2 + (x - h) = 0$	horizontal
hyperbola	$\left( \frac{y - k}{b} \right)^2 - \left( \frac{x - h}{a} \right)^2 = 1$	vertical $\frac{x - h}{a} = 0 \Rightarrow \frac{y - k}{b} = \pm 1$
	$-\left( \frac{y - k}{b} \right)^2 + \left( \frac{x - h}{a} \right)^2 = 1$	horizontal $\frac{y - k}{b} = 0 \Rightarrow \frac{x - h}{a} = \pm 1$

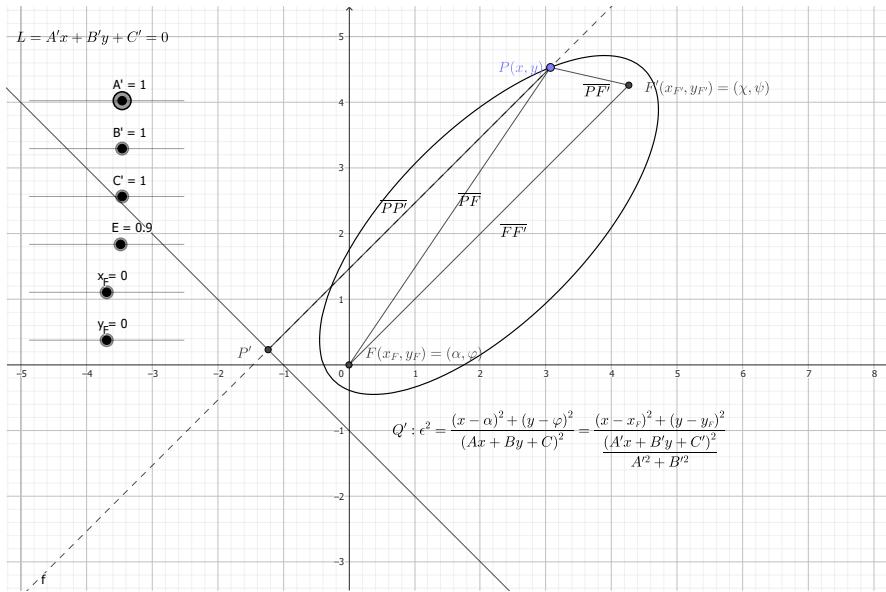


Figure 21.3: conic sections: ellipse

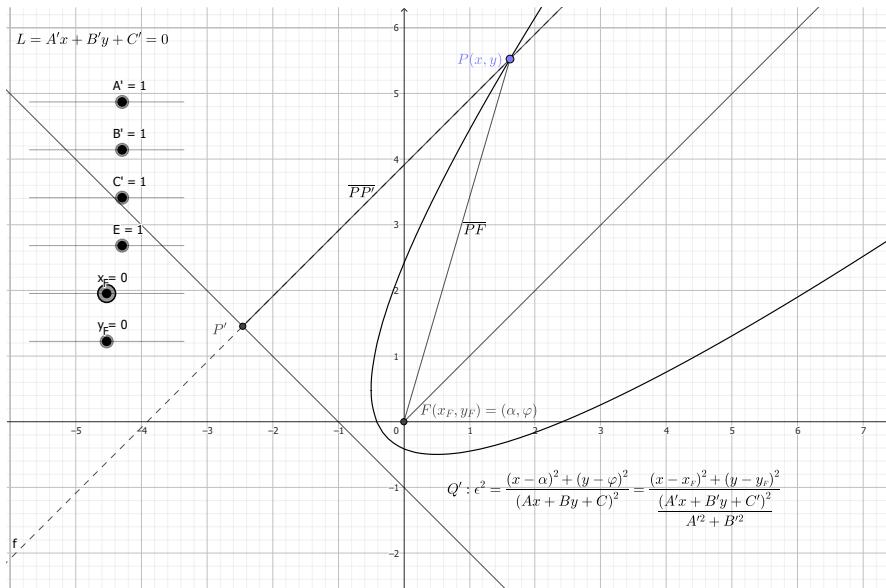


Figure 21.4: conic sections: parabola

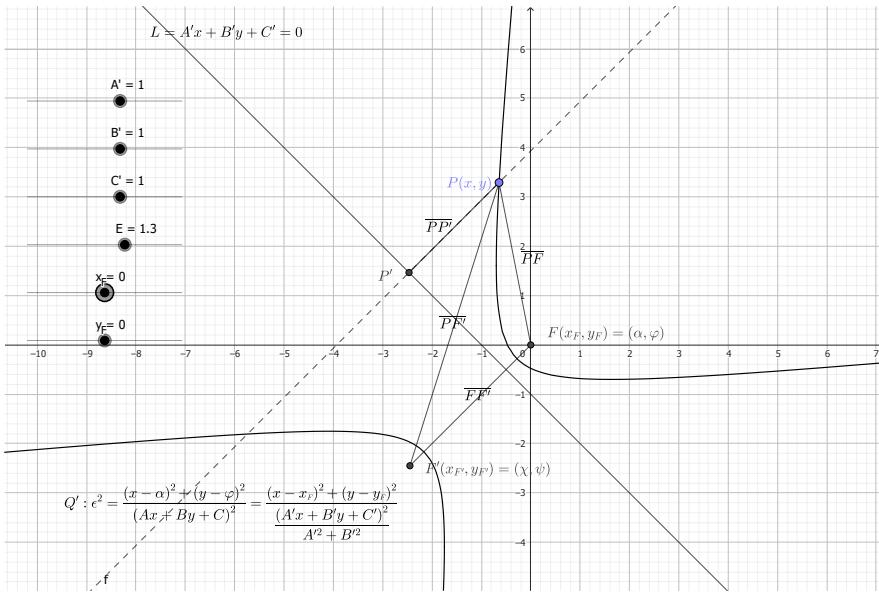


Figure 21.5: conic sections: hyperbola

## 21.4 parametric equation

circle	$\left(\frac{y-k}{a}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & a & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix}$
ellipse	$\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix}$
parabola	$(y - k) - 4c(x - h)^2 = 0$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 4c & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} t & 0 & h \\ 0 & t^2 & k \\ 0 & 0 & 1 \end{pmatrix}$
	$-4c(y - k)^2 + (x - h) = 0$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 4c & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} t^2 & 0 & h \\ 0 & t & k \\ 0 & 0 & 1 \end{pmatrix}$
hyperbola	$\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 = 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pm \cosh t \\ \sinh t \\ 1 \end{pmatrix} = \begin{pmatrix} \tan t & 0 & h \\ 0 & \sec t & k \\ 0 & 0 & 1 \end{pmatrix}$
	$-\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sinh t \\ \pm \cosh t \\ 1 \end{pmatrix} = \begin{pmatrix} \sec t & 0 & h \\ 0 & \tan t & k \\ 0 & 0 & 1 \end{pmatrix}$

tangent half-angle formula<sup>[24]</sup>

## 21.5 polar coordinate

$$(x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$(r \cos \theta - \alpha)^2 + (r \sin \theta - \varphi)^2 = [\epsilon (Ar \cos \theta + Br \sin \theta + C)]^2$$

If  $\begin{cases} F = (x_F, y_F) = (\alpha, \varphi) = (0, 0) \\ L = Ax + By + C = x + p = 0 \end{cases}$

$$(r \cos \theta)^2 + (r \sin \theta)^2 = [\epsilon (r \cos \theta + p)]^2$$

$$r^2 =$$

$$r = \pm \epsilon (r \cos \theta + p)$$

$$= \pm (r \epsilon \cos \theta + \epsilon p)$$

$$r (1 \mp \epsilon \cos \theta) = \epsilon p$$

$$r = \frac{\epsilon p}{1 \mp \epsilon \cos \theta}$$

<https://www.geogebra.org/calculator/azksjxbq>

$r = \frac{\epsilon p}{1 - \epsilon \cos \theta}$  will not cross  $L = x + p = 0$  on graphs, so maybe it is a more correct solution

$$r = \frac{\epsilon p}{1 - \epsilon \cos \theta}$$

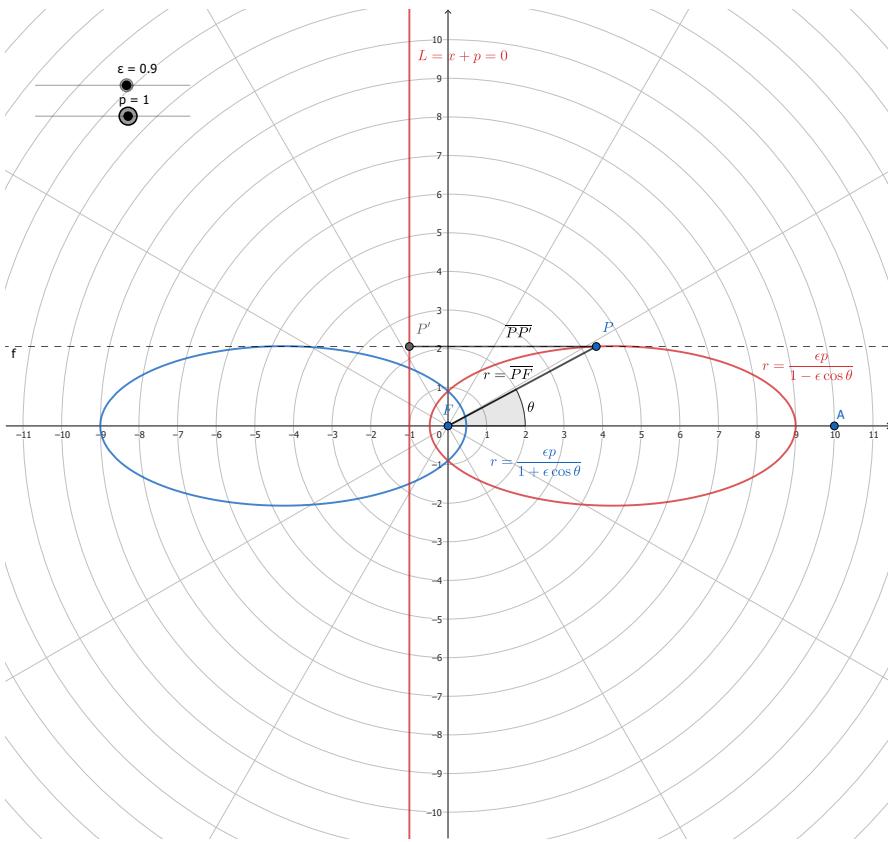


Figure 21.6: polar conic sections: ellipse

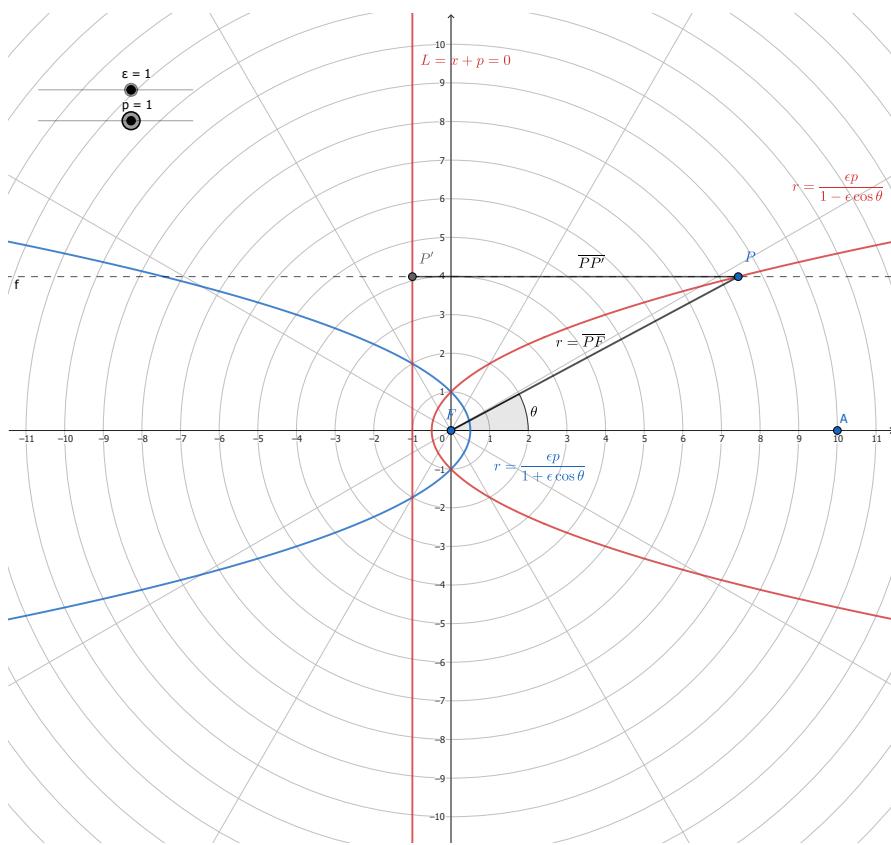


Figure 21.7: polar conic sections: parabola

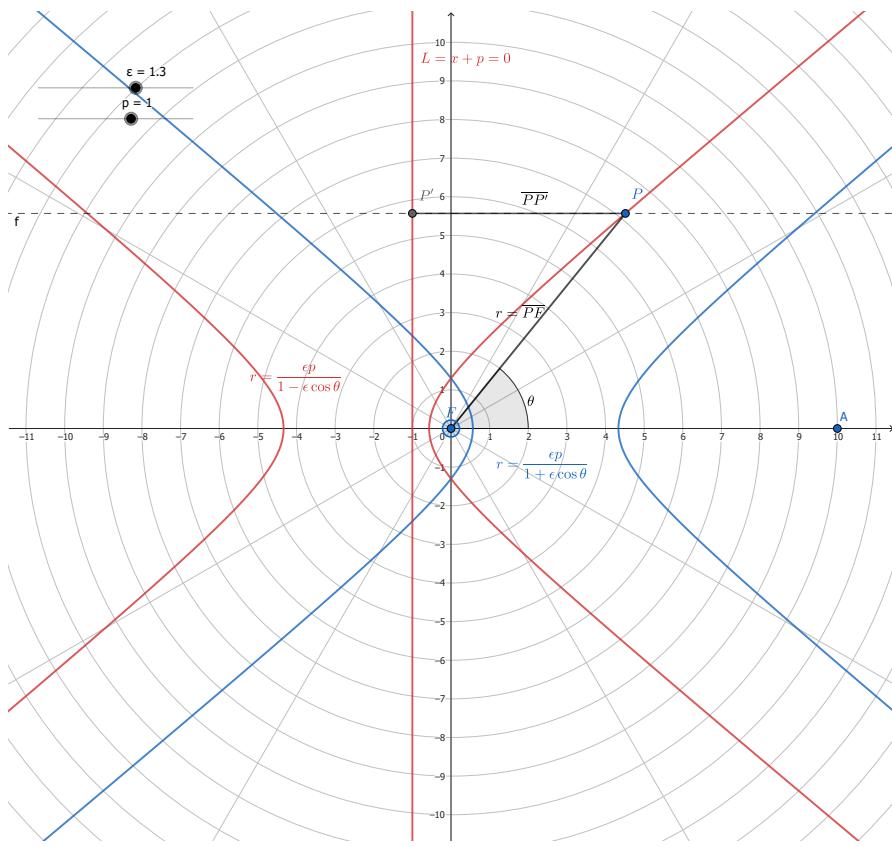


Figure 21.8: polar conic sections: hyperbola

## 21.6 Cartesian coordinate: general form / quadratic equation

<https://ccjou.wordpress.com/2013/05/24/%E5%9C%A8%E9%9D%A2/>

[https://en.wikipedia.org/wiki/Matrix\\_representation\\_of\\_conic\\_sections](https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections)

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$(x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y) \begin{pmatrix} ax + (b/2)y \\ (b/2)x + cy \end{pmatrix} = ax^2 + bxy + cy^2$$

$$\begin{aligned} 0 &= (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f \\ &= \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f, \quad \begin{cases} A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} & A \text{ real symmetric} \\ \mathbf{b} = \begin{pmatrix} d \\ e \end{pmatrix} \\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases} \end{aligned}$$

real symmetric matrix diagonalizable<sup>[23]</sup>

## 21.7 homogeneous coordinate

X homogeneous coordinate

[homogeneous coordinate](#) O: HTML, X: PDF becoming web link

O homogeneous coordinate<sup>[25]</sup>

X homogeneous coordinate

X homogeneous coordinate<sup>[21.7]</sup>

<https://ccjou.wordpress.com/2013/05/24/%E5%9C%A8%E9%9D%A2/>

$$(x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{aligned} (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} &= (x \ y \ 1) \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \kappa \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} \alpha x + \beta y + \gamma \\ \delta x + \epsilon y + \zeta \\ \eta x + \theta y + \kappa \end{pmatrix}, \quad \begin{cases} \gamma + \eta = d \\ \zeta + \theta = e \end{cases} \\ &= (x \ y \ 1) \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \zeta \\ \eta & \theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} 0 & 0 & d/2 \\ 0 & 0 & e/2 \\ d/2 & e/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
0 &= ax^2 + bxy + cy^2 + dx + ey + f \\
&= (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f = \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f \\
&= (x \ y \ 1) \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (\mathbf{x}^\top \ 1) M \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
0 &= ax^2 + bxy + cy^2 + dx + ey + f \\
&= (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f = \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f \\
&= (x \ y \ 1) \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (\mathbf{x}^\top \ 1) M \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}
\end{aligned}$$

[https://en.wikipedia.org/wiki/Matrix\\_representation\\_of\\_conic\\_sections](https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections)

$$\begin{aligned}
0 &= Q = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \\
&= [x \ y \ 1] \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{x}_h^\top A_Q \mathbf{x}_h \\
&= [x \ y] \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [D \ E] \begin{bmatrix} x \\ y \end{bmatrix} + F = \mathbf{x}^\top A_{Q,33} \mathbf{x} + \mathbf{b}^\top \mathbf{x} + F
\end{aligned}$$



# Chapter 22

## distance from a point to a line

點到直線距離

**Theorem 22.1.**

$$\begin{cases} P = P(x_0, y_0) \\ L = L(x, y) = Ax + By + C = 0, A^2 + B^2 \neq 0 \end{cases} \Downarrow d(P, L) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

[https://en.wikipedia.org/wiki/Distance\\_from\\_a\\_point\\_to\\_a\\_line](https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line)

<https://highscope.ch.ntu.edu.tw/wordpress/?p=47407>

<https://web.math.sinica.edu.tw/mathmedia/HTMLArticle18.jsp?mID=40312>

Proofs:

### 22.1 by shortest $\overline{PP'}$

$$\begin{aligned} P' &= P'(x, y) \in L = Ax + By + C = 0 \\ \Rightarrow y &= \frac{-1}{B}(Ax + C) \end{aligned}$$

$$\begin{aligned} \overline{PP'}^2(x, y) &= (x_0 - x)^2 + (y_0 - y)^2 \\ &= (x_0 - x)^2 + \left( y_0 - \frac{-1}{B}(Ax + C) \right)^2 \\ &= (x - x_0)^2 + \left( \frac{A}{B}x + \frac{C}{B} + y_0 \right)^2 = \overline{PP'}^2(x) \end{aligned}$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial x} \overline{PP'}^2(x) = 2(x - x_0) + 2 \left( \frac{A}{B}x + \frac{C}{B} + y_0 \right) \frac{A}{B} \\
&= \frac{2}{B^2} (B^2(x - x_0) + A^2x + AC + ABy_0) \\
&= \frac{2}{B^2} [(A^2 + B^2)x - (B^2x_0 - ABy_0 - AC)] \\
x &= \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2}
\end{aligned}$$

or by completing the square to find  $x$ .

$$\begin{aligned}
&\overline{PP'}^2 \left( x = \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} \right) \\
&= \left( \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} - x_0 \right)^2 + \left( \frac{A}{B} \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} + \frac{C}{B} + y_0 \right)^2 \\
&= \left( \frac{-A^2x_0 - ABy_0 - AC}{A^2 + B^2} \right)^2 + \left( \frac{A(B^2x_0 - ABy_0 - AC) + C(A^2 + B^2) + B(A^2 + B^2)y_0}{B(A^2 + B^2)} \right)^2 \\
&= \left( \frac{-A(Ax_0 + By_0 + C)}{A^2 + B^2} \right)^2 + \left( \frac{AB^2x_0 + B^3y_0 + B^2C}{B(A^2 + B^2)} \right)^2 \\
&= \frac{A^2(Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2} + \frac{B^2(Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2} \\
&= \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2} \\
\overline{PP'} &= \overline{PP'} \left( x = \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} \right) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

## 22.2 by perpendicular foot

$$y = \frac{-A}{B}x - \frac{C}{B} = \frac{-1}{B}(Ax + C), \text{ if } B \neq 0$$

$$L_{\perp} : \left( y = \frac{B}{A}x + K \right) \perp \left( y = \frac{-A}{B}x - \frac{C}{B} \right) : L$$

$$L_{\perp} = L_{\perp}(x, y) = Bx - Ay + K = 0$$

$$P = P(x_0, y_0) \in L_{\perp} = B(x - x_0) - A(y - y_0) = 0$$

$$L_{\perp} = Bx - Ay - (Bx_0 - Ay_0) = 0$$

perpendicular foot = foot of the perpendicular  $P'$

$$\begin{aligned}
P' \in (L^\perp \cap L) &= \begin{cases} L = Ax + By + C = 0 \\ L^\perp = Bx - Ay - (Bx_0 - Ay_0) = 0 \end{cases} \\
&= \begin{cases} Ax + By = -C \\ Bx - Ay = Bx_0 - Ay_0 \end{cases} \\
P' = P'(x, y) &= \left( \frac{\begin{vmatrix} -C & B \\ Bx_0 - Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}}, \frac{\begin{vmatrix} A & -C \\ B & Bx_0 - Ay_0 \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}} \right) \\
&= \left( \frac{\begin{vmatrix} C & B \\ -Bx_0 + Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}}, \frac{\begin{vmatrix} A & C \\ B & -Bx_0 + Ay_0 \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}} \right) \\
&= \left( \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2}, \frac{-ABx_0 + A^2y_0 - BC}{A^2 + B^2} \right)
\end{aligned}$$

$$\begin{aligned}
d(P, L) &= \overline{PP'} \\
&= \left\| (x_0, y_0) - \left( \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2}, \frac{-ABx_0 + A^2y_0 - BC}{A^2 + B^2} \right) \right\| \\
&= \sqrt{\left( x_0 - \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} \right)^2 + \left( y_0 - \frac{-ABx_0 + A^2y_0 - BC}{A^2 + B^2} \right)^2} \\
&= \sqrt{\left( \frac{A^2x_0 + ABy_0 + AC}{A^2 + B^2} \right)^2 + \left( \frac{ABx_0 + B^2y_0 + BC}{A^2 + B^2} \right)^2} \\
&= \sqrt{\frac{A^2(Ax_0 + By_0 + C)^2 + B^2(Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2}} = \sqrt{\frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}} \\
&= \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

## 22.3 by normal vector

$$\begin{cases} \vec{n} = (A, B) \perp L = Ax + By + C = 0 \\ \vec{PP'} = P' - P = (x - x_0, y - y_0) \end{cases}$$

$P$ 到 $L$ 的距離 $d(P, L)$ 即為 $L$ 線上一點 $P'$ 對應之 $\vec{PP'}$ 在 $L$ 法向量 $\vec{n}$ 方向上的投影長

$$\begin{aligned}
\vec{PP'} \cdot \vec{n} &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| \cos \theta \\
\left| \vec{PP'} \cdot \vec{n} \right| &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| |\cos \theta| \\
\left\| \vec{PP'} \right\| |\cos \theta| &= \left| \vec{PP'} \cdot \hat{n} \right| = \frac{\left| \vec{PP'} \cdot \vec{n} \right|}{\left\| \vec{n} \right\|} = \frac{|(x - x_0, y - y_0) \cdot (A, B)|}{\|(A, B)\|} \\
&= \frac{|A(x - x_0) + B(y - y_0)|}{\sqrt{A^2 + B^2}} = \frac{|-Ax_0 - By_0 + Ax + By|}{\sqrt{A^2 + B^2}} \\
\frac{Ax + By + C = 0}{Ax + By = -C} &= \frac{|-Ax_0 - By_0 - C|}{\sqrt{A^2 + B^2}} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

PDF LaTeX \usepackage{fdsymbol} to have \overrightharpoon vector; however, there are too many side effects, including ugly mathptmx  $\sum, \dots$

```

\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMylargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMylargesymbols}{50}

```

## 22.4 by Cauchy inequality

$$\begin{aligned}
Ax + By + C &= 0 \\
Ax + By &= -C \\
(Ax + By) - (Ax_0 + By_0) &= -C - (Ax_0 + By_0) \\
A(x - x_0) + B(y - y_0) &= -(Ax_0 + By_0 + C) \\
\overline{PP'}^2 &= (x_0 - x)^2 + (y_0 - y)^2 \\
[A^2 + B^2] \overline{PP'}^2 &= [A^2 + B^2] \left[ (x_0 - x)^2 + (y_0 - y)^2 \right] \\
&\geq [A(x - x_0) + B(y - y_0)]^2 \\
&= [- (Ax_0 + By_0 + C)]^2 = (Ax_0 + By_0 + C)^2 \\
\overline{PP'}^2 &\geq \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2} \\
\overline{PP'} &\geq \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

# Chapter 23

## real symmetric matrix diagonalizable

<https://ccjou.wordpress.com/2011/02/09/%E5%80%BC%E5%AE%9A%E7%9F%A5%E6%9C%8D%E6%9E%9A%E7%9F%A5%E5%80%BC/>

<https://tex.stackexchange.com/questions/30619/what-is-the-best-symbol-for-vector-matrix-transpose>

**Theorem 23.1.**

實對稱矩陣的特徵值皆是實數，且對應特徵向量是實向量。

$$\begin{aligned}
 & \left\{ \begin{array}{l} A \in \mathcal{M}_{n \times n}(\mathbb{R}) \\ A^\top = A \end{array} \right. \quad \begin{array}{l} \text{real matrix} \\ \text{symmetric matrix} \end{array} \quad \left. \begin{array}{l} \text{real symmetric matrix} \\ \lambda \in \mathbb{C} \\ \mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n \end{array} \right. \quad \begin{array}{l} \text{complex eigenvalue} \\ \text{complex eigenvector} \end{array} \\
 & \left\{ \begin{array}{l} Ax = \lambda x \end{array} \right. \quad \Downarrow \quad \left\{ \begin{array}{l} \lambda \in \mathbb{R} \\ \mathbf{x} \in \mathbb{R}^n \end{array} \right. \quad \begin{array}{l} \text{real eigenvalue (1)} \\ \text{real eigenvector (2)} \end{array}
 \end{aligned}$$

*Proof.* (1)

$$\begin{aligned}
 & Ax = \lambda x \\
 & \overline{Ax} = \overline{\overline{Ax}} = \overline{\lambda x} = \bar{\lambda} \bar{x} \\
 & \bar{x}^\top \overline{A^\top} = (\overline{Ax})^\top = (\bar{\lambda} \bar{x})^\top = \bar{\lambda} \bar{x}^\top \\
 & \bar{x}^\top A \stackrel{\text{symmetric}}{=} \bar{x}^\top A^\top \stackrel{\text{real}}{=} \\
 & \quad \bar{x}^\top A = \bar{\lambda} \bar{x}^\top \\
 & \lambda \bar{x}^\top x = \bar{x}^\top (\lambda x) \stackrel{Ax=\lambda x}{=} \bar{x}^\top Ax = \bar{\lambda} \bar{x}^\top x \\
 & \quad \lambda \bar{x}^\top x = \bar{\lambda} \bar{x}^\top x \\
 & (\lambda - \bar{\lambda}) \bar{x}^\top x = 0 \wedge \begin{cases} \bar{x}^\top x = \sum_{i=1}^n |x_i|^2 \\ x \neq 0 \end{cases} \Rightarrow \bar{x}^\top x \neq 0 \\
 & \lambda - \bar{\lambda} = 0 \\
 & \lambda = \bar{\lambda} \Leftrightarrow \lambda \in \mathbb{R}
 \end{aligned}$$

□

*Proof.* (1) fast concept

$$\begin{aligned}
 (\bar{A}\bar{x})^\top \mathbf{x} &= (\bar{x}^\top \bar{A}^\top) \mathbf{x} \stackrel{\text{symmetric}}{=} (\bar{x}^\top \bar{A}) \mathbf{x} = \bar{x}^\top (\bar{A}\mathbf{x}) \\
 (L) &= (\bar{A}\bar{x})^\top \mathbf{x} = \bar{x}^\top (\bar{A}\mathbf{x}) = (R) \\
 (L) &= (\bar{A}\bar{x})^\top \mathbf{x} \stackrel{Ax=\lambda x}{=} (\bar{\lambda}\bar{x})^\top \mathbf{x} = \bar{\lambda}\bar{x}^\top \mathbf{x} \\
 (R) &= \bar{x}^\top (\bar{A}\mathbf{x}) \stackrel{\text{real}}{=} \bar{x}^\top (Ax) \stackrel{Ax=\lambda x}{=} \bar{x}^\top (\lambda\mathbf{x}) = \lambda\bar{x}^\top \mathbf{x} \\
 \bar{\lambda}\bar{x}^\top \mathbf{x} &= (\bar{A}\bar{x})^\top \mathbf{x} = \bar{x}^\top (\bar{A}\mathbf{x}) = \lambda\bar{x}^\top \mathbf{x} \\
 \bar{\lambda}\bar{x}^\top \mathbf{x} &= \lambda\bar{x}^\top \mathbf{x}
 \end{aligned}$$

□

*Proof.* (2)

???

推論特徵空間  $N(A - \lambda I)$  ( $A - \lambda I$  的零空間) 為  $\mathbb{R}^n$  的子空間，故  $\mathbf{x} \in N(A - \lambda I)$  是一個非零實向量。

□

### Theorem 23.2.

實對稱矩陣對應相異特徵值的特徵向量互為正交。

$$\left\{
 \begin{array}{ll}
 \begin{cases} A \in \mathcal{M}_{n \times n}(\mathbb{R}) & \text{real matrix} \\ A^\top = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\
 Ax = \lambda x & \\
 \begin{cases} Ax_1 = \lambda_1 x_1 & (e_1) \\ Ax_2 = \lambda_2 x_2 & (e_2) \end{cases} & \begin{array}{ll} \lambda \in \mathbb{R} & \text{real eigenvalue} \\ x \in \mathbb{R}^n & \text{real eigenvector} \end{array} \\
 \lambda_1 \neq \lambda_2 & \\
 \Downarrow & \\
 x_1^\top x_2 = 0 \Leftrightarrow x_1 \perp x_2 &
 \end{array}
 \right.$$

*Proof.* (1)

$$\begin{aligned}
A\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2 \\
\mathbf{x}_1^\top A \mathbf{x}_2 &\stackrel{\mathbf{x}_1^\top}{=} \mathbf{x}_1^\top \lambda_2 \mathbf{x}_2 = \lambda_2 \mathbf{x}_1^\top \mathbf{x}_2 = (1) \\
A\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1 \\
\mathbf{x}_1^\top A^\top &= (A\mathbf{x}_1)^\top = (\lambda_1 \mathbf{x}_1)^\top = \lambda_1 \mathbf{x}_1^\top \\
\mathbf{x}_1^\top A^\top &= \lambda_1 \mathbf{x}_1^\top \\
\mathbf{x}_1^\top A \mathbf{x}_2 &\stackrel{\text{symmetric}}{=} \mathbf{x}_1^\top A^\top \mathbf{x}_2 \stackrel{\mathbf{x}_2}{=} \lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 = (2) \\
\lambda_2 \mathbf{x}_1^\top \mathbf{x}_2 &\stackrel{(1)}{=} \mathbf{x}_1^\top A \mathbf{x}_2 \stackrel{(2)}{=} \lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 \\
\lambda_2 \mathbf{x}_1^\top \mathbf{x}_2 &= \lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 \\
(\lambda_2 - \lambda_1) \mathbf{x}_1^\top \mathbf{x}_2 &= 0 \wedge \lambda_1 \neq \lambda_2 \\
\mathbf{x}_1^\top \mathbf{x}_2 &= 0
\end{aligned}$$

□

*Proof.* (1) fast concept

$$\begin{aligned}
(A\mathbf{x}_1)^\top \mathbf{x}_2 &= (\mathbf{x}_1^\top A^\top) \mathbf{x}_2 \stackrel{\text{symmetric}}{=} (\mathbf{x}_1^\top A) \mathbf{x}_2 = \mathbf{x}_1^\top (A\mathbf{x}_2) \\
(L) &= (A\mathbf{x}_1)^\top \mathbf{x}_2 = \mathbf{x}_1^\top (A\mathbf{x}_2) = (R) \\
(L) &= (A\mathbf{x}_1)^\top \mathbf{x}_2 \stackrel{(e_1)}{=} (\lambda_1 \mathbf{x}_1)^\top \mathbf{x}_2 = \lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 \\
(R) &= \mathbf{x}_1^\top (A\mathbf{x}_2) \stackrel{(e_2)}{=} \mathbf{x}_1^\top (\lambda_2 \mathbf{x}_2) = \lambda_2 \mathbf{x}_1^\top \mathbf{x}_2 \\
\lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 &= (A\mathbf{x}_1)^\top \mathbf{x}_2 = \mathbf{x}_1^\top (A\mathbf{x}_2) = \lambda_2 \mathbf{x}_1^\top \mathbf{x}_2 \\
\lambda_1 \mathbf{x}_1^\top \mathbf{x}_2 &= \lambda_2 \mathbf{x}_1^\top \mathbf{x}_2
\end{aligned}$$

□

### Theorem 23.3.

$$\left\{
\begin{array}{ll}
\begin{cases} A \in \mathcal{M}_{n \times n}(\mathbb{R}) & \text{real matrix} \\ A^\top = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\
\begin{cases} A\mathbf{x}_1 = \lambda \mathbf{x}_1 \\ \mathbf{x}_2^\top \mathbf{x}_1 = 0 \Leftrightarrow \mathbf{x}_2 \perp \mathbf{x}_1 \end{cases} & \begin{array}{l} (e) \\ (o) \end{array} \\
\Downarrow \\
A\mathbf{x}_2 \perp \mathbf{x}_1 \Leftrightarrow (A\mathbf{x}_2)^\top \mathbf{x}_1 = 0
\end{array}
\right.$$

*Proof.*

$$\begin{aligned}
(A\mathbf{x}_2)^\top \mathbf{x}_1 &= (\mathbf{x}_2^\top A^\top) \mathbf{x}_1 \stackrel{\text{symmetric}}{=} (\mathbf{x}_2^\top A) \mathbf{x}_1 \\
&= \mathbf{x}_2^\top (A\mathbf{x}_1) \stackrel{(e)}{=} \mathbf{x}_2^\top (\lambda \mathbf{x}_1) \\
&= \lambda \mathbf{x}_2^\top \mathbf{x}_1 \stackrel{(o)}{=} \lambda \cdot 0 = 0 \\
(A\mathbf{x}_2)^\top \mathbf{x}_1 &= 0 \Leftrightarrow A\mathbf{x}_2 \perp \mathbf{x}_1
\end{aligned}$$

□



## Chapter 24

# tangent half-angle formula

[https://en.wikipedia.org/wiki/Tangent\\_half-angle\\_formula](https://en.wikipedia.org/wiki/Tangent_half-angle_formula)

<https://zh.wikipedia.org/zh-tw/正切半角公式>

正切半形公式又稱萬能公式

以切表弦公式，簡稱以切表弦



## **Chapter 25**

# **homogeneous coordinate**

<https://youtu.be/EKN7dTJ4ep8?si=8woajZxbqPfEXhdK&t=2263>

<https://youtu.be/1z1S2kQKXD8?si=71o339yBtIQYhWtj&t=3082>



# Chapter 26

## Archimedean property

### 26.1 integer Archimedean property

### 26.2 rational Archimedean property

<https://math.stackexchange.com/questions/3699023/proof-the-the-field-of-rational-numbers-has-the-archimedean-property>

<https://math.stackexchange.com/questions/1919829/proving-the-archimedean-properties-of-rational-numbers>

### 26.3 real Archimedean property



# Chapter 27

## Matplotlib / matplotlib

- tikzplotlib<sup>[13.5]</sup>: Python<sup>[12]</sup> matplotlib<sup>[27]</sup> export to TikZ<sup>[13]</sup> .tex

### 27.1 Timothy H. Wu

巫孟叡

- API = application programming interface
  - functional<sup>[27.1.1]</sup>
  - object-oriented<sup>[27.1.2]</sup>
    - \* figure
    - \* axes
    - \* subplot

[https://matplotlib.org/stable/tutorials/introductory/quick\\_start.html](https://matplotlib.org/stable/tutorials/introductory/quick_start.html)

<https://pbpython.com/effective-matplotlib.html>

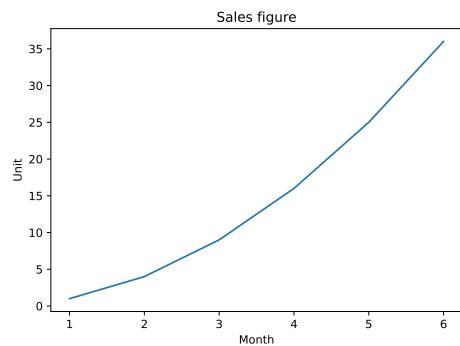
<https://tex.stackexchange.com/questions/84847/can-i-use-webp-images-in-latex>

You probably need to convert the image to png.

#### 27.1.1 funcitonal API

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2) # this doesn't
↪ actually show the plot.
# plt.show() This is automatically
↪ called for Jupyter notebook.
```



---

To plot a scatterplot, call `scatter()` instead of `plot()`.

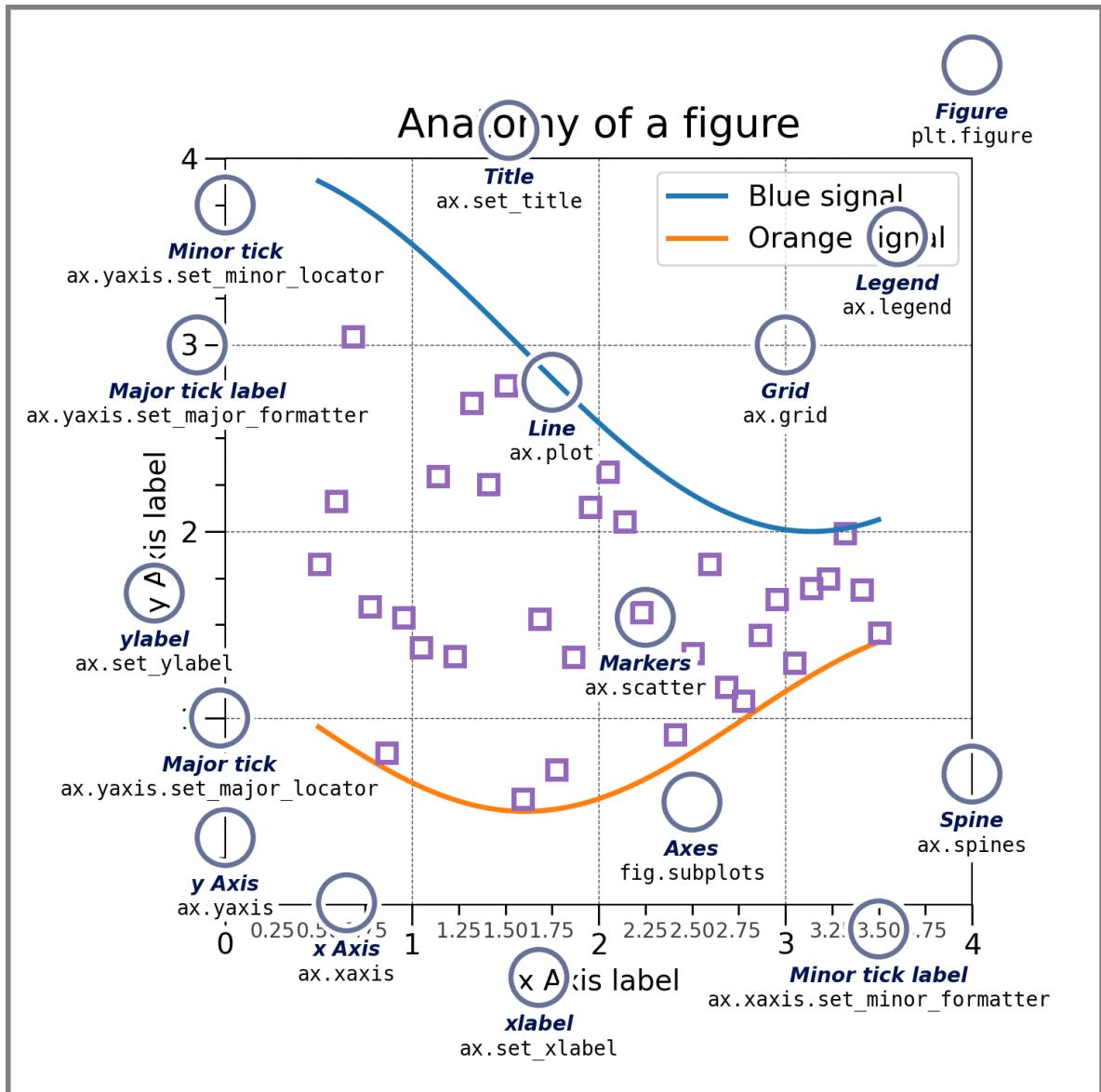


Figure 27.1: matplotlib figure anatomy

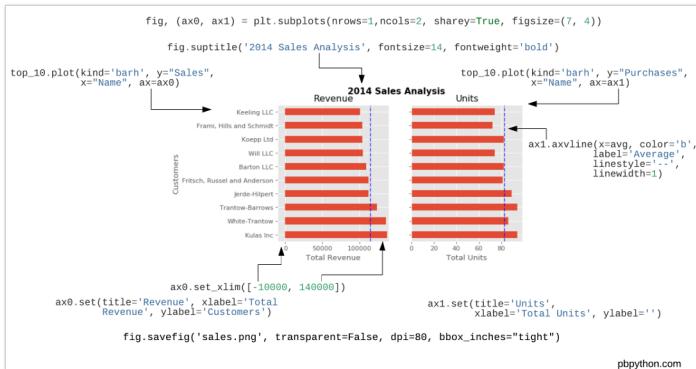


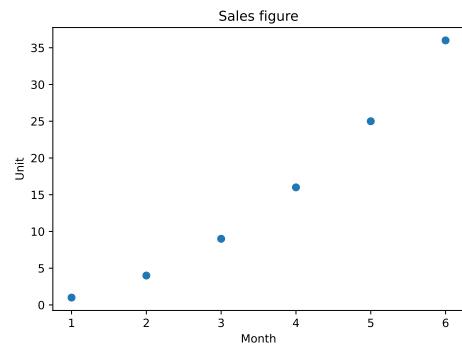
Figure 27.2: matplotlib subplot anatomy

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.scatter(a1, a2) # instead of
↪ plot.plot(), use scatter() to show
↪ scatter plot

```

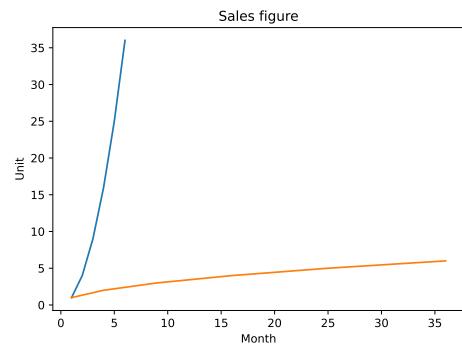


```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2)
plt.plot(a2, a1)

```



The behavior of the functional API is stateful. What's stateful? An example is when you read a text file. When you `open()` a text file to read, the library read the next line every time you call `readline()`. It remembers where you left off, despite the fact that you do not give it the position to read from. This behavior of the library is called **stateful**. The way we've used Matplotlib is also **stateful**. And everytime, `plot.show()` is called (and it automatically gets called on cell ends), some state about plots is reset. We can see that here:

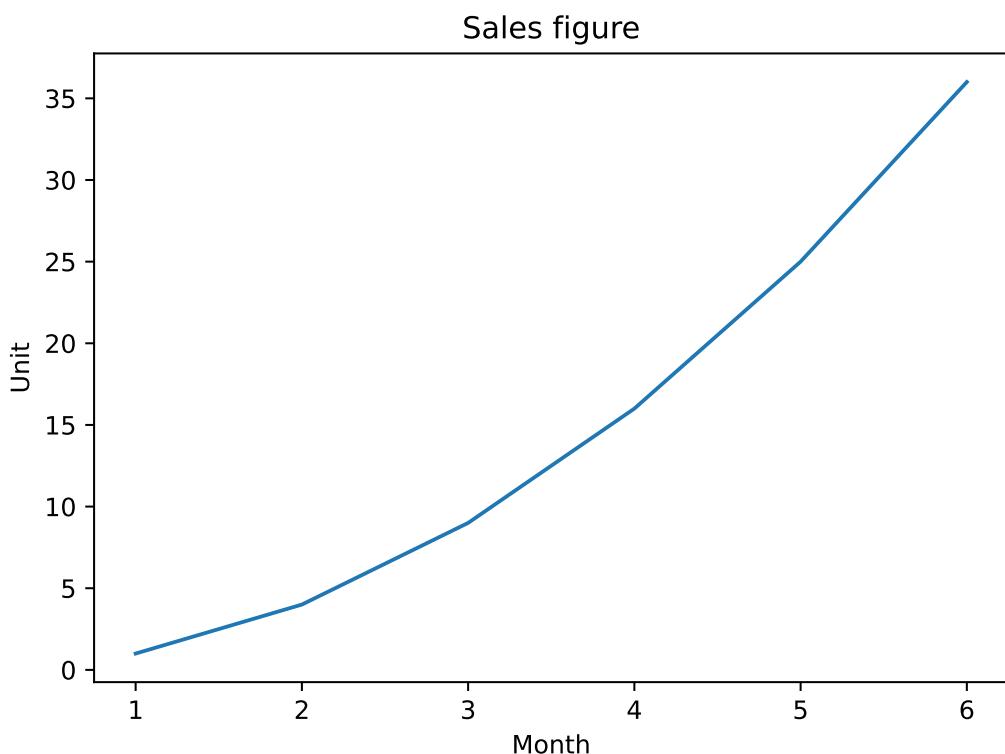
```

import matplotlib.pyplot as plt

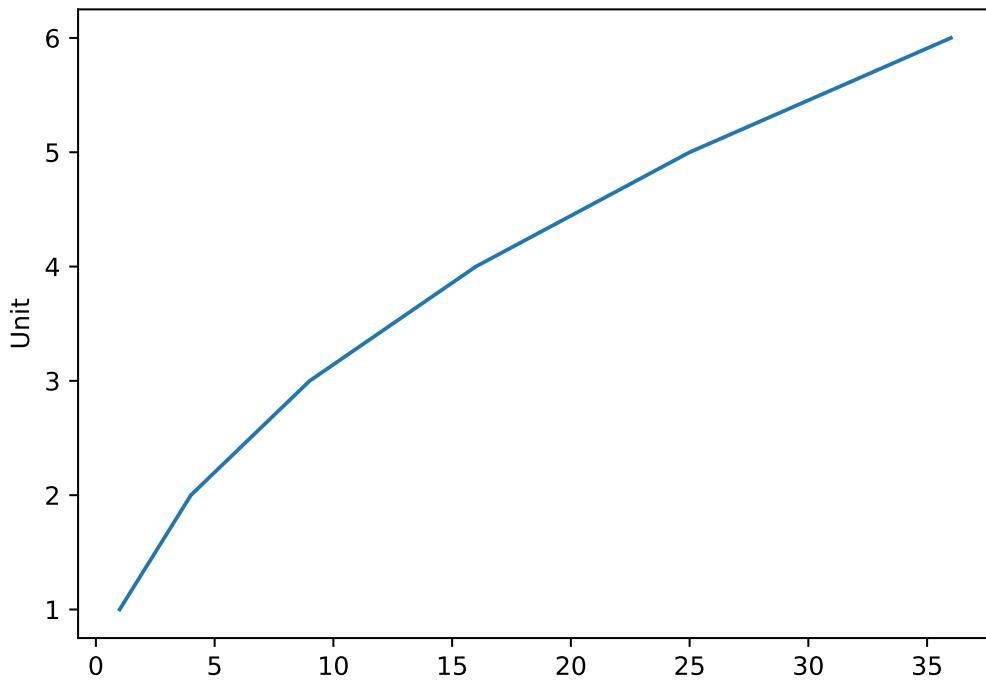
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]

```

```
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2)
```



```
plt.plot(a2, a1)
plt.ylabel('Unit')
```



It makes two graphs instead of one. Also note that `ylabel()` was called after `plot()`, and it is still shown before `plot.show()` but Sales Figure plot title and other labels don't show up on this graph. Because every time `plot.show()` is called, things are reset. This is a **stateful API** we're using. The functional APIs are used when you plot Matplotlib by calling on `pyplot` module level API (module level functions).

### 27.1.2 object-oriented API

In object-oriented API, we're getting two type of objects. One is `Figure`, the other one is `Axes`. `Figure` is the *canvas* of the plot. In English, axes is the plural form of axis. We're talking about the axis in x axis and y axis. Since one `plot` consists of both axis, in Matplotlib the object that represents one plot is called `Axes`. Since it's an object. We'll call it "a" axes.

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
print("fig is of type:", type(fig))

## fig is of type: <class 'matplotlib.figure.Figure'>
ax1 = fig.add_axes([0, 0, 1, 1]) # [left, bottom, width, height]
print("ax1 is of type:", type(ax1))

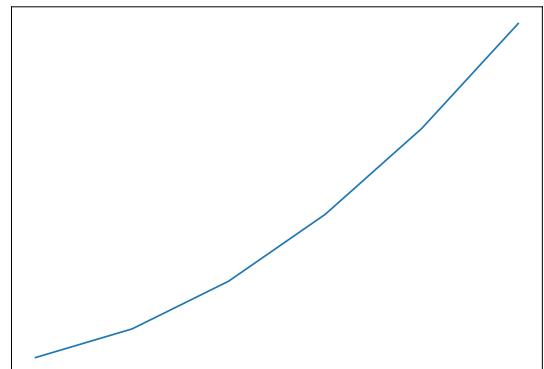
## ax1 is of type: <class 'matplotlib.axes._axes.Axes'>
```

```
# ax1.plot(a1, a2)
```

1. Call `figure()` to get a Figure type
2. Call `add_axes()` to get a `ax1` type `ax1 = fig.add_axes([0, 0, 1, 1]) # [left, bottom, width, height]`
  - The list given to `add_axes()` is the rectangular region of where to show the plot:
    - Bottom left corner at  $x=0, y=0$ , width and height of both 1, 1

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
# print("fig is of type:", type(fig))
ax1 = fig.add_axes([0, 0, 1, 1]) #
↪ [left, bottom, width, height]
# print("ax1 is of type:", type(ax1))
ax1.plot(a1, a2)
```

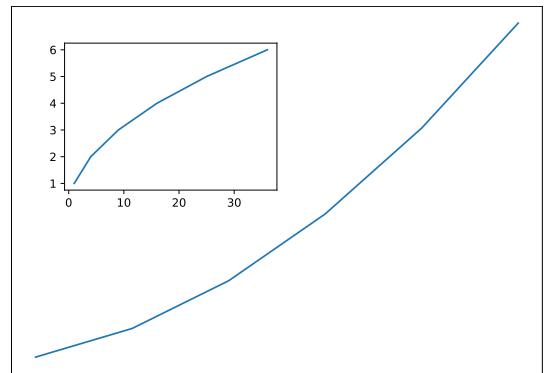


```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax1 = fig.add_axes([0, 0, 1, 1])
ax1.plot(a1, a2)
ax2 = fig.add_axes([0.1, 0.5, .4, .4])
ax2.plot(a2, a1)

```

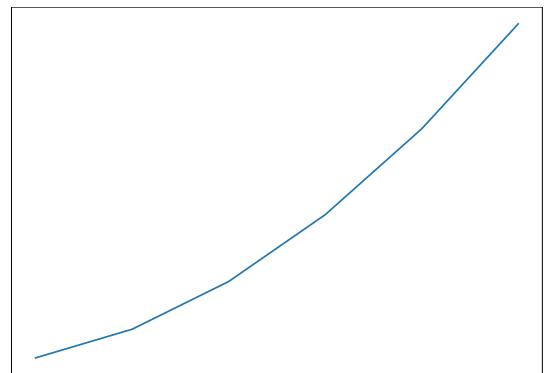


```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
ax.set_xlabel('Time')
ax.set_ylabel('Unit')
ax.set_title('Sales figure')
# alternatively:
# ax.set(xlabel='Time', ylabel='Unit',
↪ title='Sales figure')

```



### 27.1.2.1 Configure the figure size and DPI

Get image size for the figure object. 6 by 4 is the default.

```

fig.get_size_inches()

import matplotlib.pyplot as plt

fig = plt.figure()
fig.get_size_inches()

## array([6.5, 4.5])

```

---

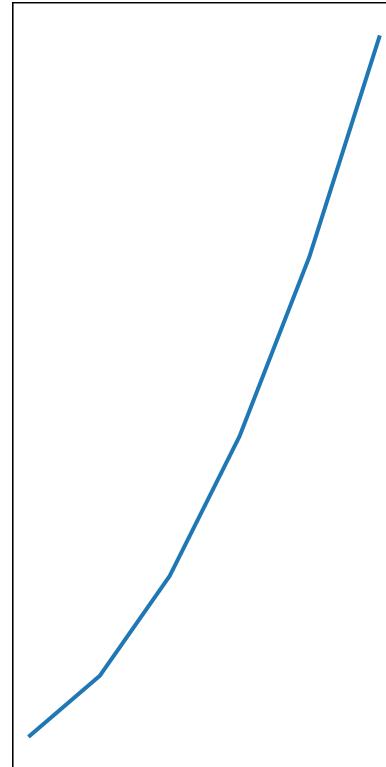
```
fig = plt.figure(figsize=(2, 4))
```

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure(figsize=(2, 4))
# you can also set after getting the
↪ figure
# fig.set_size_inches((12, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)

```




---

```
fig.set_size_inches((4, 2))
```

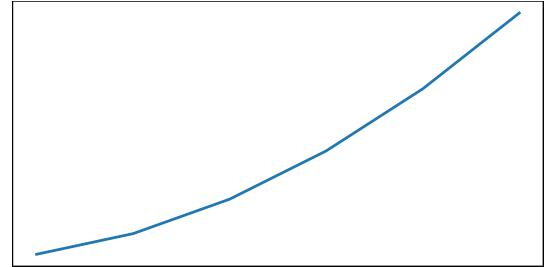
```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
# you can also set after getting the
↪ figure
fig.set_size_inches((4, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)

```

---



DPI = dots per inch

Get image DPI for the figure object. 100 is the default here.

```

fig.get_dpi()
import matplotlib.pyplot as plt

fig = plt.figure()
fig.get_dpi()

## 100.0

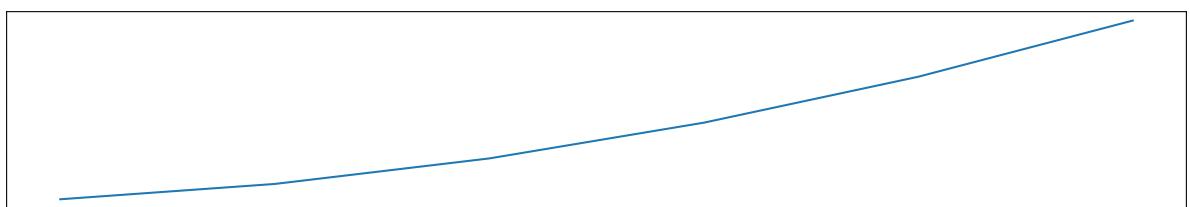
```

---

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
# you can also set after getting the figure
fig.set_size_inches((12, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
plt.title('Sales figure')
fig.get_dpi()

## 100.0
```



### 27.1.2.2 subplot

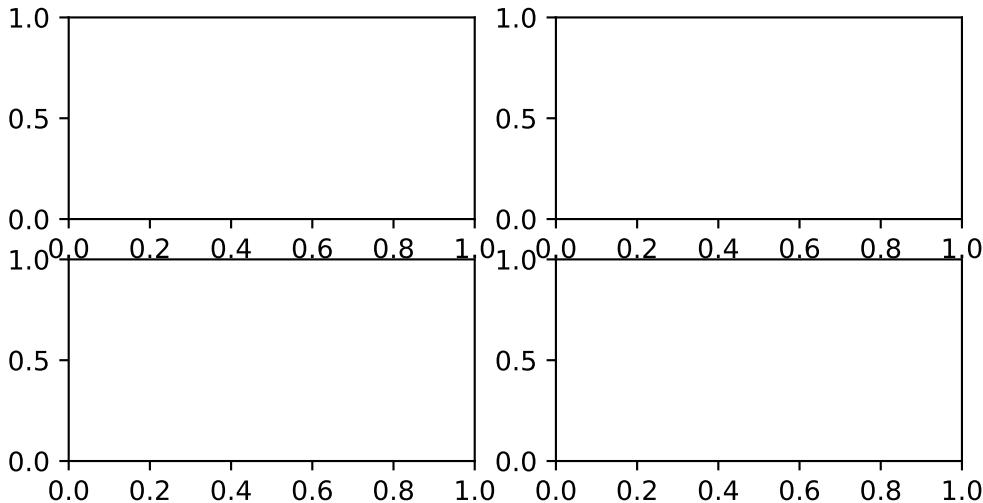
```

import matplotlib.pyplot as plt

# Subplots handles add_axes for you according to the number of rows and columns
fig, axes = plt.subplots(nrows=2, ncols=2, figsize=(6, 3))
# axes is a numpy array, you can use it like using a list.
print(axes)

## [[<Axes: > <Axes: >]
##  [<Axes: > <Axes: >]]

```

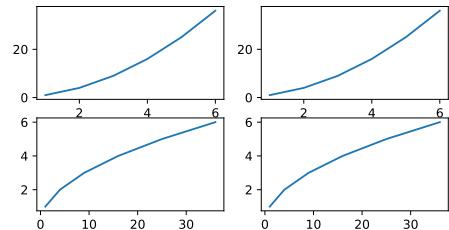


```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
# Subplots handles add_axes for you
↪ according to the number of rows
↪ and columns
fig, axes = plt.subplots(nrows=2,
↪ ncols=2, figsize=(6, 3))
# axes is a numpy array, you can use
↪ it like using a list.
# print(axes)
axes[0][0].plot(a1, a2)
axes[0][1].plot(a1, a2)
axes[1][0].plot(a2, a1)
axes[1][1].plot(a2, a1)

```



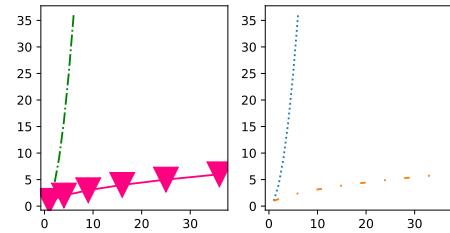
### 27.1.2.3 color and linestyle

- Color
  - <https://matplotlib.org/stable/tutorials/colors/colors.html>

- line-style (ls)
  - [https://matplotlib.org/stable/gallery/lines\\_bars\\_and\\_markers/linestyles.html](https://matplotlib.org/stable/gallery/lines_bars_and_markers/linestyles.html)
- marker
  - [https://matplotlib.org/stable/api/markers\\_api.html](https://matplotlib.org/stable/api/markers_api.html)
- linewidth (lw)

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
# Subplots handles add_axes for you
↪ according to the number of rows
↪ and columns
fig, axes = plt.subplots(nrows=1,
↪ ncols=2, figsize=(6, 3))
# axes is a numpy array
axes[0].plot(a1, a2, color='green',
↪ linestyle='-.')
axes[0].plot(a2, a1, color=(1, 0,
↪ 0.5), marker='v', markersize=20)
axes[1].plot(a1, a2,
↪ linestyle='dotted')
axes[1].plot(a2, a1, linestyle=(0, (3,
↪ 10, 1, 10)))
```



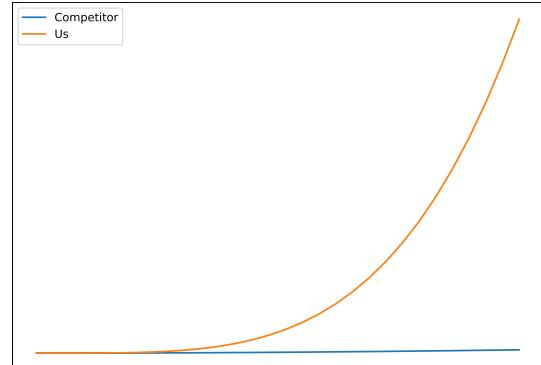
#### 27.1.2.4 other inputs

- NumPy array
- Pandas series

#### 27.1.2.5 legend

```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace(0, 10, 30)
y = x * x
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
line1 = ax.plot(x, y,
↪ label="Competitor")
line2 = ax.plot(x, y**2, label="Us")
ax.legend()
```



#### 27.1.2.6 customize style

predefined styles

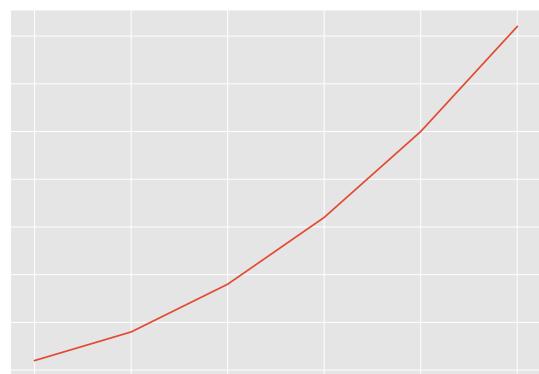
```
import matplotlib.pyplot as plt

print(plt.style.available)
```

```
## ['Solarize_Light2', '_classic_test_patch', '_mpl-gallery', '_mpl-gallery-nogrid', 'bmh']

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4,
↪ 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
plt.style.use('ggplot')
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
```



restore default style

```
import matplotlib.pyplot as plt

plt.style.use('default') # But strangely enough figure size gets changed still

import matplotlib.pyplot as plt

plt.rcParams["figure.figsize"] = (6, 4)
plt.rcParams["figure.dpi"] = 100
```

### 27.1.2.7 save to file

`savefig` saves image to file. We also set the `bbox_inches` parameter to `tight` to make sure the image doesn't get out of the image bound.

save to .png

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
fig.savefig('out.png', bbox_inches = 'tight')
```

save to .pdf

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
fig.savefig('out.pdf', bbox_inches = 'tight')
```

### 27.1.3 Seaborn

- Kimberly Fessel
  - visually explained
  - Seaborn
  - Matplotlib
  - Pandas
  - iPyWidgets

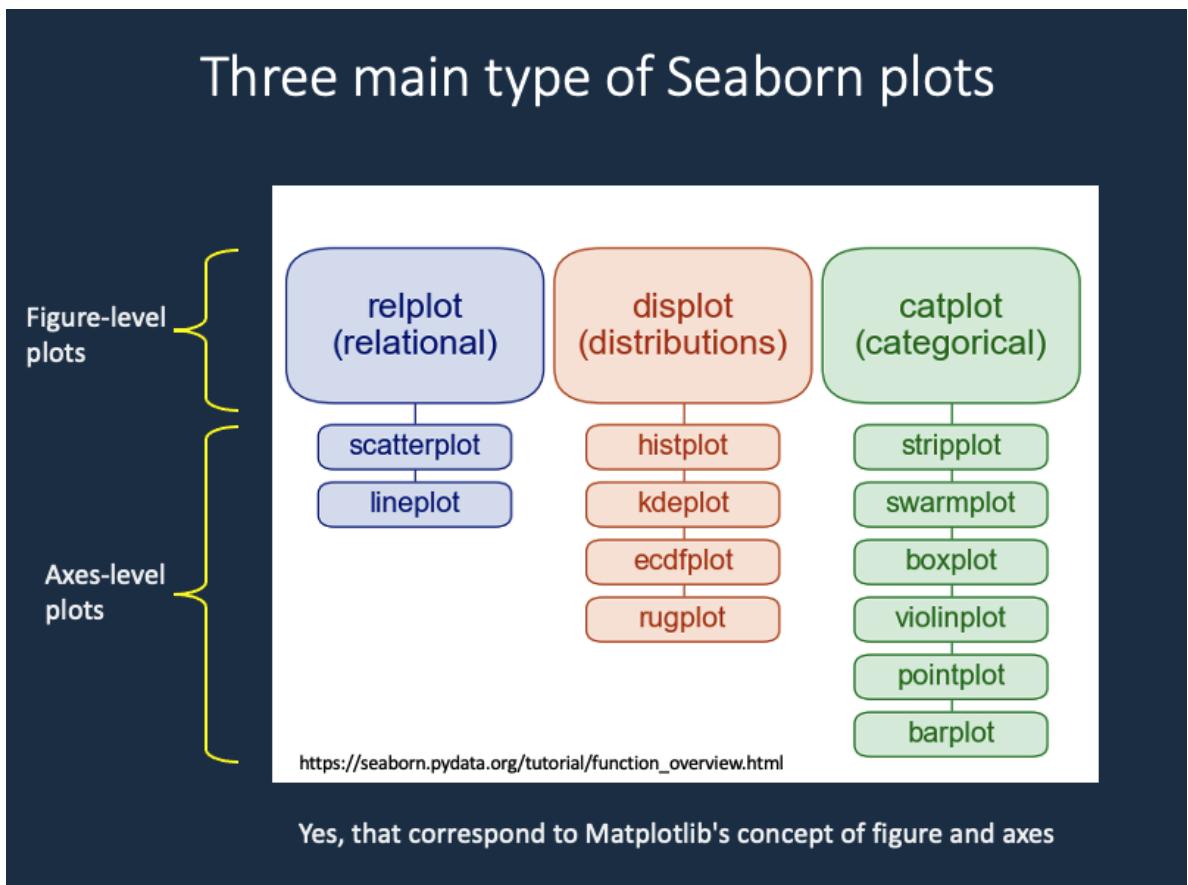


Figure 27.3: seaborn plot type

- Seaborn
  - figure-level plot
  - axes-level plot

R or RStudio run Python with installing packages or modules by using `reticulate`, R package and directly using Anaconda `conda` environment for convenience, instead of `virtualenv`

[https://rstudio.github.io/reticulate/articles/python\\_packages.html](https://rstudio.github.io/reticulate/articles/python_packages.html)

```
library(reticulate)
```

```
## Warning: package 'reticulate' was built under R version 4.3.3
```

```
conda_list()
```

```
## name
## 1 base
## 2 mm
## 3 mmr
## 4 monai
## 5 pytorch
## 6 pytorch_1.12.1_cuda_11.6
## 7 r-reticulate
## 8 sandbox
## 9 sandbox-3.9
## 10 sandbox_py_3.10
## 11 v51
## python
## 1 C:\\\\Users\\\\RW\\\\anaconda3\\\\python.exe
## 2 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\mm\\\\python.exe
## 3 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\mmr\\\\python.exe
## 4 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\monai\\\\python.exe
## 5 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\pytorch\\\\python.exe
## 6 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\pytorch_1.12.1_cuda_11.6\\\\python.exe
## 7 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\r-reticulate\\\\python.exe
## 8 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox\\\\python.exe
## 9 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox-3.9\\\\python.exe
## 10 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\sandbox_py_3.10\\\\python.exe
## 11 C:\\\\Users\\\\RW\\\\anaconda3\\\\envs\\\\v51\\\\python.exe
```

```
use_condaenv(condaenv = 'sandbox-3.9')
```

```
## install Seaborn
# conda_install("r-reticulate", "seaborn")
```

```
## import Seaborn (it will be automatically discovered in "r-reticulate")
seaborn <- import("seaborn")
```

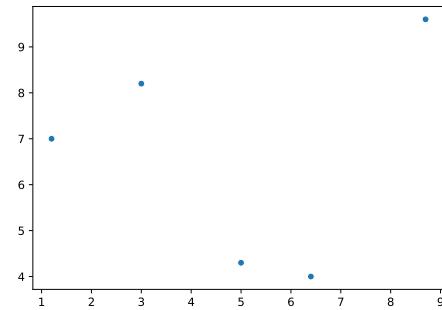
```
import matplotlib.pyplot as plt # need it sometimes
```

```
import seaborn as sns
```

```
sns.set_theme() # set the default theme
```

```
import seaborn as sns

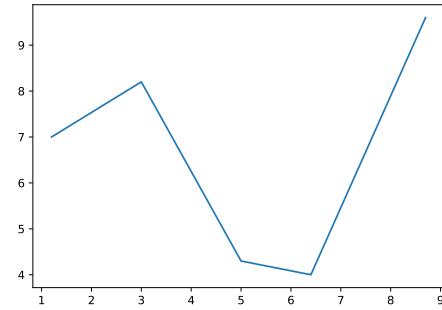
x = [3, 5, 1.2, 8.7, 6.4]
y = [8.2, 4.3, 7, 9.6, 4]
sns.scatterplot(x=x, y=y)
```




---

```
import seaborn as sns

x = [3, 5, 1.2, 8.7, 6.4]
y = [8.2, 4.3, 7, 9.6, 4]
sns.lineplot(x=x, y=y)
```



### 27.1.3.2 data frame

```
sns.load_dataset
```

```
import seaborn as sns

tips = sns.load_dataset("tips")
tips
```

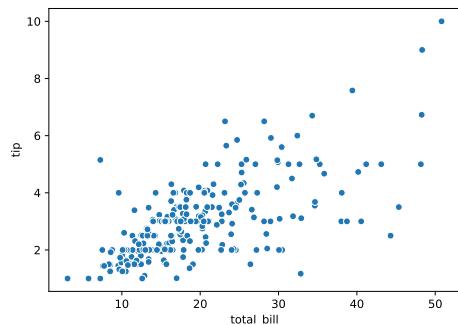
	total_bill	tip	sex	smoker	day
## 0	16.99	1.01	Female	No	Su
## 1	10.34	1.66	Male	No	Su
## 2	21.01	3.50	Male	No	Su
## 3	23.68	3.31	Male	No	Su
## 4	24.59	3.61	Female	No	Su
## ..	...	...	...	...	...
## 239	29.03	5.92	Male	No	Sa
## 240	27.18	2.00	Female	Yes	Sa
## 241	22.67	2.00	Male	Yes	Sa
## 242	17.82	1.75	Male	No	Sa
## 243	18.78	3.00	Female	No	Thu
##					
##	[244 rows x 7 columns]				

---

```
data
```

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
                 x='total_bill',
                 y='tip')
```

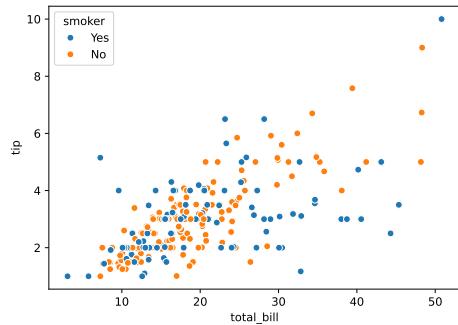



---

hue

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
                 x='total_bill',
                 y='tip',
                 hue='smoker')
```

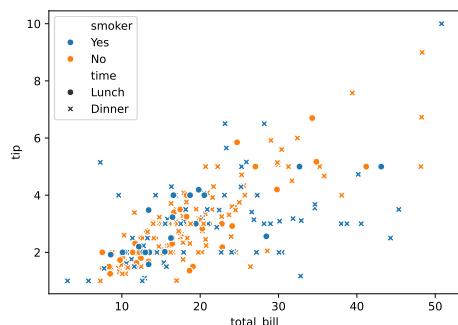



---

style

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
                 x='total_bill',
                 y='tip',
                 hue='smoker',
                 style='time')
```

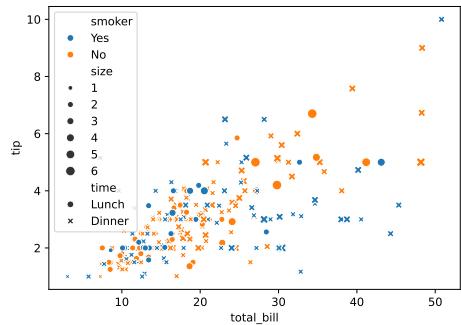



---

size

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
                 x='total_bill',
                 y='tip',
                 hue='smoker',
                 style='time',
                 size='size')
```



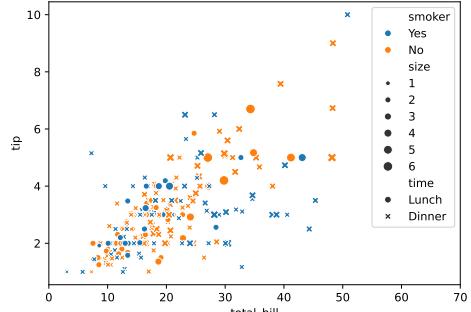
- Legend is covering up the graph, it's getting out of hand. Let's tune the range of x
- `sns.scatterplot()` actually returns something that resembles Matplotlib axis. So we use a Matplotlib axis function:

```
ax = sns.scatterplot(...)
```

<https://stackoverflow.com/questions/26597116/seaborn-plots-not-showing-up>

```
import matplotlib.pyplot as plt
import seaborn as sns

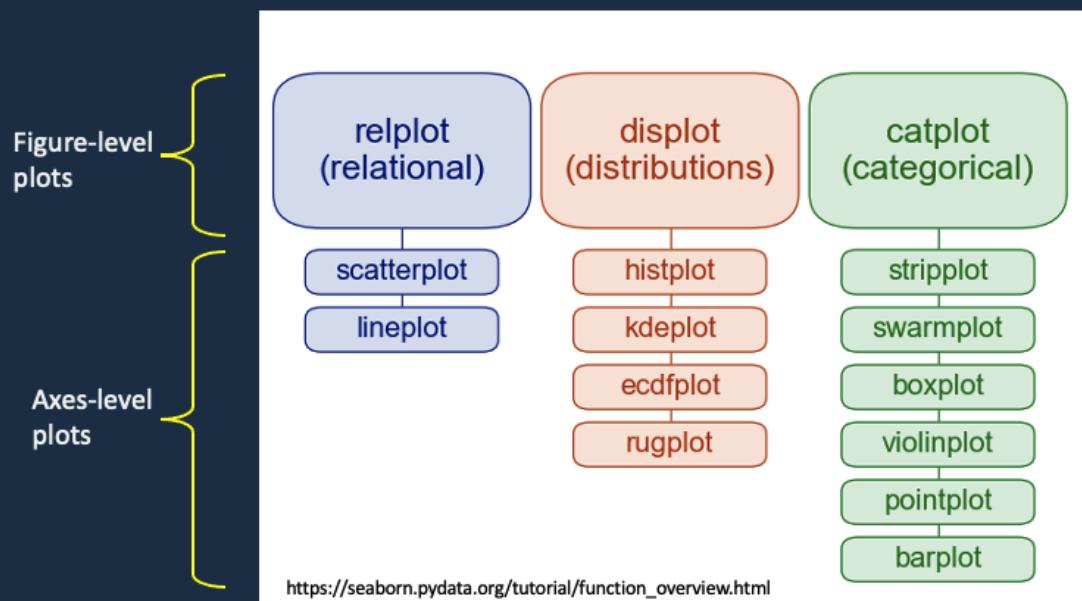
tips = sns.load_dataset("tips")
ax = sns.scatterplot(data=tips,
                      x='total_bill',
                      y='tip',
                      hue='smoker',
                      style='time',
                      size='size')
ax.set_xlim((0, 70))  
## (0.0, 70.0)
# alternatively:
# ax.set(xlim=(0, 70))
plt.show()
```



### 27.1.3.3 axis-level plot and figure-level plot

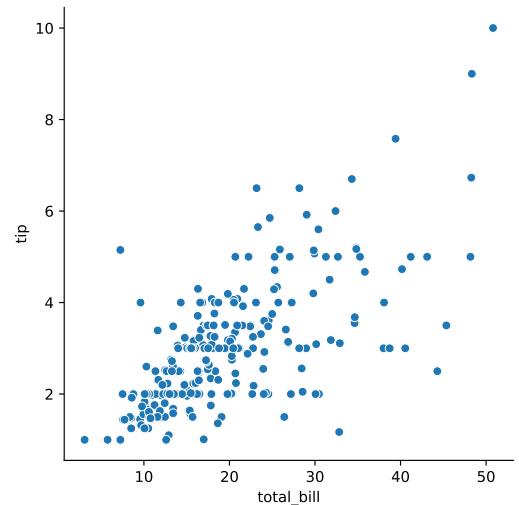
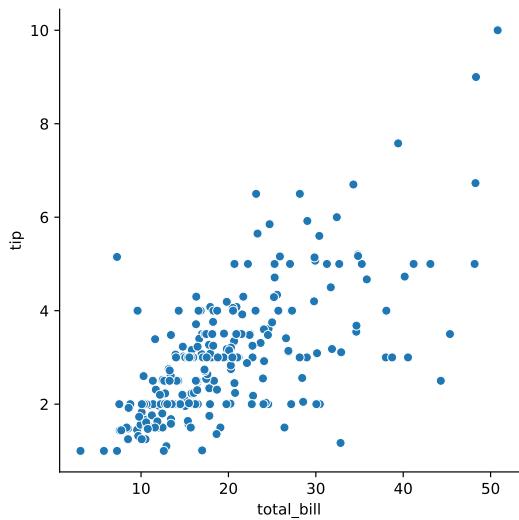
`sns.relpolt`

## Three main type of Seaborn plots



Yes, that correspond to Matplotlib's concept of figure and axes

Figure 27.4: seaborn plot type

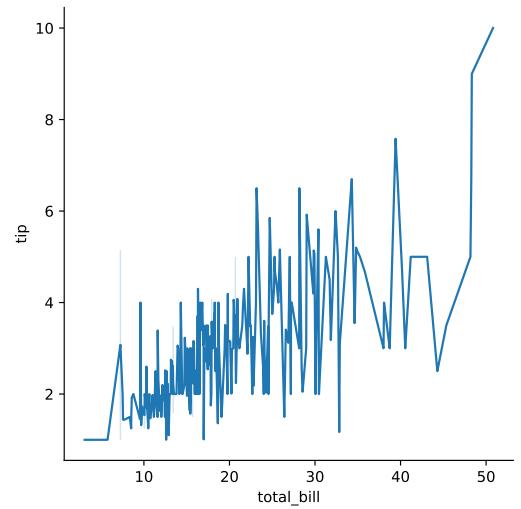
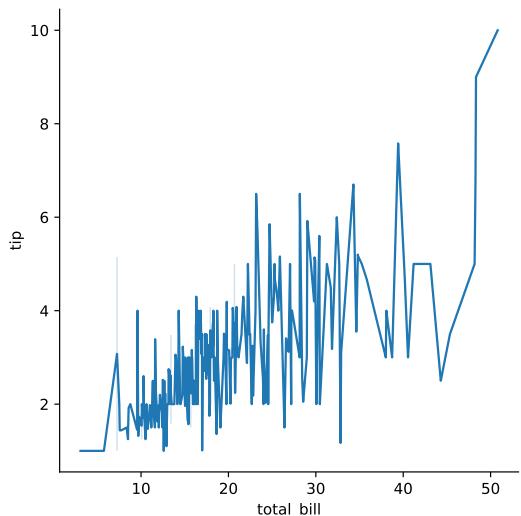


```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
            data=tips,
            x='total_bill',
            y='tip')
```

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='line',
            data=tips,
            x='total_bill',
            y='tip')
```



---

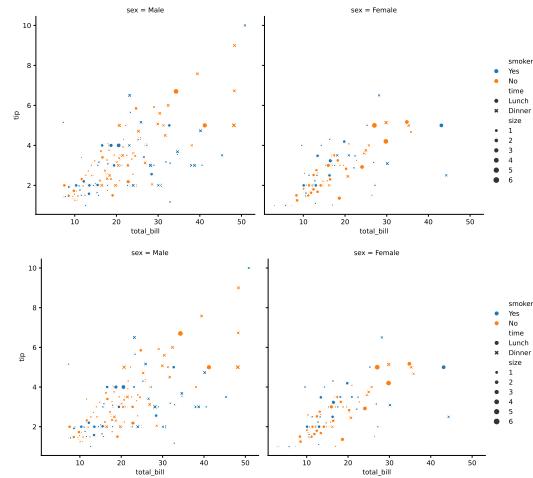
With figure-level plot, we can draw more than one plot (one `axes`).

Here we specify that different `sex` be on different column by specifying `col=sex`.

`col=sex`

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
            data=tips,
            x='total_bill',
            y='tip',
            hue='smoker',
            style='time',
            size='size',
            col='sex')
```

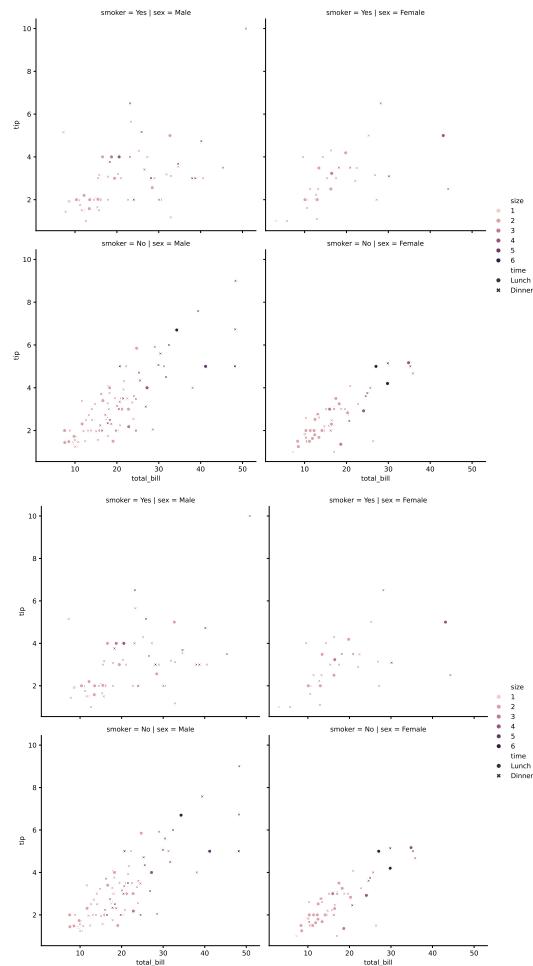



---

```
col=sex row=smoker
```

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
            data=tips,
            x='total_bill',
            y='tip',
            style='time',
            hue='size',
            row='smoker',
            col='sex')
```



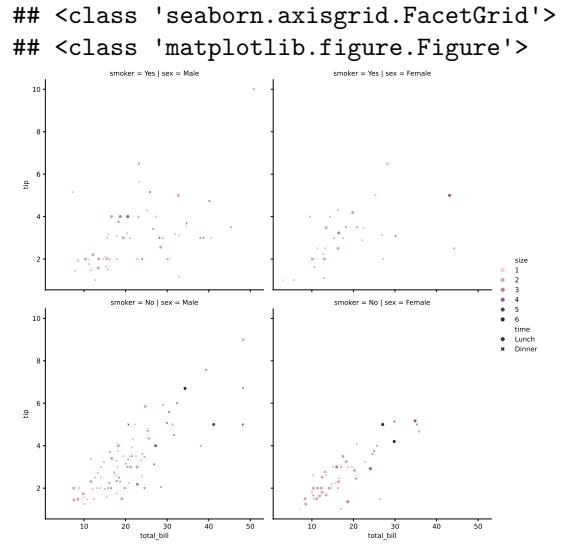
### 27.1.3.4 accessing figure and axes objects

Recall that Seaborn uses Matplotlib to draw the graphics. So underneath the Seaborn library, you can still access Matplotlib's figure object and axes objects if necessary. The call to figure-level plot returns an object.

```
import seaborn as sns

tips = sns.load_dataset("tips")
g = sns.relplot(kind='scatter',
                 data=tips,
                 x='total_bill',
                 y='tip',
                 style='time',
                 hue='size',
                 row='smoker',
                 col='sex')

print(type(g))
print(type(g.fig)) # g.fig gets you
                    ← the Figure
g.fig
```



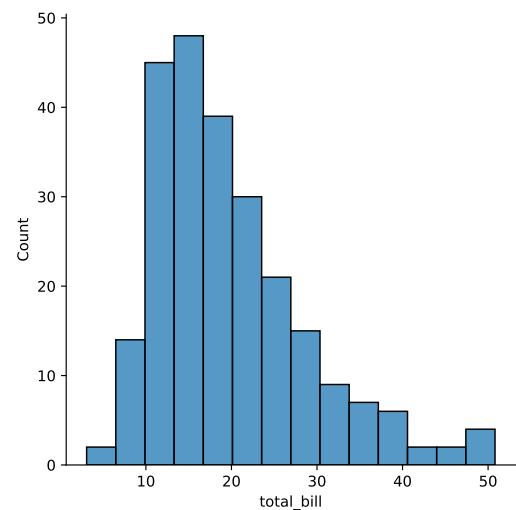
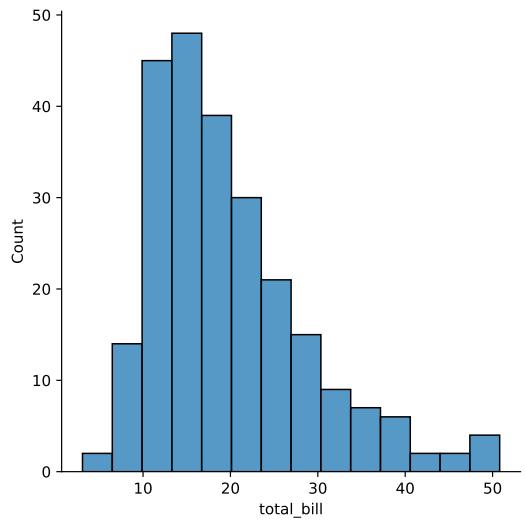
### 27.1.3.5 distribution plot

```
sns.displot
```

#### 27.1.3.5.1 histogram

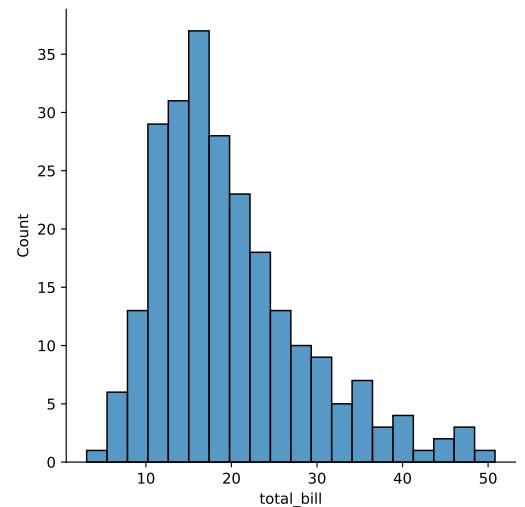
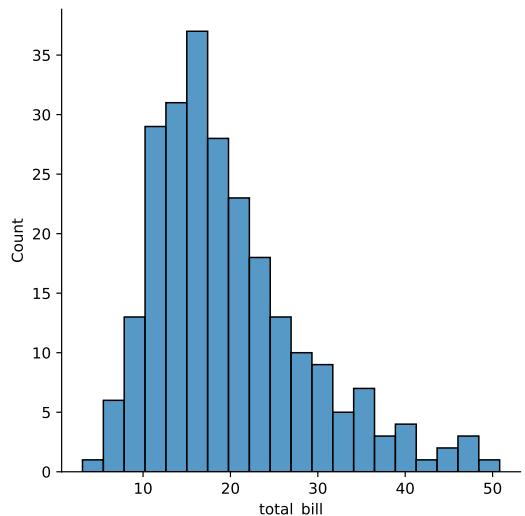
```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist',
            data=tips,
            x='total_bill')
```



```
import seaborn as sns

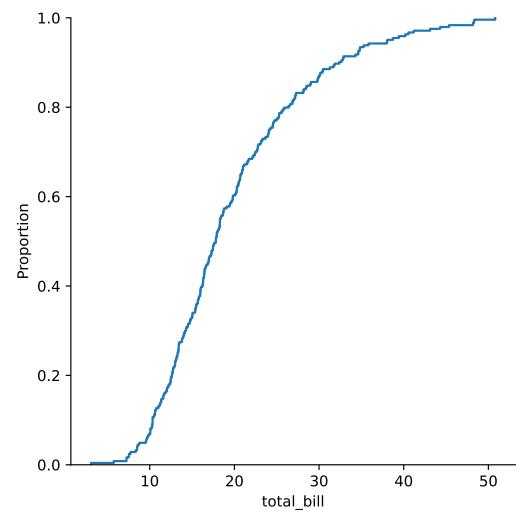
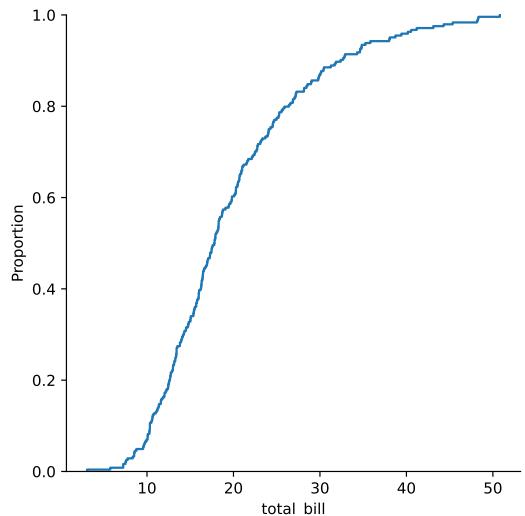
tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20,
            data=tips,
            x='total_bill')
```



#### 27.1.3.5.2 ECDF = empirical cumulative distrutive function

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='ecdf',
            data=tips,
            x='total_bill')
```

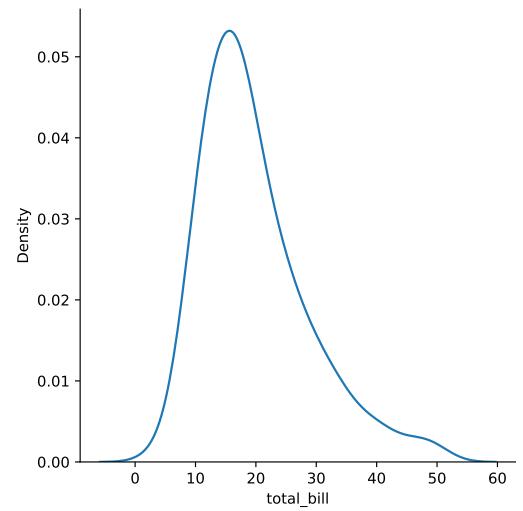
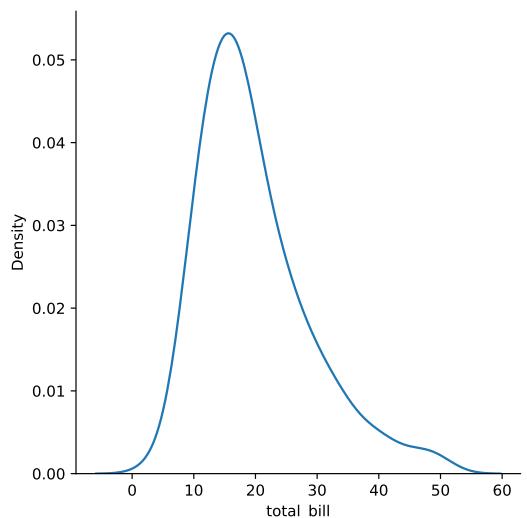


#### 27.1.3.5.3 KDE = kernel density estimation

---

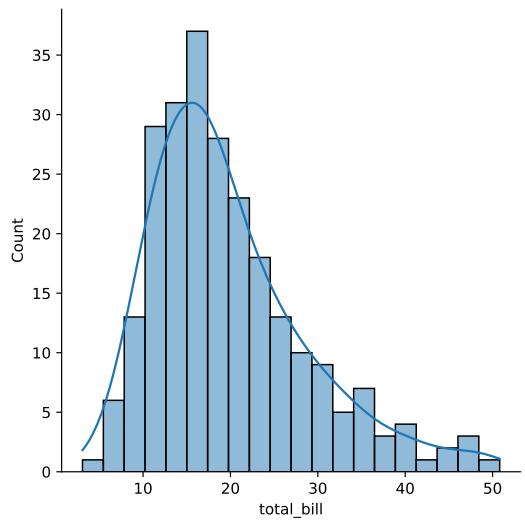
```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='kde',
            data=tips,
            x='total_bill')
```

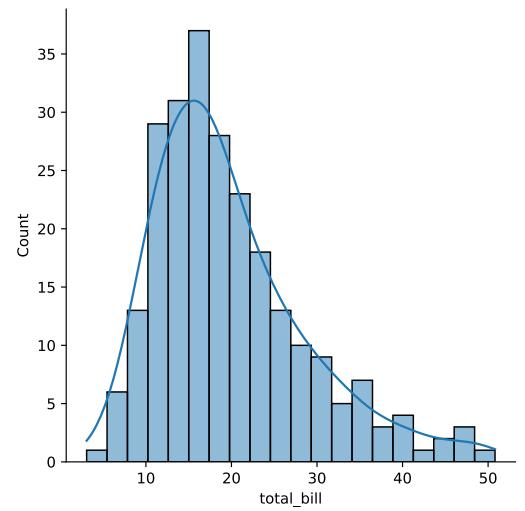


```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20,
            kde=True,
            data=tips,
            x='total_bill')
```



## '% This file was created with tikzplotlib'




---

TikZ / `tikzpicture` with PGFplots `axis` by transforming Matplotlib-based Seaborn plot to .tex via Python package `tikzplotlib`

`xlabel={total_bill}`, changed to `xlabel={total bill}`, without `_` in text

---

`X ylabel={Count}`, changed to `ylabel={$\text{Count}$}`,: not necessary to be changed, the problem is `_` in xlabel name

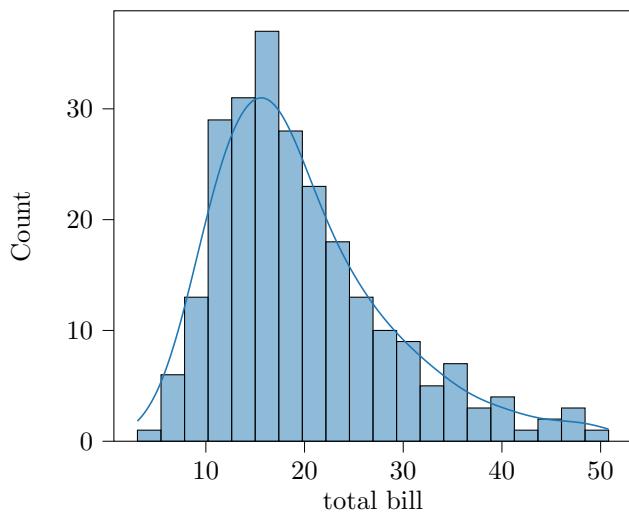
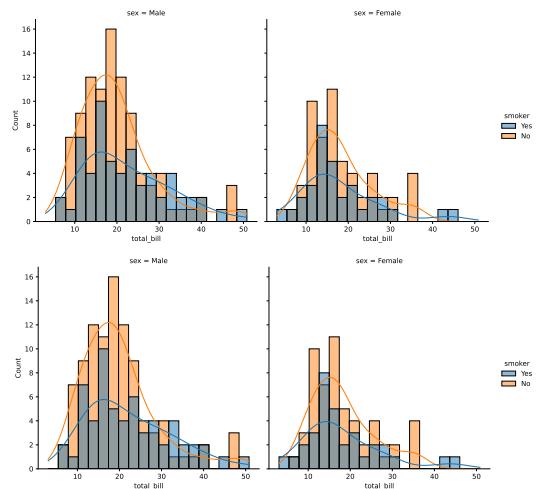


Figure 27.5: tikzplotlib

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20,
            kde=True,
            data=tips,
            x='total_bill',
            hue='smoker',
            col='sex'
            )
```



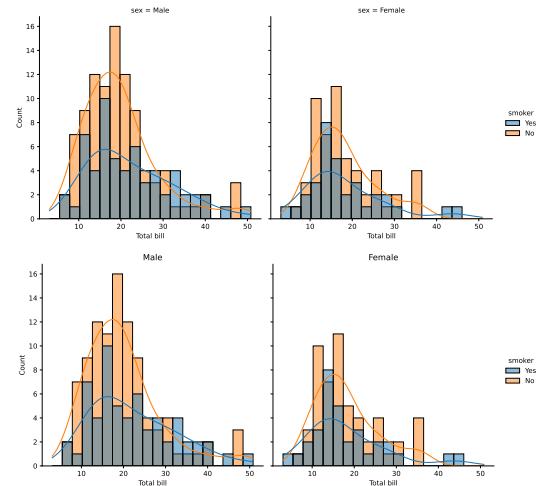
Let's customize the label and title.

```

import seaborn as sns

tips = sns.load_dataset("tips")
g = sns.displot(kind='hist', bins=20,
                 kde=True,
                 data=tips,
                 x='total_bill',
                 hue='smoker',
                 col='sex'
)
g.set(xlabel='Total bill')
g.axes[0][0].set(title='Male')
g.axes[0][1].set(title='Female')
g.fig

```

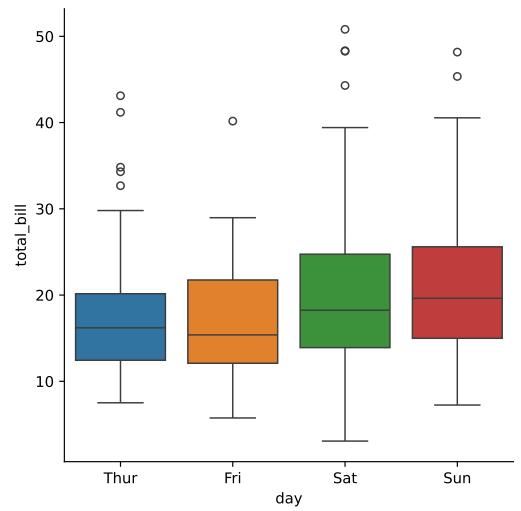
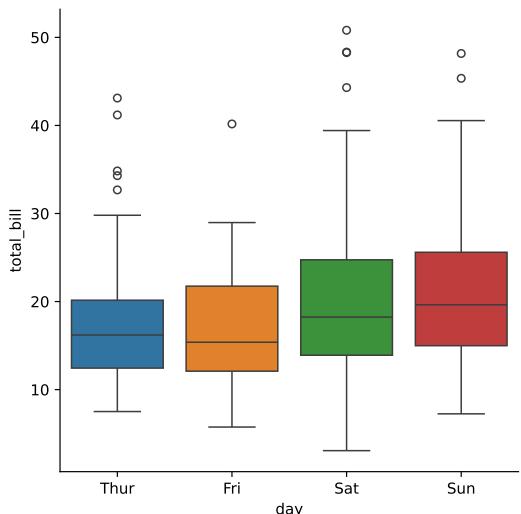


### 27.1.3.6 categorical plot

`sns.catplot`

```
import seaborn as sns

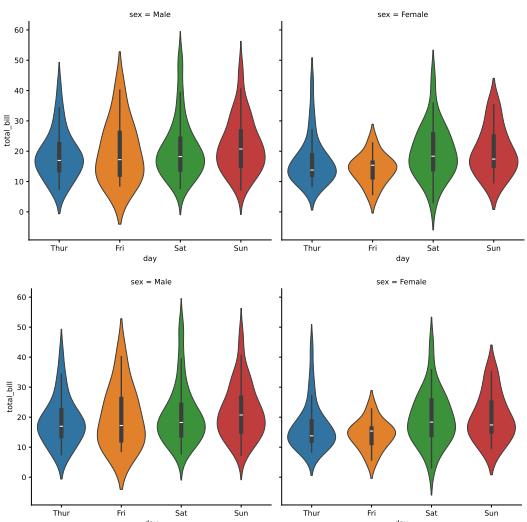
tips = sns.load_dataset("tips")
sns.catplot(kind='box',
            data=tips,
            x="day", hue="day",
            y="total_bill"
            )
```




---

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.catplot(kind='violin',
            data=tips,
            x="day", hue="day",
            y="total_bill",
            col='sex'
            )
```

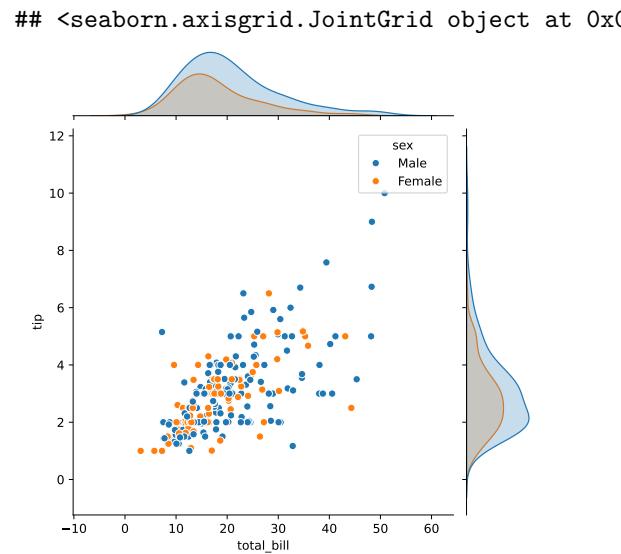


### 27.1.3.7 joint plot

```
sns.jointplot
```

```
import matplotlib.pyplot as plt
import seaborn as sns

tips = sns.load_dataset("tips")
sns.jointplot(data=tips,
               x="total_bill",
               y="tip",
               hue='sex'
              )
plt.show()
```

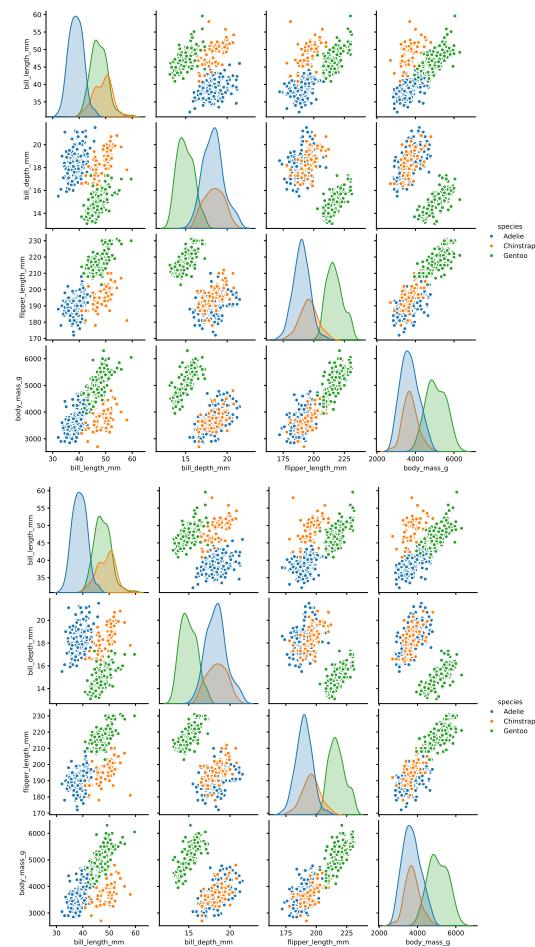


### 27.1.3.8 pair plot

```
sns.pairplot
```

```
import seaborn as sns

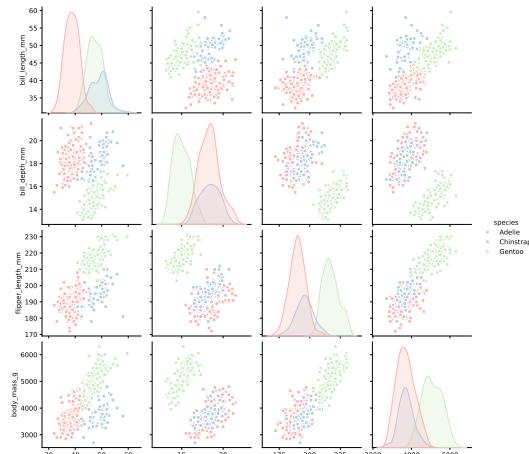
penguins =
    sns.load_dataset("penguins")
sns.pairplot(data=penguins,
              hue='species')
```



### 27.1.3.9 palette

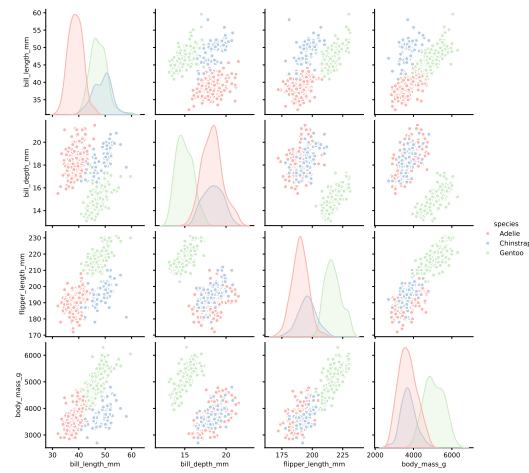
Changing the color with Matplotlib's color

A list of color: <https://matplotlib.org/stable/tutorials/colors/colormaps.html>



```
import seaborn as sns

penguins =
    sns.load_dataset("penguins")
sns.pairplot(data=penguins,
             hue='species',
             palette='Pastel1')
```



### 27.1.3.10 volcano plot

### 27.1.3.11 heatmap

## 27.2 export

### 27.2.1 .svg

<https://stackoverflow.com/questions/24525111/how-can-i-get-the-output-of-a-matplotlib-plot-as-an-svg>

```
import matplotlib.pyplot as plt
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
             hue='species')

plt.savefig("test.svg")
# plt.savefig("test.svg", dpi=1200)
```

## 27.2.2 .eps

<https://stackoverflow.com/questions/16183462/saving-images-in-python-at-a-very-high-quality>

The PostScript backend does not support transparency; partially transparent artists will be rendered opaque.

```
import matplotlib.pyplot as plt
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
              hue='species')

plt.savefig("test.eps")
# plt.savefig("test.eps", dpi=1200)
```



# Chapter 28

## survival analysis

28.1 Python package `tableone`

28.2 Python package `lifelines`



# Chapter 29

## Manim

### 29.1 VSCode extension: Manim Sideview

<https://marketplace.visualstudio.com/items?itemName=Rickaym.manim-sideview>

ffmpeg.exe placed in the same folder with .py

VSCode Ctrl + Shift + P: open Mobject gallery

### 29.2 installation

<https://docs.manim.community/en/stable/installation.html>

#### 29.2.1 Conda

conda install -c conda-forge manim

### 29.3 quickstart

<https://docs.manim.community/en/stable/tutorials/quickstart.html>

[https://www.w3schools.com/tags/att\\_video\\_autoplay.asp](https://www.w3schools.com/tags/att_video_autoplay.asp)

[https://www.w3schools.com/tags/att\\_video\\_loop.asp](https://www.w3schools.com/tags/att_video_loop.asp)

```
from manim import *

class CreateCircle(Scene):
    def construct(self):
        circle = Circle() # create a
        ← circle
        circle.set_fill(PINK,
        ← opacity=0.5) # set the color and
        ← transparency
        self.play(Create(circle)) # ← show the circle on screen
```

manim -pql scene.py CreateCircle



# Chapter 30

## ggplot2

<https://bookdown.org/xiangyun/msg/system.html#chap:system>

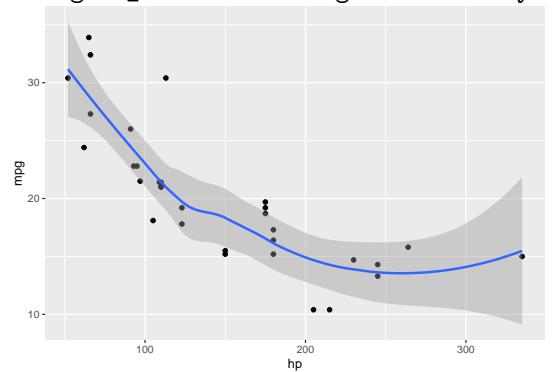
Modern Statistical Graphics [section 5.1](#)

- <https://www.rdocumentation.org/> to search function
- ggplot2
  - <https://ggplot2.tidyverse.org/index.html> to search ggplot2 function
  - panel = layer
    - \* geom = geometric objects / geometry = element
      - [element](#)
    - \* statistic
    - \* scale
    - \* coordinate system
    - \* facet

```
library(ggplot2)

p <- ggplot(aes(x = hp, y = mpg), data
             = mtcars) +
  geom_point() # layer of scatterplot
  p + geom_smooth(method = "loess") # # add layer of smooth
```

```
## Warning: package 'ggplot2' was built under R version 3.5.2
## `geom_smooth()` using formula = 'y ~ x'
```

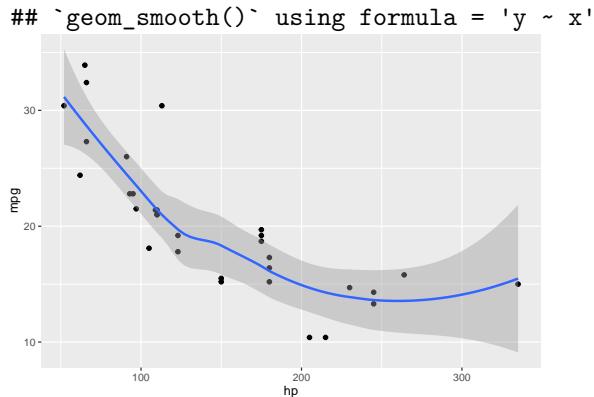


### 30.1 geom

<https://bookdown.org/xiangyun/msg/system.html#section-13>

```
library(ggplot2)

ggplot(aes(x = hp, y = mpg), data =
  mtcars) +
  geom_point() +
  geom_smooth(method = "loess")
```



points [https://ggplot2.tidyverse.org/reference/geom\\_point.html?q=geom\\_point#null](https://ggplot2.tidyverse.org/reference/geom_point.html?q=geom_point#null)

### geom\_point

smoothed conditional means [https://ggplot2.tidyverse.org/reference/geom\\_smooth.html?q=geom\\_sm#null](https://ggplot2.tidyverse.org/reference/geom_smooth.html?q=geom_sm#null)

Aids the eye in seeing patterns in the presence of overplotting. `geom_smooth()` and `stat_smooth()` are effectively aliases: they both use the same arguments. Use `stat_smooth()` if you want to display the results with a non-standard geom.

### geom\_smooth

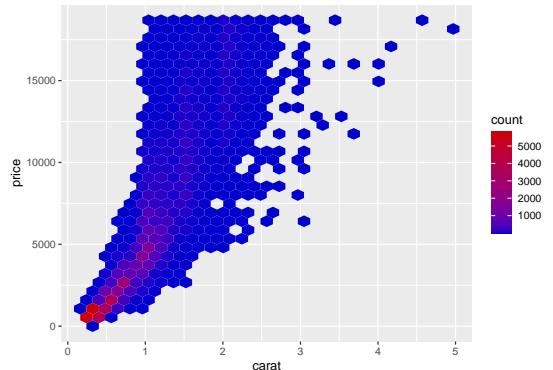
### stat\_smooth

`method` Smoothing method (function) to use, accepts either NULL or a character vector, e.g. "lm", "glm", "gam", "loess" or a function, e.g. MASS::rlm or mgcv::gam, `stats::lm`, or `stats::loess`. "auto" is also accepted for backwards compatibility. It is equivalent to NULL.

```
# install.packages("hexbin")
```

```
library(ggplot2)

ggplot(aes(x = carat, y = price), data
  = diamonds) +
  geom_hex() +
  scale_fill_gradient(low = "blue3",
  high = "red3")
```

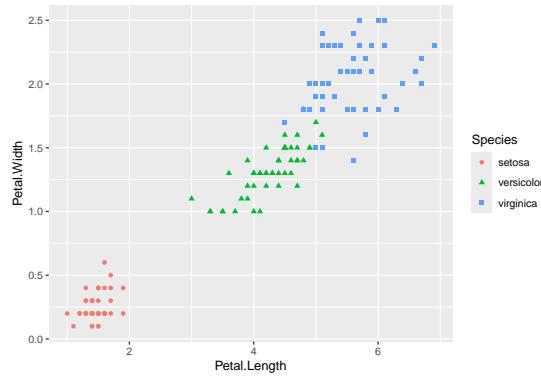


gradient color scales [https://ggplot2.tidyverse.org/reference/scale\\_gradient.html?q=scale\\_fill\\_gradie#ref-usage](https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradie#ref-usage)

[https://ggplot2.tidyverse.org/reference/scale\\_gradient.html?q=scale\\_fill\\_gradient#ref-examples](https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradient#ref-examples)

```
library(ggplot2)

ggplot(aes(x = Petal.Length, y =
           Petal.Width), data = iris) +
  geom_point(aes(color = Species,
                 shape = Species))
```



basic plot system

<https://bookdown.org/xiangyun/msg/elements.html#sec:points>

```
# iris species converted to type
#   integer 1, 2, 3 for further using
#   vectors
idx <- as.integer(iris[["Species"]])
plot(iris[, 3:4],
     pch = c(24, 21, 25)[idx],
     col = c("black", "red",
            "blue")[idx], panel.first =
     grid())
)
legend("topleft",
       legend = levels(iris[["Species"]]),
       col = c("black", "red", "blue"), pch
       = c(24, 21, 25), bty = "n"
)
```

plot <https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/plot.default>

**pch** a vector of **plotting characters** or symbols: see **points**.

**col** The **colors** for lines and points. Multiple colors can be specified so that each point can be given its own color. If there are fewer colors than points they are recycled in the standard fashion. Lines will all be plotted in the first colour specified.

**panel.first** an ‘expression’ to be evaluated after the plot axes are set up but before any plotting takes place. This can be useful for drawing **background grids** or **scatterplot smooths**. Note that this works by lazy evaluation: passing this argument from other plot methods may well not work since it may be evaluated too early.

**legend** <https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/legend>

**bty** the **type of box** to be drawn around the legend. The allowed values are “o” (the default) and “n”.

<https://stackoverflow.com/questions/10108073/plot-legends-without-border-and-with-white-background>

Use option **bty = "n"** in legend to remove the box around the legend.

**legend** a character or expression vector of length  $\geq 1$  to appear in the legend. Other objects will be coerced by [as.graphicsAnnot](#)



### 30.1.1 basic plot system decomposition

	Sepal.Length	Sepal.Width	Petal.Length
## 1	5.1	3.5	1.4
## 2	4.9	3.0	1.4
## 3	4.7	3.2	1.3
## 4	4.6	3.1	1.5
## 5	5.0	3.6	1.4
## 6	5.4	3.9	1.7
## 7	4.6	3.4	1.4
## 8	5.0	3.4	1.5
## 9	4.4	2.9	1.4
## 10	4.9	3.1	1.5
## 11	5.4	3.7	1.5
## 12	4.8	3.4	1.6
## 13	4.8	3.0	1.4
## 14	4.3	3.0	1.1
## 15	5.8	4.0	1.2
## 16	5.7	4.4	1.5
## 17	5.4	3.9	1.3
## 18	5.1	3.5	1.4
## 19	5.7	3.8	1.7
## 20	5.1	3.8	1.5
## 21	5.4	3.4	1.7
## 22	5.1	3.7	1.5
## 23	4.6	3.6	1.0
## 24	5.1	3.3	1.7
## 25	4.8	3.4	1.9
## 26	5.0	3.0	1.6
## 27	5.0	3.4	1.6
## 28	5.2	3.5	1.5
## 29	5.2	3.4	1.4
## 30	4.7	3.2	1.6
## 31	4.8	3.1	1.6
## 32	5.4	3.4	1.5
## 33	5.2	4.1	1.5
## 34	5.5	4.2	1.4
## 35	4.9	3.1	1.5
## 36	5.0	3.2	1.2
## 37	5.5	3.5	1.3
## 38	4.9	3.6	1.4
## 39	4.4	3.0	1.3
## 40	5.1	3.4	1.5
## 41	5.0	3.5	1.3
## 42	4.5	2.3	1.3
## 43	4.4	3.2	1.3
## 44	5.0	3.5	1.6
## 45	5.1	3.8	1.9
## 46	4.8	3.0	1.4
## 47	5.1	3.8	1.6
## 48	4.6	3.2	1.4
## 49	5.3	3.7	1.5
## 50	5.0	3.3	1.4
## 51	7.0	3.2	4.7
## 52	6.4	3.2	4.5
## 53	6.9	3.1	4.9
## 54	5.5	3.0	4.6
## 55	5.0	3.4	4.5
## 56	5.4	3.9	4.7
## 57	4.9	3.1	4.6
## 58	5.8	4.0	1.2
## 59	5.2	3.5	1.5
## 60	4.7	3.2	1.3
## 61	5.4	3.7	1.5
## 62	5.1	3.3	1.3
## 63	5.7	4.4	1.5
## 64	5.4	3.9	1.3
## 65	5.1	3.5	1.5
## 66	4.8	3.4	1.6
## 67	5.0	3.0	1.6
## 68	5.5	4.2	1.4
## 69	5.1	3.4	1.3
## 70	5.7	4.0	1.2
## 71	5.4	3.7	1.3
## 72	5.1	3.3	1.5
## 73	4.8	3.0	1.6
## 74	5.0	3.4	1.4
## 75	4.5	2.3	1.3
## 76	4.4	3.2	1.3
## 77	5.0	3.5	1.6
## 78	5.1	3.8	1.9
## 79	4.8	3.0	1.4
## 80	5.5	4.2	1.5
## 81	5.0	3.3	1.3
## 82	5.7	4.0	1.2
## 83	5.4	3.7	1.3
## 84	5.1	3.3	1.5
## 85	4.8	3.0	1.6
## 86	5.0	3.4	1.4
## 87	4.5	2.3	1.3
## 88	4.4	3.2	1.3
## 89	5.0	3.5	1.6
## 90	5.1	3.8	1.9
## 91	4.8	3.0	1.4
## 92	5.5	4.2	1.5
## 93	5.0	3.3	1.3
## 94	5.7	4.0	1.2
## 95	5.4	3.7	1.3
## 96	5.1	3.3	1.5
## 97	4.8	3.0	1.6
## 98	5.0	3.4	1.4
## 99	4.5	2.3	1.3
## 100	4.4	3.2	1.3
## 101	5.0	3.5	1.6
## 102	5.1	3.8	1.9
## 103	4.8	3.0	1.4
## 104	5.5	4.2	1.5
## 105	5.0	3.3	1.3
## 106	5.7	4.0	1.2
## 107	5.4	3.7	1.3
## 108	5.1	3.3	1.5
## 109	4.8	3.0	1.6
## 110	5.0	3.4	1.4
## 111	4.5	2.3	1.3
## 112	4.4	3.2	1.3
## 113	5.0	3.5	1.6
## 114	5.1	3.8	1.9
## 115	4.8	3.0	1.4
## 116	5.5	4.2	1.5
## 117	5.0	3.3	1.3
## 118	5.7	4.0	1.2
## 119	5.4	3.7	1.3
## 120	5.1	3.3	1.5
## 121	4.8	3.0	1.6
## 122	5.0	3.4	1.4
## 123	4.5	2.3	1.3
## 124	4.4	3.2	1.3
## 125	5.0	3.5	1.6
## 126	5.1	3.8	1.9
## 127	4.8	3.0	1.4
## 128	5.5	4.2	1.5
## 129	5.0	3.3	1.3
## 130	5.7	4.0	1.2
## 131	5.4	3.7	1.3
## 132	5.1	3.3	1.5
## 133	4.8	3.0	1.6
## 134	5.0	3.4	1.4
## 135	4.5	2.3	1.3
## 136	4.4	3.2	1.3
## 137	5.0	3.5	1.6
## 138	5.1	3.8	1.9
## 139	4.8	3.0	1.4
## 140	5.5	4.2	1.5
## 141	5.0	3.3	1.3
## 142	5.7	4.0	1.2
## 143	5.4	3.7	1.3
## 144	5.1	3.3	1.5
## 145	4.8	3.0	1.6
## 146	5.0	3.4	1.4
## 147	4.5	2.3	1.3
## 148	4.4	3.2	1.3
## 149	5.0	3.5	1.6
## 150	5.1	3.8	1.9
## 151	4.8	3.0	1.4
## 152	5.5	4.2	1.5
## 153	5.0	3.3	1.3
## 154	5.7	4.0	1.2
## 155	5.4	3.7	1.3
## 156	5.1	3.3	1.5
## 157	4.8	3.0	1.6
## 158	5.0	3.4	1.4
## 159	4.5	2.3	1.3
## 160	4.4	3.2	1.3
## 161	5.0	3.5	1.6
## 162	5.1	3.8	1.9
## 163	4.8	3.0	1.4
## 164	5.5	4.2	1.5
## 165	5.0	3.3	1.3
## 166	5.7	4.0	1.2
## 167	5.4	3.7	1.3
## 168	5.1	3.3	1.5
## 169	4.8	3.0	1.6
## 170	5.0	3.4	1.4
## 171	4.5	2.3	1.3
## 172	4.4	3.2	1.3
## 173	5.0	3.5	1.6
## 174	5.1	3.8	1.9
## 175	4.8	3.0	1.4
## 176	5.5	4.2	1.5
## 177	5.0	3.3	1.3
## 178	5.7	4.0	1.2
## 179	5.4	3.7	1.3
## 180	5.1	3.3	1.5
## 181	4.8	3.0	1.6
## 182	5.0	3.4	1.4
## 183	4.5	2.3	1.3
## 184	4.4	3.2	1.3
## 185	5.0	3.5	1.6
## 186	5.1	3.8	1.9
## 187	4.8	3.0	1.4
## 188	5.5	4.2	1.5
## 189	5.0	3.3	1.3
## 190	5.7	4.0	1.2
## 191	5.4	3.7	1.3
## 192	5.1	3.3	1.5
## 193	4.8	3.0	1.6
## 194	5.0	3.4	1.4
## 195	4.5	2.3	1.3
## 196	4.4	3.2	1.3
## 197	5.0	3.5	1.6
## 198	5.1	3.8	1.9
## 199	4.8	3.0	1.4
## 200	5.5	4.2	1.5
## 201	5.0	3.3	1.3
## 202	5.7	4.0	1.2
## 203	5.4	3.7	1.3
## 204	5.1	3.3	1.5
## 205	4.8	3.0	1.6
## 206	5.0	3.4	1.4
## 207	4.5	2.3	1.3
## 208	4.4	3.2	1.3
## 209	5.0	3.5	1.6
## 210	5.1	3.8	1.9
## 211	4.8	3.0	1.4
## 212	5.5	4.2	1.5
## 213	5.0	3.3	1.3
## 214	5.7	4.0	1.2
## 215	5.4	3.7	1.3
## 216	5.1	3.3	1.5
## 217	4.8	3.0	1.6
## 218	5.0	3.4	1.4
## 219	4.5	2.3	1.3
## 220	4.4	3.2	1.3
## 221	5.0	3.5	1.6
## 222	5.1	3.8	1.9
## 223	4.8	3.0	1.4
## 224	5.5	4.2	1.5
## 225	5.0	3.3	1.3
## 226	5.7	4.0	1.2
## 227	5.4	3.7	1.3
## 228	5.1	3.3	1.5
## 229	4.8	3.0	1.6
## 230	5.0	3.4	1.4
## 231	4.5	2.3	1.3
## 232	4.4	3.2	1.3
## 233	5.0	3.5	1.6
## 234	5.1	3.8	1.9
## 235	4.8	3.0	1.4
## 236	5.5	4.2	1.5
## 237	5.0	3.3	1.3
## 238	5.7	4.0	1.2
## 239	5.4	3.7	1.3
## 240	5.1	3.3	1.5
## 241	4.8	3.0	1.6
## 242	5.0	3.4	1.4
## 243	4.5	2.3	1.3
## 244	4.4	3.2	1.3
## 245	5.0	3.5	1.6
## 246	5.1	3.8	1.9
## 247	4.8	3.0	1.4
## 248	5.5	4.2	1.5
## 249	5.0	3.3	1.3
## 250	5.7	4.0	1.2
## 251	5.4	3.7	1.3
## 252	5.1	3.3	1.5
## 253	4.8	3.0	1.6
## 254	5.0	3.4	1.4
## 255	4.5	2.3	1.3
## 256	4.4	3.2	1.3
## 257	5.0	3.5	1.6
## 258	5.1	3.8	1.9
## 259	4.8	3.0	1.4
## 260	5.5	4.2	1.5
## 261	5.0	3.3	1.3
## 262	5.7	4.0	1.2
## 263	5.4	3.7	1.3
## 264	5.1	3.3	1.5
## 265	4.8	3.0	1.6
## 266	5.0	3.4	1.4
## 267	4.5	2.3	1.3
## 268	4.4	3.2	1.3
## 269	5.0	3.5	1.6
## 270	5.1	3.8	1.9
## 271	4.8	3.0	1.4
## 272	5.5	4.2	1.5
## 273	5.0	3.3	1.3
## 274	5.7	4.0	1.2
## 275	5.4	3.7	1.3
## 276	5.1	3.3	1.5
## 277	4.8	3.0	1.6
## 278	5.0	3.4	1.4
## 279	4.5	2.3	1.3
## 280	4.4	3.2	1.3
## 281	5.0	3.5	1.6
## 282	5.1	3.8	1.9
## 283	4.8	3.0	1.4
## 284	5.5	4.2	1.5
## 285	5.0	3.3	1.3
## 286	5.7	4.0	1.2
## 287	5.4	3.7	1.3
## 288	5.1	3.3	1.5
## 289	4.8	3.0	1.6
## 290	5.0	3.4	1.4
## 291	4.5	2.3	1.3
## 292	4.4	3.2	1.3
## 293	5.0	3.5	1.6
## 294	5.1	3.8	1.9
## 295	4.8	3.0	1.4
## 296	5.5	4.2	1.5
## 297	5.0	3.3	1.3
## 298	5.7	4.0	1.2
## 299	5.4	3.7	1.3
## 300	5.1	3.3	1.5

```
iris$Species
```

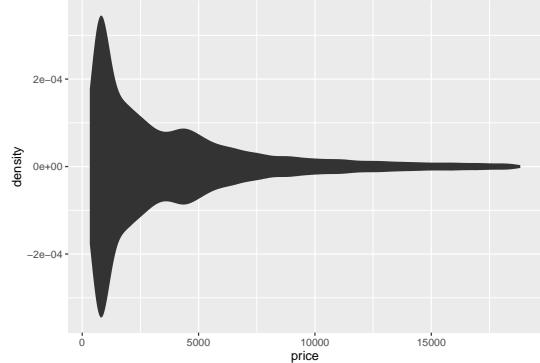
```
## [1] setosa      setosa      setosa      setosa      se
## [7] setosa      setosa      setosa      setosa      se
## [13] setosa     setosa      setosa      setosa      se
## [19] setosa     setosa      setosa      setosa      se
## [25] setosa     setosa      setosa      setosa      se
## [31] setosa     setosa      setosa      setosa      se
## [37] setosa     setosa      setosa      setosa      se
## [43] setosa     setosa      setosa      setosa      se
## [49] setosa     setosa      setosa      versicolor ve
## [55] versicolor versicolor versicolor versicolor ve
## [61] versicolor versicolor versicolor versicolor ve
## [67] versicolor versicolor versicolor versicolor ve
## [73] versicolor versicolor versicolor versicolor ve
## [79] versicolor versicolor versicolor versicolor ve
## [85] versicolor versicolor versicolor versicolor ve
## [91] versicolor versicolor versicolor versicolor ve
## [97] versicolor versicolor versicolor versicolor ve
## [103] virginica virginica virginica virginica vi
## [109] virginica virginica virginica virginica vi
## [115] virginica virginica virginica virginica vi
## [121] virginica virginica virginica virginica vi
## [127] virginica virginica virginica virginica vi
## [133] virginica virginica virginica virginica vi
## [139] virginica virginica virginica virginica vi
## [145] virginica virginica virginica virginica vi
## Levels: setosa versicolor virginica
```

## 30.2 statistic

```
library(ggplot2)

ggplot(diamonds, aes(x = price)) +
  stat_density(aes(ymax = ..density..,
                  ymin = -..density..),
               geom = "ribbon", position =
                 "identity"
  )
```

## Warning: The dot-dot notation (`..density..`)  
 ## i Please use `after\_stat(density)` instead.  
 ## This warning is displayed once every 8 hours.  
 ## Call `lifecycle::last\_lifecycle\_warnings()`  
 ## generated.



smoothed density estimates [https://ggplot2.tidyverse.org/reference/geom\\_density.html](https://ggplot2.tidyverse.org/reference/geom_density.html)

Computes and draws kernel density estimate, which is a smoothed version of the histogram. This is a useful alternative to the histogram for continuous data that comes from an underlying smooth distribution.

`geom_density`

`stat_density`

[https://ggplot2.tidyverse.org/reference/geom\\_density.html#ref-examples](https://ggplot2.tidyverse.org/reference/geom_density.html#ref-examples)

### 30.3 scale

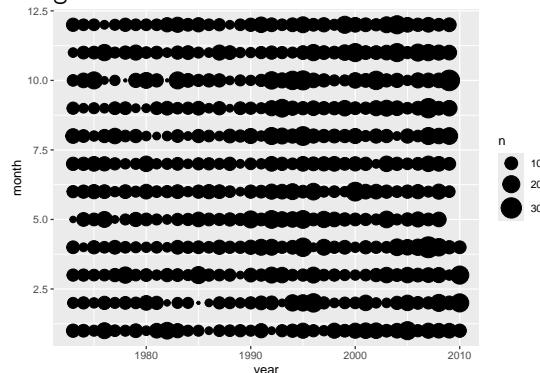
`ggplot2`

<https://bookdown.org/xiangyun/msg/system.html#section-15>

```
library(ggplot2)

data(quake6, package = "MSG")
ggplot(quake6, aes(x = year, y =
  month)) +
  stat_sum(aes(size = ..n..)) +
  scale_size(range = c(1, 8))
```

## Warning: package 'ggplot2' was built under R 3.6.3  
 ## Warning: The dot-dot notation (`..n..`) will be removed in ggplot2 3.4.0.  
 ## i Please use `after\_stat(n)` instead.  
 ## This warning is displayed once every 8 hours.  
 ## Call `lifecycle::last\_lifecycle\_warnings()`  
 ## generated.



count overlapping points [https://ggplot2.tidyverse.org/reference/geom\\_count.html?q=stat\\_sum#ref-usage](https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-usage)

This is a variant `geom_point()` that counts the number of observations at each location, then maps the count to point area. It useful when you have discrete data and overplotting.

`geom_count`

`stat_sum`

[https://ggplot2.tidyverse.org/reference/geom\\_count.html?q=stat\\_sum#ref-examples](https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-examples)

scales for area or radius [https://ggplot2.tidyverse.org/reference/scale\\_size.html?q=scale\\_size#null](https://ggplot2.tidyverse.org/reference/scale_size.html?q=scale_size#null)

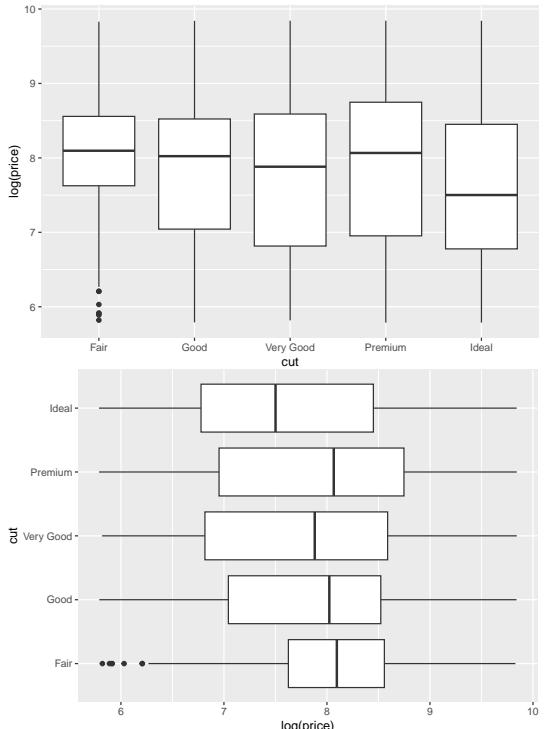
`scale_size`

## 30.4 coordinate system

<https://bookdown.org/xiangyun/msg/system.html#section-16>

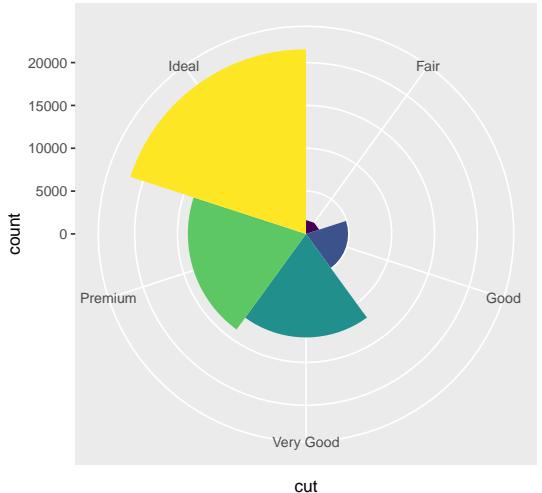
```
library(ggplot2)

p <- ggplot(aes(x = cut, y =
  log(price)), data = diamonds) +
  geom_boxplot()
p
p + coord_flip()
```



```
library(ggplot2)

ggplot(aes(x = cut, fill = cut), data =
  diamonds) +
  coord_polar() +
  geom_bar(width = 1, show.legend =
    FALSE)
```

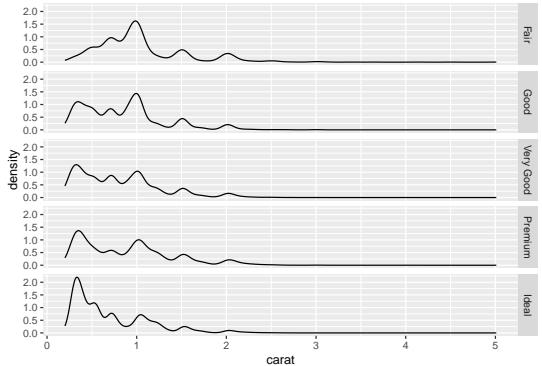


## 30.5 facet

<https://bookdown.org/xiangyun/msg/system.html#subsec:facet>

```
library(ggplot2)

ggplot(aes(x = carat), data =
  diamonds) +
  geom_density() +
  facet_grid(cut ~ .)
```

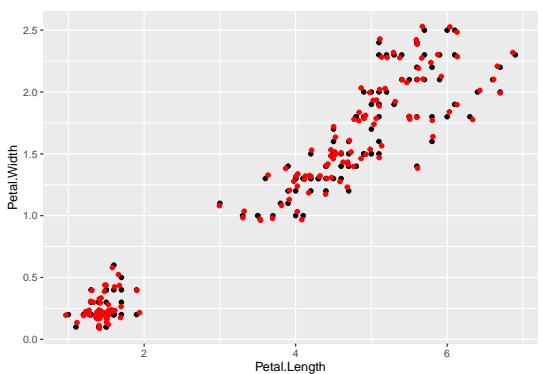


## 30.6 jitter

<https://bookdown.org/xiangyun/msg/system.html#section-17>

```
library(ggplot2)

ggplot(aes(x = Petal.Length, y =
  Petal.Width), data = iris) +
  geom_point() +
  geom_jitter(color = "red")
```



## 30.7 font

<https://bookdown.org/xiangyun/msg/system.html#subsec:font>



# Chapter 31

## notational system for design

<sup>9</sup> p.51

NSD = notational system for design

<sup>5</sup>

### 31.1 graphic notation

<sup>9</sup> p.51

- $X$ : treatment or exposure to an agent or an event of interest
- $P$ : **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$ : **observation** or process of measurement
- $R$ : **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $g$ : **groups**
  - $k$ : **kinds** of treatments, exposures, or placebos
  - $t$ : **time** or sequential order

<https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-markdown>

```
(ref:rudolph) *nice* cite: [@Lam94].  
(ref:campbell1963) *nice* cite: [@campbell1963].  
(ref:campbell1963) ([@campbell1963]  
(ref:campbell1963) \ [@campbell1963]
```

<sup>5</sup> 5

### 31.2 pre-experimental design

<sup>5</sup> p.6

	Sources of Invalidity											
	Internal				External							
	History	Maturat	Testing	Instrumentation	Regression	Selection	Mortality	Interaction of Selection and Maturat, etc.	Interaction of Testing and X	Interaction of Selection and X	Reactive Arrangements	Multiple-X Interference
<i>Pre-Experimental Designs:</i>												
1. One-Shot Case Study	-	-				-	-			-		
	X	O										
2. One-Group Pretest-Posttest Design	-	-	-	-	?	+	+	-	-	-	-	?
	O	X	O									
3. Static-Group Comparison	+	?	+	+	+	-	-	-		-		
	X	O										
<i>True Experimental Designs:</i>												
4. Pretest-Posttest Control Group Design	+	+	+	+	+	+	+	+	-	?	?	?
	R	O	X	O								
	R	O										
5. Solomon Four-Group Design	+	+	+	+	+	+	+	+	+	?	?	?
	R	O	X	O								
	R	O										
6. Posttest-Only Control Group Design	+	+	+	+	+	+	+	+	+	?	?	?
	R	X	O									
	R	O										

Figure 31.1: pre- and true experimental designs ( <sup>5</sup> p.8)

### 31.2.1 one-shot case study

$X \quad O$

### 31.2.2 one-group pretest-posttest design

$O \quad X \quad O$

paired  $t$  test

<sup>9</sup> p.62

$O \quad X \quad O$   
 $O_t \quad X \quad O_t$   
 $O_0 \quad X \quad O_1$

---

$O \quad X \quad O$   
 $O_{gt} \quad X_g \quad O_{gt}$   
 $O_{10} \quad X \quad O_{11}$

### 31.2.3 static-group comparison

$X \quad O$   
 $X_g \quad O_{gt}$   
 $X \quad O_{11}$   
 $O_{21}$

---

	Sources of Invalidity											
	Internal							External				
	History	Maturation	Testing	Instrumentation	Regression	Selection	Mortality	Interaction of Selection and Maturation, etc.	Interaction of Testing and X	Interaction of Selection and X	Reactive Arrangements	Multiple-X Interference
<i>Quasi-Experimental Designs:</i>												
7. Time Series $O \ O \ O \ O X \ O \ O \ O$	-	+	+	?	+	+	+	+	-	?	?	
8. Equivalent Time Samples Design $X_1O \ X_2O \ X_3O \ X_4O$ , etc.	+	+	+	+	+	+	+	+	-	?	-	-
9. Equivalent Materials Samples Design $M_1X_1O \ M_2X_2O \ M_3X_3O \ M_4X_4O$ , etc.	+	+	+	+	+	+	+	+	-	?	?	-
10. Nonequivalent Control Group Design $O \ X \ O$	+	+	+	+	?	+	+	-	-	?	?	
$\frac{O}{O}$												
11. Counterbalanced Designs $X_1O \ X_2O \ X_3O \ X_4O$ $X_2O \ X_3O \ X_4O \ X_1O$ $X_3O \ X_4O \ X_1O \ X_2O$ $X_4O \ X_1O \ X_2O \ X_3O$	+	+	+	+	+	+	+	?	?	?	-	
12. Separate-Sample Pretest-Posttest Design $R \ O \ (X)$ $R \ X \ O$	-	-	+	?	+	+	-	-	+	+	+	
12a. $R \ O \ (X)$ $R \ X \ O$ $\frac{R}{R} \ O \ (X)$ $R \ X \ O$	+	-	+	?	+	+	-	+	+	+	+	
12b. $R \ O_1 \ (X)$ $R \ O_2 \ (X)$ $R \ X \ O_1$	-	+	+	?	+	+	-	?	+	+	+	
12c. $R \ O_1 \ X \ O_2$ $R \ X \ O_2$	-	-	+	?	+	+	+	-	+	+	+	

Figure 31.2: quasi-experimental designs ( <sup>5</sup> p.40)

	Sources of Invalidity								
	Internal					External			
	History	Maturity	Testing	Instrumentation	Regression	Selection	Mortality	Selection and Maturation, etc.	
<i>Quasi-Experimental Designs</i>									
<i>Continued:</i>									
13. Separate-Sample Pretest-Posttest Control Group Design	+ + + + + + + -							+ + +	
$R \ O \ (X)$ $R \ X \ O$ $\hline R \ O$ $R \ O$									
13a. { R O (X) R X O (R') R O (X) R X O { R O (X) R X O { R O R O (R') R O { R O R O { R O R O R O R O	+ + + + + + + +							+ + +	
14. Multiple Time-Series	+ + + + + + + +							- - ?	
$O \ O \ O \ O \ O \ O$ $\hline O \ O \ O \ O \ O \ O$									
15. Institutional Cycle Design									
Class A X $\bar{O}_1$									
Class B <sub>1</sub> RO <sub>2</sub> X O <sub>3</sub>									
Class B <sub>2</sub> R X O <sub>4</sub>									
Class C O <sub>5</sub> X									
*Gen. Pop. Con. Cl. B O <sub>6</sub>									
*Gen. Pop. Con. Cl. C O <sub>7</sub>									
$O_2 < O_1 \}$	+ - + + ? - ?							+ ? +	
$O_5 < O_4 \}$	- - - ? ? + +							- ? +	
$O_2 < O_4$	- - + ? ? + ?							+ ? ?	
$O_5 = O_7 \}$	+							-	
16. Regression Discontinuity	+ + + ? + + ? +							+ - + +	

• General Population Controls for Class B, etc.

Figure 31.3: quasi-experimental designs continued ( <sup>5</sup> p.56)

$$\begin{array}{ccc}
 X & O \\
 X_g & O_{gt} \\
 X & O_{11} \\
 & O_{01}
 \end{array}$$

### 31.3 true experimental design

<sup>5</sup> p.13

#### 31.3.1 posttest-only control group design

basic experimental design

two-sample  $t$  test

<sup>9</sup> p.53

$$\begin{array}{ccc}
 R & X & O \\
 R & & O
 \end{array}$$

or, with a placebo or an active control,

$$\begin{array}{ccc}
 R & X & O \\
 R & P & O
 \end{array}$$


---

$$\begin{array}{ccc}
 R & X_g & O_{gt}
 \end{array}$$


---

$$\begin{array}{lll}
 R & X_g = X_1 = X & O_{gt} = O_{11} \\
 R & X_g = X_2 = \emptyset & O_{gt} = O_{21}
 \end{array}$$

or, with a placebo or an active control

$$\begin{array}{lll}
 R & X_g = X_1 = X & O_{gt} = O_{11} \\
 R & X_g = X_2 = P & O_{gt} = O_{21}
 \end{array}$$


---

$$\begin{array}{ccc}
 R & X & O_{11} \\
 R & & O_{21}
 \end{array}$$

or, with a placebo or an active control

$$\begin{array}{ccc}
 R & X & O_{11} \\
 R & P & O_{21}
 \end{array}$$


---

$$\begin{array}{ccc}
 R & X_g & O_{gt} \\
 R & X & O_{11} \\
 R & P & O_{21}
 \end{array}$$


---

$$\begin{array}{ccc}
 R & X & O \\
 R & X_g & O_{gt} \\
 R & X & O_{11} \\
 R & P & O_{21}
 \end{array}$$

### 31.3.2 pretest-posttest control group design

$R$	$O$	$X$	$O$
$R$	$O_{gt}$	$X_g$	$O_{gt}$
$R$	$O_{10}$	$X$	$O_{11}$
$R$	$O_{20}$		$O_{21}$

### 31.3.3 Solomon four-group design

<sup>9</sup> p.52

Solomon 4-group design = pretest-posttest + posttest-only control group design

$R$	$O_{gt}$	$X_g$	$O_{gt}$
$R$	$O_{10}$	$X$	$O_{11}$
$R$	$O_{20}$		$O_{21}$
$R$		$X$	$O_{31}$
$R$			$O_{41}$

---

$R$	$O$	$X$	$O$
$R$	$O_{gt}$	$X_g$	$O_{gt}$
$R$	$O_{10}$	$X$	$O_{11}$
$R$	$O_{20}$		$O_{21}$
$R$		$X$	$O_{31}$
$R$			$O_{41}$

## 31.4 quasi-experimental design

<sup>5</sup> p.34

## 31.5 correlational and ex post facto designs

<sup>5</sup> p.64

## 31.6 graphic notation, advanced

<sup>9</sup> p.74

- $X$ : treatment or exposure to an agent or an event of interest
- $P$ : **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$ : **observation** or process of measurement
- $R$ : **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $_g$ : **groups**
  - $_k$ : **kinds** of treatments, exposures, or placebos
  - $_t$ : **time** or sequential order
- $V$ : **variable(s)**
  - $B(V)$ : **blocking** by the variable(s)
  - $M(V)$ : **matching** by the variable(s)

- $S(V)$ : **stratifying** by the variable(s)
- $L(V/L)$ : **limiting** to the level(s) of the variable(s)
- $M^*$ : research **material(s)** selected
- –: cohort



# Chapter 32

## design of experiment

experimental design = experiment design = design of experiments = DoE

<sup>9</sup> p.72

question-design-analysis loop

### 32.1 notational system for design<sup>[31]</sup>

graphic notation, advanced<sup>[31.6]</sup>

### 32.2 terminology

- population
  - sample
    - \* subsample
- unit
  - experimental unit
    - \* response
    - \* block: group of similar experimental unit (<sup>10</sup> p.74)
  - observational unit / measurement unit <sup>1</sup>
- replication (<sup>9</sup> p.76): an independent observation of the treatment (<sup>10</sup> p.74)
  - treatment replication: experimental-unit-to-experimental-unit variation
  - measurement replication = subsample: measurement-to-measurement variation
- replicate
  - experimental replicate
  - biological replicate
  - technical replicate

---

<sup>10</sup> p.73

$Y_{ij}$ : the response observed from the  $j^{\text{th}}$  experimental unit assigned to the  $i^{\text{th}}$  treatment

$\mu_i$ : the mean response to the  $i^{\text{th}}$  treatment

$\varepsilon_{ij}$ : the noise from other possible natural variation or nonrandom and random error

---

<sup>1</sup><https://passel2.unl.edu/view/lesson/2e09f0055f13/6>

$$Y_{ij} = \mu_i + \varepsilon_{ij}, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ treatments} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ experimental units per treatment} \end{cases}$$

Each treatment has  $n_j$  experimental units, so there are totally  $n_i n_j$  experimental units.

If experimental units cannot be homogeneous, we can try to

- stratify them
- group them, and measure group to group variation
- block them

here  $n_j$  blocks each with  $n_i$  experimental units where **each treatment occurs once in each block**

$$\begin{aligned} Y_{ij} &= \mu_i + \varepsilon_{ij} \\ &= \mu_i + b_j + \varepsilon_{ij}^*, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ experimental units per block} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ blocks} \end{cases} \end{aligned}$$

where

$$\varepsilon_{ij} = b_j + \varepsilon_{ij}^*$$

i.e. the variation between groups or blocks of experimental units has been identified and isolated from  $\varepsilon_{ij}^*$ , which represents the variability of experimental units within a block. By isolating the block effect from the experimental units, the within-block variation can be used to compare treatment effects, which involves computing the estimated standard errors of contrasts of the treatments.

$$\begin{aligned} Y_{ij} - Y_{i'j} &= (\mu_i + b_j + \varepsilon_{ij}^*) \\ &\quad - (\mu_{i'} + b_j + \varepsilon_{i'j}^*) \\ &= (\mu_i - \mu_{i'}) + (\varepsilon_{ij}^* - \varepsilon_{i'j}^*) \end{aligned}$$

which does not depend on the block effect  $b_j$  or free of block effects. The result of this difference is that the variance of the difference of two treatment responses within a block depends on the within-block variation among the experimental units and not the between-block variation.

### 32.2.1 replication vs. subsample

It is very important to distinguish between a **subsample** and a **replication** since the error variance estimated from between subsamples is in general considerably smaller than the error variance estimated from replications or between experimental units. (<sup>10</sup> p.77)

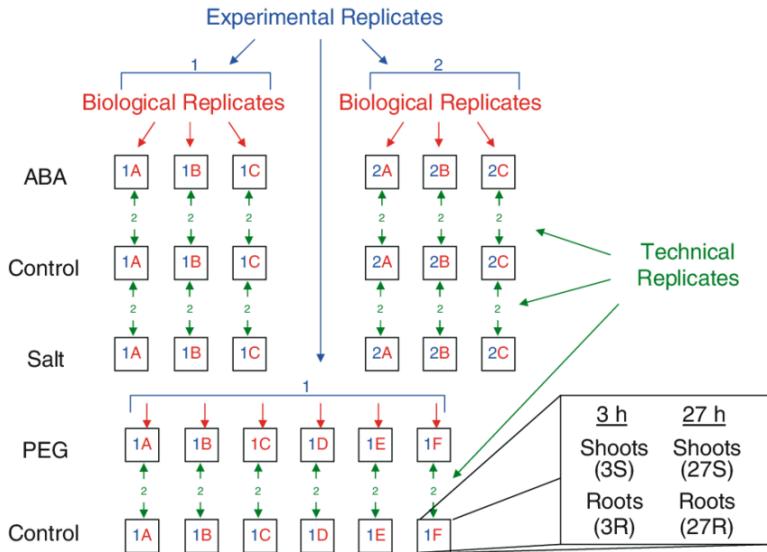
<https://www.researchgate.net/post/What-is-Experimental-Unit-Replicate-Total-sample-size-treatment-size>

### 32.2.2 replication vs. repeated measurements

#### 32.2.3 replication, replicate

##### 32.2.3.1 technical replicate, biological replicate

[https://www.youtube.com/watch?v=c\\_cpl5YsBV8](https://www.youtube.com/watch?v=c_cpl5YsBV8)

Figure 32.1: experimental, biological, technical replicates ( <sup>11</sup> )

### 32.2.4 Latin square design

LSD = Latin square design

<sup>6</sup> p.505~507

<https://tex.stackexchange.com/questions/501671/how-to-get-math-mode-curly-braces-in-tikz>

```
\usepackage{pgfplots} in engine.opts=list(extra.preamble={"\usepackage{pgfplots}"}))  
\usetikzlibrary{decorations}
```

$p = 4$ columns			
$p = 4$ rows	A	B	C
	B	C	D
	C	D	A
	D	A	B

$p = |\{A, B, C, D\}| = 4$  treatments

Figure 32.2: Latin square example

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \mathcal{E}_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases}$$

$$\mathcal{E}_{ijk} \stackrel{\text{i.i.d.}}{\sim} n(0, \sigma^2)$$

$\rho_i$ :  $i^{\text{th}}$  row

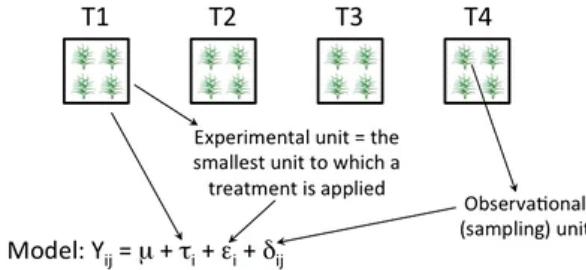
$\kappa_j$ :  $j^{\text{th}}$  column

$\tau_k$ :  $k^{\text{th}}$  treatment

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases}$$

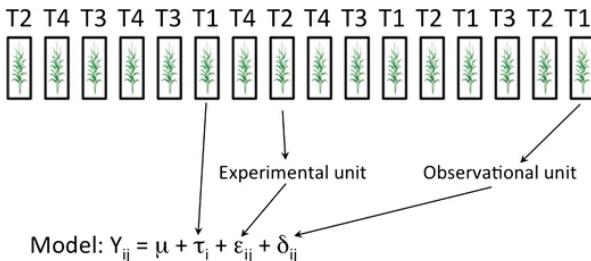
$$= \mu + \rho_i + \kappa_j + \tau_k + \varepsilon_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \end{cases}$$

### 32.2.5 model assumption and experimental unit, measurement/observational unit



ANOVA Source of variation	df
Treatments + Experimental error	3 (fixed)
Observational error	12

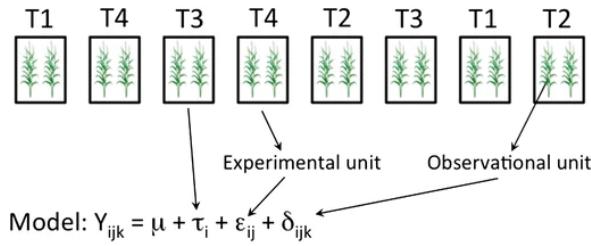
Figure 32.3: model assumption and experimental unit 1 ( [12](#) fig.1)



ANOVA Source of variation	df
Treatments	3 (fixed)
Error (experimental + observational)	12

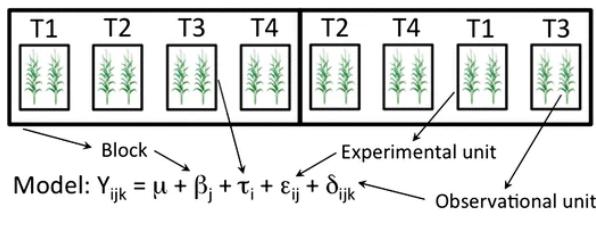
Figure 32.4: model assumption and experimental unit 2 ( [12](#) fig.2)

$$Y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ij} + \Delta_{ijk}$$



ANOVA Source of variation	df
Treatments	3 (fixed)
Experimental error	4
Observational error	8

Figure 32.5: model assumption and experimental unit 3 ( <sup>12</sup> fig.3)



ANOVA Source of variation	df
Blocks	1
Treatments	3 (fixed)
Experimental error	3
Observational error	8

Figure 32.6: model assumption and experimental unit 4 ( <sup>12</sup> fig.4)

## 32.3 experiment structure

### 32.3.1 treatment structure

<sup>10</sup> p.77

- 1-way treatment structure
- 2-way treatment structure
- factorial arrangement treatment structure
- *fractional* factorial arrangement treatment structure
- factorial arrangement with one or more controls

### 32.3.2 design structure

<sup>10</sup> p.77

- CRD = completely randomized design
- RCBD = randomized complete block design
  - ? why not called CRBD = completely randomized block design
- LSD = Latin square design<sup>[32.2.4]</sup>
- IBD = incomplete block design
  - BIBD = balanced IBD
- various combinations and generalizations

### 32.3.3 size of experimental unit

- split-plot design
  - split-split-plot design
  - split-split-split-plot design
- repeated measures design
  - cross-over design
  - change-over design
- nested design = hierarchical design
- variations and combinations
  - SSEU = several sizes of experimental units

## 32.4 approach to experimentation

<sup>9</sup> p.75

- approach to experimentation
  - best-guess approach
  - one-factor-at-a-time approach = OFAT
  - factorial approach

## 32.5 sample size estimation

## 32.6 statistical analysis plan

## 32.7 protocol

<sup>9</sup> p.95

- study objective
- study endpoint
  - primary endpoint
  - secondary endpoint(s)
- experimental unit(s)
- treatment structure<sup>[32.3.1]</sup>
- design structure<sup>[32.3.2]</sup>
- potential confounder(s)
- randomization
- blinding
- chance reduction
- sample size estimation<sup>[32.5]</sup>
- data collection
- data management system
- statistical analysis plan<sup>[32.6]</sup>
- DSMB / DSMC = data and safety monitoring board / committee

## 32.8 DoE course with six sigma and Minitab

<https://zhuanlan.zhihu.com/p/265914617>

<https://www.zhihu.com/question/416312693/answer/1426399810>

### 32.8.1 evolution

- Fisher
- Rao



# Chapter 33

## quine

```
s = 's = %r\nprint(s%%s)'  
print(s%s)
```

```
## s = 's = %r\nprint(s%%s)'  
## print(s%s)
```

This snippet is a clever example of a quine. A quine is a computer program that takes no input and produces a copy of its own source code as its output. The given code in Python is written to print its own source when executed. Let's break it down:

`s = 's = %r\nprint(s%%s)':` This line defines a string `s` that contains a format string. `%r` is a placeholder that gets replaced with the `repr()` of the argument provided to the `%` operator, which in this case will be the string `s` itself. This means it will insert the string representation of `s` into the format string at `%r`.

`print(s%s):` This line prints the result of `s%s`. Here, the `%` operator is used to format the string `s` with itself. The `%s` inside the print statement is replaced by the string `s`, leading to the entire string being printed out, including the print statement itself.

This is because the format operation replaces `%r` with the representation of the string `s`, and `%%` is a way to escape the `%` sign in format strings, resulting in a single `%` in the output. This output is exactly the same as the source code, making it a quine.

### 33.1 `%r`

The `%r` in Python string formatting represents the “representation” of a value, which is typically the way you would see it if you were to type it into a Python interpreter. It uses the `repr()` function to convert the value to a string. This is useful for debugging, among other things, because it shows strings with quotes around them and escapes special characters. Essentially, `%r` gives you the “developer’s view” of what a variable looks like.

Here's a simple example to illustrate `%r` versus `%s` in string formatting:

```
my_str = "Hello, World!\nNew line character is represented with \\n"  
print("Using %%s: %s" % my_str)
```

```
## Using %s: Hello, World!  
## New line character is represented with \n
```

```
print("Using %%r: %r" % my_str)
## Using %r: 'Hello, World!\nNew line character is represented with \\n'
```

In this example:

The `%s` specifier tells Python to convert the object using `str()`, which is designed to be readable and outputs the string "Hello, World!\nNew line character is represented with \n", interpreting the escape character `\n` as a newline.

The `%r` specifier tells Python to convert the object using `repr()`, which aims to generate output that could be used to recreate the object, outputting the string 'Hello, World!\nNew line character is represented with \\n', preserving the actual escape characters in the output.

Notice how `%r` preserves the string exactly as it is, including the quotes and escaped characters, making it clear it's a string and showing the escape sequence explicitly.

# Chapter 34

## quaternion

### 34.1 TaylorCatAlice

[https://en.wikipedia.org/wiki/Blackboard\\_bold](https://en.wikipedia.org/wiki/Blackboard_bold)

#### 34.1.1 complex / dionion / bionion

$$\begin{aligned} c &= a + bi = a + ib, \begin{cases} c \in \mathbb{C} \\ i^2 = -1 \\ a, b \in \mathbb{R} \Leftrightarrow \langle a, b \rangle \in \mathbb{R}^2 \end{cases} \\ &= z = x + yi = x + iy, \begin{cases} z \in \mathbb{C} \\ i^2 = -1 \\ x, y \in \mathbb{R} \Leftrightarrow \langle x, y \rangle \in \mathbb{R}^2 \end{cases} \\ &= \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}} i \right) = r (\cos \theta + i \sin \theta) = re^{i\theta} \end{aligned}$$

Also see complex group representation<sup>[37.4]</sup>.

#### 34.1.2 trionion / triernion / triplex / ternion

<https://zh.wikipedia.org/zh-tw/%E4%B8%89%E5%85%83%E6%95%B8>

<https://math.stackexchange.com/questions/1784166/why-are-there-no-triernions-3-dimensional-analogue-of-complex-numbers-quote>

<https://math.stackexchange.com/questions/32100/is-there-a-third-dimension-of-numbers/4453131>

$$\begin{aligned}
t &= a + bi + cj = a + ib + jc, \quad \begin{cases} t \in \mathbb{T} \\ i^2 = -1 \\ j^2 = -1 \end{cases} \\
&= w = x + yi + zj = x + iy + jz, \quad \begin{cases} w \in \mathbb{T} \\ i^2 = -1 \\ j^2 = -1 \end{cases} \\
&= \sqrt{x^2 + y^2 + z^2} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}i + \frac{z}{\sqrt{x^2 + y^2 + z^2}}j \right) = ?
\end{aligned}$$


---

$$\begin{cases} A(BC) = (AB)C & (a) \text{ associativity} \\ A(B+C) = AB + AC & (d) \text{ distributivity} \end{cases}$$

$$\begin{aligned}
&\mathbb{T} \ni ij = X + Yi + Zj \in \mathbb{T} \\
-ij &= (i^2)j \stackrel{(a)}{=} i(ij) = i(X + Yi + Zj) \stackrel{(d)}{=} -Y + Xi + Zij \\
ij &= \frac{Y}{Z} - \frac{X}{Z}i - \frac{1}{Z}j \Rightarrow \begin{cases} X = \frac{Y}{Z} \\ Y = -\frac{X}{Z} \\ Z = -\frac{1}{Z} \end{cases} \Rightarrow Z^2 = -1 \Rightarrow Z \notin \mathbb{R} \\
-ij &= i(j^2) \stackrel{(a)}{=} (ij)j = (X + Yi + Zj)j \stackrel{(d)}{=} -Z + Xj + Yij \\
ij &= \frac{Z}{Y} - \frac{1}{Y}i - \frac{X}{Y}j \Rightarrow \begin{cases} X = \frac{Z}{Y} \\ Y = -\frac{1}{Y} \\ Z = -\frac{X}{Y} \end{cases} \Rightarrow Y^2 = -1 \Rightarrow Y \notin \mathbb{R}
\end{aligned}$$

### 34.1.3 quaternion

<https://en.wikipedia.org/wiki/Quaternion>

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \quad \begin{cases} q \in \mathbb{H} \\ a, b, c, d \in \mathbb{R} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \end{cases} \\
&= w = t + xi + yj + zk = t + ix + jy + kz, \quad \begin{cases} w \in \mathbb{H} \\ t, x, y, z \in \mathbb{R} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \end{cases} \\
&= ?
\end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\
w &= t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\
&= t + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (e_1 \ e_2 \ e_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + x, \begin{cases} e_1 = i = i \\ e_2 = j = j \\ e_3 = k = k \end{cases} \\
&= t + \frac{ix + jy + kz}{r} r, \begin{cases} r^2 = x^2 + y^2 + z^2 \\ \|n\|^2 = \left(\frac{ix + jy + kz}{r}\right)^2 = -1 \end{cases} |q|^2 = t^2 + r^2 \\
&= \sqrt{t^2 + r^2} \left( \frac{t}{\sqrt{t^2 + r^2}} + n \frac{r}{\sqrt{t^2 + r^2}} \right) = |q| \left( \cos \frac{\theta}{2} + n \sin \frac{\theta}{2} \right) = |q| e^{n \frac{\theta}{2}}
\end{aligned}$$


---

The quaternion set is denoted  $\mathbb{H}$  for Sir R.W. **Hamilton**, because he suddenly and strikingly realized

$$\begin{cases} ij = k \\ k \in \mathbb{H} \end{cases} \Rightarrow ij \in \mathbb{H} \text{ for closure property}$$


---

for the sake of rigorosity, see [group theory](#)<sup>[37]</sup>

$$k^2 = -1$$

$$\begin{aligned}
ij &= k \\
ijk &= i(jk) \stackrel{(a)}{=} (ij)k = kk = k^2 = -1 \\
kij &= (ki)j \stackrel{(a)}{=} k(ij) = kk = k^2 = -1
\end{aligned}$$

$$\begin{aligned}
ij &= k \\
-j &= (i^2)j \stackrel{(a)}{=} i(ij) = ik \\
-i &= i(j^2) \stackrel{(a)}{=} (ij)j = kj
\end{aligned}$$

$$\begin{aligned}
-j &= (i^2) j \stackrel{(a)}{=} i(ij) = ik \\
1 &= -j^2 = j(-j) = j(ik) \stackrel{(a)}{=} (ji)k \\
k &= [1]k = [(ji)k]k \stackrel{(a)}{=} (ji)(k^2) = (ji)(-1) \\
&\quad -k = ji \\
-i &= i(j^2) \stackrel{(a)}{=} (ij)j = kj \\
1 &= (-i)i = (kj)i \stackrel{(a)}{=} k(ji) = kji \\
1 &= kji
\end{aligned}$$

There is no more **commutativity**<sup>[34.1.3.2.1]</sup>, i.e.

$$AB \not\equiv BA$$

but

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

satisfying **anticommutativity**<sup>[34.1.3.2.2]</sup>.

$$\begin{cases} ij = k \\ ji = -k \end{cases} \Leftrightarrow ji = -k = -ij \\
\Leftrightarrow ji = -ij \\
\Leftrightarrow ij + ji = 0$$

$$\begin{cases} kij = -1 \\ kji = 1 \end{cases} \Leftrightarrow kij = -1 = -kji \\
\Leftrightarrow kij = -kji \\
\Leftrightarrow kij + kji = 0$$


---

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\
w &= t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\
&= t + (i \quad j \quad k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (i \quad j \quad k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (e_1 \quad e_2 \quad e_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + x, \begin{cases} e_1 = i = i \\ e_2 = j = j \\ e_3 = k = k \end{cases}
\end{aligned}$$

.	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

.	1	i	j	k	-1	-i	-j	-k
1	1	i	j	k				
i	i	-1	k	-j				
j	j	-k	-1	i				
k	k	j	-i	-1				
-1								
-i								
-j								
-k								

Figure 34.1: quaternion basis group table

.	1	-1	i	-i	j	-j	k	-k
1								
-1								
i								
-i								
j								
-j								
k								
-k								

Figure 34.2: quaternion basis group table 2

### 34.1.3.1 true origin of ( dot product & cross product ) / ( inner product & outer product )

product of two pure imaginary quaternions

$$\begin{aligned}
\mathbf{x}_1 \mathbf{x}_2 &= (x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k})(x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k}) \\
&= x_{11}x_{21}\mathbf{i}^2 + x_{11}x_{22}\mathbf{ij} + x_{11}x_{23}\mathbf{ik} \\
&\quad + x_{12}x_{21}\mathbf{ji} + x_{12}x_{22}\mathbf{j}^2 + x_{12}x_{23}\mathbf{jk} \\
&\quad + x_{13}x_{21}\mathbf{ki} + x_{13}x_{22}\mathbf{kj} + x_{13}x_{23}\mathbf{k}^2 \\
&= -(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}) \\
&\quad + (x_{12}x_{23} - x_{13}x_{22})\mathbf{jk} + (x_{13}x_{21} - x_{11}x_{23})\mathbf{ki} + (x_{11}x_{22} - x_{12}x_{21})\mathbf{ij} \\
&= -(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}) \\
&\quad + (x_{12}x_{23} - x_{13}x_{22})\mathbf{i} + (x_{13}x_{21} - x_{11}x_{23})\mathbf{j} + (x_{11}x_{22} - x_{12}x_{21})\mathbf{k} \\
&= -\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{x}_2 \\
&= -(\mathbf{x}_1 \cdot \mathbf{x}_2) + (\mathbf{x}_1 \times \mathbf{x}_2), \quad \begin{cases} \mathbf{x}_1 \cdot \mathbf{x}_2 = x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} \\ \mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} \end{cases} \\
&= -\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{x}_2
\end{aligned}$$

product of two general quaternions / ordinary quaternions = Grassmann product

$$\begin{aligned}
q_1 q_2 &= (q_{10} + q_{11}\mathbf{i} + q_{12}\mathbf{j} + q_{13}\mathbf{k})(q_{20} + q_{21}\mathbf{i} + q_{22}\mathbf{j} + q_{23}\mathbf{k}) \\
&= (x_{10} + \mathbf{x}_1)(x_{20} + \mathbf{x}_2), \quad \begin{cases} x_{i\mu} = q_{i\mu} & \mu \in \{0\} \cup (\mathbb{N} \cap [1, 3]) \\ \mathbf{x}_i = x_{ij}\mathbf{e}_j & i, j \in \mathbb{N} \cap [1, 3], \begin{cases} \mathbf{e}_1 = \mathbf{i} \\ \mathbf{e}_2 = \mathbf{j} \\ \mathbf{e}_3 = \mathbf{k} \end{cases} \end{cases} \\
&= x_{10}x_{20} + x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1 \mathbf{x}_2 \\
&= x_{10}x_{20} + x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 - \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{x}_2 \\
&= (x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2) \\
x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2 &= (q_{10} \quad q_{11} \quad q_{12} \quad q_{13}) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = q_1^\mu \eta_{\mu\nu} q_2^\nu \\
&= (q_{10} \quad q_{11} \quad q_{12} \quad q_{13}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = \mathbf{q}_1^\top H \mathbf{q}_2, H = [\eta_{\mu\nu}]_{4 \times 4} = \eta_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
q_1 q_2 &= (x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2) \\
QP &= (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})
\end{aligned}$$

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

Minkowski metric tensor

$$\eta = H = [\eta_{\mu\nu}]_{4 \times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta_{\mu\nu}$$

and quaternions as 4-vectors or four-vectors

$$\mathbf{q}_1^T = (q_{10} \quad q_{11} \quad q_{12} \quad q_{13})$$

$$\mathbf{q}_2 = \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix}$$

### 34.1.3.2 commutativity vs. anticommutativity

#### 34.1.3.2.1 commutativity 交換律 = 交換性 = 對易性

$$AB = BA \Leftrightarrow AB - BA = 0$$

$$AB = BA \Rightarrow AB \equiv BA$$

#### 34.1.3.2.2 anticommutativity 反交換律 = 反交換性 = 反對易性

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

$$AB = -BA \Rightarrow AB \neq BA$$

### 34.1.3.3 bracket

#### 34.1.3.3.1 self-invented bracket 自創括號 = 自創括

##### 34.1.3.3.1.1 commutative bracket 交換括號 = 對易式

$$[X, Y] = \frac{XY - YX}{2}$$

##### 34.1.3.3.1.2 anticommutative bracket 反交換括號 = 反對易式

$$\{X, Y\} = \frac{XY + YX}{2}$$

#### 34.1.3.3.2 Poisson bracket [https://en.wikipedia.org/wiki/Poisson\\_bracket](https://en.wikipedia.org/wiki/Poisson_bracket)

#### 34.1.3.3.3 Lagrange bracket

#### 34.1.3.3.4 Lie bracket

### 34.1.3.4 triple product

product = double product = Grassmann product

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

pure imaginary

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

$$\begin{cases} a_0 = 0 \\ b_0 = 0 \\ = (00 - \mathbf{a} \cdot \mathbf{b}) + (0\mathbf{b} + 0\mathbf{a} + \mathbf{a} \times \mathbf{b}) \\ = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b} \end{cases}$$

pure imaginary product can get both ( real & imaginary ) / ( scalar & vector ) parts

$$ab = \mathbf{a}\mathbf{b} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}, \begin{cases} a = 0 + \mathbf{a} = \mathbf{a} \\ b = 0 + \mathbf{b} = \mathbf{b} \end{cases}$$


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triple product

[https://en.wikipedia.org/wiki/Triple\\_product](https://en.wikipedia.org/wiki/Triple_product)

pure imaginary

$$\begin{cases} a = 0 + \mathbf{a} = \mathbf{a} \\ b = 0 + \mathbf{b} = \mathbf{b} \\ c = 0 + \mathbf{c} = \mathbf{c} \end{cases}$$

$$\begin{aligned} abc &= \mathbf{a}\mathbf{b}\mathbf{c} = (\mathbf{a}\mathbf{b})\mathbf{c} = (-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= \mathbf{a}(\mathbf{b}\mathbf{c}) = \mathbf{a}(-\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \times \mathbf{c}) \end{aligned}$$

$$\begin{aligned} abc &= (\mathbf{a}\mathbf{b})\mathbf{c} = (-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (-(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \\ &= [-(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}] + [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \\ &= \mathbf{a}(\mathbf{b}\mathbf{c}) = \mathbf{a}(-\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \times \mathbf{c}) \\ &= -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{b} \times \mathbf{c}) \\ &= -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + (-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times (\mathbf{b} \times \mathbf{c})) \\ &= [-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] + [\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})] \end{aligned}$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

$$\begin{aligned} &\begin{cases} -(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \end{cases} \\ \Rightarrow &\begin{cases} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) & (s) \\ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) & (v) \end{cases} \end{aligned}$$

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permutation

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots \\ \sigma(x_1) & \sigma(x_2) & \cdots \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$


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$$(s) \Rightarrow \left\{ \begin{array}{ll} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) & \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, s_1 \\ (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) & \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, s_2 \\ (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) & \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, s_3 \\ (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) & \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, s_4 \\ (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) & \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, s_5 \\ (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) & \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, s_6 \end{array} \right.$$

$$\stackrel{\text{commutative}}{\Rightarrow} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \stackrel{s_1}{=} (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} \stackrel{s_2}{=} (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \stackrel{s_3}{=} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$\stackrel{\text{anticommutative}}{=} -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} \stackrel{s_6}{=} -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} \stackrel{s_5}{=} -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \stackrel{s_4}{=} -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$$

$$\Leftrightarrow \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$= -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$


---

$$(v) \Rightarrow \left\{ \begin{array}{l} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \\ (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} - (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\ (\mathbf{b} \times \mathbf{a}) \times \mathbf{c} - (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} = \mathbf{b} \times (\mathbf{a} \times \mathbf{c}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) \\ (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b}) - \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) \\ (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} - (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} = \mathbf{c} \times (\mathbf{b} \times \mathbf{a}) - \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) \end{array} \right. \quad \left\{ \begin{array}{l} \begin{pmatrix} a & b & c \\ a & b & c \\ a & b & c \\ b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}, v_1 \\ \begin{pmatrix} a & b & c \\ a & b & c \\ a & b & c \\ b & c & a \\ a & b & c \\ b & a & c \end{pmatrix}, v_2 \\ \begin{pmatrix} a & b & c \\ a & b & c \\ a & b & c \\ c & a & b \\ a & b & c \\ a & c & b \end{pmatrix}, v_3 \\ \begin{pmatrix} a & b & c \\ a & b & c \\ a & b & c \\ b & a & c \\ a & b & c \\ b & a & c \end{pmatrix}, v_4 \\ \begin{pmatrix} a & b & c \\ a & b & c \\ a & c & b \\ a & b & c \\ a & c & b \\ a & b & c \end{pmatrix}, v_5 \\ \begin{pmatrix} a & b & c \\ a & b & c \\ c & b & a \\ c & b & a \end{pmatrix}, v_6 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} -Z - C = X - A \quad v_1 \\ -X - A = Y - B \quad v_2 \\ -Y - B = Z - C \quad v_3 \\ Z - C = -Y - B \quad v_4 \\ Y - B = -X - A \quad v_5 \\ X - A = -Z - C \quad v_6 \end{array} \right., \left\{ \begin{array}{ll} \cdot \text{ and scalar-vector product} & \text{commutative} \\ \times & \text{anticommutative} \end{array} \right.,$$

$$\left\{ \begin{array}{l} X = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} \\ Y = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = -(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} \\ Z = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{b} \times \mathbf{a}) = (\mathbf{b} \times \mathbf{a}) \times \mathbf{c} \\ A = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} = (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} \\ B = \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} \\ C = \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} -Z - C = X - A \quad v_1 = v_6 \\ -X - A = Y - B \quad v_2 = v_5 \\ -Y - B = Z - C \quad v_3 = v_4 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} Z + X = A - C \\ X + Y = B - A \\ Y + Z = C - B \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} X + Y = B - A \\ Y + Z = C - B \\ Z + X = A - C \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} 2(X + Y + Z) = 0 \Rightarrow X + Y + Z = 0 \\ Y + Z = C - B \Rightarrow X = B - C \Leftrightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \text{ "back cab"} \\ Z + X = A - C \Rightarrow Y = C - A \Leftrightarrow \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \\ X + Y = B - A \Rightarrow Z = A - B \Leftrightarrow \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \end{array} \right.$$

### 34.1.3.5 differential operator

[https://en.wikipedia.org/wiki/Differential\\_operator](https://en.wikipedia.org/wiki/Differential_operator)

**34.1.3.5.1 4-differential operator** 4-differential operator / four-differential operator = d'Alembert operator

$$\begin{aligned}
D &= \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} = \partial_t + i \partial_x + j \partial_y + k \partial_z \\
&= \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} = \partial_t + i \partial_x + j \partial_y + k \partial_z \\
&= \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = \partial_0 + e_i \partial_i = \partial_0 + \nabla
\end{aligned}$$

$$D = \partial_0 + i \partial_1 + j \partial_2 + k \partial_3 = \partial_0 + \nabla = D_0 + \mathbf{D}$$

**34.1.3.5.2 nabla** nabla = spatial differential operator = 3-differential operator / three-differential operator

$$\nabla = e_i \partial_i = \sum_{i=1}^3 e_i \partial_i = \sum_{i=1}^3 e_i \frac{\partial}{\partial x_i} = \left( \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)^\top = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

**34.1.3.5.3 Laplace operator** Laplace operator = Laplacian

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

**34.1.3.5.4 d'Alembert operator**

$$\square = \square_c = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\square_1 = \square_{c=1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial t^2} - \nabla^2$$

**34.1.3.6 electromagnetism**

Maxwell

**34.1.3.6.1 4-potential** electromagnetic 4-potential / four-potential

$$A = A_0 + i A_1 + j A_2 + k A_3 = A_0 + \mathbf{A}$$

4-differential operator<sup>[34.1.3.5.1]</sup>

$$D = \partial_0 + i \partial_1 + j \partial_2 + k \partial_3 = \partial_0 + \nabla = D_0 + \mathbf{D}$$

$$QP = (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})$$

commutative bracket<sup>[34.1.3.3.1.1]</sup>

$$\begin{aligned}
[\mathbf{D}, \mathbf{A}] &= \frac{\mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D}}{2} \\
2[\mathbf{D}, \mathbf{A}] &= \mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D} \\
&= (\partial_0 + i\partial_1 + j\partial_2 + k\partial_3)(A_0 + iA_1 + jA_2 + kA_3) \\
&\quad - (A_0 + iA_1 + jA_2 + kA_3)(\partial_0 + i\partial_1 + j\partial_2 + k\partial_3) \\
\mathbf{D}\mathbf{A} &= (\mathbf{D}_0\mathbf{A}_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0\mathbf{A} + \mathbf{A}_0\mathbf{D} + \mathbf{D} \times \mathbf{A}) \\
&= (\mathbf{D}_0\mathbf{A}_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0\mathbf{A} + \mathbf{D}\mathbf{A}_0 + \mathbf{D} \times \mathbf{A}) \\
&= \mathbf{D}_0(\mathbf{A}_0 + \mathbf{A}) - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 + \mathbf{D} \times \mathbf{A} \\
&= \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 + \mathbf{D} \times \mathbf{A} \\
&= \partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0 + \nabla \times \mathbf{A} \\
\mathbf{A}\mathbf{D} &= (A_0\mathbf{D}_0 - \mathbf{A} \cdot \mathbf{D}) + (A_0\mathbf{D} + \mathbf{D}_0\mathbf{A} + \mathbf{A} \times \mathbf{D}) \\
&= (\mathbf{D}_0\mathbf{A}_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}\mathbf{A}_0 + \mathbf{D}_0\mathbf{A} - \mathbf{D} \times \mathbf{A}) \\
&= (\mathbf{D}_0\mathbf{A}_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0\mathbf{A} + \mathbf{D}\mathbf{A}_0 - \mathbf{D} \times \mathbf{A}) \\
&= \mathbf{D}_0(\mathbf{A}_0 + \mathbf{A}) - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 - \mathbf{D} \times \mathbf{A} \\
&= \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 - \mathbf{D} \times \mathbf{A} \\
&= \partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0 - \nabla \times \mathbf{A}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\mathbf{A} &= \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 + \mathbf{D} \times \mathbf{A} = \partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0 + \nabla \times \mathbf{A} \\
\mathbf{A}\mathbf{D} &= \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 - \mathbf{D} \times \mathbf{A} = \partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0 - \nabla \times \mathbf{A}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D} &= 2\mathbf{D} \times \mathbf{A} = 2\nabla \times \mathbf{A} \\
[\mathbf{D}, \mathbf{A}] &= \frac{\mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D}}{2} = \mathbf{D} \times \mathbf{A} = \nabla \times \mathbf{A}
\end{aligned}$$

anticommutative bracket<sup>[34.1.3.3.1.2]</sup>

$$\begin{aligned}
\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D} &= 2(\mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0) = 2(\partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0) \\
\{\mathbf{D}, \mathbf{A}\} &= \frac{\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}}{2} = \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 = \partial_0\mathbf{A} - \nabla \cdot \mathbf{A} + \nabla\mathbf{A}_0
\end{aligned}$$

commutation and anticommutation on differential operator and any quaternion

$$\begin{aligned}
[\mathbf{D}, \mathbf{Q}] &= \nabla \times \mathbf{Q} \\
\{\mathbf{D}, \mathbf{Q}\} &= \partial_0\mathbf{Q} - \nabla \cdot \mathbf{Q} + \nabla Q_0
\end{aligned}$$

or more evident

$$\begin{aligned}
[\mathbf{D}, \mathbf{Q}] &= \nabla \times \mathbf{Q} \\
\{\mathbf{D}, \mathbf{Q}\} &= \partial_0\mathbf{Q} - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\
&= \partial_0(Q_0 + \mathbf{Q}) - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\
&= (\partial_0 Q_0 - \nabla \cdot \mathbf{Q}) + (\partial_0 \mathbf{Q} + \nabla Q_0) \\
&= \left( \frac{\partial Q_0}{\partial t} - \nabla \cdot \mathbf{Q} \right) + \left( \frac{\partial \mathbf{Q}}{\partial t} + \nabla Q_0 \right)
\end{aligned}$$

### 34.1.3.6.2 Maxwell compromise for both quaternion and 3-vector electric potential and vector potential

$$A = A_0 + iA_1 + jA_2 + kA_3 = A_0 + \mathbf{A} = U + \mathbf{A}$$

electric quaternion and electric field

$$\begin{aligned} E &= -\{\mathbf{D}, A\} \\ &= -(\partial_0 A - \nabla \cdot \mathbf{A} + \nabla A_0) \\ &= -\partial_0 A + \nabla \cdot \mathbf{A} - \nabla A_0 \\ &= -\partial_t (U + \mathbf{A}) + \nabla \cdot \mathbf{A} - \nabla U \\ &= -\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{A} - \nabla U - \frac{\partial \mathbf{A}}{\partial t} \\ &= E_0 + \mathbf{E}, \quad \begin{cases} E_0 = -\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{A} \\ \mathbf{E} = -\nabla U - \frac{\partial \mathbf{A}}{\partial t} \end{cases} \quad \text{electric field 3-vector} \end{aligned}$$

magnetic field

$$B = [\mathbf{D}, A] = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B}$$

Work on time? Yes.

$$qE = qE_0 + q\mathbf{E} = qE_0 + \mathbf{F}_E$$

force equivalent on time

$$qE_0$$


---

$$\begin{cases} E = -\{\mathbf{D}, A\} = E_0 + \mathbf{E} \\ B = +[\mathbf{D}, A] = B_0 + \mathbf{B} = 0 + \mathbf{B} = \mathbf{B} \quad B_0 = 0 \end{cases}$$


---

for any quaternion commuting and anticommutating with differential operator

$$\begin{aligned} [\mathbf{D}, Q] &= \nabla \times \mathbf{Q} \\ \{\mathbf{D}, Q\} &= \partial_0 Q - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\ &= \partial_0 (Q_0 + \mathbf{Q}) - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\ &= (\partial_0 Q_0 - \nabla \cdot \mathbf{Q}) + (\partial_0 \mathbf{Q} + \nabla Q_0) \\ &= \left( \frac{\partial Q_0}{\partial t} - \nabla \cdot \mathbf{Q} \right) + \left( \frac{\partial \mathbf{Q}}{\partial t} + \nabla Q_0 \right) \end{aligned}$$


---

$$\begin{cases} [\mathbf{D}, \mathbf{E}] = \nabla \times \mathbf{E} = (0) + (\nabla \times \mathbf{E}) \\ \{\mathbf{D}, \mathbf{E}\} = (\partial_0 E_0 - \nabla \cdot \mathbf{E}) + (\partial_0 \mathbf{E} + \nabla E_0) = \left( \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ [\mathbf{D}, \mathbf{B}] = \nabla \times \mathbf{B} = (0) + (\nabla \times \mathbf{B}) \\ \{\mathbf{D}, \mathbf{B}\} = (\partial_0 B_0 - \nabla \cdot \mathbf{B}) + (\partial_0 \mathbf{B} + \nabla B_0) \stackrel{B_0=0}{=} -\nabla \cdot \mathbf{B} + \partial_0 \mathbf{B} = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases}$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

Maxwell equations without source terms

$$\begin{aligned} \begin{cases} [\mathbf{D}, \mathbf{B}] = +\{\mathbf{D}, \mathbf{E}\} \Leftrightarrow (0) + (\nabla \times \mathbf{B}) = \left( \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ \{\mathbf{D}, \mathbf{E}\} = -\{\mathbf{D}, \mathbf{B}\} \Leftrightarrow (0) + (\nabla \times \mathbf{E}) = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases} \\ \Leftrightarrow \begin{cases} \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} = 0 \Leftrightarrow \nabla \cdot \mathbf{E} = \frac{\partial E_0}{\partial t} \\ \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 = \nabla \times \mathbf{B} \Leftrightarrow \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \text{ 動電生磁} \\ -\nabla \cdot \mathbf{B} = 0 \Leftrightarrow \nabla \cdot \mathbf{B} = 0 \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \Leftrightarrow \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \text{ 動磁生電} \end{cases} \end{aligned}$$

### 34.1.3.7 Joule heat vs. Thomson heat (Kelvin heat?)

The Lord Kelvin = William Thomson

**34.1.3.7.1 thermoelectric effect** thermoelectric effect = Seebeck effect = Peltier effect = Thomson effect

### 34.1.3.8 source term

$$J = J_0 + iJ_1 + jJ_2 + kJ_3 = J_0 + \mathbf{J} = \rho + \mathbf{J}$$

Maxwell equations with source terms

$$\begin{aligned} \begin{cases} [\mathbf{D}, \mathbf{B}] = J + \{\mathbf{D}, \mathbf{E}\} \Leftrightarrow (0) + (\nabla \times \mathbf{B}) = \left( \rho + \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ \{\mathbf{D}, \mathbf{E}\} = 0 - \{\mathbf{D}, \mathbf{B}\} \Leftrightarrow (0) + (\nabla \times \mathbf{E}) = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases} \\ \Leftrightarrow \begin{cases} \rho + \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} = 0 \Leftrightarrow \nabla \cdot \mathbf{E} = \rho + \frac{\partial E_0}{\partial t} \\ \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 = \nabla \times \mathbf{B} \Leftrightarrow \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \text{ 動電生磁} \\ -\nabla \cdot \mathbf{B} = 0 \Leftrightarrow \nabla \cdot \mathbf{B} = 0 \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \Leftrightarrow \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \text{ 動磁生電} \end{cases} \end{aligned}$$

### 34.1.4 quaternion group

[https://en.wikipedia.org/wiki/Quaternion\\_group](https://en.wikipedia.org/wiki/Quaternion_group)

group theory<sup>[37]</sup>

or please first see quaternion group representation<sup>[37.5]</sup>.

### 34.1.4.1 2D rotation

#### 34.1.4.1.1 matrix

$$\mathbf{r} = (x, y) = \langle x, y \rangle = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$$

$$\mathbf{r}' = (x', y') = \langle x', y' \rangle = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix}$$

$$\begin{aligned} \mathbf{r}' &= \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} = r \begin{pmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{pmatrix} \\ &= r \begin{pmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta \\ \sin \alpha \cos \theta + \cos \alpha \sin \theta \end{pmatrix} = r \begin{pmatrix} \cos \theta \cos \alpha - \sin \theta \sin \alpha \\ \sin \theta \cos \alpha + \cos \theta \sin \alpha \end{pmatrix} \\ &= r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\ &= R \mathbf{r}, \quad \begin{cases} R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = R(\theta) = R_\theta \\ \mathbf{r} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}, \mathbf{r}' = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} \end{cases} \end{aligned}$$

orthonormal matrix

$$\mathbf{r}' = O \mathbf{r}$$

$$\begin{aligned} |\mathbf{r}'|^2 &= |\mathbf{r}|^2 \\ \mathbf{r}' \cdot \mathbf{r}' &= \mathbf{r} \cdot \mathbf{r} \\ \mathbf{r}'^\top \mathbf{r}' &= \mathbf{r}^\top \mathbf{r} \\ (O \mathbf{r})^\top (O \mathbf{r}) &= \\ \mathbf{r}^\top O^\top O \mathbf{r} &= \\ \mathbf{r}^\top O^\top O \mathbf{r} &= \mathbf{r}^\top \mathbf{r} \\ O^\top O &= 1 = I = I_2 \end{aligned}$$

$$\begin{aligned} R^\top R &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^\top \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = I \end{aligned}$$

$$R^\top R = 1 \Rightarrow R \in \{O | O^\top O = 1\}$$

reflection matrix

$$\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & P_x^\top P_x = P_x^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_x \in \{O | O^\top O = 1\} \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & P_y^\top P_y = P_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_y \in \{O | O^\top O = 1\} \end{cases}$$

translation matrix?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$O(2)$  group

$$\begin{aligned} O(2) &= \{1, R, P_x, P_y\} \\ &= \{I_2, R_\theta, P_x, P_y\} \subseteq \{O | O^\top O = 1\} \end{aligned}$$

$$\begin{aligned} 1 &= O^\top O \\ 1 &= \det 1 = \det I = \det(I_2) \\ &= \det(O^\top O) = (\det O^\top)(\det O) = (\det O)(\det O) = (\det O)^2 \\ 1 &= (\det O)^2 \\ \det O &= \pm 1 \end{aligned}$$

$$\det R = \det R_\theta = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta - (-\sin^2 \theta) = \cos^2 \theta + \sin^2 \theta = 1$$

$$\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \det P_x = -1 \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \det P_y = -1 \end{cases}$$

special orthonormal group of degree 2

$$\begin{aligned} SO(2) &= \{1, R\} = \{I_2, R_\theta\} \subseteq \left\{ O \middle| \begin{cases} O^\top O = 1 & O \\ \det O = 1 & S \end{cases} \right\} \\ &\subset \{1, R, P_x, P_y\} = O(2) \subseteq \{O | O^\top O = 1\} \end{aligned}$$

**34.1.4.1.2 complex**

$$z = r(\cos \alpha + i \sin \alpha) = r e^{i\alpha}$$

$$z' = r(\cos(\alpha + \theta) + i \sin(\alpha + \theta)) = r e^{i(\alpha+\theta)}$$

$$z' = z_\theta z$$

$$z_\theta = \frac{z'}{z} = \frac{r' e^{i(\alpha+\theta)}}{r e^{i\alpha}} = \frac{r'}{r} e^{i\theta} = \frac{r'}{r} (\cos \theta + i \sin \theta)$$

$$\begin{aligned} z_\theta z &= \left[ \frac{r'}{r} (\cos \theta + i \sin \theta) \right] [r (\cos \alpha + i \sin \alpha)] \\ &= r' [(\cos \theta \cos \alpha - \sin \theta \sin \alpha) + i (\sin \theta \cos \alpha + \cos \theta \sin \alpha)] \\ &= r' [\cos(\alpha + \theta) + i \sin(\alpha + \theta)] = z' \end{aligned}$$

$$\hat{z}_\theta = z_\theta \left( \frac{r'}{r} = 1 \right) = e^{i\theta} = \cos \theta + i \sin \theta$$

$$\hat{z}_\theta^* = \overline{\hat{z}_\theta} = e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\hat{z}_\theta^* \hat{z}_\theta = e^{i\theta} e^{-i\theta} = e^{i\theta + (-i\theta)} = e^{i0} = e^0 = 1$$

unitary group of degree 1

$$U(1) = \{1, \hat{z}_\theta\} = \{e^{i0}, e^{i\theta}\}$$

**34.1.4.1.3**  $SO(2) \cong U(1) \quad \mathbb{C} \leftrightarrow \mathcal{M}_{2 \times 2}(\mathbb{R}) = \mathcal{M}_2(\mathbb{R})$  complex group representation<sup>[37.4]</sup>

$$x + yi \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

$$\begin{aligned} U(1) &= \{1, \hat{z}_\theta\} = \{e^{i0}, e^{i\theta}\} \\ &= \{\cos 0 + i \sin 0, \cos \theta + i \sin \theta\} \\ &\leftrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos 0 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin \theta \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 1 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin \theta \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\} = \{I_2, R_\theta\} = \{1, R\} = SO(2) \end{aligned}$$

$$U(1) \cong SO(2) \Leftrightarrow SO(2) \cong U(1)$$

unitary group of degree 1 and special orthonormal group of degree 2 are isomorphism

**34.1.4.2 3D rotation****34.1.4.2.1 matrix**

### 34.1.4.2.1.1 construction with 2D rotation matrix

$$\begin{aligned}
 \mathbf{r}' &= \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \\ z' \end{pmatrix} \stackrel{z' = z}{=} \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \\ z \end{pmatrix} = \begin{pmatrix} R(\theta) & \\ & 1 \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} \\
 &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} = R_z(\theta) \mathbf{r}, \quad \begin{cases} R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} \end{cases} \\
 \mathbf{r}' &= \begin{pmatrix} x' \\ r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} \stackrel{x' = x}{=} \begin{pmatrix} x \\ r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} = \begin{pmatrix} 1 & \\ & R(\theta) \end{pmatrix} \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} = R_x(\theta) \mathbf{r}, \quad \begin{cases} R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} \end{cases} \\
 \mathbf{r}' &= \begin{pmatrix} r \sin(\alpha + \theta) \\ y' \\ r \cos(\alpha + \theta) \end{pmatrix} \stackrel{y' = y}{=} \begin{pmatrix} r \sin(\alpha + \theta) \\ y \\ r \cos(\alpha + \theta) \end{pmatrix} \\
 &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} r \sin \alpha \\ y \\ r \cos \alpha \end{pmatrix} = R_y(\theta) \mathbf{r}, \quad \begin{cases} R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} r \sin \alpha \\ y \\ r \cos \alpha \end{pmatrix} \end{cases}
 \end{aligned}$$

### 34.1.4.2.1.2 Euler angle $z \rightarrow x \rightarrow z : \alpha \rightarrow \beta \rightarrow \gamma$

$$\begin{aligned}
 &R_z(\gamma) R_x(\beta) R_z(\alpha) \\
 &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \cos \beta \sin \alpha & \cos \beta \cos \alpha & -\sin \beta \\ \sin \beta \sin \alpha & \sin \beta \cos \alpha & \cos \beta \end{pmatrix} \\
 &= \begin{pmatrix} \cos \gamma \cos \alpha - \sin \gamma \cos \beta \sin \alpha & -\cos \gamma \sin \alpha - \sin \gamma \cos \beta \cos \alpha & \sin \gamma \sin \beta \\ \sin \gamma \cos \alpha + \cos \gamma \cos \beta \sin \alpha & -\sin \gamma \sin \alpha - \cos \gamma \cos \beta \cos \alpha & -\cos \gamma \sin \beta \\ \sin \beta \sin \alpha & \sin \beta \cos \alpha & \cos \beta \end{pmatrix}
 \end{aligned}$$

$x \rightarrow y \rightarrow z : \alpha \rightarrow \beta \rightarrow \gamma$

$$\begin{aligned}
& R_z(\gamma) R_y(\beta) R_x(\alpha) \\
&= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta \sin \alpha & \sin \beta \cos \alpha \\ 0 & \cos \alpha & -\sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos \gamma \cos \beta & \cos \gamma \sin \beta \sin \alpha - \sin \gamma \cos \alpha & \cos \gamma \sin \beta \cos \alpha + \sin \gamma \sin \alpha \\ \sin \gamma \cos \beta & \sin \gamma \sin \beta \sin \alpha + \cos \gamma \cos \alpha & \sin \gamma \sin \beta \cos \alpha - \cos \gamma \sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{pmatrix}
\end{aligned}$$

#### 34.1.4.2.1.3 3D rotation about an arbitrary axis <https://math.stackexchange.com/questions/4550704/rotation-around-an-arbitrary-axis>

spherical coordinate unit vector

$$\begin{cases} \hat{x} = r \sin \theta \cos \phi & \stackrel{r=1}{=} \sin \theta \cos \phi \\ \hat{y} = r \sin \theta \sin \phi & \stackrel{r=1}{=} \sin \theta \sin \phi \\ \hat{z} = r \cos \theta & \stackrel{r=1}{=} \cos \theta \end{cases}$$

although I prefer  $\theta$  and  $\phi$  switched back to be compatible with 2D coordinate

$$\begin{cases} \hat{x} = r \sin \phi \cos \theta & \stackrel{r=1}{=} \sin \phi \cos \theta \\ \hat{y} = r \sin \phi \sin \theta & \stackrel{r=1}{=} \sin \phi \sin \theta \\ \hat{z} = r \cos \phi & \stackrel{r=1}{=} \cos \phi \end{cases}$$

or cos first in  $x, y$ -plane

$$\begin{cases} \hat{x} = r \cos \phi \cos \theta & \stackrel{r=1}{=} \cos \phi \cos \theta \\ \hat{y} = r \cos \phi \sin \theta & \stackrel{r=1}{=} \cos \phi \sin \theta \\ \hat{z} = r \sin \phi & \stackrel{r=1}{=} \sin \phi \end{cases}$$


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still use the most convention

$$\hat{\mathbf{n}} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$


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$$\left\{ \begin{array}{l} \hat{\mathbf{n}} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \\ \hat{\mathbf{u}} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} \Leftrightarrow \cos \theta \sin \theta - \cos \theta \sin \theta = 0 \Rightarrow \hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = 0 \Leftrightarrow \hat{\mathbf{u}} \perp \hat{\mathbf{n}} \\ \hat{\mathbf{v}} = \hat{\mathbf{n}} \times \hat{\mathbf{u}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \end{vmatrix} = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \hat{\mathbf{v}} \perp \hat{\mathbf{n}} \\ \hat{\mathbf{v}} \perp \hat{\mathbf{u}} \end{cases} \end{array} \right.$$

$S = \{\hat{\mathbf{n}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}\} = \{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}}\}$  is a basis of the spherical coordinate

$$[\mathbf{V}]_S = \begin{pmatrix} u \\ v \\ n \end{pmatrix}$$

$$\begin{aligned} \mathbf{V} &= (\hat{\mathbf{u}} \quad \hat{\mathbf{v}} \quad \hat{\mathbf{n}}) \begin{pmatrix} u \\ v \\ n \end{pmatrix} = u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + n\hat{\mathbf{n}} \\ &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} u \\ v \\ n \end{pmatrix} = S [\mathbf{V}]_S \\ \begin{pmatrix} u \\ v \\ n \end{pmatrix} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^{-1} \mathbf{V} \\ [\mathbf{V}]_S &= S^{-1} \mathbf{V} \end{aligned}$$

$$\begin{aligned} S^{-1} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^{-1}, S \in \{O | O^T O = 1\} \\ &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^T \\ &= \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}^T \\ \hat{\mathbf{v}}^T \\ \hat{\mathbf{n}}^T \end{pmatrix} \\ S^{-1} S &= \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \end{aligned}$$

$\hat{\mathbf{n}}$  as  $z$  direction

$$\begin{aligned}
[\mathbf{V}]_S &= S^{-1} \mathbf{V} \\
[\mathbf{V}']_S &= R_z(\gamma) [\mathbf{V}]_S = R_z(\gamma) S^{-1} \mathbf{V} \\
\mathbf{V}' &= S [\mathbf{V}']_S = S R_z(\gamma) [\mathbf{V}]_S = S R_z(\gamma) S^{-1} \mathbf{V} \\
\mathbf{V}' &= S R_z(\gamma) S^{-1} \mathbf{V}
\end{aligned}$$

<https://www.symbolab.com/>

$$\begin{aligned}
&SR_z(\gamma) S^{-1} \\
&= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \\
&= \begin{pmatrix} c_1 c_2 & -s_2 & s_1 c_2 \\ c_1 s_2 & c_2 & s_1 s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 c_2 & c_1 s_2 & -s_1 \\ -s_2 & c_2 & 0 \\ s_1 c_2 & s_1 s_2 & c_1 \end{pmatrix}, \begin{cases} c_1 = \cos \theta & s_1 = \sin \theta \\ c_2 = \cos \phi & s_2 = \sin \phi \\ c_3 = \cos \gamma & s_3 = \sin \gamma \end{cases} \\
&= \begin{pmatrix} c_1 c_2 & -s_2 & s_1 c_2 \\ c_1 s_2 & c_2 & s_1 s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3 c_1 c_2 + s_3 s_2 & c_3 c_1 s_2 - c_2 s_3 & -c_3 s_1 \\ c_1 c_2 s_3 - c_3 s_2 & c_3 c_2 + c_1 s_3 s_2 & -s_3 s_1 \\ c_2 s_1 & s_2 s_1 & c_1 \end{pmatrix} \\
&= \begin{pmatrix} c_1^2 c_2^2 c_3 + c_3 s_2^2 + c_2^2 s_1^2 & c_1 s_2 (c_1 c_2 c_3 - s_2 s_3) + c_2 (-c_1 c_2 s_3 - c_3 s_2) + c_2 s_2 s_1^2 \\ c_1 c_2 (c_1 c_3 s_2 + c_2 s_3) - s_2 (c_2 c_3 - c_1 s_2 s_3) + c_2 s_2 s_1^2 & c_1^2 c_3 s_2^2 + s_2^2 s_1^2 + c_2^2 c_3 \\ -c_1 c_2 c_3 s_1 - s_2 s_1 s_3 + c_1 c_2 s_1 & -c_1 c_3 s_2 s_1 + c_2 s_1 s_3 + c_1 s_2 s_1 \end{pmatrix}
\end{aligned}$$

#### 34.1.4.2.2 quaternion <https://math.stackexchange.com/questions/328117/how-does-one-derive-this-rotation-quaternion-formula>

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\
w &= t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\
t + (\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= x_0 + (\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + \mathbf{x}, \begin{cases} \mathbf{e}_1 = \mathbf{i} = \mathbf{i} \\ \mathbf{e}_2 = \mathbf{j} = \mathbf{j} \\ \mathbf{e}_3 = \mathbf{k} = \mathbf{k} \end{cases}
\end{aligned}$$

$$q_1 q_2 = (x_{10} x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10} \mathbf{x}_2 + x_{20} \mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2)$$

$$QP = (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})$$

$$q = q_0 + \mathbf{q} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$$

$$\begin{aligned}
q^* &= \bar{q} = \overline{q_0 + \mathbf{q}} = q_0 - \mathbf{q} \\
&= \overline{q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}} = q_0 - q_1 \mathbf{i} - q_2 \mathbf{j} - q_3 \mathbf{k} \\
&= q_0 - q_1 \mathbf{i} - q_2 \mathbf{j} - q_3 \mathbf{k}
\end{aligned}$$

$$v = v_0 + \mathbf{v} = v_0 + v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} = v_0 + v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

$$\begin{aligned}
qv &= (q_0 v_0 - \mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + v_0 \mathbf{q} + \mathbf{q} \times \mathbf{v}) \\
&\stackrel{v_0=0}{=} (q_0 0 - \mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + 0 \mathbf{q} + \mathbf{q} \times \mathbf{v}) \\
&= (-\mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v}) \\
q\mathbf{v} &= (-\mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v})
\end{aligned}$$

$$\begin{aligned}
v\bar{q} &= (v_0 q_0 - \mathbf{v} \cdot \bar{\mathbf{q}}) + (v_0 \bar{\mathbf{q}} + q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\
&\stackrel{v_0=0}{=} (0 q_0 - \mathbf{v} \cdot \bar{\mathbf{q}}) + (0 \bar{\mathbf{q}} + q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\
&= (-\mathbf{v} \cdot \bar{\mathbf{q}}) + (q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\
&= (-\mathbf{v} \cdot (-\mathbf{q})) + (q_0 \mathbf{v} + \mathbf{v} \times (-\mathbf{q})) \\
&= (\mathbf{v} \cdot \mathbf{q}) + (q_0 \mathbf{v} - \mathbf{v} \times \mathbf{q}) \\
v\bar{q} &= (\mathbf{v} \cdot \mathbf{q}) + (q_0 \mathbf{v} - \mathbf{v} \times \mathbf{q})
\end{aligned}$$

<https://math.stackexchange.com/questions/41574/can-eulers-identity-be-extended-to-quaternions>

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd \\
&= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\
&= a + \theta \frac{ib + jc + kd}{\theta}, \theta^2 = b^2 + c^2 + d^2
\end{aligned}$$

$$\begin{aligned}
\left( \frac{ib + jc + kd}{r} \right)^2 &= \frac{-b^2 - c^2 - d^2 + bc(ij + ji) + cd(jk + kj) + db(ki + ik)}{r^2} \\
&= \frac{-b^2 - c^2 - d^2 + bc(k - k) + cd(i - i) + db(j - j)}{r^2} \\
&= \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\
&= \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\
\frac{ib + jc + kd}{r} &= \pm \sqrt{-1}
\end{aligned}$$

$$\begin{aligned}
e^q &= e^{a+ib+jc+kd} \\
&= e^{a+r\sqrt{-1}} = e^a e^{r\sqrt{-1}} = e^{a+\theta\sqrt{-1}} = e^a e^{\theta\sqrt{-1}} \\
&= e^a (\cos r + \sqrt{-1} \sin r) = e^a (\cos \theta + \sqrt{-1} \sin \theta) \\
&= e^a \left( \cos r + \frac{ib+jc+kd}{r} \sin r \right) = e^a \left[ \cos r + (ib+jc+kd) \frac{\sin r}{r} \right] \\
&= e^a \left( \cos \theta + \frac{ib+jc+kd}{\theta} \sin \theta \right) = e^a \left[ \cos \theta + (ib+jc+kd) \frac{\sin \theta}{\theta} \right]
\end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd \\
&= a + r \frac{ib+jc+kd}{r}, r^2 = b^2 + c^2 + d^2 \\
&= \sqrt{a^2 + r^2} \left( \frac{a}{\sqrt{a^2 + r^2}} + \frac{ib+jc+kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right) \\
&= \rho (\cos \phi + \sqrt{-1} \sin \phi), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} \end{cases} \\
&= \rho e^{\phi\sqrt{-1}}
\end{aligned}$$

$$q = \rho e^{\phi\sqrt{-1}}, \begin{cases} q = a + ib + jc + kd \\ \rho = \sqrt{a^2 + r^2} = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \arctan \frac{\pm \sqrt{b^2 + c^2 + d^2}}{a} \end{cases}$$

$$\begin{aligned}
\rho e^{-\phi\sqrt{-1}} &= \rho [\cos(-\phi) + \sqrt{-1} \sin(-\phi)] \\
&= \rho [\cos \phi - \sqrt{-1} \sin \phi] \\
&= \sqrt{a^2 + r^2} \left[ \frac{a}{\sqrt{a^2 + r^2}} - \frac{ib+jc+kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right] \\
&= a - (ib + jc + kd) = a - ib - jc - kd = \bar{q} = q^*
\end{aligned}$$

### 34.1.5 octonion

## 34.2 Krasjet

<https://github.com/Krasjet/quaternion>

<https://krasjet.github.io/quaternion/>

[https://krasjet.github.io/quaternion/bonus\\_gimbal\\_lock.pdf](https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf)

### 34.2.1 Rodrigues rotation

$$\mathbf{v} \rightarrow \mathbf{v}'$$

$$\mathbf{v} \xrightarrow{\text{rotate about } \mathbf{u}} \mathbf{v}'$$

$$\mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}'$$

$$\begin{cases} \mathbf{v} = \mathbf{v}_{\parallel n} + \mathbf{v}_{\perp n} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} & \Rightarrow \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} \\ \mathbf{v}' = \mathbf{v}'_{\parallel n} + \mathbf{v}'_{\perp n} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} & \Rightarrow \mathbf{v}'_{\perp} = \mathbf{v}' - \mathbf{v}'_{\parallel} \end{cases}$$

$$\begin{aligned} \mathbf{v}_{\parallel n} &= \mathbf{v}_{\parallel} = \text{proj}_n \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \hat{\mathbf{n}} \\ &= \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \end{aligned}$$

$$\begin{aligned} \mathbf{v}_{\parallel n} &= \mathbf{v}_{\parallel} = \text{proj}_{\hat{\mathbf{n}}} \mathbf{v} = \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} \hat{\mathbf{n}} \stackrel{\|\hat{\mathbf{n}}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \\ &= \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} \frac{\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} = \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|^2} \hat{\mathbf{n}} \stackrel{\|\hat{\mathbf{n}}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \end{aligned}$$

$$\mathbf{n} = \hat{\mathbf{n}}$$

$$\begin{aligned} \mathbf{v}_{\parallel n} &= \mathbf{v}_{\parallel} = \text{proj}_n \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \hat{\mathbf{n}} \stackrel{\|\mathbf{n}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \\ &= \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \stackrel{\|\mathbf{n}\|=1}{=} (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} = \mathbf{v} \cdot \mathbf{n} \mathbf{n} \end{aligned}$$

$$\mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$$

$$\mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}' = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp}$$

$$\mathbf{v}_{\parallel} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}'_{\parallel}$$

$$\mathbf{v}_{\parallel} = \mathbf{v}_{\parallel n} = \mathbf{v}'_{\parallel n} = \mathbf{v}'_{\parallel}$$

$$\mathbf{v}_{\parallel} = \mathbf{v}'_{\parallel}$$

$$\mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel}$$

$$\begin{cases} \mathbf{u} = \mathbf{n} \times \mathbf{v}_{\perp} \\ \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \end{cases} \quad \begin{cases} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \\ \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{cases}$$

$$\begin{aligned}
\mathbf{v}'_{\perp} &= \mathbf{v}'_{\parallel v_{\perp}} + \mathbf{v}'_{\parallel u} \\
&= (\cos \theta) \mathbf{v}_{\perp} + (\sin \theta) \mathbf{u}, \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \\
&= \cos \theta \mathbf{v}_{\perp} + \sin \theta \mathbf{u} \\
&= \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}) \\
\mathbf{v}'_{\perp} &= \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp})
\end{aligned}$$

$$\mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp}$$


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$$\mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp})$$


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$$\begin{aligned}
\mathbf{v}' &= \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp}, \begin{cases} \mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \end{cases} \\
&= \mathbf{v}_{\parallel} + \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \\
&= \mathbf{v}_{\parallel} + \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \begin{cases} \mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \\ \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \end{cases} \\
&= (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \cos \theta [\mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}] + \sin \theta [\mathbf{n} \times (\mathbf{v} - \mathbf{v}_{\parallel})] \\
&= (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \cos \theta \mathbf{v} - \cos \theta (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta [\mathbf{n} \times \mathbf{v} - \mathbf{n} \times \mathbf{v}_{\parallel}] \\
&= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta [\mathbf{n} \times \mathbf{v} - \mathbf{0}] \\
&= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v}
\end{aligned}$$


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$$\mathbf{v}' = \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v}$$


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$$\begin{aligned}
\mathbf{v}' &= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + \left[ 1 - \left( 1 - 2 \sin^2 \frac{\theta}{2} \right) \right] (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left[ 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + \left[ 2 \sin^2 \frac{\theta}{2} \right] (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left[ 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} \right] \\
&= \cos \theta \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} + \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \right] \\
&= \left[ 2 \cos^2 \frac{\theta}{2} - 1 \right] \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} + \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \right]
\end{aligned}$$

### 34.2.2 quaternion operation

$$q = a + bi + cj + dk$$

$$\varrho = \alpha + \beta i + \gamma j + \delta k$$

$$\begin{aligned}
& \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \leftarrow q\varrho = (a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k) \\
& = a(\alpha + \beta i + \gamma j + \delta k) \\
& + bi(\alpha + \beta i + \gamma j + \delta k) \\
& + cj(\alpha + \beta i + \gamma j + \delta k) \\
& + dk(\alpha + \beta i + \gamma j + \delta k) \\
& = a\alpha + a\beta i + a\gamma j + a\delta k \\
& + b\alpha i - b\beta + b\gamma k - b\delta j \\
& + c\alpha j - c\beta k - c\gamma + c\delta i \\
& + d\alpha k + d\beta j - d\gamma i - d\delta \\
& = (a\alpha - b\beta - c\gamma - d\delta) \\
& + (b\alpha + a\beta - d\gamma + c\delta)i \\
& + (c\alpha + d\beta + a\gamma - b\delta)j \\
& + (d\alpha - c\beta + b\gamma + a\delta)k \\
& \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \leftarrow \\
& = (\alpha a - \beta b - \gamma c - \delta d) \\
& + (\beta a + \alpha b + \delta c - \gamma d)i \\
& + (\gamma a - \delta b + \alpha c + \beta d)j \\
& + (\delta a + \gamma b - \beta c + \alpha d)k \\
& \begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \leftarrow \\
& qv \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_i v
\end{aligned}$$

concept like quaternion group representation<sup>[37.5]</sup>

$$\begin{aligned}
L(q) = Q_i &= \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} a + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} b \\
&+ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} c + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} d \\
&\leftrightarrow 1a + ib + jc + kd
\end{aligned}$$

$$\begin{aligned}
ij \leftrightarrow & \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow k \\
ji \leftrightarrow & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -k \\
ki \leftrightarrow & \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow j
\end{aligned}$$

$$Q_i = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & \begin{pmatrix} -b & c & d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{pmatrix} = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix}, \quad \begin{cases} a = q_0 \\ \mathbf{q} = \begin{pmatrix} b \\ c \\ d \end{pmatrix} \\ Q_i = \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{cases}$$

Grassmann product

$$qv \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_i v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q}^\top v \\ \mathbf{q} v_0 + Q_i v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot v \\ \mathbf{q} v_0 + q_0 v + \mathbf{q} \times v \end{pmatrix}$$

$$vq \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_r v$$

$$\begin{aligned}
R(q) = Q_r = & \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
& + c \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
& \leftrightarrow a1 + bi + cj + dk
\end{aligned}$$

$$ij \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -k$$

$$ji \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow k$$

$$ki \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow -j$$

$$\begin{aligned} R(q) = Q_r &= \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \leftrightarrow \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} \\ &= \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \begin{cases} \alpha = a+bi & \bar{\alpha} = a-bi \\ \beta = c+di & \bar{\beta} = c-di \end{cases} \end{aligned}$$

$$Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & \begin{pmatrix} -b & c & d \\ a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \end{pmatrix} = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix}, \begin{cases} a = q_0 \\ b \\ c \\ d \\ Q_i = \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{cases}$$

$$vq \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{q}^\top v \\ v_0 \mathbf{q} + Q_i^\top v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{v} \cdot \mathbf{q} \\ v_0 \mathbf{q} + q_0 \mathbf{v} + \mathbf{v} \times \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \end{pmatrix}$$

$$\begin{cases} qv \leftrightarrow Q_l v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q}^\top v \\ \mathbf{q} v_0 + Q_i v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v} \end{pmatrix} \\ vq \leftrightarrow Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{q}^\top v \\ v_0 \mathbf{q} + Q_i^\top v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{v} \cdot \mathbf{q} \\ v_0 \mathbf{q} + v_0 q_0 + \mathbf{v} \times \mathbf{q} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} - \mathbf{q} \times \mathbf{v} \end{pmatrix} \end{cases}$$

$$\begin{cases} qv = \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = L(q)v = Q_l v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} \\ vq = \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = R(q)v = Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} \end{cases}$$

$$\begin{aligned}
& \left\{ \begin{array}{l} \mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}' \\ \left\{ \begin{array}{l} \mathbf{v} = \mathbf{v}_{\parallel n} + \mathbf{v}_{\perp n} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \Rightarrow \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} \\ \mathbf{v}' = \mathbf{v}'_{\parallel n} + \mathbf{v}'_{\perp n} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \Rightarrow \mathbf{v}'_{\perp} = \mathbf{v}' - \mathbf{v}'_{\parallel} \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{u} = \mathbf{n} \times \mathbf{v}_{\perp} \\ \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \\ \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \end{array} \right. \end{array} \right. \\
& \Leftrightarrow \left\{ \begin{array}{l} v \xrightarrow{\text{rotate about } n} v' \\ \left\{ \begin{array}{l} v = \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}_{\perp n} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} = v_{\parallel} + v_{\perp} \Rightarrow v_{\perp} = v - v_{\parallel} \\ v' = \begin{pmatrix} 0 \\ \mathbf{v}' \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp n} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp} \end{pmatrix} = v'_{\parallel} + v'_{\perp} \Rightarrow v'_{\perp} = v' - v'_{\parallel} \end{array} \right. \\ \left\{ \begin{array}{l} u = \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ 00 - 0 \end{pmatrix} = \begin{pmatrix} n_0 v_{\perp 0} - \mathbf{n} \cdot \mathbf{v}_{\perp} \\ \mathbf{n} v_{\perp 0} + n_0 \mathbf{v}_{\perp} + \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = n v_{\perp} \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \\ \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{array} \right. \\ \left\{ \begin{array}{l} \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \\ \mathbf{v}'_{\parallel} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} = v_{\parallel} \\ \mathbf{v}'_{\perp} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp} \end{pmatrix} = \cos \theta \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} + \sin \theta \begin{pmatrix} 0 \\ \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \cos \theta v_{\perp} + \sin \theta n v_{\perp} = (\cos \theta + \sin \theta n) v_{\perp} \end{array} \right. \end{array} \right. \end{aligned}$$

$$v'_{\perp} = (\cos \theta + \sin \theta n) v_{\perp}$$

$$v'_{\perp} = p v_{\perp}, p = \cos \theta + \sin \theta n = \cos \theta \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + \sin \theta \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}$$

$$p = \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} = \cos \theta + \sin \theta n, \|\mathbf{n}\| = 1$$

$$|p| = \left\| \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}^T \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}} = \sqrt{\cos^2 \theta + \sin^2 \theta \|\mathbf{n}\|^2} = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$|p| = \left\| \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} \right\| = 1$$

$$p^* = \bar{p} = \begin{pmatrix} p_0 \\ -\mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -\sin \theta \mathbf{n} \end{pmatrix} = \cos \theta - \sin \theta n$$

$$\begin{aligned}
p^* p &= \bar{p} p = \begin{pmatrix} p_0 \\ -\mathbf{p} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} p_0 p_0 - (-\mathbf{p}) \cdot \mathbf{p} \\ -\mathbf{p} p_0 + p_0 \mathbf{p} + (-\mathbf{p}) \times \mathbf{p} \end{pmatrix} \\
&= \begin{pmatrix} p_0 p_0 + \mathbf{p} \cdot \mathbf{p} \\ \mathbf{0} + \mathbf{0} \end{pmatrix} = p_0 p_0 + \mathbf{p} \cdot \mathbf{p} = p_0^2 + \|\mathbf{p}\|^2 = |p|^2 = 1^2 = 1 \\
&= \begin{pmatrix} p_0 p_0 - \mathbf{p} \cdot (-\mathbf{p}) \\ \mathbf{p} p_0 + p_0 (-\mathbf{p}) + \mathbf{p} \times (-\mathbf{p}) \end{pmatrix} = p \bar{p} = pp^*
\end{aligned}$$

$$p^* p = \bar{p} p = 1 = p \bar{p} = pp^* = |p|^2 = p_0^2 + \|\mathbf{p}\|^2 \Rightarrow p^* = \frac{1}{p} = p^{-1}$$

$$\begin{aligned}
p^2 &= pp = \begin{pmatrix} p_0 p_0 - \mathbf{p} \cdot \mathbf{p} \\ \mathbf{p} p_0 + p_0 \mathbf{p} + \mathbf{p} \times \mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos^2 \theta - \sin^2 \theta \|\mathbf{n}\|^2 \\ 2(\sin \theta \mathbf{n}) \cos \theta + \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2 \theta - \sin^2 \theta \\ 2 \sin \theta \cos \theta \mathbf{n} \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \mathbf{n} \end{pmatrix} \\
&\left( \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} \right)^2 = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \mathbf{n} \end{pmatrix} \\
q^2 &= \left( \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{n} \end{pmatrix} \right)^2 = \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} = p
\end{aligned}$$

$$qv_{\parallel} = v_{\parallel} q$$

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel \mathbf{n}} &= \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel} = \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} \alpha 0 - \beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \beta \mathbf{n} 0 + \alpha \mathbf{v}_{\parallel} + \beta \mathbf{n} \times \mathbf{v}_{\parallel} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} + \mathbf{0} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} 0\alpha - \mathbf{v}_{\parallel} \cdot \beta \mathbf{n} \\ \mathbf{v}_{\parallel} \alpha + 0\beta \mathbf{n} + \mathbf{v}_{\parallel} \times \beta \mathbf{n} \end{pmatrix} = \begin{pmatrix} -\mathbf{v}_{\parallel} \cdot \beta \mathbf{n} \\ \alpha \mathbf{v}_{\parallel} + \mathbf{0} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} = v_{\parallel} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} = v_{\parallel \mathbf{n}} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \\
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel} &= v_{\parallel} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}
\end{aligned}$$

$$qv_{\perp} = v_{\perp} q^*$$

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp \mathbf{n}} &= \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp} = \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} \\
&= \begin{pmatrix} \alpha 0 - \beta \mathbf{n} \cdot \mathbf{v}_{\perp} \\ \beta \mathbf{n} 0 + \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \mathbf{v}_{\perp} + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} \\
&= \begin{pmatrix} 0\alpha - \mathbf{v}_{\perp} \cdot (-\beta \mathbf{n}) \\ \mathbf{v}_{\perp} \alpha + 0(-\beta \mathbf{n}) + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha \mathbf{v}_{\perp} + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp \mathbf{n}} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} \\
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp} &= v_{\perp} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}^* = v_{\perp} \overline{\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}}
\end{aligned}$$

$$\begin{aligned}
q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n &\in \left\{ p \left| \begin{array}{l} p = \cos \theta + \sin \theta n \\ n = \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix}, \|n\| = 1 \end{array} \right. \right\} \subset \left\{ \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \left| \begin{array}{l} \alpha, \beta \in \mathbb{R} \\ \|\mathbf{n}\| = 1 \end{array} \right. \right\} \\
&\Rightarrow \begin{cases} q^* q = \bar{q} q = 1 = q \bar{q} = q q^* = |q|^2 = q_0^2 + \|\mathbf{q}\|^2 \Rightarrow q^* = \frac{1}{q} = q^{-1} & (u) \text{ unit quaternion} \\ q v_{\parallel} = v_{\parallel} q \\ q v_{\perp} = v_{\perp} q^* = v_{\perp} \bar{q} \Leftrightarrow \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{n} \end{pmatrix} v_{\perp} = v_{\perp} \begin{pmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \mathbf{n} \end{pmatrix} & (c) \text{ commutativity} \\ & (a) \text{ anticommutativity} \end{cases}
\end{aligned}$$

$$\begin{aligned}
v' &= v'_{\parallel} + v'_{\perp}, \begin{cases} v'_{\parallel} = v_{\parallel} & q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, |n| = 1 \\ v'_{\perp} = p v_{\perp} = q^2 v_{\perp} & p = q^2 = \cos \theta + \sin \theta n \end{cases} \\
&= v_{\parallel} + q^2 v_{\perp} \stackrel{(u)}{=} 1 v_{\parallel} + q q v_{\perp} \\
&\stackrel{(a)}{=} q q^* v_{\parallel} + q v_{\perp} q^* \stackrel{(c)}{=} q v_{\parallel} q^* + q v_{\perp} q^* \\
&= q (v_{\parallel} + v_{\perp}) q^* = q (v) q^* = q v q^*
\end{aligned}$$


---

$$\begin{aligned}
v' &= q v q^{-1}, q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, |n| = 1 \\
&= q v q^* = q (v_{\parallel} + v_{\perp}) q^* \\
&= q v_{\parallel} q^* + q v_{\perp} q^* = q q^* v_{\parallel} + q q v_{\perp} = 1 v_{\parallel} + q^2 v_{\perp} \\
&= v_{\parallel} + p v_{\perp}, p = q^2 = \cos \theta + \sin \theta n
\end{aligned}$$


---

$$\begin{aligned}
v' &= q v q^* = q (v q^*) \leftrightarrow L(q) R(q^*) v = Q_l Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \\
&= (q v) q^* \leftrightarrow R(q^*) L(q) v = Q_r Q_l v = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \\
\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} &= \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i^T \\ \mathbf{q} q_0 + Q_i \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i Q_i^T \end{pmatrix} = L(q) R(q^*) \\
\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} &= \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i \\ \mathbf{q} q_0 + Q_i^T \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i^T Q_i \end{pmatrix} = R(q^*) L(q)
\end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n = \cos \frac{\theta}{2} + bi + cj + dk \\
\Rightarrow Q_i Q_i^T &= \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} = \begin{pmatrix} a^2 + d^2 + c^2 & -cb & -db \\ -cb & d^2 + a^2 + b^2 & -dc \\ -db & -dc & c^2 + b^2 + a^2 \end{pmatrix} \\
&= Q_i^T Q_i = \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} = \begin{pmatrix} a^2 + d^2 + c^2 & -cb & -db \\ -cb & d^2 + a^2 + b^2 & -dc \\ -db & -dc & c^2 + b^2 + a^2 \end{pmatrix} \\
\mathbf{q}^T Q_i^T &= (b \quad c \quad d) \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} = (ba \quad ca \quad da) \\
&= \mathbf{q}^T Q_i = (b \quad c \quad d) \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} = (ba \quad ca \quad da) \\
Q_i \mathbf{q} &= \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ab \\ ac \\ ad \end{pmatrix} \\
&= Q_i^T \mathbf{q} = \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ab \\ ac \\ ad \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} = \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i^T \\ \mathbf{q} q_0 + Q_i \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i Q_i^T \end{pmatrix} = L(q) R(q^*) \\
&= \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} = \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i \\ \mathbf{q} q_0 + Q_i^T \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i^T Q_i \end{pmatrix} = R(q^*) L(q) \\
&\Downarrow \\
L(q) R(q^*) &= R(q^*) L(q) \\
&\Updownarrow \\
v' &= qvq^* = q(vq^*) = (qv)q^*
\end{aligned}$$


---

$$a = \cos \frac{\theta}{2} \Rightarrow \theta = 2 \arccos a = 2 \cos^{-1} a$$

$$\sin \frac{\theta}{2} n = \sin \frac{2 \arccos a}{2} n = \sin (\cos^{-1} a) n$$

$$n = \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\mathbf{n}} \end{pmatrix}, \|n\| = 1, \mathbf{n} = \begin{pmatrix} \sin \gamma \cos \alpha \\ \sin \gamma \sin \alpha \\ \cos \gamma \end{pmatrix}, \begin{cases} \alpha = \angle \hat{\mathbf{n}}_{xy} \hat{\mathbf{x}} \\ \gamma = \angle \hat{\mathbf{n}} \hat{\mathbf{z}} \end{cases}$$


---

$$q = a + bi + cj + dk$$

$$\begin{aligned}
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, n = \begin{pmatrix} 0 \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \sin \gamma \cos \alpha \\ \sin \gamma \sin \alpha \\ \cos \gamma \end{pmatrix}, \|n\| = 1, \begin{cases} \alpha = \angle \hat{n}_{xy} \hat{x} \\ \gamma = \angle \hat{n} \hat{z} \end{cases} \\
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{n}, \mathbf{n} = (\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k} \\
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\
&= \cos \frac{\theta}{2} + \mathbf{n} \sin \frac{\theta}{2} \\
q^{-1} &= q^* = \cos \frac{\theta}{2} - \mathbf{n} \sin \frac{\theta}{2} \\
&= \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\
\mathbf{v}' &= v' = qvq^{-1} = q\mathbf{v}q^{-1}, \begin{cases} v = \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \\ v' = \mathbf{v}' = v'_1 \mathbf{i} + v'_2 \mathbf{j} + v'_3 \mathbf{k} \end{cases} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \\
&= q(\theta, \gamma, \alpha) \mathbf{v} q^{-1}(\theta, \gamma, \alpha) \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= q(\theta, n_z, n_x) \mathbf{v} q^{-1}(\theta, n_z, n_x)
\end{aligned}$$


---

$$\mathbf{v}' = v' = qvq^{-1} = q\mathbf{v}q^{-1}, \begin{cases} v = \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \\ v' = \mathbf{v}' = v'_1 \mathbf{i} + v'_2 \mathbf{j} + v'_3 \mathbf{k} \\ q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\ q^{-1} = \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \end{cases}$$

### 34.2.3 matrix form

$$q \stackrel{|q|=1}{=} a + bi + cj + dk, \begin{cases} a = \cos \frac{\theta}{2} \\ b = \sin \frac{\theta}{2} \sin \gamma \cos \alpha \\ c = \sin \frac{\theta}{2} \sin \gamma \sin \alpha \\ d = \sin \frac{\theta}{2} \cos \gamma \end{cases}$$

$$L(q) = Q_l = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

$$R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

$$R(q^*) = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

$$\begin{aligned} v' &= qvq^* = (qv)q^* \leftrightarrow R(q^*)L(q)v \\ &= q(vq^*) \leftrightarrow L(q)R(q^*)v \\ &= \left( \begin{array}{cccc} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{array} \right) \left( \begin{array}{cccc} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{array} \right) v \\ &= \left( \begin{array}{cccc} a^2 + b^2 + c^2 + d^2 & 0 & 0 & 0 \\ 0 & b^2 + a^2 - d^2 - c^2 & -2ad + 2bc & 2ac + 2bd \\ 0 & 2ad + 2bc & c^2 - d^2 + a^2 - b^2 & 2cd - 2ab \\ 0 & -2ac + 2bd & 2cd + 2ab & d^2 - c^2 - b^2 + a^2 \end{array} \right) v \\ &= \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 0 & 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 0 & 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{array} \right) v \\ \left( \begin{array}{c} 0 \\ v' \end{array} \right) &= \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 0 & 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 0 & 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{array} \right) \left( \begin{array}{c} 0 \\ v \end{array} \right) \end{aligned}$$


---

$$\mathbf{v}' = \left( \begin{array}{ccc} 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{array} \right) \mathbf{v}$$


---

$$q \stackrel{|q|=1}{=} a + bi + cj + dk, \quad \begin{cases} a = \cos \frac{\theta}{2} \\ b = \sin \frac{\theta}{2} \sin \gamma \cos \alpha \\ c = \sin \frac{\theta}{2} \sin \gamma \sin \alpha \\ d = \sin \frac{\theta}{2} \cos \gamma \end{cases}$$

### 34.2.4 exponential form

$$\begin{aligned}
 q &= a + bi + cj + dk = a + ib + jc + kd \\
 &= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\
 &\stackrel{|q|=1}{=} \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{n}, \begin{cases} a = \cos \frac{\theta}{2} & \cos \frac{\theta}{2} = a \\ r = \sin \frac{\theta}{2} & \sin \frac{\theta}{2} = r \\ \frac{ib + jc + kd}{r} = \mathbf{n} & \mathbf{n} = \frac{ib + jc + kd}{r} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \|\mathbf{n}\|^2 &= \left( \frac{ib + jc + kd}{r} \right)^2 = \frac{-b^2 - c^2 - d^2 + bc(ij + ji) + cd(jk + kj) + db(ki + ik)}{r^2} \\
 &= \frac{-b^2 - c^2 - d^2 + bc(k - k) + cd(i - i) + db(j - j)}{r^2} \\
 &= \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\
 &= \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\
 \mathbf{n} &= \frac{ib + jc + kd}{r} = \pm \sqrt{-1}
 \end{aligned}$$

$$\begin{aligned}
 q &= a + bi + cj + dk = a + ib + jc + kd \\
 &= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\
 &= \sqrt{a^2 + r^2} \left( \frac{a}{\sqrt{a^2 + r^2}} + \frac{ib + jc + kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right) \\
 &= \rho (\cos \phi + \sqrt{-1} \sin \phi), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \frac{\theta}{2} \end{cases} \\
 &= \rho e^{\phi \sqrt{-1}} \stackrel{|q|=1}{=} |q| e^{i \frac{\theta}{2}} = e^{i \frac{\theta}{2}}
 \end{aligned}$$

$$q = \rho e^{\phi \sqrt{-1}} \stackrel{|q|=1}{=} e^{i \frac{\theta}{2}}, \begin{cases} q = a + ib + jc + kd \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} = 1 \\ \tan \frac{\theta}{2} = \frac{r}{a} \Leftrightarrow \frac{\theta}{2} = \arctan \frac{r}{a} = \arctan \frac{\pm \sqrt{b^2 + c^2 + d^2}}{a} \end{cases}$$

$$\begin{aligned}
\mathbf{v}' &= v' = qvq^{-1} = q\mathbf{v}q^{-1}, \quad \left\{ \begin{array}{l} v = \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \\ v' = \mathbf{v}' = v'_1\mathbf{i} + v'_2\mathbf{j} + v'_3\mathbf{k} \end{array} \right. , \quad \mathbf{n} = \begin{pmatrix} \sin \gamma \cos \alpha \\ \sin \gamma \sin \alpha \\ \cos \gamma \end{pmatrix}, \quad \left\{ \begin{array}{l} \alpha = \angle \hat{\mathbf{n}}_{xy} \hat{\mathbf{x}} \\ \gamma = \angle \hat{\mathbf{n}} \hat{\mathbf{z}} \end{array} \right. \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \alpha) \mathbf{i} + (\sin \gamma \sin \alpha) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \\
&= q(\theta, \gamma, \alpha) \mathbf{v} q^{-1}(\theta, \gamma, \alpha) = q\left(\frac{\theta}{2}, \gamma, \alpha\right) \mathbf{v} q^{-1}\left(\frac{\theta}{2}, \gamma, \alpha\right) \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \left[ n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k} \right] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \left[ n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k} \right] \right\} \\
&= q(\theta, n_z, n_x) \mathbf{v} q^{-1}(\theta, n_z, n_x) = q\left(\frac{\theta}{2}, n_z, n_x\right) \mathbf{v} q^{-1}\left(\frac{\theta}{2}, n_z, n_x\right) \\
&= e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}}
\end{aligned}$$


---

$$\mathbf{v}' = e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}}$$


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general quaternion exponential form

$$\begin{aligned}
q &= \rho e^{\phi \sqrt{-1}} = |q| e^{\mathbf{n} \phi}, \quad \left\{ \begin{array}{l} q = a + ib + jc + kd = a + r \frac{ib + jc + kd}{r} = a + r\mathbf{n} \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \arctan \frac{\pm \sqrt{b^2 + c^2 + d^2}}{a} \end{array} \right. \\
&= e^{\ln|q|} e^{\mathbf{n} \phi} = e^{\ln|q| + \mathbf{n} \phi}
\end{aligned}$$

### 34.2.5 double cover

<https://www.bilibili.com/video/BV1rj41117VW/?t=15m15s>

$$\begin{array}{ccccccccccccc}
\theta & 0 & \rightarrow & \frac{\pi}{2} & \rightarrow & \pi & \rightarrow & \frac{3\pi}{2} & \rightarrow & 2\pi & \rightarrow & \frac{5\pi}{2} & \rightarrow & 3\pi & \rightarrow & \frac{7\pi}{2} \\
\theta & 0 & \rightarrow & \frac{\pi}{4} & \rightarrow & \frac{\pi}{2} & \rightarrow & \frac{3\pi}{4} & \rightarrow & \pi & \rightarrow & \frac{5\pi}{4} & \rightarrow & \frac{3\pi}{2} & \rightarrow & \frac{7\pi}{4} \\
e^{\mathbf{n} \frac{\theta}{2}} & 1 & \rightarrow & e^{\mathbf{n} \frac{\pi}{4}} & \rightarrow & \mathbf{n} & \rightarrow & e^{\mathbf{n} \frac{3\pi}{4}} & \rightarrow & -1 & \rightarrow & e^{\mathbf{n} \frac{5\pi}{4}} & \rightarrow & -\mathbf{n} & \rightarrow & e^{\mathbf{n} \frac{7\pi}{4}} \\
e^{-\mathbf{n} \frac{\theta}{2}} & 1 & \rightarrow & e^{\mathbf{n} \frac{-\pi}{4}} & \rightarrow & -\mathbf{n} & \rightarrow & e^{\mathbf{n} \frac{-3\pi}{4}} & \rightarrow & -1 & \rightarrow & e^{\mathbf{n} \frac{-5\pi}{4}} & \rightarrow & \mathbf{n} & \rightarrow & e^{\mathbf{n} \frac{-7\pi}{4}} \\
\mathbf{v}' = e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}} & \mathbf{v} & \rightarrow & & \rightarrow & -\mathbf{n} \mathbf{v} \mathbf{n} & \rightarrow & & \rightarrow & \mathbf{v} & \rightarrow & & \rightarrow & -\mathbf{n} \mathbf{v} \mathbf{n} & \rightarrow
\end{array}$$

$$SU(2) = \left\{ 1(\cdot) 1, e^{\mathbf{n} \frac{\theta}{2}}(\cdot) e^{-\mathbf{n} \frac{\theta}{2}} \right\}$$

3D rotation about an arbitrary axis<sup>[34.1.4.2.1.3]</sup>

$$\mathbf{V}' = SR_z(\gamma) S^{-1} \mathbf{V}$$

$$\mathbf{v}' = SR_z(\theta) S^{-1} \mathbf{v}$$

$$SO(3) = \{1, SR_z(\theta) S^{-1}\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, SR_z(\theta) S^{-1} \right\}$$

$$SU(2) \cong 2 \cdot SO(3)$$

Special unitary group of degree 2  $SU(2)$  is the boss of special orthonormal group of degree 3  $SO(3)$ ,  $SU(2)$  double covers  $SO(3)$ .

[https://en.wikipedia.org/wiki/Special\\_unitary\\_group#The\\_group\\_SU\(2\)](https://en.wikipedia.org/wiki/Special_unitary_group#The_group_SU(2))

[https://en.wikipedia.org/wiki/Unitary\\_group](https://en.wikipedia.org/wiki/Unitary_group)

“Particle is group representation, spin is square root of 4-vector.”?

「粒子是群表示，自旋是四維向量開根號。」？

### 34.2.6 right-hand vs. left-hand coordinates

Krasjet 四元數與三維旋轉 p.63 73

#### 34.2.6.1 clock

$$\mathbf{v}' = e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}}$$

$$\mathbf{v}' = e^{-\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{\mathbf{n} \frac{\theta}{2}}$$

right-hand = anticlockwise

$$[\mathbf{v}']_R = \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_R [\mathbf{v}]_R \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_R$$

$$[\mathbf{v}']_L = \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_R [\mathbf{v}]_L \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_R$$

$$[\mathbf{v}']_L = \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_L [\mathbf{v}]_L \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_L$$

### 34.2.6.2 Rodrigues rotation in right-hand vs. left-hand coordinates

$$\mathbf{u} = \mathbf{n} \times \mathbf{v}_\perp$$

$$[\mathbf{u}]_R = [\mathbf{n}]_R \times_R [\mathbf{v}_\perp]_R = [\mathbf{n}]_L \times_L [\mathbf{v}_\perp]_L = [\mathbf{u}]_L$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[[\mathbf{u}]_L]_R = -[\mathbf{u}]_R$$

$$\left[ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_L \right]_R = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_R$$

$$\det O = \pm 1$$

$\det O = -1$  will change handedness of the coordinate.

### 34.2.7 gimbal lock

[https://krasjet.github.io/quaternion/bonus\\_gimbal\\_lock.pdf](https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf)

### 34.2.8 Lie group

#### 34.2.8.1 special unitary group

### 34.2.9 geometric algebra and Clifford algebra

[https://en.wikipedia.org/wiki/Geometric\\_algebra](https://en.wikipedia.org/wiki/Geometric_algebra)

[https://en.wikipedia.org/wiki/Clifford\\_algebra](https://en.wikipedia.org/wiki/Clifford_algebra)

John Vince\_2009\_Geometric Algebra\_ An Algebraic System for Computer Games and Animation

<https://www.amazon.com/Geometric-Algebra-Algebraic-Computer-Animation/dp/1848823789>

### 34.2.10 dual quaternion

### 34.2.11 interpolation

#### 34.2.11.1 Lerp

#### 34.2.11.2 Nlerp

#### 34.2.11.3 Slerp

## 34.3 3Blue1Brown

### 34.3.1 Ben Eater

<https://eater.net/quaternions/video/intro>

<https://eater.net/quaternions>

### 34.3.2 3B1B

### 34.3.3 Sutrabla

<https://www.newscientist.com/article/mg20427391-600-alices-adventures-in-algebra-wonderland-solved/>

<https://threejs.org/>

## 34.4 CCJou: LA Revelation

<https://ccjou.wordpress.com/2014/04/21/四元數/>



# Chapter 35

## DICOM

### 35.1 Innolitics: DICOM Standard Browser

<https://dicom.innolitics.com/ciods>

<https://dicom.innolitics.com/ciods/parametric-map/parametric-map-multi-frame-functional-groups/52009229/0048021a/0040072a>

### 35.2 David Clunie: Medical Image Format Site

<https://www.dclunie.com/>



# **Chapter 36**

## **tendon pathophysiology**

Mark E. Schweitzer, MD

<https://www.youtube.com/watch?v=uIo58pQgxhY>

### **36.1 histology**



# Chapter 37

## group theory

[https://en.wikipedia.org/wiki/Group\\_\(mathematics\)](https://en.wikipedia.org/wiki/Group_(mathematics))

### 37.1 matrix group

subset of two-by-two matrices at least excluding zero matrix

$$\mathcal{M} = (\mathcal{M}, \cdot) \subset (\mathcal{M}_{2 \times 2}(\mathbb{C}) - \{0\}, \cdot) = \mathcal{M}_{2 \times 2}(\mathbb{C}) - \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

matrix multiplication

$$\begin{aligned} & \forall \langle M_1, M_2 \rangle \in \mathcal{M}^2, \exists M_1 M_2 \in \mathcal{M} [M_1 M_2 = M_1 \cdot M_2] \\ \Leftrightarrow & \cdot : \mathcal{M} \times \mathcal{M} = \mathcal{M}^2 \rightarrow \mathcal{M} \\ \Leftrightarrow & \cdot : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \\ \Leftrightarrow & \mathcal{M} \times \mathcal{M} \stackrel{\cdot}{\rightarrow} \mathcal{M} \\ \Leftrightarrow & \mathcal{M}^2 \stackrel{\cdot}{\rightarrow} \mathcal{M} \end{aligned}$$

matrix group

$$\begin{cases} \forall \langle M_1, M_2, M_3 \rangle \in \mathcal{M}^3 [M_1 (M_2 M_3) = (M_1 M_2) M_3] & \text{associativity} \\ \exists I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}, \forall M \in \mathcal{M} [IM = M] & \text{left unit element} \\ \forall M \in \mathcal{M}, \exists M^{-1} \in \mathcal{M} [M^{-1} M = I] & \text{left inverse (element)} \end{cases}$$

$\Rightarrow \mathcal{M} = (\mathcal{M}, \cdot)$  is a matrix group

### 37.2 group definition and basic theorem

[https://en.wikipedia.org/wiki/Group\\_\(mathematics\)#Elementary\\_consequences\\_of\\_the\\_group\\_axioms](https://en.wikipedia.org/wiki/Group_(mathematics)#Elementary_consequences_of_the_group_axioms)

**Definition 37.1** (group). group definition by a set and a binary operation on the set

$$\begin{cases}
 \circ : G \times G = G^2 \rightarrow G & \text{binary operation} \\
 \forall \langle g_1, g_2, g_3 \rangle \in G^3 [g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3] & \text{associativity} \\
 \exists e = \in G, \forall g \in G [e \circ g = g] & \text{left unit element} \\
 \forall g \in G, \exists g^{-1} \in G [g^{-1} \circ g = e] & \text{left inverse (element)} \\
 \Leftrightarrow G = (G, \circ) \text{ is a group}
 \end{cases}$$

**Theorem 37.1.** *group left inverses equal right inverses*

Proof:

to be proved

□

### 37.3 Polya enumeration theorem

### 37.4 complex group representation

#### 37.4.1 complex basis group

$$\begin{aligned}
 G &= \{1, i, -1, -i\} \\
 &= \{i^0, i^1, i^2, i^3\}
 \end{aligned}$$

$$\begin{aligned}
 &\forall \langle g_1, g_2 \rangle \in G^2, \exists g_1 g_2 \in G [g_1 g_2 = g_1 \cdot g_2] \\
 \Leftrightarrow &\cdot : G \times G = G^2 \rightarrow G
 \end{aligned}$$

<https://tex.stackexchange.com/questions/627708/tikz-how-to-put-tables-within-arbitrarily-placed-nodes>

.	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

.	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Figure 37.1: complex basis group table

**37.4.2**  $\mathbb{C} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = I$$

$$\begin{aligned} c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ &= xI + yJ, J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R}), \langle x, y \rangle \in \mathbb{R}^2 \end{aligned}$$


---

$$J^2 = -I$$


---

$$J^2 = -I$$

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ca + cd & cb + d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} a^2 + bc = -1 & b = 0 \Rightarrow a^2 = -1 \Leftrightarrow a \in \mathbb{R} \Rightarrow b \neq 0 \\ ab + bd = 0 & (b = 0) \vee (a = -d) \xrightarrow[b \neq 0]{\text{if } a = d} a = -d \\ ca + cd = 0 & \\ cb + d^2 = -1 & a^2 = d^2 \Rightarrow (a = d) \vee (a = -d) \end{cases} \Rightarrow a = d = 0 \Rightarrow bc = -1$$

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} = J(a, b), b \neq 0$$

$$J(a, b) = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix}, b \neq 0$$


---

$$J(a = 1, b) = \begin{pmatrix} 1 & b \\ -2 & -1 \end{pmatrix} \Rightarrow J^2(a = 1, b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$xI + yJ(a = 1, b) = \begin{pmatrix} x + y & yb \\ y \cdot \frac{-2}{b} & x - y \end{pmatrix}$$


---

$$J(a = 0, b) = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}$$


---

$$J(a = 0, b = 1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$


---

$$\begin{aligned} J(a=0, b=-1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J(a=0, b=1) \\ \Rightarrow J^2(a=0, b=-1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \end{aligned}$$

$$\begin{aligned} J = J(a=0, b=-1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \Rightarrow &\begin{cases} 1 \leftrightarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \leftrightarrow J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases} \\ \Rightarrow x + yi &\leftrightarrow xI + yJ \\ = x &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{aligned}$$


---

$$x + yi \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

realizing

$$\mathbb{C} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R}) = \mathcal{M}_2(\mathbb{R})$$

### 37.4.3 ( determinant of complex group representation ) equivalent to ( squared modulus of complex number )

$$\det(xI + yJ) = \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \left| \begin{matrix} x & -y \\ y & x \end{matrix} \right| = x^2 + y^2 = |x + yi|^2$$

#### 37.4.3.1 Lagrange identity

Lagrange identity

generalization of Brahmagupta–Fibonacci identity

specialization of Binet–Cauchy identity

cf. Euler identity<sup>[37.5.1.1]</sup>

$$\begin{aligned} &\det[(aI + bJ)(cI + dJ)] \\ &= \det \left[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \right] \\ &= \det \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix} = |(ac - bd) + (ad + bc)i|^2 = (ac - bd)^2 + (ad + bc)^2 \\ &= \left[ \det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right] \left[ \det \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \right] = |a + bi|^2 |c + di|^2 = (a^2 + b^2)(c^2 + d^2) \end{aligned}$$

$$|a + bi|^2 |c + di|^2 = (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

$$\begin{aligned}
& \det [(x_1 I + y_1 J)(x_2 I + y_2 J)] \\
&= \det \left[ \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \right] \\
&= \det \begin{pmatrix} x_1 x_2 - y_1 y_2 & -x_1 y_2 - y_1 x_2 \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix} = |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i|^2 = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2 \\
&= \left[ \det \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \right] \left[ \det \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \right] = |x_1 + y_1 i|^2 |x_2 + y_2 i|^2 = (x_1^2 + y_1^2) (x_2^2 + y_2^2)
\end{aligned}$$

$$|x_1 + y_1 i|^2 |x_2 + y_2 i|^2 = (x_1^2 + y_1^2) (x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2$$

### 37.4.4 Euler formula proved by complex group representation

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \cos n \frac{\theta}{n} & -\sin n \frac{\theta}{n} \\ \sin n \frac{\theta}{n} & \cos n \frac{\theta}{n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left( \cos \frac{\theta}{n} \quad -\sin \frac{\theta}{n} \right)^n \begin{pmatrix} x \\ y \end{pmatrix} = R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} &= \lim_{n \rightarrow \infty} \begin{pmatrix} \cos \frac{\theta}{n} & -\sin \frac{\theta}{n} \\ \sin \frac{\theta}{n} & \cos \frac{\theta}{n} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n} \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} \\
&= I + \frac{\theta}{n} R_{\frac{\pi}{2}}
\end{aligned}$$


---

$$\lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} = \lim_{n \rightarrow \infty} \begin{pmatrix} \cos \frac{\theta}{n} & -\sin \frac{\theta}{n} \\ \sin \frac{\theta}{n} & \cos \frac{\theta}{n} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} = I + \frac{\theta}{n} R_{\frac{\pi}{2}}$$


---

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \lim_{n \rightarrow \infty} \begin{pmatrix} x' \\ y' \end{pmatrix} = \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix} = \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \right] \left[ \lim_{n \rightarrow \infty} \begin{pmatrix} x \\ y \end{pmatrix} \right] = \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \right] \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n \rightarrow \infty} \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \rightarrow \infty} \left[ I + \frac{\theta}{n} R_{\frac{\pi}{2}} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n \rightarrow \infty} \left[ I + \frac{\theta}{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \rightarrow \infty} \left[ I + \frac{\theta J}{n} \right]^n \begin{pmatrix} x \\ y \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= e^{J\theta} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

$$\begin{aligned} \begin{cases} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} x' \\ y' \end{pmatrix} = e^{J\theta} \begin{pmatrix} x \\ y \end{pmatrix} \end{cases} \\ \Rightarrow e^{J\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \xrightarrow{x+y \leftrightarrow x-y} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ \\ e^{i\theta} = \cos \theta + i \sin \theta \end{aligned}$$

□

## 37.5 quaternion group representation

[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Two-dimensional\\_irreducible\\_representation\\_over\\_a\\_splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Two-dimensional_irreducible_representation_over_a_splitting_field)

$$\begin{aligned} q = a + bi + cj + dk &= a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ w = t + xi + yj + zk &= t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\ a1 + bi + cj + dk &= t1 + xi + yj + zk = x_0 1 + e_i x_i \end{aligned}$$

### 37.5.1 $\mathbb{H} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C})$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_2 = 1$$

$$\begin{aligned} e = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ -a^2 - 1 & -a \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} a & b \\ -a^2 - 1 & -a \end{pmatrix} = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \Rightarrow e_2 = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = j & a = 0 \\ \begin{pmatrix} a & b \\ -a^2 - 1 & -a \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} & a \neq 0 \end{cases} \end{aligned}$$

$$\begin{pmatrix} \alpha^2 + \beta^2 & 0 \\ 0 & \beta^2 + \alpha^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = e^2 = -1 = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Downarrow$

$$\alpha^2 + \beta^2 = -1 \Leftrightarrow \beta^2 + \alpha^2 = -1$$

$$\alpha^2 + \beta^2 = -1 \Rightarrow \langle \alpha, \beta \rangle \notin \mathbb{R}^2 \Rightarrow \langle \alpha, \beta \rangle \in \mathbb{R}^2 \Rightarrow \alpha^2 + \beta^2 \geq 0$$

quaternion group has no irreducible two-dimensional representation over the reals <sup>1</sup>

<sup>1</sup>[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Four-dimensional\\_irreducible\\_representation\\_over\\_a\\_non-splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field)

$$\langle \alpha, \beta \rangle \in \mathbb{C}^2 - \mathbb{R}^2$$

$$\alpha^2 + \beta^2 = -1 = \beta^2 + \alpha^2$$

$$\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} \begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} \beta^2 + \alpha^2 & 0 \\ 0 & \alpha^2 + \beta^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$ij = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} = k$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, k = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

$$jk = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = i$$


---

	$\alpha^2 + \beta^2 = -1$	$\begin{cases} \alpha = \sqrt{-1} \\ \beta = 0 \end{cases}$	$\begin{cases} \alpha = \sqrt{-2} \\ \beta = 1 \end{cases}$	$\beta = \alpha^2, n \in \{1, 2, 4, 5\}$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
-1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
i	$\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \sqrt{-2} & 1 \\ 1 & -\sqrt{-2} \end{pmatrix}$	$\begin{pmatrix} e^{\pi \frac{n}{3}\sqrt{-1}} & e^{\pi \frac{2n}{3}\sqrt{-1}} \\ e^{\pi \frac{2n}{3}\sqrt{-1}} & -e^{\pi \frac{n}{3}\sqrt{-1}} \end{pmatrix}$
-i	$\begin{pmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$	$\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} -\sqrt{-2} & -1 \\ -1 & \sqrt{-2} \end{pmatrix}$	$\begin{pmatrix} -e^{\pi \frac{n}{3}\sqrt{-1}} & -e^{\pi \frac{2n}{3}\sqrt{-1}} \\ -e^{\pi \frac{2n}{3}\sqrt{-1}} & e^{\pi \frac{n}{3}\sqrt{-1}} \end{pmatrix}$
j	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
-j	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
k	$\begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -\sqrt{-2} \\ -\sqrt{-2} & -1 \end{pmatrix}$	$\begin{pmatrix} e^{\pi \frac{2n}{3}\sqrt{-1}} & -e^{\pi \frac{n}{3}\sqrt{-1}} \\ -e^{\pi \frac{n}{3}\sqrt{-1}} & -e^{\pi \frac{2n}{3}\sqrt{-1}} \end{pmatrix}$
-k	$\begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & \sqrt{-2} \\ \sqrt{-2} & 1 \end{pmatrix}$	$\begin{pmatrix} -e^{\pi \frac{2n}{3}\sqrt{-1}} & e^{\pi \frac{n}{3}\sqrt{-1}} \\ e^{\pi \frac{n}{3}\sqrt{-1}} & e^{\pi \frac{2n}{3}\sqrt{-1}} \end{pmatrix}$

---

$$\begin{aligned}
-1 &= \alpha^2 + \beta^2 \\
&\stackrel{\beta=\alpha^2}{=} \alpha^2 + \alpha^4 \\
\alpha^4 + \alpha^2 + 1 &= 0, \alpha^4 + \alpha^2 + 1 = (\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1) \\
(\alpha^2 - 1)(\alpha^4 + \alpha^2 + 1) &= 0 \\
\alpha^6 - 1 &= 0 \\
\alpha^6 &= 1 = e^{2\pi k \sqrt{-1}}, k \in \mathbb{Z} \\
\alpha &= e^{2\pi \frac{n}{6} \sqrt{-1}}, n \in \{0, 1, 2, 3, 4, 5\} - \{0, 3\} \\
&= e^{\pi \frac{n}{3} \sqrt{-1}}, n \in \{1, 2, 4, 5\}
\end{aligned}$$


---

$$\begin{array}{llll}
\alpha^2 + \beta^2 = -1 & \begin{cases} \alpha = i \\ \beta = 0 \end{cases} & \begin{cases} \alpha = \sqrt{2}i \\ \beta = 1 \end{cases} & \omega = e^{i\pi \frac{n}{3}}, n \in \{1, 2, 4, 5\} \\
1 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
-1 & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
i & \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} \sqrt{2}i & 1 \\ 1 & -\sqrt{2}i \end{pmatrix} \\
-i & \begin{pmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -\sqrt{2}i & -1 \\ -1 & \sqrt{2}i \end{pmatrix} \\
j & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
-j & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
k & \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 1 & -\sqrt{2}i \\ -\sqrt{2}i & -1 \end{pmatrix} \\
-k & \begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -1 & \sqrt{2}i \\ \sqrt{2}i & 1 \end{pmatrix}
\end{array}$$

realizing

$$\mathbb{H} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C}) = \mathcal{M}_2(\mathbb{C})$$

### 37.5.1.1 Euler identity

cf. Lagrange identity<sup>[37.4.3.1]</sup>

$$\begin{aligned}
&\det(a + bi + cj + dk) \\
&= \det \left[ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\
&= \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} = \begin{vmatrix} a + bi & -c - di \\ c - di & a - bi \end{vmatrix} \\
&= (a^2 + b^2) + (c^2 + d^2) = a^2 + b^2 + c^2 + d^2
\end{aligned}$$

$$\det(a + bi + cj + dk) = \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} = a^2 + b^2 + c^2 + d^2$$

$$\begin{aligned}
& \det [(q_{10} + q_{11}\mathbf{i} + q_{12}\mathbf{j} + q_{13}\mathbf{k})(q_{20} + q_{21}\mathbf{i} + q_{22}\mathbf{j} + q_{23}\mathbf{k})] = \det [(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k})] \\
&= \det \left\{ \left[ q_{10} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_{11} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + q_{12} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + q_{13} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right. \\
&\quad \left. \left[ q_{20} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_{21} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + q_{22} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + q_{23} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right\} \\
&= \det \left\{ \left[ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right. \\
&\quad \left. \left[ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \gamma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right\} \\
&= \det \left\{ \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} \begin{pmatrix} \alpha + \beta i & -\gamma - \delta i \\ \gamma - \delta i & \alpha - \beta i \end{pmatrix} \right\} \\
&= \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} \det \begin{pmatrix} \alpha + \beta i & -\gamma - \delta i \\ \gamma - \delta i & \alpha - \beta i \end{pmatrix} = (a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\
&= \det \left\{ \begin{pmatrix} [a + bi][\alpha + \beta i] - [c + di][\gamma - \delta i] & [a + bi][- \gamma - \delta i] - [c + di][\alpha - \beta i] \\ [c - di][\alpha + \beta i] + [a - bi][\gamma - \delta i] & [c - di][- \gamma - \delta i] + [a - bi][\alpha - \beta i] \end{pmatrix} \right\} \\
&= \det \left\{ \begin{pmatrix} (a\alpha - b\beta - c\gamma - d\delta) + i(a\beta + b\alpha + c\delta - d\gamma) & -(a\gamma - b\delta + c\alpha + d\beta) - i(a\delta + b\gamma - c\beta + d\alpha) \\ (a\gamma - b\delta + c\alpha + d\beta) - i(a\delta + b\gamma - c\beta + d\alpha) & (a\alpha - b\beta - c\gamma - d\delta) - i(a\beta + b\alpha + c\delta - d\gamma) \end{pmatrix} \right\} \\
&= \det \{(a\alpha - b\beta - c\gamma - d\delta) + (a\beta + b\alpha + c\delta - d\gamma)\mathbf{i} + (a\gamma - b\delta + c\alpha + d\beta)\mathbf{j} + (a\delta + b\gamma - c\beta + d\alpha)\mathbf{k}\} \\
&= (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2 \\
&= (q_{10}q_{20} - q_{11}q_{21} - q_{12}q_{22} - q_{13}q_{23})^2 + (q_{10}q_{21} + q_{11}q_{20} + q_{12}q_{23} - q_{13}q_{22})^2 \\
&\quad + (q_{10}q_{22} - q_{11}q_{23} + q_{12}q_{20} + q_{13}q_{21})^2 + (q_{10}q_{23} + q_{11}q_{22} - q_{12}q_{21} + q_{13}q_{20})^2 \\
&\quad (a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\
&\quad = (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 \\
&\quad \quad + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2
\end{aligned}$$

**Theorem 37.2.** For any two integers greater than zero, their multiplication can be the summation of squared four integers greater than zero.

$$\forall \langle m_1, m_2 \rangle \in (\mathbb{N} \cup \{0\})^2, \exists \langle k_1, k_2, k_3, k_4 \rangle \in (\mathbb{N} \cup \{0\})^4 [m_1 m_2 = k_1^2 + k_2^2 + k_3^2 + k_4^2]$$

Proof:

$$\text{Let } \begin{cases} m_1 = a^2 + b^2 + c^2 + d^2 & \langle a, b, c, d \rangle \in (\mathbb{N} \cup \{0\})^4 \\ m_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & \langle \alpha, \beta, \gamma, \delta \rangle \in (\mathbb{N} \cup \{0\})^4 \end{cases} \xrightarrow{\text{closure property}} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 + d^2 \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{pmatrix} \in (\mathbb{N} \cup \{0\})^2,$$

$$\begin{aligned}
m_1 m_2 &= (a^2 + b^2 + c^2 + d^2) (a^2 + \beta^2 + \gamma^2 + \delta^2) \\
&\stackrel{\text{Euler identity}}{=} (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 \\
&\quad + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2 \\
&= |a\alpha - b\beta - c\gamma - d\delta|^2 \\
&\quad + |a\beta + b\alpha + c\delta - d\gamma|^2 \\
&\quad + |a\gamma - b\delta + c\alpha + d\beta|^2 \\
&\quad + |a\delta + b\gamma - c\beta + d\alpha|^2 \\
&= k_1^2 + k_2^2 + k_3^2 + k_4^2, \quad \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} |a\alpha - b\beta - c\gamma - d\delta| \\ |a\beta + b\alpha + c\delta - d\gamma| \\ |a\gamma - b\delta + c\alpha + d\beta| \\ |a\delta + b\gamma - c\beta + d\alpha| \end{pmatrix} \in (\mathbb{N} \cup \{0\})^4 \\
&\therefore \begin{cases} \langle a, b, c, d \rangle \in (\mathbb{N} \cup \{0\})^4 \\ \langle \alpha, \beta, \gamma, \delta \rangle \in (\mathbb{N} \cup \{0\})^4 \end{cases} \\
&\xrightarrow{\text{closure property}} \begin{pmatrix} |a\alpha - b\beta - c\gamma - d\delta| \\ |a\beta + b\alpha + c\delta - d\gamma| \\ |a\gamma - b\delta + c\alpha + d\beta| \\ |a\delta + b\gamma - c\beta + d\alpha| \end{pmatrix} \in (\mathbb{N} \cup \{0\})^4
\end{aligned}$$

□

### 37.5.2 $\mathbb{H} \rightarrow \mathcal{M}_{4 \times 4}(\mathbb{R})$

[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Four-dimensional\\_irreducible\\_representation\\_over\\_a\\_non-splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field)

$$\mathbb{H} \rightarrow \mathcal{M}_2(\mathbb{C}) \xrightarrow{\mathbb{C} \rightarrow \mathcal{M}_2(\mathbb{R})} \mathcal{M}_{4 \times 4}(\mathbb{R}) = \mathcal{M}_4(\mathbb{R})$$

$$\mathbb{C} \rightarrow \mathcal{M}_2(\mathbb{R}) \Leftarrow \begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$$


---

$$1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$-1 \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



	$\begin{cases} \alpha = i \\ \beta = 0 \end{cases}$	$\begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$	$\begin{cases} \alpha = \sqrt{2}i \\ \beta = 1 \end{cases}$	$\begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
-1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
i	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \sqrt{2}i & 1 \\ 1 & -\sqrt{2}i \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \\ 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix}$
-i	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\sqrt{2}i & -1 \\ -1 & \sqrt{2}i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$
j	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
-j	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
k	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -\sqrt{2}i \\ -\sqrt{2}i & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ -\sqrt{2} & 0 & 0 & -1 \end{pmatrix}$
-k	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & \sqrt{2}i \\ \sqrt{2}i & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \end{pmatrix}$

some examinations

$$ij \rightarrow \begin{pmatrix} 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \\ 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ -\sqrt{2} & 0 & 0 & -1 \end{pmatrix} \leftarrow k$$

# Chapter 38

## tensor

### 38.1 Einstein summation convention

#### 38.1.1 dummy index

#### 38.1.2 free index

### 38.2 Elliot Schneider

- Elliot Schneider: Physics with Elliot

#### 38.2.1 Fundamentals of Cartesian Tensors

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2176662131>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076341>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076344>

#### 38.2.2 Fundamentals of Curvilinear Tensors

#### 38.2.3 Fundamentals of Spacetime Tensors



# Chapter 39

## dual space

dual space and linear functional

<https://ccjou.wordpress.com/2011/06/13/%E7%94%9F%E6%84%85%E6%96%AF%E4%BD%A0%E5%8D%95/>

### 39.1 linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \end{aligned}$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (b_i) = b_i$$

$$(a_{i1} \quad \cdots \quad a_{ij} \quad \cdots \quad a_{in}) \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = (b_i) = b_i$$

$$\begin{pmatrix} \cdots & a_{ij} & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix} = b_i$$

$$\mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$

$$a_{ij}x_j = \mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$


---

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots$$

if finite,

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

## 39.2 matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

$$AX = B$$

$$(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}) \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix} = (b_{ik}) = b_{ik}$$

$$(a_{i1} \quad \cdots \quad a_{ij} \quad \cdots \quad a_{in}) \begin{pmatrix} x_{1k} \\ \vdots \\ x_{jk} \\ \vdots \\ x_{nk} \end{pmatrix} = (b_{ik}) = b_{ik}$$

$$(\cdots \quad a_{ij} \quad \cdots) \begin{pmatrix} \vdots \\ x_{jk} \\ \vdots \end{pmatrix} = b_{ik}$$

$$a_{ij}x_j = \mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j$$

\iddots in MathJax

<https://math.meta.stackexchange.com/questions/23273/mathjax-and-iddots-udots-or-reflectbox>

$$b_{ik} = a_{ij}x_{jk} = \mathbf{a}_i \cdot \mathbf{x}_k = \mathbf{a}_i^\top \mathbf{x}_k$$

$$\text{row}(A) \text{ col}(X) = b_{\text{row}, \text{col}}$$

$$\mathbf{a}_i \cdot \mathbf{x}_k = \mathbf{a}_i^\top \mathbf{x}_k = a_{ij}x_{jk}$$

---


$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj}$$

$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = \cdots + a_{ik}x_{kj} + \cdots$$

if finite,

$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = a_{i1}x_{1j} + \cdots + a_{ik}x_{kj} + \cdots + a_{in}x_{nj}$$

### 39.3 functional

( inner product or dot product ) or linear equations

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots$$

if finite,

---


$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

actually, several rows

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j & = & \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j & = & a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

in functional aspect,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$


---

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\cdots, x_j, \cdots) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \cdots, x_j, \cdots, x_n) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$


---

or simply

$$f_i(\cdots, x_j, \cdots) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = \cdots + a_{ij}x_j + \cdots$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^\infty \rightarrow \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^\infty \rightarrow \mathbb{F}$$

or more abstract notation,

$$f_i : F^\infty \rightarrow F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^\infty \rightarrow \mathbb{R}$$

if finite,

$$f_i(x_1, \dots, x_j, \dots, x_n) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ij}x_j = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n$$

$$\begin{aligned} f_i(x_1, x_2, \dots, x_n) &= f_i(x_1, \dots, x_j, \dots, x_n) = f_i(x_j) = f_i(\mathbf{x}) \\ &= \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ij}x_j = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n \end{aligned}$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^n \rightarrow \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^n \rightarrow \mathbb{F}$$

or more abstract notation,

$$f_i : F^n \rightarrow F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^n \rightarrow \mathbb{R}$$

functionals are a set of functions mapping  $n$ -dimensional vectors to scalars.

$$f_i : F^n \rightarrow F$$

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\cdots, x_j, \cdots) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j = \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^\infty \rightarrow F\} = \left\{ \begin{array}{c} f_i(\dots, x_j, \dots) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij} x_j, \\ \vdots \\ \vdots \end{array} \right\}$$

if finite,

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \dots, x_j, \dots, x_n) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^n \rightarrow F\} = \left\{ \begin{array}{c} \vdots \\ f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n, \\ \vdots \end{array} \right\}$$


---

linear functionals are a set of functions mapping vectors to scalars linearly

$$f = \{f_i | f_i : F^n \rightarrow F\} = \left\{ \begin{array}{c} \vdots \\ f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n, \\ \vdots \end{array} \right\}$$

## 39.4 definition of linear functional

functionals generalized to general vector space

$$f = \{f_i | f_i : V \rightarrow F\}$$

linear functionals generalized to general vector space is a linear transformation

$$f = \left\{ f_i \left| \begin{array}{l} f_i : V \rightarrow F \\ \forall \langle \mathbf{x}, \mathbf{y} \rangle \in V^2 [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : V \rightarrow F \\ \forall \mathbf{x}, \mathbf{y} \in V [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$\text{if } \begin{cases} F = \mathbb{C} \\ V = \mathbb{C}^n \end{cases},$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : \mathbb{C}^n \rightarrow \mathbb{C} \\ \forall \langle \mathbf{x}, \mathbf{y} \rangle \in (\mathbb{C}^n)^2 [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : \mathbb{C}^n \rightarrow \mathbb{C} \\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

then

$$f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n$$

satisfying

$$\begin{aligned}
f_i(\mathbf{x} + \mathbf{y}) &= a_{i1}(x+y)_1 + \cdots + a_{ij}(x+y)_j + \cdots + a_{in}(x+y)_n \\
&= a_{i1}(x_1+y_1) + \cdots + a_{ij}(x_j+y_j) + \cdots + a_{in}(x_n+y_n) \\
&= (a_{i1}x_1 + a_{i1}y_1) + \cdots + (a_{ij}x_j + a_{ij}y_j) + \cdots + (a_{in}x_n + a_{in}y_n) \\
&= (a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n) + (a_{i1}y_1 + \cdots + a_{ij}y_j + \cdots + a_{in}y_n) \\
&= f_i(\mathbf{x}) + f_i(\mathbf{y})
\end{aligned}$$

$$\begin{aligned}
f_i(c\mathbf{x}) &= a_{i1}(cx)_1 + \cdots + a_{ij}(cx)_j + \cdots + a_{in}(cx)_n \\
&= a_{i1}(cx_1) + \cdots + a_{ij}(cx_j) + \cdots + a_{in}(cx_n) \\
&= c(a_{i1}x_1) + \cdots + c(a_{ij}x_j) + \cdots + c(a_{in}x_n) \\
&= c(a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n) \\
&= cf_i(\mathbf{x})
\end{aligned}$$


---

different functional has different  $a_{ij}$

let

$$a_{ij} = f_i(\mathbf{e}_j), \mathbf{e}_j = \left\langle \overbrace{0, \dots, 0}^{j-1}, 1, 0, \dots, 0 \right\rangle = (0 \ \cdots \ 0 \ \ 1 \ \ 0 \ \ \cdots \ \ 0)^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

if  $f_i$  is a linear functional, then

$$\begin{aligned}
f_i(\mathbf{x}) &= f_i(x_1\mathbf{e}_1 + \cdots + x_j\mathbf{e}_j + \cdots + x_n\mathbf{e}_n) \\
&= f_i(x_1\mathbf{e}_1) + \cdots + f_i(x_j\mathbf{e}_j) + \cdots + f_i(x_n\mathbf{e}_n) \\
&= x_1f_i(\mathbf{e}_1) + \cdots + x_jf_i(\mathbf{e}_j) + \cdots + x_nf_i(\mathbf{e}_n) \\
&= x_1a_{i1} + \cdots + x_ja_{ij} + \cdots + x_na_{in} \\
&= a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\
&= f_i(\mathbf{x})
\end{aligned}$$

## 39.5 set of all linear transformations is a vector space

<https://ccjou.wordpress.com/2011/04/08/%E7%94%A8%E6%88%80%E6%95%99%E7%9A%84%E5%8F%8B%E8%80%85/>

vector space<sup>[40]</sup>

<https://math.stackexchange.com/questions/2381942/the-set-of-all-linear-maps-tv-w-is-a-vector-space>

$$T : V \rightarrow W \Leftrightarrow \forall \mathbf{v} \in V, \exists! \mathbf{w} \in W [\mathbf{w} = T(\mathbf{v})]$$

$$\left\{ \begin{array}{l} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \left\{ \begin{array}{l} \forall \mathbf{u}, \mathbf{v} \in V [T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})] \\ \forall \mathbf{v} \in V, c \in F [T(c\mathbf{v}) = cT(\mathbf{v})] \end{array} \right. \end{array} \right. \quad \text{linearity}$$

$\Leftrightarrow T$  is a linear transformation

$$\left\{ \begin{array}{l} V, W \text{ are vector spaces, both over } F \\ T, U : V \rightarrow W \\ \left\{ \begin{array}{l} T : V \rightarrow W \\ U : V \rightarrow W \end{array} \right. \\ T, U \text{ are both linear transformations} \\ \mathbf{v} \in V \\ c \in F \end{array} \right.$$

There is still linearity over linear transformations

(va)

$$\begin{aligned} (T + U)(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u} + \mathbf{v}) + U(\mathbf{u} + \mathbf{v}) \\ &= [T(\mathbf{u}) + T(\mathbf{v})] + [U(\mathbf{u}) + U(\mathbf{v})] \\ &= [T(\mathbf{u}) + U(\mathbf{u})] + [T(\mathbf{v}) + U(\mathbf{v})] \\ &= (T + U)(\mathbf{u}) + (T + U)(\mathbf{v}) \end{aligned}$$

(sm)

$$\begin{aligned} (T + U)(c\mathbf{v}) &= T(c\mathbf{v}) + U(c\mathbf{v}) \\ &= cT(\mathbf{v}) + cU(\mathbf{v}) \\ &= c[T(\mathbf{v}) + U(\mathbf{v})] \\ &= c(T + U)(\mathbf{v}) \end{aligned}$$

so we can define

---


$$\left\{ \begin{array}{ll} (T + U)(\mathbf{v}) = T(\mathbf{v}) + U(\mathbf{v}) & \text{linear transformation addition} \\ (cT)(\mathbf{v}) = cT(\mathbf{v}) & \text{scalar linear transformation multiplication} \end{array} \right.$$

the set of all linear tranformations is a vector space

$\mathcal{T}$  is the set of all linear tranformations

$$\left\{ \begin{array}{ll}
 F & (f) F \text{ is a field} \\
 \mathcal{T} \neq \emptyset & (ne) \text{ nonempty set} \\
 + : \mathcal{T} \times \mathcal{T} = \mathcal{T}^2 \xrightarrow{\cdot} \mathcal{T} \Leftrightarrow \forall T, U \in \mathcal{T}, \exists S \in \mathcal{T} [S = T + U] & (va) \text{ vector addition} \\
 \cdot : F \times \mathcal{T} \xrightarrow{\cdot} \mathcal{T} \Leftrightarrow \forall c \in F, \forall T \in \mathcal{T}, \exists U \in \mathcal{T} [U = cT = c \cdot T] & (sm) \text{ scalar multiplication} \\
 \left\{ \begin{array}{ll}
 \forall S, T, U \in \mathcal{T} [S + (T + U) = (S + T) + U] & (a) \\
 \forall T, U \in \mathcal{T} [T + U = U + T] & (c) \\
 \exists! O \in \mathcal{T}, \forall T \in \mathcal{T} [O + T = T] & (e) \\
 \forall T \in \mathcal{T}, \exists! -T \in \mathcal{T} [(-T) + T = O] & (i) \\
 \forall b, c \in F, T \in \mathcal{T} [b(cT) = (bc)T] & (a) \\
 \exists! 1 \in F, \forall T \in \mathcal{T} [1T = T] & (e) \\
 \forall c \in F, T, U \in \mathcal{T} [c(T + U) = cT + cU] & (dv) \\
 \forall b, c \in F, T \in \mathcal{T} [(b + c)T = bT + cT] & (ds)
 \end{array} \right. & \begin{array}{l}
 (va) \text{ vector addition axioms} \\
 (sm) \text{ scalar multiplication axioms}
 \end{array}
 \end{array} \right.$$

$\Leftrightarrow \mathcal{T} = \mathcal{T}(F, +, \cdot) = (\mathcal{T}, F, +, \cdot)$  is a vector space over the field  $F$

$\Leftrightarrow \mathcal{T}$  is a vector space

Selected proofs of 8 vector space axioms due to some trivial field and vector space properties:

(va) (a)

$$\begin{aligned}
 (S + (T + U))(\mathbf{v}) &= S(\mathbf{v}) + (T + U)(\mathbf{v}) \\
 &= S(\mathbf{v}) + T(\mathbf{v}) + U(\mathbf{v}) \\
 &= (S + T)(\mathbf{v}) + U(\mathbf{v}) \\
 &= ((S + T) + U)(\mathbf{v})
 \end{aligned}$$

(va) (c)

$$\begin{aligned}
 (T + U)(\mathbf{v}) &= T(\mathbf{v}) + U(\mathbf{v}) \\
 &= U(\mathbf{v}) + T(\mathbf{v}) \\
 &= (U + T)(\mathbf{v})
 \end{aligned}$$

(va) (e)

$$O(\mathbf{v}) = 0\mathbf{w} \in W$$

$$\begin{aligned}
 (O + T)(\mathbf{v}) &= O(\mathbf{v}) + T(\mathbf{v}) \\
 &= 0\mathbf{w} + T(\mathbf{v}) \\
 &= T(\mathbf{v})
 \end{aligned}$$

$$O_1(\mathbf{v}) - O_2(\mathbf{v}) = 0\mathbf{w} - 0\mathbf{w} = 0\mathbf{w} \Rightarrow O_1(\mathbf{v}) = O_2(\mathbf{v})$$

(sm) (dv)

$$\begin{aligned}
 (c(T+U))(\mathbf{v}) &= c(T+U)(\mathbf{v}) \\
 &= c[T(\mathbf{v}) + U(\mathbf{v})] \\
 &= cT(\mathbf{v}) + cU(\mathbf{v}) \\
 &= (cT + cU)(\mathbf{v})
 \end{aligned}$$

The set of all linear tranformations  $\mathcal{T}$  is a vector space.

□

## 39.6 definition of dual space

$$V^* = L(V, F)$$

$$= f = \left\{ f_i \left| \begin{array}{l} \left\{ \begin{array}{l} f_i : V \rightarrow F \\ \forall \mathbf{x}, \mathbf{y} \in V [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(cx) = cf_i(\mathbf{x})] \end{array} \right\} \\ \text{(L) linearity} \end{array} \right. \right\}$$

functional mapping vector to field scalar

$\Leftrightarrow V^*$  is a dual space, a set of linear functionals  $f_i$  mapping vectors in the vector space  $V$  to scalars in the field  $F$

vector space<sup>[40]</sup>

<https://web.math.sinica.edu.tw/mathmedia/HTMLArticle18.jsp?mID=31304>

[https://web.math.sinica.edu.tw/mathmedia/author18.jsp?query\\_filter=%E9%BE%94%E6%98%87](https://web.math.sinica.edu.tw/mathmedia/author18.jsp?query_filter=%E9%BE%94%E6%98%87)

## 39.7 double dual

double dual = second dual

<https://ccjou.wordpress.com/2014/04/10/%E9%BE%94%E6%98%87/>

# Chapter 40

## vector space

<https://ccjou.wordpress.com/2010/04/15/%E5%90%8C%E6%9E%9A%E7%9F%A5%E9%97%A8/>

### 40.1 What is a vector?

What is a vector? or What is an element in a vector space?

Binary operations defined on a vector space satisfying some properties is more important than what is a vector.

ultimate answer: double dual concept<sup>[40.4.1.2]</sup>

### 40.2 vector space definition

<https://tex.stackexchange.com/a/141489> multiline node

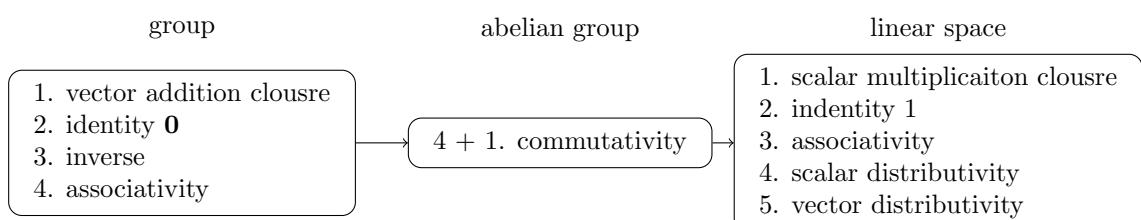


Figure 40.1: vector space construction

$$\left\{ \begin{array}{ll} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ + : V \times V = V^2 \xrightarrow{\perp} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (va) \text{ vector addition} \\ \cdot : F \times V \rightarrow V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \left\{ \begin{array}{ll} \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{array} \right. & (va) \text{ axioms} \\ \left\{ \begin{array}{ll} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{array} \right. & (sm) \text{ axioms} \end{array} \right.$$

$\Leftrightarrow V = V(F, +, \cdot) = (V, F, +, \cdot)$  is a vector space over the field  $F$

$\Leftrightarrow V$  is a vector space

### 40.2.1 commutative group structure of vector space

(va) axioms = vector addition axioms

$V = (V, +)$  is a commutative group  $\Leftrightarrow V = (V, +)$  is an abelian group

$$\Leftrightarrow \left\{ \begin{array}{ll} V = (V, +) = (V, +_V) \text{ is a group} & (g) \text{ group} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{ll} + : V \times V = V^2 \xrightarrow{\perp} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (cl) \text{ closure} \\ \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \end{array} \right. \quad (g) \quad (c)$$

$V$  is a vector space

$\Leftrightarrow V = V(F, +, \cdot) = (V, F, +, \cdot)$  is a vector space over the field  $F$

$$\Leftrightarrow \begin{cases} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ V = (V, +) \text{ is a commutative group} \Leftrightarrow V = (V, +) \text{ is an abelian group} & (va) \text{ vector addition} \\ \cdot : F \times V \rightarrow V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \begin{cases} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{cases} & (sm) \text{ axioms} \end{cases}$$
  
 $\Leftrightarrow \begin{cases} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ \begin{cases} V = (V, +) = (V, +_V) \text{ is a group} & (g) \text{ group} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{cases} & (va) \text{ vector addition} \\ \cdot = \cdot_{F \times V} : F \times V \rightarrow V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \begin{cases} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{cases} & (sm) \text{ axioms} \end{cases}$ 
  
 $\Leftrightarrow \begin{cases} F = F(+_F, \cdot_F) = (F, +_F, \cdot_F) = (F, +, \cdot) \text{ is a field} & (f) \\ V \neq \emptyset & (ne) \\ \begin{cases} + : V \times V = V^2 \xrightarrow{\perp} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (cl) \text{ closure} \\ \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \end{cases} & (va) \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \\ \cdot : F \times V \rightarrow V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (cl) \text{ closure} \\ \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = s \cdot_{F \times V} (t \cdot_{F \times V} \mathbf{v}) = (s \cdot_F t) \cdot_{F \times V} \mathbf{v} = (st)\mathbf{v}] & (a) \text{ associativity} & (sm) \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = (s +_F t)\mathbf{v} = s\mathbf{v} +_V t\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{cases}$

## 40.2.2 scalar distributivity

(sm) (ds)

$$\forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}]$$

$$\forall s, t \in F, \mathbf{v} \in V [(s +_F t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}]$$

$$\forall s, t \in F, \mathbf{v} \in V [(s +_F t)\mathbf{v} = s\mathbf{v} +_V t\mathbf{v}]$$

## 40.3 linearity

$$\begin{aligned} & \begin{cases} f(x+y) = f(x) + f(y) & \text{additivity} \\ f(\lambda x) = \lambda f(x) & \text{homogeneity} \end{cases} \\ \Leftrightarrow & f(\lambda x + y) = \lambda f(x) + f(y) \\ \Leftrightarrow & f \text{ is linear} \end{aligned}$$

### 40.3.1 linear structure of vector space

$$\forall s \in F, \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\forall s \in F, \forall \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\forall s \in F, \langle \mathbf{u}, \mathbf{v} \rangle \in V^2 [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\begin{aligned} & \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + \mathbf{v} \in V ] & \text{vector addition closure} \\ \forall s \in F, \mathbf{v} \in V [ s\mathbf{v} \in V ] & \text{scalar multiplication closure} \end{cases} \\ \Leftrightarrow & \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + \mathbf{v} \in V ] & (a) \text{ additivity} \\ \forall s \in F, \mathbf{v} \in V [ s\mathbf{v} \in V ] & (h) \text{ homogeneity} \end{cases} \\ \Leftrightarrow & \forall s \in F, \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ] \quad (l) \text{ linearity} \end{aligned}$$

### 40.3.2 linear transformation or linear map

$$\begin{aligned} & \begin{cases} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) ] & (a) \text{ additivity} \\ \forall \mathbf{v} \in V, c \in F [ T(c\mathbf{v}) = cT(\mathbf{v}) ] & (h) \text{ homogeneity} \end{cases} \end{cases} \quad (L) \\ \Leftrightarrow & \begin{cases} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \forall \mathbf{u}, \mathbf{v} \in V, c \in F [ T(\mathbf{u} + c\mathbf{v}) = T(\mathbf{u}) + cT(\mathbf{v}) ] \quad (l) \text{ linearity} \end{cases} \\ \Leftrightarrow & T \text{ is a linear map from } V \text{ to } W \\ \Leftrightarrow & T \text{ is a linear transformation} \end{aligned}$$

## 40.4 vector space example

- arrow vector
- number
  - integer
  - real
  - complex
  - quaternion
- function
  - polynomial function
  - continuous function

- matrix
  - real matrix
  - complex matrix
- reciprocal space

applications in different disciplines

- math
  - recursive number series
  - Fourier series
- physics
  - electrical circuit: linear response / [superposition theorem](#) in [linear circuit](#) / linear network
- chemistry
  - [balancing chemical equation](#)

#### 40.4.1 reciprocal space

reciprocal space = 倒易空間

$$\begin{cases} \mathbf{e}_1 = \mathbf{a} & \mathbf{a} \times \mathbf{b} \neq \mathbf{0} \\ \mathbf{e}_2 = \mathbf{b} & \mathbf{b} \times \mathbf{c} \neq \mathbf{0} \\ \mathbf{e}_3 = \mathbf{c} & \mathbf{c} \times \mathbf{a} \neq \mathbf{0} \end{cases} \Rightarrow \begin{cases} \mathbf{e}'_1 = \frac{\mathbf{b} \times \mathbf{c}}{\Omega} \\ \mathbf{e}'_2 = \frac{\mathbf{c} \times \mathbf{a}}{\Omega} \\ \mathbf{e}'_3 = \frac{\mathbf{a} \times \mathbf{b}}{\Omega} \end{cases},$$

$$\Omega = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

reciprocal space as dual space and contravariant vector

$$\begin{aligned} \text{span} \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} &= \text{span} \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} = V \\ &= \mathbb{R}^3 = \text{span} \{ \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 \} = \text{span} \left\{ \frac{\mathbf{b} \times \mathbf{c}}{\Omega}, \frac{\mathbf{c} \times \mathbf{a}}{\Omega}, \frac{\mathbf{a} \times \mathbf{b}}{\Omega} \right\} \\ &= \text{span} \{ \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \} = \text{span} \{ \mathbf{e}^* \}_{* \in \{1, 2, 3\}} = V^* \end{aligned}$$

##### 40.4.1.1 Kronecker delta

$$\begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{pmatrix} = [\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}^1 \cdot \mathbf{e}_1 & \mathbf{e}^1 \cdot \mathbf{e}_2 & \mathbf{e}^1 \cdot \mathbf{e}_3 \\ \mathbf{e}^2 \cdot \mathbf{e}_1 & \mathbf{e}^2 \cdot \mathbf{e}_2 & \mathbf{e}^2 \cdot \mathbf{e}_3 \\ \mathbf{e}^3 \cdot \mathbf{e}_1 & \mathbf{e}^3 \cdot \mathbf{e}_2 & \mathbf{e}^3 \cdot \mathbf{e}_3 \end{pmatrix}$$

Kronecker delta

$$\mathbf{e}_i \cdot \mathbf{e}'_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker delta tensor = Kronecker tensor

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\mathbf{v} = v_a \mathbf{a} + v_b \mathbf{b} + v_c \mathbf{c} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$

$$\mathbf{e}^1 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_1 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_1 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_1 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_1 = v_1$$

$$\mathbf{e}^2 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_2 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_2 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_2 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_2 = v_2$$

$$\mathbf{e}^3 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_3 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_3 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_3 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_3 = v_3$$

$$\begin{aligned}\mathbf{v} &= v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \\ &= (\mathbf{v} \cdot \mathbf{e}'_1) \mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}'_2) \mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}'_3) \mathbf{e}_3 \\ &= (\mathbf{e}^1 \cdot \mathbf{v}) \mathbf{e}_1 + (\mathbf{e}^2 \cdot \mathbf{v}) \mathbf{e}_2 + (\mathbf{e}^3 \cdot \mathbf{v}) \mathbf{e}_3 \\ &= \mathbf{e}^1(\mathbf{v}) \mathbf{e}_1 + \mathbf{e}^2(\mathbf{v}) \mathbf{e}_2 + \mathbf{e}^3(\mathbf{v}) \mathbf{e}_3\end{aligned}$$

$$\begin{cases} \mathbf{e}^1(\mathbf{v}) = \mathbf{e}^1 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_1 = v_1 \\ \mathbf{e}^2(\mathbf{v}) = \mathbf{e}^2 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_2 = v_2 \\ \mathbf{e}^3(\mathbf{v}) = \mathbf{e}^3 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_3 = v_3 \end{cases}$$

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i = \delta_{j,i} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

reciprocal space is a dual space of its original vector space

$$\begin{aligned}V &= \text{span} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3\} \\ &= \left\{ \sum_{j=1}^3 v_j \mathbf{e}_j \right\} = \left\{ v_j \mathbf{e}_j \middle| \begin{array}{l} v_j \in F \\ \mathbf{e}_j \in F^3 \end{array} \right\} = \{\mathbf{v} | \mathbf{v} \in V\} \\ V^* &= \text{span} \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\} = \{v^{*1} \mathbf{e}^1 + v^{*2} \mathbf{e}^2 + v^{*3} \mathbf{e}^3\} \\ &= \left\{ \sum_{i=1}^3 v^{*i} \mathbf{e}^i \right\} = \left\{ v^{*i} \mathbf{e}^i \middle| \begin{array}{l} v^{*i} \in F \\ \mathbf{e}^i \in F^3 \end{array} \right\} = \{\mathbf{v}^* | \mathbf{v}^* \in V^*\} \\ \mathbf{v}^*(\mathbf{v}) &= (v^{*i} \mathbf{e}^i)(\mathbf{v}), \mathbf{v} \in V \\ &= (v^{*1} \mathbf{e}^1 + v^{*2} \mathbf{e}^2 + v^{*3} \mathbf{e}^3)(\mathbf{v}) \\ &= v^{*1} \mathbf{e}^1(\mathbf{v}) + v^{*2} \mathbf{e}^2(\mathbf{v}) + v^{*3} \mathbf{e}^3(\mathbf{v}) \\ &= v^{*1} v_1 + v^{*2} v_2 + v^{*3} v_3 \in F\end{aligned}$$

element in dual space is a functional or mapping from its original vector space to the field

$$\mathbf{v}^* : V \rightarrow F$$

$$V \xrightarrow{\mathbf{v}^*} F$$

$$V^* = \{\mathbf{v}^* | \mathbf{v}^* : V \rightarrow F\}$$

$$\begin{array}{ccccccc}
 & & V & = & \{ & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{v} & \cdots \} \\
 \begin{matrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \\ \mathbf{v}^* \end{matrix} & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 V^* = \{ & F & \supseteq & \{ & 1 & 0 & 0 & v_1 & \cdots \} \\
 & V & = & \{ & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{v} & \cdots \} \\
 & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & F & \supseteq & \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i} v_i & \cdots \} \\
 \vdots & & & & & & & & 
 \end{array}$$


---

$$\begin{aligned}
 V^* &= \{\mathbf{v}^* | \mathbf{v}^* \in V^*\} = \{\mathbf{v}^* | \mathbf{v}^* : V \rightarrow F\} \\
 &= \left\{ \mathbf{v}^* \middle| V \xrightarrow{\mathbf{v}^*} F \right\} \\
 &= \{\omega | \omega : V \rightarrow F\} \\
 &= \left\{ \omega^i e^i \middle| \begin{cases} \omega^i \in F \\ e^i \in F^3 \end{cases} \right\}
 \end{aligned}$$

By defining vector addition and scalar multiplication on the dual space

$$\left\{ \begin{array}{l} + : V^* \times V^* \rightarrow V^* \Leftrightarrow \forall \omega_1, \omega_2 \in V^*, \exists! (\omega_1 + \omega_2) \in V^* [(\omega_1 + \omega_2)(\mathbf{v}) = \omega_1(\mathbf{v}) + \omega_2(\mathbf{v})] \\ \cdot : F \times V^* \rightarrow V^* \Leftrightarrow \forall k \in F, \forall \omega \in V^*, \exists! (k\omega) \in V^* [(k\omega)(\mathbf{v}) = k \cdot \omega(\mathbf{v})] \\ \forall \omega \in V^*, \exists! \mathbf{0} \in V^* [(\omega + \mathbf{0})(\mathbf{v}) = \omega(\mathbf{v}) + \mathbf{0}(\mathbf{v}) = \omega(\mathbf{v})] \end{array} \right.$$

the dual space also becomes a vector space.

#### 40.4.1.2 double dual concept

double dual space = second dual space

$$\begin{aligned}
 V^{**} &= (V^*)^* \\
 &= \{\omega^* | \omega^* : V^* \rightarrow F\} \\
 &= \{\omega^* | \omega^* \in V^{**}\}
 \end{aligned}$$

$$\begin{aligned}
 V^{**} &= (V^*)^* = \text{span} \{e^\mu\}_{\mu \in \{1, 2, 3\}}^* \\
 &= \text{span} \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}^* \\
 &= \text{span} \{e^{1*}, e^{2*}, e^{3*}\} \\
 &= \text{span} \{e^{\nu*}\}_{\nu \in \{1, 2, 3\}}
 \end{aligned}$$

$$\begin{aligned}
 \omega^*(\omega) &= (\omega^{*\nu} e^{\nu*})(\omega), \omega \in V^* \\
 &= (\omega^{*1} e^{1*} + \omega^{*2} e^{2*} + \omega^{*3} e^{3*})(\omega) \\
 &= \omega^{*1} e^{1*}(\omega) + \omega^{*2} e^{2*}(\omega) + \omega^{*3} e^{3*}(\omega) \\
 &= \omega^{*1} \omega_1 + \omega^{*2} \omega_2 + \omega^{*3} \omega_3 \in F
 \end{aligned}$$


---

$$V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\}$$

$$\begin{array}{ccccccc} & & V^* & = & \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\ V^{**} = \{ & \begin{matrix} e^{1*} \\ e^{2*} \\ e^{3*} \\ \omega^* \\ \vdots \end{matrix} & : & \begin{matrix} \downarrow \\ F \\ V^* \\ \vdots \end{matrix} & \supseteq & \{ & 1 & 0 & 0 & \omega^1 & \dots \} \\ & & & & & \{ & e_1 & e_2 & e_3 & v & \dots \} \\ & & & & & & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*} \omega^\mu & \dots \} \end{array}$$


---

$$\begin{cases} e^{1*}(\omega) = e^{1*} \cdot \omega = \omega \cdot e^{1*} = \omega(e^{1*}) \\ e^{2*}(\omega) = e^{2*} \cdot \omega = \omega \cdot e^{2*} = \omega(e^{2*}) \\ e^{3*}(\omega) = e^{3*} \cdot \omega = \omega \cdot e^{3*} = \omega(e^{3*}) \end{cases}$$

$$\omega^*(\omega) = \omega^* \cdot \omega = \omega \cdot \omega^* = \omega(\omega^*)$$

i.e.  $f$  acts on  $x$  equivalent to  $x$  acts on  $f$

$$x(f) = x \cdot f = f \cdot x = f(x)$$

$$e^\mu(e_\nu) = e^\mu \cdot e_\nu = \delta_\nu^\mu = \delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$$\begin{array}{c} e^{\nu*}(e^\mu) \stackrel{\text{def.}}{=} e_\nu \cdot e^\mu = e^\mu \cdot e_\nu = e^\mu(e_\nu) \\ \Downarrow \\ V^{**} = \text{span}\{e^{\nu*}\}_{\nu \in \{1, 2, 3\}} \cong \text{span}\{e_\nu\}_{\nu \in \{1, 2, 3\}} = V \\ V^{**} \cong V \\ \Downarrow \begin{cases} V^{**} \cong V & V, V^{**} \text{ are isomorphism} \\ & \text{independent of choice of bases} \\ V, V^{**} \text{ are naturally isomorphism} \end{cases} \end{array}$$


---

$$V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\} \cong V = \{v | v : V^* \rightarrow F\}$$

$$\begin{array}{ccccccc} & & V^* & = & \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\ V^{**} = \{ & \begin{matrix} e^{1*} \\ e^{2*} \\ e^{3*} \\ \omega^* \\ \vdots \end{matrix} & : & \begin{matrix} \downarrow \\ F \\ V^* \\ \vdots \end{matrix} & \supseteq & \{ & 1 & 0 & 0 & \omega^1 & \dots \} \\ & & & & & \{ & e^1 & e^2 & e^3 & v^* & \dots \} \\ & & & & & & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*} \omega^\mu & \dots \} \\ & & & & & \cong V = \{ & \begin{matrix} e_1 \\ e_2 \\ e_3 \\ v \end{matrix} & : & \begin{matrix} \downarrow \\ F \\ V^* \\ \vdots \end{matrix} & \supseteq & \{ & 1 & 0 & 0 & v^{*1} & \dots \} \\ & & & & & & \{ & e^1 & e^2 & e^3 & v^* & \dots \} \\ & & & & & & & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & & & v_1 & v_2 & v_3 & v_\mu v^{*\mu} & \dots \} \end{array}$$

---

$$V \cong V^{**}$$

---

$$V \cong V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\}$$

$$V = \{v | v : V^* \rightarrow F\}$$

i.e. **vector space is a set of functionals or mappings from its dual space to the field**, answering **What is a vector?**<sup>[40.1]</sup>, and satifying Fig: 40.1.

## 40.5 field

[https://web.math.sinica.edu.tw/math\\_media/d312/31202.pdf](https://web.math.sinica.edu.tw/math_media/d312/31202.pdf)

## 40.6 module

<https://web.math.sinica.edu.tw/mathmedia/HTMLArticle18.jsp?mID=31304>

## 40.7 subspace



# Chapter 41

$\mathrm{d}f$

## 41.1 $\mathrm{d}f$ decomposed with partials as a set of basis in vector space

$$f = \{f_i\} = \{f_1, f_2, \dots\} = \{f, g, \dots\}$$

$$\mathbf{v} : f \rightarrow F$$

$$\mathbf{v}(af + bg) = a\mathbf{v}(f) + b\mathbf{v}(g)$$

$$\mathbf{v}(fg) = f|_P \mathbf{v}(g) + \mathbf{v}(f) g|_P$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [f(x)g(x)]|_{x=x_0} = f(x_0) \frac{\mathrm{d}}{\mathrm{d}x} g(x)|_{x=x_0} + \frac{\mathrm{d}}{\mathrm{d}x} f(x)|_{x=x_0} g(x_0)$$

$$V = \{\mathbf{v} | \mathbf{v} : f \rightarrow F\}$$

$$\begin{aligned} f &= f(\mathbf{x}) \\ &= f(x_1, \dots, x_j, \dots, x_n) \\ &= f(x^1, \dots, x^j, \dots, x^n) \end{aligned}$$

$$\mathbf{x} = \langle x^1, \dots, x^j, \dots, x^n \rangle$$

$$\mathbf{x}(t) = \langle x^1(t), \dots, x^j(t), \dots, x^n(t) \rangle$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\mathrm{d}x^1}{\mathrm{d}t} \frac{\partial f}{\partial x^1} + \dots + \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} + \dots + \frac{\mathrm{d}x^n}{\mathrm{d}t} \frac{\partial f}{\partial x^n} \\ &= \dots + \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} + \dots = \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} = \frac{\mathrm{d}x^j}{\mathrm{d}t} \partial_j f \end{aligned}$$

$$\begin{aligned}
V &= \text{span} \{ \mathbf{e}_1, \dots, \mathbf{e}_j, \dots, \mathbf{e}_n \} \\
&= \text{span} \left\{ \frac{\partial}{\partial x^1}|_P, \dots, \frac{\partial}{\partial x^j}|_P, \dots, \frac{\partial}{\partial x^n}|_P \right\} \\
&= \text{span} \{ \boldsymbol{\partial}_1, \dots, \boldsymbol{\partial}_j, \dots, \boldsymbol{\partial}_n \} \\
&= \left\{ \boldsymbol{\partial}_t \left| \boldsymbol{\partial}_t = a_j \mathbf{e}_j = a_j \boldsymbol{\partial}_j = a_j \frac{\partial}{\partial x^j}|_P \right. \right\} \\
&= \left\{ \frac{\partial}{\partial t}|_P \left| \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \dots + a_j \frac{\partial}{\partial x^j}|_P + \dots + a_n \frac{\partial}{\partial x^n}|_P \right. \right\}
\end{aligned}$$

## 41.2 dual space of span of partials

$$V^* = \{ \omega_f | \omega_f : V \rightarrow F \}$$

$$\omega_f(\mathbf{e}_j) = \omega_f(\boldsymbol{\partial}_j) = \omega_f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F$$

$$\begin{aligned}
\omega_{fg}(\boldsymbol{\partial}_j) &= \frac{\partial fg}{\partial x^j}|_P = f|_P \frac{\partial g}{\partial x^j}|_P + \frac{\partial f}{\partial x^j}|_P g|_P \\
&= f|_P \omega_g(\boldsymbol{\partial}_j) + \omega_f(\boldsymbol{\partial}_j) g|_P
\end{aligned}$$

$$\omega_{x^i}(\boldsymbol{\partial}_j) = \omega_{x^i} \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$V^* = \{ \omega_f | \omega_f : V \rightarrow F \} = \left\{ \omega_f \left| \begin{array}{l} \omega_f(\mathbf{e}_j) = \omega_f(\boldsymbol{\partial}_j) = \omega_f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ \omega_{fg}(\boldsymbol{\partial}_j) = f|_P \omega_g(\boldsymbol{\partial}_j) + \omega_f(\boldsymbol{\partial}_j) g|_P \\ \omega_{x^i}(\boldsymbol{\partial}_j) = \omega_{x^i} \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\}$$

$$\{ df | df : V \rightarrow F \} = \left\{ df \left| \begin{array}{l} df(\mathbf{e}_j) = df(\boldsymbol{\partial}_j) = df \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ dfg(\boldsymbol{\partial}_j) = f|_P (dg) + (df) g|_P \\ dx^i(\boldsymbol{\partial}_j) = dx^i \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\}$$

$$dx^i \left( \frac{\partial}{\partial x^j}|_P \right) = \delta_{ij} = \mathbf{e}^i \cdot \mathbf{e}_j \Rightarrow \begin{cases} \mathbf{e}^i = dx^i \\ \mathbf{e}_j = \frac{\partial}{\partial x^j}|_P \end{cases}$$

$$V^* = \{ df | df : V \rightarrow F \} = \left\{ df \left| \begin{array}{l} df(\mathbf{e}_j) = df(\boldsymbol{\partial}_j) = df \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ dfg(\boldsymbol{\partial}_j) = f|_P (dg) + (df) g|_P \\ dx^i(\boldsymbol{\partial}_j) = dx^i \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\}$$

$$= \text{span} \{ dx^1, \dots, dx^i, \dots, dx^n \} = \text{span} \{ \mathbf{e}^1, \dots, \mathbf{e}^j, \dots, \mathbf{e}^n \}$$

### 41.3 directional derivative

$$\begin{aligned}
 df(\mathbf{v}) &= df(v_j \mathbf{e}_j) = v_j df(\mathbf{e}_j) \\
 &= v_j df(\partial_j) = v_j \frac{\partial f}{\partial x^j}|_P \\
 &= v_1 \frac{\partial f}{\partial x^1}|_P + v_2 \frac{\partial f}{\partial x^2}|_P + \cdots + v_n \frac{\partial f}{\partial x^n}|_P \\
 &= (v_1 \quad \cdots \quad v_j \quad \cdots \quad v_n) \nabla f
 \end{aligned}$$


---

$$\widehat{PQ} = C(t) - C(0) = Q - P$$

$$\mathbf{v} = \frac{\partial}{\partial t}|_P$$

$$\begin{aligned}
 df(s\mathbf{v}) &= df\left(s \frac{\partial}{\partial t}|_P\right) = s \frac{\partial f}{\partial t}|_P \\
 &= s\mathbf{v}(f) = s \cdot \lim_{t \rightarrow 0} \frac{f(C(t)) - f(C(0))}{t} \\
 &\approx s \cdot \frac{f(Q) - f(P)}{s} = f(Q) - f(P) = \Delta f
 \end{aligned}$$

### 41.4 coefficient of linear combination for vector space and dual space

$$\begin{aligned}
 V &= \{\mathbf{v} | \mathbf{v} : f \rightarrow F\} \\
 &= \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_j, \dots, \mathbf{e}_n\} \\
 &= \text{span}\left\{\frac{\partial}{\partial x^1}|_P, \dots, \frac{\partial}{\partial x^j}|_P, \dots, \frac{\partial}{\partial x^n}|_P\right\} = \text{span}\{\partial_1, \dots, \partial_j, \dots, \partial_n\} \\
 &= \left\{\partial_t \middle| \partial_t = a_j \mathbf{e}_j = a_j \partial_j = a_j \frac{\partial}{\partial x^j}|_P\right\} \\
 &= \left\{\frac{\partial}{\partial t}|_P \middle| \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \cdots + a_j \frac{\partial}{\partial x^j}|_P + \cdots + a_n \frac{\partial}{\partial x^n}|_P\right\}
 \end{aligned}$$

$$\begin{aligned}
 V^* &= \{df|df : V \rightarrow F\} \\
 &= \text{span}\{\mathbf{e}^1, \dots, \mathbf{e}^i, \dots, \mathbf{e}^n\} \\
 &= \text{span}\{dx^1, \dots, dx^i, \dots, dx^n\} \\
 &= \{df|df = b^i \mathbf{e}^i = b^i dx^i\} \\
 &= \{df|df = b^1 dx^1 + \cdots + b^i dx^i + \cdots + b^n dx^n\}
 \end{aligned}$$


---

or more simply to be comparison

$$\begin{array}{lll}
 V &= \text{span}\{ & \mathbf{e}_j = \partial_j \quad \} = \{ & \mathbf{v} = \partial_t|_P = a_j \mathbf{e}_j = a_j \partial_j|_P : f \rightarrow F \quad \} \\
 V^* &= \text{span}\{ & \mathbf{e}^i = dx^i \quad \} = \{ & \omega = df = b^i \mathbf{e}^i = b^i dx^i : V \rightarrow F \quad \}
 \end{array}$$


---

$$\begin{aligned}
& \left\{ \begin{array}{l} dx^i(\partial_j) = dx^i\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \\ \partial_t = a_j e_j = a_j \partial_j \Leftrightarrow \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \cdots + a_j \frac{\partial}{\partial x^j}|_P + \cdots + a_n \frac{\partial}{\partial x^n}|_P \end{array} \right. \\
& \Rightarrow \left\{ \begin{array}{l} dx^i(\partial_t) = dx^i\left(\frac{\partial}{\partial t}|_P\right) = \frac{\partial x^i}{\partial t}|_P \\ dx^i(\partial_t) = dx^i(a_j \partial_j) = a_j dx^i(\partial_j) = a_j \delta_{ij} = a_i \end{array} \right. \Rightarrow a_i = dx^i(\partial_t) = \frac{\partial x^i}{\partial t}|_P \\
& \Rightarrow a_i = \frac{\partial x^i}{\partial t}|_P \Rightarrow a_j = \frac{\partial x^j}{\partial t}|_P = \partial_t x^j|_P \\
& \Rightarrow \frac{\partial}{\partial t}|_P = a_i \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t}|_P \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}|_P \Rightarrow \frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \\
& \Rightarrow \partial_t|_P = \frac{\partial x^j}{\partial t} \partial_j|_P \Leftrightarrow \partial_t|_P = \partial_t x^j \partial_j|_P
\end{aligned}$$

$$\begin{aligned}
df &= b^i e^i = b^i dx^i \\
\frac{\partial f}{\partial x^j} &= df(\partial_j) = df(e_j) = b^i e^i \cdot e_j = b^i \delta_{ij} = b^j \\
b^j &= \frac{\partial f}{\partial x^j} \\
b^i &= \frac{\partial f}{\partial x^i} = \partial_i f \\
df &= b^i e^i = b^i dx^i = \frac{\partial f}{\partial x^i} dx^i \\
df &= \frac{\partial f}{\partial x^i} dx^i \\
df &= \partial_i f dx^i
\end{aligned}$$


---

$$\begin{array}{lll}
V &= \text{span}\{ & e_j = \partial_j \} = \{ & v = \partial_t|_P = a_j e_j = \partial_t x^j \partial_j|_P : f \rightarrow F \} \\
V^* &= \text{span}\{ & e^i = dx^i \} = \{ & \omega = df = b^i e^i = \partial_i f dx^i : V \rightarrow F \}
\end{array}$$

## 41.5 change of basis / change of coordinate

$$\frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \stackrel{t=x'^j}{\Rightarrow} \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} = \frac{\partial x^1}{\partial x'^j} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^j} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^j} \frac{\partial}{\partial x^3}$$

$$\begin{aligned}
df &= \frac{\partial f}{\partial x^i} dx^i \\
f &= x'^j \Downarrow \\
dx'^j &= \frac{\partial x'^j}{\partial x^i} dx^i
\end{aligned}$$


---

$$\begin{cases} \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} = \sum_i \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} \\ dx'^j = \frac{\partial x'^j}{\partial x^i} dx^i = \sum_i \frac{\partial x'^j}{\partial x^i} dx^i \end{cases}$$

# Chapter 42

## determinant

### 42.1 induction of determinant axioms

### 42.2 determinant axioms

determinant axioms

$$\left\{ \begin{array}{l} \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u} + s\mathbf{v}, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u}, \mathbf{v} + s\mathbf{u}) \end{array} \right. \quad \begin{array}{l} \text{translation invariance} \\ \text{scaling} \end{array}$$
$$\left\{ \begin{array}{l} \det(s\mathbf{u}, \mathbf{v}) = s \det(\mathbf{u}, \mathbf{v}) \\ \det(\mathbf{u}, s\mathbf{v}) = s \det(\mathbf{u}, \mathbf{v}) \end{array} \right. \quad \text{decomposition}$$
$$\Leftrightarrow \left\{ \begin{array}{l} \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u} + s\mathbf{v}, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u}, \mathbf{v} + s\mathbf{u}) \end{array} \right. \quad \begin{array}{l} \text{translation invariance} \\ \text{linearity} \end{array}$$
$$\left\{ \begin{array}{l} \det(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) = \det(\mathbf{u}_1, \mathbf{v}) + \det(\mathbf{u}_2, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2) = \det(\mathbf{u}, \mathbf{v}_1) + \det(\mathbf{u}, \mathbf{v}_2) \end{array} \right.$$

### 42.3 determinant theorems or properties

### 42.4 ellipse area

### 42.5 Cramer rule geometry perspective



# **Chapter 43**

## **hypergeometric function**

**43.1 linear space of function**

**43.2 beta function**

**43.3 gamma function**

**43.4 recursion**

**43.5 mean and variance of discrete probability distributions**



# **Chapter 44**

## **Feynman method**

**44.1 Feynman method of derivative**

**44.2 Feynman method of integral**

**44.3 path integral**

<https://www.youtube.com/watch?v=Sp5SvdDh2u8>

[https://en.wikipedia.org/wiki/Path\\_integral\\_formulation](https://en.wikipedia.org/wiki/Path_integral_formulation)

<https://www.youtube.com/watch?v=7yjv-gLHFqg>



# Chapter 45

## Hilbert space

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knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
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### 45.1 Taylor expansion or Taylor series

**Lemma 45.1.** Newton-Leibniz formula = N-LF, equivalent to first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1

牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF, equivalent to 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 ??

$$\int_a^b f'(x) dx = f(b) - f(a) \quad (45.1)$$

$$\left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \\ f \text{ differentiable on } (a, b) & (2) \\ f' : [a, b] \rightarrow \mathbb{R} & (4) \\ f' \text{ continuous on } [a, b] & (5) \end{array} \right. \quad a < b \quad (3) \Rightarrow \exists c \in (a, b) \left( f'(c) = \frac{\int_a^b f'(x) dx}{b-a} \right) \quad \text{MVTi1 ??}$$

$$\exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \} \quad 45.1$$

$$\Rightarrow \quad \Updownarrow$$

$$\int_a^b f'(x) dx = f(b) - f(a) \quad 45.1$$

( $\Downarrow$ ):

$$\left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \} \quad \text{MVTd1 ??} \\ f \text{ differentiable on } (a, b) & (2) \\ [a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \dots \cup [x_{n-1}, x_n] \end{array} \right. \quad 45.2$$

$$\stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a)$$

(↑):

$$\left\{ \begin{array}{l} \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \end{array} \right. \xrightarrow[?]{} \int_a^b f'(x) dx = f(b) - f(a) \\ \left\{ \begin{array}{ll} f' : [a, b] \rightarrow \mathbb{R} & (4) \\ f' \text{ continuous on } [a, b] & (5) \end{array} \right. \xrightarrow[?]{} a < b \quad (3) \Rightarrow \exists c \in (a, b) \left( f'(c) = \frac{\int_a^b f'(x) dx}{b-a} \right) \end{array} \right. \begin{array}{l} \text{N-LF 45.1} \\ \text{MVTi1 ??} \end{array}$$

$$\Rightarrow \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \}$$

Proof: (↓)

重疊端點分割  $[a, b]$  成  $n$  部分聯集 ( $n \in \mathbb{N}$ )

$$[a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \quad (45.2)$$

$$\begin{aligned}
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \exists c \in (a, b) \{f'(c)(b-a) = f(b) - f(a)\} \quad \text{MVTd1 ??} \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 & [a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \quad 45.2
 \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{ll} (f : [x_0, x_n] \rightarrow \mathbb{R}) = (f : [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \rightarrow \mathbb{R}) \Leftarrow (0) \\ f \text{ continuous on } [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \Leftarrow (1) \\ f \text{ differentiable on } (x_0, x_1) \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \Leftarrow (2) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} f : [x_0, x_1] \rightarrow \mathbb{R}, f : [x_0, x_2] \rightarrow \mathbb{R}, \dots, f : [x_{n-1}, x_n] \rightarrow \mathbb{R} \\ f \text{ continuous on } [x_0, x_1], [x_0, x_2], \dots, [x_{n-1}, x_n] \\ f \text{ differentiable on } (x_0, x_1), [x_0, x_2], \dots, [x_{n-1}, x_n] \Rightarrow f \text{ differentiable on } (x_0, x_1), (x_0, x_2), \dots, (x_{n-1}, x_n) \end{array} \right.$$

$$\stackrel{??}{\Rightarrow} \left\{ \begin{array}{ll} \exists c_1 \in (x_0, x_1) \{f'(c_1)(x_1 - x_0) = f(x_1) - f(x_0)\} \\ \exists c_2 \in (x_1, x_2) \{f'(c_2)(x_2 - x_1) = f(x_2) - f(x_1)\} \\ \vdots \\ \exists c_k \in (x_{k-1}, x_k) \{f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1})\} \quad \forall k \in \mathbb{N} \cap [1, n] \\ \vdots \\ \exists c_n \in (x_{n-1}, x_n) \{f'(c_n)(x_n - x_{n-1}) = f(x_n) - f(x_{n-1})\} \end{array} \right.$$

$$\Rightarrow f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$$

$$\Rightarrow \sum_k f'(c_k)(x_k - x_{k-1}) = \sum_k f(x_k) - f(x_{k-1})$$

$$\Rightarrow \sum_{k=1}^n f'(c_k)(x_k - x_{k-1}) = \sum_{k=1}^n f(x_k) - f(x_{k-1})$$

$$= [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \cdots + [f(x_n) - f(x_{n-1})]$$

$$= f(x_n) - f(x_0) = f(b) - f(a)$$

$$\Rightarrow \sum_{k=1}^n f'(c_k)(x_k - x_{k-1}) = f(b) - f(a) \stackrel{\Delta x_k = x_k - x_{k-1}}{\Leftrightarrow} \sum_{k=1}^n f'(c_k) \Delta x_k = f(b) - f(a)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \lim_{\Delta x_k = \frac{x_n - x_0}{n}} \sum_{k=1}^n f'(c_k) \Delta x_k = \lim_{n \rightarrow \infty} \lim_{\Delta x_k = \frac{b-a}{n} \rightarrow 0} f(b) - f(a) = f(b) - f(a)$$

$$\Rightarrow \int_a^b f'(x) dx = \int_{x_0}^{x_n} f'(x) dx = f(b) - f(a)$$

$$\Rightarrow \int_a^b f'(x) dx = f(b) - f(a)$$

$$\int_a^b f'(x) dx = f(b) - f(a)$$

得到 牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF [eq:N-LF]

□

Proof: (↑)

$$\begin{aligned}
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1} \\
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & a < b \ (3) \\ f \text{ continuous on } [a, b] & (1) \end{array} \right. \Rightarrow \exists c \in (a, b) \left( f(c) = \frac{\int_a^b f(x) dx}{b-a} \right) \quad \text{MVTi1 ??}
 \end{aligned}$$
  

$$\Rightarrow \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1}$$
  

$$\Rightarrow \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & a < b \ (3) \\ f \text{ continuous on } [a, b] & (1) \end{array} \right. \Rightarrow \exists c \in (a, b) \left\{ f(c)(b-a) = \int_a^b f(x) dx \right\} \quad \text{MVTi1 ??}$$
  

$$\Rightarrow \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1}$$
  

$$\Rightarrow \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 \left\{ \begin{array}{ll} \text{if } f' : [a, b] \rightarrow \mathbb{R} & (4) \\ \text{if } f' \text{ continuous on } [a, b] & (5) \end{array} \right. \Rightarrow \exists c \in (a, b) \left\{ f'(c)(b-a) = \int_a^b f'(x) dx \right\} \quad \text{MVTi1 ??}$$
  

$$\Rightarrow \exists c \in (a, b) \left\{ f'(c)(b-a) = \int_a^b f'(x) dx = f(b) - f(a) \right\}$$
  

$$\Rightarrow \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \}$$

$$\exists c \in (a, b) [f'(c)(b-a) = f(b) - f(a)]$$

得到 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 [thm:MVTd1]

□

$$\begin{aligned}
 \int_a^b f'(x) dx &= f(b) - f(a) \\
 \int_{x_0}^x f'(t) dt &= f(x) - f(x_0) \\
 f(x) &= f(x_0) + \int_{x_0}^x f'(t) dt
 \end{aligned}$$


---

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt$$


---

**Lemma 45.2.** univariable product rule

$$d(uv) = (du)v + u dv = v du + u dv$$

$$\frac{d(uv)}{dt} = v \frac{du}{dt} + u \frac{dv}{dt}$$

**Lemma 45.3.** *integration by parts*

$$\begin{aligned}\int \frac{d(uv)}{dt} dt &= \int v \frac{du}{dt} + u \frac{dv}{dt} dt \\ \int d(uv) &= \int v \frac{du}{dt} dt + \int u \frac{dv}{dt} dt \\ uv &= \int v du + \int u dv \\ \int u \frac{dv}{dt} dt &= uv - \int v \frac{du}{dt} dt \text{ "switching derivatives"}\end{aligned}\tag{45.3}$$

$$\int u dv = uv - \int v du \text{ "switching differentials"}\tag{45.4}$$

**Theorem 45.1.** *univariable Taylor theorem with the remainder in integral form*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt$$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

Proof:

$$f(x) - f(a) \quad (45.5)$$

$$\stackrel{45.1}{=} \int_a^x f'(t) dt = \int_a^x f'(t) \frac{d(t-x)}{dt} dt \stackrel{45.3}{=} [f'(t)(t-x)]_{t=a}^x - \int_a^x (t-x) \frac{df'(t)}{dt} dt \quad (45.6)$$

$$= [-f'(a)(a-x)] - \int_a^x f''(t)(t-x) dt = f'(a)(x-a) - \int_a^x f''(t) \frac{d\frac{(t-x)^2}{2}}{dt} dt \quad (45.7)$$

$$= f'(a)(x-a) - \left( \left[ f''(t) \frac{(t-x)^2}{2} \right]_{t=a}^x - \int_a^x \frac{(t-x)^2}{2} \frac{df''(t)}{dt} dt \right) \quad (45.8)$$

$$= f'(a)(x-a) - \left( \left[ -f''(a) \frac{(x-a)^2}{2} \right] - \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \right) \quad (45.9)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \quad (45.10)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{d\frac{(t-x)^3}{2 \cdot 3}}{dt} dt \quad (45.11)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(t) \frac{(t-x)^3}{2 \cdot 3} \right]_{t=a}^x - \int_a^x \frac{(t-x)^3}{2 \cdot 3} \frac{df'''(t)}{dt} dt \right) \quad (45.12)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(a) \frac{(x-a)^3}{2 \cdot 3} \right] - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \right) \quad (45.13)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \quad (45.14)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{d\frac{(t-x)^4}{2 \cdot 3 \cdot 4}}{dt} dt \quad (45.15)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} \quad (45.16)$$

$$- \left( \left[ f^{(4)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} \right]_{t=a}^x - \int_a^x \frac{(t-x)^4}{2 \cdot 3 \cdot 4} \frac{df^{(4)}(t)}{dt} dt \right) \quad (45.17)$$

$$= f'(t)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} \quad (45.18)$$

$$- \left( \left[ -f^{(4)}(a) \frac{(x-a)^4}{2 \cdot 3 \cdot 4} \right] - \int_a^x f^{(5)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} dt \right) \quad (45.19)$$

$$= f'(t)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(a) \frac{(x-a)^3}{2 \cdot 3} + f^{(4)}(a) \frac{(x-a)^4}{2 \cdot 3 \cdot 4} \quad (45.20)$$

$$\vdots + \int_a^x f^{(5)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} dt \quad (45.21)$$

$$= \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt \text{ remainder in integral form} \quad (45.22)$$

□

Proof:

$$\begin{aligned}
& f(x) - f(a) \\
\stackrel{45.1}{=} & \int_a^x f'(t) dt = \int_a^x f'(t) \frac{d(t-x)}{dt} dt \stackrel{45.3}{=} [f'(t)(t-x)]_{t=a}^x - \int_a^x (t-x) \frac{df'(t)}{dt} dt \\
= & [-f'(a)(a-x)] - \int_a^x f''(t)(t-x) dt = f'(a)(x-a) + \int_a^x f''(t)(x-t) dt \\
= & f'(a)(x-a) - \int_a^x f''(t)(t-x) dt = \sum_{k=1}^1 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f''(t)(x-t) dt \\
= & f'(a)(x-a) - \int_a^x f''(t) \frac{d\frac{(t-x)^2}{2}}{dt} dt \\
= & f'(a)(x-a) - \left( \left[ f''(t) \frac{(t-x)^2}{2} \right]_{t=a}^x - \int_a^x \frac{(t-x)^2}{2} \frac{df''(t)}{dt} dt \right) \\
= & f'(a)(x-a) - \left( \left[ -f''(a) \frac{(x-a)^2}{2} \right] - \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \right) \\
= & f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \\
= & \sum_{k=1}^2 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt = \sum_{k=1}^2 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f'''(t) \frac{(x-t)^2}{2!} dt \\
= & f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{d\frac{(t-x)^3}{2 \cdot 3}}{dt} dt \\
= & f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(t) \frac{(t-x)^3}{2 \cdot 3} \right]_{t=a}^x - \int_a^x \frac{(t-x)^3}{2 \cdot 3} \frac{df'''(t)}{dt} dt \right) \\
= & f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(a) \frac{(x-a)^3}{2 \cdot 3} \right] - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \right) \\
= & f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \\
= & \sum_{k=1}^3 \frac{f^{(k)}(a)}{k!} (x-a)^k - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt = \sum_{k=1}^3 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(4)}(t) \frac{(x-t)^3}{3!} dt \\
\vdots & \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
f(x) - f(a) = & \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
f(x) = & f(a) + \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
= & \frac{f^{(0)}(a)}{0!} (x-a)^0 + \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
= & \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt
\end{aligned}$$

□

**Theorem 45.2.** univariable Taylor theorem with the remainder in Lagrange form

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

Proof:

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem  
 = CFEVT / extreme value theorem = EVT ?? and 連續函數 介值定理 / 中間值定理 continuous function intermediate value theorem = CFIVT / intermediate value theorem = IVT ??

$$\text{let } f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + r_n(x) \quad (45.23)$$

$$r_n(x) = \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt \quad (45.24)$$

$$\text{let } n \in \{2k-1 | k \in \mathbb{N}\} \quad (45.25)$$

$$\text{let } f(t) \stackrel{??}{\in} [m, M] \subseteq f((a, x)) \text{ when } a < x \quad (45.26)$$

$$\int_a^x m \frac{(t-x)^n}{n!} dt \leq r_n(x) \leq \int_a^x M \frac{(t-x)^n}{n!} dt \quad (45.27)$$

$$m \int_a^x \frac{(t-x)^n}{n!} dt = M \int_a^x \frac{(t-x)^n}{n!} dt \quad (45.28)$$

$$m \left[ \frac{(t-x)^{n+1}}{(n+1)!} \right]_{t=a}^x = M \left[ \frac{(t-x)^{n+1}}{(n+1)!} \right]_{t=a}^x \quad (45.29)$$

$$m \frac{(x-a)^{n+1}}{(n+1)!} = M \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.30)$$

$$\Downarrow \quad ?? \quad (45.31)$$

$$r_n(x) \stackrel{\exists \xi \in (a, x)}{=} f^{(n+1)}(\xi) \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.32)$$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + f^{(n+1)}(\xi) \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.33)$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \quad (45.34)$$

□

**Theorem 45.3.** univariable Taylor theorem with the remainder in big O form

$$f(a+h) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + O(h^{n+1})$$

Proof:

$$\text{if } \left| f^{(n+1)}(t) \right| \leq K \forall t \in (a, x) \quad (45.35)$$

$$|r_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \right| \leq \frac{K}{(n+1)!} |x-a|^{n+1} \quad (45.36)$$

$$r_n(x) \in O((x-a)^{n+1}) \quad (45.37)$$

$$\text{let } R_n(h) = r_n(a+h) \quad (45.38)$$

$$R_n(h) \in O(h^{n+1}) \quad (45.39)$$

$$f(a+h) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \quad (45.40)$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + O(h^{n+1}) \quad (45.41)$$

□

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$f(x) = \sin(x)$$

$$\sin(x) = \sum_{k=0}^n \frac{\sin^{(k)}(a)}{k!} (x-a)^k + \int_a^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$a = 0$$

$$\sin(x) = \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} x^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \frac{\sin^{(0)}(0)}{0!} x^0 + \frac{\sin^{(1)}(0)}{1!} x^1 + \frac{\sin^{(2)}(0)}{2!} x^2 + \frac{\sin^{(3)}(0)}{3!} x^3 \\
&\quad + \int_0^x \sin^{(3+1)}(t) \frac{(x-t)^3}{3!} dt \\
&= \frac{0}{0!} x^0 + \frac{\cos(0)}{1!} x^1 + \frac{-\sin(0)}{2!} x^2 + \frac{-\cos(0)}{3!} x^3 \\
&\quad + \int_0^x \sin^{(4)}(t) \frac{(x-t)^3}{3!} dt \\
&= 0 + \frac{1}{1} x + 0 - \frac{1}{6} x^3 + \int_0^x \sin(t) \frac{(x-t)^3}{6} dt \\
&= x - \frac{1}{6} x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt \\
\sin(x) &= x - \frac{1}{6} x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt
\end{aligned}$$

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} x^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \frac{\sin^{(0)}(0)}{0!} x^0 + \frac{\sin^{(1)}(0)}{1!} x^1 + \frac{\sin^{(2)}(0)}{2!} x^2 + \frac{\sin^{(3)}(0)}{3!} x^3 + \frac{\sin^{(4)}(0)}{4!} x^4 \\
&\quad + \int_0^x \sin^{(4+1)}(t) \frac{(x-t)^4}{4!} dt \\
&= \frac{0}{0!} x^0 + \frac{\cos(0)}{1!} x^1 + \frac{-\sin(0)}{2!} x^2 + \frac{-\cos(0)}{3!} x^3 + \frac{\sin(0)}{4!} x^4 \\
&\quad + \int_0^x \sin^{(5)}(t) \frac{(x-t)^4}{4!} dt \\
&= 0 + \frac{1}{1} x + 0 - \frac{1}{6} x^3 + 0 + \int_0^x \cos(t) \frac{(x-t)^4}{24} dt \\
&= x - \frac{1}{6} x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\
\sin(x) &= x - \frac{1}{6} x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt
\end{aligned}$$

$$\begin{aligned}
 \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt \\
 &= x - \frac{1}{6}x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\
 &\quad \Downarrow \\
 \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt &= \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\
 &\quad \Downarrow \\
 \int_0^x (x-t)^4 \cos(t) dt &= 4 \int_0^x (x-t)^3 \sin(t) dt
 \end{aligned}$$


---

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$f(x) = e^x = \exp(x)$$

$$e^x = \sum_{k=0}^n \frac{\exp^{(k)}(a)}{k!} (x-a)^k + \int_a^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$a = 0$$

$$e^x = \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} (x-0)^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$\begin{aligned}
 e^x &= \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} (x-0)^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
 &= \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} x^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
 &= \sum_{k=0}^n \frac{\exp(0)}{k!} x^k + \int_0^x \exp(t) \frac{(x-t)^n}{n!} dt \\
 &= \sum_{k=0}^n \frac{e^0}{k!} x^k + \int_0^x e^t \frac{(x-t)^n}{n!} dt = \sum_{k=0}^n \frac{1}{k!} x^k + \int_0^x e^t \frac{(x-t)^n}{n!} dt \\
 &= \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\
 e^x &= \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt
 \end{aligned}$$


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$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt$$


---

$$\lim_{n \rightarrow \infty} \frac{1+n+\frac{n^2}{2}+\cdots+\frac{n^n}{n!}}{e^n}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1+n+\frac{n^2}{2}+\cdots+\frac{n^n}{n!}}{e^n}, \wedge e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\ &= \lim_{n \rightarrow \infty} \frac{1+n+\frac{n^2}{2}+\cdots+\frac{n^n}{n!}}{\left[ \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \right]_{x=n}} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \frac{n^k}{k!}}{\sum_{k=0}^n \frac{n^k}{k!} + \frac{1}{n!} \int_0^n (n-t)^n e^t dt} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{\frac{1}{n!} \int_0^n (n-t)^n e^t dt}{\sum_{k=0}^n \frac{n^k}{k!}}} \end{aligned}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1+n+\frac{n^2}{2}+\cdots+\frac{n^n}{n!}}{e^n}, \wedge e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \frac{n^k}{k!}}{e^n}, \wedge e^n = \sum_{k=0}^n \frac{n^k}{k!} + \frac{1}{n!} \int_0^n (n-t)^n e^t dt \Rightarrow \sum_{k=0}^n \frac{n^k}{k!} = e^n - \frac{1}{n!} \int_0^n (n-t)^n e^t dt \\ &= \lim_{n \rightarrow \infty} \frac{e^n - \frac{1}{n!} \int_0^n (n-t)^n e^t dt}{e^n} = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt \right], \text{ if } \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (t-n)^n e^{t-n} dt \in \mathbb{R} \\ &= \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt = 1 - \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt, x = n-t \\ &= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_{t=0}^{t=n} (n-t)^n e^{t-n} dt, t = n-x \\ &= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_{n-x=0}^{n-x=n} (n-(n-x))^n e^{(n-x)-n} d(n-x) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n!} \left( - \int_{x=n}^{x=0} x^n e^{-x} dx \right) = \lim_{n \rightarrow \infty} \frac{1}{n!} \left( \int_{x=0}^{x=n} x^n e^{-x} dx \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n x^n e^{-x} dx \end{aligned}$$

$$\int x^n e^{-x} dx$$

difficult process

$$\begin{aligned}
\int x^n e^{-x} dx &= - \int x^n de^{-x} \\
&= - \left[ x^n e^{-x} - \int e^{-x} dx^n \right] \\
&= - \left[ x^n e^{-x} - \int e^{-x} n x^{n-1} dx \right] \\
&= - x^n e^{-x} + n \int x^{n-1} e^{-x} dx
\end{aligned}$$

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx$$

$$\begin{aligned}
\int_0^n x^n e^{-x} dx &= [-x^n e^{-x}]_{x=0}^n + n \int_0^n x^{n-1} e^{-x} dx \\
&= [-n^n e^{-n} - (-0^n e^{-0})] + n \int_0^n x^{n-1} e^{-x} dx \\
&= -n^n e^{-n} + n \int_0^n x^{n-1} e^{-x} dx \\
\int_0^n x^n e^{-x} dx &= -n^n e^{-n} + n \int_0^n x^{n-1} e^{-x} dx
\end{aligned}$$

more formal process with gamma function

<https://www.wolframalpha.com/input/?i2d=true&i=Divide%5BIntegrate%5BPower%5Bx%2Cn%5DPower%5Be%2C-x%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D>

<https://www.wolframalpha.com/input/?i2d=true&i=Limit%5BDivide%5BIntegrate%5BPower%5Bx%2Cn%5DPower%5Be%2C-x%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D%2Cn-%3E%E2%88%9E%5D>

to be proved

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#### 45.1.1 Elliot Schneider

<https://www.youtube.com/watch?v=HQsZG8Yxb7w>

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=12m25s>

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

$$f(x_0 + \epsilon) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1}), \epsilon = x - x_0$$

$$f(x + \epsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \epsilon^n$$

$$\begin{aligned}
f(x + \epsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \epsilon^n = \sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n} f(x)}{n!} \epsilon^n = \sum_{n=0}^{\infty} \frac{\left(\frac{d}{dx}\right)^n f(x)}{n!} \epsilon^n \\
&= \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n f(x)}{n!} = \left( \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n}{n!} \right) f(x) = e^{\epsilon \frac{d}{dx}} f(x)
\end{aligned}$$


---

univariable Taylor operator

$$f(x + \epsilon) = e^{\epsilon \frac{d}{dx}} f(x)$$

$$e^{\epsilon \frac{d}{dx}} = \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n}{n!} = 1 + \epsilon \frac{d}{dx} + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2} + \dots$$


---

$$f(x) = mx + b,$$

$$\begin{aligned}
e^{\epsilon \frac{d}{dx}} f(x) &= \left( 1 + \epsilon \frac{d}{dx} + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2} + \dots \right) (mx + b) \\
&= 1(mx + b) + \epsilon \frac{d}{dx}(mx + b) + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2}(mx + b) + \dots \\
&= (mx + b) + (\epsilon m) + \left( \frac{1}{2} \epsilon^2 0 \right) + (0 + 0 + \dots) \\
&= (mx + b) + (\epsilon m) + (0 + 0 + \dots) \\
&= (mx + b) + (\epsilon m) + 0 \\
&= mx + b + m\epsilon = m(x + \epsilon) + b = f(x + \epsilon)
\end{aligned}$$


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<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=15m34s>

multivariable ...

$$\begin{aligned}
f(x + \epsilon) &= e^{\epsilon \frac{d}{dx}} f(x) \\
f(x + \epsilon_x, y + \epsilon_y, z + \epsilon_z) &= f(x, y + \epsilon_y, z + \epsilon_z) \\
&\quad + \epsilon \frac{\partial}{\partial x} f(x, y + \epsilon_y, z + \epsilon_z) \\
&\quad + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial x^2} f(x, y + \epsilon_y, z + \epsilon_z) \\
&\quad + \dots
\end{aligned}$$

$$\begin{aligned}
f(x + \epsilon) &= e^{\epsilon \frac{d}{dx}} f(x) \\
f(x + \epsilon_x, y + \epsilon_y, z + \epsilon_z) &= f(\mathbf{r} + \boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon} \cdot \nabla} f(\mathbf{r}), \quad \begin{cases} \mathbf{r} = (x, y, z) \\ \boldsymbol{\epsilon} = (\epsilon_x, \epsilon_y, \epsilon_z) \\ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{cases}
\end{aligned}$$

$$\boldsymbol{\epsilon} \cdot \nabla = (\epsilon_x, \epsilon_y, \epsilon_z) \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \epsilon_x \frac{\partial}{\partial x} + \epsilon_y \frac{\partial}{\partial y} + \epsilon_z \frac{\partial}{\partial z}$$


---

multivariable Taylor operator

$$f(\mathbf{r} + \boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon} \cdot \nabla} f(\mathbf{r})$$


---

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=17m36s>

physics

#### 45.1.1.1 making complicated equation simple

linearize

simple pendulum

$\sin(x)$  to  $x$  linearization

potential energy

$$f(x_0 + \epsilon) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1})$$

$$U(x_0 + \epsilon) = \sum_{k=0}^n \frac{U^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1}) = U(x_0) + U'(x_0)x + \frac{1}{2}U''(x_0)x^2 + \dots$$

$$F = \frac{-dU}{dx} = -U'(x_0) - U''(x_0)x - \dots \stackrel{\text{if } U'(x_0)=0}{=} -U''(x_0)x = -kx, k = U''(x_0)$$

#### 45.1.1.2 non-relativistic limit

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=22m32s>

$$\begin{aligned} E &= \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{m^2c^4 + p^2c^2} \\ &= mc^2 \sqrt{1 + \left(\frac{pc}{mc^2}\right)^2} = mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2} = mc^2 \left(1 + \left(\frac{p}{mc}\right)^2\right)^{\frac{1}{2}} \\ &= mc^2 \left[ \frac{1}{0!} + \frac{1}{1!} \left(\frac{1}{2}\right) \left(\left(\frac{p}{mc}\right)^2\right)^1 + \frac{1}{2!} \left(\frac{1}{2}\right) \left(\frac{1}{2} - 1\right) \left(\left(\frac{p}{mc}\right)^2\right)^2 + \dots \right] \\ &= mc^2 \left[ 1 + \frac{1}{2} \left(\frac{p}{mc}\right)^2 - \frac{1}{8} \left(\frac{p}{mc}\right)^4 + \dots \right] = mc^2 \left[ 1 + \frac{1}{2} \frac{p^2}{m^2c^2} - \frac{1}{8} \frac{p^4}{m^4c^4} + \dots \right] \\ &= mc^2 + \frac{1}{2} \frac{p^2}{m} - \frac{1}{8} \frac{p^4}{m^3c^2} + \dots = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots \\ &= E_0 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots, \quad \begin{cases} E_0 = mc^2 \\ \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{mv^2}{2} = \frac{1}{2}mv^2 \end{cases} \end{aligned}$$

slippy here  $p = \gamma mv$

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=24m50s>

leading relativistic correction

$$-\frac{p^4}{8m^3c^2}$$

binding energy of hydrogen atom

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=25m40s>

fine-structure constant

#### 45.1.1.3 quantum momentum operator

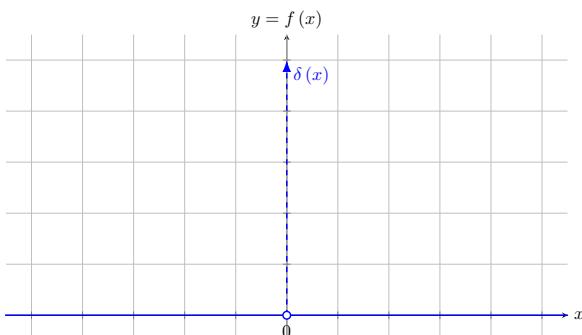
<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=29m13s>

## 45.2 Dirac delta function

Dirac function = Dirac delta function

[https://tikz.net/delta\\_function/](https://tikz.net/delta_function/)

<https://tikz.dev/tikz-arrows>

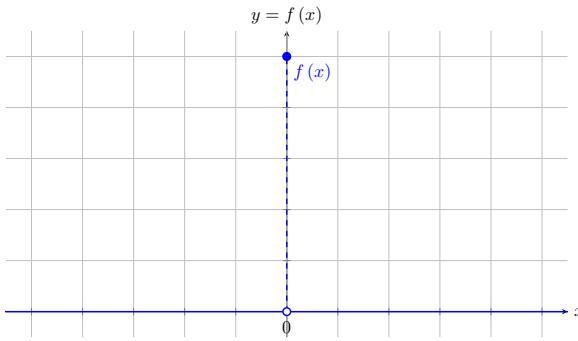


$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(x) dx = 1$$

Figure 45.1: Dirac function = Dirac delta function

$$\text{supp}(f) = \left\{ x \left| \begin{array}{l} x \in \mathcal{D} \\ f(x) \neq 0 \end{array} \right. \right\}$$

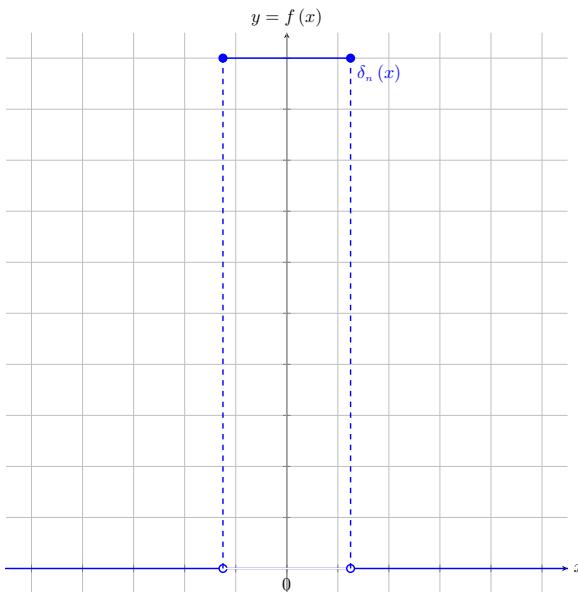
$$\text{supp}(\delta) = \left\{ x \left| \begin{array}{l} x \in \mathcal{D} = \mathbb{R} \\ \delta(x) \neq 0 \end{array} \right. \right\} = \{0\}$$



$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Figure 45.2:  $f(x) = 1$  if  $x = 0$  else 0

<https://tex.stackexchange.com/questions/45275/tikz-get-values-for-predefined-dash-patterns>



$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| \leq \frac{1}{2n} \end{cases}, \forall n \in \mathbb{N}$$

$$\delta(x) = \{\delta_n(x) | n \in \mathbb{N}\} = \{\delta_1(x), \delta_2(x), \dots\} = \lim_{n \rightarrow \infty} \delta_n(x)$$

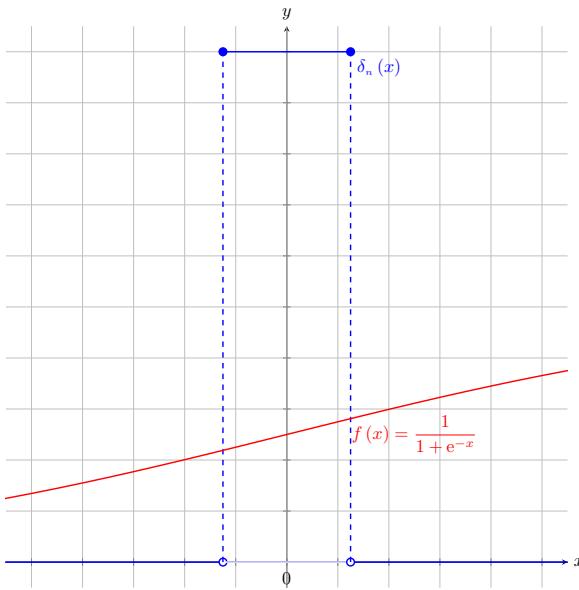
Figure 45.3:  $\delta_n(x)$ 

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \Leftrightarrow \begin{cases} x > \frac{1}{2n} \\ x < -\frac{1}{2n} \end{cases} \\ n & |x| \leq \frac{1}{2n} \Leftrightarrow -\frac{1}{2n} \leq x \leq \frac{1}{2n} \Leftrightarrow x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right] \end{cases}, \forall n \in \mathbb{N}$$

$$\int_{-\infty}^{\infty} \delta_n(x) dx = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} dx = n [x]_{-\frac{1}{2n}}^{\frac{1}{2n}} = n \left[ \frac{1}{2n} - \left( -\frac{1}{2n} \right) \right] = n \cdot \frac{1}{n} = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx \\
&= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x) dx \\
&= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx, \begin{cases} f(x) \in [m, M] \subseteq f\left(\left[\frac{-1}{2n}, \frac{1}{2n}\right]\right) \\ \downarrow \\ m \leq f(x) \leq M \end{cases} \\
\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n m dx &\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n M dx \\
m = m \cdot 1 &= m \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx \leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq M \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = M \cdot 1 = M \\
&\Downarrow ?? \\
\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx &\stackrel{\exists \xi_n \in (\frac{-1}{2n}, \frac{1}{2n})}{=} f(\xi_n) \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = f(\xi_n) \cdot 1 = f(\xi_n) \\
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx = \lim_{n \rightarrow \infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\
&= f(0) \\
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= f(0)
\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$



$$\begin{aligned} \int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \\ &= \lim_{n \rightarrow \infty} f(\xi_n), \quad \xi_n \in \left( \frac{-1}{2n}, \frac{1}{2n} \right) \end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-0) f(x) dx = f(0)$$

Figure 45.4:  $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-0) f(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x) dx = f(x_0)$$

$$\int_{-\infty}^{\infty} \delta(x-x') f(x) dx = f(x')$$

<https://www.youtube.com/watch?v=nDa3cqFk80o>

$$\begin{aligned} &\left\{ \{f_n(x)|n \in \mathbb{N}\} \middle| \begin{cases} f_n : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} f_n(x) dx = 1 \end{cases} \right\} \\ &= \left\{ \left\{ \frac{n}{2} e^{-n|x|} \middle| n \in \mathbb{N} \right\}, \right. \\ &\quad \left\{ \frac{1}{\pi} \frac{n}{n^2 x^2 + 1} \middle| n \in \mathbb{N} \right\}, \\ &\quad \left\{ \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \middle| n \in \mathbb{N} \right\}, \\ &\quad \left. \left\{ \frac{1}{\pi} \frac{\sin(nx)}{x} \middle| n \in \mathbb{N} \right\}, \dots \right\} \end{aligned}$$

$$\left\{ \delta(x) \middle| \begin{cases} \delta : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} \delta(x) \cdot 1 dx = 1 \end{cases} \right\}$$

$$\left\{ \delta(x) \middle| \begin{cases} \delta : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} \delta(x-0) \cdot f(x) dx = f(0) \end{cases} \right\}$$

In measure theory, we can define the distance of two functions by

$$d(f, g) = \sqrt{\int_{-\infty}^{\infty} [f(x) - g(x)]^2 dx}$$

for real distance of two square delta function approximations,

$$\begin{aligned} d(\delta_m, \delta_n) &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x) - \delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2 - 2\delta_m(x)\delta_n(x) + [\delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2 dx - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \int_{-\infty}^{\infty} [\delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} \delta_m(x)\delta_m(x) dx - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \int_{-\infty}^{\infty} \delta_n(x)\delta_n(x) dx} \\ &= \sqrt{\delta_m(\xi_m) - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \delta_n(\xi_n)}, \begin{cases} \xi_m \in \left(\frac{-1}{2m}, \frac{1}{2m}\right) \\ \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \end{cases} \\ &= \sqrt{m - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + n}, \text{ if } m > n \\ &= \sqrt{m - 2\delta_n(\xi_n) + n}, \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\ &= \sqrt{m - 2n + n} = \sqrt{m - n} \in \mathbb{R} \\ d(\delta_m, \delta_n) &\stackrel{m \geq n}{=} \sqrt{m - n} \in \mathbb{R} \end{aligned}$$

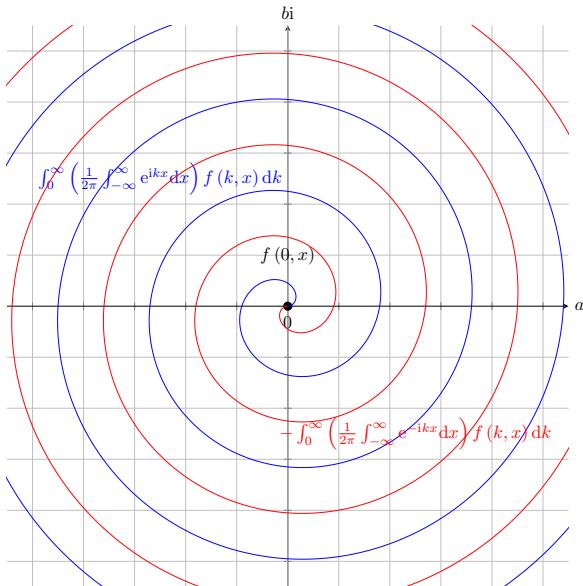
$\langle d(\delta_m, \delta_n) \rangle_{n \in \mathbb{N}} = \langle \sqrt{m - n} \rangle_{n \in \mathbb{N}}$  is not a Cauchy series, not even mentioned convergence

#### 45.2.1 complex delta function

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \\
&= \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^{k'-} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'+}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^0 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i0x} dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^0 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ - \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ - \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \right) f(x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \infty \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{k'-}^{k'+} \frac{1}{dk} f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} - e^{-ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ f(0, x) + \int_0^{\infty} \left( \frac{1}{2\pi} 0 \right) f(k, x) dk \right] \approx f(0, x) \\
f(0, x) &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk = \int_{-\infty}^{\infty} \delta(k) f(k, x) dk = \int_{-\infty}^{\infty} \delta(k - 0) f(k, x) dk
\end{aligned}$$

<https://tex.stackexchange.com/questions/150138/how-can-i-create-a-polar-plot-on-a-cartesian-grid>



$$\begin{aligned}
f(0, x) &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \\
&= \int_{-\infty}^{\infty} \delta(k - 0) f(k, x) dk
\end{aligned}$$

Figure 45.5:  $\int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk$

According to Fourier transform and inverse transform, Fourier analysis<sup>[50]</sup>

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\cdot\omega} \left( \int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s) ds \right) d\omega$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixk} \left( \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right) dk$$

or

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\cdot\omega} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s) ds \right) d\omega$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right) dk$$


---

complex function if analytic always with better properties than real function

$$\begin{aligned} & \left\{ \delta(k) \mid k \in \mathbb{R} \right\} \left| \left\{ \delta : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk = f(0, x) \right\} \right. \\ &= \left\{ \left\{ K \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk \mid k \in \mathbb{R} \right\}, \dots \right\} \end{aligned}$$


---

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\cdot) dx$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk$$

$$\delta(x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx'} (\cdot) dk$$


---

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk$$

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x) dx$$

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ikx} e^{-ikx'} \psi(x') dx' \right) dk \\
&\stackrel{\text{Fubini}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk \right) dx' \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk \right) dx' \\
&= \int_{-\infty}^{\infty} \psi(x') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk \right) dx' \\
&= \int_{-\infty}^{\infty} \psi(x') \delta(x - x') dx' = \int_{-\infty}^{\infty} \delta(x - x') \psi(x') dx' = \psi(x)
\end{aligned}$$

---


$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk$$

$$\int_{-\infty}^{\infty} |\psi(k)|^2 dk = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$

$$\begin{cases} \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk \\ \psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \end{cases}$$

$$\begin{aligned}
\psi^*(k) &= \overline{\psi}(k) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{-ikx} \psi(x)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{-ikx}} \overline{\psi(x)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx
\end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} |\psi(k)|^2 dk = \int_{-\infty}^{\infty} \psi^*(k) \psi(k) dk \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx \right) \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx \right) \left( \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} e^{ikx} e^{-ikx'} \psi(x') dx' dx dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dx' dx dk \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dx' dx dk \\
&\stackrel{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \psi(x') \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \psi(x') \delta(x-x') dx' dx \\
&= \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} \psi(x') \delta(x-x') dx' dx \\
&= \int_{-\infty}^{\infty} \psi^*(x) \psi(x) dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx
\end{aligned}$$

$$\int_{-\infty}^{\infty} |\psi(k)|^2 dk = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$


---

convolution

$$\psi(x) = \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \Rightarrow \psi(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)$$

$$\psi(x) = \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1$$

$$\psi_2(x-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2$$

$$\begin{cases} \psi(x) = & \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \\ \psi_1(y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \\ \psi_2(x-y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \end{cases}$$

$$\begin{aligned}
\psi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \right) \left( \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \right) dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik_1 y} e^{-ik_2 y} \psi_1(k_1) e^{-ikx} e^{ik_2 x} \psi_2(k_2) dk_1 dk_2 dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) e^{i(k_2-k)x} \psi_2(k_2) dk_1 dk_2 dy dx \\
&\stackrel{\text{Fubini}}{=} \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) e^{i(k_2-k)x} \psi_2(k_2) dy dx dk_1 dk_2 \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right) \left( \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right) dk_1 dk_2 \\
&\quad \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right) \left( \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right) dk_1 dk_2 \\
&= \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right] dk_1 dk_2 \\
&= \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \psi_1(k_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} dy \right] \left[ \psi_2(k_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_2-k)x} dx \right] dk_1 dk_2 \\
&= \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\psi_1(k_1) \delta(k_1 - k_2)] [\psi_2(k_2) \delta(k_2 - k)] dk_1 dk_2 \\
&\stackrel{\text{Fubini}}{=} \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\psi_1(k_1) \delta(k_1 - k_2)] [\psi_2(k_2) \delta(k_2 - k)] dk_2 dk_1 \\
&= \sqrt{2\pi} \int_{-\infty}^{\infty} [\psi_1(k_1)] \left[ \int_{-\infty}^{\infty} \delta(k_1 - k_2) \psi_2(k_2) \delta(k_2 - k) dk_2 \right] dk_1 \\
&= \sqrt{2\pi} \int_{-\infty}^{\infty} [\psi_1(k_1)] [\delta(k_1 - k) \psi_2(k)] dk_1 = \sqrt{2\pi} \psi_2(k) \int_{-\infty}^{\infty} \psi_1(k_1) \delta(k_1 - k) dk_1 \\
&= \sqrt{2\pi} \psi_2(k) \psi_1(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)
\end{aligned}$$

$$\begin{cases} \psi(x) = & \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \\ \psi_1(y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \\ \psi_2(x-y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \end{cases} \Downarrow \psi(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)$$

### 45.2.2 3D delta function

<https://www.youtube.com/watch?v=Y8y965ZAmQE>

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x') \delta(y - y') \delta(z - z')$$

Poisson equation

$$\begin{aligned}
\Delta \left( \frac{1}{r} \right) &= \nabla^2 \left( \frac{1}{\sqrt{\mathbf{r}^2}} \right) = \nabla \cdot \nabla \left( \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \\
&= \nabla \cdot \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) = \nabla \cdot \frac{-\mathbf{r}}{r^3} \\
&= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\
&= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\
&= \frac{+2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0
\end{aligned}$$

$$\Delta \left( \frac{1}{r} \right) = -4\pi\delta(\mathbf{r})$$

<https://math.stackexchange.com/questions/3774483/derivatives-of-frac1r-and-dirac-delta-function>

$$\begin{aligned}
\Delta \frac{1}{r(\epsilon)} &= \nabla^2 \frac{1}{\sqrt{\mathbf{r}^2 + \epsilon^2}} = \nabla \cdot \nabla \frac{1}{\sqrt{x^2 + y^2 + z^2 + \epsilon^2}} \\
&= \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2 + \epsilon^2}} \right) \\
&= \nabla \cdot \left( \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \right) \\
&= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \\
&= \frac{+2x^2 - y^2 - z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} + \frac{-x^2 + 2y^2 - z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} + \frac{-x^2 - y^2 + 2z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} \\
&= \frac{-3\epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} = \frac{-3\epsilon^2}{(\mathbf{r}^2 + \epsilon^2)^{\frac{5}{2}}}
\end{aligned}$$

$$\Delta \left( \frac{1}{r} \right) = \nabla^2 \left( \frac{1}{r} \right) = \nabla^2 \frac{1}{|\mathbf{r}|} = \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{0}|} = \Delta \frac{1}{|\mathbf{r} - \mathbf{0}|} = \Delta(\mathbf{r} - \mathbf{0}) = \Delta(\mathbf{r})$$

$$\Delta \frac{1}{r(\epsilon)} = \Delta_\epsilon(r) = \Delta_\epsilon(\mathbf{r}) = \Delta_\epsilon(\mathbf{r} - \mathbf{0})$$

$$\lim_{\epsilon \rightarrow 0} \Delta_\epsilon(\mathbf{r}) = \lim_{\epsilon \rightarrow 0} \Delta \frac{1}{r(\epsilon)} = \Delta \left( \frac{1}{r} \right) = \Delta(\mathbf{r})$$

$$K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r} - \mathbf{0}) d\mathbf{r} = f(\mathbf{0})$$

$$K_1 \int_{\mathbb{R}^1} f(x) \delta(x) dx = \lim_{n \rightarrow \infty} K_1 \int_{\mathbb{R}^1} f(x) \delta_n(x) dx = K_1 \int_{\mathbb{R}^1} f(x) \delta_n(x - 0) dx = f(0)$$

$$\begin{aligned}
\int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} &= \int_{\mathbb{R}^3} f(\mathbf{r}) \frac{-3\epsilon^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} d\mathbf{r}, \quad \begin{cases} \mathbf{r} = r\mathbf{n}_s \\ \mathbf{n}_s = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} \end{cases} \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(r\mathbf{n}_s) \frac{-3\epsilon^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} r^2 \sigma(d\mathbf{n}_s) dr, \quad \begin{cases} \mathbb{S}^2 = \text{unit sphere centered at the origin} \\ \sigma(d\mathbf{n}_s) \text{ is the surface measure of } \mathbb{S}^2 \\ r^2 \sigma(d\mathbf{n}_s) = r^2 \sin \phi d\phi d\theta \\ r = \epsilon s \end{cases} \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{-3\epsilon^2}{((\epsilon s)^2 + \epsilon^2)^{\frac{5}{2}}} (\epsilon s)^2 \sigma(d\mathbf{n}_s) d\epsilon s \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{-3\epsilon^5}{\epsilon^5 (s^2 + 1)^{\frac{5}{2}}} s^2 \sigma(d\mathbf{n}_s) ds = - \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} \sigma(d\mathbf{n}_s) ds \\
&= - \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} \sin \phi d\phi d\theta ds = - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds \\
&\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} \\
&= \lim_{\epsilon \rightarrow 0} \left[ - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds \right] \\
&= - 4\pi f(\mathbf{0}) \int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds
\end{aligned}$$

<https://www.integral-calculator.com/>

$$\begin{aligned}
&\int \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds, s = \tan u \\
&= \int \frac{3 \tan^2 u}{(\tan^2 u + 1)^{\frac{5}{2}}} d \tan u = \int \frac{3 \tan^2 u}{(\tan^2 u + 1)^{\frac{5}{2}}} \sec^2 u du \\
&= \int \frac{3 \tan^2 u}{(\sec^2 u)^{\frac{5}{2}}} \sec^2 u du = \int \frac{3 \tan^2 u}{\sec^5 u} \sec^2 u du = \int \frac{3 \tan^2 u}{\sec^3 u} du \\
&= \int \frac{3 \left( \frac{\sin u}{\cos u} \right)^2}{\left( \frac{1}{\cos u} \right)^3} du = 3 \int \sin^2 u \cos u du = 3 \int \sin^2 u d \sin u = 3 \frac{\sin^3 u}{3} + C \\
&= \sin^3 u + C = \left( \frac{s}{\sqrt{s^2 + 1}} \right)^3 + C = \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} + C \stackrel{s \geq 0}{=} \frac{1}{(1 + s^{-2})^{\frac{3}{2}}} + C
\end{aligned}$$

$$\int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds = \left[ \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} \right]_0^\infty = \left[ \frac{1}{(1 + s^{-2})^{\frac{3}{2}}} \right]_{s=\infty} - \left[ \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} \right]_{s=0} = 1 - 0 = 1$$

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} \\
&= \lim_{\epsilon \rightarrow 0} \left[ - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_S) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds \right] \\
&= -4\pi f(\mathbf{0}) \int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds = -4\pi f(\mathbf{0}) 1 = -4\pi f(\mathbf{0})
\end{aligned}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r}) \Delta(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} = -4\pi f(\mathbf{0}) = -4\pi \int_{\mathbb{R}^3} f(\mathbf{r}) \delta(\mathbf{r}) d\mathbf{r}$$

$$\Delta\left(\frac{1}{r}\right) = \Delta(\mathbf{r}) = -4\pi\delta(\mathbf{r})$$

□

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r})$$

<https://math.stackexchange.com/questions/368155/where-does-the-relation-nabla21-r-4-pi-delta3-bf-r-between-laplacian>

$$\begin{aligned}
& \left\{ \delta(k) \mid k \in \mathbb{R} \right\} \left| \left\{ \delta : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk = f(0, x) \right\} \right. \\
&= \left\{ \left\{ K \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk \mid k \in \mathbb{R} \right\}, \dots \right\}
\end{aligned}$$

[34.1.3]

$$\begin{aligned}
& \left\{ \delta(\mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^3 \right\} \left| \left\{ \delta : \mathbb{R}^3 \rightarrow \mathbb{H} \mid \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{k}-\mathbf{0}) f(\mathbf{k}, \mathbf{r}) d^3\mathbf{r} = f(\mathbf{0}, \mathbf{r}) \right\} \right. \\
&= \left\{ \left\{ K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mathbf{k} \cdot \mathbf{r}} (\cdot) d^3\mathbf{k} \mid \mathbf{k} \in \mathbb{R}^3 \right\}, \dots \right\}
\end{aligned}$$

Coulomb law

$$F_e = k \frac{Qq}{r^2}$$

$$\mathbf{F}_e = k \frac{Qq}{r^2} \hat{\mathbf{r}}$$

electric field from point electric charge

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \mathbf{E}(r) = \mathbf{E} = k \frac{q}{r^2} \hat{\mathbf{r}} = k_e \frac{q}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} = k_e q \frac{\mathbf{r}}{|\mathbf{r}|^3} = k_e q \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} \\ &= k_e q \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} = k_e q \left( -\nabla \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = -k_e q \nabla \frac{1}{\|\mathbf{r}\|}\end{aligned}$$

$$\mathbf{E}(\mathbf{r}) = -k_e q \nabla \frac{1}{\|\mathbf{r}\|}$$

electric field from tiny point electric charge

$$\begin{aligned}d\mathbf{E}(\mathbf{r}) &= d\mathbf{E}(r) = d\mathbf{E} = k \frac{dq}{r^2} \hat{\mathbf{r}} = k \frac{dq(r)}{r^2} \hat{\mathbf{r}} = k_e dq(\mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} \\ &= k_e dq(\mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} = k_e dq(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|}\end{aligned}$$

$$d\mathbf{E}(\mathbf{r}) = -k_e dq(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|}$$

$$dq = \rho dV \Leftrightarrow \rho = \frac{dq}{dV}$$

$$dq(\mathbf{r}) = \rho(\mathbf{r}) dV \Leftrightarrow \rho(\mathbf{r}) = \frac{dq(\mathbf{r})}{dV}$$

$$q = \int_q dq(\mathbf{r}) = \int_V \rho(\mathbf{r}) dV = \iiint_V \rho(\mathbf{r}) d^3r$$

$$\begin{aligned}\mathbf{E} &= \iiint_V \mathbf{E}(\mathbf{r}) d^3r = \iiint_V \left( -k_e \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|} \right) d^3r \\ &= -k_e \iiint_V \left( \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|} \right) d^3r \\ &= -k_e \iiint_V \left( \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r} - \mathbf{0}\|} \right) d^3r = \mathbf{E}(\mathbf{0})\end{aligned}$$

$$\Downarrow \begin{cases} Q \text{ at } \mathbf{r} \\ dq \text{ or } \rho \text{ at } \mathbf{r}', \text{ totally } V \end{cases}$$

$$\mathbf{E}(\mathbf{r}) = \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r'$$

$$\nabla_r \cdot \mathbf{E}(\mathbf{r}) = \nabla_r \cdot \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \cdot \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r'$$

$$= \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r^2 \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \Delta_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r'$$

$$= \iiint_V f(\mathbf{r}') \Delta(\mathbf{r} - \mathbf{r}') d^3 r', f(\mathbf{r}') = -k_e \rho(\mathbf{r}')$$

$$= \iiint_V f(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') d^3 r', f(\mathbf{r}') = -k_e \rho(\mathbf{r}')$$

$$= -4\pi f(\mathbf{r}' = \mathbf{r}) = -4\pi [-k_e \rho(\mathbf{r}' = \mathbf{r})]$$

$$= -4\pi f(\mathbf{r}) = -4\pi [-k_e \rho(\mathbf{r})] = 4\pi k_e \rho(\mathbf{r})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi k_e \rho(\mathbf{r})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi k_e \rho(\mathbf{r})$$


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<https://www.bilibili.com/video/BV1qu411578c/?t=7m26s>

[https://en.wikipedia.org/wiki/Helmholtz\\_equation](https://en.wikipedia.org/wiki/Helmholtz_equation)

Helmholtz equation

$$(\Delta + k^2) \frac{e^{\pm i \mathbf{k} \cdot \mathbf{r}}}{r} = -4\pi \delta(\mathbf{r})$$

$$(\nabla_r^2 + \mathbf{k}^2) \frac{e^{\pm i \mathbf{k} \cdot \mathbf{r}}}{\|\mathbf{r}\|} = -4\pi \delta(\mathbf{r})$$

### 45.2.3 quantum state

<https://www.bilibili.com/video/BV1qu411578c/?t=7m45s>

### 45.2.4 rigorous definition of delta function

$$\int_{-\infty}^{\infty} \delta(x - 0) \cdot f(x) dx = f(0)$$

$$\int \delta(x) f(x) dx = f(0)$$

$$\int \delta(x)(\cdot) dx : f(x) \rightarrow f(0)$$

$$f(x) \xrightarrow{\int \delta(x)(\cdot) dx} f(0)$$

$f$  must be with good properties

$$\text{continuous linear functional} \begin{cases} \text{function-like} & \eta_\varphi(f) = \int \varphi(x) f(x) dx \\ \text{non-function-like, e.g.} & \delta_{x'}(f) = f(x') \end{cases}$$

general function with “formally” integral notation

$$\delta_{x'}(f) = \int \delta(x - x') f(x) dx = f(x')$$

<https://www.bilibili.com/video/BV1qu411578c/?t=11m36s>

Quantum mechanics is founded on Hilbert space; However, Dirac delta function is not a linear functional in Hilbert space.?

One of the reason is that we cannot find 1-1 of Dirac delta function, but many-to-one or one-to-many functions.

量子力学本不需 Dirac delta function, 其卻仍大行其道, 實廣義函數為其負重前行.

### 45.3 Fourier expansion or Fourier series

<https://www.bilibili.com/video/BV1m24y1A74K/?t=3m2s>

Taylor vs. Fourier

$$f(x) = x_0 x^0 + x_1 x^1 + x_2 x^2 + \dots = \sum_{k=0}^{\infty} x_k x^k, x_k = \frac{f^{(k)}(0)}{k!}$$

$$f(r) = r_0 r^0 + r_1 r^1 + r_2 r^2 + \dots = \sum_{k=0}^{\infty} r_k r^k, r_k = \frac{f^{(k)}(0)}{k!}$$

$$f(\theta) = a_0 \cos(0\theta) + a_1 \cos(1\theta) + a_2 \cos(2\theta) + \dots = \sum_{k=0}^{\infty} a_k \cos(k\theta)$$

$$a_k = \int_{-\infty}^{\infty} \cos(k\theta) f(\theta) d\theta$$

$$\begin{aligned} f(\theta) &= a_0 \cos(0\theta) + a_1 \cos(1\theta) + a_2 \cos(2\theta) + \dots = \sum_{k=0}^{\infty} a_k \cos(k\theta) \\ &= a_0 \Re[e^{i0\theta}] + a_1 \Re[e^{i1\theta}] + a_2 \Re[e^{i2\theta}] + \dots = \sum_{k=0}^{\infty} a_k \Re[e^{ik\theta}] = \sum_{k=0}^{\infty} a_k \Re[(e^{i\theta})^k] \\ &= \Re[a_0 e^{i0\theta} + a_1 e^{i1\theta} + a_2 e^{i2\theta} + \dots] = \Re \left[ \sum_{k=0}^{\infty} a_k e^{ik\theta} \right] = \Re \left[ \sum_{k=0}^{\infty} a_k (e^{i\theta})^k \right] \end{aligned}$$

$$f(x) = x_0 x^0 + x_1 x^1 + x_2 x^2 + \dots = \sum_{k=0}^{\infty} x_k x^k$$

$$f(z) = z_0 z^0 + z_1 z^1 + z_2 z^2 + \cdots = \sum_{k=0}^{\infty} z_k z^k$$

$$z = r e^{i\theta}$$

$$\begin{aligned} f(z) &= z_0 z^0 + z_1 z^1 + z_2 z^2 + \cdots = \sum_{k=0}^{\infty} z_k z^k \\ f(re^{i\theta}) &= z_0 (re^{i\theta})^0 + z_1 (re^{i\theta})^1 + z_2 (re^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} z_k (re^{i\theta})^k \\ &= f(r) = z_0 (e^{i\theta})^0 r^0 + z_1 (e^{i\theta})^1 r^1 + z_2 (e^{i\theta})^2 r^2 + \cdots = \sum_{k=0}^{\infty} z_k (e^{i\theta})^k r^k \\ &= r_0 r^0 + r_1 r^1 + r_2 r^2 + \cdots = \sum_{k=0}^{\infty} r_k r^k, \quad r_k = z_k (e^{i\theta})^k \\ &= f(e^{i\theta}) = z_0 r^0 (e^{i\theta})^0 + z_1 r^1 (e^{i\theta})^1 + z_2 r^2 (e^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} z_k r^k (e^{i\theta})^k \\ &= (e^{i\theta})_0 (e^{i\theta})^0 + (e^{i\theta})_1 (e^{i\theta})^1 + (e^{i\theta})_2 (e^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} (e^{i\theta})_k (e^{i\theta})^k \\ &, (e^{i\theta})_k = z_k r^k \end{aligned}$$

## 45.4 Hilbert space

**Definition 45.1.** sequence limit

$$\lim_{n \rightarrow \infty} a_n = a \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} (n > N \Rightarrow |a_n - a| < \epsilon)$$

$$x_1, x_2, \dots, x_n, x_{n+1}, \dots$$

- to find convergence
  - monotone convergence theorem
  - split into known sequence
  - fraction: Stolz theorem
  - sequence itself
    - \* ratio  $\frac{x_m}{x_n}$ , but not good for some terms being zero
    - \* difference  $x_m - x_n$

**Theorem 45.4.** Cauchy convergence theorem

**45.5** Dirac bracket

**45.6** quantum operator

**45.7** Gram-Schmidt algorithm to find orthonormal basis

**45.8** Hermite polynomials and Legendre polynomials



## Chapter 46

# Lagrange inversion

Lagrange inversion theorem

拉格朗日反演



## Chapter 47

# quantum color dynamics



# Chapter 48

## Lagrangian

- Elliot Schneider: Physics with Elliot
  - Lagrangian fundamentals
  - Lagrangian mechanics

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458875>

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<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912464>

# Chapter 49

## Hamiltonian

- Elliot Schneider: Physics with Elliot
  - Hamiltonian mechanics



# Chapter 50

## Fourier analysis

### 50.1 basic calculation

$$\int_{\frac{-\tau}{2} + \psi}^{\frac{\tau}{2} + \psi} \cos\left(n \frac{2\pi}{\tau} t + \phi\right) dt = \left[ \frac{\sin\left(n \frac{2\pi}{\tau} t + \phi\right)}{n \frac{2\pi}{\tau}} \right]_{\frac{-\tau}{2} + \psi}^{\frac{\tau}{2} + \psi} = 0$$

$$\begin{aligned}\cos \alpha \cos \beta &= \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\ \sin \alpha \sin \beta &= \frac{-\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\ \sin \alpha \cos \beta &= \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} \\ \cos \alpha \sin \beta &= \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}\end{aligned}$$

$$\begin{aligned}
& \int \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} \int \cos\phi_0 dt & m = n = 0 \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos\phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n) + \cos(-\phi_n)}{2} dt & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos\phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n) + \cos\phi_n}{2} dt & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos\phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n)}{2} dt + \frac{t \cos\phi_n}{2} & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\cos\alpha \cos\beta &= \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\
\sin\alpha \sin\beta &= \frac{-\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\
\sin\alpha \cos\beta &= \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} \\
\cos\alpha \sin\beta &= \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}
\end{aligned}$$

$$\begin{aligned}
& \int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} \tau \cos\phi_0 & m = n = 0 \\ \frac{\tau \cos\phi_n}{2} & m = n \\ 0 & m \neq n \end{cases} \\
&= \begin{cases} \tau \cos\phi_0 & n = 0 \\ \frac{\tau \cos\phi_n}{2} & n \neq 0 \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{\tau \cos\phi_n} \int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \end{cases} & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \sin\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\ = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \sin\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\ = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$


---

$$\frac{2}{\tau \cos \phi_n} \int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt = \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \\ 0 & m \neq n \end{cases}$$

$$\frac{2}{\tau \cos \phi_n} \int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \sin\left(m\frac{2\pi}{\tau}t\right) \sin\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt = \begin{cases} 1 & m = n \neq 0 \\ 0 & \neg(m = n \neq 0) \end{cases}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \sin\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \sin\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$


---

## 50.2 Simon Xu

**50.2.1 DFT = discrete Fourier transform**

**50.2.2 FFT = fast Fourier transform**

**50.2.3 wavelet**



# Chapter 51

## axiom of choice

### 51.1 Zorn lemma



# **Chapter 52**

## **linear algebra**

### **52.1 CCJou**

[https://www.youtube.com/playlist?list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAc3t\\_Ju](https://www.youtube.com/playlist?list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju)

### **52.2 Chi, Chen-Yu**

<https://www.youtube.com/playlist?list=PLJWAeYEa8SXBej3kuQMz8vV41VabZUILb>



# **Chapter 53**

## **analysis**

### **53.1 Chi, Chen-Yu**

<https://www.youtube.com/playlist?list=PLVJXJebpO4PhAc21JW-cYbzT3sq4s7Qg8>

<https://www.youtube.com/playlist?list=PLil-R4o6jmGihq7XzdNzb0d5hHqEJbr6L>

[https://www.youtube.com/playlist?list=PLil-R4o6jmGhUqtKbZf0LIFKd-xN\\_g\\_M](https://www.youtube.com/playlist?list=PLil-R4o6jmGhUqtKbZf0LIFKd-xN_g_M)

[https://www.youtube.com/playlist?list=PLil-R4o6jmGhkuZPmKL\\_A5Y7N4HOsa1nX](https://www.youtube.com/playlist?list=PLil-R4o6jmGhkuZPmKL_A5Y7N4HOsa1nX)

### **53.2 Chen, Jin-Tzu**

<https://www.youtube.com/playlist?list=PLil-R4o6jmGjoxAWZurHXAY0q9yxwXv5F>



# Chapter 54

## matrix calculus

13

$$\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle = [x_1 \ x_2 \ \dots \ x_n]^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(x_1, x_2, \dots, x_n) = f(\langle x_1, x_2, \dots, x_n \rangle) = f(\mathbf{x})$$
$$\mathbf{y} = \langle y_1, y_2, \dots, y_m \rangle = [y_1 \ y_2 \ \dots \ y_m]^\top = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

### 54.1 vector-by-scalar

$$\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$

### 54.2 scalar-by-vector

$$\nabla f = \frac{\partial}{\partial \mathbf{x}} f = \frac{\partial f}{\partial \mathbf{x}} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}^\top = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$f_i = f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x})$$

$$\mathbf{f} = \langle f_1, f_2, \dots, f_m \rangle = [f_1 \ f_2 \ \dots \ f_m]^\top = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

## 54.3 vector-by-vector

### 54.3.1 numerator-layout notation

分子布局

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} \\ \frac{\partial y_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial y_p}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} & \frac{\partial \mathbf{y}}{\partial x_2} & \dots & \frac{\partial \mathbf{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \left[ \frac{\partial y_i}{\partial x_j} \right]_{m \times n}$$

### 54.3.2 denominator-layout notation

分母布局

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} & \frac{\partial y_2}{\partial \mathbf{x}} & \dots & \frac{\partial y_m}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \left[ \frac{\partial y_j}{\partial x_i} \right]_{n \times m}$$

# Chapter 55

## genetics

### 55.1 term

- chromosome
- locus (sl.) loci (pl.)
- marker
- allele
- haplotype
- genotype
- phenotype / trait
  - endophenotype

### 55.2 marker

- large marker: STRP = short tandem repeat polymorphism, STRs = short tandem repeats
  - linkage analysis
  - paternity testing
  - taxonomy
- small marker: SNP = single-nucleotide polymorphism
  - association analysis
  - disease diagnosis
  - pharmacodynamics? drug design / pharmacogenomics = custom drug
- RFLP = restriction fragment length polymorphism
- platform
  - customized: MALDI-TOF MS(= mass spectrometry)
  - whole-genome gene chip
    - \* Affymetrix
      - Taiwan BioBank: TWB2.0
    - \* Illumina
  - NGS = next-generation sequencing

### 55.3 genome project

- HGP = Human Genome Project
  - 20~25K genes

- 3 billion bps(base pairs)
- ELSI = ethical, legal, and social issues
- 99.9% bps are exactly the same in all people
- germline mutation
  - \* male : female = 2 : 1
- HapMap = International HapMap Project
  - SNP, haplotype, tag SNP
    - \* haplotypes are combination of SNPs
    - \* tag SNPs can identify unique haplotypes
  - HapMap 1 & 2
    - \* between-ancestry SNP: 1 common SNP per 5 Kb to 1 common SNP per 1 Kb
      - African
      - European
      - East Asian
        - Han Chinese
        - Japanese
    - HapMap 3: more ancestries
  - 1000 Genomes Project
    - NGS
    - identify 95% genetic variants with frequencies at least 1%
    - final phase 77M SNPs
    - browser
      - \* Ensembl GRCh37
      - \* Ensembl GRCh38 [http://asia.ensembl.org/Homo\\_sapiens/Info/Index](http://asia.ensembl.org/Homo_sapiens/Info/Index)
  - TWB = Taiwan BioBank
    - browser: TaiwanView <https://taiwanview.twbiobank.org.tw/index>
    - pricing: [https://www.biobank.org.tw/about\\_price.php](https://www.biobank.org.tw/about_price.php)
  - TPMI = Taiwan Precision Medicine Initiative <https://tpmi.ibms.sinica.edu.tw/www/precision-medicine/>

## 55.4 linkage analysis

- Mendel 1<sup>st</sup> & 2<sup>nd</sup> laws
  - law of segregation ~ 3 : 1
  - law of assortment ~ 9 : 3 : 3 : 1
- phenotypic model by R.A. Fisher
  -

$$P = G + E$$

- \*  $G$  is the genotypic component
- \*  $E$  is the environmental component

$$P = G + E + G \cdot E$$

- \*  $G \cdot E$  is the interaction between the genotypic component and environmental component
- linkage = co-segregation = cosegregation
  - $\theta$  = recombination fraction: 1% recombination = 1 crossover per 100 meioses = 1 cM(centiMorgan) on genetic map
- statistical hypothesis testing for linkage mapping
  - statistical hypothesis testing for categorical trait in linkage mapping: PLA = parametric linkage analysis
    - \*  $H_0$ : no linkage  $\theta = 0.5$
    - \*  $H_1$ : linkage  $\theta < 0.5$

- statistical hypothesis testing for quantitative trait in linkage mapping: VCLA = variance component linkage analysis
  - \*  $H_0$ : no linkage  $\sigma_q^2 = 0$
  - \*  $H_1$ : linkage  $\sigma_q^2 > 0$
- study design
  - case control
  - trio
  - affected / discordant sib-pair
  - extended pedigree
- data format: linkage format
  - family-based
    - \* PID = pedigree Id
    - \* IID = individual Id
    - \* FID = father Id
    - \* MID = mother Id
    - \* gender
    - \* affection
    - \* marker
      - M1 = marker 1
      - M2 = marker 2
      - ...
- single major locus model
  - a two-allele  $A$  and  $a$  locus influences a dichotomous trait
  - allele frequency
    - \*  $p = P(A)$
    - \*  $q = P(a) = 1 - p$
  - penetrance
    - \*  $f_{AA} = P(\text{affected} \mid AA)$
    - \*  $f_{Aa} = P(\text{affected} \mid Aa) = f_{aA}$
    - \*  $f_{aa} = P(\text{affected} \mid aa)$
  - disease mode of inheritance
    - \* dominant model
      - $\begin{cases} f_{AA} = 1 \\ f_{Aa} = 1 \\ f_{aa} = 0 \end{cases}$
    - \* recessive model
      - $\begin{cases} f_{AA} = 0 \\ f_{Aa} = 0 \\ f_{aa} = 1 \end{cases}$
    - \* additive model
      - $\begin{cases} f_{AA} = 1 \\ f_{Aa} = \frac{1}{2} \\ f_{aa} = 0 \end{cases}$
    - \* phenocopy model
      - $f_{aa} > 0$  perhaps due to environmental cause
    - \* liability model
      - e.g.  $f_{AA}, f_{Aa}, f_{aa}$  are age-dependent

#### 55.4.1 PLA = parametric linkage analysis

- LOD score

### 55.4.2 VCLA = variance component linkage analysis

- allele-sharing: IBS and IBD
  - IBS = identity-by-state
  - IBD = identity-by-descent

## 55.5 association analysis

- LD = linkage disequilibrium
- genotype & allele frequency
  - diallelic marker
    - \*  $p_{AA}$
    - \*  $p_{Aa}$
    - \*  $p_{aA}$
    - \*  $p_{aa}$
    - \*  $p_A$
    - \*  $p_a$
- HWC = Hardy-Weinberg condition
  - HWE = Hardy-Weinberg equilibrium
  - HWD = Hardy-Weinberg disequilibrium
- gametic or haplotype frequency
  - LE = linkage equilibrium
  - LD = linkage disequilibrium

# Chapter 56

## multivariate normal distribution

**Definition 56.1.** probability density function (PDF) of normal distribution (= Gaussian distribution)

$$\mathcal{N}(x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

If a continuous random variable  $X$  follows a normal distribution with mean 0 and variance  $\sigma^2$

$$\begin{aligned} X &\sim n(\mu, \sigma^2) = \mathcal{N}(\mu, \sigma^2) \\ \Leftrightarrow f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = n(x | \mu, \sigma^2) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \mathcal{N}(x | \mu, \sigma^2) \end{aligned}$$

A random variable  $X$  can be standardized by subtracting the mean  $\mu$  and dividing by the standard deviation  $\sigma$ , resulting in the standardized random variable  $Z$

$$Z = \frac{X - \mu}{\sigma} \text{ or } z = \frac{x - \mu}{\sigma}$$

The standardized random variable  $Z$  follows the standard normal distribution

$$\begin{aligned} Z &\sim n(0, 1^2) = \mathcal{N}(0, 1^2) \\ \Leftrightarrow f_Z(z) &= \frac{e^{-\frac{1}{2}\left(\frac{z-0}{1}\right)^2}}{1 \cdot \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \\ &= n(z | 0, 1^2) = \mathcal{N}(z | 0, 1^2) \end{aligned}$$

To generalize from univariate random variables to multivariate random vectors, a random vector<sup>14</sup>

$$\mathbf{Z} = \langle Z_1, Z_2, \dots, Z_p \rangle = [Z_1 \ Z_2 \ \dots \ Z_p]^\top = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}$$

with  $p$  random variable components is said to follow the standard multivariate normal distribution if and only if its joint PDF is given by

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{p/2}} \exp \left\{ -\frac{\mathbf{z}^T \mathbf{z}}{2} \right\} = \frac{1}{(2\pi)^{p/2}} \exp \left\{ -\frac{\mathbf{z} \cdot \mathbf{z}}{2} \right\} \quad (56.1)$$

(56.1) can be rewritten as the following

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= \underbrace{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \cdots \frac{1}{\sqrt{2\pi}}}_{p \text{ times}} \exp \left\{ -\frac{z_1^2}{2} - \frac{z_2^2}{2} - \cdots - \frac{z_p^2}{2} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_1^2}{2} \right\} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_2^2}{2} \right\} \cdots \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_p^2}{2} \right\} \\ &= f(z_1) f(z_2) \cdots f(z_p) \end{aligned}$$

where

$$f_{Z_i}(z_i) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_i^2}{2} \right\} = f(z_i) \Rightarrow Z_i \sim \mathcal{N}(0, 1^2) = \text{n}(0, 1^2) \quad (56.2)$$

$$\begin{aligned} f_{Z_i}(z_i) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{Z}}(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_p) dz_1 \cdots dz_{i-1} dz_{i+1} \cdots dz_p \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1) \cdots f(z_{i-1}) f(z_i) f(z_{i+1}) \cdots f(z_p) dz_1 \cdots dz_{i-1} dz_{i+1} \cdots dz_p \\ &= f(z_i) \int_{-\infty}^{\infty} f(z_1) dz_1 \cdots \int_{-\infty}^{\infty} f(z_{i-1}) dz_{i-1} \int_{-\infty}^{\infty} f(z_{i+1}) dz_{i+1} \cdots \int_{-\infty}^{\infty} f(z_p) dz_p \\ &= f(z_i) \stackrel{(56.2)}{=} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_i^2}{2} \right\} \end{aligned}$$

**Definition 56.2.** covariance matrix of a random vector<sup>8</sup>

$$\text{C}[\mathbf{X}] = \text{Cov}[\mathbf{X}] = \text{V}[\mathbf{X}] = \mathbb{E}[(\mathbf{X} - \mathbb{E}(\mathbf{X}))(\mathbf{X} - \mathbb{E}(\mathbf{X}))^T]$$

$$\mathbf{X} = \langle X_1, X_2, \dots, X_p \rangle = [X_1 \ X_2 \ \cdots \ X_p]^T = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$$

$$\mathbb{E}[\mathbf{X}] = \langle \mathbb{E}[X_1], \mathbb{E}[X_2], \dots, \mathbb{E}[X_p] \rangle = [\mathbb{E}[X_1] \ \mathbb{E}[X_2] \ \cdots \ \mathbb{E}[X_p]]^T = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_p] \end{bmatrix}$$

$$\mathbf{X} - \mathbb{E}[\mathbf{X}] = \begin{bmatrix} X_1 - \mathbb{E}[X_1] \\ X_2 - \mathbb{E}[X_2] \\ \vdots \\ X_p - \mathbb{E}[X_p] \end{bmatrix}$$

$$\begin{aligned}
& [\mathbf{X} - \mathbb{E}(\mathbf{X})] [\mathbf{X} - \mathbb{E}(\mathbf{X})]^\top \\
&= \begin{bmatrix} X_1 - \mathbb{E}[X_1] \\ X_2 - \mathbb{E}[X_2] \\ \vdots \\ X_p - \mathbb{E}[X_p] \end{bmatrix} \begin{bmatrix} X_1 - \mathbb{E}[X_1] & X_2 - \mathbb{E}[X_2] & \cdots & X_p - \mathbb{E}[X_p] \end{bmatrix} \\
&= \begin{bmatrix} (X_1 - \mathbb{E}[X_1])(X_1 - \mathbb{E}[X_1]) & (X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2]) & \cdots & (X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p]) \\ (X_2 - \mathbb{E}[X_2])(X_1 - \mathbb{E}[X_1]) & (X_2 - \mathbb{E}[X_2])(X_2 - \mathbb{E}[X_2]) & \cdots & (X_2 - \mathbb{E}[X_2])(X_p - \mathbb{E}[X_p]) \\ \vdots & \vdots & \ddots & \vdots \\ (X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1]) & (X_p - \mathbb{E}[X_p])(X_2 - \mathbb{E}[X_2]) & \cdots & (X_p - \mathbb{E}[X_p])(X_p - \mathbb{E}[X_p]) \end{bmatrix} \\
&= \begin{bmatrix} (X_1 - \mathbb{E}[X_1])^2 & \cdots & (X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p]) \\ \vdots & \ddots & \vdots \\ (X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1]) & \cdots & (X_p - \mathbb{E}[X_p])^2 \end{bmatrix}
\end{aligned}$$

$$\mathbb{E} [[\mathbf{X} - \mathbb{E}(\mathbf{X})][\mathbf{X} - \mathbb{E}(\mathbf{X})]^T] \quad (56.3)$$

$$= \mathbb{E} \begin{bmatrix} (X_1 - \mathbb{E}[X_1])^2 & \cdots & (X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p]) \\ \vdots & \ddots & \vdots \\ (X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1]) & \cdots & (X_p - \mathbb{E}[X_p])^2 \end{bmatrix} \quad (56.4)$$

$$= \begin{bmatrix} \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] & \cdots & \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p])] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[(X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1])] & \cdots & \mathbb{E}[(X_p - \mathbb{E}[X_p])^2] \end{bmatrix} \quad (56.5)$$

$$= \begin{bmatrix} V(X_1, X_1) & \cdots & V(X_1, X_p) \\ \vdots & \ddots & \vdots \\ V(X_p, X_1) & \cdots & V(X_p, X_p) \end{bmatrix} = \begin{bmatrix} V(X_1, X_1) & V(X_1, X_2) & \cdots & V(X_1, X_p) \\ V(X_2, X_1) & V(X_2, X_2) & \cdots & V(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ V(X_p, X_1) & V(X_p, X_2) & \cdots & V(X_p, X_p) \end{bmatrix} \quad (56.6)$$

$$= \begin{bmatrix} V(X_1) & \cdots & V(X_1, X_p) \\ \vdots & \ddots & \vdots \\ V(X_p, X_1) & \cdots & V(X_p) \end{bmatrix} = \begin{bmatrix} V(X_1) & V(X_1, X_2) & \cdots & V(X_1, X_p) \\ V(X_2, X_1) & V(X_2) & \cdots & V(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ V(X_p, X_1) & V(X_p, X_2) & \cdots & V(X_p) \end{bmatrix} \quad (56.7)$$

$$= \begin{bmatrix} V(X_1) & \cdots & C(X_1, X_p) \\ \vdots & \ddots & \vdots \\ C(X_p, X_1) & \cdots & V(X_p) \end{bmatrix} = \begin{bmatrix} V(X_1) & C(X_1, X_2) & \cdots & C(X_1, X_p) \\ C(X_2, X_1) & V(X_2) & \cdots & C(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ C(X_p, X_1) & C(X_p, X_2) & \cdots & V(X_p) \end{bmatrix} \quad (56.8)$$

$$= \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_p^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = [\sigma_{ij}]_{p \times p} = \Sigma \quad (56.9)$$

$$\mathbf{X} \sim \mathcal{D}(\boldsymbol{\mu}, \Sigma) = d(\boldsymbol{\mu}_{\mathbf{X}}, \Sigma_{\mathbf{X}}) = d(\mathbb{E}[\mathbf{X}], \mathbb{C}[\mathbf{X}]) = d(\mathbb{E}[\mathbf{X}], \mathbb{V}[\mathbf{X}])$$

$$\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}}) = n(\mathbb{E}[\mathbf{Z}], \mathbb{V}[\mathbf{Z}])$$

$$\mathbb{E}[\mathbf{Z}] = \begin{bmatrix} \mathbb{E}[Z_1] \\ \mathbb{E}[Z_2] \\ \vdots \\ \mathbb{E}[Z_p] \end{bmatrix} = [\mathbb{E}[Z_i]]_{p \times 1}$$

$$\Rightarrow \mathbb{E}[Z_i] = \int_{-\infty}^{\infty} z_i f_{Z_i}(z_i) dz_i \stackrel{(56.2)}{=} \int_{-\infty}^{\infty} z_i \frac{e^{-\frac{1}{2}z_i^2}}{\sqrt{2\pi}} dz_i = 0$$

$$\Rightarrow \mathbb{E}[\mathbf{Z}] = \mathbf{0}$$

$$\Rightarrow \mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}} = \mathbf{0}, \Sigma_{\mathbf{Z}}) = n(\mathbf{0}, \mathbb{V}[\mathbf{Z}])$$

$$\mathbb{V}(Z_i) = \int_{-\infty}^{\infty} (z_i - \mu_{Z_i})^2 f_{Z_i}(z_i) dz_i \stackrel{(56.2)}{=} \int_{-\infty}^{\infty} (z_i - 0)^2 \frac{e^{-\frac{1}{2}z_i^2}}{\sqrt{2\pi}} dz_i = 1$$

$$\text{V}(Z_i, Z_j) \stackrel{i \neq j \Rightarrow Z_i, Z_j \text{ are independent}}{=} 0 \quad (56.10)$$

$$\begin{aligned} \text{V}[\mathbf{Z}] &= \begin{bmatrix} \text{V}(Z_1) & \text{V}(Z_1, Z_2) & \cdots & \text{V}(Z_1, Z_p) \\ \text{V}(Z_2, Z_1) & \text{V}(Z_2) & \cdots & \text{V}(Z_2, Z_p) \\ \vdots & \vdots & \ddots & \vdots \\ \text{V}(Z_p, Z_1) & \text{V}(Z_p, Z_2) & \cdots & \text{V}(Z_p) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix} \stackrel{56.10}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_{p \times p} = I_p = I \end{aligned}$$

$$\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}}) = \text{n}(\text{E}[\mathbf{Z}], \text{V}[\mathbf{Z}]) = \mathcal{N}(\mathbf{0}, I) \Leftrightarrow \begin{cases} \boldsymbol{\mu}_{\mathbf{Z}} = \text{E}[\mathbf{Z}] = \mathbf{0} = [0]_p = [0]_{p \times 1} \\ \Sigma_{\mathbf{Z}} = \text{V}[\mathbf{Z}] = I = I_p = I_{p \times p} \end{cases}$$

$$\begin{aligned} \mathbf{Z} &= \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \begin{bmatrix} \frac{X_1 - \mu_1}{\sigma_1} \\ \frac{X_2 - \mu_2}{\sigma_2} \\ \vdots \\ \frac{X_p - \mu_p}{\sigma_p} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_p - \mu_p \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \left( \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \right) = B^{-1}(\mathbf{X} - \boldsymbol{\mu}) \end{aligned}$$

$$\Rightarrow \mathbf{X} = B\mathbf{Z} + \boldsymbol{\mu}$$

$$\mathbf{X} = B\mathbf{Z} + \boldsymbol{\mu} = T(\mathbf{Z})$$

by  $\text{V}[A\mathbf{X} + \mathbf{b}] = A\text{V}[\mathbf{X}]A^{\top}$

$$\Sigma = \Sigma_{\mathbf{X}} = \text{V}[\mathbf{X}] = \text{V}[B\mathbf{Z} + \boldsymbol{\mu}] = B\text{V}[\mathbf{Z}]B^{\top} = BIB^{\top} = BB^{\top} \quad (56.11)$$

Consider two infinitesimal volumes of  $p$ -dimensional parallelepipeds in the different  $\mathbb{R}^p$  spaces<sup>15</sup>

$$V_{\mathbf{x}} = [x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2] \times \cdots \times [x_p, x_p + dx_p]$$

and

$$V_{\mathbf{z}} = [z_1, z_1 + dz_1] \times [z_2, z_2 + dz_2] \times \cdots \times [z_p, z_p + dz_p]$$

Their relationship under linear transformation is

$$\begin{aligned} V_{\mathbf{x}} = T(V_{\mathbf{z}}) &= [T(z_1), T(z_1) + T(dz_1)] \\ &\quad \times [T(z_2), T(z_2) + T(dz_2)] \\ &\quad \times \dots \\ &\quad \times [T(z_p), T(z_p) + T(dz_p)] \end{aligned}$$

and

$$dx_i = \sum_j \frac{\partial x_i}{\partial z_j} dz_j$$

For examples in 2 dimension,

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

Two element infinitesimal one-directional vectors of  $\mathbf{Z}$  transformed into another space of  $\mathbf{X}$  are

$$T(dz_1) = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} dz_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} dz_1 \\ \frac{\partial x_2}{\partial z_1} dz_1 \end{bmatrix}$$

and

$$T(dz_2) = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_2} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 \\ dz_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_2} dz_2 \\ \frac{\partial x_2}{\partial z_2} dz_2 \end{bmatrix}$$

Their corresponding area(volume) in the space of  $\mathbf{X}$  is

$$\begin{aligned} \int_{A_{\mathbf{x}}} dA_{\mathbf{x}} &= \int_{A_{\mathbf{x}}} dx_1 dx_2 = \int_{T(A_{\mathbf{z}})} dx_1 dx_2 \\ &= \int_{A_{\mathbf{z}}} |[T(dz_1) \quad T(dz_2)]| = \int_{A_{\mathbf{z}}} \left| \begin{array}{cc} \frac{\partial x_1}{\partial z_1} dz_1 & \frac{\partial x_1}{\partial z_2} dz_2 \\ \frac{\partial x_2}{\partial z_1} dz_1 & \frac{\partial x_2}{\partial z_2} dz_2 \end{array} \right| \\ &= \int_{A_{\mathbf{z}}} \left| \begin{array}{cc} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{array} \right| dz_1 dz_2 = \int_{A_{\mathbf{z}}} |J| dA_{\mathbf{z}} \end{aligned}$$

To generalize for volumes in  $p$  dimension,

$$\begin{aligned}
\int_{V_x} dV_x &= \int_{V_x} dx_1 dx_2 \cdots dx_p = \int_{T(V_z)} dx_1 dx_2 \cdots dx_p \\
&= \int_{A_z} \left| \begin{bmatrix} T(dx_1) & T(dx_2) & \cdots & T(dx_p) \end{bmatrix} \right| = \int_{V_z} \left| \begin{bmatrix} \frac{\partial x_i}{\partial z_j} dz_j \end{bmatrix}_{p \times p} \right| \\
&= \int_{V_z} \left| \begin{bmatrix} \frac{\partial x_i}{\partial z_j} \end{bmatrix}_{p \times p} \right| dz_1 dz_2 \cdots dz_p = \int_{V_z} |J| dV_z
\end{aligned}$$

i.e.

$$\int_{V_x} dV_x = \int_{V_z} |J| dV_z \quad (56.12)$$

where  $J$  is a Jacobian matrix

$$J = \left[ \frac{\partial x_i}{\partial z_j} \right]_{p \times p} = \frac{\partial \mathbf{x}}{\partial \mathbf{z}}$$

or  $|J|$  is a Jacobian determinant or simply Jacobian

$$|J| = \left| \frac{\partial x_i}{\partial z_j} \right|_{p \times p} = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right|$$

The probability of the same event should be invariant under transformation.

$$\begin{aligned}
\int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_x &= \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dx_1 dx_2 \cdots dx_p \\
&= \int_{T(V_z)} f_{\mathbf{X}}(\mathbf{x}) dx_1 dx_2 \cdots dx_p \\
&= \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_z = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dz_1 dz_2 \cdots dz_p
\end{aligned}$$

i.e.

$$\int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_x = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_z \quad (56.13)$$

$$\begin{cases} \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_x = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_z & 56.13 \\ \int_{V_x} dV_x = \int_{V_z} |J| dV_z & 56.12 \end{cases}$$

$$\mathbf{Z} = B^{-1}(\mathbf{X} - \boldsymbol{\mu}) \quad (56.14)$$

$$\mathbf{z} = B^{-1}(\mathbf{x} - \boldsymbol{\mu})$$

$$\mathbf{X} = B\mathbf{Z} + \boldsymbol{\mu}$$

$$\mathbf{x} = B\mathbf{z} + \boldsymbol{\mu}$$

$$J = \left[ \frac{\partial x_i}{\partial z_j} \right]_{p \times p} = \frac{\partial \mathbf{x}}{\partial \mathbf{z}} = B \quad (56.15)$$

$$|J| = \left| \frac{\partial x_i}{\partial z_j} \right|_{p \times p} = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right| = |B|$$

$$\begin{aligned} & \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_z \stackrel{56.13}{=} \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_x = \int_{V_x} dV_x f_{\mathbf{X}}(\mathbf{x}) \\ & \stackrel{56.12}{=} \int_{V_z} |J| dV_z f_{\mathbf{X}}(\mathbf{x}(\mathbf{z})) = \int_{V_z} f_{\mathbf{X}}(\mathbf{x}(\mathbf{z})) |J| dV_z \\ & f_{\mathbf{Z}}(\mathbf{z}) \stackrel{\Downarrow}{=} f_{\mathbf{X}}(\mathbf{x}(\mathbf{z})) |J| \\ & f_{\mathbf{X}}(\mathbf{x}(\mathbf{z})) \stackrel{\Downarrow}{=} |J|^{-1} f_{\mathbf{Z}}(\mathbf{z}) \stackrel{56.1}{=} |J|^{-1} \frac{1}{(2\pi)^{p/2}} \exp \left\{ \frac{-\mathbf{z}^\top \mathbf{z}}{2} \right\} \\ & f_{\mathbf{X}}(\mathbf{x}) \stackrel{\Downarrow}{=} |J|^{-1} f_{\mathbf{Z}}(\mathbf{z}(\mathbf{x})) \stackrel{56.15, 56.14}{=} |B|^{-1} f_{\mathbf{Z}}(B^{-1}(\mathbf{x} - \boldsymbol{\mu})) \\ & = |B|^{-1} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} [B^{-1}(\mathbf{x} - \boldsymbol{\mu})]^\top [B^{-1}(\mathbf{x} - \boldsymbol{\mu})] \right\} \\ & = |B|^{-1/2} |B|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (B^{-1})^\top B^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\ & = |B|^{-1/2} |B^\top|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (B^\top)^{-1} B^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\ & = |BB^\top|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (BB^\top)^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\ & \stackrel{56.11}{=} |\Sigma|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\ & = (|\Sigma| (2\pi)^p)^{-1/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \end{aligned}$$

**Definition 56.3.** probability density function (PDF) of multivariate normal distribution (= multivariate Gaussian distribution)

$$\mathcal{N}(\mathbf{x} | \boldsymbol{\mu}, \Sigma) = f_{\mathbf{X}}(\mathbf{x}) = (|\Sigma| (2\pi)^p)^{-1/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

**Definition 56.4.** correlation coefficient

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}} \sqrt{\sigma_{jj}}} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2} \sqrt{\sigma_j^2}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\text{V}(X_i, X_j)}{\sqrt{\text{V}(X_i)} \sqrt{\text{V}(X_j)}} = \text{R}(X_i, X_j)$$

## 56.1 bivariate normal distribution

$p = 2$  is the case of bivariate normal distribution

$$\Sigma = [\sigma_{ij}]_{2 \times 2} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}$$

$$\rho_{12} = \rho = \rho_{21}$$

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2(1 - \rho_{12}\rho_{21}) = \sigma_1^2\sigma_2^2(1 - \rho^2)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|a - b|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} \Sigma^{-1} &= \frac{1}{|\Sigma|} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{\sigma_1^2\sigma_2^2(1 - \rho^2)} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{(1 - \rho^2)} \begin{bmatrix} \frac{1}{\sigma_2^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathcal{N} \left( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix} \right) \\ = \left( |\Sigma| (2\pi)^{p=2} \right)^{-1/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\ = \left( \sigma_1^2\sigma_2^2(1 - \rho^2)(2\pi)^2 \right)^{-1/2} \exp \left\{ \frac{-1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \frac{1}{(1 - \rho^2)} \begin{bmatrix} \frac{1}{\sigma_2^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{2(1 - \rho^2)} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{2(1 - \rho^2)} \left[ \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1}{\sigma_1} \right) \left( \frac{x_2 - \mu_2}{\sigma_2} \right) + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2 \right] \right\} \end{aligned}$$

**Definition 56.5.** probability density function (PDF) of bivariate normal distribution (= bivariate Gaussian distribution)

$$\begin{aligned}\mathcal{N} \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix} \right) \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2(1-\rho^2)} \left[ \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1}{\sigma_1} \right) \left( \frac{x_2 - \mu_2}{\sigma_2} \right) + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2 \right] \right\}\end{aligned}$$

# **Chapter 57**

## **logic**

### **57.1 Shai Ben-David**

<https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1p0eE5LEtv3N>



# **Chapter 58**

## **algorithm**

### **58.1 Chen, Vivian**

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUMcBaEuGyWbBXBn21seO3vA>

### **58.2 The Bubble Sort Curve**

[https://www.youtube.com/watch?v=Gm8v\\_MR7TGk](https://www.youtube.com/watch?v=Gm8v_MR7TGk)

<https://linesthatconnect.github.io/blog/a-rigorous-derivation-of-the-bubble-sort-curve/>



# Chapter 59

# computer graphics

## 59.1 Cem Yuksel

[https://www.youtube.com/playlist?list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW\\_Hh](https://www.youtube.com/playlist?list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh)

<https://www.youtube.com/playlist?list=PLplnkTzzqsZS3R5DjmCQsqupu43oS9CFN>



# Chapter 60

## autoregression in time series

張翔老師. 2015. “ARMA Part1.” <https://www.youtube.com/watch?v=G-0dR57W-fo>.

張翔老師. 2015. “ARMA Part2.” <https://www.youtube.com/watch?v=fQaZzO7E6FE>.

張翔老師. 2015. “ARMA Part3.” <https://www.youtube.com/watch?v=Ocw4NXoO8Xo>.  
time series [data]

$$\dots, Y_{t-2}, Y_{t-1}, Y_t, Y_{t+1}, Y_{t+2}, \dots$$

- lag
  - 1<sup>st</sup> lag = lag 1  
 $Y_{t-1}$
  - $k^{\text{th}}$  lag = lag  $k$   
 $Y_{t-k}$

1<sup>st</sup> difference

$$\Delta Y_t = Y_t - Y_{t-1}$$

approximation for RoR = rate of return

$$\begin{aligned}\Delta \ln Y_t &= \ln Y_t - \ln Y_{t-1} = \ln \frac{Y_t}{Y_{t-1}} = \ln \left( 1 + \frac{Y_t - Y_{t-1}}{Y_{t-1}} \right) \\ &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} + O\left(\left(\frac{Y_t - Y_{t-1}}{Y_{t-1}}\right)^2\right) \\ &\approx \frac{Y_t - Y_{t-1}}{Y_{t-1}} = \text{RoR}\end{aligned}$$

$$\begin{cases} \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\ \ln(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \end{cases}$$

$$\begin{aligned}\frac{1}{1+t} &= 1 - t + t^2 - t^3 + \dots \\ \ln(1+x) &= \int_0^x \frac{1}{1+t} dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\end{aligned}$$

**Definition 60.1.** autocorrelation = serial correlation

$$\exists t_1 \neq t_2 [V(Y_{t_1}, Y_{t_2}) \neq 0]$$

$$\begin{cases} E(Y_t) \triangleq \mu_t \\ V(Y_t) \triangleq \sigma_t^2 \end{cases}$$

**Definition 60.2.**  $k^{\text{th}}$  order autocovariance

$$V(Y_t, Y_{t-k}) \triangleq \gamma(t, k)$$

$$\sigma_t^2 \triangleq V(Y_t) = V(Y_t, Y_t) = V(Y_t, Y_{t-0}) = \gamma(t, k=0) = \gamma(t, 0)$$

**Definition 60.3.**  $k^{\text{th}}$  order autocorrelation

$$\rho_{t,t-k} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_{tt}}\sqrt{\sigma_{t-k,t-k}}} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_t^2}\sqrt{\sigma_{t-k}^2}} = \frac{\sigma_{t,t-k}}{\sigma_t \sigma_{t-k}} = \frac{V(Y_t, Y_{t-k})}{\sqrt{V(Y_t)}\sqrt{V(Y_{t-k})}} \triangleq R(Y_t, Y_{t-k}) \triangleq \rho(t, k)$$

**Definition 60.4.** stationary time series

$$\begin{cases} E(Y_t) \triangleq \mu_t = \mu & (1) \text{ independent of } t \\ V(Y_t) \triangleq \sigma_t^2 = \sigma^2 < \infty & (2) \text{ independent of } t \\ V(Y_t, Y_{t-k}) \triangleq \gamma(t, k) = \gamma(k) & (3) \text{ independent of } t \end{cases}$$

properties

$$\begin{cases} V(Y_t) \triangleq \sigma_t^2 = \gamma(t, k=0) = \gamma(k=0) \stackrel{(3)}{=} \gamma(0) \stackrel{(2)}{=} \sigma^2 & (4) \Rightarrow V(Y_{t-k}) = \gamma(0) \\ R(Y_t, Y_{t-k}) \triangleq \rho(t, k) \triangleq \frac{V(Y_t, Y_{t-k})}{\sqrt{V(Y_t)}\sqrt{V(Y_{t-k})}} \\ = \frac{\gamma(t, k)}{\sqrt{\gamma(0)}\sqrt{\gamma(0)}} \stackrel{(3)}{=} \frac{\gamma(k)}{\gamma(0)} \stackrel{(4)}{=} \frac{\gamma_k}{\gamma_0} \triangleq \rho(k) \triangleq \rho_k & (5) \\ \gamma(k) = \gamma(-k) & (6) \Rightarrow \rho(k) = \rho(-k) \end{cases}$$

$$\gamma(k) \stackrel{(3)}{=} V(Y_t, Y_{t-k}) = V(Y_{t-k}, Y_t) = V(Y_{t'}, Y_{t'+k}) = V(Y_{t'}, Y_{t'-(k)}) \stackrel{(3)}{=} \gamma(-k) \Rightarrow (6)$$

point estimation

$$\left\{ \begin{array}{l} \widehat{E}(Y_t) \triangleq \bar{Y} \triangleq \widehat{\mu} = \frac{1}{T} \sum_{t=1}^T Y_t \rightarrow E(Y_t) = \mu \\ \widehat{V}(Y_t) \triangleq \widehat{\gamma}_0 = \frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2 \rightarrow V(Y_t) = \sigma^2 = \gamma_0 \\ \widehat{V}(Y_t, Y_{t-k}) \triangleq \widehat{\gamma}_k \\ = \frac{1}{T} \sum_{t=k+1}^T (Y_t - \bar{Y})(Y_{t-k} - \bar{Y}) = \frac{1}{T} \sum_{t=1}^{T-k} (Y_t - \bar{Y})(Y_{t+k} - \bar{Y}) \rightarrow V(Y_t, Y_{t-k}) = \gamma_k \\ \frac{\widehat{V}(Y_t, Y_{t-k})}{\sqrt{\widehat{V}(Y_t)} \sqrt{\widehat{V}(Y_{t-k})}} \triangleq \widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0} = \frac{\frac{1}{T} \sum_{t=k+1}^T (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2} \rightarrow R(Y_t, Y_{t-k}) = \rho_k \end{array} \right.$$

$$Y_1 Y_{1+k} + Y_2 Y_{2+k} + \cdots + Y_t Y_{t+k} + \cdots + Y_{T-k} Y_T$$

$1^{\text{st}}$ -order autocorrelation estimation

$$\widehat{\rho}_1 = \frac{\widehat{\gamma}_1}{\widehat{\gamma}_0} = \frac{\frac{1}{T} \sum_{t=1+1=2}^T (Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2}$$

## 60.1 $AR(1) = 1^{\text{st}}\text{-order autoregressive model} = \text{first-order autoregressive model}$

張翔老師. 2015. “ARMA Part3.” <https://www.youtube.com/watch?v=Ocw4NXoO8Xo>.

**Definition 60.5.**  $AR(1) = 1^{\text{st}}$ -order autoregressive model\$

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 Y_{t-1} + \varepsilon_t \\ &= \beta_0 + \beta_1 (\beta_0 + \beta_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \beta_0 (1 + \beta_1) + \beta_1^2 Y_{t-2} + \varepsilon_t + \beta_1 \varepsilon_{t-1} \\ &= \beta_0 (1 + \beta_1) + \beta_1^2 (\beta_0 + \beta_1 Y_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \beta_1 \varepsilon_{t-1} \\ &= \beta_0 (1 + \beta_1 + \beta_1^2) + \beta_1^3 Y_{t-3} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_1^2 \varepsilon_{t-2} \\ &\quad \vdots \\ &= \beta_0 (1 + \beta_1 + \beta_1^2 + \cdots) + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_1^2 \varepsilon_{t-2} + \cdots \\ &\stackrel{|\beta_1| < 1}{=} \frac{\beta_0}{1 - \beta_1} + \sum_{k=0}^{\infty} \beta_1^k \varepsilon_{t-k} \end{aligned}$$

$$\left\{ \begin{array}{l} \mu = E(Y_t) = E\left(\frac{\beta_0}{1-\beta_1} + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) \\ = \frac{\beta_0}{1-\beta_1} + \sum_{i=0}^{\infty} \beta_1^i E(\mathcal{E}_{t-i}) \stackrel{\mathcal{E}_t \sim \mathcal{N}(0, \sigma_{\mathcal{E}}^2)}{=} \frac{\beta_0}{1-\beta_1} \quad \mu = \frac{\beta_0}{1-\beta_1} \\ \gamma_0 = \sigma^2 = V(Y_t) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) = V\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) \\ = \sum_{i=0}^{\infty} (\beta_1^i)^2 V(\mathcal{E}_{t-i}) \stackrel{\mathcal{E}_t \sim \mathcal{N}(0, \sigma_{\mathcal{E}}^2)}{=} \sum_{i=0}^{\infty} (\beta_1^2)^i \sigma_{\mathcal{E}}^2 = \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \quad \gamma_0 = \sigma^2 = \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\ \gamma_k = V(Y_t, Y_{t-k}) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) \\ = V\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) = \dots \quad \gamma_k = \frac{\beta_1^k \sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\ \rho_k = R(Y_t, Y_{t-k}) = \frac{\gamma_k}{\gamma_0} = \frac{\frac{\beta_1^k \sigma_{\mathcal{E}}^2}{1-\beta_1^2}}{\sigma_{\mathcal{E}}^2} = \beta_1^k \quad \rho_k = \frac{\gamma_k}{\gamma_0} = \beta_1^k \end{array} \right.$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \beta_1^1 = \beta_1$$

$$\begin{aligned} \gamma_k &= V(Y_t, Y_{t-k}) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) \\ &= V\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) = \dots \end{aligned}$$

$$\mathbf{Y} = \langle Y_1, Y_2, \dots, Y_T \rangle = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_T \end{bmatrix}^T = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix}$$

AR(1) covariance matrix

by (56.3)

$$\begin{aligned}
 V(\mathbf{Y}) &= E \left[ [\mathbf{Y} - E(\mathbf{Y})] [\mathbf{Y} - E(\mathbf{Y})]^T \right] \\
 &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_T^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_{TT} \end{bmatrix} = [\sigma_{ij}]_{T \times T} = \Sigma \\
 &= \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\beta_1^1 \sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-1} \sigma_{\varepsilon}^2}{1-\beta_1^2} \\ \frac{\beta_1^1 \sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-2} \sigma_{\varepsilon}^2}{1-\beta_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_1^{T-1} \sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\beta_1^{T-2} \sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} \end{bmatrix} \\
 &= \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} \begin{bmatrix} 1 & \beta_1 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \cdots & \beta_1^{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \beta_1 & \cdots & \beta_1^{T-2} \\ \beta_1^2 & \beta_1 & 1 & \cdots & \beta_1^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \beta_1^{T-3} & \cdots & 1 \end{bmatrix} \\
 &= \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{T-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{T-1} & \rho_{T-2} & \rho_{T-3} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}
 \end{aligned}$$

$$\mathbf{Y} \sim \mathcal{D}(\boldsymbol{\mu}, \Sigma) = d(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{Y}}) = d(E[\mathbf{Y}], V[\mathbf{Y}])$$

where

$$\Sigma = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}$$

## 60.2 $AR(2) = 2^{\text{nd}}\text{-order autoregressive model} = \text{second-order autoregressive model}$

**Definition 60.6.**  $AR(2) = 2^{\text{nd}}$ -order autoregressive model

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim \mathcal{N}(0, \sigma_{\varepsilon}^2)$$

### 60.3 AR( $p$ ) = $p^{\text{th}}$ -order autoregressive model

$$Y_t = \beta_0 + \sum_{i=1}^p \beta_i Y_{t-i} + \varepsilon_t$$

### 60.4 MA( $q$ ) = $q^{\text{th}}$ -order moving-average model

$$Y_t = \mu + \varepsilon_t + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}$$

### 60.5 ARMA( $p,q$ ) = $p^{\text{th}}, q^{\text{th}}$ -order autoregressive-moving-average model

$$Y_t = \varepsilon_t + \sum_{i=1}^q \alpha_i \varepsilon_{t-i} + \sum_{i=1}^p \beta_i Y_{t-i}$$

# Chapter 61

# conductivity

## 61.1 resistance

[https://www.bilibili.com/video/BV1tN41147SK/?spm\\_id\\_from=333.999.section.playall&vd\\_source=2b462dd516a0bbcdcb7d97d0fc60b38](https://www.bilibili.com/video/BV1tN41147SK/?spm_id_from=333.999.section.playall&vd_source=2b462dd516a0bbcdcb7d97d0fc60b38)

## 61.2 semiconductor

<https://www.youtube.com/watch?v=-lHXZk5M6cI>



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