# math

Joey Yu Hsu

2024-05-26

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math on bookdown started on 2024/01/28

Joey Yu Hsu, MD

"If you are doing research, you may not get a bunch of money but $$." by Joey Yu Hsu, MD

# Part I

# by descipline

# Chapter 1

# mathematics

## 1.1 tool

- formula typesetting
  - TeX
    - LaTeX
      - pdfLaTeX
      - XeLaTeX
      - editor/tool:
        - LyX
        - OverLeaf
        - MathPix Snip
        - Micro$oft Office Word
          - WordTeX https://tomwildenhain.com/wordtex/
            - Pandoc dependent
          - https://superuser.com/questions/1114697/select-a-different-math-font-in-microsoft-word
          - https://www.youtube.com/watch?v=jlX_pThh7z8
        - Micro$oft Office PowerPoint
          - IguanaTeX https://www.jonathanleroux.org/software/iguanatex/
  - MathML
  - MathJax: JavaScript
- plot[3]
- symbolic computing
  - Maple: by MapleSoft
  - Mathematica: by Wolfram
- numeric computing
  - MatLab: by MathWorks

equivalence relation[11]

equivalence class[10]

partition[9]

## 1.2 discipline

# Chapter 2

# physics

## 2.1 discipline

- relativity
  - special relativity
    - Lorentz transformation[17]
  - general relativity
- analytic mechanics
  - Lagrangian mechanics
  - Hamiltonian mechanics
- electromagnetism
- quantum mechanics
- field theory

# Chapter 3

# plot

- LaTeX
  - [TikZ][??]
    * https://tikz.dev/
    * TikZ-3Dplot
    * [PGFplots][??]
      · https://tikz.dev/pgfplots/
      · https://pgfplots.sourceforge.net/gallery.html
      · https://pgfplots.net/
    * editor / export
      · https://zhuanlan.zhihu.com/p/660371706
      · offline
        · TikzEdt: WYSIWYG and live preview
        · TikZiT
      · online
        · OverLeaf
        · MathCha
        · GeoGebra Classic
      · Python
        · TikZplotLib / tikzplotlib[??]
          · matplotlib export to TikZ .tex
          · PyPI
          · GitHub
      · R
        · TikZDevice / tikzDevice
          · r chunk `engine='tikz'` knitr `out.width=if (knitr:::is_html_output()) '100%'`
          · CRAN
            · reference manual
            · vignette: TikZDevice - LaTeX Graphics for R
          · GitHub
    * TikZ library
  - xypic = xy-pic[13]
- OverLeaf
- MathCha
- GeoGebra
  - GeoGebra Classic: to export TikZ
  - GeoGebra Calculator Suite
- Python
  - MatPlotLib / matplotlib[??]
  - [Seaborn] / seaborn[??]
  - Plotly
  - Manim
- R
  - Modern Statistical Graphics
  - ggplot2[28]
    * Modern Statistical Graphics section 5.1
  - GraphViz `.gv`
  - Mermaid `.mmd`

    * about
    * JavaScript based diagramming and charting tool that renders Markdown-inspired text definitions to create and modify diagrams dynamically
  - Shiny
    * R Markdown Guide section 5.1
  - tool
    * Jamovi

neural network plot/draw https://github.com/ashishpatel26/Tools-to-Design-or-Visualize-Architecture-of-Neural-Network

# Chapter 4

# programming language

## 4.1 discipline

- Python[12]
- JavaScript
- SQL = structured query language
- R[18]
  - RMarkdown
    - Bookdown
  - knitr: engine
    - [TikZ][??]
  - reticulate: Python
  - Jamovi
- C#
  - web
    - MVC
    - .NET
  - desktop
    - UWP = Universal Windows Platform
    - WPF = Windows Presentation Foundation
    - WinForms = Windows Forms
  - 3D/game
    - Unity

## 4.2 learning map

- W3School
- SoloLearn
- Codecademy

# Chapter 5

# machine learning

## 5.1 Shai Ben-David

https://www.youtube.com/playlist?list=PLPW2keNyw-usgvmR7FTQ3ZRjfLs5jT4BO

## 5.2 deep learning

我妻幸長

Esc = Einstein summation convention

$$W\boldsymbol{x} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{\nu=0}^{n} w_{0\nu} x_\nu \\ \sum_{\nu=0}^{n} w_{1\nu} x_\nu \\ \vdots \\ \sum_{\nu=0}^{n} w_{m\nu} x_\nu \end{pmatrix} \overset{\text{Esc}}{=} \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = w_{\mu\nu} x_\nu$$

$$\boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}$$

$$\boldsymbol{y}^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal = y_\mu^\intercal = (w_{\mu\nu} x_\nu)^\intercal = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\intercal = \left[ \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right]^\intercal = [W\boldsymbol{x}]^\intercal$$

$$= x_\nu^\intercal w_{\mu\nu}^\intercal = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\intercal = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \boldsymbol{x}^\intercal W^\intercal$$

$$\boldsymbol{x}^\intercal W^\intercal = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\intercal = x_\nu^\intercal w_{\mu\nu}^\intercal = y_\mu^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal = \boldsymbol{y}^\intercal$$

$$
W\boldsymbol{x} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum\limits_{\nu=0}^{n} w_{0\nu}x_\nu \\ \sum\limits_{\nu=0}^{n} w_{1\nu}x_\nu \\ \vdots \\ \sum\limits_{\nu=0}^{n} w_{m\nu}x_\nu \end{pmatrix} \overset{\text{Esc}}{=} \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = w_{\mu\nu}x_\nu, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases}
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$$
= W\boldsymbol{x} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_0 + \sum\limits_{j=1}^{n} w_{0j}x_j \\ b_1 + \sum\limits_{j=1}^{n} w_{1j}x_j \\ \vdots \\ b_m + \sum\limits_{j=1}^{n} w_{mj}x_j \end{pmatrix} \overset{\text{Esc}}{=} \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = b_\mu + w_{\mu j}x_j
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\boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu}x_\nu = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases}
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= \boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu j}x_j + b_\mu = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases}
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$$
= b_\mu^\intercal + x_j^\intercal w_{\mu j}^\intercal = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \boldsymbol{x}^\intercal W^\intercal
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$$= W\boldsymbol{x} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_0 + \sum\limits_{j=1}^{n} w_{0j}x_j \\ b_1 + \sum\limits_{j=1}^{n} w_{1j}x_j \\ \vdots \\ b_m + \sum\limits_{j=1}^{n} w_{mj}x_j \end{pmatrix} \overset{\text{Esc}}{=} \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = b_\mu + w_{\mu j}x_j$$

$$\boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu}x_\nu = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases}$$

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$$= x_\nu^\intercal w_{\mu\nu}^\intercal = x_\nu^\intercal w_{\nu\mu} = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\intercal = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \boldsymbol{x}^\intercal W^\intercal$$

$$= b_\mu^\intercal + x_j^\intercal w_{\mu j}^\intercal = b_\mu^\intercal + x_j^\intercal w_{j\mu} = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal$$

$$\boldsymbol{x}^\intercal W^\intercal = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\intercal = x_\nu^\intercal w_{\nu\mu} = y_\mu^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal = \boldsymbol{y}^\intercal$$

$$\boldsymbol{x}^\intercal W^\intercal = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal = b_\mu^\intercal + x_j^\intercal w_{j\mu} = y_\mu^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal$$

$$
\begin{aligned}
\boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu =& w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
= \boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu =& w_{\mu j} x_j + b_\mu = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\boldsymbol{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\boldsymbol{x}^\intercal W^\intercal =& \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\intercal = x_\nu^\intercal w_{\mu\nu}^\intercal = x_\nu^\intercal w_{\nu\mu} = y_\mu^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal = \boldsymbol{y}^\intercal \\
= \boldsymbol{x}^\intercal W^\intercal =& \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal = b_\mu^\intercal + x_j^\intercal w_{j\mu} = y_\mu^\intercal = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal = \boldsymbol{y}^\intercal
\end{aligned}
$$

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$$
\begin{aligned}
\sigma(\boldsymbol{y}) = \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma(y_\mu) =& \sigma(w_{\mu\nu} x_\nu) = \sigma \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \sigma \left( \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \sigma(W\boldsymbol{x}), \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
= \sigma(\boldsymbol{y}) = \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma(y_\mu) =& \sigma(w_{\mu j} x_j + b_\mu) = \sigma \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \sigma \left( \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \sigma(W\boldsymbol{x}), \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
(\boldsymbol{x}^\intercal W^\intercal)\varsigma =& \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal \varsigma = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\intercal \varsigma = \left(x_\nu^\intercal w_{\mu\nu}^\intercal\right)\varsigma = \left(x_\nu^\intercal w_{\nu\mu}\right)\varsigma = \left(y_\mu^\intercal\right)\varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal \varsigma = (\boldsymbol{y}^\intercal)\varsigma \\
= (\boldsymbol{x}^\intercal W^\intercal)\varsigma =& \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal \varsigma = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal \varsigma = \left(b_\mu^\intercal + x_j^\intercal w_{j\mu}\right)\varsigma = \left(y_\mu^\intercal\right)\varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal \varsigma = (\boldsymbol{y}^\intercal)\varsigma
\end{aligned}
$$

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$$
\begin{aligned}
\sigma\left(\boldsymbol{y}\right) = \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma\left(y_\mu\right) = \sigma\left(w_{\mu\nu}x_\nu\right) = \sigma \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \sigma\left(\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \sigma\left(W\boldsymbol{x}\right), \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
= \sigma\boldsymbol{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma_\mu y_\mu = \sigma_\mu w_{\mu\nu}x_\nu = \sigma_\mu \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \sigma \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W\boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
= \sigma\left(\boldsymbol{y}\right) = \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma\left(y_\mu\right) = \sigma\left(w_{\mu j}x_j + b_\mu\right) = \sigma \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \sigma\left(\begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \sigma\left(W\boldsymbol{x}\right), \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
= \sigma\boldsymbol{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma_\mu y_\mu = \sigma_\mu\left(w_{\mu j}x_j + b_\mu\right) = \sigma_\mu \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \sigma \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W\boldsymbol{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\left(\boldsymbol{x}^\mathsf{T} W^\mathsf{T}\right)\varsigma &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\mathsf{T} \varsigma = \left(x_\nu^\mathsf{T} w_{\mu\nu}^\mathsf{T}\right)\varsigma = \left(x_\nu^\mathsf{T} w_{\nu\mu}\right)\varsigma = \left(y_\mu^\mathsf{T}\right)\varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma = \left(\boldsymbol{y}^\mathsf{T}\right)\varsigma \\
= \boldsymbol{x}^\mathsf{T} W^\mathsf{T}\varsigma &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\mathsf{T} \varsigma = x_\nu^\mathsf{T} w_{\mu\nu}^\mathsf{T}\varsigma_\mu = x_\nu^\mathsf{T} w_{\nu\mu}\varsigma_\mu = y_\mu^\mathsf{T}\varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma_\mu = \boldsymbol{y}^\mathsf{T}\varsigma \\
= \left(\boldsymbol{x}^\mathsf{T} W^\mathsf{T}\right)\varsigma &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\mathsf{T} \varsigma = \left(b_\mu^\mathsf{T} + x_j^\mathsf{T} w_{j\mu}\right)\varsigma = \left(y_\mu^\mathsf{T}\right)\varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma = \left(\boldsymbol{y}^\mathsf{T}\right)\varsigma \\
= \boldsymbol{x}^\mathsf{T} W^\mathsf{T}\varsigma &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\mathsf{T} \varsigma = \left(b_\mu^\mathsf{T} + x_j^\mathsf{T} w_{j\mu}\right)\varsigma_\mu = y_\mu^\mathsf{T}\varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma_\mu = \boldsymbol{y}^\mathsf{T}\varsigma
\end{aligned}
$$

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$$
\begin{aligned}
\sigma\boldsymbol{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma_\mu y_\mu = \sigma_\mu w_{\mu\nu}x_\nu = \sigma_\mu \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \sigma \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W\boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
= \sigma\boldsymbol{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} &= \sigma_\mu y_\mu = \sigma_\mu\left(w_{\mu j}x_j + b_\mu\right) = \sigma_\mu \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \sigma \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W\boldsymbol{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\boldsymbol{x}^\mathsf{T} W^\mathsf{T}\varsigma &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\mathsf{T} \varsigma = x_\nu^\mathsf{T} w_{\mu\nu}^\mathsf{T}\varsigma_\mu = x_\nu^\mathsf{T} w_{\nu\mu}\varsigma_\mu = y_\mu^\mathsf{T}\varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma_\mu = \boldsymbol{y}^\mathsf{T}\varsigma \\
= \boldsymbol{x}^\mathsf{T} W^\mathsf{T}\varsigma &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\mathsf{T} \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\mathsf{T} \varsigma = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\mathsf{T} \varsigma = \left(b_\mu^\mathsf{T} + x_j^\mathsf{T} w_{j\mu}\right)\varsigma_\mu = y_\mu^\mathsf{T}\varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\mathsf{T} \varsigma_\mu = \boldsymbol{y}^\mathsf{T}\varsigma
\end{aligned}
$$

$$\boldsymbol{z} = \sigma \boldsymbol{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma_\mu y_\mu = \sigma_\mu w_{\mu\nu} x_\nu = \sigma_\mu \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix} = \sigma \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W \boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases}$$

$$= \boldsymbol{z} = z_\mu = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma_\mu y_\mu = \sigma_\mu \left( w_{\mu j} x_j + b_\mu \right) = \sigma_\mu \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix} = \sigma \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W \boldsymbol{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases}$$

$$\boldsymbol{x}^\intercal W^\intercal \varsigma = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal \varsigma = \begin{pmatrix} w_{0\nu}x_\nu \\ w_{1\nu}x_\nu \\ \vdots \\ w_{m\nu}x_\nu \end{pmatrix}^\intercal \varsigma = x_\nu^\intercal w_{\mu\nu}^\intercal \varsigma_\mu = x_\nu^\intercal w_{\nu\mu} \varsigma_\mu = y_\mu^\intercal \varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal \varsigma_\mu = \boldsymbol{y}^\intercal \varsigma = \boldsymbol{z}^\intercal$$

$$= \boldsymbol{x}^\intercal W^\intercal \varsigma = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\intercal \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\intercal \varsigma = \begin{pmatrix} b_0 + w_{0j}x_j \\ b_1 + w_{1j}x_j \\ \vdots \\ b_m + w_{mj}x_j \end{pmatrix}^\intercal \varsigma = \left( b_\mu^\intercal + x_j^\intercal w_{j\mu} \right) \varsigma_\mu = y_\mu^\intercal \varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\intercal \varsigma_\mu = z_\mu^\intercal = \boldsymbol{z}^\intercal$$

$$\begin{aligned} \boldsymbol{z} =& \sigma \boldsymbol{y} = \sigma_\mu w_{\mu\nu} x_\nu = \sigma W \boldsymbol{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\ = z_\mu =& \sigma_\mu y_\mu = \sigma_\mu \left( w_{\mu j} x_j + b_\mu \right), \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\ \boldsymbol{x}^\intercal W^\intercal \varsigma =& x_\nu^\intercal w_{\mu\nu}^\intercal \varsigma_\mu = x_\nu^\intercal w_{\nu\mu} \varsigma_\mu = \boldsymbol{y}^\intercal \varsigma = \boldsymbol{z}^\intercal \\ =& \left( b_\mu^\intercal + x_j^\intercal w_{j\mu} \right) \varsigma_\mu = y_\mu^\intercal \varsigma_\mu = z_\mu^\intercal \end{aligned}$$

matrix calculus[52]

4-15

wrong or incompatible transpose

$$\begin{aligned} \boldsymbol{x}^\intercal W &= \begin{pmatrix} x_0 & x_1 & \cdots & x_m \end{pmatrix} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} \sum_{\mu=1}^{m} x_\mu w_{\mu 0} \\ \sum_{\mu=1}^{m} x_\mu w_{\mu 1} \\ \vdots \\ \sum_{\mu=1}^{m} x_\mu w_{\mu n} \end{pmatrix}^\intercal \\ &\overset{\text{Einstein summation convention}}{=} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\intercal \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^\intercal \\ &= x_\mu^\intercal w_{\mu\nu} = \left( x_\mu w_{\mu\nu} \right)^\intercal ? \end{aligned}$$

4-18

wrong or incompatible transpose

$$\begin{aligned}
\boldsymbol{x}^{\intercal}W &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^{\intercal} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^{\intercal}, \begin{cases} x_0 = 1 \\ w_{0\nu} = b_\nu \end{cases} \\
&= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^{\intercal} \begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_i w_{i0} + b_0 \\ x_i w_{i1} + b_1 \\ \vdots \\ x_i w_{in} + b_n \end{pmatrix}^{\intercal}, \begin{cases} 1 = x_0 \\ b_\nu = w_{0\nu} \end{cases}
\end{aligned}$$

wrong or incompatible transpose

$$\begin{aligned}
\sigma\left(\boldsymbol{x}^{\intercal}W\right) =& \sigma\left(\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^{\intercal} \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^{\intercal}\right) = \begin{pmatrix} \sigma_0\left(x_\mu w_{\mu 0}\right) \\ \sigma_1\left(x_\mu w_{\mu 1}\right) \\ \vdots \\ \sigma_n\left(x_\mu w_{\mu n}\right) \end{pmatrix}^{\intercal}, \begin{cases} x_0 = 1 \\ w_{0\nu} = b_\nu \end{cases} \\
=& \sigma\left(\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^{\intercal} \begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} x_i w_{i0} + b_0 \\ x_i w_{i1} + b_1 \\ \vdots \\ x_i w_{in} + b_n \end{pmatrix}^{\intercal}\right) = \sigma_\nu\left(x_\mu w_{\mu\nu}\right), \begin{cases} 1 = x_0 \\ b_\nu = w_{0\nu} \end{cases}
\end{aligned}$$

# Part II

# by date

# Chapter 6

# A Minimal Book Example

## 6.1 About

This is a *sample* book written in **Markdown**. You can use anything that Pandoc's Markdown supports; for example, a math equation $a^2 + b^2 = c^2$.

### 6.1.1 Usage

Each **bookdown** chapter is an .Rmd file, and each .Rmd file can contain one (and only one) chapter. A chapter *must* start with a first-level heading: `# A good chapter`, and can contain one (and only one) first-level heading.

Use second-level and higher headings within chapters like: `## A short section` or `### An even shorter section`.

The `index.Rmd` file is required, and is also your first book chapter. It will be the homepage when you render the book.

### 6.1.2 Render book

You can render the HTML version of this example book without changing anything:

1. Find the **Build** pane in the RStudio IDE, and
2. Click on **Build Book**, then select your output format, or select "All formats" if you'd like to use multiple formats from the same book source files.

Or build the book from the R console:

```
bookdown::render_book()
```

To render this example to PDF as a `bookdown::pdf_book`, you'll need to install XeLaTeX. You are recommended to install TinyTeX (which includes XeLaTeX): https://yihui.org/tinytex/.

### 6.1.3 Preview book

As you work, you may start a local server to live preview this HTML book. This preview will update as you edit the book when you save individual .Rmd files. You can start the server in a work session by using the RStudio add-in "Preview book", or from the R console:

```
bookdown::serve_book()
```

## 6.2 Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (`#`) per .Rmd file.

### 6.2.1 A section

All chapter sections start with a second-level (`##`) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

Table 6.1: Here is a nice table!

| temperature | pressure |
|---|---|
| 0 | 0.0002 |
| 20 | 0.0012 |
| 40 | 0.0060 |
| 60 | 0.0300 |
| 80 | 0.0900 |
| 100 | 0.2700 |
| 120 | 0.7500 |
| 140 | 1.8500 |
| 160 | 4.2000 |
| 180 | 8.8000 |

**An unnumbered section**

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

## 6.3 Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

### 6.3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
    - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world` = `# Hello world {#hello-world}`.
    - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 6.3.
    - If you prefer text as the link instead of a numbered reference use: any text you want can go here.

### 6.3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 6.1.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

Figure 6.1: Here is a nice figure!

Don't miss Table 6.1.

```
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

## 6.4 Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

## 6.5 Footnotes and citations

### 6.5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

### 6.5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package[1] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr**[2] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

## 6.6 Blocks

### 6.6.1 Equations

Here is an equation.

$$f\left(k\right) = \binom{n}{k} p^k\left(1-p\right)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

### 6.6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

**Theorem 6.1.** *For a right triangle, if $c$ denotes the* length *of the hypotenuse and $a$ and $b$ denote the lengths of the **other** two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

### 6.6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

## 6.7 Sharing your book

### 6.7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

### 6.7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

[1] This is a footnote.

### 6.7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book- all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```

# Chapter 7

# test

## 7.1 RStudio

### 7.1.1 RMarkdown preview LaTeX equation options

https://tex.stackexchange.com/questions/474458/r-markdown-preview-latex-by-scrolling-over

### 7.1.2 writer options

https://rstudio.github.io/visual-markdown-editing/markdown.html#writer-options

#### 7.1.2.1 line wrapping

https://rstudio.github.io/visual-markdown-editing/markdown.html#line-wrapping

#### 7.1.2.2 ensuring the same markdown between source / visual mode

https://stackoverflow.com/questions/71775027/rstudio-switch-markdown-editing-mode-between-source-and-visual-changes-special

**canonical mode**

https://rstudio.github.io/visual-markdown-editing/markdown.html#canonical-mode

```
---
title: "My Document"
editor_options:
  markdown:
    wrap: 72
    references:
      location: block
    canonical: true
---
```

### 7.1.3 Rtools

Rtools43 for Windows https://cran.r-project.org/bin/windows/Rtools/rtools43/rtools.html

### 7.1.4 addins

https://github.com/rstudio/addinexamples

```
if (!requireNamespace("devtools", quietly = TRUE))
  install.packages("devtools")

devtools::install_github("rstudio/htmltools")
devtools::install_github("rstudio/shiny")
devtools::install_github("rstudio/miniUI")
```

### 7.1.5 Git

commit: filename or extension is too long

https://stackoverflow.com/questions/22575662/filename-too-long-in-git-for-windows

https://stackoverflow.com/questions/55327408/how-to-fix-git-for-windows-error-could-not-lock-config-file-c-file-path-to-g

## 7.2 RMarkdown

R Markdown 指南 https://cosname.github.io/rmarkdown-guide/index.html

https://www.rstudio.com/wp-content/uploads/2015/02/rmarkdown-cheatsheet.pdf

https://slides.yihui.org/2020-taipei-satrday-rmarkdown.html#1

### 7.2.1 verbatim

https://community.rstudio.com/t/continued-issues-with-new-verbatim-in-rstudio/139737

https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html

### 7.2.2 Pandoc link

https://pandoc.org/chunkedhtml-demo/8.16-links-1.html

https://stackoverflow.com/questions/39281266/use-internal-links-in-rmarkdown-html-output

https://community.rstudio.com/t/how-to-hyperlink-between-different-rmd-files-in-rmarkdown/62289

### 7.2.3 URL

https://stackoverflow.com/questions/29787850/how-do-i-add-a-url-to-r-markdown

```
[I'm an inline-style link](https://www.google.com)

[I'm an inline-style link with title](https://www.google.com "Google's Homepage")

[I'm a reference-style link][Arbitrary case-insensitive reference text]

[I'm a relative reference to a repository file](../blob/master/LICENSE)

[You can use numbers for reference-style link definitions][1]

Or leave it empty and use the [link text itself]

Some text to show that the reference links can follow later.

[arbitrary case-insensitive reference text]: https://www.mozilla.org
[1]: http://slashdot.org
[link text itself]: http://www.reddit.com
```

### 7.2.4 arrow

https://reimbar.org/dev/arrows/

Up arrow: `↑`

Down arrow: `↓`

Left arrow: `←`

Right arrow: `→`

Double headed arrow: `↔`

### 7.2.5 superscript and subscript

script$^{\text{superscript}}$$_{\text{subscript}}$

```
script^superscript^
```

script$^{\text{superscript}}$

```
~subscript~
```

script$_{\text{subscript}}$

#### 7.2.5.1 LaTeX

https://tex.stackexchange.com/questions/580824/subscript-not-distinguished-enough

https://tex.stackexchange.com/questions/262295/make-subscript-size-smaller-always

#### 7.2.5.2 equation

https://stackoverflow.com/questions/26049762/erroneous-nesting-of-equation-structures-in-using-beginalign-in-a-multi-l

#### 7.2.5.3 aligned

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= b^2 + a^2 && + \text{commutavie} \\ &= b^2 + a^2 && \quad + \text{commutavie} \\ &= b^2 + a^2 && \Box \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= b^2 + a^2 & +\text{commutavie} \\ &= a^2 + b^2 & +\text{commutavie} \\ &= b^2 + a^2 & \Box \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= b^2 + a^2 & +\text{commutavie} + \text{commutavie} \\ &= a^2 + b^2 & +\text{commutavie} + \text{commutavie} \\ &= b^2 + a^2 & \Box + \text{commutavie} \end{aligned} \qquad \begin{aligned} c^2 &= a^2 + b^2 \\ &= b^2 + a^2 & +\text{commutavie} \\ &= a^2 + b^2 & +\text{commutavie} \\ &= b^2 + a^2 & \Box \end{aligned}$$

#### 7.2.5.4 proof QED

https://math.meta.stackexchange.com/questions/3582/qed-for-mathjax-here-on-stackexchange

`\tag*{$\Box$}`

$$a^2 + b^2 = c^2 \tag*{$\Box$}$$

`\tag*{$\blacksquare$}`

$$a^2 + b^2 = c^2 \tag*{$\blacksquare$}$$

### 7.2.6 image

https://stackoverflow.com/questions/25166624/insert-picture-table-in-r-markdown

#### 7.2.6.1 DiagrammeR / mermaid flowchart

```
Error: Functions that produce HTML output found in document targeting latex output.
Please change the output type of this document to HTML.
If your aiming to have some HTML widgets shown in non-HTML format as a screenshot,
please install webshot or webshot2 R package for knitr to do the screenshot.
Alternatively, you can allow HTML output in non-HTML formats
by adding this option to the YAML front-matter of
```

```
your rmarkdown file:

  always_allow_html: true

Note however that the HTML output will not be visible in non-HTML formats.
```

https://bookdown.org/yihui/rmarkdown-cookbook/diagrams.html#diagrams

https://stackoverflow.com/questions/40803017/how-to-include-diagrammer-mermaid-flowchart-in-a-rmarkdown-file

```
{r}
library(DiagrammeR)
mermaid("
graph LR
    A-->B
",
width = 100
)
```

https://github.com/rich-iannone/DiagrammeR/issues/364

https://stackoverflow.com/questions/55994210/how-to-solve-diagrammer-waste-of-space-issue-in-rmarkdown

#### 7.2.6.2 multiple images / figures in the same line

https://cosname.github.io/rmarkdown-guide/rmarkdown-base.html#element-figure

```
{r, fig.show = "hold", out.width = "50%"}
plot(cars)
plot(nhtemp)
```

```r
plot(cars)
plot(nhtemp)
```

cf.

```
{r}
plot(cars)
plot(nhtemp)
```

```r
plot(cars)
```

202401280001-test_files/figure-latex/unnamed-chunk-5-1.pdf

```r
plot(nhtemp)
```

202401280001-test_files/figure-latex/unnamed-chunk-5-2.pdf

#### 7.2.6.3 figure size

https://sebastiansauer.github.io/figure_sizing_knitr/

YAML in index.Rmd

```yaml
---
title: "My Document"
output: html_document:
fig_width: 6
```

```
fig_height: 4
---
```

first R-chunk in your RMD document

`knitr::opts_chunk$set(fig.width=12, fig.height=8)`

width, height and options

```{r fig.height = 3, fig.width = 5

plot(pressure)

```

{r fig.height = 3, fig.width = 5

```
plot(pressure)
```

202401280001-test_files/figure-latex/unnamed-chunk-7-1.pdf

{r fig.height = 3, fig.width = 3, fig.align = "center"

```
plot(pressure)
```

202401280001-test_files/figure-latex/unnamed-chunk-8-1.pdf

{r fig.width = 5, fig.asp = .62

```
plot(pressure)
```

```
<center>
![](https://bookdown.org/yihui/rmarkdown-cookbook/images/cover.png){width=20%}
</center>
```

#### 7.2.6.3.1 knitr https://yihui.org/knitr/options/

https://bookdown.org/yihui/rmarkdown/tufte-figures.html

```
par(mar = c(4, 4, .1, .2)); plot(sunspots)
```

202401280001-test_files/figure-latex/unnamed-chunk-11-1.pdf

```
plot(cars)
```

202401280001-test_files/figure-latex/fig-margin-1.pdf

```
We know from _the first fundamental theorem of calculus_ that
for $x$ in $[a, b]$:
$$\frac{d}{dx}\left( \int_{a}^{x} f(u)\,du\right)=f(x).$$
```

##### 7.2.6.3.2 `out.width` vs. `fig.width` https://stackoverflow.com/questions/29657777/how-to-make-fig-width-and-out-width-consistent-with-knitr

when chunk option `cache=FALSE` is set, then `out.width` has no effect because no PDF output is created. Hence one has to specify exact measures in inches for `fig.width` and `fig.height` for each chunk

https://stackoverflow.com/questions/59567235/a-ggmap-too-small-when-rendered-within-a-rmd-file

```
plot(pressure)
```

202401280001-test_files/figure-latex/unnamed-chunk-14-1.pdf

problem: `out.width='100%'` causing `LaTeX Error: Not in outer par mode.`

solution: `out.width=if (knitr:::is_html_output()) '100%'`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

202401280001-test_files/figure-latex/unnamed-chunk-16-1.pdf

`fig.width=10, fig.height=2`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

202401280001-test_files/figure-latex/unnamed-chunk-18-1.pdf

`out.width=if (knitr:::is_html_output()) '100%'`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

202401280001-test_files/figure-latex/unnamed-chunk-20-1.pdf

#### 7.2.6.4 dynamic knitr plot width and height

https://stackoverflow.com/questions/15365829/dynamic-height-and-width-for-knitr-plots

```
plot(pressure)
```

202401280001-test_files/figure-latex/unnamed-chunk-22-1.pdf

#### 7.2.6.5 web image in PDF

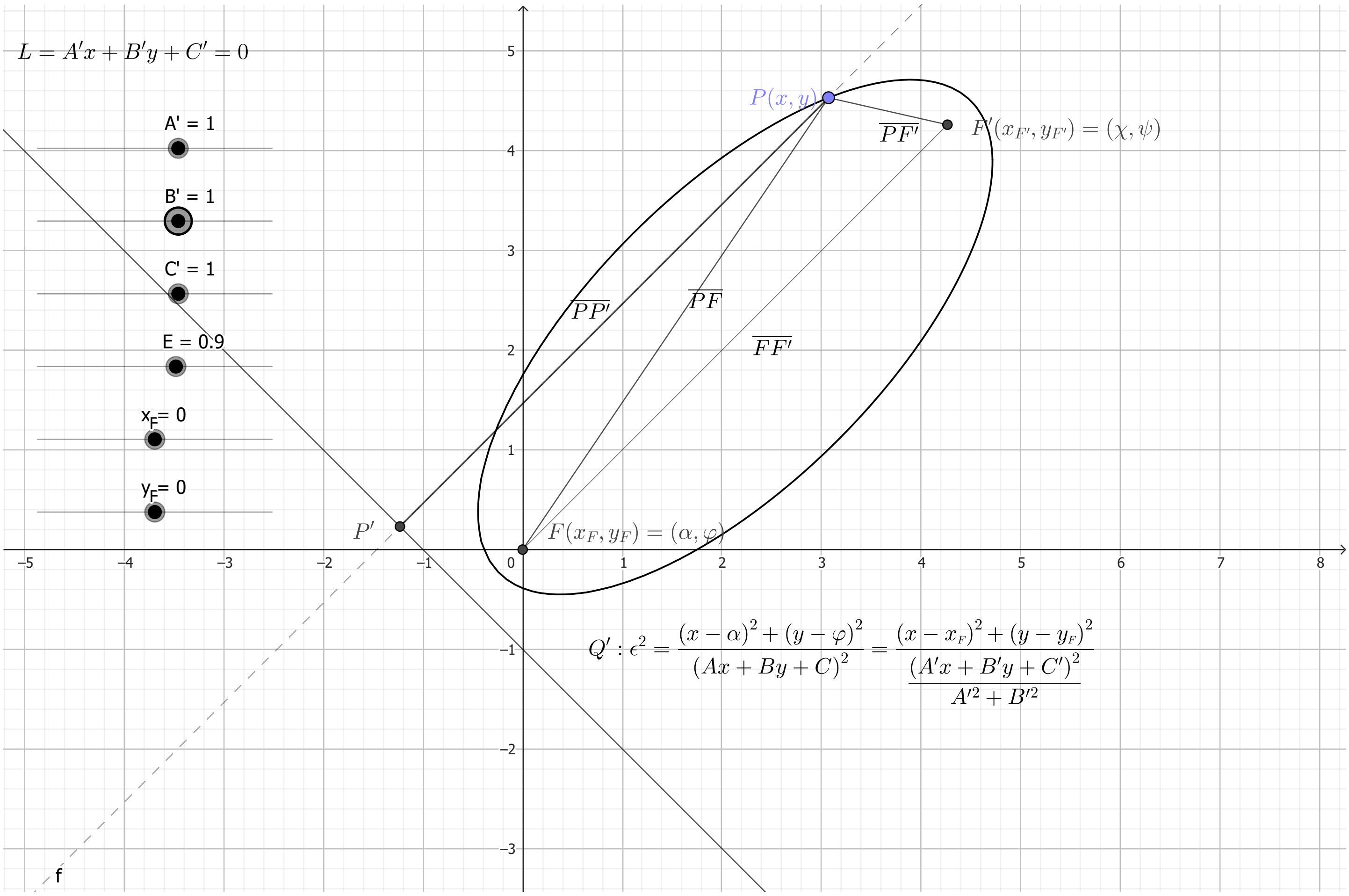
https://stackoverflow.com/questions/46331896/how-can-i-insert-an-image-from-internet-to-the-pdf-file-produced-by-r-bookdown-i

```
cover_url = 'https://bookdown.org/yihui/bookdown/images/cover.jpg'
if (!file.exists(cover_file <- xfun::url_filename(cover_url)))
  xfun::download_file(cover_url)
knitr::include_graphics(if (knitr::pandoc_to('html')) cover_url else cover_file)
```

Figure 7.1: conic sections

#### 7.2.6.6 SVG

https://stackoverflow.com/questions/50165404/how-to-make-a-pdf-using-bookdown-including-svg-images




https://stackoverflow.com/questions/34064292/is-it-possible-to-include-svg-image-in-pdf-document-rendered-by-rmarkdown

### 7.2.7 horizontal rule

```
***
```

horizontal rule (or slide break)

---

```r
dim(iris)
```

```
## [1] 150   5
```

### 7.2.8 footnote

### 7.2.9 hyperlink

PDF pandoc internal link will lose focus

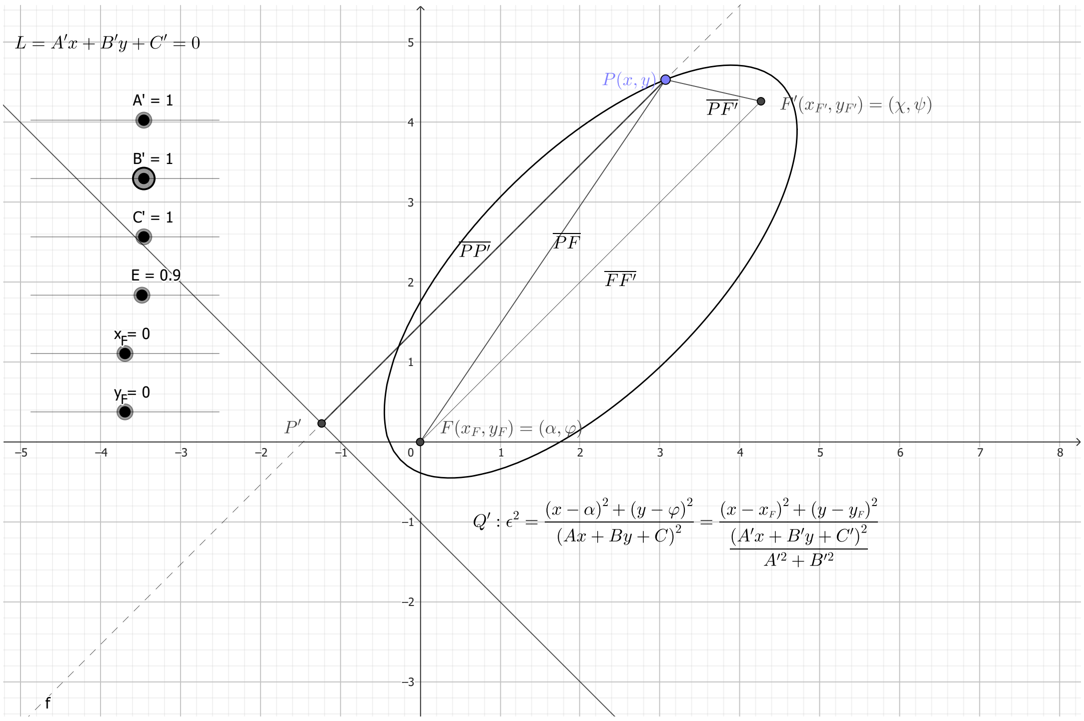



Figure 7.2: conic sections

equivalence relation [11] equivalence relation[1] equivalence relation[11]

equivalence class [10] equivalence class[2] equivalence class[10]

partition [9] partition[3] partition[9]

- LaTeX
  - [TikZ][??]
    - TikZ-3Dplot
    - PGFplots
  - xypic = xy-pic[4]
- OverLeaf
- MathCha
- GeoGebra
- Python
  - MatPlotLib
  - Seaborn
  - Plotly

### 7.2.10 code chunk

#### 7.2.10.1 code folding

https://cosname.github.io/rmarkdown-guide/rmarkdown-document.html#html-code-folding

### 7.2.11 xaringan

slide realtime preview with RStudio addin Infinite Moon Reader in RStudio viewer

https://github.com/yihui/xaringan

https://www.youtube.com/watch?v=3n9nASHg9gc

[1] {11} equivalence relation
[2] {10} equivalence class
[3] {9} partition
[4] {13} xy-pic

### 7.2.12 embed a web page

#### 7.2.12.1 iframe

#### 7.2.12.2 include URL

https://bookdown.org/yihui/rmarkdown-cookbook/include-url.html

```
knitr::include_url("https://yihui.org")
```

```
## Google Chrome was not found. Try setting the `CHROMOTE_CHROME` environment variable to the executable of a
```

#### 7.2.12.3 htmltools

https://stackoverflow.com/questions/36524238/include-html-files-in-r-markdown-file

```
## Warning: `includeHTML()` was provided a `path` that appears to be a complete HTML document.
## x Path: ThreeJS/06.html
## i Use `tags$iframe()` to include an HTML document. You can either ensure `path` is accessible in your app
```

however `out.height` etc. not works

```
## Warning: `includeHTML()` was provided a `path` that appears to be a complete HTML document.
## x Path: ThreeJS/06.html
## i Use `tags$iframe()` to include an HTML document. You can either ensure `path` is accessible in your app
```

## 7.3 Bookdown

### 7.3.1 system locale

https://bookdown.org/tpemartin/ntpu-programming-for-data-science/appendix-d-.html

```
Sys.getlocale()
```

Windows

```
Sys.setlocale(category = "LC_ALL", locale = "UTF-8")
```

MacOS

```
Sys.setlocale(category = "LC_ALL", locale = "en_US.UTF-8")
```

https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html

### 7.3.2 `render_book()`

https://bookdown.org/yihui/bookdown/build-the-book.html

```
render_book(input = ".", output_format = NULL, ..., clean = TRUE,
  envir = parent.frame(), clean_envir = !interactive(),
  output_dir = NULL, new_session = NA, preview = FALSE,
  config_file = "_bookdown.yml")
```

### 7.3.3 `serve_book()`

https://bookdown.org/yihui/bookdown/serve-the-book.html

```
serve_book(dir = ".", output_dir = "_book", preview = TRUE,
  in_session = TRUE, quiet = FALSE, ...)
```

### 7.3.4 LaTeX

#### 7.3.4.1 hyperlink, URL, href

https://www.baeldung.com/cs/latex-hyperref-url-hyperlinks

https://www.omdte.com/小技巧讓-facebook和-line顯示中文網址，網址不再變亂碼/

#### 7.3.4.2 ugly mathptmx $\sum$

PDF LaTeX `\usepackage{fdsymbol}` to have `\overrightharpoon` vector; however, there are too many side effects, including ugly mathptmx $\sum$, ...

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}
```

https://tex.stackexchange.com/questions/315102/different-sum-signs

https://tex.stackexchange.com/questions/275038/how-to-replace-mathptmx-sum-with-cm-sum

https://tex.stackexchange.com/questions/391410/calligraphic-symbols-are-too-fancy-with-mathptmx-package

https://blog.csdn.net/kongtaoxing/article/details/131005044

In `preamble.tex`, add

```
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}

\DeclareMathAlphabet{\mathcal}{OMS}{cmsy}{m}{n}
\DeclareSymbolFont{largesymbols}{OMX}{cmex}{m}{n}
```

#### 7.3.4.3 LaTeX package in HTML document

https://github.com/rstudio/rmarkdown/issues/1829

```
---
title: "assignment"
author: "author"
output: html_document
---

$$
  \require{cancel}
  \cancel{x}
$$
```

$$\cancel{x}$$

https://stackoverflow.com/questions/18189175/how-to-use-textup-with-mathjax

> `\textup` is not available in MathJax. You can replace it with `\mathrm`, **but `\mathrm` does not interpret spaces**.

### 7.3.5 depth of table of contents `toc_depth`

https://stackoverflow.com/questions/49009212/how-to-change-toc-depth-in-r-bookdown-gitbook

```
bookdown::gitbook:
    toc_depth: 2
```

https://stackoverflow.com/questions/68537309/how-can-i-specific-the-initial-level-to-have-my-table-of-contents-be-expanded-to

```
toc:
  collapse: section
```

### 7.3.6 multi-column layout / two columns

https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html

#### 7.3.6.1 for both HTML and PDF

figure size[7.2.6.3]

> Below is a Div containing three child Divs side by side. The Div in the middle is empty, just to add more space between the left and right Divs.

```
:::::: {.cols data-latex=""}

::: {.col data-latex="{0.55\textwidth}"}
![](202401280001-test_files/figure-latex/unnamed-chunk-35-1.pdf)<!-- -->
:::

::: {.col data-latex="{0.05\textwidth}"}
\
<!-- an empty Div (with a white space), serving as
a column separator -->
:::

::: {.col data-latex="{0.4\textwidth}"}
The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do
eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut
enim ad minim veniam, quis nostrud exercitation ullamco laboris
nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor
in reprehenderit in voluptate velit esse cillum dolore eu fugiat
nulla pariatur.
:::
::::::
```

`{r, echo=FALSE, fig.width=5, fig.height=4}`

`202401280001-test_files/figure-latex/unnamed-chunk-36-1.pdf`

The figure on the left-hand side shows the `cars` data.
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

`{r, echo=FALSE, fig.width=10, fig.height=8, out.width = "100%"}`

The figure on the left-hand side shows the `cars` data.
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

`{r, echo=FALSE, fig.width=10, fig.height=8}`

`202401280001-test_files/figure-latex/unnamed-chunk-38-1.pdf`

The figure on the left-hand side shows the `cars` data.
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

#### 7.3.6.2 multi-column `fig.cap` must use `fig.pos="H"`

https://community.rstudio.com/t/adding-fig-cap-caption-text-to-code-chunk-causes-figure-to-print-at-top-of-page-instead-of-where-it-should-be/30297

https://bookdown.org/yihui/rmarkdown-cookbook/figure-placement.html

to avoid `LaTeX Error: Not in outer par mode` for caption in multi-column LaTeX PDF

in `output.yml` add `extra_dependencies: ["float"]` under `bookdown::pdf_book:`

include first chunk `knitr::opts_chunk$set(fig.pos = "H", out.extra = "")` in .Rmd

add `out.width=if (knitr:::is_html_output()) '50%'` for TikZ chunk

thus complete chunk beginning with `{r, echo=FALSE, cache=TRUE, engine='tikz', fig.ext=if (knitr:::is_latex_output( 'pdf' else 'png', fig.width=10, fig.height=2, out.width=if (knitr:::is_html_output()) '100%', fig.cap=''}`

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

202401280001-test_files/figure-latex/unnamed-chunk-

LaTeX package `caption`

https://tex.stackexchange.com/questions/128485/how-to-make-a-caption-via-captionof-and-extra-margins-adhere-to-minipage-marg

What is the different between using `\captionof{table}{ABC}` and `\caption{ABC}`?

https://tex.stackexchange.com/questions/514286/what-is-the-different-between-using-captionoftableabc-and-captionabc

side-by-side table

https://stackoverflow.com/questions/73745714/how-to-print-gt-tbl-tables-side-by-side-with-knitr-kable

R ternary operator

https://stackoverflow.com/questions/8790143/does-the-ternary-operator-exist-in-r

#### 7.3.6.3 caption above figure

https://stackoverflow.com/questions/56979022/caption-above-figure-in-html-rmarkdown

`fig.topcaption=TRUE`

#### 7.3.6.4 for only HTML

##### 7.3.6.4.1 CSS flex

Here is the **first** Div.

```
str(iris)
```

```
## 'data.frame':    150 obs. of  5 variables:
##  $ Sepal.Length: num  5.1 4.9 4.7 4.6 5 5.4 4.6 5 4.4 4.9 ...
##  $ Sepal.Width : num  3.5 3 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 ...
##  $ Petal.Length: num  1.4 1.4 1.3 1.5 1.4 1.7 1.4 1.5 1.4 1.5 ...
##  $ Petal.Width : num  0.2 0.2 0.2 0.2 0.2 0.4 0.3 0.2 0.2 0.1 ...
##  $ Species     : Factor w/ 3 levels "setosa","versicolor",..: 1 1 1 1 1 1 1 1 1 1 ...
```

And this block will be put on the right:

```
plot(iris[, -5])
```

202401280001-test_files/figure-latex/unnamed-chunk-42-1.pdf

#### 7.3.6.4.2 CSS grid https://github.com/yihui/knitr/issues/1743

https://medium.com/enjoy-life-enjoy-coding/css-所以我說那個版能不能好切一點-grid-基本用法-cd763091cf70

```r
head(iris)
```

```
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species
## 1          5.1         3.5          1.4         0.2  setosa
## 2          4.9         3.0          1.4         0.2  setosa
## 3          4.7         3.2          1.3         0.2  setosa
## 4          4.6         3.1          1.5         0.2  setosa
## 5          5.0         3.6          1.4         0.2  setosa
## 6          5.4         3.9          1.7         0.4  setosa
```

```r
plot(iris)
```

202401280001-test_files/figure-latex/unnamed-chunk-44-1.pdf

## 7.4 conditional block/chunk for either HTML or PDF, and Chinese issue

https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html

等價關係 equivalence relation

$$
\begin{aligned}
& R \text{ is an equivalence relation over } A \times B \\
\Leftrightarrow & \begin{cases} R = \sim = \{\langle x, y\rangle | x \sim y\} \subseteq A \times B & \text{(e) equivalence 等價} \\ \vdots & \vdots \end{cases} \\
\Leftrightarrow & \begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & (R)\,\text{relation} \\ \forall \langle x, y\rangle \in R\,(xRx) & (r)\,\text{reflexive} \\ \forall \langle x, y\rangle \in R\,(xRy \Rightarrow yRx) & (s)\,\text{symmetric} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R \left( \begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (t)\,\text{transitive} \end{cases} \Leftrightarrow \begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & \text{關係} \\ \forall \langle x, y\rangle \in R\,(\langle x, x\rangle \in R) & \text{自反} \\ \forall \langle x, y\rangle \in R\,(\langle y, x\rangle \in R) & \text{對稱} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R\,(\langle x, z\rangle \in R) & \text{遞移} \end{cases}
\end{aligned}
$$

## 7.5 video embedding

https://stackoverflow.com/questions/42543206/r-markdown-compile-error

always_allow_html: true

```r
install.packages("webshot")
webshot::install_phantomjs()
```

however webshot not work

Error: cannot find bilibili.com

https://cran.r-project.org/web/packages/vembedr/vignettes/embed.html

```
## embed_youtube("0LFg5dvP0oc")
```

### 7.5.1 timestamp

- YouTube: https://www.youtube.com/embed/%7BvideoID%7D?start=%7Bsecond%7D
- BiliBili: https://player.bilibili.com/player.html?bvid=%7BvideoID%7D&autoplay=0&t=%7Bsecond%7D

## 7.6 equation term coloring

LaTeX annotation by TikZ[60]

### 7.6.1 font color

RegEx replacement in RStudio for `{\\color{(\w+)}` in LyX to be replaced with `\\color{$1}{` in HTML document, and remain the same for PDF document

In HTML document, if no `{}` for text range, only the first following term will take effect

`\color{orange}x=y`

$$\color{orange}x=y$$

`\color{orange}` and `\color{cyan}` are better color for HTML GitBook `White` and `Night` themes and PDF

`\color{cyan}{x=y}`

$$\color{orange}{x=y}$$

`\color{cyan}{x=y}`

$$\color{cyan}{x=y}$$

```
::: {show-in="html"}

$$
\dfrac{\colorbox{#FFD1DC}{$\epsilon^{2}\left(y_{{\scriptscriptstyle F}}-y_{{\scriptscriptstyle
↪  L}}\right)^{2}$}}{1-\epsilon^{2}}
$$

:::

::: {show-in="pdf"}

$$
\dfrac{\colorbox{red!50}{\text{\ensuremath{\epsilon^{2}\left(y_{{\scriptscriptstyle
↪  F}}-y_{{\scriptscriptstyle L}}\right)^{2}}}}}{1-\epsilon^{2}}
$$

:::
```

### 7.6.2 background color

https://bookdown.org/yihui/rmarkdown-cookbook/font-color.html

LaTex color

https://latexcolor.com/

https://www.overleaf.com/learn/latex/Using_colors_in_LaTeX

https://latex-tutorial.com/color-latex/#:~:text=To%20summarize%2C%20pyellow!50efined%20colors%20in,when%20loading%20the%

LaTex color methods

color frame

https://tex.stackexchange.com/questions/582748/highlight-equation-with-boxes-and-arrows

color box

https://tex.stackexchange.com/questions/567739/how-to-move-and-size-colorbox

color box with round corners

https://tex.stackexchange.com/questions/568880/color-box-with-rounded-corners

highlighting

https://tex.stackexchange.com/questions/318991/highlighting-math https://forum.remnote.io/t/highlighting-latex-formulas/149

LyX

https://tex.stackexchange.com/questions/250069/create-a-color-box https://latexlyx.blogspot.com/2013/12/lyx.html

https://tex.stackexchange.com/questions/635486/prevent-lyx-from-escaping-math-in-color-box-title

Bookdown - conditional display of text and code blocks (LaTeX/PDF vs. HTML) https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html

$$F = ma$$

https://community.rstudio.com/t/highlighting-text-inline-in-rmarkdown-or-bookdown-pdf/35118/4

$$F = ma$$

$$F = F$$

$$F = ma \tag{7.1}$$

$$F = ma$$

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_n X_n$$

## 7.7 link and reference

https://stackoverflow.com/questions/57469501/cross-referencing-bookdownhtml-document2-not-working

$$E = mc^2 \tag{7.2}$$

`\@ref(nice-label)` 7.8

`[link to partition][partition]` link to partition

`[partition] \@ref(partition)`

partition [#partition] (9) @ref(#partition)

`[equivalence class] \@ref(equivalence-class)`

equivalence class (10)

equivalence class [#equivalence class] (@ref(equivalence class)) @ref(#equivalence class)

[equivalence-class] [#equivalence-class] (10) @ref(#equivalence-class)

X [equivalence-class.html] [equivalence-class.html#equivalence-class] (@ref(equivalence-class.html)) @ref(equivalence-class.html#equivalence-class)

equivalence relation [#equivalence relation] (@ref(equivalence relation)) @ref(#equivalence relation)

[equivalence-relation] [#equivalence-relation] (11) @ref(#equivalence-relation)

X [equivalence-relation.html] [equivalence-relation.html#equivalence-relation] (@ref(equivalence-relation.html)) @ref(equivalence-relation.html#equivalence-relation)

## 7.8 number and reference equations

https://stackoverflow.com/questions/71595882/rstudio-error-in-windows-running-pdflatex-exe-on-file-name-tex-exit-code-10

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#equations

`\#eq:emc \@ref(eq:emc)`

https://stackoverflow.com/questions/55923290/consistent-math-equation-numbering-in-bookdown-across-pdf-docx-html-output

$$\begin{aligned}
&C \text{ is an equivalence class of } a \text{ on } A \\
&\Leftrightarrow [a]_\sim = C = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{cases} \right\} \subseteq A \neq \emptyset \\
&\Leftrightarrow [a] = [a]_\sim = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{cases} \right\} \subseteq A \neq \emptyset \\
&\Rightarrow [a]_\sim = \{x | x \sim a\} \subseteq A \neq \emptyset
\end{aligned} \tag{7.3}$$

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#cross-referencing

This cross reference is the Fig. 7.4

https://stackoverflow.com/questions/51595939/bookdown-cross-reference-figure-in-another-file

I ran into the same issue and came up with this solution if you aim at compiling 2 different pdfs. It relies on LaTeX's xr package for cross references: https://stackoverflow.com/a/52532269/576684

## 7.9 footnote

```
noun^[This is a footnote]

noun[^202401260000-test-cross-link-1]

[^202401260000-test-cross-link-1]: This is a footnote.
```

noun[5]

## 7.10 citation

https://stackoverflow.com/questions/48965247/use-csl-file-for-pdf-output-in-bookdown/49145699#49145699

citation 1[3] citation 2[3]

citation 3[4] citation 4[4]

### 7.10.1 citation in `fig.cap`

https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-rmarkdown

`(ref:rudolph) *nice* cite: [@Lam94].`

`(ref:campbell1963) *nice* cite: [@campbell1963].`

`(ref:campbell1963) ([@campbell1963]`

`(ref:campbell1963) \ [@campbell1963]`

[5]This is a footnote.

Sources of Invalidity

| | Internal | | | | | | | | External | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | History | Maturation | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| *Pre-Experimental Designs:*<br>1. One-Shot Case Study<br>X O | − | − | | | | − | − | | | − | | |
| 2. One-Group Pretest-Posttest Design<br>O X O | − | − | − | − | ? | + | + | − | − | − | ? | |
| 3. Static-Group Comparison<br>X O<br>- - - - - - -<br>O | + | ? | + | + | + | − | − | − | | − | | |
| *True Experimental Designs:*<br>4. Pretest-Posttest Control Group Design<br>R O X O<br>R O O | + | + | + | + | + | + | + | + | − | ? | ? | |
| 5. Solomon Four-Group Design<br>R O X O<br>R O O<br>R X O<br>R O | + | + | + | + | + | + | + | + | + | ? | ? | |
| 6. Posttest-Only Control Group Design<br>R X O<br>R O | + | + | + | + | + | + | + | + | + | ? | ? | |

Figure 7.3: pre- and true experimental designs ( [5] p.8)

### 7.10.2 backreference

https://community.rstudio.com/t/how-to-create-a-backreference-to-place-of-citation-in-rmarkdown/84866

https://blog.csdn.net/RobertChenGuangzhi/article/details/50455429

https://latex.org/forum/viewtopic.php?t=3722

## 7.11 environment for definition, theorem, proof

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html

https://github.com/rstudio/rstudio/issues/5264

`@howthebodyworks` Ideally, previews of such equations should also work inside a theorem, although I could survive without that.

https://github.com/rstudio/rstudio/issues/8773

**Theorem 7.1** (Theorem Name)**.** *Here is my theorem.*

*Proof Name.* Here is my proof. □

**Theorem 7.2** (Pythagorean theorem)**.** *For a right triangle, if $c$ denotes the length of the hypotenuse and $a$ and $b$ denote the lengths of the other two sides, we have*

$$a^2 + b^2 \overset{7.2}{=} c^2$$

**Definition 7.1** (Definition Name)**.** Here is my definition.

number and reference equations

(7.3)

(7.2)

7.2

Figure 7.4: parabola arc with points

### 7.11.1 collapsible panel or HTML component

https://stackoverflow.com/questions/52576626/rmarkdown-collapsible-panel

**Theorem 7.3.** *This is the description of a theorem*

$$c^2 = a^2 + b^2$$

Proof:

$$c^2 = a^2 + b^2$$

□

#### 7.11.1.1 nested collapsible panel or HTML component

## 7.12 slide or presentation

### 7.12.1 Xaringan and Infinite Moon Reader

https://rpubs.com/RW1304/xarigan-zh

https://slides.yihui.org/xaringan/#1

https://slides.yihui.org/xaringan/zh-CN.html#1

https://github.com/yihui/xaringan/tree/master

https://bookdown.org/yihui/rmarkdown/xaringan.html

### 7.12.2 ioslides

https://www.youtube.com/watch?v=gkyjTcpCITM

https://bookdown.org/yihui/rmarkdown/ioslides-presentation.html

https://stackoverflow.com/questions/63749683/how-to-set-up-theorem-environment-in-the-rmarkdown-presentation

```
---
title: "Theorem demo"
output:
  ioslides_presentation:
    css: style.css
---
```

```

/* theorem environment _ plain */

/*
.theorem {
  display: block;
  font-style: italic;
  font-size: 24px;
  font-family: "Times New Roman";
  color: black;
}
.theorem::before {
  content: "Theorem. ";
  font-weight: bold;
  font-style: normal;
```

```
}
.theorem[text]::before {
  content: "Theorem (" attr(text) ") ";
}
.theorem p {
  display: inline;
}
*/

/* theorem environment _ Copenhagen style */

/*
.theorem {
  display: block;
  font-style: italic;
  font-size: 24px;
  font-family: "Times New Roman";
  color: black;
  border-radius: 10px;
  background-color: rgb(222,222,231);
  box-shadow: 5px 10px 8px #888888;
}
.theorem::before {
  content: "Theorem. ";
  font-weight: bold;
  font-style: normal;
  display: inline-block;
  width: -webkit-fill-available;
  color: white;
  border-radius: 10px 10px 0 0;
  padding: 10px 5px 5px 15px;
  background-color: rgb(38, 38, 134);
}
.theorem p {
  padding: 15px 15px 15px 15px;
}
*/
```

### 7.12.3 PowerPoint

https://bookdown.org/yihui/rmarkdown/powerpoint-presentation.html

# Chapter 8

# test2

https://www.overleaf.com/learn/latex/Commands

`\newcommand`

`\renewcommand` to replace existing command

https://www.physicsread.com/latex-newcommand/

font color[7.6.1]

$$\sin\left(\alpha\right), \sin^{\mathrm{n}}\left(\beta\right), \sin^{\mathrm{m}}\left(\gamma\right)$$

$$\cos\left(2\theta\right) - \sin\left(2\theta\right) = \cos\left(4\theta\right)$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\tan\left(\alpha + \beta\right) = \frac{\tan\left(\alpha\right) + \tan\left(\beta\right)}{1 - \tan\left(\alpha\right)\tan\left(\beta\right)}$$

https://www.alanshawn.com/latex3-tutorial/#latex3-regular-expression-xxviii

`\regex_replace_all`

https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon

```
\documentclass{article}

\usepackage{tikz,xparse}

\ExplSyntaxOn

\tl_new:N \l_bob_func_tl

\NewDocumentCommand \makelinesegment { m }
 {
  \pgfextra
  \tl_set:Nn \l_bob_func_tl { #1 }
  \regex_replace_all:nnN { ([0-9])- } { \1)--( } \l_bob_func_tl
  \regex_replace_all:nnN { : } { , } \l_bob_func_tl
  \exp_last_unbraced:NNV % this is not necessary, but I find it cleaner, correct me if I'm wrong
  \endpgfextra
  (\l_bob_func_tl)
 }

\ExplSyntaxOff

\begin{document}
```

```
% \makelinesegment{1:1--1:2-2:2} % do not use outside `\tikz` or `{tikzpicture}`

\tikz\draw\makelinesegment{1:1--1:2-2:2};

\end{document}
```

https://stackoverflow.com/questions/41655383/r-markdown-similar-feature-to-newcommand-in-latex

$\operatorname{Var}(X)$

$$\begin{aligned}\operatorname{Var}[Y] &= x \\ &= 3\end{aligned}$$

$$0 = \frac{\partial}{\partial z_l}\left(\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda\|h(z_l) \cdot w_{l+1} - z_{l+1}\|\right)$$

https://tex.stackexchange.com/questions/353114/latex-equations-colour-all-instances-of-symbol

$$0 = \frac{\partial}{\partial z_l}\left(\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda\|h(z_l) \cdot w_{l+1} - z_{l+1}\|\right)$$

$$0 = \frac{\partial}{\partial z_l}\left(\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda\|h(z_l) \cdot w_{l+1} - z_{l+1}\|\right)$$

$$\begin{aligned}&\left(y_F^2 - \epsilon^2 y_L^2\right)\left(1-\epsilon^2\right) - \left(y_F - \epsilon^2 y_L\right)^2 \\ &= \left(1-\epsilon^2\right) y_F^2 - \left(\epsilon^2 - \epsilon^4\right) y_L^2 - y_F^2 + 2\epsilon^2 y_F y_L - \epsilon^4 y_L^2 \\ &= -\epsilon^2 y_F^2 - \epsilon^2 y_L^2 + 2\epsilon^2 y_F y_L = -\epsilon^2\left(y_F - y_L\right)^2\end{aligned}$$

# Chapter 9

# partition

$$\{A_i\}_{i \in I} = \{A_i | i \in I\} \text{ is a partition of a set } A$$
$$\Leftrightarrow \begin{cases} \forall i \in I \, (A_i \neq \emptyset) \\ A = \bigcup_{i \in I} A_i \\ \forall i, j \in I \, (i \neq j \Rightarrow A_i \cap A_j = \emptyset) \end{cases}$$

https://proofwiki.org/wiki/Definition:Set_Partition

# Chapter 10

# equivalence class

$$
\begin{aligned}
&C \text{ is an equivalence class of } a \text{ on } A \\
\Leftrightarrow & [a]_\sim = C = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{cases} \right\} \subseteq A \neq \emptyset \\
\Leftrightarrow & [a] = [a]_\sim = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{cases} \right\} \subseteq A \neq \emptyset \\
\Rightarrow & [a]_\sim = \{x | x \sim a\} \subseteq A \neq \emptyset
\end{aligned}
$$

where the definition of equivalence relation can be found in 11.

# Chapter 11

# equivalence relation

等價關係 equivalence relation

$$
\begin{aligned}
&R \text{ is an equivalence relation over } A \times B \\
\Leftrightarrow &\begin{cases} R = \sim = \{\langle x, y\rangle | x \sim y\} \subseteq A \times B & \text{(e) equivalence 等價} \\ \vdots & \vdots \end{cases} \\
\Leftrightarrow &\begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & (R) \text{ relation} \\ \forall \langle x, y\rangle \in R\,(xRx) & (r) \text{ reflexive} \\ \forall \langle x, y\rangle \in R\,(xRy \Rightarrow yRx) & (s) \text{ symmetric} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R \left(\begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz\right) & (t) \text{ transitive} \end{cases}
\Leftrightarrow \begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & \text{關係} \\ \forall \langle x, y\rangle \in R\,(\langle x, x\rangle \in R) & \text{自反} \\ \forall \langle x, y\rangle \in R\,(\langle y, x\rangle \in R) & \text{對稱} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R\,(\langle x, z\rangle \in R) & \text{遞移} \end{cases}
\end{aligned}
$$

# Chapter 12

# Python

## 12.1 using Python in R / RMarkdown

https://bookdown.org/yihui/rmarkdown/language-engines.html

```
names(knitr::knit_engines$get())
```

```
##  [1] "awk"         "bash"        "coffee"      "gawk"        "groovy"
##  [6] "haskell"     "lein"        "mysql"       "node"        "octave"
## [11] "perl"        "php"         "psql"        "Rscript"     "ruby"
## [16] "sas"         "scala"       "sed"         "sh"          "stata"
## [21] "zsh"         "asis"        "asy"         "block"       "block2"
## [26] "bslib"       "c"           "cat"         "cc"          "comment"
## [31] "css"         "ditaa"       "dot"         "embed"       "eviews"
## [36] "exec"        "fortran"     "fortran95"   "go"          "highlight"
## [41] "js"          "julia"       "python"      "R"           "Rcpp"
## [46] "sass"        "scss"        "sql"         "stan"        "targets"
## [51] "tikz"        "verbatim"    "theorem"     "lemma"       "corollary"
## [56] "proposition" "conjecture"  "definition"  "example"     "exercise"
## [61] "hypothesis"  "proof"       "remark"      "solution"
```

https://rstudio.github.io/reticulate/articles/python_packages.html

```
x = 'hello, python world!'
print(x.split(' '))
```

```
## ['hello,', 'python', 'world!']
```

```
library(reticulate)
virtualenv_python()
```

```
library(reticulate)
# conda_list()
```

```
library(reticulate)
virtualenv_list()
```

https://rstudio.github.io/reticulate/reference/install_python.html

```
library(reticulate)
version <- "3.9.12"
# install_python(version)

## create a new environment
# virtualenv_create("r-reticulate", version = version)

# use_virtualenv("r-reticulate")

## install MatPlotLib
# virtualenv_install("r-reticulate", "matplotlib")
```

```
## import MatPlotLib (it will be automatically discovered in "r-reticulate")
matplotlib <- import("matplotlib")
```

copy `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl\tcl8.6` and `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl\tk8.6` two folders to the folder `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\Lib`

```
# library(reticulate)
# use_virtualenv("r-reticulate")
# # matplotlib <- import("matplotlib")
# matplotlib$use("Agg", force = TRUE)
```

```
import matplotlib.pyplot as plt
plt.plot([0, 2, 1, 4])
plt.show()
```

202401292317-Python_files/figure-latex/unnamed-chunk-9-1.pdf

## 12.2 SoloLearn

https://www.sololearn.com/

https://www.sololearn.com/en/learn/courses/python-intermediate

## 12.3 list comprehension

https://www.sololearn.com/en/learn/courses/python-intermediate/lesson/1188906590?p=1

```
cubes = [i**3 for i in range(5)]

print(cubes)
```

```
## [0, 1, 8, 27, 64]
```

## 12.4 functional programming

- pure function
- lambda
- map
- filter
- generator
- decorator
- recursion
- *args
- **kwargs

## 12.5 object-oriented programming = OOP

- class
- inheritance
- magic method
- operator overloading
- data hiding
- static method
- property

# Chapter 13

# xy-pic

https://bookdown.org/yihui/rmarkdown-cookbook/install-latex-pkgs.html

`tinytex::install_tinytex()`

## 13.1 pure LaTeX equation environment not rendering xy-pic in HTML

the following xymatrix from LaTeX package xy for xy-pic is not shown or rendered in HTML:

`$\LaTeX$` can only be used in HTML, not PDF

$$\begin{tikzcd}
U \arrow[drr, "\varphi"] \arrow[dr, "{(x,y)}"'] \arrow[ddr, "\psi"'] & & \\
& X \times_Z Y \arrow[r, "p"'] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

## 13.2 R figure knit with TikZ engine

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

https://bookdown.org/yihui/rmarkdown-cookbook/html-scroll.html

https://tex.stackexchange.com/questions/669083/how-can-i-draw-the-following-commutative-diagram

Figure 13.1: xy-pic or xymatrix

Figure 13.2: tikz-cd

```
\xymatrix{
  1 \ar[r] &
  H \ar[r]^{\alpha} \ar[d]_{\mathrm{id}} &
  G \ar[r]^{\beta} \ar[d]_{\theta} &
  K \ar[r] \ar[d]_{\mathrm{id}} &
  1
  \\
  1 \ar[r] &
  H \ar[r] &
  H\times K \ar[r] &
  K \ar[r] &
  1
}
```

Figure 13.3: xy-pic or xymatrix

# Chapter 14

# statistics

## 14.1 Hung Hung

https://www.youtube.com/playlist?list=PLTpF-A8hKVUOqfNyA6mOD6lo2cc6clZZP

https://www.youtube.com/watch?v=3S2r4XBzKts

population

普查 vs. 統計

random variable

$$X$$

sample has randomness

probability function

$$\mathrm{P}_X(E) \in [0, 1]$$

event = subset of sample space

$$E$$

input event with output probability in 0 to 1

$$\mathrm{P}_X : \{E_i\}_{i \in I} \to [0, 1]$$

target of interest

- probability function

but events are hard to be listed or enumerated

$$X : \{\omega_i\}_{i \in I} \to \mathbb{R}$$

CDF = cumulative distribution function

$$F_X(x) = \mathrm{P}_X((-\infty, x]) = \mathrm{P}_X(X \leq x)$$

real function is much easier to be operable, there is differentiation or difference operation

target of interest

- CDF = cumulative distribution function
- probability function

target of interest

- $\mathrm{P}_X(\cdot)$ PF = probability function
  - $f_X(x) = \mathrm{P}_X(X = x)$ PMF = probability mass function
  - $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}_X(X \leq x)$ PDF = probability density function
- $F_X(x)$ CDF = cumulative distribution function[14.1.2.1]
- $M_X(\xi)$ MGF = moment generating function[14.1.2.7.1]
- $\varphi_X(\xi)$ CF = characteristic function[14.1.2.7.2]

$$\begin{matrix} X & \sim & F_X(x) & \overset{\text{FToC}}{\longleftrightarrow} & f_X(x) & \leftrightarrow & \mathrm{P}_X \\ & \text{inversion formula} : & \updownarrow & \forall \xi \approx 0 \left[M_X(\xi) \in \mathbb{R}\right] \wedge & \uparrow\downarrow & \searrow\nwarrow & \hookleftarrow \wedge \operatorname{supp}(f_X) \text{ is bounded} \\ & & \varphi_X(\xi) & \leftrightarrows & M_X(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \end{matrix}$$

In population,

$$X \sim F_X(x)$$

by sampling,

$$X_1, \cdots, X_i, \cdots, X_n = X_i \sim F_X(x)$$

or

$$X_1, \cdots, X_i, \cdots, X_n = X_i \overset{\text{i.i.d.}}{\sim} F_X(x)$$

i.i.d. = independently identically distributed

and inference back

parametrically

$$\widehat{X} \sim \widehat{F}_X(x) = \widehat{F}_X(x|\theta)$$

or nonparametrically

$$\hat{X} \sim \hat{F}_X(x)$$

inference is function of samples, or called random function, to estimate unknown parameters

$$\widehat{\Theta} \leftarrow (X_1, \cdots, X_i, \cdots, X_n) = T(X_1, \cdots, X_n) = T(\cdots, X_i, \cdots) = T(X_i)$$

correspondng CDF for inference or estimation function of sampling random variables

$$T(X_1, \cdots, X_n) = T \sim F_T(t)$$

wish to be unbiased and consistent

$$\begin{cases} \mathrm{E}\left(\widehat{\Theta}\right) = \theta \Leftrightarrow \mathrm{E}\left(\widehat{\Theta}\right) - \theta = 0 & \text{unbiasedness} \\ \mathrm{V}\left(\widehat{\Theta}\right) = 0 & \text{consistency} \end{cases}$$

unbiasedness usually harder than consistency, thus usually first considered consistency.

modeling or parameterizing with unknown parameter $\theta$

$$F_X(x) \overset{M}{=} F_X(x|\theta) = F_X(x;\theta)$$

parameterization is to reduce unknown parameters from infinite ones to finite ones

e.g. for normally distributed data

$$f_X(x) \overset{M}{=} f_X(x|\theta) = \frac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} = f_X\left(x|\mu,\sigma^2\right) = f_X(x|\mu,\sigma)$$

the price of parameterization is guess wrong model.

For some non-negative data, instead of normal distribution, consider distributions skewed to the right

- gamma
- exponential
- Weibull
- log-normal

topics

1. $\mathrm{P}_X$ probability theory
2. $f_X(x) \overset{M}{=} f_X(x|\theta) = f_X(x;\theta)$ various univariable distribution
3. $f_{\boldsymbol{X}}(\boldsymbol{x}) \overset{M}{=} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) = f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})$ multivariable distribution
4. $T(X_1, \cdots, X_n)$ inference
   - point estimation
     $$\widehat{\mu} = \begin{cases} \overline{X} & \to \mu \\ \text{median}(X_1, \cdots, X_n) & \to \mu \\ \vdots \end{cases}$$
   - interval estimation = hypothesis testing
     $$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases} \leftarrow T \in \{0, 1\}$$
5. how to find $T$
6. behavior of random function $T(X_1, \cdots, X_n) = T \sim F_T(t)$
   - statistical properties of $T$
   - asymptotic properties
     $$n \to \infty \begin{cases} \text{CLT} = \text{central limit theorem} \\ \text{LLN} = \text{law of large number} \end{cases}$$

### 14.1.1 probability theory

https://www.youtube.com/watch?v=HBmTDtMBr3c

**Definition 14.1.** sample space: The set $S$ of all possible outcomes of an experiment is called the sample space

$$S = \{\omega_i\}_{i \in I}$$

**Definition 14.2.** event: An event $E$ is any collection of possible outcomes of an experiment, i.e. any subset of $S$

$$E \subseteq S$$

set operation

commutativity, associativity, distributivity

De Morgan law

pairwise disjoint = mutually exclusive

partition

#### 14.1.1.1 probability function

probability function axioms = probability function definition

Kolmogorov axioms of probability[6] p.72

**Definition 14.3.** probability function: Given a sample space $S$ and its event $E$, a probability function is a function P satisfying

$$\begin{cases} \mathrm{P}\left(S\right) = 1 \\ \forall E \subseteq S \left(\mathrm{P}\left(E\right) \geq 0\right) \\ E_1, \cdots, E_i, \cdots \text{ are pairwise disjoint} \Rightarrow \mathrm{P}\left(\bigcup\limits_{i \in I} E_i\right) = \sum\limits_{i \in I} E_i \end{cases}$$

tossing a dice

theorems

$$\mathrm{P}\left(\emptyset\right) = 0$$

$$\mathrm{P}\left(E\right) \leq 1$$

$$\mathrm{P}\left(E^{\mathrm{C}}\right) = \mathrm{P}\left(\overline{E}\right) = 1 - \mathrm{P}\left(E\right)$$

$$\mathrm{P}\left(E_2 \cap \overline{E}_1\right) = \mathrm{P}\left(E_2\right) - \mathrm{P}\left(E_2 \cap E_1\right)$$

$$E_1 \subseteq E_2 \Rightarrow \mathrm{P}\left(E_1\right) \leq \mathrm{P}\left(E_2\right)$$

addition rule[6] p.75 and extended addition rule[6] p.76

inclusion-exclusion principle = sieve principle

$$\mathrm{P}\left(E_1 \cup E_2\right) = \mathrm{P}\left(E_1\right) + \mathrm{P}\left(E_2\right) - \mathrm{P}\left(E_1 \cap E_2\right)$$

$$\mathrm{P}\left(E_1 \cup E_2 \cup E_3\right) = \mathrm{P}\left(E_1\right) + \mathrm{P}\left(E_2\right) + \mathrm{P}\left(E_3\right) - \mathrm{P}\left(E_1 \cap E_2\right) - \mathrm{P}\left(E_2 \cap E_3\right) - \mathrm{P}\left(E_3 \cap E_1\right) + \mathrm{P}\left(E_1 \cap E_2 \cap E_3\right)$$

$$\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n}\left(\left(-1\right)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathrm{P}\left(\bigcap_{i \in \{i_1, \ldots, i_k\}} E_i\right)\right)$$

symmetric difference[6] p.75

union probability upper-bounded by sum of indivisual probability

$$\mathrm{P}\left(E_1 \cup E_2\right) = \mathrm{P}\left(E_1\right) + \mathrm{P}\left(E_2\right) - \mathrm{P}\left(E_1 \cap E_2\right) \leq \mathrm{P}\left(E_1\right) + \mathrm{P}\left(E_2\right)$$

$$E_1 \cap E_2 = \emptyset \Leftrightarrow \mathrm{P}\left(E_1 \cup E_2\right) = \mathrm{P}\left(E_1\right) + \mathrm{P}\left(E_2\right)$$

Boole inequality

$$\mathrm{P}\left(\bigcup_{i \in I} E_i\right) \leq \sum_{i \in I} E_i$$

$$\mathrm{P}\left(\overbrace{H_0} \middle| H_0\right) = \mathrm{P}\left(\text{reject } H_0 | H_0 \text{ is true}\right) = \alpha = \text{type 1 error}$$

multiple hypothesis testing

How to control the family-wise error rate?

Ideally,

FWER = family-wise error rate

$$
\begin{aligned}
\alpha =& \mathrm{P}\left(\overbrace{H_0^1} \cup \cdots \cup \overbrace{H_0^m} \middle| H_0^1 \cap \cdots \cap H_0^m\right) = \mathrm{P}\left(\text{reject any } H_0^i | \text{any } H_0^j \text{ is true}\right) \\
=& \mathrm{P}\left(\bigcup_{i=1}^{m} \overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = 1 - \mathrm{P}\left(\overbrace{\bigcup_{i=1}^{m} \overbrace{H_0^i}} \middle| \bigcap_{j=1}^{m} H_0^j\right) \\
=& 1 - \mathrm{P}\left(\bigcap_{i=1}^{m} \underbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = 1 - \mathrm{P}\left(\text{not to reject any } H_0^i \middle| \bigcap_{j=1}^{m} H_0^j\right) \\
& \overset{H_0^j \text{ pairwise independent}}{=} 1 - \prod_{i=1}^{m} \mathrm{P}\left(\underbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = 1 - \prod_{i=1}^{m}\left(1 - \mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right)\right) \\
& \overset{\forall i,j \left[\mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \alpha_0\right]}{=} 1 - \prod_{i=1}^{m}(1-\alpha_0) = 1 - (1-\alpha_0)^m \\
\alpha =& 1 - (1-\alpha_0)^m \\
\alpha_0 =& 1 - (1-\alpha)^{\frac{1}{m}} = 1 - \sqrt[m]{1-\alpha} \\
\Downarrow & \\
\text{set } & \mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \alpha_0 = 1 - \sqrt[m]{1-\alpha}
\end{aligned}
$$

But condition $H_0^j$ pairwise independent is too strong.

Practically,

$$
\begin{aligned}
\alpha =& \mathrm{P}\left(\overbrace{H_0^1} \cup \cdots \cup \overbrace{H_0^m} \middle| H_0^1 \cap \cdots \cap H_0^m\right) = \mathrm{P}\left(\text{reject any} H_0^i | \text{any } H_0^j \text{ is true}\right) \\
=& \mathrm{P}\left(\bigcup_{i=1}^{m} \overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) \underset{\text{Boole inequality}}{\overset{\mathrm{P}\left(\bigcup_{i \in I} E_i\right) \leq \sum_{i \in I} E_i}{\leq}} \sum_{i=1}^{m} \mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) \underset{\Uparrow}{=} \sum_{i=1}^{m} \alpha_0 = m\alpha_0 \underset{\Downarrow}{=} \alpha \\
\text{let } \forall i,j & \left[\mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \alpha_0\right] \Rightarrow \sum_{i=1}^{m} \mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \sum_{i=1}^{m} \alpha_0 = m\alpha_0 \Rightarrow \alpha_0 = \frac{\alpha}{m} \\
\Downarrow & \\
\text{set } & \mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \alpha_0 = \frac{\alpha}{m}
\end{aligned}
$$

Bonferroni correction

$$
\mathrm{P}\left(\overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) = \frac{\alpha}{m} \Rightarrow \mathrm{P}\left(\bigcup_{i=1}^{m} \overbrace{H_0^i} \middle| \bigcap_{j=1}^{m} H_0^j\right) \leq \alpha
$$

Bonferroni inequality[6] p.77

Bonferroni inequality and Boole inequality are equivalent inequalities

birthday problem[6] p.78

#### 14.1.1.2 conditional probability

### 14.1.2 univariable distribution

$$
\begin{cases}
\mathrm{P}_X\left(X \in E\right) & \forall E \subseteq S \\
\mathrm{P}_X\left(X \leq x\right) & \forall x \in \mathbb{R}
\end{cases}
$$

$$\begin{cases} \mathrm{P}_X(X \in E) & \forall E \subseteq S \\ \mathrm{P}_X(X \le x) = \mathrm{P}_X((-\infty, x]) = F_X(x) & \forall x \in \mathbb{R} \end{cases}$$

$$\begin{aligned} &\mathrm{P}_X(X \le x) = \mathrm{P}_X((-\infty, x]) \\ =&\mathrm{P}_X\left(\bigcup_{\epsilon>0} (-\infty, x-\epsilon]\right) = \lim_{\epsilon\to 0} \mathrm{P}_X((-\infty, x-\epsilon]) \\ \leftrightarrow&\mathrm{P}_X((-\infty, x)) = \mathrm{P}_X(E), E = (-\infty, x) \end{aligned}$$

#### 14.1.2.1 cumulative distribution function

CDF = cumulative distribution function

$$F_X(x) = \mathrm{P}_X((-\infty, x]) = \mathrm{P}_X(X \le x)$$
$$X \sim \mathrm{P}_X \leftrightarrow F_X(x)$$

$$X \sim F_X(x) \leftrightarrow \mathrm{P}_X$$

---

**Definition 14.4.** CDF = cumulative distribution function: A cumulative distribution function is a function $F : \mathbb{R} \to [0, 1]$ satisfying

$$F_X(x) = \mathrm{P}_X((-\infty, x]) = \mathrm{P}_X(X \le x)$$

**Theorem 14.1.** *CDF = cumulative distribution function: $F(x)$ is a cumulative distribution function iff*

$$\begin{cases} \begin{cases} \lim\limits_{x\to-\infty} F(x) = 0 & \lim\limits_{x\to+\infty} F(x) = 1 \end{cases} & (01)\ [0,1] \\ \forall x_1 < x_2 \left[F(x_1) \le F(x_2)\right] & (nd)\ \text{non-decreasing} \\ \lim\limits_{x\to x_0^+} F(x) = F(x_0) & (rc)\ \text{right-continuous} \end{cases}$$

---

**Definition 14.5.** RV = r.v. = random variable

$$\begin{cases} X \text{ is a continuous RV} & \lim\limits_{x\to x_0} F_X(x) = F_X(x_0) \\ X \text{ is a discrete RV} & F_X \text{ is a step function of } x \end{cases}$$

[6] p.103

**Definition 14.6.** RV = r.v. = random variable

[6] p.104

**Definition 14.7.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned} \mathcal{R}_X &= \left\{ x \middle| \begin{cases} \omega \in S \\ x \in X(\omega) \end{cases} \right\} \\ &= \{x | \forall \omega \in S \left[x \in X(\omega)\right]\} \\ &= \{x | x \in X(\Omega)\} = X(\Omega) \end{aligned}$$

#### 14.1.2.2 probability density function

$$\begin{cases} \mathrm{P}_X(X \le x) = \mathrm{P}_X((-\infty, x]) = F_X(x) \\ \mathrm{P}_X(X = x) = \mathrm{P}_X(x) = ? \end{cases}$$

**Definition 14.8.** PDF = probability density function

PMF = probability mass function

$$\begin{cases} f_X(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} F_X(x) & X \text{ continuous RV} \\ f_X(x) = F_X(x) - F_X(x^-) & X \text{ discrete RV} \end{cases}$$

$$\begin{cases} f_X(x) = \text{derivative of } F_X(x) & X \text{ continuous} \\ f_X(x) = \text{difference of } F_X(x) & X \text{ discrete} \end{cases}$$

$$\begin{cases} f_X(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} F_X(x) & \Leftrightarrow F_X(x) = \int\limits_{-\infty}^{x} f_X(t)\,\mathrm{d}t \\ f_X(x) = F_X(x) - F_X(x^-) & \Leftrightarrow F_X(x) = \sum\limits_{t \le x} f_X(t) \end{cases}$$

$$\begin{cases} X \sim \mathrm{P}_X \leftrightarrow F_X(x) \leftrightarrow f_X(x) & \text{e.g. probability theory} \\ X \sim F_X(x) \leftrightarrow \mathrm{P}_X & \Rightarrow F_X(x) \overset{M}{=} F_X(x|\theta) \text{ e.g. survial analysis} \\ X \sim f_X(x) \leftrightarrow F_X(x) \leftrightarrow \mathrm{P}_X & \Rightarrow f_X(x) \overset{M}{=} f_X(x|\theta) \text{ e.g. general statistics} \end{cases}$$

---

**Theorem 14.2.** *PDF = probability density function or PMF = probability mass function: $f(x)$ is a probability density function or probability mass function iff*

$$\begin{cases} \forall x \in \mathbb{R}\,[f(x) \ge 0] \\ \begin{cases} \int\limits_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1 & t \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} f(x) = 1 & t \text{ discrete} \end{cases} \end{cases}$$

---

$$\forall E \subseteq S \left[ \mathrm{P}_X(X \in E) = \begin{cases} \int_{x \in E} f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x \in E} f_X(x) & X \text{ discrete} \end{cases} \right]$$

---

https://www.youtube.com/watch?v=KIXBlj-3M2k

$$\begin{aligned} \mathrm{P}_X(X = x) &= \lim_{\epsilon \to 0} \mathrm{P}_X([x-\epsilon, x+\epsilon]) \\ &= \lim_{\epsilon \to 0} \mathrm{P}_X(x - \epsilon \le X \le x + \epsilon) \\ &= \lim_{\epsilon \to 0} [F_X(x+\epsilon) - F_X(x-\epsilon)] \\ &= \begin{cases} F_X(x) - F_X(x) = 0 & X \text{ continuous} \\ F_X(x) - F_X(x^-) = f_X(x) & X \text{ discrete} \end{cases} \end{aligned}$$

$$X \sim F_X(x) \leftrightarrow \mathrm{P}_X$$

$$Y = g(X)$$

$$\begin{cases} Y \sim F_Y(y) \leftrightarrow f_Y(y) & \Rightarrow F_Y(y) \overset{M}{=} F_Y(y|\theta) \\ Y \sim f_Y(y) \leftrightarrow F_Y(y) \leftrightarrow \mathrm{P}_Y & \Rightarrow f_Y(y) \overset{M}{=} f_Y(y|\theta) \end{cases}$$

#### 14.1.2.3 range vs. support

**Definition 14.9.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned} \mathcal{R}_X &= \left\{ x \,\middle|\, \begin{cases} \omega \in S \\ x \in X(\omega) \end{cases} \right\} \\ &= \{x | \forall \omega \in S\,[x \in X(\omega)]\} \\ &= \{x | x \in X(\Omega)\} = X(\Omega) \end{aligned}$$

**Definition 14.10.** support

$$\operatorname{supp}(f) = \left\{ x \middle| \begin{cases} f : D \to \mathcal{R} \\ x \in D \\ f_X(x) \neq 0 \end{cases} \right\}$$

**Definition 14.11.** support of r.v. = support of RV = the support of a random variable

$$\operatorname{supp}(f_X) = \left\{ x \middle| \begin{cases} x \in X(\Omega) \\ f_X(x) \neq 0 \end{cases} \right\} \overset{f_X(x) \geq 0}{=} \left\{ x \middle| \begin{cases} x \in X(\Omega) \\ f_X(x) > 0 \end{cases} \right\}$$

#### 14.1.2.4 continuous monotone transformation

**Theorem 14.3.** *Random variable $Y$ is monotone transformation of random variable $X$, i.e.* $\begin{cases} X \sim F_X(x) \leftrightarrow f_X(x) \\ Y = g(X) \begin{cases} \forall x_1 < x_2 \left[ g(x_1) < g(x_2) \right] \\ \forall x_1 < x_2 \left[ g(x_1) > g(x_2) \right] \end{cases} \end{cases}$

*then*

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \right|$$

Proof:

$$\begin{aligned}
F_Y(y) =& \mathrm{P}_Y(Y \leq y) \\
=& \mathrm{P}(g(X) \leq y) \begin{cases} \forall x_1 < x_2 \left[ g(x_1) < g(x_2) \right] & \Leftrightarrow \forall g(x_1) < g(x_2) \left[ x_1 < x_2 \right] \\ \forall x_1 < x_2 \left[ g(x_1) > g(x_2) \right] & \Leftrightarrow \forall g(x_1) > g(x_2) \left[ x_1 < x_2 \right] \end{cases} \\
=& \begin{cases} \mathrm{P}_X\left(X \leq g^{-1}(y) = x\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) < g^{-1}(y_2) \right] \\ \mathrm{P}_X\left(X \geq g^{-1}(y) = x\right) & \forall y_1 > y_2 \left[ g^{-1}(y_1) < g^{-1}(y_2) \right] \end{cases} \\
=& \begin{cases} \mathrm{P}_X\left(X \leq g^{-1}(y) = x\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) < g^{-1}(y_2) \right] \\ \mathrm{P}_X\left(X \geq g^{-1}(y) = x\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) > g^{-1}(y_2) \right] \end{cases} \\
=& \begin{cases} F_X\left(g^{-1}(y)\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) < g^{-1}(y_2) \right] \\ 1 - F_X\left(g^{-1}(y)\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) > g^{-1}(y_2) \right] \end{cases} \\
F_Y(y) =& \begin{cases} F_X\left(g^{-1}(y)\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) < g^{-1}(y_2) \right] \\ 1 - F_X\left(g^{-1}(y)\right) & \forall y_1 < y_2 \left[ g^{-1}(y_1) > g^{-1}(y_2) \right] \end{cases}
\end{aligned}$$

$$
\begin{aligned}
f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) &= \begin{cases} \frac{\mathrm{d}}{\mathrm{d}y}F_X\left(g^{-1}(y)\right) & \forall y_1 < y_2\left[g^{-1}(y_1) < g^{-1}(y_2)\right] \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[1 - F_X\left(g^{-1}(y)\right)\right] & \forall y_1 < y_2\left[g^{-1}(y_1) > g^{-1}(y_2)\right] \end{cases} \\
&= \begin{cases} \frac{\mathrm{d}F_X\left(g^{-1}(y)\right)}{\mathrm{d}g^{-1}(y)}\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \forall y_1 < y_2\left[g^{-1}(y_1) < g^{-1}(y_2)\right] \\ \frac{-\mathrm{d}F_X\left(g^{-1}(y)\right)}{\mathrm{d}g^{-1}(y)}\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \forall y_1 < y_2\left[g^{-1}(y_1) > g^{-1}(y_2)\right] \end{cases} \\
&= \begin{cases} \frac{\mathrm{d}F_X\left(g^{-1}(y)\right)}{\mathrm{d}g^{-1}(y)}\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \forall y_1 < y_2\left[g^{-1}(y_1) < g^{-1}(y_2)\right] \\ \frac{\mathrm{d}F_X\left(g^{-1}(y)\right)}{\mathrm{d}g^{-1}(y)}\frac{-\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \forall y_1 < y_2\left[g^{-1}(y_1) > g^{-1}(y_2)\right] \end{cases} \\
&= \begin{cases} f_X\left(g^{-1}(y)\right)\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \begin{cases} \frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \geq 0 \\ f_X\left(g^{-1}(y)\right) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \\ f_X\left(g^{-1}(y)\right)\frac{-\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \begin{cases} \frac{-\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \geq 0 \\ f_X\left(g^{-1}(y)\right) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \end{cases} \\
f_Y(y) &= \begin{cases} f_X\left(g^{-1}(y)\right)\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \geq 0 \\ f_X\left(g^{-1}(y)\right)\frac{-\mathrm{d}g^{-1}(y)}{\mathrm{d}y} & \frac{-\mathrm{d}g^{-1}(y)}{\mathrm{d}y} \geq 0 \end{cases} \\
f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y}\right|
\end{aligned}
$$

□

segment $g(X)$ into monotone functions

For example, $\begin{cases} g(x) = x^2 \\ Y = g(X) \end{cases} \Rightarrow Y = g(X) = X^2$,

$$
\begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases}
$$

$$
\begin{aligned}
Y &= g(X) = X^2 \\
\Rightarrow Y &= \begin{cases} X^2 = g(X) & X \geq 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2\left[X_1^2 < X_2^2\right] \\ X^2 = g(X) & X < 0 \Leftrightarrow X \in (-\infty, 0) \Rightarrow \forall X_1 < X_2\left[X_1^2 > X_2^2\right] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & X \geq 0 \Rightarrow \forall X_1^2 < X_2^2\left[X_1 < X_2\right] \Rightarrow \forall Y_1 < Y_2\left[X_1 < X_2\right] \\ -\sqrt{Y} = g^{-1}(Y) & X < 0 \Rightarrow \forall X_1^2 < X_2^2\left[X_1 > X_2\right] \Rightarrow \forall Y_1 < Y_2\left[X_1 > X_2\right] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \geq 0 \Rightarrow \forall Y_1 < Y_2\left[g^{-1}(Y_1) < g^{-1}(Y_2)\right] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X < 0 \Rightarrow \forall Y_1 < Y_2\left[g^{-1}(Y_1) > g^{-1}(Y_2)\right] \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) =& \mathrm{P}_Y(Y \le y) = \mathrm{P}\left(X^2 \le y\right) \\
=& \mathrm{P}\left(\left\{X^2 \le y\right\} \cap (\{X < 0\} \cup \{X \ge 0\})\right) \\
=& \mathrm{P}\left(\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) \cup \left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right)\right) \\
=& \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) + \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right) \\
& - \mathrm{P}\left(\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) \cap \left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right)\right) \\
=& \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) + \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right) - \mathrm{P}(\emptyset) \\
=& \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) + \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right) - 0 \\
=& \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) + \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right) \\
=& \mathrm{P}(\{-X \le \sqrt{y}\} \cap \{X < 0\}) + \mathrm{P}(\{X \le \sqrt{y}\} \cap \{X \ge 0\}) \\
=& \mathrm{P}(\{X \ge -\sqrt{y}\} \cap \{X < 0\}) + \mathrm{P}(\{X \le \sqrt{y}\} \cap \{X \ge 0\}) \\
=& \mathrm{P}_X(-\sqrt{y} \le X < 0) + \mathrm{P}_X(0 \le X \le \sqrt{y}) \\
=& [F_X(0) - F_X(-\sqrt{y})] + [F_X(\sqrt{y}) - F_X(0)] \\
=& F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}
$$

□

Another example, $\begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases}$,

$$
\begin{aligned}
& Y = g(X) = X^2 \\
\Rightarrow & Y = \begin{cases} X^2 = g(X) & X \ge 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2 \left[X_1^2 < X_2^2\right] \\ X^2 = g(X) & -1 \le X < 0 \Leftrightarrow X \in [-1, 0) \Rightarrow \forall X_1 < X_2 \left[X_1^2 > X_2^2\right] \end{cases} \\
\Rightarrow & X = \begin{cases} \sqrt{Y} = g^{-1}(Y) & X \in [0, \infty) \Rightarrow \forall X_1^2 < X_2^2 [X_1 < X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 < X_2] \\ -\sqrt{Y} = g^{-1}(Y) & X \in [-1, 0) \Rightarrow \forall X_1^2 < X_2^2 [X_1 > X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 > X_2] \end{cases} \\
\Rightarrow & X = \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \in [0, \infty) \Rightarrow \forall Y_1 < Y_2 \left[g^{-1}(Y_1) < g^{-1}(Y_2)\right] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 \left[g^{-1}(Y_1) > g^{-1}(Y_2)\right] \end{cases} \\
\Rightarrow & X = \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \Rightarrow \forall Y_1 < Y_2 \left[g^{-1}(Y_1) < g^{-1}(Y_2)\right] \\ \sqrt{Y} = g^{-1}(Y) & Y \in [0, 1] \Rightarrow X \in [0, 1] \Rightarrow \forall Y_1 < Y_2 \left[g^{-1}(Y_1) < g^{-1}(Y_2)\right] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 \left[g^{-1}(Y_1) > g^{-1}(Y_2)\right] \end{cases}
\end{aligned}
$$

$$F_Y(y) = \mathrm{P}_Y(Y \le y) = \mathrm{P}\left(X^2 \le y\right) \begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases}$$

$$= \mathrm{P}\left(\left\{X^2 \le y\right\} \cap (\{X < 0\} \cup \{X \ge 0\})\right) = \cdots \text{ as } \begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases}$$

$$= \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X < 0\}\right) + \mathrm{P}\left(\left\{X^2 \le y\right\} \cap \{X \ge 0\}\right)$$

$$= \mathrm{P}\left(\{-X \le \sqrt{y}\} \cap \{X < 0\}\right) + \mathrm{P}\left(\{X \le \sqrt{y}\} \cap \{X \ge 0\}\right)$$

$$= \begin{cases} \mathrm{P}\left(\{-X \le \sqrt{y}\} \cap \{X < 0\} \cap \{X > 1\}\right) + \mathrm{P}\left(\{X \le \sqrt{y}\} \cap \{X \ge 0\} \cap \{X > 1\}\right) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ \mathrm{P}\left(\{-X \le \sqrt{y}\} \cap \{X < 0\} \cap \{X \ge -1\}\right) + \mathrm{P}\left(\{X \le \sqrt{y}\} \cap \{X \ge 0\} \cap \{X \le 1\}\right) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0) \end{cases} \end{cases}$$

$$= \begin{cases} \mathrm{P}(\emptyset) + \mathrm{P}_X\left(1 < X \le \sqrt{y}\right) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ \mathrm{P}\left(\{X \ge -\sqrt{y}\} \cap \{X < 0\} \cap \{X \ge -1\}\right) + \mathrm{P}_X\left(0 \le X \le \min\left\{\sqrt{y}, 1\right\}\right) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0) \end{cases} \end{cases}$$

$$= \begin{cases} 0 + \left[F_X\left(\sqrt{y}\right) - F_X(1)\right] & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ \mathrm{P}_X\left(\max\left\{-1, -\sqrt{y}\right\} \le X < 0\right) + \mathrm{P}_X\left(0 \le X \le \sqrt{y}\right) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0) \end{cases} \end{cases}$$

$$= \begin{cases} F_X\left(\sqrt{y}\right) - F_X(1) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ \mathrm{P}_X\left(-\sqrt{y} \le X < 0\right) + \left[F_X\left(\sqrt{y}\right) - F_X(0)\right] & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0) \end{cases} \end{cases}$$

$$= \begin{cases} F_X\left(\sqrt{y}\right) - F_X(1) & y > 1 \\ F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) & -1 \le y \le 1 \end{cases}$$

□

#### 14.1.2.5 discrete monotone transformation

$$\begin{cases} Y = g(X) = X^2 \\ X \text{ discrete} \Rightarrow \qquad Y \text{ discrete} \end{cases}$$

$$\begin{aligned} f_Y(y) =& \mathrm{P}_Y(Y = y) \\ =& \mathrm{P}\left(X^2 = y\right) \\ =& \mathrm{P}\left(\{X = \sqrt{y}\} \cup \{X = -\sqrt{y}\}\right) \\ =& \mathrm{P}\left(\{X = \sqrt{y}\}\right) + \mathrm{P}\left(\{X = -\sqrt{y}\}\right) - \mathrm{P}\left(\{X = \sqrt{y}\} \cap \{X = -\sqrt{y}\}\right) \\ =& \mathrm{P}_X\left(X = \sqrt{y}\right) + \mathrm{P}_X\left(X = -\sqrt{y}\right) - \mathrm{P}(\emptyset) \\ =& \mathrm{P}_X\left(X = \sqrt{y}\right) + \mathrm{P}_X\left(X = -\sqrt{y}\right) - 0 \\ =& \mathrm{P}_X\left(X = \sqrt{y}\right) + \mathrm{P}_X\left(X = -\sqrt{y}\right) \\ =& f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right) \end{aligned}$$

□

**Theorem 14.4.** *discrete monotone transformation*

$$\begin{cases} Y = g(X) \\ X \text{ discrete} \Rightarrow \quad Y \text{ discrete} \end{cases}$$
$$\Downarrow$$
$$f_Y(y) = \sum_{\{x | g(x) = y\}} f_X(x) = \sum_{\{x | x = g^{-1}(y)\}} f_X(x)$$

Proof:

$$
\begin{aligned}
f_Y(y) =& \mathrm{P}_Y(Y=y) \\
&= \mathrm{P}(g(X)=y) = \sum_{t\in\{x|g(x)=y\}} f_X(t) = \sum_{x\in\{x|g(x)=y\}} f_X(x) = \sum_{\{x|g(x)=y\}} f_X(x) \\
&= \mathrm{P}_X\left(X=g^{-1}(y)\right) = \sum_{t\in\{x|x=g^{-1}(y)\}} f_X(t) = \sum_{x\in\{x|x=g^{-1}(y)\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x)
\end{aligned}
$$

$$
\begin{array}{rcccccccc}
f_Y(y) & = & \mathrm{P}_Y(Y=y) & & & & & & \\
& = & \mathrm{P}(g(X)=y) & = & \sum_{t\in\{x|g(x)=y\}} f_X(t) & = & \sum_{x\in\{x|g(x)=y\}} f_X(x) & = & \sum_{\{x|g(x)=y\}} f_X(x) \\
& = & \mathrm{P}_X\left(X=g^{-1}(y)\right) & = & \sum_{t\in\{x|x=g^{-1}(y)\}} f_X(t) & = & \sum_{x\in\{x|x=g^{-1}(y)\}} f_X(x) & = & \sum_{\{x|x=g^{-1}(y)\}} f_X(x)
\end{array}
$$

□

**Theorem 14.5.** *probability integral transformation*

$$
\begin{gathered}
\begin{cases}
\begin{cases}
X \text{ continuous} & (c) \\
X \sim F_X(x) & (d)
\end{cases} \\
Y = F_X(X) & (t)
\end{cases} \\
\Downarrow \\
F_Y(y) = y, \forall y \in [0,1] \\
\Updownarrow \text{def.} \\
Y \sim U = U(y) \Leftrightarrow Y \sim U(y) \Leftrightarrow Y \text{ is uniformly distributed on } [0,1]
\end{gathered}
$$

Proof:

$$
\begin{aligned}
F_Y(y) =& \mathrm{P}_Y(Y \le y) \overset{(t)}{=} \mathrm{P}(F_X(X) \le y), \forall x_1 < x_2\left[F_X(x_1) < F_X(x_2)\right] \Rightarrow \begin{cases} \exists F_X^{-1} : Y \to X \\ \forall y_1 < y_2 \left[F_X^{-1}(y_1) < F_X^{-1}(y_2)\right] \end{cases} \\
&= \mathrm{P}_X\left(X \le F_X^{-1}(y) = x\right) = \mathrm{P}_X(X \le x), x = F_X^{-1}(y) \\
&= \mathrm{P}_X(X \le x) = F_X(x) \overset{x=F_X^{-1}(y)}{=} F_X\left(F_X^{-1}(y)\right) = y \\
F_Y(y) =& y
\end{aligned}
$$

□

Note:

According to Theorem 14.5,

$$
\begin{aligned}
& U = F_X(X) \overset{14.5}{\sim} U(u) \text{ on } [0,1] \\
&\Rightarrow X = F_X^{-1}(U) \wedge X \sim F_X(x) \Rightarrow F_X^{-1}(U) = X \sim F_X(x) \Rightarrow F_X^{-1}(U) \sim F_X(x) \\
&\Rightarrow X = F_X^{-1}(U) \sim F_X(x)
\end{aligned}
$$

i.e. uniform random variables subsituted into the inverse of $F_X$,
we can get random variables follow $F_X(x)$

#### 14.1.2.6 expected value

$$
\mathrm{E}(g(X)) = \mathrm{E}[g(X)] = \mathrm{E}g(X) = \mathbb{E}[g(X)] = \mathbb{E}g(X)
$$

**Definition 14.12.** expected value: The expected value of a random variable $g(X)$ is

$$
\mathrm{E}(g(X)) = \mathrm{E}[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum_{x\in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}
$$

[6] p.126

**Definition 14.13.** expected value or expectation function: The expected value of a random variable $X$ is

$$\mathrm{E}(X) = \mathrm{E}[X] = \begin{cases} \int\limits_{-\infty}^{+\infty} x f_X(x)\,\mathrm{d}x = 1 & X \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} x f_X(x) = 1 & X \text{ discrete} \end{cases}$$

**Theorem 14.6.** *the rule of the lazy statistician*

*the law of the unconscious statistician = the LOTUS*

$$\mathrm{E}(g(X)) = \mathrm{E}[g(X)] = \begin{cases} \int\limits_{-\infty}^{+\infty} g(x) f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}$$

Proof:[7] p.162 for p.119

Discrete case:

to be proved

Continuous case:

to be proved

□

---

By linearity of $\int$ and $\sum$, expected values have the following properties or theorems,

- $\mathrm{E}[a_1 g_1(X_1) + a_2 g_2(X_2) + c] = a_1 \mathrm{E}[g_1(X_1)] + a_2 \mathrm{E}[g_2(X_2)] + c$
- $\forall x \in \mathbb{R}\, [g(x) \geq 0] \Rightarrow \mathrm{E}[g(X)] \geq 0$
- $\forall x \in \mathbb{R}\, [g_1(x) \geq g_2(x)] \Rightarrow \mathrm{E}[g_1(X)] \geq \mathrm{E}[g_2(X)]$
- $\forall x \in \mathbb{R}\, [a \leq g(x) \leq b] \Rightarrow a \leq \mathrm{E}[g(X)] \leq b$

---

**Theorem 14.7.** $\mathrm{E}[X]$ *minimizes Euclidean distance* $\mathrm{E}\left[(X-b)^2\right]$ *over* $b$*, i.e.*

$$\mathrm{E}[X] = \arg\min_b \mathrm{E}\left[(X-b)^2\right]$$

Proof:

$$
\begin{aligned}
\mathrm{E}\left[(X-b)^2\right] =& \mathrm{E}\left[(X-\mathrm{E}[X]+\mathrm{E}[X]-b)^2\right] \\
=& \mathrm{E}\left[\{(X-\mathrm{E}[X])+(\mathrm{E}[X]-b)\}^2\right] \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2+2(X-\mathrm{E}[X])(\mathrm{E}[X]-b)+(\mathrm{E}[X]-b)^2\right] \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+2(\mathrm{E}[X]-b)\,\mathrm{E}[(X-\mathrm{E}[X])]+\mathrm{E}\left[(\mathrm{E}[X]-b)^2\right] \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+2(\mathrm{E}[X]-b)\,\mathrm{E}[X-\mathrm{E}[X]]+(\mathrm{E}[X]-b)^2 \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+2(\mathrm{E}[X]-b)(\mathrm{E}[X]-\mathrm{E}[X])+(\mathrm{E}[X]-b)^2 \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+2(\mathrm{E}[X]-b)\,0+(\mathrm{E}[X]-b)^2 \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+0+(\mathrm{E}[X]-b)^2 \\
=& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right]+(\mathrm{E}[X]-b)^2 \overset{(\mathrm{E}[X]-b)^2\geq 0}{\geq} \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] \\
\mathrm{E}\left[(X-b)^2\right] \geq& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] \\
& \Downarrow \\
\mathrm{E}\left[(X-b)^2\right] =& \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] \text{ holds if } (\mathrm{E}[X]-b)^2=0 \Rightarrow b=\mathrm{E}[X] \Rightarrow \mathrm{E}[X]=\arg\min_b \mathrm{E}\left[(X-b)^2\right]
\end{aligned}
$$

$\square$

Note:

When $b=\mathrm{E}[X]$, $\mathrm{E}\left[(X-b)^2\right]$ has minimum loss $\mathrm{E}\left[(X-\mathrm{E}[X])^2\right]=\mathrm{V}[X]=\mathrm{V}(X)$, i.e. defintion of variance appears.

---

**Theorem 14.8.** median $[X]$ *minimizes* $\mathrm{E}[|X-b|]$ *over* $b$, *i.e.*

$$\mathrm{median}[X]=\arg\min_b \mathrm{E}[|X-b|]$$

Proof:

to be proved

$\square$

Note:

When $b=\mathrm{median}[X]$, $\mathrm{E}\left[(X-b)^2\right]$ has minimum loss $\mathrm{E}[|X-\mathrm{median}[X]|]$, i.e. defintion of MAD(mean absolute deviation) in robust statistics appears.

---

**Definition 14.14.** indicator function

$$
\begin{aligned}
1(E)=1(x\in E)=1(\{x\in E\})=1(\{x|x\in E\}) &= \begin{cases}1 & E\\ 0 & \overline{E}\end{cases} = \begin{cases}1 & \text{if } E\\ 0 & \text{if } \overline{E}=E^{\mathrm{C}}\end{cases} \\
&= \begin{cases}1 & \text{if event } E \text{ occurs}\\ 0 & \text{if event } E \text{ does not occur}\end{cases}
\end{aligned}
$$

Note:

**Theorem 14.9.** *probability as expected value*

$$\mathrm{P}_X(E) = \mathrm{P}(x \in E) = \mathrm{E}[1(X \in E)]$$

Proof:

$$\begin{aligned}
\mathrm{P}_X(E) = \mathrm{P}(x \in E) &= \int_{x \in E} f_X(x)\,\mathrm{d}x = \int_E f_X(x)\,\mathrm{d}x \\
&= \int 1(x \in E) f_X(x)\,\mathrm{d}x \\
&= \int g(x) f_X(x)\,\mathrm{d}x, g(x) = 1(x \in E) \\
&= \mathrm{E}[g(X)], g(X) = 1(X \in E) \\
&= \mathrm{E}[1(X \in E)] \\
\mathrm{P}_X(E) = \mathrm{P}(x \in E) &= \mathrm{E}[1(X \in E)]
\end{aligned}$$

$\square$

---

Iverson bracket https://en.wikipedia.org/wiki/Iverson_bracket

$$\begin{cases} v(p(x)) = \mathrm{T} & \Leftrightarrow [p(x)] = 1 \\ v(p(x)) = \mathrm{F} & \Leftrightarrow [p(x)] = 0 \end{cases}$$

$$[p(x)] = \begin{cases} 1 & v(p(x)) = \mathrm{T} \\ 0 & v(\neg p(x)) = \mathrm{T} \end{cases} = \begin{cases} 1 & p(x) \\ 0 & \neg p(x) \end{cases}$$

negation = NOT

$$[\neg p] = 1 - [p]$$

in set theory or domain of events,

$$1\left(\overline{E}\right) = 1 - 1(E)$$

conjunction = AND

$$[p \wedge q] = [p][q]$$

in set theory or domain of events,

$$1(E_1 \cap E_2) = 1(E_1)\,1(E_2)$$

disjunction = OR

$$[p \vee q] = [p] + [q] - [p][q] = [p] + [q] - [p \wedge q]$$

Proof:

in set theory or domain of events,

$$\begin{aligned}
1(E_1 \cup E_2) &\overset{\text{de Moivre}}{=} 1\left(\overline{\overline{E}_1 \cap \overline{E}_2}\right) \\
&= 1 - 1\left(\overline{E}_1 \cap \overline{E}_2\right) = 1 - 1\left(\overline{E}_1\right) 1\left(\overline{E}_2\right) \\
&= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\
&= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\
&= 1 - [1 - 1(E_1) - 1(E_2) + 1(E_1)\,1(E_2)] \\
&= 1(E_1) + 1(E_2) - 1(E_1)\,1(E_2) \\
&= 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)
\end{aligned}$$

$$1(E_1 \cup E_2) = 1(E_1) + 1(E_2) - 1(E_1)\,1(E_2) = 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)$$

$\square$

implication = conditional

$$\begin{aligned}[p \to q] &= [\neg p \vee q] \\ &= [\neg p] + [q] - [\neg p]\,[q] \\ &= 1 - [p] + [q] - (1 - [p])\,[q] \\ &= 1 - [p] + [p]\,[q]\end{aligned}$$

exclusive disjunction = XOR

$$\begin{aligned}[p \veebar q] = [p \oplus q] &= |[p] - [q]| = ([p] - [q])^2 \\ &= [p]\,(1 - [q]) + (1 - [p])\,[q]\end{aligned}$$

biconditional = XNOR

$$[p \leftrightarrow q] = [p \odot q] = [\neg\,(p \oplus q)] = [\neg\,(p \veebar q)] = ([p] + (1 - [q]))\,((1 - [p]) + [q])$$

Kronecker delta

$$\delta_{ij} = [i = j]$$

single-argument notation

$$\delta_i = \delta_{i0} = \begin{cases} 1 & i = j = 0 \\ 0 & i \neq j = 0 \end{cases}$$

sign function

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 = [x > 0] - [x < 0] \\ -1 & x < 0 \end{cases}$$

absolute function

$$\begin{aligned}|x| &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases} = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases} \\ &= \begin{cases} x \cdot 1 & x > 0 \\ x \cdot 0 & x = 0 \\ x \cdot (-1) & x < 0 \end{cases} = \begin{cases} x \cdot \operatorname{sgn}(x) & x > 0 \\ x \cdot \operatorname{sgn}(x) & x = 0 \\ x \cdot \operatorname{sgn}(x) & x < 0 \end{cases} \\ &= x \cdot \operatorname{sgn}(x) = x\,([x > 0] - [x < 0]) = x\,[x > 0] - x\,[x < 0]\end{aligned}$$

binary min and max function

$$\max(x, y) = x\,[x > y] + y\,[x \leq y]$$

$$\min(x, y) = x\,[x \leq y] + y\,[x > y]$$

binary max function

$$\max(x,y) = \frac{x+y+|x-y|}{2}$$

floor and ceiling functions

floor function

$$\begin{aligned}\lfloor x \rfloor &= n,\ n \le x < n+1 \\ &= \sum_{n\in\mathbb{N}} n\,[n \le x < n+1]\end{aligned}$$

ceiling function

$$\begin{aligned}\lceil x \rceil &= n,\ n-1 < x \le n \\ &= \sum_{n\in\mathbb{N}} n\,[n-1 < x \le n]\end{aligned}$$

Heaviside step function

$$\mathrm{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} = [x > 0] = 1_{(0,\infty)}(x)$$

or conveniently define “unit step function”

$$u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases} = [x \ge 0] = 1_{[0,\infty)}(x)$$

ramp function = rectified linear unit activation function = ReLU

$$\mathrm{ReLU}(x) = \begin{cases} x & x \ge 0 \\ 0 & x < 0 \end{cases} = x\,[x \ge 0]$$

indicator function

$$A \subseteq X \Rightarrow \begin{cases} 1_A : X \to \{0,1\} & \Leftrightarrow x \in X \overset{1_A}{\to} \{0,1\} \\ 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} & = [x \in A] = \begin{cases} 1 & v(x \in A) = \mathrm{T} \\ 0 & v(\neg(x \in A)) = \mathrm{T} \end{cases} \end{cases}$$

$A, B \subseteq \Omega$,

$$A = B \Leftrightarrow 1_A = 1_B$$

$$A = \Omega \Leftrightarrow 1_A(x) = 1$$

$$A = \emptyset \Leftrightarrow 1_A(x) = 0$$

**Theorem 14.10.** *subset indicator order*

$$A \subset B \Rightarrow 1_A(x) \le 1_B(x)$$

Proof:

$$\begin{aligned}
&\forall x \left(1_A(x) = 1 \Rightarrow 1_B(x) = 1\right)\\
\Leftrightarrow&\forall x \left(\neg 1_A(x) = 1 \vee 1_B(x) = 1\right)\\
\Leftrightarrow&\forall x \neg \left(1_A(x) = 1 \wedge \neg 1_B(x) = 1\right)\\
\Rightarrow&\neg\exists x \left(1_A(x) = 1 \wedge 1_B(x) = 0\right)\\
\Rightarrow&\neg\exists x \left(1_B(x) = 0 < 1 = 1_A(x)\right)\\
\Rightarrow&\neg\exists x \left(1_B(x) < 1_A(x)\right)\\
\Rightarrow&\forall x \left(1_B(x) \geq 1_A(x)\right)
\end{aligned}$$

□

---

in set theory or domain of events,

$$1\left(E_1 \cap E_2\right) = 1\left(E_1\right) 1\left(E_2\right)$$

$$1\left(\overline{E}\right) = 1 - 1\left(E\right)$$

$$1\left(E_1 \cup E_2\right) = 1\left(E_1\right) + 1\left(E_2\right) - 1\left(E_1\right) 1\left(E_2\right) = 1\left(E_1\right) + 1\left(E_2\right) - 1\left(E_1 \cap E_2\right)$$

---

expectation in many perspectives

$$Y = g\left(X\right)$$

$$\int\limits_{-\infty}^{+\infty} y\, f_Y\left(y\right) \mathrm{d}y = \mathrm{E}\left[Y\right] = \mathrm{E}\left[g\left(X\right)\right] = \int\limits_{-\infty}^{+\infty} g\left(x\right) f_X\left(x\right) \mathrm{d}x$$

$$\mathrm{E}_Y\left[Y\right] = \int\limits_{-\infty}^{+\infty} y\, f_Y\left(y\right) \mathrm{d}y = \mathrm{E}\left[Y\right] = \mathrm{E}\left[g\left(X\right)\right] = \int\limits_{-\infty}^{+\infty} g\left(x\right) f_X\left(x\right) \mathrm{d}x = \mathrm{E}_X\left[g\left(X\right)\right]$$

$$\mathrm{E}_Y\left[Y\right] = \mathrm{E}_X\left[g\left(X\right)\right]$$

---

$$\begin{aligned}
&\mathrm{E}\left[a_1 g_1\left(X_1\right) + a_2 g_2\left(X_2\right) + c\right], \begin{cases} Y_1 = g_1\left(X_1\right) \\ Y_2 = g_2\left(X_2\right) \end{cases}\\
=&\mathrm{E}\left[a_1 Y_1 + a_2 Y_2 + c\right]
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}\left[a_1 g_1\left(X_1\right) + a_2 g_2\left(X_2\right) + c\right] = a_1 \mathrm{E}\left[g_1\left(X_1\right)\right] + a_2 \mathrm{E}\left[g_2\left(X_2\right)\right] + c\\
=&\mathrm{E}\left[a_1 Y_1 + a_2 Y_2 + c\right] = a_1 \mathrm{E}\left[Y_1\right] + a_2 \mathrm{E}\left[Y_2\right] + c
\end{aligned}$$

$$a_1 \mathrm{E}\left[g_1\left(X_1\right)\right] + a_2 \mathrm{E}\left[g_2\left(X_2\right)\right] + c = a_1 \mathrm{E}_{X_1}\left[g_1\left(X_1\right)\right] + a_2 \mathrm{E}_{X_2}\left[g_2\left(X_2\right)\right] + c$$

$$a_1 \mathrm{E}\left[Y_1\right] + a_2 \mathrm{E}\left[Y_2\right] + c = a_1 \mathrm{E}_{Y_1}\left[Y_1\right] + a_2 \mathrm{E}_{Y_1}\left[Y_2\right] + c$$

$$a_1 \mathrm{E}_{X_1}\left[g_1\left(X_1\right)\right] + a_2 \mathrm{E}_{X_2}\left[g_2\left(X_2\right)\right] + c = a_1 \mathrm{E}_{Y_1}\left[Y_1\right] + a_2 \mathrm{E}_{Y_1}\left[Y_2\right] + c$$

#### 14.1.2.7 moment

**Definition 14.15.** $n^{\text{th}}$ moment: For each integer $n$, the $n^{\text{th}}$ moment of $X$ is $\mathrm{E}[X^n]$.

The $n^{\text{th}}$ central moment of $X$ is $\mu_n = \mathrm{E}[(X - \mathrm{E}[X])^n]$.

$$\mathrm{E}[X^n] = \begin{cases} \int\limits_{-\infty}^{+\infty} x^n f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x^n f_X(x) & X \text{ discrete} \end{cases}$$

$$\mu = \mathrm{E}\left[X^1\right] = \mathrm{E}[X] = \begin{cases} \int\limits_{-\infty}^{+\infty} x^1 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int\limits_{-\infty}^{+\infty} x\, f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x\, f_X(x) & X \text{ discrete} \end{cases}$$

$$\mu_n = \mathrm{E}[(X - \mathrm{E}[X])^n] = \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu)^n f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} (x-\mu)^n f_X(x) & X \text{ discrete} \end{cases}$$

---

$1^{\text{st}}$ moment of $X$ = mean

$$\mu = \mathrm{E}\left[X^1\right] = \mathrm{E}[X] = \begin{cases} \int\limits_{-\infty}^{+\infty} x^1 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int\limits_{-\infty}^{+\infty} x\, f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x\, f_X(x) & X \text{ discrete} \end{cases}$$

$1^{\text{st}}$ central moment of $X = 0$

$$\begin{aligned}
\mu_1 = \mathrm{E}\left[(X - \mathrm{E}[X])^1\right] &= \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu)^1 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} (x-\mu)^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu) f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} (x-\mu) f_X(x) & X \text{ discrete} \end{cases} \\
&= \begin{cases} \int\limits_{-\infty}^{+\infty} x\, f_X(x)\,\mathrm{d}x - \int\limits_{-\infty}^{+\infty} \mu\, f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x\in X(\Omega)} x\, f_X(x) - \sum\limits_{x\in X(\Omega)} \mu\, f_X(x) & X \text{ discrete} \end{cases} \\
&= \begin{cases} \mathrm{E}[X] - \mu \int\limits_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \mathrm{E}[X] - \mu \sum\limits_{x\in X(\Omega)} f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \mathrm{E}[X] - \mu \cdot 1 & X \text{ continuous} \\ \mathrm{E}[X] - \mu \cdot 1 & X \text{ discrete} \end{cases} \\
&= \begin{cases} \mathrm{E}[X] - \mu & X \text{ continuous} \\ \mathrm{E}[X] - \mu & X \text{ discrete} \end{cases} = \begin{cases} \mathrm{E}[X] - \mathrm{E}[X] & X \text{ continuous} \\ \mathrm{E}[X] - \mathrm{E}[X] & X \text{ discrete} \end{cases} \\
&= \begin{cases} 0 & X \text{ continuous} \\ 0 & X \text{ discrete} \end{cases} = 0 \\
\mathrm{E}[(X - \mathrm{E}[X])] &= 0 \\
\mathrm{E}[X - \mathrm{E}[X]] &= 0
\end{aligned}$$

$$\forall X\, (\mathrm{E}[X - \mathrm{E}[X]] = 0)$$

For normal distribution, actually for any distribution,

$$\begin{gathered} X \sim \mathrm{n}(0,1) = \mathcal{N}\left(0, 1^2\right) \\ \Downarrow \\ \mathrm{E}[X - \mathrm{E}[X]] = 0 \end{gathered}$$

---

$2^{\text{nd}}$ central moment of $X$ = variance

$$\mu_2 = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} (x-\mu)^2 f_X(x) & X \text{ discrete} \end{cases}$$
$$= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{V}[X] = \mathrm{V}(X)$$

For normal distribution,

$$X \sim \mathrm{n}(0,1) = \mathcal{N}\left(0, 1^2\right) = \mathcal{N}\left(\mu = 0, \mathrm{V}^2[X] = 1^2\right)$$
$$\Downarrow$$
$$\mathrm{V}[X] = \mathrm{V}(X) = 1$$

variance properties

$$\mathrm{V}[aX + b] = a^2 \mathrm{V}[X]$$

Proof:

to be proved

$\square$

---

$3^{\text{rd}}$ central moment of $X$

$$\mu_3 = \mathrm{E}\left[(X - \mathrm{E}[X])^3\right] = \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu)^3 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} (x-\mu)^3 f_X(x) & X \text{ discrete} \end{cases}$$

skewness
偏度

$$\text{skewness}[X] = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^3\right]}{(\mathrm{V}[X])^{\frac{3}{2}}} = \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^3\right]}{\left(\mathrm{E}\left[(X - \mathrm{E}[X])^2\right]\right)^{\frac{3}{2}}}$$
$$= \begin{cases} \dfrac{\int\limits_{-\infty}^{+\infty} (x-\mu)^3 f_X(x)\,\mathrm{d}x}{\left(\int\limits_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,\mathrm{d}x\right)^{\frac{3}{2}}} & X \text{ continuous} \\ \dfrac{\sum\limits_{x \in X(\Omega)} (x-\mu)^3 f_X(x)}{\left(\sum\limits_{x \in X(\Omega)} (x-\mu)^2 f_X(x)\right)^{\frac{3}{2}}} & X \text{ discrete} \end{cases}$$

For normal distribution,

$$X \sim \mathrm{n}(0,1) = \mathcal{N}\left(0, 1^2\right) = \mathcal{N}\left(\mu = 0, \mathrm{V}^2[X] = 1^2\right)$$
$$\Downarrow$$
$$\text{skewness}[X] = \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^3\right]}{(\mathrm{V}[X])^{\frac{3}{2}}} = \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^3\right]}{1^{\frac{3}{2}}} = 0$$

Proof:

to be proved

□

---

$4^{\text{th}}$ central moment of $X$

$$\mu_4 = \mathrm{E}\left[(X - \mathrm{E}[X])^4\right] = \begin{cases} \int\limits_{-\infty}^{+\infty} (x-\mu)^4 f_X(x)\,\mathrm{d}x & X \text{ continuous} \\ \sum\limits_{x \in X(\Omega)} (x-\mu)^4 f_X(x) & X \text{ discrete} \end{cases}$$

kurtosis

峰度

$$\begin{aligned} \text{kurtosis}[X] =& \frac{\mu_4}{\mu_2^2} = \frac{\mathrm{E}\left[(X-\mathrm{E}[X])^4\right]}{(\mathrm{V}[X])^2} = \frac{\mathrm{E}\left[(X-\mathrm{E}[X])^4\right]}{\left(\mathrm{E}\left[(X-\mathrm{E}[X])^2\right]\right)^2} \\ =& \begin{cases} \dfrac{\int\limits_{-\infty}^{+\infty} (x-\mu)^4 f_X(x)\,\mathrm{d}x}{\left(\int\limits_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,\mathrm{d}x\right)^2} & X \text{ continuous} \\ \dfrac{\sum\limits_{x\in X(\Omega)} (x-\mu)^4 f_X(x)}{\left(\sum\limits_{x\in X(\Omega)} (x-\mu)^2 f_X(x)\right)^2} & X \text{ discrete} \end{cases} \end{aligned}$$

For normal distribution,

$$\begin{gathered} X \sim \mathrm{n}(0,1) = \mathcal{N}\left(0,1^2\right) = \mathcal{N}\left(\mu=0, \mathrm{V}^2[X]=1^2\right) \\ \Downarrow \\ \text{kurtosis}[X] = \frac{\mathrm{E}\left[(X-\mathrm{E}[X])^4\right]}{(\mathrm{V}[X])^2} = \frac{\left[(X-\mathrm{E}[X])^4\right]}{1^2} = 3 \end{gathered}$$

Proof:

to be proved

□

---

For normal distribution,

$$\begin{gathered} X \sim \mathrm{n}(0,1) = \mathcal{N}\left(0,1^2\right) = \mathcal{N}\left(\mu=0, \mathrm{V}^2[X]=1^2\right) \\ \Downarrow \\ \begin{cases} \mu = \mathrm{E}[X] & = 0 \\ \mu_1 = \mathrm{E}[X - \mathrm{E}[X]] & = 0 \\ \text{variance}[X] = \mathrm{V}[X] = \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] & = 1 \\ \text{skewness}[X] = \dfrac{\mathrm{E}\left[(X-\mathrm{E}[X])^3\right]}{(\mathrm{V}[X])^{\frac{3}{2}}} & = 0 \\ \text{kurtosis}[X] = \dfrac{\mathrm{E}\left[(X-\mathrm{E}[X])^4\right]}{(\mathrm{V}[X])^2} & = 3 \end{cases} \end{gathered}$$

$$\begin{aligned}
\mu =& \mathrm{E}[X] = 0 \\
\mu_1 =& \mathrm{E}[X - \mathrm{E}[X]] = 0 \\
\text{variance}[X] =& \mathrm{V}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = 1 \\
\text{skewness}[X] =& \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^3\right]}{(\mathrm{V}[X])^{\frac{3}{2}}} = 0 \\
\text{kurtosis}[X] =& \frac{\mathrm{E}\left[(X - \mathrm{E}[X])^4\right]}{(\mathrm{V}[X])^2} = 3
\end{aligned}$$

---

$$X \sim F_X(x) \leftrightarrow f_X(x) \to \{\mu_n | n \in \mathbb{N}\} = \left\{\mu_n \middle| \begin{cases} n \in \mathbb{N} \\ \mu_n = \mathrm{E}[(X - \mathrm{E}[X])^n] \end{cases}\right\}$$

---

#### 14.1.2.7.1 moment generating function

**Definition 14.16.** MGF = moment generating function: The moment generating function of $X$ is $M(\xi) = M_X(\xi) = \mathrm{E}\left[\mathrm{e}^{\xi X}\right]$, provided that the expression exists for $t \approx 0$.

$$M(t) = M_X(t) = \mathrm{E}\left[\mathrm{e}^{tX}\right]$$

$$M(\xi) = M_X(\xi) = \mathrm{E}\left[\mathrm{e}^{\xi X}\right]$$

**Theorem 14.11.** *moment generating function(MGF) generating moment*

$$M_X^{(n)}(\xi) = \mathrm{E}[X^n]$$

where

$$M_X^{(n)}(\xi) = \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} M_X(\xi)$$

Proof:

to be proved

□

$$X \sim F_X(x) \leftrightarrow f_X(x) \to \{\mu_n | n \in \mathbb{N}\} = \left\{\mu_n \middle| \begin{cases} n \in \mathbb{N} \\ \mu_n = \mathrm{E}[(X - \mathrm{E}[X])^n] \end{cases}\right\}$$

$$\begin{array}{ccccccc}
X & \sim & F_X(x) & \leftrightarrow & f_X(x) & \to & \{\mu_n | n \in \mathbb{N}\} \\
 & & & & \downarrow & \nearrow & \\
 & & & & M_X(\xi) & &
\end{array}$$

$$\begin{array}{ccccccc}
X & \sim & F_X(x) & \leftrightarrow & f_X(x) & & \\
 & & & & \downarrow & \searrow & \\
 & & & & M_X(\xi) & \to & \{\mu_n | n \in \mathbb{N}\}
\end{array}$$

**Theorem 14.12.** *If $X$ and $Y$ have bounded support, then $\forall u\, [F_X(u) = F_Y(u)]$ iff $\forall n \in \mathbb{N}\, (\mathrm{E}[X^n] = \mathrm{E}[Y^n])$.*

$$\forall u\left[F_X\left(u\right)=F_Y\left(u\right)\right]\Rightarrow\forall n\in\mathbb{N}\left(\mathrm{E}\left[X^n\right]=\mathrm{E}\left[Y^n\right]\right)$$

$$\begin{cases}\forall n\in\mathbb{N}\left(\mathrm{E}\left[X^n\right]=\mathrm{E}\left[Y^n\right]\right)\\ \begin{cases}\operatorname{supp}\left(f_X\right)\text{ is bounded}\\ \operatorname{supp}\left(f_Y\right)\text{ is bounded}\end{cases}\end{cases}\Rightarrow\forall u\left[F_X\left(u\right)=F_Y\left(u\right)\right]$$

Proof:

to be proved

□

**Theorem 14.13.** *If $M_X\left(t\right)$ and $M_Y\left(t\right)$ exist, then $\forall u\left[F_X\left(u\right)=F_Y\left(u\right)\right]$ iff $\forall t\approx0\left[M_X\left(t\right)=M_Y\left(t\right)\right]$.*

$$\forall u\left[F_X\left(u\right)=F_Y\left(u\right)\right]\Rightarrow\forall t\approx0\left[M_X\left(t\right)=M_Y\left(t\right)\right]$$

$$\begin{cases}\forall t\approx0\left[M_X\left(t\right)=M_Y\left(t\right)\right]\\ \begin{cases}\exists M_X\left(t\right)\in\mathbb{R}\\ \exists M_Y\left(t\right)\in\mathbb{R}\end{cases}\end{cases}\Rightarrow\forall u\left[F_X\left(u\right)=F_Y\left(u\right)\right]$$

Proof:

to be proved

□

$$\begin{array}{ccccccc} X & \sim & F_X\left(x\right) & \leftrightarrow & f_X\left(x\right) & & \\ & & & \nrightarrow & \uparrow\downarrow & \searrow\nwarrow & \nleftarrow\wedge\operatorname{supp}\left(f_X\right)\text{ is bounded}\\ & & \forall\xi\approx0\left[M_X\left(\xi\right)\in\mathbb{R}\right] & \wedge & M_X\left(\xi\right) & \rightarrow & \left\{\mu_n|n\in\mathbb{N}\right\}\end{array}$$

#### 14.1.2.7.2 characteristic function

**Definition 14.17.** CF = characteristic function: The characteristic function of $X$ is $\varphi\left(\xi\right)=\varphi_X\left(\xi\right)=\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\xi X}\right]$, provided that the expression always exists.

$$\varphi\left(t\right)=\varphi_X\left(t\right)=\mathrm{E}\left[\mathrm{e}^{\mathrm{i}tX}\right]$$

$$\varphi\left(\xi\right)=\varphi_X\left(\xi\right)=\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\xi X}\right]$$

Note:

1. $\varphi\left(\xi\right)=\varphi_X\left(\xi\right)=\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\xi X}\right]$ always exists.

$$\forall X\left(\varphi\left(\xi\right)=\varphi_X\left(\xi\right)=\mathrm{E}\left[\mathrm{e}^{\mathrm{i}\xi X}\right]\in\mathbb{R}\right)$$

2. moment generating function to characteristic function

$$M\left(\xi\right)=M_X\left(\xi\right)=\mathrm{E}\left[\mathrm{e}^{\xi X}\right]\in\mathbb{R}\Rightarrow M_X\left(\mathrm{i}\xi\right)=\varphi_X\left(\xi\right)$$

3. inversion theorem or inversion formula

For $a < b$,

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{+T}\frac{\mathrm{e}^{-ita}-\mathrm{e}^{-itb}}{\mathrm{i}t}\varphi_X(t)\,\mathrm{d}t\\
=&\mathrm{P}(a<X<b)+\frac{1}{2}\left[\mathrm{P}(X=a)+\mathrm{P}(X=b)\right]
\end{aligned}
$$

i.e. $\varphi_X(\xi)$ determines $F_X(x)$.

$$
\begin{array}{ccccccccc}
X & \sim & F_X(x) & \leftrightarrow & f_X(x) & & \\
 & & \updownarrow & \forall\xi\approx 0\left[M_X(\xi)\in\mathbb{R}\right]\ \wedge & \uparrow\downarrow & \searrow\nwarrow & \looparrowleft\wedge\operatorname{supp}(f_X)\ \text{ is bounded}\\
 & & \varphi_X(\xi) & \leftrightarrows & M_X(\xi) & \rightarrow & \{\mu_n|n\in\mathbb{N}\}
\end{array}
$$

$$
\begin{array}{ccccccccc}
X & \sim & & F_X(x) & \xleftrightarrow{\text{FToC}} & f_X(x) & \leftrightarrow & \mathrm{P}_X\\
 & \text{inversion formula} & : & \updownarrow & \forall\xi\approx 0\left[M_X(\xi)\in\mathbb{R}\right]\ \wedge & \uparrow\downarrow & \searrow\nwarrow & \looparrowleft\wedge\operatorname{supp}(f_X)\ \text{ is bounded}\\
 & & & \varphi_X(\xi) & \leftrightarrows & M_X(\xi) & \rightarrow & \{\mu_n|n\in\mathbb{N}\}
\end{array}
$$

---

https://www.youtube.com/watch?v=fSbs6im6wqY

MGF theorems

https://www.youtube.com/watch?v=fSbs6im6wqY&t=524s

**Theorem 14.14.** *If*

CLT

#### 14.1.2.8 common families of distributions

mean and variance of discrete probability distributions[41.5]

##### 14.1.2.8.1 discrete distribution

###### 14.1.2.8.1.1 discrete uniform distribution

$$
\begin{gathered}
X\sim\mathcal{DU}(1,N)\\
\Updownarrow\\
\begin{cases}
f_X(x|N)=\dfrac{1}{N}\\
x\in X(\Omega)=\{1,2,\cdots,N\}
\end{cases}\\
\Downarrow\\
\begin{cases}
\mathrm{E}[X]= & \dfrac{N+1}{2}\\
\mathrm{V}[X]= & \dfrac{(N+1)(N-1)}{12}
\end{cases}
\end{gathered}
$$

$$
\begin{aligned}
\mathrm{E}[X]=&\sum_{x\in X(\Omega)}x\,f_X(x|N)=\sum_{x=1}^{N}x\frac{1}{N}\\
=&\frac{1}{N}\sum_{x=1}^{N}x=\frac{1}{N}\frac{N(N+1)}{2}=\frac{N+1}{2}\\
\mathrm{E}[X]=&\frac{N+1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}[X] &= \sum_{x \in X(\Omega)} (x - \mathrm{E}[X])^2 f_X(x|N) = \sum_{x=1}^{N} \left(x - \frac{N+1}{2}\right)^2 \frac{1}{N} \\
&= \frac{1}{N} \sum_{x=1}^{N} \left[x^2 - (N+1)x + \left(\frac{N+1}{2}\right)^2\right] \\
&= \frac{1}{N} \left[\sum_{x=1}^{N} x^2 - (N+1)\sum_{x=1}^{N} x + N\left(\frac{N+1}{2}\right)^2\right] \\
&= \frac{1}{N} \left[\frac{N(N+1)(2N+1)}{6} - (N+1)\frac{N(N+1)}{2} + N\left(\frac{N+1}{2}\right)^2\right] \\
&= (N+1)\left[\frac{2N+1}{6} - \frac{N+1}{2} + \frac{N+1}{4}\right] = (N+1)\frac{4N+2-(3N+3)}{12} \\
&= \frac{(N+1)(N-1)}{12} \\
\mathrm{V}[X] &= \frac{(N+1)(N-1)}{12}
\end{aligned}
$$

---

$$
X \sim \mathcal{DU}(a,b) \Updownarrow \begin{cases} f_X(x|N) = \dfrac{1}{N} = \dfrac{1}{b-a+1} & N = b-a+1 \\ x \in X(\Omega) = \{a, a+1, \cdots, b\} \end{cases} \Updownarrow F_X(x|a,b) = \frac{\lfloor x \rfloor - a + 1}{b-a+1}
$$

##### 14.1.2.8.1.2 hypergeometric distribution

$$
\begin{gathered}
X \sim \mathcal{HG}(N,M,K) \\
\Updownarrow \\
\begin{cases} f_X(x|N,M,K) = \dfrac{\dbinom{N-M}{K-x}\dbinom{M}{x}}{\dbinom{N}{K}} \\ x \in X(\Omega) = \{\max\{0, K-(N-M)\}, \cdots, \min\{K,M\}\} \end{cases} \\
\Downarrow K \ll N, M \\
x \in X(\Omega) = \{0, 1, \cdots, K\} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = & \dfrac{KM}{N} \\ \mathrm{V}[X] = & \dfrac{KM}{N}\dfrac{(N-M)(N-K)}{N(N-1)} \end{cases}
\end{gathered}
$$

##### 14.1.2.8.1.3 Bernoulli distribution

$$
\begin{gathered}
X \sim \mathcal{B}(p), p = \mathrm{P}(X=1) \\
\Updownarrow \\
\begin{cases} f_X(x|p) = (1-p)^{1-x} p^x \\ x \in X(\Omega) = \{0,1\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = & p \\ \mathrm{V}[X] = & p(1-p) \\ M_X(\xi) = & (1-p) + p\mathrm{e}^{\xi} \end{cases}
\end{gathered}
$$

##### 14.1.2.8.1.4 binomial distribution

independent and identical Bernoulli trials

$$
\begin{gathered}
X \sim \mathrm{b}(n,p), p = \mathrm{P}(X=1) \\
\Updownarrow \\
\begin{cases} f_X(x|n,p) = \binom{n}{x}(1-p)^{n-x}p^x \\ x \in X(\Omega) = \{0,1,\cdots,n\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = & np \\ \mathrm{V}[X] = & np(1-p) \\ M_X(\xi) = & \left[(1-p)+p\mathrm{e}^{\xi}\right]^n \end{cases}
\end{gathered}
$$

reparameterization technique

$$
\begin{aligned}
M_X(\xi) =& \mathrm{E}\left[\mathrm{e}^{\xi X}\right] = \sum_{x\in X(\Omega)} \mathrm{e}^{\xi x} f_X(x|n,p) \\
=& \sum_{x=1}^{n} \mathrm{e}^{\xi x}\binom{n}{x}(1-p)^{n-x}p^x = \sum_{x=1}^{n}\binom{n}{x}(1-p)^{n-x}\left(p\mathrm{e}^{\xi}\right)^x \\
=& \sum_{x=1}^{n}\binom{n}{x}\left[\frac{1-p}{(1-p)+p\mathrm{e}^{\xi}}\right]^{n-x}\left[\frac{p\mathrm{e}^{\xi}}{(1-p)+p\mathrm{e}^{\xi}}\right]^{x}\left[(1-p)+p\mathrm{e}^{\xi}\right]^{n-x}\left[(1-p)+p\mathrm{e}^{\xi}\right]^{x} \\
=& \sum_{x=1}^{n}\binom{n}{x}\left[\frac{1-p}{(1-p)+p\mathrm{e}^{\xi}}\right]^{n-x}\left[\frac{p\mathrm{e}^{\xi}}{(1-p)+p\mathrm{e}^{\xi}}\right]^{x}\left[(1-p)+p\mathrm{e}^{\xi}\right]^{n} \\
=& \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n}\sum_{x=1}^{n}\binom{n}{x}\left[\frac{1-p}{(1-p)+p\mathrm{e}^{\xi}}\right]^{n-x}\left[\frac{p\mathrm{e}^{\xi}}{(1-p)+p\mathrm{e}^{\xi}}\right]^{x} \\
=& \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n}\sum_{x=1}^{n}\binom{n}{x}\left[p^*\right]^{n-x}\left[1-p^*\right]^{x}, p^* = \frac{1-p}{(1-p)+p\mathrm{e}^{\xi}} \\
=& \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n}\sum_{x=1}^{n} f_X(x|n,p^*), X\sim\mathrm{b}(n,p^*) \\
=& \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n}\cdot 1 = \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n} \\
M_X(\xi) =& \left[(1-p)+p\mathrm{e}^{\xi}\right]^{n}
\end{aligned}
$$

**14.1.2.8.1.5 Poisson distribution** count = number of events

an unbounded discrete distribution we first see or met

$$
\begin{gathered}
X \sim \mathcal{P}(\lambda) \\
\Updownarrow \\
\begin{cases} f_X(x|\lambda) = \dfrac{\mathrm{e}^{-\lambda}\lambda^x}{x!} \\ x \in X(\Omega) = \{0,1,\cdots\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = & \lambda \\ \mathrm{V}[X] = & \lambda = \mathrm{E}[X] \\ M_X(\xi) = & \exp\left[\lambda\left(\mathrm{e}^{\xi}-1\right)\right] = \mathrm{e}^{\lambda\left(\mathrm{e}^{\xi}-1\right)} \end{cases}
\end{gathered}
$$

the Poisson postulates

$$\begin{cases} N_t & \text{a r.v. denoting the of events in } [0,t] \\ N_0 = N_{t=0} = 0 & \text{reset the count at the initial point} \\ \forall s < t \left[ N_s \perp N_t - N_s \right] & \text{disjoint intervals independent} \\ N_s = N_{t+s} - N_t & \text{depends on length instead of initial point} \\ \lim\limits_{t \to 0} \dfrac{\mathrm{P}\left(N_t = 1\right)}{t} = \lambda & \Rightarrow \forall t \approx 0 \left[ \mathrm{P}\left(N_t = 1\right) \approx \lambda t \right] \\ \lim\limits_{t \to 0} \dfrac{\mathrm{P}\left(N_t > 1\right)}{t} = 0 & \text{no coincidence for small } t \end{cases}$$
$$\text{solve the differential equations} \Downarrow \text{with probability axioms}$$
$$f_X\left(x|\lambda t\right) = \mathrm{P}\left(N_t = x\right) = \frac{\mathrm{e}^{-\lambda t}\left(\lambda t\right)^x}{x!}$$

**14.1.2.8.1.6 negative binomial distribution** also an unbounded discrete distribution

Count the number of independent and identical Bernoulli trials until $r$ a fixed number of success.

$$\begin{gathered} X \sim \mathcal{NB}\left(r,p\right) \\ \Updownarrow \\ \begin{cases} f_X\left(x|r,p\right) = \dbinom{x-1}{r-1}\left(1-p\right)^{x-r} p^r \\ x \in X\left(\Omega\right) = \left\{r, r+1, \cdots\right\} \end{cases} \\ \Downarrow \\ \begin{cases} \mathrm{E}\left[X\right] = \\ \mathrm{V}\left[X\right] = \\ M_X\left(\xi\right) = \end{cases} \end{gathered}$$

$$\begin{gathered} Y = X - r, X \sim \mathcal{NB}\left(r,p\right) \Rightarrow X = Y + r \\ \Updownarrow \\ \begin{cases} f_Y\left(y|r,p\right) = \dbinom{r+y-1}{r-1}\left(1-p\right)^y p^r \\ y \in Y\left(\Omega\right) = \left\{0, 1, \cdots\right\} \end{cases} \\ \Downarrow \\ \begin{cases} \mathrm{E}\left[Y\right] = & r\dfrac{1-p}{p} = r\left(\dfrac{1}{p} - 1\right) \\ \mathrm{V}\left[Y\right] = & r\dfrac{1-p}{p^2} = r\left(\dfrac{1}{p^2} - \dfrac{1}{p}\right) \\ M_Y\left(\xi\right) = & \left[\dfrac{p}{1-\left(1-p\right)\mathrm{e}^{\xi}}\right]^r \end{cases} \end{gathered}$$

Note:

$$\begin{aligned} Y = X - r \Rightarrow f_Y\left(y|r,p\right) = & \mathrm{P}\left(Y = y\right) \\ = & \mathrm{P}\left(X - r = y\right) \\ = & \mathrm{P}\left(X = r + y\right) \end{aligned}$$

reparameterization technique

$$
\begin{aligned}
M_Y(\xi) =& \mathrm{E}\left[\mathrm{e}^{\xi Y}\right] = \sum_{y\in Y(\Omega)} \mathrm{e}^{\xi y} f_Y(y|r,p) \\
=& \sum_{y=0}^{\infty} \mathrm{e}^{\xi y} \binom{r+y-1}{r-1} (1-p)^y p^r \\
=& p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} \left[(1-p)\,\mathrm{e}^{\xi}\right]^y \\
=& p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} \left[1-p^*\right]^y \left[p^*\right]^r \frac{1}{\left[p^*\right]^r}, 1-p^* = (1-p)\,\mathrm{e}^{\xi} \\
=& \left[\frac{p}{p^*}\right]^r \sum_{y=0}^{\infty} f_Y(y|r,p^*), Y \sim \mathcal{NB}(r,p^*), p^* = 1-(1-p)\,\mathrm{e}^{\xi} \\
=& \left[\frac{p}{p^*}\right]^r \cdot 1 = \left[\frac{p}{p^*}\right]^r = \left[\frac{p}{1-(1-p)\,\mathrm{e}^{\xi}}\right]^r
\end{aligned}
$$

Note:

For sufficiently small $t$ such that

$$0 \le p^* = (1-p)\,\mathrm{e}^{\xi} \le 1$$

or else $t \gg 1$

$$p^* = (1-p)\,\mathrm{e}^{\xi} > 1$$

#### 14.1.2.8.1.7 geometric distribution

$$
\begin{gathered}
X \sim \mathcal{NB}(r,p) \\
\Updownarrow \\
\begin{cases} f_X(x|r,p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \\ x \in X(\Omega) = \{r, r+1, \cdots\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = \\ \mathrm{V}[X] = \\ M_X(\xi) = \end{cases}
\end{gathered}
$$

$r = 1$,

$$
\begin{gathered}
X \sim \mathcal{NB}(r=1,p) = \mathcal{G}(p) \\
\Updownarrow \\
\begin{cases} f_X(x|r=1,p) = \left[\binom{x-1}{r-1} (1-p)^{x-r} p^r\right]_{r=1} = \binom{x-1}{1-1} (1-p)^{x-1} p^1 = (1-p)^{x-1} p \\ x \in X(\Omega) = \{r, r+1, \cdots\}_{r=1} = \{1, 2, \cdots\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathrm{E}[X] = \\ \mathrm{V}[X] = \\ M_X(\xi) = \end{cases}
\end{gathered}
$$

the only “memoryless” discrete distribution, and there is also a “memoryless” continuous distribution.

memoryless property

$$\forall s > t \left[\mathrm{P}(X > s|X > t) = \mathrm{P}(X > s-t)\right]$$

Survival depends on length instead of initial point; it might be proper assumption for stuff survival, but might not be proper for human suvival.

#### 14.1.2.8.2 continuous distribution

##### 14.1.2.8.2.1 uniform distribution = continuous uniform distribution

$$X \sim \mathcal{U}(a,b)$$
$$\Updownarrow$$
$$\begin{cases} f_X(x|a,b) = \dfrac{1(a \le x \le b)}{b-a} = \dfrac{1(x \in [a,b])}{b-a} \\ x \in X(\Omega) = [a,b] \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}[X] = & \dfrac{a+b}{2} \\ \mathrm{V}[X] = & \dfrac{(b-a)^2}{12} \\ M_X(\xi) = & \dfrac{\mathrm{e}^{b\xi} - \mathrm{e}^{a\xi}}{(b-a)\xi} \end{cases}$$

$a = 0, b = 1$

$$X \sim \mathcal{U}(a=0,b=1) = \mathcal{U}(0,1)$$
$$\Updownarrow$$
$$\begin{cases} f_X(x|a=0,b=1) = \left[\dfrac{1(a \le x \le b)}{b-a}\right]_{a=0,b=1} = 1(0 \le x \le 1) = 1(x \in [0,1]) \\ x \in X(\Omega) = [a,b]\,|_{a=0,b=1} = [0,1] \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}[X] = & \dfrac{1}{2} \\ \mathrm{V}[X] = & \dfrac{1}{12} \\ M_X(\xi) = & \dfrac{\mathrm{e}^{\xi} - 1}{\xi} \end{cases}$$

##### 14.1.2.8.2.2 gamma distribution

##### 14.1.2.8.2.3 exponential distribution

##### 14.1.2.8.2.4 Chi-square distribution

##### 14.1.2.8.2.5 Weibull distribution https://www.youtube.com/watch?v=ojZb6nZWdvI

##### 14.1.2.8.2.6 normal distribution

$$X \sim \mathcal{N}\left(\mu, \sigma^2\right)$$
$$\Updownarrow$$
$$\begin{cases} f_X\left(x|\mu,\sigma^2\right) = \dfrac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \dfrac{\mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = \dfrac{\mathrm{e}^{-(x-\mu)^2/\left(2\sigma^2\right)}}{\sigma\sqrt{2\pi}} = \dfrac{\exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}[X] = & \mu \\ \mathrm{V}[X] = & \sigma^2 \\ M_X(\xi) = & \mathrm{e}^{\left(\mu\xi + \frac{\sigma^2}{2}\xi^2\right)} = \mathrm{e}^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases}$$

##### 14.1.2.8.2.7 beta distribution random success probability

##### 14.1.2.8.2.8 Cauchy distribution https://www.youtube.com/watch?v=ojZb6nZWdvI&t=33m34s

https://en.wikipedia.org/wiki/Cauchy_distribution

$$X \sim \mathcal{C}\left(\theta, \sigma\right)$$
$$\Updownarrow$$
$$\begin{cases} f_X\left(x|\theta,\sigma\right) = \dfrac{1}{\pi\sigma}\left[1+\left(\dfrac{x-\theta}{\sigma}\right)^2\right]^{-1} = \dfrac{1}{\pi\sigma\left[1+\left(\dfrac{x-\theta}{\sigma}\right)^2\right]} = \dfrac{1}{\pi}\left[\dfrac{\sigma}{\left(x-\theta\right)^2+\sigma^2}\right] \\ x \in X\left(\Omega\right) = \mathbb{R} \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}\left[X\right] & \text{diverges} \\ \mathrm{V}\left[X\right] & \text{diverges} \\ M_X\left(\xi\right) & \text{diverges} \end{cases}$$

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Figure 14.1: not completed: Cauchy distribution vs. normal distribution

heavy tail

$t$ distribution is also heavy tail

box plot

##### 14.1.2.8.2.9 log-normal distribution

#### 14.1.2.9 exponential family

**Definition 14.18.** exponential family: A family of PDF/PMF is called exponential family if

$$f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum\limits_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)} = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)$$

with $\boldsymbol{\theta} = \boldsymbol{\theta}\left(\theta_1,\cdots,\theta_k\right) = \left(\theta_1,\cdots,\theta_k\right)$ for some $h\left(x\right), c\left(\boldsymbol{\theta}\right), w_j\left(\boldsymbol{\theta}\right), t_j\left(x\right)$, where

$$h\left(x\right)c\left(\boldsymbol{\theta}\right) \geq 0 \Rightarrow f\left(x|\theta\right) \geq 0$$

and parameters $\boldsymbol{\theta}$ and statistic or real number $x$ can be separated.

$$\mathcal{E}^f = \left\{ f \,\middle|\, f = f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum\limits_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)} = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right) \right\}$$

- exponential family
  - normal distribution[14.1.2.8.2.6]
  - gamma distribution[14.1.2.8.2.2]
  - log-normal distribution[14.1.2.8.2.9]
- non-exponential family
  - Cauchy distribution[14.1.2.8.2.8] https://www.youtube.com/watch?v=ojZb6nZWdvI&t=1h7m43s

##### 14.1.2.9.1 binomial distribution with known $n$

$$f\left(x|p\right) = \binom{n}{x}\left(1-p\right)^{n-x}p^x$$

or

$$f\left(x|p\right)=\binom{n}{x}\left(1-p\right)^{n-x}p^{x}=f\left(x|n=n,p\right)$$

not

$$f\left(x|n,p\right)=\binom{n}{x}\left(1-p\right)^{n-x}p^{x}$$

$$\begin{aligned}
f\left(x|p\right)=&\binom{n}{x}\left(1-p\right)^{n-x}p^{x}\\
=&\binom{n}{x}\left(1-p\right)^{n}\left(\frac{p}{1-p}\right)^{x}\\
=&\binom{n}{x}\left(1-p\right)^{n}\mathrm{e}^{\left(\ln\frac{p}{1-p}\right)x}\\
=&h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum_{j=1}^{k}w_{j}\left(\boldsymbol{\theta}\right)t_{j}\left(x\right)},\begin{cases}
h\left(x\right)= & \binom{n}{x}\\
c\left(\boldsymbol{\theta}\right)=c\left(\theta_{1}\right)=c\left(p\right)= & \left(1-p\right)^{n}\\
w_{1}\left(\boldsymbol{\theta}\right)=w\left(\theta_{1}\right)=w\left(p\right)= & \ln\frac{p}{1-p}\\
t_{1}\left(x\right)= & x\\
k= & 1
\end{cases}
\end{aligned}$$

$$\Downarrow$$

$$f\left(x|p\right)=\binom{n}{x}\left(1-p\right)^{n-x}p^{x}\in\mathcal{E}^{f}=\left\{f\middle|f=f\left(x|\boldsymbol{\theta}\right)=h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k}w_{j}\left(\boldsymbol{\theta}\right)t_{j}\left(x\right)\right)\right\}$$

why known $n$?

known $n$

$$\binom{n}{x}=h\left(x\right)$$

unknown $n$

$$\binom{n}{x}=h\left(x,n\right)\neq h_{1}\left(n\right)h_{2}\left(x\right)$$

#### 14.1.2.9.2 continuous uniform distribution not in exponential family

$$X\sim\mathcal{U}\left(a,b\right)$$

$$\begin{aligned}
f_{X}\left(x|a,b\right)=&\frac{1\left(x\in\left[a,b\right]\right)}{b-a}\\
=&\frac{1}{b-a},x\in\left[a,b\right]
\end{aligned}$$

$$\frac{1}{b-a}=h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum_{j=1}^{k}w_{j}\left(\boldsymbol{\theta}\right)t_{j}\left(x\right)},\begin{cases}
h\left(x\right)= & 1\\
c\left(\boldsymbol{\theta}\right)=c\left(\theta_{1},\theta_{2}\right)=c\left(a,b\right)= & \dfrac{1}{b-a}\\
w_{1}\left(\theta_{1}\right)=w_{2}\left(\theta_{2}\right)= & 0\\
t_{1}\left(x\right)=t_{2}\left(x\right)= & x\\
k= & 2
\end{cases}$$

however,

$$1\left(x\in\left[a,b\right]\right)\neq h\left(x\right)c\left(\boldsymbol{\theta}\right)=h\left(x\right)c\left(a,b\right)$$

thus

$$f_X(x|a,b) = \frac{1(x \in [a,b])}{b-a} \notin \mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x)\,c(\boldsymbol{\theta}) \exp\left( \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x) \right) \right\}$$

#### 14.1.2.9.3 normal distribution is in exponential family

$$X \sim \mathcal{N}\left(\mu, \sigma^2\right)$$
$$\Updownarrow$$
$$\begin{cases} f_X\left(x|\mu,\sigma^2\right) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \dfrac{e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = \dfrac{e^{-(x-\mu)^2/\left(2\sigma^2\right)}}{\sigma\sqrt{2\pi}} = \dfrac{\exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}[X] = & \mu \\ \mathrm{V}[X] = & \sigma^2 \\ M_X(\xi) = & e^{\left(\mu\xi + \frac{\sigma^2}{2}\xi^2\right)} = e^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases}$$

---

$$\begin{aligned} f_X\left(x|\mu,\sigma^2\right) = f_X(x|\mu,\sigma) =& \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\ =& \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}\left(x^2 - 2x\mu + \mu^2\right)} = \frac{e^{\frac{-1}{2}\left(\frac{\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} = \frac{e^{\frac{-1}{2}\left(\frac{\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} e^{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2} \\ =& h(x)\,c(\boldsymbol{\theta})\, e^{\sum_{j=1}^{k} w_j(\boldsymbol{\theta}) t_j(x)}, \begin{cases} h(x) = & 1 \\ c(\boldsymbol{\theta}) = c(\theta_1,\theta_2) = c(\mu,\sigma) = & \dfrac{e^{\frac{-1}{2}\left(\frac{\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} \\ w_1(\theta_1,\theta_2) = w_1(\mu,\sigma) = & \frac{\mu}{\sigma^2} \\ w_2(\theta_2) = w_2(\sigma) = & \frac{-1}{2\sigma^2} \\ t_1(x) = & x \\ t_2(x) = & x^2 \\ k = & 2 \end{cases} \end{aligned}$$
$$\Downarrow$$
$$f_X(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \in \mathcal{E}^f = \left\{ f \middle| f = f(x|\boldsymbol{\theta}) = h(x)\,c(\boldsymbol{\theta}) \exp\left( \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x) \right) \right\}$$

#### 14.1.2.9.4 normal distribution with unknown equal mean and standard deviation in curved exponential family

$$X \sim \mathcal{N}\left(\mu, \sigma^2 = \mu^2\right) = \mathcal{N}\left(\mu, \mu^2\right)$$
$$\Updownarrow$$
$$\begin{cases} f_X\left(x|\mu,\mu^2\right) = f_X(x|\mu) = \left[\dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right]_{\sigma=\mu} = \dfrac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\mu}\right)^2} \\ x \in X(\Omega) = \mathbb{R} \end{cases}$$
$$\Downarrow$$
$$\begin{cases} \mathrm{E}[X] = & \mu \\ \mathrm{V}[X] = & \mu^2 \\ M_X(\xi) = & e^{\left(\mu\xi + \frac{\mu^2}{2}\xi^2\right)} = e^{\mu\xi + \frac{\mu^2}{2}\xi^2} \end{cases}$$

---

$$
\begin{aligned}
f_X\left(x|\mu,\mu^2\right) = f_X\left(x|\mu\right) =& \frac{1}{\mu\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\mu}\right)^2} \\
=& \frac{1}{\mu\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2\mu^2}\left(x^2-2x\mu+\mu^2\right)} = \frac{\mathrm{e}^{\frac{-1}{2}}}{\mu\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2\mu^2}x^2+\frac{\mu}{\mu^2}x} = \frac{\mathrm{e}^{\frac{-1}{2}}}{\mu\sqrt{2\pi}}\mathrm{e}^{\frac{1}{\mu}x-\frac{1}{2\mu^2}x^2} \\
=& h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)}, \begin{cases}
h\left(x\right) = & 1 \\
c\left(\boldsymbol{\theta}\right) = c\left(\theta_1\right) = c\left(\mu\right) = & \frac{\mathrm{e}^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} \\
w_1\left(\theta_1\right) = w_1\left(\mu\right) = & \frac{1}{\mu} \\
w_2\left(\theta_1\right) = w_2\left(\mu\right) = & \frac{-1}{2\mu^2} \\
t_1\left(x\right) = & x \\
t_2\left(x\right) = & x^2 \\
k = & 2 > 1 = p
\end{cases} \\
& \Downarrow \begin{cases} p = \dim\boldsymbol{\theta} \\ k = \dim\boldsymbol{w} \end{cases} \\
f_X\left(x|\mu,\sigma\right) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \in& \mathcal{C}^f = \left\{f \middle| f = f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k>p} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)\right\} \\
\subset& \mathcal{E}^f = \left\{f \middle| f = f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)\right\}
\end{aligned}
$$

---

curved exponential family

$$
\mathcal{C}^f = \left\{f \middle| f = f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k>p} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)\right\}
$$

---

exponential family

$$
\mathcal{E}^f = \left\{f \middle| f = f\left(x|\boldsymbol{\theta}\right) = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k} w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)\right\}
$$

---

https://tex.stackexchange.com/questions/145969/filling-specified-area-by-random-dots-in-tikz

202402170031-statistics_files/figure-latex/unnamed-chunk-33-1.pd

Figure 14.2: curved exponential family vs. exponential family

#### 14.1.2.9.5 properties of exponential family

##### 14.1.2.9.5.1 fundamentals of statistical inference

**Lemma 14.1.** *Leibnitz rule*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a(\theta)}^{b(\theta)} g(x,\theta)\,\mathrm{d}x =& \frac{\mathrm{d}\int_{a(\theta)}^{b(\theta)} g(x,\theta)\,\mathrm{d}x}{\mathrm{d}\theta} \\
=& g(x,b(\theta))\frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta} - g(x,a(\theta))\frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta} g(x,\theta)\,\mathrm{d}x \\
=& g(x,b(\theta))\frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta} - g(x,a(\theta))\frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x
\end{aligned}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a(\theta)}^{b(\theta)} g(x,\theta)\,\mathrm{d}x = g(x,b(\theta))\frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta} - g(x,a(\theta))\frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x
$$

---

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a(\theta)=a}^{b(\theta)=b} g(x,\theta)\,\mathrm{d}x =& g(x,b(\theta))\frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta} - g(x,a(\theta))\frac{\mathrm{d}a(\theta)}{\mathrm{d}\theta} + \int_{a(\theta)}^{b(\theta)} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x \\
=& g(x,b)\frac{\mathrm{d}b}{\mathrm{d}\theta} - g(x,a)\frac{\mathrm{d}a}{\mathrm{d}\theta} + \int_a^b \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x \\
=& g(x,b)\,0 - g(x,a)\,0 + \int_a^b \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x = 0 + 0 + \int_a^b \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x \\
=& \int_a^b \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{X(\Omega)\perp\theta} g(x,\theta)\,\mathrm{d}x =& \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{a(\theta)=a}^{b(\theta)=b} g(x,\theta)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x = \int_{X(\Omega)\perp\theta} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{X(\Omega)\perp\theta} g(x,\theta)\,\mathrm{d}x =& \int_{X(\Omega)\perp\theta} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x
\end{aligned}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{X(\Omega)\perp\theta} g(x,\theta)\,\mathrm{d}x = \int_{X(\Omega)\perp\theta} \frac{\mathrm{d}g(x,\theta)}{\mathrm{d}\theta}\mathrm{d}x
$$

---

point estimation

**Lemma 14.2.** *parameter-independent expectation:*

*Assume the domain of $X$ independent of $\theta$, then*

$$
\mathrm{E}\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right] = 0, \forall i = 1, \cdots, p
$$

Proof:

$$
\begin{aligned}
\mathrm{E}\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right] =& \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial\theta_i} f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
=& \int \left[\frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial\theta_i}\right] f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
=& \int \left[\frac{1}{f(x|\boldsymbol{\theta})}\frac{\partial f(x|\boldsymbol{\theta})}{\partial\theta_i}\right] f(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \frac{1}{f(x|\boldsymbol{\theta})}\frac{\partial f(x|\boldsymbol{\theta})}{\partial\theta_i} f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
=& \int \frac{f(x|\boldsymbol{\theta})}{f(x|\boldsymbol{\theta})}\frac{\partial f(x|\boldsymbol{\theta})}{\partial\theta_i}\mathrm{d}x = \int 1\frac{\partial f(x|\boldsymbol{\theta})}{\partial\theta_i}\mathrm{d}x = \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial\theta_i}\mathrm{d}x \\
\overset{\begin{cases}\text{Leibnitz rule}\\ x\perp\boldsymbol{\theta}\end{cases}}{=}& \frac{\partial}{\partial\theta_i}\int f(x|\boldsymbol{\theta})\,\mathrm{d}x
\end{aligned}
$$

□

Note:

$$
\begin{aligned}
\mathrm{E}\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] &= \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right) f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
X \sim f_X(x|\boldsymbol{\theta}) \Rightarrow \mathrm{E}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta})}{\partial \theta_i}\right] &= \int \frac{\partial \ln f_X(x|\boldsymbol{\theta})}{\partial \theta_i} f_X(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \left(\frac{\frac{\partial f_X(x|\boldsymbol{\theta})}{\partial \theta_i}}{f_X(x|\boldsymbol{\theta})}\right) f_X(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}\mathrm{d}x \\
X \sim f_X(x|\boldsymbol{\theta}) \Rightarrow \mathrm{E}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] &= \int \frac{\partial \ln f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_X(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \left(\frac{\frac{\partial f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i}}{f_X(x|\boldsymbol{\theta}^*)}\right) f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= \int \frac{\partial f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i}\left(\frac{f_X(x|\boldsymbol{\theta})}{f_X(x|\boldsymbol{\theta}^*)}\right)\mathrm{d}x
\end{aligned}
$$

---

$$
\mathrm{E}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] = \mathrm{E}_{\boldsymbol{\theta}^*}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] = \int \frac{\partial \ln f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_X(x|\boldsymbol{\theta}^*)\,\mathrm{d}x
$$

$$
\mathrm{E}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] = \mathrm{E}_{\boldsymbol{\theta}}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] = \int \frac{\partial \ln f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_X(x|\boldsymbol{\theta})\,\mathrm{d}x
$$

$$
\mathrm{E}_{\boldsymbol{\theta}}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] = \int \frac{\partial \ln f_X(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_X(x|\boldsymbol{\theta})\,\mathrm{d}x
$$

$$
\mathrm{E}_{\boldsymbol{\theta}}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] \not\equiv 0
$$

$$
\mathrm{E}_{\boldsymbol{\theta}}\left[\frac{\partial \ln f_X(X|\boldsymbol{\theta}^*)}{\partial \theta_i}\right] \overset{\boldsymbol{\theta}^*=\boldsymbol{\theta}}{\equiv} 0
$$

as the fundamental to estimate parameters.

interval estimation

**Lemma 14.3.** *parameter-independent variance:*

*Assume the domain of $X$ independent of $\theta$, then*

$$
\mathrm{E}\left[\frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2}\right] = -\mathrm{E}\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right)^2\right], \forall i = 1, \cdots, p
$$

Proof:

$$
\begin{aligned}
&\mathrm{E}\left[\frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2}\right] = \int \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \frac{\partial}{\partial \theta_i}\frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= \int \left\{\frac{\partial}{\partial \theta_i}\left(\frac{1}{f(x|\boldsymbol{\theta})}\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}\right)\right\} f(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \left\{\frac{\partial}{\partial \theta_i}\left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)\right\} f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta}) - \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{[f(x|\boldsymbol{\theta})]^2} f(x|\boldsymbol{\theta})\,\mathrm{d}x = \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2}[f(x|\boldsymbol{\theta})]^2 - \left[\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}\right]^2 [f(x|\boldsymbol{\theta})]}{[f(x|\boldsymbol{\theta})]^2}\mathrm{d}x \\
&= \int \frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2}\mathrm{d}x - \int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x \overset{\begin{cases}\text{Leibnitz rule}\\ X(\Omega) \perp \boldsymbol{\theta}\end{cases}}{=} \frac{\partial^2}{\partial \theta_i^2}\int f(x|\boldsymbol{\theta})\,\mathrm{d}x - \int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= \frac{\partial^2}{\partial \theta_i^2}1 - \int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x = 0 - \int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x = -\int \left(\frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x \\
&= -\int \left(\frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i}\right)^2 f(x|\boldsymbol{\theta})\,\mathrm{d}x = -\mathrm{E}\left[\left(\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right)^2\right]
\end{aligned}
$$

□

---

$$\begin{cases} \mathrm{E}\left[\dfrac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = & 0 \\ \mathrm{E}\left[\dfrac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2}\right] = & -\mathrm{E}\left[\left(\dfrac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right)^2\right] \end{cases}, \forall i = 1, \cdots, p$$

exponential family expectation and variance

**Theorem 14.15.** *exponential family expectation*

$$X \in \mathcal{E}^f = \left\{ f \middle| f = h(x)\, c(\boldsymbol{\theta}) \exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x)\right)\right\}$$
$$\Downarrow \forall i = 1, \cdots, p$$
$$\mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] = \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}$$

Proof:

$$\begin{aligned}
f(x|\boldsymbol{\theta}) =& h(x)\, c(\boldsymbol{\theta})\, \mathrm{e}^{\sum_{j=1}^{k} w_j(\boldsymbol{\theta}) t_j(x)} = h(x)\, c(\boldsymbol{\theta}) \exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x)\right) \\
\ln f(x|\boldsymbol{\theta}) =& \ln h(x)\, c(\boldsymbol{\theta})\, \mathrm{e}^{\sum_{j=1}^{k} w_j(\boldsymbol{\theta}) t_j(x)} \\
=& \ln h(x) + \ln c(\boldsymbol{\theta}) + \ln \mathrm{e}^{\sum_{j=1}^{k} w_j(\boldsymbol{\theta}) t_j(x)} \\
=& \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x) \\
\ln f(x|\boldsymbol{\theta}) =& \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x) \\
\ln f(X|\boldsymbol{\theta}) =& \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(X) \\
\frac{\partial}{\partial \theta_i} \ln f(X|\boldsymbol{\theta}) =& \frac{\partial}{\partial \theta_i}\left[\ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(X)\right] \\
=& \frac{\partial}{\partial \theta_i} \ln h(X) + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \frac{\partial}{\partial \theta_i} \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(X) \\
=& 0 + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \sum_{j=1}^{k} \frac{\partial}{\partial \theta_i} \{w_j(\boldsymbol{\theta})\, t_j(X)\} \\
=& \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{\left(\frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta})\right) t_j(X) + w_j(\boldsymbol{\theta}) \left(\frac{\partial}{\partial \theta_i} t_j(X)\right)\right\} \\
=& \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{\left(\frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta})\right) t_j(X) + w_j(\boldsymbol{\theta})\, 0\right\} \\
=& \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{\left(\frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta})\right) t_j(X)\right\} \\
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} =& \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{\frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right\}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \\
0 \overset{\text{lemma: } \mathrm{E}\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right]=0}{=} \mathrm{E}\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] &= \mathrm{E}\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}\right] \\
&= \mathrm{E}\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}\right] + \mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&\overset{\mathrm{E}[c]=c}{=} \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \mathrm{E}\left[\frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} \mathrm{E}\left[t_j(X)\right] \\
0 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
\mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] &= \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}
\end{aligned}
$$

$\square$

**Theorem 14.16.** *exponential family variance*

$$
\begin{gathered}
X \in \mathcal{E}^f = \left\{ f \middle| f = h(x)\, c(\boldsymbol{\theta}) \exp\left( \sum_{j=1}^{k} w_j(\boldsymbol{\theta})\, t_j(x) \right) \right\} \\
\Downarrow \forall i = 1, \cdots, p \\
\mathrm{V}\left[\sum_{j=1}^{k} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] = \frac{-\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \mathrm{E}\left[\sum_{j=1}^{k} \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X)\right]
\end{gathered}
$$

Proof:

same as the above

$$
\begin{aligned}
f\left(x|\boldsymbol{\theta}\right) =& h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum\limits_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)} = h\left(x\right)c\left(\boldsymbol{\theta}\right)\exp\left(\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\right)\\
\ln f\left(x|\boldsymbol{\theta}\right) =& \ln h\left(x\right)c\left(\boldsymbol{\theta}\right)\mathrm{e}^{\sum\limits_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)}\\
=& \ln h\left(x\right)+\ln c\left(\boldsymbol{\theta}\right)+\ln\mathrm{e}^{\sum\limits_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)}\\
=& \ln h\left(x\right)+\ln c\left(\boldsymbol{\theta}\right)+\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\\
\ln f\left(x|\boldsymbol{\theta}\right) =& \ln h\left(x\right)+\ln c\left(\boldsymbol{\theta}\right)+\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(x\right)\\
\ln f\left(X|\boldsymbol{\theta}\right) =& \ln h\left(X\right)+\ln c\left(\boldsymbol{\theta}\right)+\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(X\right)\\
\frac{\partial}{\partial\theta_i}\ln f\left(X|\boldsymbol{\theta}\right) =& \frac{\partial}{\partial\theta_i}\left[\ln h\left(X\right)+\ln c\left(\boldsymbol{\theta}\right)+\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(X\right)\right]\\
=& \frac{\partial}{\partial\theta_i}\ln h\left(X\right)+\frac{\partial}{\partial\theta_i}\ln c\left(\boldsymbol{\theta}\right)+\frac{\partial}{\partial\theta_i}\sum_{j=1}^{k}w_j\left(\boldsymbol{\theta}\right)t_j\left(X\right)\\
=& 0+\frac{\partial}{\partial\theta_i}\ln c\left(\boldsymbol{\theta}\right)+\sum_{j=1}^{k}\frac{\partial}{\partial\theta_i}\left\{w_j\left(\boldsymbol{\theta}\right)t_j\left(X\right)\right\}\\
=& \frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\left\{\left(\frac{\partial}{\partial\theta_i}w_j\left(\boldsymbol{\theta}\right)\right)t_j\left(X\right)+w_j\left(\boldsymbol{\theta}\right)\left(\frac{\partial}{\partial\theta_i}t_j\left(X\right)\right)\right\}\\
=& \frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\left\{\left(\frac{\partial}{\partial\theta_i}w_j\left(\boldsymbol{\theta}\right)\right)t_j\left(X\right)+w_j\left(\boldsymbol{\theta}\right)0\right\}\\
=& \frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\left\{\left(\frac{\partial}{\partial\theta_i}w_j\left(\boldsymbol{\theta}\right)\right)t_j\left(X\right)\right\}\\
\frac{\partial\ln f\left(X|\boldsymbol{\theta}\right)}{\partial\theta_i} =& \frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\left\{\frac{\partial w_j\left(\boldsymbol{\theta}\right)}{\partial\theta_i}t_j\left(X\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\ln f\left(X|\boldsymbol{\theta}\right)}{\partial\theta_i} =& \frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\frac{\partial w_j\left(\boldsymbol{\theta}\right)}{\partial\theta_i}t_j\left(X\right)\\
\mathrm{V}\left[\frac{\partial\ln f\left(X|\boldsymbol{\theta}\right)}{\partial\theta_i}\right] =& \mathrm{V}\left[\frac{\partial\ln c\left(\boldsymbol{\theta}\right)}{\partial\theta_i}+\sum_{j=1}^{k}\frac{\partial w_j\left(\boldsymbol{\theta}\right)}{\partial\theta_i}t_j\left(X\right)\right]\\
\overset{\mathrm{V}[aX+b]=a^2\mathrm{V}[X]}{=}& \mathrm{V}\left[\sum_{j=1}^{k}\frac{\partial w_j\left(\boldsymbol{\theta}\right)}{\partial\theta_i}t_j\left(X\right)\right]\\
\mathrm{V}\left[\sum_{j=1}^{k}\frac{\partial w_j\left(\boldsymbol{\theta}\right)}{\partial\theta_i}t_j\left(X\right)\right] =& \mathrm{V}\left[\frac{\partial\ln f\left(X|\boldsymbol{\theta}\right)}{\partial\theta_i}\right]
\end{aligned}
$$

$$\begin{aligned}
\mathrm{V}\left[\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right] &= \mathrm{V}\left[\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right] = \mathrm{E}\left[\left(\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}-\mathrm{E}\left[\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right]\right)^2\right]\\
&\overset{\text{lemma: }\mathrm{E}\left[\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right]=0}{=} \mathrm{E}\left[\left(\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}-0\right)^2\right] = \mathrm{E}\left[\left(\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right)^2\right]\\
&\overset{\text{lemma: }\mathrm{E}\left[\frac{\partial^2\ln f(X|\boldsymbol{\theta})}{\partial\theta_i^2}\right]=-\mathrm{E}\left[\left(\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right)^2\right]}{=} -\mathrm{E}\left[\frac{\partial^2\ln f(X|\boldsymbol{\theta})}{\partial\theta_i^2}\right] = -\mathrm{E}\left[\frac{\partial}{\partial\theta_i}\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}\right]\\
&\overset{\frac{\partial\ln f(X|\boldsymbol{\theta})}{\partial\theta_i}=\frac{\partial\ln c(\boldsymbol{\theta})}{\partial\theta_i}+\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)}{=} -\mathrm{E}\left[\frac{\partial}{\partial\theta_i}\left\{\frac{\partial\ln c(\boldsymbol{\theta})}{\partial\theta_i}+\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right\}\right]\\
&= -\mathrm{E}\left[\frac{\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}+\frac{\partial}{\partial\theta_i}\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right]\\
&= \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\frac{\partial}{\partial\theta_i}\left\{\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right\}\right]\\
&= \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\left\{\left(\frac{\partial}{\partial\theta_i}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}\right)t_j(X)+\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial}{\partial\theta_i}t_j(X)\right\}\right]\\
&= \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\left\{\frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial\theta_i^2}t_j(X)+\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}0\right\}\right]\\
&= \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial\theta_i^2}t_j(X)\right]\\
\mathrm{V}\left[\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right] &= \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial\theta_i^2}t_j(X)\right]
\end{aligned}$$

□

---

exponential family expectation and variance

sense of downgrading by differentiation instead of integration

$$X\in\mathcal{E}^f=\left\{f\,\middle|\,f=h(x)\,c(\boldsymbol{\theta})\exp\left(\sum_{j=1}^{k}w_j(\boldsymbol{\theta})\,t_j(x)\right)\right\}$$
$$\Rightarrow\begin{cases}\mathrm{E}\left[\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right] = \frac{-\partial\ln c(\boldsymbol{\theta})}{\partial\theta_i}\\ \mathrm{V}\left[\sum_{j=1}^{k}\frac{\partial w_j(\boldsymbol{\theta})}{\partial\theta_i}t_j(X)\right] = \frac{-\partial^2\ln c(\boldsymbol{\theta})}{\partial\theta_i^2}-\mathrm{E}\left[\sum_{j=1}^{k}\frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial\theta_i^2}t_j(X)\right]\end{cases},\forall i=1,\cdots,p$$

### 14.1.3 multivariable distribution

https://www.youtube.com/watch?v=y-Oi5voWQKo

univariable random vector

discrete: PMF equals probability function

$$f_X(x)=\mathrm{P}(X=x)$$

continuous: PDF equals probability intensity

$$f_X(x)\,\mathrm{d}x = \mathrm{dP}(X \le x)$$

Given $(X,Y) = (X_1, X_2) = \langle X, Y\rangle = \langle X_1, X_2\rangle \sim f_{XY} = f_{XY}(x,y) = f_{X_1X_2} = f_{X_1X_2}(x_1,x_2)$

discrete:

**Definition 14.19.** JPMF = joint probability mass function: the JPMF of $(X_1, X_2) = \langle X_1, X_2\rangle$ is

$$f_{X_1X_2} = f_{X_1X_2}(x_1,x_2) = \mathrm{P}(X_1 = x_1, X_2 = x_2)$$

joint vs. marginal

**Theorem 14.17.** *joint probability mass function can inference marginal probability mass function: The marginal PMF of $X_1$, $f_{X_1}(x_1) = \mathrm{P}(X_1 = x_1)$ is given by*

$$f_{X_1}(x_1) = \sum_{x_2\in X_2(\Omega)} f_{X_1X_2}(x_1,x_2) = \sum_{x_2\in X_2(\Omega)} \mathrm{P}(X_1 = x_1, X_2 = x_2)$$

Proof:

$$\begin{aligned}
f_{X_1}(x_1) =& \mathrm{P}(X_1 = x_1)\\
=& \mathrm{P}(\{X_1 = x_1\} \cap \{X_2 \in (-\infty,\infty)\})\\
=& \mathrm{P}\left(\bigcup_{x_2\in(-\infty,\infty)} \{X_1 = x_1 \wedge X_2 = x_2\}\right)\\
=& \bigcup_{x_2\in(-\infty,\infty)} \mathrm{P}(X_1 = x_1 \wedge X_2 = x_2) = \bigcup_{x_2\in X_2(\Omega)} \mathrm{P}(X_1 = x_1, X_2 = x_2)\\
=& \bigcup_{x_2\in X_2(\Omega)} \mathrm{P}(X_1 = x_1, X_2 = x_2) = \sum_{x_2\in X_2(\Omega)} f_{X_1X_2}(x_1,x_2)
\end{aligned}$$

□

continuous:

**Definition 14.20.** JCDF = joint cumulative distribution function: the JCDF of $(X_1, X_2) = \langle X_1, X_2\rangle$ is

$$F_{X_1X_2} = F_{X_1X_2}(x_1,x_2) = \mathrm{P}(X_1 \le x_1, X_2 \le x_2)$$

**Definition 14.21.** JPDF = joint probability density function: the JPDF of $(X_1, X_2) = \langle X_1, X_2\rangle$ is

$$f_{X_1X_2} = f_{X_1X_2}(x_1,x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1X_2}(x_1,x_2)$$

joint vs. marginal

**Theorem 14.18.** *joint cumulative distribution function can inference marginal cumulative distribution function: The marginal CDF of $X_1$ is*

$$F_{X_1}(x_1) = F_{X_1X_2}(x_1,\infty)$$

Proof:

$$\begin{aligned}
F_{X_1}(x_1) =& \mathrm{P}(X_1 \le x_1)\\
=& \mathrm{P}(X_1 \le x_1 \wedge X_2 \le \infty)\\
=& \mathrm{P}(X_1 \le x_1, X_2 \le \infty)\\
=& F_{X_1X_2}(x_1,\infty)
\end{aligned}$$

□

Note:

$F_{X_1}(x_1)$

$$F_{X_1}(x_1) = F_{X_1X_2}(x_1,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f_{X_1X_2}(u_1,u_2)\,\mathrm{d}u_1\mathrm{d}u_2$$

---

$$F_{X_1}(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f_{X_1X_2}(u_1,u_2)\,\mathrm{d}u_1\mathrm{d}u_2$$

---

$f_{X_1}(x_1)$

According to the fundamental theorem of calculus,

$$f_{X_1}(x_1) = \frac{\mathrm{d}}{\mathrm{d}x_1}F_{X_1}(x_1) = \frac{\mathrm{d}}{\mathrm{d}x_1}F_{X_1X_2}(x_1,\infty) = \frac{\mathrm{d}}{\mathrm{d}x_1}\int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X_1X_2}(u_1,u_2)\,\mathrm{d}u_1\mathrm{d}u_2 = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,u_2)\,\mathrm{d}u_2$$

---

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,u_2)\,\mathrm{d}u_2$$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\,\mathrm{d}x_2$$

---

$$f_{X_1}(x_1) = \begin{cases} \int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\,\mathrm{d}x_2 & \text{continuous} \\ \sum\limits_{x_2\in X_2(\Omega)} f_{X_1X_2}(x_1,x_2) & \text{discrete} \end{cases}$$

**Theorem 14.19.** *A function $f(x,y)$ is a joint PDF/PMF or JPDF/JPMF iff*

$$\begin{cases} f(x,y) \geq 0 & (ne)\,\text{non-negative} \\ \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\mathrm{d}y = 1 & \text{continuous} \\ \sum\limits_{x\in X(\Omega_X)}\sum\limits_{y\in Y(\Omega_Y)} f(x,y) = 1 & \text{discrete} \end{cases} & (1)\,\text{total event} \end{cases}$$

**Definition 14.22.** expected value: The expected value of a random vector $g(X,Y)$ is

$$\mathrm{E}_{X,Y}[g(X,Y)] = \mathrm{E}[g(X,Y)] = \begin{cases} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} g(x,y)\,f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y & \text{continuous} \\ \sum\limits_{x\in X(\Omega_X)}\sum\limits_{y\in Y(\Omega_Y)} g(x,y)\,f_{XY}(x,y) & \text{discrete} \end{cases}$$

Def: 14.14

$$\begin{aligned} \mathrm{P}((X,Y)\in E) = \mathrm{E}[1((X,Y)\in E)] &= \begin{cases} \int_{y\in Y(\Omega_Y)}\int_{x\in X(\Omega_X)} 1((X,Y)\in E)\,f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y & \text{continuous} \\ \sum\limits_{x\in X(\Omega_X)}\sum\limits_{y\in Y(\Omega_Y)} 1((X,Y)\in E)\,g(x,y)\,f_{XY}(x,y) & \text{discrete} \end{cases} \\ &= \begin{cases} \iint_{(X,Y)\in E} f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y & \text{continuous} \\ \sum\sum\limits_{(X,Y)\in E} f_{XY}(x,y) & \text{discrete} \end{cases} \end{aligned}$$

---

example

$f_{XY}(x,y) = \mathrm{e}^{-y} 1(0<x<y<\infty) \Rightarrow \mathrm{P}(X+Y\le 1)$

Check if

$$f_{XY}(x,y) = \mathrm{e}^{-y} 1(0<x<y<\infty)$$

is a JPDF:

$$\begin{cases} f_{XY}(x,y) = \mathrm{e}^{-y} 1(0<x<y<\infty) \begin{cases} \mathrm{e}^{-y} > 0 \\ 1(0<x<y<\infty) = \begin{cases} 0 \\ 1 \end{cases} \end{cases} \ge 0 & (ne)\,\text{non-negative} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y = ? & (1)\,\text{total event} \end{cases}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-y} 1(0<x<y<\infty)\,\mathrm{d}x\mathrm{d}y, x\in(0,\infty) \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-y} 1(0<x<y<\infty)\,\mathrm{d}x\mathrm{d}y, y > x \in (0,\infty) \\
&= \int_{0}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-y} 1(0<x<y<\infty)\,\mathrm{d}x\mathrm{d}y \overset{\text{Fubini}}{=} \int_{0}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-y} 1(0<x<y<\infty)\,\mathrm{d}y\mathrm{d}x, y > x \\
&= \int_{0}^{\infty}\int_{x}^{\infty} \mathrm{e}^{-y} 1\mathrm{d}y\mathrm{d}x = \int_{0}^{\infty}\int_{x}^{\infty} \mathrm{e}^{-y}\mathrm{d}y\mathrm{d}x = \int_{0}^{\infty} \left[-\mathrm{e}^{-y}\right]_{y=x}^{\infty} \mathrm{d}x = \int_{0}^{\infty} \left[-\mathrm{e}^{-\infty} - \left(-\mathrm{e}^{-x}\right)\right] \mathrm{d}x \\
&= \int_{0}^{\infty} \left[-0 - \left(-\mathrm{e}^{-x}\right)\right] \mathrm{d}x = \int_{0}^{\infty} \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_{x=0}^{\infty} = \left[-\mathrm{e}^{-\infty} - \left(-\mathrm{e}^{-0}\right)\right] = [-0-(-1)] = 1
\end{aligned}$$

$$\begin{cases} f_{XY}(x,y) = \mathrm{e}^{-y} 1(0<x<y<\infty) \ge 0 & (ne)\,\text{non-negative} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y = 1 & (1)\,\text{total event} \end{cases}$$

$\mathrm{P}(X+Y\le 1)$

https://tex.stackexchange.com/questions/75933/how-to-draw-the-region-of-inequality

https://tex.stackexchange.com/questions/352511/how-to-fill-in-inequality-where-all-inequalities-overlap

[interpolation dashed lines][??]

202402170031-statistics_files/figure-latex/unnamed-chunk-46-1.pd

Figure 14.3: $X+Y\le 1$

$$\begin{aligned}
\mathrm{P}(X+Y\le 1) &= \iint_{(X,Y)\in E} f_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y, E = \{X+Y\le 1\} \\
&= \iint_{E} \mathrm{e}^{-y} 1(0<x<y<\infty)\,\mathrm{d}x\mathrm{d}y, E = \{X+Y\le 1\} \\
&= \int_{0}^{0.5}\int_{x}^{1-x} \mathrm{e}^{-y}\mathrm{d}y\mathrm{d}x = \int_{0}^{0.5} \left[-\mathrm{e}^{-y}\right]_{y=x}^{1-x} \mathrm{d}x = \int_{0}^{0.5} \left[-\mathrm{e}^{-(1-x)} - \left(-\mathrm{e}^{-x}\right)\right] \mathrm{d}x \\
&= \int_{0}^{0.5} \left[\mathrm{e}^{-x} - \mathrm{e}^{x-1}\right] \mathrm{d}x = \left[-\mathrm{e}^{-x} - \mathrm{e}^{x-1}\right]_{x=0}^{0.5} = \left[-\mathrm{e}^{-0.5} - \mathrm{e}^{0.5-1}\right] - \left[-\mathrm{e}^{-0} - \mathrm{e}^{0-1}\right] \\
&= 1 + \mathrm{e}^{-1} - 2\mathrm{e}^{-0.5} \fallingdotseq 0.154818\cdots
\end{aligned}$$

□

https://www.youtube.com/watch?v=r-bsPw2-sRg https://www.youtube.com/watch?v=EwMFsvERFVw

https://www.youtube.com/watch?v=Os6HJLDtYgM

https://www.youtube.com/watch?v=8GefzYp6iBE

https://www.youtube.com/watch?v=nOSXlzxXrPY

https://www.youtube.com/watch?v=hwU7JLCRCgo

https://www.youtube.com/watch?v=duKMBB1bdCU

https://www.youtube.com/watch?v=H7wBXhvjZfg

https://www.youtube.com/playlist?list=PLTp0eSi9MdkNZB4kyLSzIXIUy9JQOJ5AM

https://www.youtube.com/watch?v=qgef6G9rzts

https://www.youtube.com/watch?v=telisdm9Aus

https://www.youtube.com/watch?v=qeRsAId5f5U

https://www.youtube.com/watch?v=cuR-HsEq_fs

https://www.youtube.com/watch?v=Ue1mgEVDwq0

https://www.youtube.com/watch?v=1meJoxJ5_UA

https://www.youtube.com/watch?v=UB0kwppDucI

https://www.youtube.com/watch?v=YrqcdCPM1nw

https://www.youtube.com/watch?v=GCw1T_lUunw

https://www.youtube.com/watch?v=MF0mZ5MpcSw

https://www.youtube.com/watch?v=zX8L8yGlYaU

https://www.youtube.com/watch?v=XJgvphHKXwY

https://www.youtube.com/watch?v=8Qzqf51O6ZE

https://www.youtube.com/watch?v=7x8QN1pYT7c

https://www.youtube.com/watch?v=oTf3n-OD7EI

https://www.youtube.com/watch?v=Igo2EJPz3sU

https://www.youtube.com/watch?v=NVemOifoMqw

https://www.youtube.com/watch?v=HrrBA3v-xTQ

https://www.youtube.com/watch?v=9HbtJ3ZIPxQ

https://www.youtube.com/watch?v=elPnSU4AF1o

https://www.youtube.com/watch?v=_OVHGnQ7Rug

https://www.youtube.com/watch?v=pqHEXAAW_vk

https://www.youtube.com/watch?v=BTK_1Lz5ox8

https://www.youtube.com/watch?v=iATKGYnlomU

https://www.youtube.com/watch?v=5LlTeUeAqDc

https://www.youtube.com/watch?v=usCaJRQ2i6E

https://www.youtube.com/watch?v=p8NXibyDKDo

## 14.2 Chen, Lin-An

https://www.youtube.com/playlist?list=PLTpF-A8hKVUPXtNAX9lro-leGgEK0OSEW

https://en.wikipedia.org/wiki/Inverse_distance_weighting Shepard interpolation

# Chapter 15

# covariance matrix

## 15.1 vector direct product

- scalar = rank-0 tensor
- vector = rank-1 tensor
- matrix = rank-2 tensor
- vector direct product = rank-1 tensor times rank-1 tensor equals rank-2 tensor: increasing rank
- vector inner product = rank-1 tensor times rank-1 tensor equals rank-0 tensor: decreasing rank

scalar = rank-0 tensor

vector = rank-1 tensor

matrix = rank-2 tensor

### 15.1.1 vector direct product: increasing rank

vector direct product = rank-1 tensor times rank-1 tensor equals rank-2 tensor: increasing rank

$$\begin{aligned} U \otimes V =& UV^{\intercal} = U_i V_j \\ = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \otimes \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} =& \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} U_1V_1 & U_1V_2 & U_1V_3 \\ U_2V_1 & U_2V_2 & U_2V_3 \\ U_3V_1 & U_3V_2 & U_3V_3 \end{pmatrix} \\ =& \begin{pmatrix} U_1 V^{\intercal} \\ U_2 V^{\intercal} \\ U_3 V^{\intercal} \end{pmatrix} = \begin{pmatrix} UV_1 & UV_2 & UV_3 \end{pmatrix} \end{aligned}$$

### 15.1.2 vector inner product: decreasing rank

vector inner product = rank-1 tensor times rank-1 tensor equals rank-0 tensor: decreasing rank

$$\begin{aligned} U \cdot V =& V^{\intercal} U = V_i U_i \\ = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} =& \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = V_1U_1 + V_2U_2 + V_3U_3 \end{aligned}$$

### 15.1.3 tensor direct product: increasing rank

$$S \otimes T = S_{ik} T_{jl} \tag{15.1}$$

$$(i,j),(k,l) \in \{(1,1),(1,2),(2,1),(2,2)\} \tag{15.2}$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \otimes \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} S_{11}T_{11} & S_{11}T_{12} & S_{12}T_{11} & S_{12}T_{12} \\ S_{11}T_{21} & S_{11}T_{22} & S_{12}T_{21} & S_{12}T_{22} \\ S_{21}T_{11} & S_{21}T_{12} & S_{22}T_{11} & S_{22}T_{12} \\ S_{21}T_{21} & S_{21}T_{22} & S_{22}T_{21} & S_{22}T_{22} \end{pmatrix} \tag{15.3}$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} S_{11}T & S_{12}T \\ S_{21}T & S_{22}T \end{pmatrix} \tag{15.4}$$

$$= \begin{pmatrix} S_{11} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} & S_{12} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \\ S_{21} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} & S_{22} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \end{pmatrix} \tag{15.5}$$

## 15.2 covariance matrix and its properties

8

$$\begin{aligned}
\mathrm{C}[\boldsymbol{X}] = \mathrm{Cov}[\boldsymbol{X}] = \mathrm{V}[\boldsymbol{X}] =& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^\intercal\right] \\
=& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X}^\intercal - \mathrm{E}(\boldsymbol{X})^\intercal]\right] \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^\intercal - \mathrm{E}(\boldsymbol{X})\boldsymbol{X}^\intercal - \boldsymbol{X}\mathrm{E}(\boldsymbol{X})^\intercal + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal\right] \\
=& \mathrm{E}[\boldsymbol{X}\boldsymbol{X}^\intercal] - \mathrm{E}[\mathrm{E}(\boldsymbol{X})\boldsymbol{X}^\intercal] - \mathrm{E}[\boldsymbol{X}\mathrm{E}(\boldsymbol{X})^\intercal] + \mathrm{E}[\mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal] \\
=& \mathrm{E}[\boldsymbol{X}\boldsymbol{X}^\intercal] - \mathrm{E}(\boldsymbol{X})\mathrm{E}[\boldsymbol{X}^\intercal] - \mathrm{E}[\boldsymbol{X}]\mathrm{E}(\boldsymbol{X})^\intercal + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal \\
=& \mathrm{E}[\boldsymbol{X}\boldsymbol{X}^\intercal] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal \\
=& \mathrm{E}[\boldsymbol{X}\boldsymbol{X}^\intercal] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{X} = [X]_{1\times 1} = X \Rightarrow C(X) = \mathrm{C}[\boldsymbol{X}] =& \mathrm{E}[\boldsymbol{X}\boldsymbol{X}^\intercal] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^\intercal \\
=& \mathrm{E}[XX] - \mathrm{E}(X)\mathrm{E}(X) \\
=& \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = \mathrm{V}(X)
\end{aligned}$$

### 15.2.1 $\mathrm{V}[\boldsymbol{X} + \boldsymbol{b}] = \mathrm{V}[\boldsymbol{X}]$

$$\begin{aligned}
\mathrm{V}[\boldsymbol{X} + \boldsymbol{b}] =& \mathrm{E}\left[[(\boldsymbol{X} + \boldsymbol{b}) - \mathrm{E}(\boldsymbol{X} + \boldsymbol{b})][(\boldsymbol{X} + \boldsymbol{b}) - \mathrm{E}(\boldsymbol{X} + \boldsymbol{b})]^\intercal\right] \\
\overset{\mathrm{E}(\boldsymbol{X}+\boldsymbol{b})=\mathrm{E}(\boldsymbol{X})+\boldsymbol{b}}{=}& \mathrm{E}\left[[\boldsymbol{X} + \boldsymbol{b} - \mathrm{E}(\boldsymbol{X}) - \boldsymbol{b}][\boldsymbol{X} + \boldsymbol{b} - \mathrm{E}(\boldsymbol{X}) - \boldsymbol{b}]^\intercal\right] \\
=& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^\intercal\right] = \mathrm{V}[\boldsymbol{X}]
\end{aligned}$$

### 15.2.2 $\mathrm{V}[A\boldsymbol{X}] = A\mathrm{V}[\boldsymbol{X}]A^\intercal$

$$\begin{aligned}
\mathrm{V}[A\boldsymbol{X}] =& \mathrm{E}\left[[(A\boldsymbol{X}) - \mathrm{E}(A\boldsymbol{X})][(A\boldsymbol{X}) - \mathrm{E}(A\boldsymbol{X})]^\intercal\right] \\
\overset{\mathrm{E}(A\boldsymbol{X})=A\mathrm{E}(\boldsymbol{X})}{=}& \mathrm{E}\left[[A\boldsymbol{X} - A\mathrm{E}(\boldsymbol{X})][A\boldsymbol{X} - A\mathrm{E}(\boldsymbol{X})]^\intercal\right] \\
=& \mathrm{E}\left[A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]]^\intercal\right] \\
=& \mathrm{E}\left[A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^\intercal A^\intercal\right] \\
=& A\mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^\intercal\right]A^\intercal = A\mathrm{V}[\boldsymbol{X}]A^\intercal
\end{aligned}$$

### 15.2.3 $\mathrm{V}[A\boldsymbol{X} + \boldsymbol{b}] = A\mathrm{V}[\boldsymbol{X}]A^\intercal$

$$\mathrm{V}[A\boldsymbol{X} + \boldsymbol{b}] = \mathrm{V}[A\boldsymbol{X}] = A\mathrm{V}[\boldsymbol{X}]A^\intercal$$

# Chapter 16

# Gosper algorithm

## 16.1 Zhuli

https://www.youtube.com/watch?v=0LFg5dvPOoc

## 16.2 TaylorCatAlice

https://www.bilibili.com/video/BV1ZX4y1o7EB

https://www.bilibili.com/video/BV1wv4y1H7m8

https://www.bilibili.com/video/BV1Ua4y1q7Q9

# Chapter 17

# Lorentz transformation

## 17.1 Einstein

https://www.youtube.com/watch?v=FvqutkaPmas

https://wap.hillpublisher.com/UpFile/202204/20220414165340.pdf

## 17.2 Bondi $k$-calculus

https://www.youtube.com/watch?v=Ghql2UNlWYA

https://en.wikipedia.org/wiki/Bondi_k-calculus

https://www.youtube.com/watch?v=bZnmF-UKvq0

## 17.3 wordline in Minkowski space

https://www.bilibili.com/video/BV1sV4y1Y7fX

### 17.3.1 Wick rotation

https://ncatlab.org/nlab/show/Wick+rotation

#### 17.3.1.1 Osterwalder-Schrader reconstruction theorem

https://ncatlab.org/nlab/show/Osterwalder-Schrader+theorem

# Chapter 18

# R

## 18.1 TonyKuoYJ

郭耀仁 認識 R 的美好

https://bookdown.org/tonykuoyj/eloquentr/getting-started.html

https://bookdown.org/tonykuoyj/eloquentr/easy-installation.html#about-packages

`install.pacakges()`

`library()`

https://bookdown.org/tonykuoyj/eloquentr/getting-started.html

### 18.1.1 quick intro

`Ctrl + Alt + I` to insert a new code chunk in RStudio

`Ctrl + Enter` to run the current line

`Ctrl + Shift + Enter` to run the current chunk

```r
R.version
```

```
##                _
## platform       x86_64-w64-mingw32
## arch           x86_64
## os             mingw32
## crt            ucrt
## system         x86_64, mingw32
## status
## major          4
## minor          3.2
## year           2023
## month          10
## day            31
## svn rev        85441
## language       R
## version.string R version 4.3.2 (2023-10-31 ucrt)
## nickname       Eye Holes
```

```r
a <- 23 # prime
a
```

```
## [1] 23
```

```r
combine <- c(11, 13) # twin prime
combine
```

```
## [1] 11 13
```

```
# ?c
# help(c)
```

`Ctrl + L` to clean R console

path with slash `/` in R, differing backslash `\` in M$ Windows

#### 18.1.1.1 function

```
add <- function(x, y) {
  return(x + y)
}

add(11, 13)
```

```
## [1] 24
```

$$BMI = \frac{BW\,[\mathrm{Kg}]}{BH\,[\mathrm{m}]^2}$$

```
get_bmi <- function (bw, bh) {
  return (bw/(bh/100)^2)
}

get_bmi(70, 170)
```

```
## [1] 24.22145
```

### 18.1.2 R style

https://bookdown.org/tonykuoyj/eloquentr/styleguide.html

snake_case rather than camelCase

### 18.1.3 data workflow or forward pipe

from *chaining method* in *object-oriented programming* to **functional programming**

#### 18.1.3.1 %>% operator

```
abs(-5:5)
```

```
##  [1] 5 4 3 2 1 0 1 2 3 4 5
```

```
# install.packages("magrittr")

library(magrittr)
```

```
##
## Attaching package: 'magrittr'
```

```
## The following object is masked _by_ '.GlobalEnv':
##
##     add
```

```
-5:5 %>% abs()
```

```
##  [1] 5 4 3 2 1 0 1 2 3 4 5
```

```
# with readability but too many lines
sys_date <- Sys.Date()
sys_date_yr <- format(sys_date, format = "%Y")
sys_date_num <- as.numeric(sys_date_yr)
sys_date_num
```

```
## [1] 2024
```

```
# less line but also less readability
sys_date_num <- as.numeric(format(Sys.Date(), format = "%Y"))
sys_date_num
```

```
## [1] 2024
```

```
# use %>% operator to demonstrate data workflow or forward pipe
sys_date_num <- Sys.Date() %>%
   format(format = "%Y") %>%
   as.numeric()
sys_date_num
```

```
## [1] 2024
```

### 18.1.4 data processing with dplyr

https://bookdown.org/tonykuoyj/eloquentr/dplyr.html

some functions functioning like those in **SQL**

```
library(dplyr)
```

```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```
# install.packages("gapminder")

library(gapminder)
```

```
## Warning: package 'gapminder' was built under R version 4.3.3
```

```
head(gapminder)
```

```
## # A tibble: 6 x 6
##   country     continent  year lifeExp      pop gdpPercap
##   <fct>       <fct>     <int>   <dbl>    <int>     <dbl>
## 1 Afghanistan Asia       1952    28.8  8425333      779.
## 2 Afghanistan Asia       1957    30.3  9240934      821.
## 3 Afghanistan Asia       1962    32.0 10267083      853.
## 4 Afghanistan Asia       1967    34.0 11537966      836.
## 5 Afghanistan Asia       1972    36.1 13079460      740.
## 6 Afghanistan Asia       1977    38.4 14880372      786.
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007)
```

```
## # A tibble: 142 x 6
##    country     continent  year lifeExp       pop gdpPercap
##    <fct>       <fct>     <int>   <dbl>     <int>     <dbl>
##  1 Afghanistan Asia       2007    43.8  31889923      975.
##  2 Albania     Europe     2007    76.4   3600523     5937.
##  3 Algeria     Africa     2007    72.3  33333216     6223.
##  4 Angola      Africa     2007    42.7  12420476     4797.
##  5 Argentina   Americas   2007    75.3  40301927    12779.
```

```
## 6 Australia   Oceania    2007    81.2  20434176    34435.
## 7 Austria     Europe     2007    79.8   8199783    36126.
## 8 Bahrain     Asia       2007    75.6    708573    29796.
## 9 Bangladesh  Asia       2007    64.1 150448339     1391.
## 10 Belgium    Europe     2007    79.4  10392226    33693.
## # i 132 more rows
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  select(country)
```

```
## # A tibble: 142 x 1
##    country
##    <fct>
##  1 Afghanistan
##  2 Albania
##  3 Algeria
##  4 Angola
##  5 Argentina
##  6 Australia
##  7 Austria
##  8 Bahrain
##  9 Bangladesh
## 10 Belgium
## # i 132 more rows
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  mutate(pop_in_thousands = pop / 1000)
```

```
## # A tibble: 1,704 x 7
##    country     continent  year lifeExp      pop gdpPercap pop_in_thousands
##    <fct>       <fct>     <int>   <dbl>    <int>     <dbl>            <dbl>
##  1 Afghanistan Asia       1952    28.8  8425333      779.            8425.
##  2 Afghanistan Asia       1957    30.3  9240934      821.            9241.
##  3 Afghanistan Asia       1962    32.0 10267083      853.           10267.
##  4 Afghanistan Asia       1967    34.0 11537966      836.           11538.
##  5 Afghanistan Asia       1972    36.1 13079460      740.           13079.
##  6 Afghanistan Asia       1977    38.4 14880372      786.           14880.
##  7 Afghanistan Asia       1982    39.9 12881816      978.           12882.
##  8 Afghanistan Asia       1987    40.8 13867957      852.           13868.
##  9 Afghanistan Asia       1992    41.7 16317921      649.           16318.
## 10 Afghanistan Asia       1997    41.8 22227415      635.           22227.
## # i 1,694 more rows
```

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  arrange(year)
```

```
## # A tibble: 1,704 x 6
##    country     continent  year lifeExp      pop gdpPercap
##    <fct>       <fct>     <int>   <dbl>    <int>     <dbl>
##  1 Afghanistan Asia       1952    28.8  8425333      779.
##  2 Albania     Europe     1952    55.2  1282697     1601.
```

```
##  3 Algeria     Africa     1952    43.1  9279525      2449.
##  4 Angola      Africa     1952    30.0  4232095      3521.
##  5 Argentina   Americas   1952    62.5 17876956      5911.
##  6 Australia   Oceania    1952    69.1  8691212     10040.
##  7 Austria     Europe     1952    66.8  6927772      6137.
##  8 Bahrain     Asia       1952    50.9   120447      9867.
##  9 Bangladesh  Asia       1952    37.5 46886859       684.
## 10 Belgium     Europe     1952    68    8730405      8343.
## # i 1,694 more rows
```

total population in the world in 2007

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  summarise(ttl_pop = sum(as.numeric(pop)))
```

```
## # A tibble: 1 x 1
##      ttl_pop
##        <dbl>
## 1 6251013179
```

total population group by the continents in 2007

```
library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
  filter(year == 2007) %>%
  group_by(continent) %>%
  summarise(ttl_pop = sum(as.numeric(pop)))
```

```
## # A tibble: 5 x 2
##   continent    ttl_pop
##   <fct>          <dbl>
## 1 Africa     929539692
## 2 Americas   898871184
## 3 Asia      3811953827
## 4 Europe     586098529
## 5 Oceania     24549947
```

### 18.1.5 visualization statically with ggplot2

```
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 4.3.3
```

```
library(gapminder)

gapminder_2007 <- gapminder %>%
  filter(year == 2007)
scatter_plot <- ggplot(gapminder_2007, aes(x = gdpPercap, y = lifeExp)) +
  geom_point()
scatter_plot
```

202402211401-R_files/figure-latex/unnamed-chunk-19-1.pdf

```r
library(ggplot2)
library(gapminder)

north_asia <- gapminder %>%
  filter(country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
line_plot <- ggplot(north_asia, aes(x = year, y = gdpPercap, colour = country)) +
  geom_line()
line_plot
```

202402211401-R_files/figure-latex/unnamed-chunk-20-1.pdf

```r
library(ggplot2)
library(gapminder)

hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +
  geom_histogram()
hist_plot
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

202402211401-R_files/figure-latex/unnamed-chunk-21-1.pdf

```r
hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +
  geom_histogram(bins = 20)
hist_plot
```

202402211401-R_files/figure-latex/unnamed-chunk-21-2.pdf

```r
library(ggplot2)
library(gapminder)

box_plot <- ggplot(gapminder_2007, aes(x = continent, y = gdpPercap)) +
  geom_boxplot()
box_plot
```

202402211401-R_files/figure-latex/unnamed-chunk-22-1.pdf

```r
library(ggplot2)
library(gapminder)

gdpPercap_2007_na <- gapminder %>%
  filter(year == 2007 & country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
bar_plot <- ggplot(gdpPercap_2007_na, aes(x = country, y = gdpPercap)) +
  geom_bar(stat = "identity")
```

```
bar_plot
```

202402211401-R_files/figure-latex/unnamed-chunk-23-1.pdf

### 18.1.6 loop

https://bookdown.org/tonykuoyj/eloquentr/for.html

```
month.name
```

```
##  [1] "January"   "February"  "March"     "April"     "May"       "June"
##  [7] "July"      "August"    "September" "October"   "November"  "December"
```

```
month.name[1]
```

```
## [1] "January"
```

```
for (month in month.name) {
  print(month)
}
```

```
## [1] "January"
## [1] "February"
## [1] "March"
## [1] "April"
## [1] "May"
## [1] "June"
## [1] "July"
## [1] "August"
## [1] "September"
## [1] "October"
## [1] "November"
## [1] "December"
```

### 18.1.7 variable type

https://bookdown.org/tonykuoyj/eloquentr/variable-types.html

https://www.w3schools.com/r/r_data_types.asp

- numeric
- integer
- complex = complex number
- character
- logical = boolean

```
class(2L)
```

```
## [1] "integer"
```

```
class(2.0L)
```

```
## [1] "integer"
```

```
class(2.3L)
```

```
## [1] "numeric"
```

time: POSIXct POSIXt

```
class(Sys.time())
```

```
## [1] "POSIXct" "POSIXt"
```

```
0 %in% -5:5
```

```
## [1] TRUE
```

#### 18.1.7.1 date

1970-01-01 = 0L

```
date_of_origin <- as.Date("1970-01-01")
as.integer(date_of_origin)
```

```
## [1] 0
```

check if type of `x` is `Date`

`inherits(x, what = "Date")`

convert `character` to `Date`

`as.Date("01-01-1970", format = "%m-%d-%Y")`

#### 18.1.7.2 time

1970-01-01 00:00:00 GMT = 0L

tz = time zone

```
time_of_origin <- as.POSIXct("1970-01-01 00:00:00", tz = "GMT")
as.integer(time_of_origin)
```

```
## [1] 0
```

check if type of `x` is time

`inherits(x, what = "POSIXct")`

convert `character` to time

`as.POSIXct("1970-01-01 00:00:00", tz = "GMT")`

#### 18.1.7.3 quotient `%/%` operator

https://www.w3schools.com/r/r_operators.asp

```
7 %/% 3
```

```
## [1] 2
```

### 18.1.8 data type

https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html

- 1D
  - vector[18.1.8.1]
  - factor[18.1.8.2]
- 2D
  - matrix[18.1.8.3]
  - data frame[18.1.8.4]
- $n$D
  - array[18.2.6.1]
  - list[18.2.6.2]

#### 18.1.8.1 vector

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons
```

```
## [1] "spring" "summer" "autumn" "winter"
```

```
favorite_season <- four_seasons[3]
favorite_season
```

```
## [1] "autumn"
```

```
favorite_seasons <- four_seasons[c(-2, -4)]
favorite_seasons
```

```
## [1] "spring" "autumn"
```

only one variable type for a vector

```
lucky_numbers <- c(7L, 24)
class(lucky_numbers[1])
```

```
## [1] "numeric"
```

```
lucky_numbers <- c(7L, FALSE)
lucky_numbers
```

```
## [1] 7 0
```

```
class(lucky_numbers[2])
```

```
## [1] "integer"
```

```
mixed_vars <- c(TRUE, 7L, 24, "spring")
mixed_vars
```

```
## [1] "TRUE"   "7"      "24"     "spring"
```

```
class(mixed_vars[1])
```

```
## [1] "character"
```

```
class(mixed_vars[2])
```

```
## [1] "character"
```

```
class(mixed_vars[3])
```

```
## [1] "character"
```

```

four_seasons <- c("spring", "summer", "autumn", "winter")
my_favorite_seasons <- four_seasons == "spring" | four_seasons == "autumn"
four_seasons[my_favorite_seasons]
```

##### 18.1.8.1.1 logic

```
## [1] "spring" "autumn"
```

```

rep(7L, times = 8)
```

##### 18.1.8.1.2 rep repeat

```
## [1] 7 7 7 7 7 7 7 7
```

```
rep("R", times = 10)
```

```
##  [1] "R" "R" "R" "R" "R" "R" "R" "R" "R" "R"
```

```

seq(from = 7, to = 77, by = 7)
```

##### 18.1.8.1.3 seq sequence

```
##  [1]  7 14 21 28 35 42 49 56 63 70 77
```

```
11:20
```

```
##  [1] 11 12 13 14 15 16 17 18 19 20
```

#### 18.1.8.2 factor

https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html#factor

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons
```

```
## [1] "spring" "summer" "autumn" "winter"
```

```
four_seasons_factor <- factor(four_seasons)
four_seasons_factor
```

```
## [1] spring summer autumn winter
## Levels: autumn spring summer winter
```

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons_factor <- factor(four_seasons, ordered = TRUE, levels = c("summer", "winter", "spring",
↪  "autumn"))
four_seasons_factor
```

```
## [1] spring summer autumn winter
## Levels: summer < winter < spring < autumn
```

```
temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE, levels = c("cold", "warm", "hot"))
temp_factors
```

```
## [1] warm hot  cold
## Levels: cold < warm < hot
```

if no levels specified, the levels will be specified alphabetically, sometimes not really true

```
temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE)
temp_factors
```

```
## [1] warm hot  cold
## Levels: cold < hot < warm
```

#### 18.1.8.3 matrix

https://bookdown.org/tonykuoyj/eloquentr/matrix-dataframe-more.html

```
my_mat <- matrix(1:6, nrow = 2)
my_mat
```

```
##      [,1] [,2] [,3]
## [1,]    1    3    5
## [2,]    2    4    6
```

```
class(my_mat)
```

```
## [1] "matrix" "array"
```

```
my_mat2 <- matrix(1:6, nrow = 2, byrow = TRUE)
my_mat2
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    4    5    6
```

```
my_mat2[2, 3]
```

```
## [1] 6
```

```r
my_mat2[2, ]
```

```
## [1] 4 5 6
```

```r
my_mat2[, 3]
```

```
## [1] 3 6
```

```r
filter <- my_mat2 < 6 & my_mat2 > 1
my_mat2[filter]
```

```
## [1] 4 2 5 3
```

boolean will become value in a matrix, like vector

```r
my_mat3 <- matrix(c(1, 2, TRUE, FALSE, 3, 4), nrow = 2)
my_mat3
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    3
## [2,]    2    0    4
```

```r
class(my_mat3[, 2])
```

```
## [1] "numeric"
```

#### 18.1.8.4 data frame

- variable: column
- observation: row
- value: cell

```r
team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season)
great_nba_teams
```

```
##               team_name wins losses is_champion  season
## 1         Chicago Bulls   72     10        TRUE 1995-96
## 2 Golden State Warriors   73      9       FALSE 2015-16
```

```r
great_nba_teams[1, 1]
```

```
## [1] "Chicago Bulls"
```

```r
great_nba_teams[1, ]
```

```
##       team_name wins losses is_champion  season
## 1 Chicago Bulls   72     10        TRUE 1995-96
```

```r
great_nba_teams[, 1]
```

```
## [1] "Chicago Bulls"         "Golden State Warriors"
```

stringsAsFactors = TRUE

```r
team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season, stringsAsFactors = TRUE)
great_nba_teams[, 1]
```

```
## [1] Chicago Bulls         Golden State Warriors
## Levels: Chicago Bulls Golden State Warriors
```

stringsAsFactors = FALSE

```
team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season, stringsAsFactors = FALSE)
great_nba_teams[, 1]
```

```
## [1] "Chicago Bulls"         "Golden State Warriors"
```

```
great_nba_teams$team_name
```

##### 18.1.8.4.1 selecting variable or column

```
## [1] "Chicago Bulls"         "Golden State Warriors"
```

```
great_nba_teams[, "team_name"]
```

```
## [1] "Chicago Bulls"         "Golden State Warriors"
```

```
filter <- great_nba_teams$is_champion == TRUE
great_nba_teams[filter, ]
```

##### 18.1.8.4.2 filtering observation or row

```
##       team_name wins losses is_champion  season
## 1 Chicago Bulls   72     10        TRUE 1995-96
```

```
str(great_nba_teams)
```

##### 18.1.8.4.3 check mixed data type

```
## 'data.frame':    2 obs. of  5 variables:
##  $ team_name  : chr  "Chicago Bulls" "Golden State Warriors"
##  $ wins       : num  72 73
##  $ losses     : num  10 9
##  $ is_champion: logi  TRUE FALSE
##  $ season     : chr  "1995-96" "2015-16"
```

## 18.2 W3School

https://www.w3schools.com/r/default.asp

### 18.2.1 same multiple variable

https://www.w3schools.com/r/r_variables_multiple.asp

```
# Assign the same value to multiple variables in one line
var1 <- var2 <- var3 <- "Orange"

# Print variable values
var1
```

```
## [1] "Orange"
```

```
var2
```

```
## [1] "Orange"
```

```
var3
```

```
## [1] "Orange"
```

### 18.2.2 legal variable name

https://www.w3schools.com/r/r_variables_name.asp

```
# Legal variable names:
myvar <- "John"
my_var <- "John"
myVar <- "John"
MYVAR <- "John"
myvar2 <- "John"
.myvar <- "John"

## Illegal variable names:
# 2myvar <- "John"
# my-var <- "John"
# my var <- "John"
# _my_var <- "John"
# my_v@ar <- "John"
# TRUE <- "John"
```

### 18.2.3 complex number

https://www.w3schools.com/r/r_data_types.asp

https://www.w3schools.com/r/r_numbers.asp

### 18.2.4 escape character

https://www.w3schools.com/r/r_strings_esc.asp

### 18.2.5 global assignment <<-

```
my_function <- function() {
txt <<- "fantastic"
  paste("R is", txt)
}

my_function()
```

```
## [1] "R is fantastic"
```

```
print(txt)
```

```
## [1] "fantastic"
```

```
txt <- "awesome"
my_function <- function() {
  txt <<- "fantastic"
  paste("R is", txt)
}

my_function()
```

```
## [1] "R is fantastic"
```

```
paste("R is", txt)
```

```
## [1] "R is fantastic"
```

### 18.2.6 data type

#### 18.2.6.1 array

https://www.w3schools.com/r/r_arrays.asp

```
# An array with one dimension with values ranging from 1 to 24
thisarray <- c(1:24)
thisarray
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
```

```
# An array with more than one dimension
multiarray <- array(thisarray, dim = c(4, 3, 2))
multiarray
```

```
## , , 1
##
##      [,1] [,2] [,3]
## [1,]    1    5    9
## [2,]    2    6   10
## [3,]    3    7   11
## [4,]    4    8   12
##
## , , 2
##
##      [,1] [,2] [,3]
## [1,]   13   17   21
## [2,]   14   18   22
## [3,]   15   19   23
## [4,]   16   20   24
```

```
multiarray[2, 3, 2]
```

```
## [1] 22
```

#### 18.2.6.2 list

https://www.w3schools.com/r/r_lists.asp

## 18.3 Apan Liao

R 演習室

https://www.youtube.com/playlist?list=PL5AC0ADBF65924EAD

### 18.3.1 data input

https://www.youtube.com/watch?v=STcIxf_vUWY&list=PL5AC0ADBF65924EAD&index=1

- `scan()`
- read
    - `read.table()`
    - `read.csv()`

### 18.3.2 descriptive statistics

https://www.youtube.com/watch?v=GL3Wv_45LaU&list=PL5AC0ADBF65924EAD&index=2

# Chapter 19

# Laplace transform

https://www.youtube.com/watch?v=lg90shB1TrU

https://www.youtube.com/watch?v=WEkuV55B4q4

# Chapter 20

# conic section

conic section 圓錐曲線 / 圓錐截痕

https://en.wikipedia.org/wiki/Conic_section

https://tex.stackexchange.com/questions/222882/drawing-minimal-xy-axis

Figure 20.1: parabola defined by focus, directrix, eccentricity

## 20.1 Cartesian coordinate: focus, directrix, eccentricity

focus, directrix, eccentricity 焦點, 準線, 離心率

$$\begin{cases} F = (0, y_F) \\ L = y - y_L = 0 \\ \epsilon = \dfrac{\overline{PF}}{d(P, L)} = \dfrac{\|(x, y) - (0, y_F)\|}{\|y - y_L\|} \end{cases} \qquad \begin{array}{l} F : \text{focus} \\ L : \text{directrix} \\ \begin{cases} P = (x, y) \\ \epsilon : \text{eccentricity} \end{cases} \end{array}$$

$$0 \le \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{\sqrt{(y - y_L)^2}} \tag{20.1}$$

$$\epsilon^2 = \frac{x^2 + (y - y_F)^2}{(y - y_L)^2} = \frac{x^2 + y^2 - 2y_F y + y_F^2}{y^2 - 2y_L y + y_L^2} \tag{20.2}$$

$$0 = x^2 + (1 - \epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \tag{20.3}$$

$$\overset{\epsilon \neq 1}{=} x^2 + (1 - \epsilon^2)\left[y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2}\right] \tag{20.4}$$

$$= x^2 + (1 - \epsilon^2) \tag{20.5}$$

$$\left[y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \left(\frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}\right)^2 - \left(\frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}\right)^2 + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2}\right] \tag{20.6}$$

$$= x^2 + (1 - \epsilon^2)\left[\left(y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}\right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2}{(1 - \epsilon^2)^2}\right] \tag{20.7}$$

$$= x^2 + (1 - \epsilon^2)\left(y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}\right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) \text{-} (y_F - \epsilon^2 y_L)^2}{1 - \epsilon^2} \tag{20.8}$$

$$\begin{aligned} & (y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) \text{-} (y_F - \epsilon^2 y_L)^2 \\ &= (1 - \epsilon^2) y_F^2 - (\epsilon^2 - \epsilon^4) y_L^2 - y_F^2 + 2\epsilon^2 y_F y_L - \epsilon^4 y_L^2 \\ &= -\epsilon^2 y_F^2 - \epsilon^2 y_L^2 + 2\epsilon^2 y_F y_L = -\epsilon^2 (y_F - y_L)^2 \end{aligned}$$

$$\frac{\epsilon^2 (y_F - y_L)^2}{1-\epsilon^2} \overset{\epsilon\neq 1}{=} x^2 + (1-\epsilon^2)\left(y - \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2}\right)^2$$

$$1 \overset{\epsilon\neq 0,1}{=} \begin{cases} \left(\dfrac{x-0}{\dfrac{\epsilon (y_F - y_L)}{\sqrt{1-\epsilon^2}}}\right)^2 + \left(\dfrac{y - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon (y_F - y_L)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2 > 0 \overset{\epsilon\geq 0}{\Rightarrow} 0 < \epsilon < 1 \\ -\left(\dfrac{x-0}{\dfrac{\epsilon (y_F - y_L)}{\sqrt{\epsilon^2-1}}}\right)^2 + \left(\dfrac{y - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon (y_F - y_L)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2 < 0 \overset{\epsilon\geq 0}{\Rightarrow} \epsilon > 1 \end{cases}$$

$\epsilon = 0$ or $\lim_{|y_L|\to\infty} \epsilon = 0$

$$r = \overline{PF} = \|(x,y) - (0,y_F)\| = \|(x, y - y_F)\| = \sqrt{x^2 + (y-y_F)^2}$$

$$\epsilon = \frac{r}{d(P,L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|} = \frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|} = \frac{\sqrt{x^2+(y-y_F)^2}}{|y-y_L|}$$

$$\lim_{|y_L|\to\infty} \epsilon = \lim_{|y_L|\to\infty} \frac{r}{d(P,L)} = \lim_{|y_L|\to\infty} \frac{\sqrt{x^2+(y-y_F)^2}}{|y-y_L|} = 0$$

$\epsilon = 1$

$$\begin{aligned} 0 =& x^2 + (1-\epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \\ \overset{\epsilon=1}{=}& x^2 + (1-1^2) y^2 - 2(y_F - 1^2 y_L) y + (y_F^2 - 1^2 y_L^2) \\ =& x^2 - 2(y_F - y_L) y + (y_F^2 - y_L^2) \\ =& x^2 - 2(y_F - y_L) y + (y_F + y_L)(y_F - y_L) \\ x^2 =& 2(y_F - y_L)\left(y - \frac{y_F + y_L}{2}\right) \end{aligned}$$

Let one curve vertex $P = V = (0,0)$ on the curve, and fix the directrix $L$ or $y_L$,

$\epsilon \neq 1$

$$\begin{aligned} 1 &\overset{P(x,y)=V(0,0)}{=} 0 + \left(\dfrac{0 - \dfrac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\dfrac{\epsilon (y_F - y_L)}{1-\epsilon^2}}\right)^2 \\ &\Rightarrow y_F - \epsilon^2 y_L = \pm\epsilon (y_F - y_L) \\ &\Rightarrow \begin{cases} (1-\epsilon) y_F = \epsilon(\epsilon - 1) y_L & + \\ (1+\epsilon) y_F = \epsilon(\epsilon+1) y_L & - \end{cases} \\ &\Rightarrow y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases} \end{aligned}$$

$\epsilon = 1$

$$\begin{aligned} x^2 &= 2(y_F - y_L)\left(y - \frac{y_F + y_L}{2}\right) \\ \overset{P(x,y)=V(0,0)}{\Rightarrow} 0^2 &= 2(y_F - y_L)\left(0 - \frac{y_F + y_L}{2}\right) \\ \Rightarrow 0 &= (y_F - y_L)(y_F + y_L) \\ \Rightarrow y_F &= \mp y_L \end{aligned}$$

or by definition of eccentricity (20.1)

$$0 \le \epsilon = \frac{\overline{PF}}{d(P,L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|} = \frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|} = \frac{\sqrt{x^2+(y-y_F)^2}}{\sqrt{(y-y_L)^2}}$$

$$\overset{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2+(0-y_F)^2}}{\sqrt{(0-y_L)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2}$$

$$\epsilon^2 = \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp\epsilon y_L$$

actually,

$$y_F = -\epsilon y_L$$

## 20.2 two-definition equivalence for ellipse and hyperbola

https://math.stackexchange.com/questions/1833973/prove-that-the-directrix-focus-and-focus-focus-definitions-are-equivalent

https://www.geogebra.org/calculator/zkppuxwp

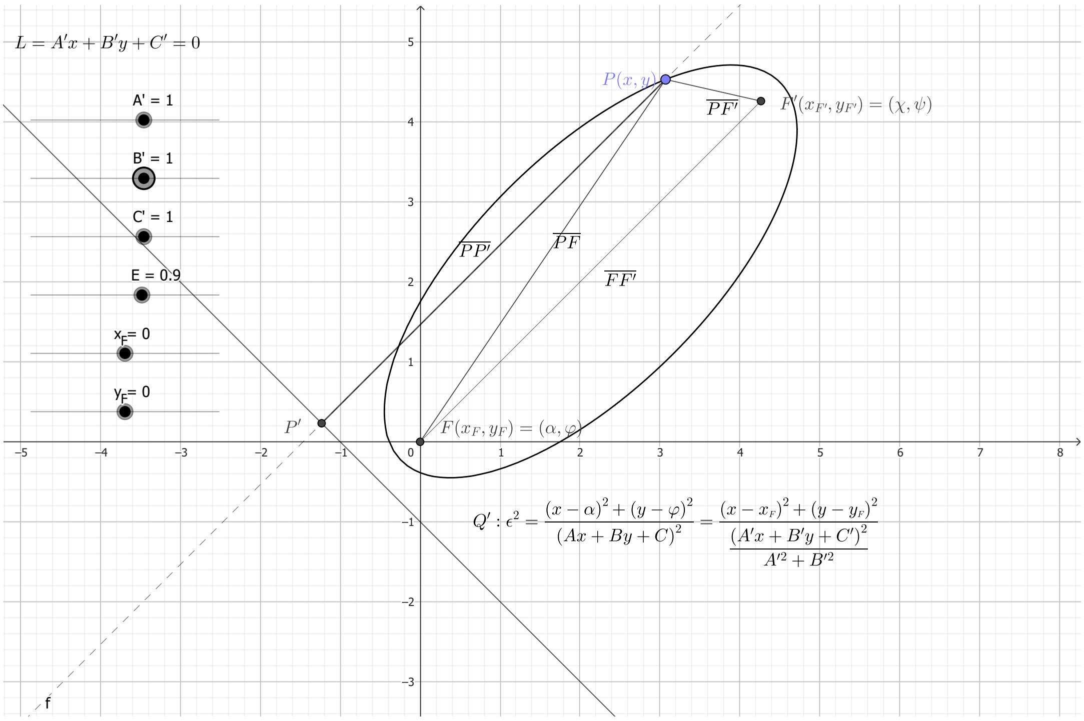


Figure 20.2: conic sections

$$\begin{cases} P = (x,y) \\ F = (x_F, y_F) = (\alpha, \varphi) \qquad F' = (x_{F'}, y_{F'}) = (\chi, \psi) \\ L = A'x + B'y + C' = 0 \end{cases}$$

### 20.2.1 first definition for conic sections including ellipses and hyperbolas

distance from a point to a line[^21^]

$$0 \le \epsilon = \frac{\overline{PF}}{d(P,L)} = \frac{\sqrt{(x-x_F)^2+(y-y_F)^2}}{\frac{|A'x+B'y+C'|}{\sqrt{A'^2+B'^2}}} = \frac{\sqrt{(x-\alpha)^2+(y-\varphi)^2}}{|Ax+By+C|}, \begin{cases} A = \frac{A'}{\sqrt{A'^2+B'^2}} \\ B = \frac{B'}{\sqrt{A'^2+B'^2}} \\ C = \frac{C'}{\sqrt{A'^2+B'^2}} \end{cases}$$

$$A^2 + B^2 = \left(\frac{A'}{\sqrt{A'^2 + B'^2}}\right)^2 + \left(\frac{B'}{\sqrt{A'^2 + B'^2}}\right)^2 = 1$$

or allowing $\epsilon < 0$ by squaring the definition

$$\epsilon^2 = \frac{(x-\alpha)^2 + (y-\varphi)^2}{(Ax + By + C)^2} = \frac{(x - x_F)^2 + (y - y_F)^2}{\frac{(A'x + B'y + C')^2}{A'^2 + B'^2}}$$

$$(x-\alpha)^2 + (y-\varphi)^2 = [\epsilon\,(Ax + By + C)]^2$$

### 20.2.2 second definition for ellipses and hyperbolas

$$\begin{aligned} 2c = \overline{FF'} &= \|(x_F, y_F) - (x_{F'}, y_{F'})\| = \|(\alpha, \varphi) - (\chi, \psi)\| \\ &= \sqrt{(\alpha-\chi)^2 + (\chi-\psi)^2} \end{aligned}$$

$$\begin{aligned} D &= \begin{cases} \sqrt{(x-x_F)^2 + (y-y_F)^2} + \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} & \text{ellipse} \\ \sqrt{(x-x_F)^2 + (y-y_F)^2} - \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} & \text{hyperbola} \end{cases} \\ &= \sqrt{(x-x_F)^2 + (y-y_F)^2} \pm \sqrt{(x-x_{F'})^2 + (y-y_{F'})^2} \\ &= \sqrt{(x-\alpha)^2 + (y-\varphi)^2} \pm \sqrt{(x-\chi)^2 + (y-\psi)^2} \end{aligned}$$

$$\begin{aligned} (x-\alpha)^2 + (y-\varphi)^2 &= \left(D \mp \sqrt{(x-\chi)^2 + (y-\psi)^2}\right)^2 \\ &= D^2 \mp 2D\sqrt{(x-\chi)^2 + (y-\psi)^2} \\ &\quad + (x-\chi)^2 + (y-\psi)^2 \end{aligned}$$

$$
\begin{aligned}
D^2 = & (x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2 \\
& \pm 2\sqrt{\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right]} \\
& (x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2 - D^2 \\
= & \mp 2\sqrt{\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right]} \\
& \left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right]^2 + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
= & 4\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] \\
& \left[(x-\alpha)^2 + (y-\varphi)^2\right]^2 + \left[(x-\chi)^2 + (y-\psi)^2\right]^2 \\
& + 2\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
= & 4\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] \\
0 = & \left[(x-\alpha)^2 + (y-\varphi)^2\right]^2 + \left[(x-\chi)^2 + (y-\psi)^2\right]^2 \\
& - 2\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
0 = & \left\{\left[(x-\alpha)^2 + (y-\varphi)^2\right] - \left[(x-\chi)^2 + (y-\psi)^2\right]\right\}^2 + D^4 \\
& - 2D^2\left\{\left[(x-\alpha)^2 + (y-\varphi)^2\right] + \left[(x-\chi)^2 + (y-\psi)^2\right]\right\} \\
0 = & \left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right]\right\}^2 + D^4 \\
& - 2D^2\left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right]\right\} \\
& - 4D^2\left[(x-\alpha)^2 + (y-\varphi)^2\right] \\
& (2D)^2\left[(x-\alpha)^2 + (y-\varphi)^2\right] \\
= & \left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right] - D^2\right\}^2 \\
= & \left\{\left[(x-\chi)^2 - (x-\alpha)^2\right] + \left[(y-\psi)^2 - (y-\varphi)^2\right] - D^2\right\}^2 \\
= & \left\{(2x-\chi-\alpha)(\alpha-\chi) + (2y-\psi-\varphi)(\varphi-\psi) - D^2\right\}^2 \\
= & \left\{2(\alpha-\chi)x - \left(\alpha^2-\chi^2\right) + 2(\varphi-\psi)y - \left(\varphi^2-\psi^2\right) - D^2\right\}^2 \\
= & \left\{2(\alpha-\chi)x + 2(\varphi-\psi)y - \left[\left(\alpha^2-\chi^2\right) + \left(\varphi^2-\psi^2\right) + D^2\right]\right\}^2 \\
D \neq 0 & \\
& (x-\alpha)^2 + (y-\varphi)^2 \\
= & \left[\frac{\alpha-\chi}{D}x + \frac{\varphi-\psi}{D}y - \left(\frac{\alpha^2-\chi^2}{2D} + \frac{\varphi^2-\psi^2}{2D} + \frac{D}{2}\right)\right]^2
\end{aligned}
$$

$$
\begin{cases}
(x-\alpha)^2 + (y-\varphi)^2 = \left[\epsilon\left(Ax+By+C\right)\right]^2 \\
(x-\alpha)^2 + (y-\varphi)^2 = \left[\dfrac{\alpha-\chi}{D}x + \dfrac{\varphi-\psi}{D}y - \left(\dfrac{\alpha^2-\chi^2}{2D} + \dfrac{\varphi^2-\psi^2}{2D} + \dfrac{D}{2}\right)\right]^2
\end{cases}
$$

$$
(A, B, C) \rightleftarrows (\chi, \psi, D)
$$

$$\begin{cases} \epsilon A = \pm\dfrac{\alpha-\chi}{D} & \chi \pm \epsilon A D = \alpha \\ \epsilon B = \pm\dfrac{\varphi-\psi}{D} & \psi \pm \epsilon B D = \varphi \\ \epsilon C = \mp\left(\dfrac{\alpha^2-\chi^2}{2D} + \dfrac{\varphi^2-\psi^2}{2D} + \dfrac{D}{2}\right) \end{cases}$$

$$\begin{aligned} 2\epsilon C &= \mp\left(\frac{\alpha-\chi}{D}\left(\alpha+\chi\right) + \frac{\varphi-\psi}{D}\left(\varphi+\psi\right) + D\right) \\ &= \mp\left(\pm\epsilon A\left(\alpha+\chi\right) \pm \epsilon B\left(\varphi+\psi\right) + D\right) \\ \mp\epsilon\left(A\alpha + B\varphi + 2C\right) &= \pm\,\epsilon A\chi \pm \epsilon B\psi + D \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A \\ 0 & 1 & \pm\epsilon B \\ \pm\epsilon A & \pm\epsilon B & 1 \end{pmatrix}\begin{pmatrix} \chi \\ \psi \\ D \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ \mp\epsilon\left(A\alpha + B\varphi + 2C\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & \pm\epsilon B & 1 \mp \epsilon^2 A^2 & \mp\epsilon\left(2A\alpha + B\varphi + 2C\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 \mp \epsilon^2 A^2 \mp \epsilon^2 B^2 & \mp\epsilon\left(2A\alpha + 2B\varphi + 2C\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 & \dfrac{\mp 2\epsilon\left(A\alpha + B\varphi + C\right)}{1 \mp \epsilon^2\left(A^2+B^2\right)} \end{pmatrix}$$

$$A^2 + B^2 = \left(\frac{A'}{\sqrt{A'^2+B'^2}}\right)^2 + \left(\frac{B'}{\sqrt{A'^2+B'^2}}\right)^2 = 1$$

$$\begin{cases} \chi = \alpha \mp \epsilon A D = \alpha \mp \epsilon\dfrac{A'}{\sqrt{A'^2+B'^2}}D \\ \psi = \varphi \mp \epsilon B D = \varphi \mp \epsilon\dfrac{B'}{\sqrt{A'^2+B'^2}}D \\ D = \dfrac{\mp 2\epsilon\left(A\alpha + B\varphi + C\right)}{1 \mp \epsilon^2\left(A^2+B^2\right)} = \dfrac{\mp 2\epsilon}{1 \mp \epsilon^2}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} \quad A^2+B^2 = 1 \end{cases}$$

actually, only one of two solutions is true

$$\begin{cases} \chi = \alpha - \epsilon A D = \alpha - \epsilon\dfrac{A'}{\sqrt{A'^2+B'^2}}D = \alpha - \dfrac{2\epsilon^2}{\epsilon^2-1}\dfrac{A'^2\alpha + A'B'\varphi + A'C'}{A'^2+B'^2} \\ \psi = \varphi - \epsilon B D = \varphi - \epsilon\dfrac{B'}{\sqrt{A'^2+B'^2}}D = \varphi - \dfrac{2\epsilon^2}{\epsilon^2-1}\dfrac{A'B'\alpha + B'^2\varphi + B'C'}{A'^2+B'^2} \\ D = \dfrac{-2\epsilon\left(A\alpha + B\varphi + C\right)}{1 - \epsilon^2\left(A^2+B^2\right)} = \dfrac{-2\epsilon}{1-\epsilon^2}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} = \dfrac{2\epsilon}{\epsilon^2-1}\dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2+B'^2}} \end{cases}$$

$$\begin{cases} \chi = \dfrac{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right)}{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)} \\ \psi = \dfrac{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\varphi - 2\epsilon^2\left(A'B'\alpha + B'^2\varphi + B'C'\right)}{\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)} \\ \left|\dfrac{D}{d\left(F,L\right)}\right| = \left|\dfrac{2\epsilon}{1-\epsilon^2}\right| \Rightarrow \left(\dfrac{D}{d\left(F,L\right)}\right)^2 = \left(\dfrac{2\epsilon}{1-\epsilon^2}\right)^2 \end{cases}$$

$$\begin{aligned} &\left(\epsilon^2-1\right)\left(A'^2+B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right) \\ &= \left(-\left(\epsilon^2+1\right)A'^2 + \left(\epsilon^2-1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \\ &= \left(-\left(\epsilon^2+1\right)A'^2 + \left(\epsilon^2-1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \end{aligned}$$

Can the above be more simplified?

$$
\begin{aligned}
\overline{FF'}^2 &= (\alpha-\chi)^2+(\varphi-\psi)^2\\
&= (\alpha-(\alpha-\epsilon AD))^2+(\varphi-(\varphi-\epsilon BD))^2\\
&= (\epsilon D)^2\left(A^2+B^2\right)\\
&= (\epsilon D)^2
\end{aligned}
$$

### 20.2.3 eccentricity and its equivalent representation

$$
\left(\frac{c}{a}\right)^2=\left(\frac{\overline{PF}}{d\,(P,L)}\right)^2=\epsilon^2=\left(\frac{\overline{FF'}}{D}\right)^2=\left(\frac{2c}{D}\right)^2\Rightarrow D=2a
$$

$$
\left(\frac{D}{d\,(F,L)}\right)^2=\left(\frac{2\epsilon}{1-\epsilon^2}\right)^2
$$

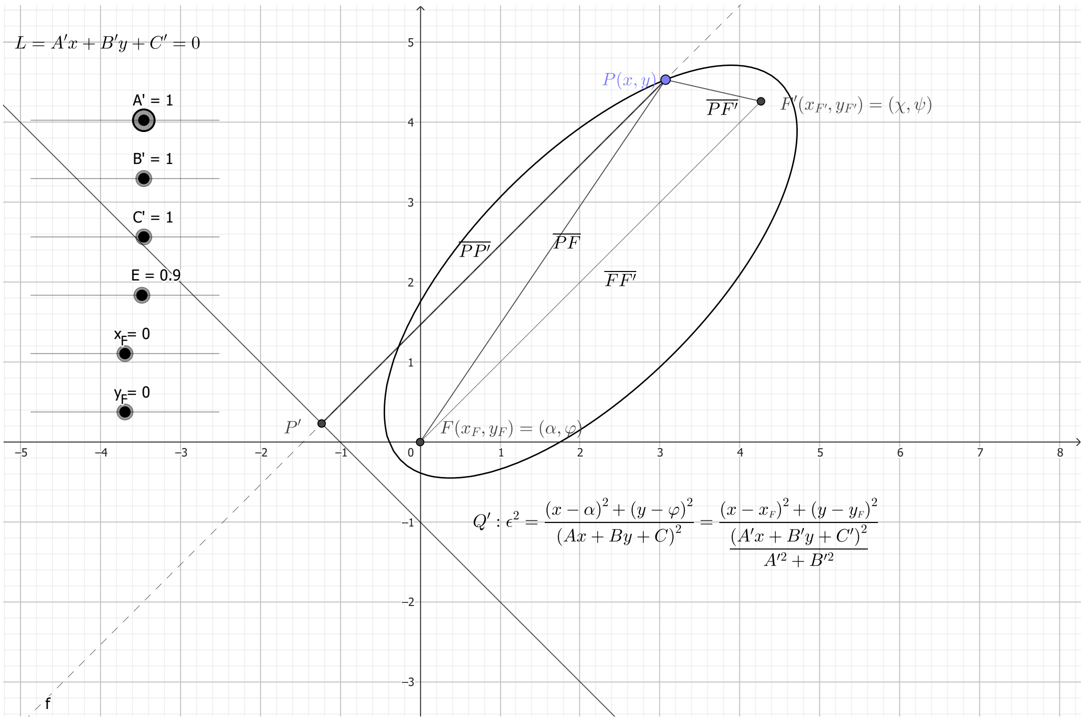


Figure 20.3: conic sections: ellipse

## 20.3 Cartesian coordinate: standard form / standard equation

$$
\begin{array}{llll}
\text{circle} & \left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2 = 1 & & b=a\\
\text{ellipse} & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 = 1 & \text{vertical} & b>a\\
 & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 = 1 & \text{horizontal} & a>b\\
\text{parabola} & (y-k)-4c\,(x-h)^2 = 0 & \text{vertical} & \\
 & -4c\,(y-k)^2+(x-h) = 0 & \text{horizontal} & \\
\text{hyperbola} & \left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2 = 1 & \text{vertical} & \frac{x-h}{a}=0\Rightarrow\frac{y-k}{b}=\pm1\\
 & -\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 = 1 & \text{horizontal} & \frac{y-k}{b}=0\Rightarrow\frac{x-h}{a}=\pm1
\end{array}
$$

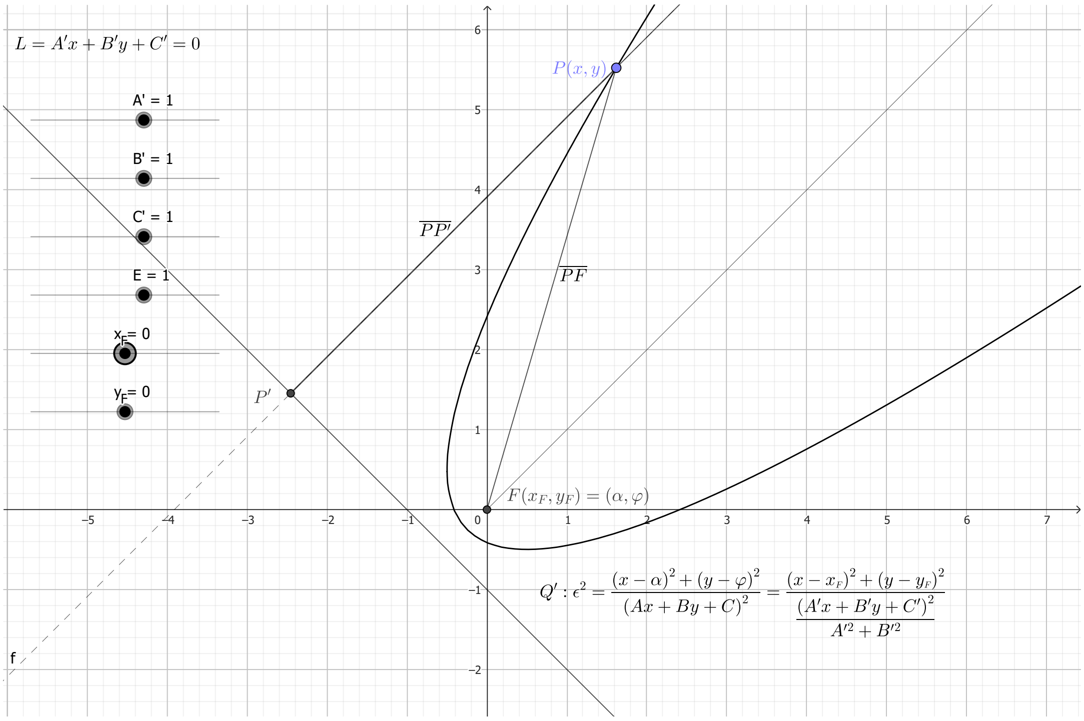


Figure 20.4: conic sections: parabola

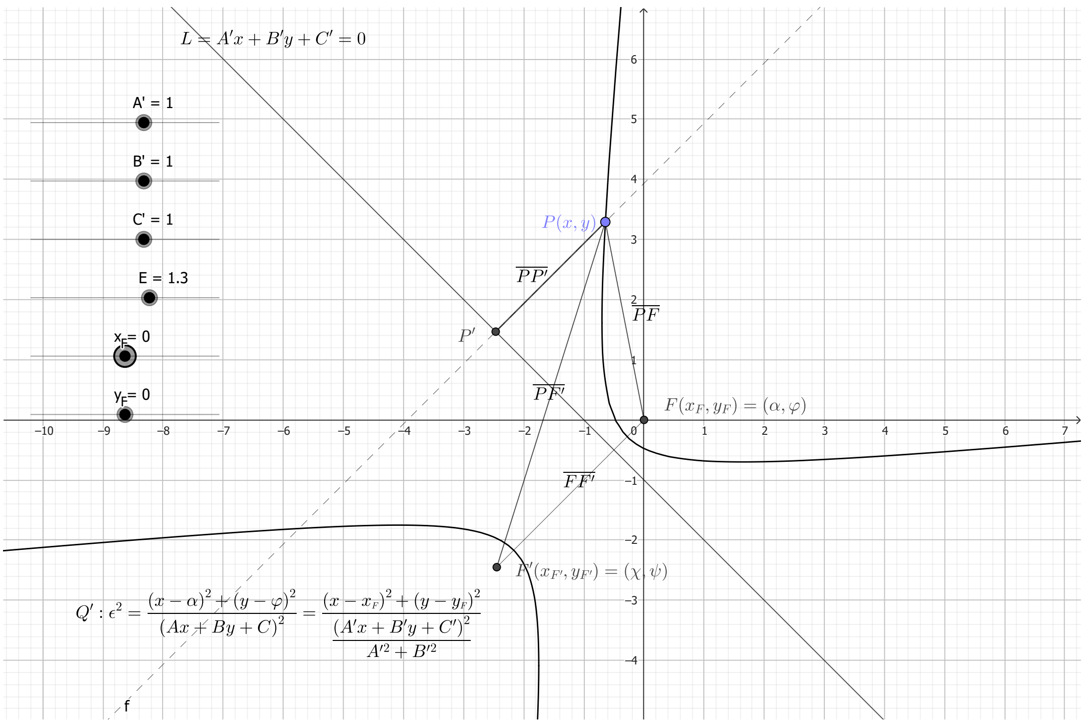


Figure 20.5: conic sections: hyperbola

## 20.4 parametric equation

$$
\begin{array}{clcccccc}
\text{circle} & \left(\frac{y-k}{a}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&a&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix} & = \begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\a\\1\end{pmatrix} \\
\text{ellipse} & \left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\cos t\\\sin t\\1\end{pmatrix} & = \begin{pmatrix}\cos t&0&h\\0&\sin t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix} \\
\text{parabola} & (y-k)-4c\,(x-h)^2 & =0 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}1&0&h\\0&4c&k\\0&0&1\end{pmatrix} & \begin{pmatrix}t\\t^2\\1\end{pmatrix} & = \begin{pmatrix}t&0&h\\0&t^2&k\\0&0&1\end{pmatrix} & \begin{pmatrix}1\\4c\\1\end{pmatrix} \\
 & -4c\,(y-k)^2+(x-h) & =0 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}4c&0&h\\0&1&k\\0&0&1\end{pmatrix} & \begin{pmatrix}t^2\\t\\1\end{pmatrix} & = \begin{pmatrix}t^2&0&h\\0&t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}4c\\1\\1\end{pmatrix} \\
\text{hyperbola} & \left(\frac{y-k}{b}\right)^2-\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\pm\cosh t\\\sinh t\\1\end{pmatrix} & = \begin{pmatrix}\tan t&0&h\\0&\sec t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix} \\
 & -\left(\frac{y-k}{b}\right)^2+\left(\frac{x-h}{a}\right)^2 & =1 & \begin{pmatrix}x\\y\\1\end{pmatrix} = & \begin{pmatrix}a&0&h\\0&b&k\\0&0&1\end{pmatrix} & \begin{pmatrix}\sinh t\\\pm\cosh t\\1\end{pmatrix} & = \begin{pmatrix}\sec t&0&h\\0&\tan t&k\\0&0&1\end{pmatrix} & \begin{pmatrix}a\\b\\1\end{pmatrix}
\end{array}
$$

tangent half-angle formula[23]

## 20.5 polar coordinate

$$(x-\alpha)^2+(y-\varphi)^2=[\epsilon\,(Ax+By+C)]^2$$

$\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$

$$(r\cos\theta-\alpha)^2+(r\sin\theta-\varphi)^2=[\epsilon\,(Ar\cos\theta+Br\sin\theta+C)]^2$$

If $\begin{cases}F=(x_F,y_F)=(\alpha,\varphi)=(0,0)\\L=Ax+By+C=x+p=0\end{cases}$

$$
\begin{aligned}
(r\cos\theta)^2+(r\sin\theta)^2 &=[\epsilon\,(r\cos\theta+p)]^2\\
r^2 &=\\
r &=\pm\,\epsilon\,(r\cos\theta+p)\\
&=\pm\,(r\epsilon\cos\theta+\epsilon p)\\
r\,(1\mp\epsilon\cos\theta) &=\epsilon p\\
r &=\frac{\epsilon p}{1\mp\epsilon\cos\theta}
\end{aligned}
$$

https://www.geogebra.org/calculator/azksjxbq

$r=\dfrac{\epsilon p}{1-\epsilon\cos\theta}$ will not cross $L=x+p=0$ on graphs, so maybe it is a more correct solution

$$r=\frac{\epsilon p}{1-\epsilon\cos\theta}$$

## 20.6 Cartesian coordinate: general form / quadratic equation

https://ccjou.wordpress.com/2013/05/24/圓錐曲線/

https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections

$$ax^2+bxy+cy^2+dx+ey+f=0$$

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}ax+(b/2)\,y\\(b/2)\,x+cy\end{pmatrix}=ax^2+bxy+cy^2$$

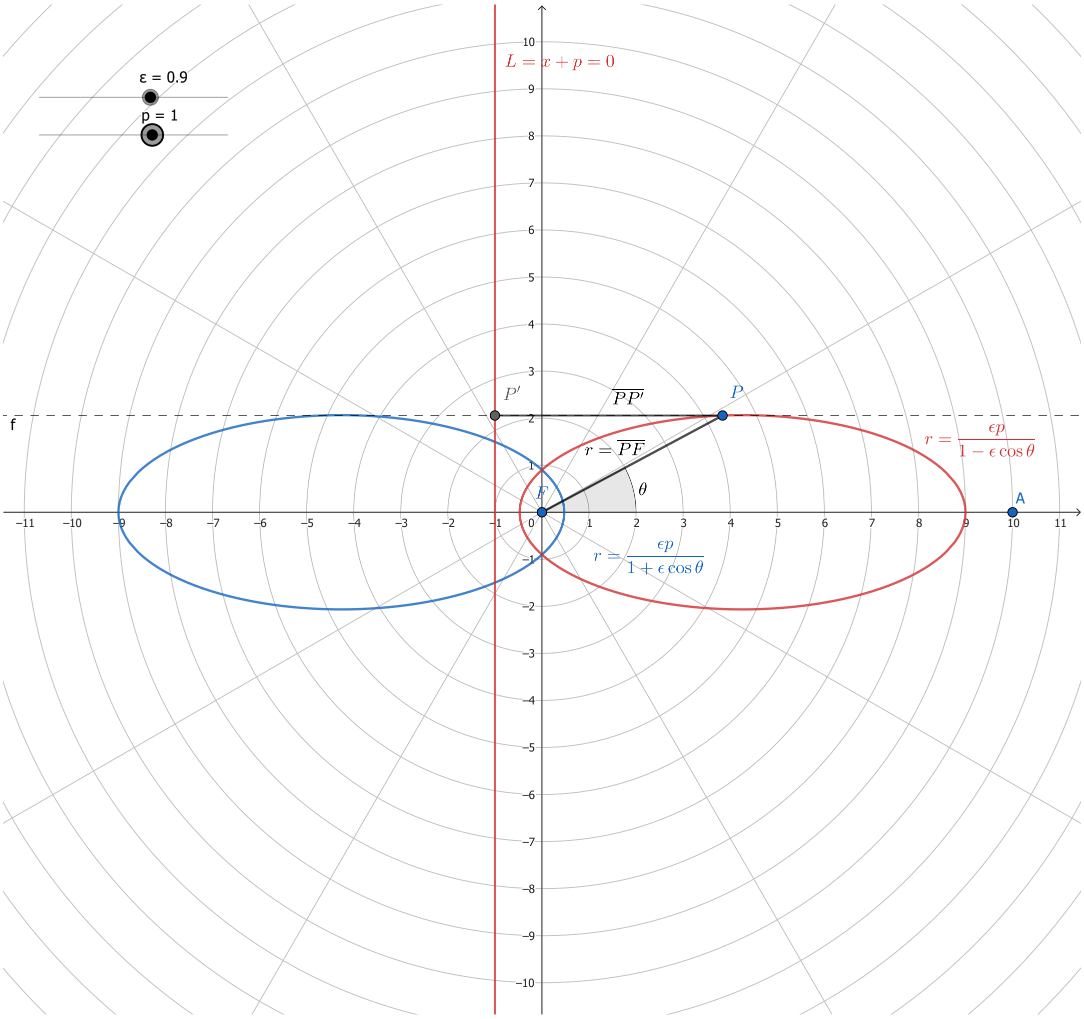


Figure 20.6: polar conic sections: ellipse

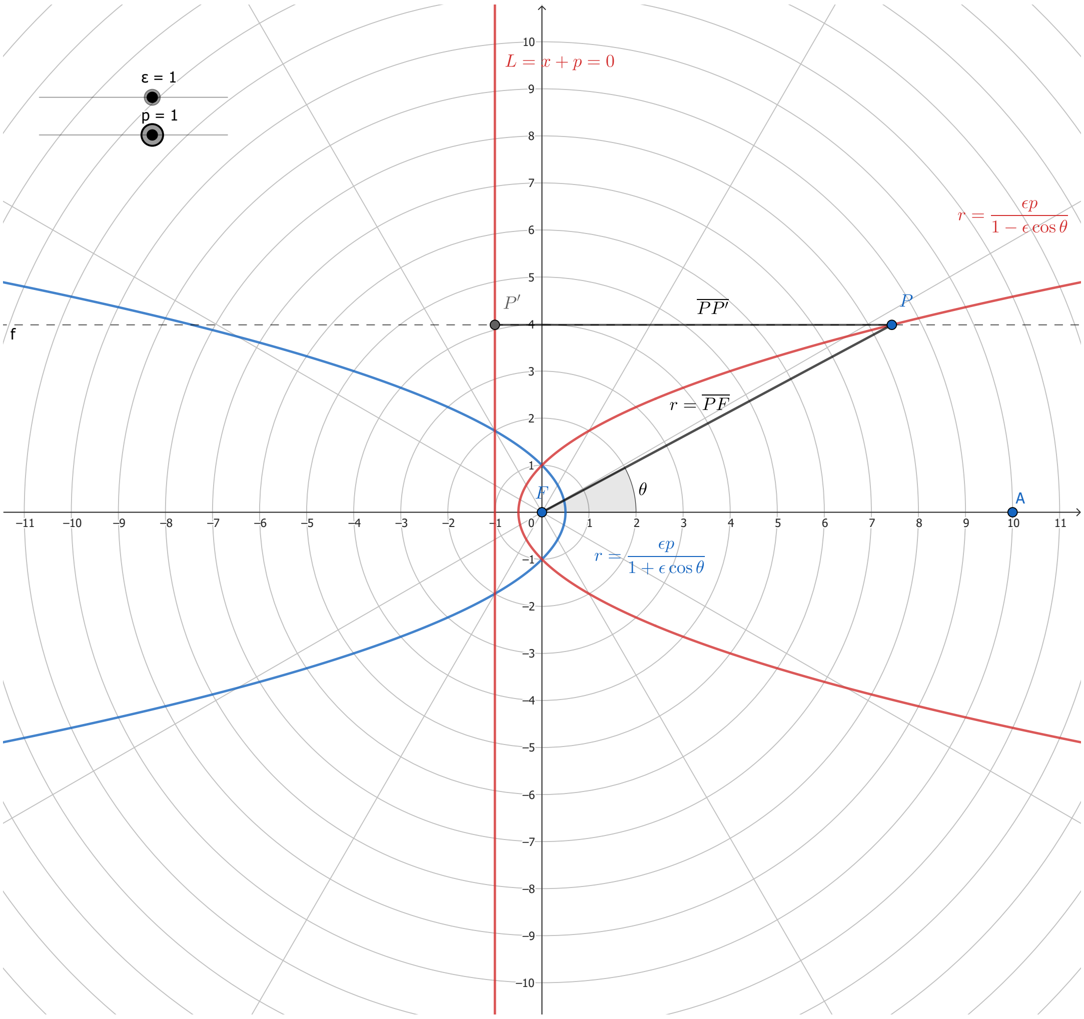


Figure 20.7: polar conic sections: parabola

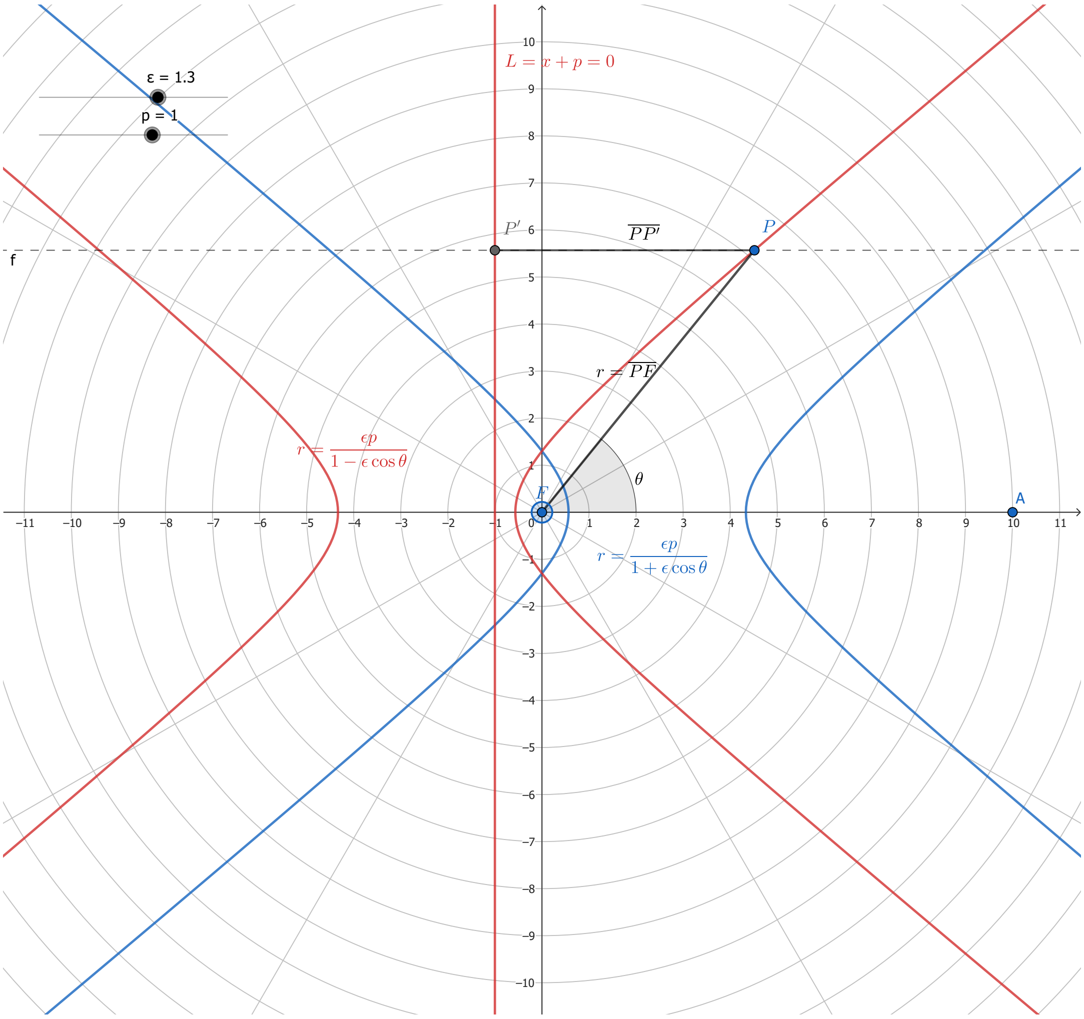


Figure 20.8: polar conic sections: hyperbola

$$0 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f$$

$$= \boldsymbol{x}^\intercal A \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + f, \begin{cases} A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} & A \text{ real symmetric} \\ \boldsymbol{b} = \begin{pmatrix} d \\ e \end{pmatrix} \\ \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases}$$

real symmetric matrix diagonalizable[22]

## 20.7 homogeneous coordinate

X homogeneous coordinate

homogeneous coordinate O: HTML, X: PDF becoming web link

O homogeneous coordinate[24]

X homogeneous coordinate

X homogeneous coordinate[20.7]

https://ccjou.wordpress.com/2013/05/24/圓錐曲線/

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & \\ b/2 & c & \\ & & \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \kappa \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \alpha x + \beta y + \gamma \\ \delta x + \epsilon y + \zeta \\ \eta x + \theta y + \kappa \end{pmatrix}, \begin{cases} \gamma + \eta = d \\ \zeta + \theta = e \end{cases} \\ &= \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \zeta \\ \eta & \theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & d/2 \\ 0 & 0 & e/2 \\ d/2 & e/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 0 =& ax^2 + bxy + cy^2 + dx + ey + f \\ =& \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = \boldsymbol{x}^\intercal A \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + f \\ =& \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^\intercal & 1 \end{pmatrix} M \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 0 =& ax^2 + bxy + cy^2 + dx + ey + f \\ =& \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = \boldsymbol{x}^\intercal A \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + f \\ =& \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^\intercal & 1 \end{pmatrix} M \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \end{aligned}$$

https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections

$$\begin{aligned} 0 = Q =& Ax^2 + Bxy + Cy^2 + Dx + Ey + F \\ =& \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boldsymbol{x}_\text{h}^\intercal A_Q \boldsymbol{x}_\text{h} \\ =& \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = \boldsymbol{x}^\intercal A_{Q,33} \boldsymbol{x} + \boldsymbol{b}^\intercal \boldsymbol{x} + F \end{aligned}$$

## 20.8 TalyorCatAlice: projective geometry

https://www.bilibili.com/video/BV1pK42117UZ

https://www.bilibili.com/video/BV1cx4y1f7w6

https://www.bilibili.com/video/BV1sy421h7aF

https://www.bilibili.com/video/BV1zv421y7iH

https://www.bilibili.com/video/BV1ZH4y1h7vC

## 20.9 double cone

https://space.bilibili.com/87052889/channel/collectiondetail?sid=1691820

## 20.10 theorem proof without analytic method

https://www.zhihu.com/question/470672139/answer/2873265380

# Chapter 21

# distance from a point to a line

點到直線距離

**Theorem 21.1.**

$$\begin{cases} P = P\left(x_0, y_0\right) \\ L = L\left(x, y\right) = Ax + By + C = 0, A^2 + B^2 \neq 0 \end{cases}$$
$$\Downarrow$$
$$d\left(P, L\right) = \frac{\left|Ax_0 + By_0 + C\right|}{\sqrt{A^2 + B^2}}$$

https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line

https://highscope.ch.ntu.edu.tw/wordpress/?p=47407

https://web.math.sinica.edu.tw/mathmedia/HTMLarticle18.jsp?mID=40312

Proofs:

## 21.1 by shortest $\overline{PP'}$

$$\begin{aligned} & P' = P'\left(x, y\right) \in L = Ax + By + C = 0 \\ \Rightarrow & y = \frac{-1}{B}\left(Ax + C\right) \end{aligned}$$

$$\begin{aligned} \overline{PP'}^2\left(x, y\right) &= \left(x_0 - x\right)^2 + \left(y_0 - y\right)^2 \\ &= \left(x_0 - x\right)^2 + \left(y_0 - \frac{-1}{B}\left(Ax + C\right)\right)^2 \\ &= \left(x - x_0\right)^2 + \left(\frac{A}{B}x + \frac{C}{B} + y_0\right)^2 = \overline{PP'}^2\left(x\right) \end{aligned}$$

$$\begin{aligned} 0 = \frac{\partial}{\partial x}\overline{PP'}^2\left(x\right) =& 2\left(x - x_0\right) + 2\left(\frac{A}{B}x + \frac{C}{B} + y_0\right)\frac{A}{B} \\ =& \frac{2}{B^2}\left(B^2\left(x - x_0\right) + A^2x + AC + ABy_0\right) \\ =& \frac{2}{B^2}\left[\left(A^2 + B^2\right)x - \left(B^2x_0 - ABy_0 - AC\right)\right] \\ x =& \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} \end{aligned}$$

or by completing the square to find $x$.

$$\begin{aligned}
&\overline{PP'}^2\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2}-x_0\right)^2+\left(\frac{A}{B}\frac{B^2x_0-ABy_0-AC}{A^2+B^2}+\frac{C}{B}+y_0\right)^2\\
=&\left(\frac{-A^2x_0-ABy_0-AC}{A^2+B^2}\right)^2+\left(\frac{A\left(B^2x_0-ABy_0-AC\right)+C\left(A^2+B^2\right)+B\left(A^2+B^2\right)y_0}{B\left(A^2+B^2\right)}\right)^2\\
=&\left(\frac{-A\left(Ax_0+By_0+C\right)}{A^2+B^2}\right)^2+\left(\frac{AB^2x_0+B^3y_0+B^2C}{B\left(A^2+B^2\right)}\right)^2\\
=&\frac{A^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}+\frac{B^2\left(Ax_0+By_0+C\right)^2}{\left(A^2+B^2\right)^2}\\
=&\frac{\left(Ax_0+By_0+C\right)^2}{A^2+B^2}\\
\overline{PP'}=&\overline{PP'}\left(x=\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

## 21.2 by perpendicular foot

$$y=\frac{-A}{B}x-\frac{C}{B}=\frac{-1}{B}\left(Ax+C\right),\ \text{if } B\neq 0$$

$$L_\perp:\left(y=\frac{B}{A}x+K\right)\perp\left(y=\frac{-A}{B}x-\frac{C}{B}\right):L$$

$$L_\perp=L_\perp\left(x,y\right)=Bx-Ay+K=0$$

$$P=P\left(x_0,y_0\right)\in L_\perp=B\left(x-x_0\right)-A\left(y-y_0\right)=0$$

$$L_\perp=Bx-Ay-\left(Bx_0-Ay_0\right)=0$$

perpendicular foot = foot of the perpendicular $P'$

$$\begin{aligned}
P'\in\left(L_\perp\cap L\right)=&\begin{cases}L=Ax+By+C=0\\ L_\perp=Bx-Ay-\left(Bx_0-Ay_0\right)=0\end{cases}\\
=&\begin{cases}Ax+By=-C\\ Bx-Ay=Bx_0-Ay_0\end{cases}\\
P'=P'\left(x,y\right)=&\left(\frac{\begin{vmatrix}-C & B\\ Bx_0-Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}},\frac{\begin{vmatrix}A & -C\\ B & Bx_0-Ay_0\end{vmatrix}}{\begin{vmatrix}A & B\\ B & -A\end{vmatrix}}\right)\\
=&\left(\frac{\begin{vmatrix}C & B\\ -Bx_0+Ay_0 & -A\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}},\frac{\begin{vmatrix}A & C\\ B & -Bx_0+Ay_0\end{vmatrix}}{\begin{vmatrix}A & -B\\ B & A\end{vmatrix}}\right)\\
=&\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)
\end{aligned}$$

$$\begin{aligned}
&d(P,L)=\overline{PP'}\\
=&\left\|(x_0,y_0)-\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)\right\|\\
=&\sqrt{\left(x_0-\frac{B^2x_0-ABy_0-AC}{A^2+B^2}\right)^2+\left(y_0-\frac{-ABx_0+A^2y_0-BC}{A^2+B^2}\right)^2}\\
=&\sqrt{\left(\frac{A^2x_0+ABy_0+AC}{A^2+B^2}\right)^2+\left(\frac{ABx_0+B^2y_0+BC}{A^2+B^2}\right)^2}\\
=&\sqrt{\frac{A^2(Ax_0+By_0+C)^2+B^2(Ax_0+By_0+C)^2}{(A^2+B^2)^2}}=\sqrt{\frac{(Ax_0+By_0+C)^2}{A^2+B^2}}\\
=&\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

## 21.3 by normal vector

$$\begin{cases}\overrightarrow{n}=(A,B)\perp L=Ax+By+C=0\\ \overrightarrow{PP'}=P'-P=(x-x_0,y-y_0)\end{cases}$$

$P$到$L$的距離$d(P,L)$即為$L$線上一點$P'$對應之$\overrightarrow{PP'}$在$L$法向量$\overrightarrow{n}$方向上的投影長

$$\begin{aligned}
\overrightarrow{PP'}\cdot\overrightarrow{n}&=\left\|\overrightarrow{PP'}\right\|\left\|\overrightarrow{n}\right\|\cos\theta\\
\left|\overrightarrow{PP'}\cdot\overrightarrow{n}\right|&=\left\|\overrightarrow{PP'}\right\|\left\|\overrightarrow{n}\right\||\cos\theta|\\
\left\|\overrightarrow{PP'}\right\||\cos\theta|=\left|\overrightarrow{PP'}\cdot\widehat{\boldsymbol{n}}\right|&=\frac{\left|\overrightarrow{PP'}\cdot\overrightarrow{n}\right|}{\left\|\overrightarrow{n}\right\|}=\frac{|(x-x_0,y-y_0)\cdot(A,B)|}{\|(A,B)\|}\\
&=\frac{|A(x-x_0)+B(y-y_0)|}{\sqrt{A^2+B^2}}=\frac{|-Ax_0-By_0+Ax+By|}{\sqrt{A^2+B^2}}\\
&\overset{Ax+By+C=0}{\underset{Ax+By=-C}{=}}\frac{|-Ax_0-By_0-C|}{\sqrt{A^2+B^2}}=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

PDF LaTeX `\usepackage{fdsymbol}` to have `\overrightharpoon` vector; however, there are too many side effects, including ugly mathptmx $\sum$, . . .

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}
```

## 21.4 by Cauchy inequality

$$\begin{aligned}
Ax+By+C&=0\\
Ax+By&=-C\\
(Ax+By)-(Ax_0+By_0)&=-C-(Ax_0+By_0)\\
A(x-x_0)+B(y-y_0)&=-(Ax_0+By_0+C)
\end{aligned}$$

$$
\begin{aligned}
\overline{PP'}^2 &= (x_0 - x)^2 + (y_0 - y)^2 \\
\left[A^2 + B^2\right] \overline{PP'}^2 &= \left[A^2 + B^2\right] \left[(x_0 - x)^2 + (y_0 - y)^2\right] \\
&\geq \left[A (x - x_0) + B (y - y_0)\right]^2 \\
&= \left[- (Ax_0 + By_0 + C)\right]^2 = (Ax_0 + By_0 + C)^2 \\
\overline{PP'}^2 &\geq \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2} \\
\overline{PP'} &\geq \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}
$$

# Chapter 22

# real symmetric matrix diagonalizable

https://ccjou.wordpress.com/2011/02/09/實對稱矩陣可正交對角化的證明/

https://tex.stackexchange.com/questions/30619/what-is-the-best-symbol-for-vector-matrix-transpose

**Theorem 22.1.**

實對稱矩陣的特徵值皆是實數，且對應特徵向量是實向量。

$$\begin{cases}\begin{cases}A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix}\\ A^\intercal = A & \text{symmetric matrix}\end{cases} & \text{real symmetric matrix}\\ A\boldsymbol{x} = \lambda\boldsymbol{x} & \begin{cases}\lambda \in \mathbb{C} & \text{complex eigenvalue}\\ \boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{C}^n & \text{complex eigenvector}\end{cases}\end{cases}$$

$$\Downarrow$$

$$\begin{cases}\lambda \in \mathbb{R} & \text{real eigenvalue (1)}\\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector (2)}\end{cases}$$

*Proof.* (1)

$$\begin{aligned}
A\boldsymbol{x} =& \lambda\boldsymbol{x}\\
\overline{A}\overline{\boldsymbol{x}} = \overline{A\boldsymbol{x}} =& \overline{\lambda\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}\\
\overline{\boldsymbol{x}}^\intercal\overline{A}^\intercal = \left(\overline{A}\overline{\boldsymbol{x}}\right)^\intercal = \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^\intercal =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\\
\overline{\boldsymbol{x}}^\intercal A \overset{\text{symmetric}}{=} \overline{\boldsymbol{x}}^\intercal A^\intercal \overset{\text{real}}{=} &\\
\overline{\boldsymbol{x}}^\intercal A =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\\
\lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} = \overline{\boldsymbol{x}}^\intercal(\lambda\boldsymbol{x}) \underset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{\overset{\cdot\boldsymbol{x}}{=}} \overline{\boldsymbol{x}}^\intercal A\boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\lambda\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} =& \overline{\lambda}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x}\\
\left(\lambda - \overline{\lambda}\right)\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} =& 0 \wedge \begin{cases}\overline{\boldsymbol{x}}^\intercal\boldsymbol{x} = \sum_{i=1}^n |x_i|^2\\ \boldsymbol{x} \neq \boldsymbol{0}\end{cases} \Rightarrow \overline{\boldsymbol{x}}^\intercal\boldsymbol{x} \neq 0\\
\lambda - \overline{\lambda} =& 0\\
\lambda =& \overline{\lambda} \Leftrightarrow \lambda \in \mathbb{R}
\end{aligned}$$

□

*Proof.* (1) fast concept

$$\left(\overline{A\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\left(\overline{\boldsymbol{x}}^{\intercal}\overline{A}^{\intercal}\right)\boldsymbol{x}\overset{\text{symmetric}}{=}\left(\overline{\boldsymbol{x}}^{\intercal}\overline{A}\right)\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)$$
$$(L)=\left(\overline{A\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)=(R)$$
$$(L)=\left(\overline{A\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}\overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=}\left(\overline{\lambda\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$
$$(R)=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)\overset{\text{real}}{=}\overline{\boldsymbol{x}}^{\intercal}\left(A\boldsymbol{x}\right)\overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=}\overline{\boldsymbol{x}}^{\intercal}\left(\lambda\boldsymbol{x}\right)=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$
$$\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}=\left(\overline{A\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$
$$\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$

□

*Proof.* (2)

???

推論特徵空間 $N(A-\lambda I)$ ($A-\lambda I$ 的零空間) 為 $\mathbb{R}^n$ 的子空間，故 $\boldsymbol{x}\in N(A-\lambda I)$ 是一個非零實向量。

□

**Theorem 22.2.**

實對稱矩陣對應相異特徵值的特徵向量互為正交。

$$\begin{cases}\begin{cases}A\in\mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix}\\ A^{\intercal}=A & \text{symmetric matrix}\end{cases} & \text{real symmetric matrix}\\ A\boldsymbol{x}=\lambda\boldsymbol{x} & 22.1\begin{cases}\lambda\in\mathbb{R} & \text{real eigenvalue}\\ \boldsymbol{x}\in\mathbb{R}^n & \text{real eigenvector}\end{cases}\\ \begin{cases}A\boldsymbol{x}_1=\lambda_1\boldsymbol{x}_1 & (e_1)\\ A\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_2 & (e_2)\end{cases} & \lambda_1\neq\lambda_2\end{cases}$$
$$\Downarrow$$
$$\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=0\Leftrightarrow\boldsymbol{x}_1\perp\boldsymbol{x}_2$$

*Proof.* (1)

$$A\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_2$$
$$\boldsymbol{x}_1^{\intercal}A\boldsymbol{x}_2\overset{\boldsymbol{x}_1^{\intercal}\cdot}{=}\boldsymbol{x}_1^{\intercal}\lambda_2\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=(1)$$
$$A\boldsymbol{x}_1=\lambda_1\boldsymbol{x}_1$$
$$\boldsymbol{x}_1^{\intercal}A^{\intercal}=(A\boldsymbol{x}_1)^{\intercal}=(\lambda_1\boldsymbol{x}_1)^{\intercal}=\lambda_1\boldsymbol{x}_1^{\intercal}$$
$$\boldsymbol{x}_1^{\intercal}A^{\intercal}=\lambda_1\boldsymbol{x}_1^{\intercal}$$
$$\boldsymbol{x}_1^{\intercal}A\boldsymbol{x}_2\overset{\text{symmetric}}{=}\boldsymbol{x}_1^{\intercal}A^{\intercal}\boldsymbol{x}_2\overset{\cdot\boldsymbol{x}_2}{=}\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=(2)$$
$$\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2\overset{(1)}{=}\boldsymbol{x}_1^{\intercal}A\boldsymbol{x}_2\overset{(2)}{=}\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$
$$\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$
$$(\lambda_2-\lambda_1)\,\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=0\wedge\lambda_1\neq\lambda_2$$
$$\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=0$$

□

*Proof.* (1) fast concept

$$(A\boldsymbol{x}_1)^{\intercal}\boldsymbol{x}_2=(\boldsymbol{x}_1^{\intercal}A^{\intercal})\,\boldsymbol{x}_2\overset{\text{symmetric}}{=}(\boldsymbol{x}_1^{\intercal}A)\,\boldsymbol{x}_2=\boldsymbol{x}_1^{\intercal}(A\boldsymbol{x}_2)$$
$$(L)=(A\boldsymbol{x}_1)^{\intercal}\boldsymbol{x}_2=\boldsymbol{x}_1^{\intercal}(A\boldsymbol{x}_2)=(R)$$
$$(L)=(A\boldsymbol{x}_1)^{\intercal}\boldsymbol{x}_2\overset{(e_1)}{=}(\lambda_1\boldsymbol{x}_1)^{\intercal}\boldsymbol{x}_2=\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$
$$(R)=\boldsymbol{x}_1^{\intercal}(A\boldsymbol{x}_2)\overset{(e_2)}{=}\boldsymbol{x}_1^{\intercal}(\lambda_2\boldsymbol{x}_2)=\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$
$$\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=(A\boldsymbol{x}_1)^{\intercal}\boldsymbol{x}_2=\boldsymbol{x}_1^{\intercal}(A\boldsymbol{x}_2)=\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$
$$\lambda_1\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2$$

□

**Theorem 22.3.**

$$\begin{cases} \begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix} \\ A^\intercal = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\ A\boldsymbol{x}_1 = \lambda \boldsymbol{x}_1 & (e) \\ \boldsymbol{x}_2^\intercal \boldsymbol{x}_1 = 0 \Leftrightarrow \boldsymbol{x}_2 \perp \boldsymbol{x}_1 & (o) \end{cases}$$
$$\Downarrow$$
$$A\boldsymbol{x}_2 \perp \boldsymbol{x}_1 \Leftrightarrow (A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 = 0$$

*Proof.*

$$\begin{aligned} (A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 &= (\boldsymbol{x}_2^\intercal A^\intercal)\,\boldsymbol{x}_1 \overset{\text{symmetric}}{=} (\boldsymbol{x}_2^\intercal A)\,\boldsymbol{x}_1 \\ &= \boldsymbol{x}_2^\intercal (A\boldsymbol{x}_1) \overset{(e)}{=} \boldsymbol{x}_2^\intercal (\lambda \boldsymbol{x}_1) \\ &= \lambda \boldsymbol{x}_2^\intercal \boldsymbol{x}_1 \overset{(o)}{=} \lambda \cdot 0 = 0 \\ (A\boldsymbol{x}_2)^\intercal \boldsymbol{x}_1 &= 0 \Leftrightarrow A\boldsymbol{x}_2 \perp \boldsymbol{x}_1 \end{aligned}$$

□

# Chapter 23

# tangent half-angle formula

https://en.wikipedia.org/wiki/Tangent_half-angle_formula

https://zh.wikipedia.org/zh-tw/正切半角公式

正切半形公式又稱萬能公式

以切表弦公式，簡稱以切表弦

# Chapter 24

# homogeneous coordinate

## 24.1 Cem Yuksel

https://youtu.be/EKN7dTJ4ep8?si=8woajZxbqPfEXhdK&t=2263

https://youtu.be/1z1S2kQKXDs?si=71o339yBtIQYhWtj&t=3082

# Chapter 25

# Archimedean property

## 25.1 integer Archimedean property

## 25.2 rational Archimedean property

https://math.stackexchange.com/questions/3699023/proof-the-the-field-of-rational-numbers-has-the-archimedean-property

https://math.stackexchange.com/questions/1919829/proving-the-archimedean-properties-of-rational-numbers

## 25.3 real Archimedean property

https://www.youtube.com/watch?v=6wBVk1fBKXQ&t=2326

# Chapter 26

# survial analysis

## 26.1 Python package `tableone`

## 26.2 Python package `lifelines`

# Chapter 27

# Manim

## 27.1 VSCode extension: Manim Sideview

https://marketplace.visualstudio.com/items?itemName=Rickaym.manim-sideview

`ffmpeg.exe` placed in the same folder with `.py`

VSCode Ctrl + Shift + P: `open Mobject gallery`

## 27.2 installation

https://docs.manim.community/en/stable/installation.html

### 27.2.1 Conda

`conda install -c conda-forge manim`

## 27.3 quickstart

https://docs.manim.community/en/stable/tutorials/quickstart.html

https://www.w3schools.com/tags/att_video_autoplay.asp

https://www.w3schools.com/tags/att_video_loop.asp

```
from manim import *

class CreateCircle(Scene):
    def construct(self):
        circle = Circle()  # create a circle
        circle.set_fill(PINK, opacity=0.5)  #
↪  set the color and transparency
        self.play(Create(circle))  # show the
            ↪  circle on screen
```

`manim -pql scene.py CreateCircle`

# Chapter 28

# ggplot2

https://bookdown.org/xiangyun/msg/system.html#chap:system

Modern Statistical Graphics section 5.1

- https://www.rdocumentation.org/ to search function
- ggplot2
  - https://ggplot2.tidyverse.org/index.html to search ggplot2 function
  - panel = layer
    * geom = geometric objects / geometry = element
      · element
    * statistic
    * scale
    * coordinate system
    * facet

```
library(ggplot2)

p <- ggplot(aes(x = hp, y = mpg), data =
↪  mtcars) +
  geom_point() # layer of scatterplot
p + geom_smooth(method = "loess") # add layer
↪  of smooth
```

```
## `geom_smooth()` using formula = 'y ~ x'
```

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## 28.1 geom

https://bookdown.org/xiangyun/msg/system.html#section-13

```
library(ggplot2)

ggplot(aes(x = hp, y = mpg), data = mtcars) +
  geom_point() +
  geom_smooth(method = "loess")
```

```
## `geom_smooth()` using formula = 'y ~ x'
```

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points https://ggplot2.tidyverse.org/reference/geom_point.html?q=geom_point#null

`geom_point`

smoothed conditional means https://ggplot2.tidyverse.org/reference/geom_smooth.html?q=geom_sm#null

> Aids the eye in seeing patterns in the presence of overplotting. `geom_smooth()` and `stat_smooth()` are effectively aliases: they both use the same arguments. Use `stat_smooth()` if you want to display the results with a non-standard geom.

`geom_smooth`

`stat_smooth`

> `method` Smoothing method (function) to use, accepts either NULL or a character vector, e.g. `"lm"`, `"glm"`, `"gam"`, `"loess"` or a function, e.g. `MASS::rlm` or `mgcv::gam`, `stats::lm`, or `stats::loess`. `"auto"` is also accepted for backwards compatibility. It is equivalent to `NULL`.

---

```
# install.packages("hexbin")
```

```
library(ggplot2)

ggplot(aes(x = carat, y = price), data =
↪  diamonds) +
  geom_hex() +
  scale_fill_gradient(low = "blue3", high =
  ↪  "red3")
```

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gradient color scales https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradient#ref-usage

https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradient#ref-examples

---

```
library(ggplot2)

ggplot(aes(x = Petal.Length, y = Petal.Width),
↪  data = iris) +
  geom_point(aes(color = Species, shape =
  ↪  Species))
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

---

basic plot system

https://bookdown.org/xiangyun/msg/elements.html#sec:points

```
# iris species converted to type integer 1, 2,
↪  3 for further using vectors
idx <- as.integer(iris[["Species"]])
plot(iris[, 3:4],
  pch = c(24, 21, 25)[idx],
  col = c("black", "red", "blue")[idx],
  ↪  panel.first = grid()
)
legend("topleft",
  legend = levels(iris[["Species"]]),
  col = c("black", "red", "blue"), pch = c(24,
  ↪  21, 25), bty = "n"
)
```

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`plot` https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/plot.default

> `pch` a vector of **plotting characters** or symbols: see `points`.
>
> `col` The **colors** for lines and points. Multiple colors can be specified so that each point can be given its own color. If there are fewer colors than points they are recycled in the standard fashion. Lines will all be plotted in the first colour specified.
>
> `panel.first` an 'expression' to be evaluated after the plot axes are set up but before any plotting takes place. This can be useful for drawing **background grids** or **scatterplot smooths**. Note that this works by lazy evaluation: passing this argument from other plot methods may well not work since it may be evaluated too early.

`legend` https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/legend

> `bty` the **type of box** to be drawn around the legend. The allowed values are "o" (the default) and "n".

https://stackoverflow.com/questions/10108073/plot-legends-without-border-and-with-white-background

> Use option `bty = "n"` in legend to remove the box around the legend.
>
> `legend` a character or expression vector of length $\geq 1$ to appear in the legend. Other objects will be coerced by `as.graphicsAnnot`

### 28.1.1 basic plot system decomposition

```
##     Sepal.Length Sepal.Width Petal.Length Petal.W
## 1            5.1         3.5          1.4
## 2            4.9         3.0          1.4
## 3            4.7         3.2          1.3
## 4            4.6         3.1          1.5
## 5            5.0         3.6          1.4
## 6            5.4         3.9          1.7
## 7            4.6         3.4          1.4
## 8            5.0         3.4          1.5
## 9            4.4         2.9          1.4
## 10           4.9         3.1          1.5
## 11           5.4         3.7          1.5
## 12           4.8         3.4          1.6
## 13           4.8         3.0          1.4
## 14           4.3         3.0          1.1
## 15           5.8         4.0          1.2
## 16           5.7         4.4          1.5
## 17           5.4         3.9          1.3
## 18           5.1         3.5          1.4
## 19           5.7         3.8          1.7
## 20           5.1         3.8          1.5
## 21           5.4         3.4          1.7
## 22           5.1         3.7          1.5
## 23           4.6         3.6          1.0
## 24           5.1         3.3          1.7
## 25           4.8         3.4          1.9
## 26           5.0         3.0          1.6
## 27           5.0         3.4          1.6
## 28           5.2         3.5          1.5
## 29           5.2         3.4          1.4
## 30           4.7         3.2          1.6
## 31           4.8         3.1          1.6
## 32           5.4         3.4          1.5
## 33           5.2         4.1          1.5
## 34           5.5         4.2          1.4
## 35           4.9         3.1          1.5
## 36           5.0         3.2          1.2
## 37           5.5         3.5          1.3
## 38           4.9         3.6          1.4
## 39           4.4         3.0          1.3
## 40           5.1         3.4          1.5
## 41           5.0         3.5          1.3
## 42           4.5         2.3          1.3
## 43           4.4         3.2          1.3
## 44           5.0         3.5          1.6
## 45           5.1         3.8          1.9
## 46           4.8         3.0          1.4
## 47           5.1         3.8          1.6
## 48           4.6         3.2          1.4
## 49           5.3         3.7          1.5
## 50           5.0         3.3          1.4
## 51           7.0         3.2          4.7
## 52           6.4         3.2          4.5
## 53           6.9         3.1          4.9
## 54           5.5         2.3          4.0
## 55           6.5         2.8          4.6
## 56           5.7         2.8          4.5
## 57           6.3         3.3          4.7
## 58           4.9         2.4          3.3
## 59           6.6         2.9          4.6
## 60           5.2         2.7          3.9
## 61           5.0         2.0          3.5
## 62           5.9         3.0          4.2
## 63           6.0         2.2          4.0
## 64           6.1         2.9          4.7
## 65           5.6         2.9          3.6
```

```r
iris$Species
```

```
##   [1] setosa     setosa     setosa     setosa
##   [7] setosa     setosa     setosa     setosa
##  [13] setosa     setosa     setosa     setosa
##  [19] setosa     setosa     setosa     setosa
##  [25] setosa     setosa     setosa     setosa
##  [31] setosa     setosa     setosa     setosa
##  [37] setosa     setosa     setosa     setosa
##  [43] setosa     setosa     setosa     setosa
##  [49] setosa     setosa     versicolor versicolor
##  [55] versicolor versicolor versicolor versicolor
##  [61] versicolor versicolor versicolor versicolor
##  [67] versicolor versicolor versicolor versicolor
##  [73] versicolor versicolor versicolor versicolor
##  [79] versicolor versicolor versicolor versicolor
##  [85] versicolor versicolor versicolor versicolor
##  [91] versicolor versicolor versicolor versicolor
##  [97] versicolor versicolor versicolor versicolor
## [103] virginica  virginica  virginica  virginica
## [109] virginica  virginica  virginica  virginica
## [115] virginica  virginica  virginica  virginica
## [121] virginica  virginica  virginica  virginica
## [127] virginica  virginica  virginica  virginica
## [133] virginica  virginica  virginica  virginica
## [139] virginica  virginica  virginica  virginica
## [145] virginica  virginica  virginica  virginica
## Levels: setosa versicolor virginica
```

```r
iris[["Species"]]
```

```
##   [1] setosa     setosa     setosa     setosa
##   [7] setosa     setosa     setosa     setosa
##  [13] setosa     setosa     setosa     setosa
##  [19] setosa     setosa     setosa     setosa
##  [25] setosa     setosa     setosa     setosa
##  [31] setosa     setosa     setosa     setosa
##  [37] setosa     setosa     setosa     setosa
##  [43] setosa     setosa     setosa     setosa
##  [49] setosa     setosa     versicolor versicolor
##  [55] versicolor versicolor versicolor versicolor
##  [61] versicolor versicolor versicolor versicolor
##  [67] versicolor versicolor versicolor versicolor
##  [73] versicolor versicolor versicolor versicolor
##  [79] versicolor versicolor versicolor versicolor
##  [85] versicolor versicolor versicolor versicolor
##  [91] versicolor versicolor versicolor versicolor
##  [97] versicolor versicolor versicolor versicolor
## [103] virginica  virginica  virginica  virginica
## [109] virginica  virginica  virginica  virginica
## [115] virginica  virginica  virginica  virginica
## [121] virginica  virginica  virginica  virginica
## [127] virginica  virginica  virginica  virginica
## [133] virginica  virginica  virginica  virginica
## [139] virginica  virginica  virginica  virginica
## [145] virginica  virginica  virginica  virginica
## Levels: setosa versicolor virginica
```

```r
as.integer(iris[["Species"]])
```

```
##   [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2
##  [75] 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
## [112] 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3
## [149] 3 3
```

```r
levels(iris[["Species"]])
```

```
## [1] "setosa"     "versicolor" "virginica"
```

```r
idx <- as.integer(iris[["Species"]])
idx
```

```
##   [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2
##  [75] 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
## [112] 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3
## [149] 3 3
```

```r
idx <- as.integer(iris[["Species"]])
c(24, 21, 25)[idx]
```

```
##   [1] 24 24 24 24 24 24 24 24 24 24 24 24 24 24 2
##  [26] 24 24 24 24 24 24 24 24 24 24 24 24 24 24 2
##  [51] 21 21 21 21 21 21 21 21 21 21 21 21 21 21 2
##  [76] 21 21 21 21 21 21 21 21 21 21 21 21 21 21 2
## [101] 25 25 25 25 25 25 25 25 25 25 25 25 25 25 2
## [126] 25 25 25 25 25 25 25 25 25 25 25 25 25 25 2
```

## 28.2 statistic

https://bookdown.org/xiangyun/msg/system.html#section-14

```r
library(ggplot2)

ggplot(diamonds, aes(x = price)) +
  stat_density(aes(ymax = ..density.., ymin =
  ↪  -..density..),
    geom = "ribbon", position = "identity"
  )
```

```
## Warning: The dot-dot notation (`..density..`) was
## i Please use `after_stat(density)` instead.
## This warning is displayed once every 8 hours.
## Call `lifecycle::last_lifecycle_warnings()` to se
## generated.
```

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smoothed density estimates https://ggplot2.tidyverse.org/reference/geom_density.html

> Computes and draws kernel density estimate, which is a smoothed version of the histogram. This is a useful alternative to the histogram for continuous data that comes from an underlying smooth distribution.

`geom_density`

`stat_density`

https://ggplot2.tidyverse.org/reference/geom_density.html#ref-examples

## 28.3 scale

ggplot2

https://bookdown.org/xiangyun/msg/system.html#section-15

```r
library(ggplot2)

data(quake6, package = "MSG")
ggplot(quake6, aes(x = year, y = month)) +
  stat_sum(aes(size = ..n..)) +
  ↪  scale_size(range = c(1, 8))
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

count overlapping points https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-usage

> This is a variant `geom_point()` that counts the number of observations at each location, then maps the count to point area. It useful when you have discrete data and overplotting.

`geom_count`

`stat_sum`

https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-examples

scales for area or radius https://ggplot2.tidyverse.org/reference/scale_size.html?q=scale_size#null

`scale_size`

## 28.4 coordinate system

https://bookdown.org/xiangyun/msg/system.html#section-16

```
library(ggplot2)

p <- ggplot(aes(x = cut, y = log(price)), data
  = diamonds) +
  geom_boxplot()
p
p + coord_flip()
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

202403211208-ggplot2_files/figure-latex/unnamed-chu

---

```
library(ggplot2)

ggplot(aes(x = cut, fill = cut), data =
  diamonds) +
  coord_polar() +
  geom_bar(width = 1, show.legend = FALSE)
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

## 28.5 facet

https://bookdown.org/xiangyun/msg/system.html#subsec:facet

```
library(ggplot2)

ggplot(aes(x = carat), data = diamonds) +
  geom_density() +
  facet_grid(cut ~ .)
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

## 28.6 jitter

https://bookdown.org/xiangyun/msg/system.html#section-17

```
library(ggplot2)

ggplot(aes(x = Petal.Length, y = Petal.Width),
  data = iris) +
  geom_point() +
  geom_jitter(color = "red")
```

202403211208-ggplot2_files/figure-latex/unnamed-chu

## 28.7 font

https://bookdown.org/xiangyun/msg/system.html#subsec:font

# Chapter 29

# notational system for design

[9] p.51

NSD = notational system for design

[5]

## 29.1 graphic notation

[9] p.51

- $X$: treatment or exposure to an agent or an event of interest
- $P$: **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$: **observation** or process of measurement
- $R$: **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $_g$: **groups**
  - $_k$: **kinds** of treatments, exposures, or placebos
  - $_t$: **time** or sequential order

https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-rmarkdown

`(ref:rudolph) *nice* cite: [@Lam94].`

`(ref:campbell1963) *nice* cite: [@campbell1963].`

`(ref:campbell1963) ([@campbell1963]`

`(ref:campbell1963) \ [@campbell1963]`

[5] [5]

## 29.2 pre-experimental design

[5] p.6

### 29.2.1 one-shot case study

$$X \quad O$$

### 29.2.2 one-group pretest-posttest design

$$O \quad X \quad O$$

paired $t$ test

[9] p.62

$$\begin{array}{ccc} O & X & O \\ O_t & X & O_t \\ O_0 & X & O_1 \end{array}$$

| | Sources of Invalidity | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Internal | | | | | | | | External | | | |
| | History | Maturation | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| *Pre-Experimental Designs:* | | | | | | | | | | | | |
| 1. One-Shot Case Study<br>*X O* | − | − | | | | − | − | | | − | | |
| 2. One-Group Pretest-Posttest Design<br>*O X O* | − | − | − | − | ? | + | + | − | − | − | ? | |
| 3. Static-Group Comparison<br>*X O*<br>- - - - - - -<br>*O* | + | ? | + | + | + | − | − | − | | − | | |
| *True Experimental Designs:* | | | | | | | | | | | | |
| 4. Pretest-Posttest Control Group Design<br>*R O X O*<br>*R O O* | + | + | + | + | + | + | + | + | − | ? | ? | |
| 5. Solomon Four-Group Design<br>*R O X O*<br>*R O O*<br>*R X O*<br>*R O* | + | + | + | + | + | + | + | + | + | ? | ? | |
| 6. Posttest-Only Control Group Design<br>*R X O*<br>*R O* | + | + | + | + | + | + | + | + | + | ? | ? | |

Figure 29.1: pre- and true experimental designs ( [5] p.8)

$$\begin{array}{ccc} O & X & O \\ O_{gt} & X_g & O_{gt} \\ O_{10} & X & O_{11} \end{array}$$

### 29.2.3 static-group comparison

$$\begin{array}{cc} X & O \\ X_g & O_{gt} \\ X & O_{11} \\ & O_{21} \end{array}$$

---

$$\begin{array}{cc} X & O \\ X_g & O_{gt} \\ X & O_{11} \\ & O_{01} \end{array}$$

## 29.3 true experimental design

[5] p.13

### 29.3.1 posttest-only control group design

basic experimental design

two-sample $t$ test

[9] p.53

$$\begin{array}{ccc} R & X & O \\ R & & O \end{array}$$

| | Internal | | | | | | | | External | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Sources of Invalidity | History | Maturation | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| ***Quasi-Experimental Designs:*** | | | | | | | | | | | | |
| 7. Time Series<br>*O O O OXO O O O* | − | + | + | ? | + | + | + | + | − | ? | ? | |
| 8. Equivalent Time Samples Design<br>$X_1O$ $X_0O$ $X_1O$ $X_0O$, etc. | + | + | + | + | + | + | + | + | − | ? | − | − |
| 9. Equivalent Materials Samples Design<br>$M_aX_1O$ $M_bX_0O$ $M_cX_1O$ $M_dX_0O$, etc. | + | + | + | + | + | + | + | + | − | ? | ? | − |
| 10. Nonequivalent Control Group Design<br>*O X O*<br>-----------<br>*O O* | + | + | + | + | ? | + | + | − | − | ? | ? | |
| 11. Counterbalanced Designs<br>$X_1O$ $X_2O$ $X_3O$ $X_4O$<br>$X_2O$ $X_4O$ $X_1O$ $X_3O$<br>$X_3O$ $X_1O$ $X_4O$ $X_2O$<br>$X_4O$ $X_3O$ $X_2O$ $X_1O$ | + | + | + | + | + | + | + | ? | ? | ? | ? | − |
| 12. Separate-Sample Pretest-Posttest Design<br>*R O (X)*<br>*R X O* | − | − | + | ? | + | + | − | − | + | + | + | |
| 12*a*. *R O (X)*<br>*R X O*<br>-----------<br>*R O (X)*<br>*R X O* | + | − | + | ? | + | + | − | + | + | + | + | |
| 12*b*. *R* $O_1$ *(X)*<br>*R* $O_2$ *(X)*<br>*R X* $O_3$ | − | + | + | ? | + | + | − | ? | + | + | + | |
| 12*c*. *R* $O_1$ *X* $O_2$<br>*R X* $O_3$ | − | − | + | ? | + | + | + | − | + | + | + | |

Figure 29.2: quasi-experimental designs ( [5] p.40)

| | Sources of Invalidity | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Internal | | | | | | | | External | | | |
| | History | Maturation | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| ***Quasi-Experimental Designs Continued:*** | | | | | | | | | | | | |
| 13. Separate-Sample Pretest-Posttest Control Group Design<br>$R$ $O$ $(X)$<br>$R$ $X$ $O$<br>- - - - - - - -<br>$R$ $O$<br>$R$ $O$ | + | + | + | + | + | + | + | − | + | + | + | |
| 13*a*. $R'$ { $R$ $O$ $(X)$ / $R$ $X$ $O$ - - - $R$ $O$ $(X)$ / $R$ $X$ $O$ - - - $R$ $O$ $(X)$ / $R$ $X$ $O$ }<br>$R'$ { $R$ $O$ / $R$ $O$ - - - $R$ $O$ / $R$ $O$ - - - $R$ $O$ / $R$ $O$ } | + | + | + | + | + | + | + | + | + | + | + | |
| 14. Multiple Time-Series<br>$O$ $O$ $OXO$ $O$ $O$<br>- - - - - - - -<br>$O$ $O$ $O$ $O$ $O$ $O$ | + | + | + | + | + | + | + | + | − | − | ? | |
| 15. Institutional Cycle Design<br>Class $A$ $X$ $O_1$<br>- - - - - - - -<br>Class $B_1$ $RO_2$ $X$ $O_3$<br>Class $B_2$ $R$ $X$ $O_4$<br>- - - - - - - -<br>Class $C$ $O_5$ $X$<br>- - - - - - - -<br>[a]Gen. Pop. Con. Cl. $B$ $O_6$<br>[a]Gen. Pop. Con. Cl. $C$ $O_7$ | | | | | | | | | | | | |
| $O_2 < O_1$ }<br>$O_5 < O_4$ } | + | − | + | + | ? | − | ? | | + | ? | + | |
| $O_2 < O_3$ | − | − | − | ? | ? | + | + | | − | ? | + | |
| $O_2 < O_4$ | − | − | + | ? | ? | + | ? | | + | ? | ? | |
| $O_6 = O_7$ }<br>$O_{2y} = O_{2o}$ } | | + | | | | | | − | | | | |
| 16. Regression Discontinuity | + | + | + | ? | + | + | ? | + | + | − | + | + |

[a] General Population Controls for Class B, etc.

Figure 29.3: quasi-experimental designs continued ( [5] p.56)

or, with a placebo or an active control,

$$\begin{array}{ccc} R & X & O \\ R & P & O \end{array}$$

---

$$R \quad X_g \quad O_{gt}$$

---

$$\begin{array}{lll} R & X_g = X_1 = X & O_{gt} = O_{11} \\ R & X_g = X_2 = \emptyset & O_{gt} = O_{21} \end{array}$$

or, with a placebo or an active control

$$\begin{array}{lll} R & X_g = X_1 = X & O_{gt} = O_{11} \\ R & X_g = X_2 = P & O_{gt} = O_{21} \end{array}$$

---

$$\begin{array}{ccc} R & X & O_{11} \\ R & & O_{21} \end{array}$$

or, with a placebo or an active control

$$\begin{array}{ccc} R & X & O_{11} \\ R & P & O_{21} \end{array}$$

---

$$\begin{array}{ccc} R & X_g & O_{gt} \\ R & X & O_{11} \\ R & P & O_{21} \end{array}$$

---

$$\begin{array}{ccc} R & X & O \\ R & X_g & O_{gt} \\ R & X & O_{11} \\ R & P & O_{21} \end{array}$$

### 29.3.2 pretest-posttest control group design

$$\begin{array}{cccc} R & O & X & O \\ R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \end{array}$$

### 29.3.3 Solomon four-group design

[9] p.52

Solomon 4-group design = pretest-posttest + posttest-only control group design

$$\begin{array}{cccc} R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \\ R & & X & O_{31} \\ R & & & O_{41} \end{array}$$

---

$$\begin{array}{cccc} R & O & X & O \\ R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \\ R & & X & O_{31} \\ R & & & O_{41} \end{array}$$

## 29.4 quasi-experimental design

[5] p.34

## 29.5 correlational and ex post facto designs

[5] p.64

## 29.6 graphic notation, advanced

[9] p.74

- $X$: treatment or exposure to an agent or an event of interest
- $P$: **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$: **observation** or process of measurement
- $R$: **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $_g$: **groups**
  - $_k$: **kinds** of treatments, exposures, or placebos
  - $_t$: **time** or sequential order
- $V$: **variable(s)**
  - $B(V)$: **blocking** by the variable(s)
  - $M(V)$: **matching** by the variable(s)
  - $S(V)$: **stratifying** by the variable(s)
  - $L(V/L)$: **limiting** to the level(s) of the variable(s)
- $M^*$: research **material(s)** selected
- $-$: cohort

# Chapter 30

# design of experiment

experimental design = experiment design = design of experiments = DoE

[9] p.72

question-design-analysis loop

## 30.1 notational system for design[29]

graphic notation, advanced[29.6]

## 30.2 terminology

- population
  - sample
    - subsample
- unit
  - experimental unit
    - response
    - block: group of similar experimental unit ([10] p.74)
  - observational unit / measurement unit [1]
- replication ([9] p.76): an independent observation of the treatment ([10] p.74)
  - treatment replication: experimental-unit-to-experimental-unit variation
  - measurement replication = subsample: measurement-to-measurement variation
- replicate
  - experimental replicate
  - biological replicate
  - technical replicate

---

[10] p.73

$Y_{ij}$: the response observed from the $j^{\text{th}}$ experimental unit assigned to the $i^{\text{th}}$ treatment

$\mu_i$: the mean response to the $i^{\text{th}}$ treatment

$\mathcal{E}_{ij}$: the noise from other possible natural variation or nonrandom and random error

$$Y_{ij} = \mu_i + \mathcal{E}_{ij}, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ treatments} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ experimental units per treatment} \end{cases}$$

Each treatment has $n_j$ experimental units, so there are totally $n_i n_j$ experimental units.

If experimental units cannot be homogeneous, we can try to

- stratify them
- group them, and measure group to group variation
- block them

[1] https://passel2.unl.edu/view/lesson/2e09f0055f13/6

here $n_j$ blocks each with $n_i$ experimental units where **each treatment occurs once in each block**

$$\begin{aligned} Y_{ij} =& \mu_i + \mathcal{E}_{ij} \\ =& \mu_i + b_j + \mathcal{E}^*_{ij}, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ experimental units per block} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ blocks} \end{cases} \end{aligned}$$

where

$$\mathcal{E}_{ij} = b_j + \mathcal{E}^*_{ij}$$

i.e. the variation between groups or blocks of experimental units has been identified and isolated from $\mathcal{E}^*_{ij}$, which represents the variability of experimental units within a block. By isolating the block effect from the experimental units, the within-block variation can be used to compare treatment effects, which involves computing the estimated standard errors of contrasts of the treatments.

$$\begin{aligned} Y_{ij} - Y_{i'j} &= \left(\mu_i + b_j + \mathcal{E}^*_{ij}\right) \\ &- \left(\mu_{i'} + b_j + \mathcal{E}^*_{i'j}\right) \\ &= \left(\mu_i - \mu_{i'}\right) + \left(\mathcal{E}^*_{ij} - \mathcal{E}^*_{i'j}\right) \end{aligned}$$

which does not depend on the block effect $b_j$ or free of block effects. The result of this difference is that the variance of the difference of two treatment responses within a block depends on the within-block variation among the experimental units and not the between-block variation.

## 30.2.1 replication vs. subsample

It is very important to distinguish between a **subsample** and a **replication** since the error variance estimated from between subsamples is in general considerably smaller than the error variance estimated from replications or between experimental units. ([10] p.77)

https://www.researchgate.net/post/What-is-Experimental-Unit-Replicate-Total-sample-size-treatment-size

## 30.2.2 replication vs. repeated measurements

## 30.2.3 replication, replicate

### 30.2.3.1 technical replicate, biological replicate

https://www.youtube.com/watch?v=c_cpl5YsBV8

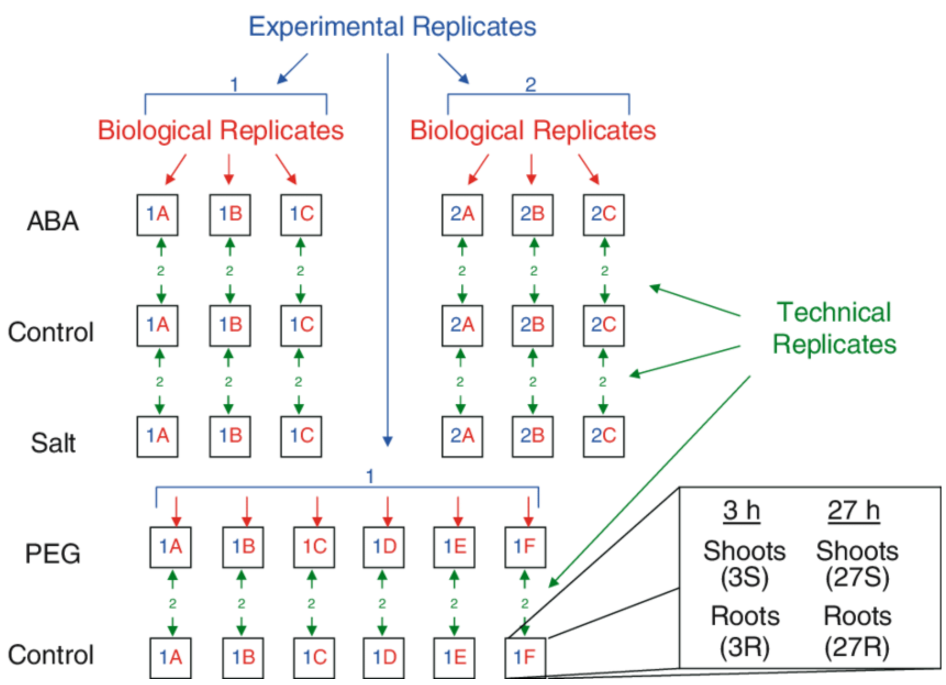



Figure 30.1: experimental, biological, technical replicates ( [11])

### 30.2.4 Latin square design

LSD = Latin square design

[6] p.505~507

https://tex.stackexchange.com/questions/501671/how-to-get-math-mode-curly-braces-in-tikz

`\\usepackage{pgfplots}` in `engine.opts=list(extra.preamble=c("\\usepackage{pgfplots}"))`

`\usetikzlibrary{decorations}`

202403232342-experimental-design_files/figure-latex/unnamed-chur

Figure 30.2: Latin square example

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \mathcal{E}_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases}$$

$$\mathcal{E}_{ijk} \overset{\text{i.i.d.}}{\sim} \text{n}\left(0, \sigma^2\right)$$

---

$\rho_i$: $i^{\text{th}}$ row

$\kappa_j$: $j^{\text{th}}$ column

$\tau_k$: $k^{\text{th}}$ treatment

$$\begin{aligned} Y_{ijk} =& \mu + \alpha_i + \beta_j + \gamma_k + \mathcal{E}_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases} \\ =& \mu + \rho_i + \kappa_j + \tau_k + \mathcal{E}_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \end{cases} \end{aligned}$$

### 30.2.5 model assumption and experimental unit, measurement/observational unit

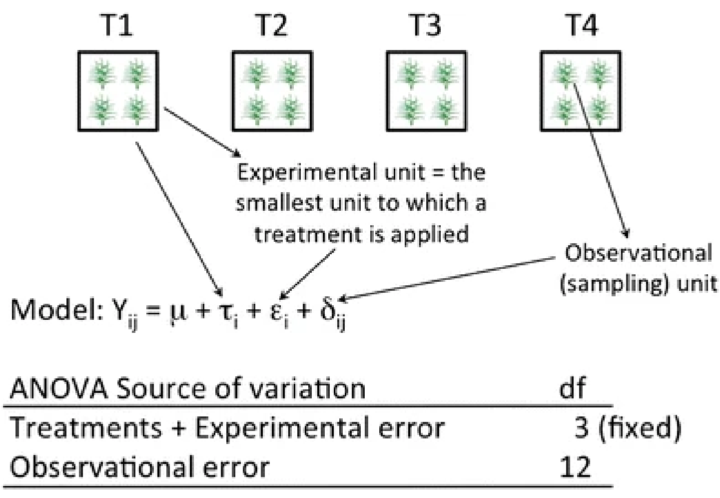
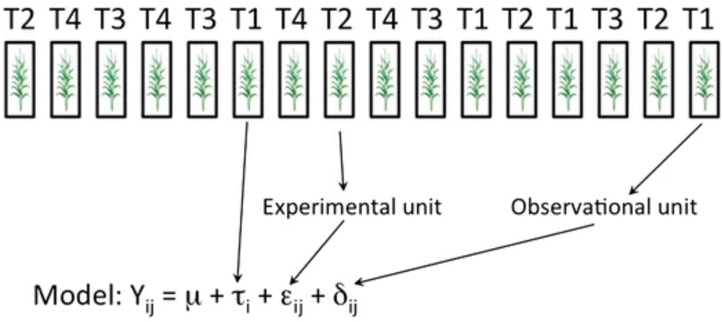


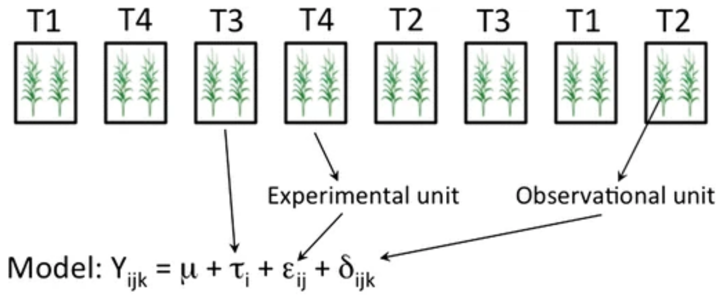

Figure 30.3: model assumption and experimental unit 1 ( [12] fig.1)

---



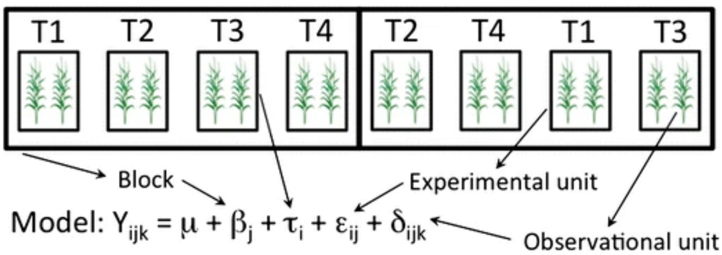

| ANOVA Source of variation | df |
|---|---|
| Treatments | 3 (fixed) |
| Error (experimental + observational) | 12 |

Figure 30.4: model assumption and experimental unit 2 ( 12 fig.2)




| ANOVA Source of variation | df |
|---|---|
| Treatments | 3 (fixed) |
| Experimental error | 4 |
| Observational error | 8 |

Figure 30.5: model assumption and experimental unit 3 ( 12 fig.3)




| ANOVA Source of variation | df |
|---|---|
| Blocks | 1 |
| Treatments | 3 (fixed) |
| Experimental error | 3 |
| Observational error | 8 |

Figure 30.6: model assumption and experimental unit 4 ( 12 fig.4)

$$Y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ij} + \Delta_{ijk}$$

## 30.3 experiment structure

### 30.3.1 treatment structure

[10] p.77

- 1-way treatment structure
- 2-way treatment structure
- factorial arrangement treatment structure
- *fractional* factorial arrangement treatment structure
- factorial arrangement with one or more controls

### 30.3.2 design structure

[10] p.77

- CRD = completely randomized design
- RCBD = randomized complete block design
  - ? why not called CRBD = completely randomized block design
- LSD = Latin square design[30.2.4]
- IBD = incomplete block design
  - BIBD = balanced IBD
- various combinations and generalizations

### 30.3.3 size of experimental unit

- split-plot design
  - split-split-plot design
  - split-split-split-plot design
- repeated measures design
  - cross-over design
  - change-over design
- nested design = hierarchical design
- variations and combinations
  - SSEU = several sizes of experimental units

## 30.4 approach to experimentation

[9] p.75

- approach to experimentation
  - best-guess approach
  - one-factor-at-a-time approach = OFAT
  - factorial approach

## 30.5 sample size estimation

## 30.6 statistical analysis plan

## 30.7 protocol

[9] p.95

- study objective
- study endpoint
  - primary endpoint
  - secondary endpoint(s)
- experimental unit(s)
- treatment structure[30.3.1]
- design structure[30.3.2]

- potential confounder(s)
- randomization
- blinding
- chance reduction
- sample size estimation[30.5]
- data collection
- data management system
- statistical analysis plan[30.6]
- DSMB / DSMC = data and safety monitoring board / committee

## 30.8 DoE course with six sigma and Mintab

https://zhuanlan.zhihu.com/p/265914617

https://www.zhihu.com/question/416312693/answer/1426399810

### 30.8.1 evolution

- Fisher
- Rao

# Chapter 31

# quine

https://www.youtube.com/watch?v=vdZr-O3nTPY

https://www.bilibili.com/video/BV1Wu4m1u7SF

```
s = 's = %r\nprint(s%%s)'
print(s%s)
```

```
## s = 's = %r\nprint(s%%s)'
## print(s%s)
```

This snippet is a clever example of a quine. A quine is a computer program that takes no input and produces a copy of its own source code as its output. The given code in Python is written to print its own source when executed. Let's break it down:

`s = 's = %r\nprint(s%%s)'`: This line defines a string `s` that contains a format string. `%r` is a placeholder that gets replaced with the `repr()` of the argument provided to the `%` operator, which in this case will be the string s itself. This means it will insert the string representation of `s` into the format string at `%r`.

`print(s%s)`: This line prints the result of `s%s`. Here, the `%` operator is used to format the string `s` with itself. The `%s` inside the print statement is replaced by the string `s`, leading to the entire string being printed out, including the print statement itself.

This is because the format operation replaces %r with the representation of the string `s`, and `%%` is a way to escape the `%` sign in format strings, resulting in a single `%` in the output. This output is exactly the same as the source code, making it a quine.

## 31.1 %r

The `%r` in Python string formatting represents the "representation" of a value, which is typically the way you would see it if you were to type it into a Python interpreter. It uses the `repr()` function to convert the value to a string. This is useful for debugging, among other things, because it shows strings with quotes around them and escapes special characters. Essentially, `%r` gives you the "developer's view" of what a variable looks like.

Here's a simple example to illustrate %r versus %s in string formatting:

```
my_str = "Hello, World!\nNew line character is represented with \\n"
print("Using %%s: %s" % my_str)
```

```
## Using %s: Hello, World!
## New line character is represented with \n
```

```
print("Using %%r: %r" % my_str)
```

```
## Using %r: 'Hello, World!\nNew line character is represented with \\n'
```

In this example:

The `%s` specifier tells Python to convert the object using `str()`, which is designed to be readable and outputs the string `"Hello, World!\nNew line character is represented with \n"`, interpreting the escape character `\n` as a newline.

The `%r` specifier tells Python to convert the object using `repr()`, which aims to generate output that could be used to recreate the object, outputting the string `'Hello, World!\nNew line character is represented with \\n'`, preserving the actual escape characters in the output.

Notice how `%r` preserves the string exactly as it is, including the quotes and escaped characters, making it clear it's a string and showing the escape sequence explicitly.

# Chapter 32

# quaternion

## 32.1 TaylorCatAlice

https://www.bilibili.com/video/BV1PV411P7w4

https://en.wikipedia.org/wiki/Blackboard_bold

### 32.1.1 complex / dionion / bionion

$$\begin{aligned}
c = a + b\mathrm{i} = a + \mathrm{i}b, &\begin{cases} c \in \mathbb{C} \\ \mathrm{i}^2 = -1 \\ a, b \in \mathbb{R} \quad \Leftrightarrow \langle a, b \rangle \in \mathbb{R}^2 \end{cases} \\
= z = x + y\mathrm{i} = x + \mathrm{i}y, &\begin{cases} z \in \mathbb{C} \\ \mathrm{i}^2 = -1 \\ x, y \in \mathbb{R} \quad \Leftrightarrow \langle x, y \rangle \in \mathbb{R}^2 \end{cases} \\
= \sqrt{x^2 + y^2} &\left( \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\mathrm{i} \right) = r\left(\cos\theta + \mathrm{i}\sin\theta\right) = r\mathrm{e}^{\mathrm{i}\theta}
\end{aligned}$$

Also see complex group representation[35.7].

### 32.1.2 trionion / triernion / triplex / ternion

https://zh.wikipedia.org/zh-tw/%E4%B8%89%E5%85%83%E6%95%B8

https://math.stackexchange.com/questions/1784166/why-are-there-no-triernions-3-dimensional-analogue-of-complex-numbers-quate

https://math.stackexchange.com/questions/32100/is-there-a-third-dimension-of-numbers/4453131

$$\begin{aligned}
t = a + b\mathrm{i} + c\mathrm{j} = a + \mathrm{i}b + \mathrm{j}c, &\begin{cases} t \in \mathbb{T} \\ \mathrm{i}^2 = -1 \\ \mathrm{j}^2 = -1 \end{cases} \\
= w = x + y\mathrm{i} + z\mathrm{j} = x + \mathrm{i}y + \mathrm{j}z, &\begin{cases} w \in \mathbb{T} \\ \mathrm{i}^2 = -1 \\ \mathrm{j}^2 = -1 \end{cases} \\
= \sqrt{x^2 + y^2 + z^2} &\left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathrm{i} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathrm{j} \right) = ?
\end{aligned}$$

---

$$\begin{cases} A\left(BC\right) = \left(AB\right)C & (a)\,\text{associativity} \\ A\left(B + C\right) = AB + AC & (d)\,\text{distributivity} \end{cases}$$

$$\mathbb{T} \ni \mathrm{ij} = X + Y\mathrm{i} + Z\mathrm{j} \in \mathbb{T}$$

$$-\mathrm{j} = \left(\mathrm{i}^2\right)\mathrm{j} \overset{(a)}{=} \mathrm{i}\,(\mathrm{ij}) = \mathrm{i}\,(X + Y\mathrm{i} + Z\mathrm{j}) \overset{(d)}{=} -Y + X\mathrm{i} + Z\mathrm{ij}$$

$$\mathrm{ij} = \frac{Y}{Z} - \frac{X}{Z}\mathrm{i} - \frac{1}{Z}\mathrm{j} \Rightarrow \begin{cases} X = \frac{Y}{Z} \\ Y = -\frac{X}{Z} \\ Z = -\frac{1}{Z} & \Rightarrow Z^2 = -1 \Rightarrow Z \notin \mathbb{R} \end{cases}$$

$$-\mathrm{i} = \mathrm{i}\left(\mathrm{j}^2\right) \overset{(a)}{=} (\mathrm{ij})\,\mathrm{j} = (X + Y\mathrm{i} + Z\mathrm{j})\,\mathrm{j} \overset{(d)}{=} -Z + X\mathrm{j} + Y\mathrm{ij}$$

$$\mathrm{ij} = \frac{Z}{Y} - \frac{1}{Y}\mathrm{i} - \frac{X}{Y}\mathrm{j} \Rightarrow \begin{cases} X = \frac{Z}{Y} \\ Y = -\frac{1}{Y} & \Rightarrow Y^2 = -1 \Rightarrow Y \notin \mathbb{R} \\ Z = -\frac{X}{Y} \end{cases}$$

### 32.1.3 quaternion

https://en.wikipedia.org/wiki/Quaternion

$$\begin{aligned} q &= a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, \begin{cases} q \in \mathbb{H} \\ a,b,c,d \in \mathbb{R} & \Leftrightarrow \langle a,b,c,d\rangle \in \mathbb{R}^4 \end{cases} \\ &= w = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z, \begin{cases} w \in \mathbb{H} \\ t,x,y,z \in \mathbb{R} & \Leftrightarrow \langle t,x,y,z\rangle \in \mathbb{R}^4 \end{cases} \\ &= ? \end{aligned}$$

$$\begin{aligned} q &= a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, \begin{cases} q \in \mathbb{H} \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ a,b,c,d \in \mathbb{R} & \Leftrightarrow \langle a,b,c,d\rangle \in \mathbb{R}^4 \end{cases} \\ &= w = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z, \begin{cases} w \in \mathbb{H} \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ t,x,y,z \in \mathbb{R} & \Leftrightarrow \langle t,x,y,z\rangle \in \mathbb{R}^4 \end{cases} \\ &= t + \begin{pmatrix}\mathrm{i} & \mathrm{j} & \mathrm{k}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = x_0 + \begin{pmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = x_0 + \begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_0 + \boldsymbol{x}, \begin{cases} \boldsymbol{e}_1 = \boldsymbol{i} = \mathrm{i} \\ \boldsymbol{e}_2 = \boldsymbol{j} = \mathrm{j} \\ \boldsymbol{e}_3 = \boldsymbol{k} = \mathrm{k} \end{cases} \\ &= t + \frac{\mathrm{i}x + \mathrm{j}y + \mathrm{k}z}{r} r, \begin{cases} r^2 = x^2 + y^2 + z^2 & |q|^2 = t^2 + r^2 \\ \|\boldsymbol{n}\|^2 = \left(\frac{\mathrm{i}x + \mathrm{j}y + \mathrm{k}z}{r}\right)^2 = -1 \end{cases} \\ &= \sqrt{t^2 + r^2}\left(\frac{t}{\sqrt{t^2 + r^2}} + \boldsymbol{n}\frac{r}{\sqrt{t^2 + r^2}}\right) = |q|\left(\cos\frac{\theta}{2} + \boldsymbol{n}\sin\frac{\theta}{2}\right) = |q|\,\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} \end{aligned}$$

---

The quaternion set is denoted $\mathbb{H}$ for Sir R.W. **Hamilton**, because he suddenly and strikingly realized

$$\begin{cases} \mathrm{ij} = \mathrm{k} \\ \mathrm{k} \in \mathbb{H} \end{cases} \Rightarrow \mathrm{ij} \in \mathbb{H} \text{ for closure property}$$

---

for the sake of rigorosity, see group theory[35]

$$\mathrm{k}^2 = -1$$

$$\mathrm{ij} = \mathrm{k}$$

$$\mathrm{ijk} = \mathrm{i}\,(\mathrm{jk}) \overset{(a)}{=} (\mathrm{ij})\,\mathrm{k} = \mathrm{kk} = \mathrm{k}^2 = -1$$

$$\mathrm{kij} = (\mathrm{ki})\,\mathrm{j} \overset{(a)}{=} \mathrm{k}\,(\mathrm{ij}) = \mathrm{kk} = \mathrm{k}^2 = -1$$

$$
\begin{aligned}
\mathrm{ij} &= \mathrm{k} \\
-\mathrm{j} = \left(\mathrm{i}^2\right)\mathrm{j} &\overset{(a)}{=} \mathrm{i}\left(\mathrm{ij}\right) = \mathrm{ik} \\
-\mathrm{i} = \mathrm{i}\left(\mathrm{j}^2\right) &\overset{(a)}{=} \left(\mathrm{ij}\right)\mathrm{j} = \mathrm{kj}
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{j} = \left(\mathrm{i}^2\right)\mathrm{j} &\overset{(a)}{=} \mathrm{i}\left(\mathrm{ij}\right) = \mathrm{ik} \\
1 = -\mathrm{j}^2 = \mathrm{j}\left(-\mathrm{j}\right) = \mathrm{j}\left(\mathrm{ik}\right) &\overset{(a)}{=} \left(\mathrm{ji}\right)\mathrm{k} \\
\mathrm{k} = [1]\,\mathrm{k} = [\left(\mathrm{ji}\right)\mathrm{k}]\,\mathrm{k} &\overset{(a)}{=} \left(\mathrm{ji}\right)\left(\mathrm{k}^2\right) = \left(\mathrm{ji}\right)\left(-1\right) \\
-\mathrm{k} &= \mathrm{ji} \\
-\mathrm{i} = \mathrm{i}\left(\mathrm{j}^2\right) &\overset{(a)}{=} \left(\mathrm{ij}\right)\mathrm{j} = \mathrm{kj} \\
1 = \left(-\mathrm{i}\right)\mathrm{i} = \left(\mathrm{kj}\right)\mathrm{i} &\overset{(a)}{=} \mathrm{k}\left(\mathrm{ji}\right) = \mathrm{kji} \\
1 &= \mathrm{kji}
\end{aligned}
$$

There is no more commutativity[32.1.3.2.1], i.e.

$$AB \neq BA$$

but

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

satisfying anticommutativity[32.1.3.2.2].

$$
\begin{aligned}
\begin{cases} \mathrm{ij} = \mathrm{k} \\ \mathrm{ji} = -\mathrm{k} \end{cases} &\Leftrightarrow \mathrm{ji} = -\mathrm{k} = -\mathrm{ij} \\
&\Rightarrow \mathrm{ji} = -\mathrm{ij} \\
&\Leftrightarrow \mathrm{ij} + \mathrm{ji} = 0
\end{aligned}
$$

$$
\begin{aligned}
\begin{cases} \mathrm{kij} = -1 \\ \mathrm{kji} = 1 \end{cases} &\Leftrightarrow \mathrm{kij} = -1 = -\mathrm{kji} \\
&\Rightarrow \mathrm{kij} = -\mathrm{kji} \\
&\Leftrightarrow \mathrm{kij} + \mathrm{kji} = 0
\end{aligned}
$$

---

$$
\begin{aligned}
q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, &\begin{cases} q \in \mathbb{H} & \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ a, b, c, d \in \mathbb{R} & \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \end{cases} \\
= w = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z, &\begin{cases} w \in \mathbb{H} & \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ t, x, y, z \in \mathbb{R} & \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \end{cases} \\
= t + \begin{pmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + \boldsymbol{x}, &\begin{cases} \boldsymbol{e}_1 = \boldsymbol{i} = \mathrm{i} \\ \boldsymbol{e}_2 = \boldsymbol{j} = \mathrm{j} \\ \boldsymbol{e}_3 = \boldsymbol{k} = \mathrm{k} \end{cases}
\end{aligned}
$$

202403251511-quaternion_files/figure-latex/unnamed-chunk-1-1.pdf

202403251511-quaternion_files/figure-latex/unnamed-chunk-2-1.pdf

Figure 32.1: quaternion basis group table

202403251511-quaternion_files/figure-latex/unnamed-chunk-3-1.pdf

Figure 32.2: quaternion basis group table 2

#### 32.1.3.1 true origin of ( dot product & cross product ) / ( inner product & outer product )

product of two pure imaginary quaternions

$$
\begin{aligned}
\boldsymbol{x}_1\boldsymbol{x}_2 =& \left(x_{11}\mathrm{i} + x_{12}\mathrm{j} + x_{13}\mathrm{k}\right)\left(x_{21}\mathrm{i} + x_{22}\mathrm{j} + x_{23}\mathrm{k}\right) \\
=& x_{11}x_{21}\mathrm{i}^2 + x_{11}x_{22}\mathrm{ij} + x_{11}x_{23}\mathrm{ik} \\
&+ x_{12}x_{21}\mathrm{ji} + x_{12}x_{22}\mathrm{j}^2 + x_{12}x_{23}\mathrm{jk} \\
&+ x_{13}x_{21}\mathrm{ki} + x_{13}x_{22}\mathrm{kj} + x_{13}x_{23}\mathrm{k}^2 \\
=& -\left(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}\right) \\
&+ \left(x_{12}x_{23} - x_{13}x_{22}\right)\mathrm{jk} + \left(x_{13}x_{21} - x_{11}x_{23}\right)\mathrm{ki} + \left(x_{11}x_{22} - x_{12}x_{21}\right)\mathrm{ij} \\
=& -\left(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}\right) \\
&+ \left(x_{12}x_{23} - x_{13}x_{22}\right)\mathrm{i} + \left(x_{13}x_{21} - x_{11}x_{23}\right)\mathrm{j} + \left(x_{11}x_{22} - x_{12}x_{21}\right)\mathrm{k} \\
=& -\left(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2\right) + \left(\boldsymbol{x}_1 \times \boldsymbol{x}_2\right), \begin{cases} \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} \\ \boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} \end{cases} \\
=& -\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{x}_1 \times \boldsymbol{x}_2
\end{aligned}
$$

product of two general quaternions / ordinary quaternions = Grassmann product

$$
\begin{aligned}
q_1q_2 =& \left(q_{10} + q_{11}\mathrm{i} + q_{12}\mathrm{j} + q_{13}\mathrm{k}\right)\left(q_{20} + q_{21}\mathrm{i} + q_{22}\mathrm{j} + q_{23}\mathrm{k}\right) \\
=& \left(x_{10} + \boldsymbol{x}_1\right)\left(x_{20} + \boldsymbol{x}_2\right), \begin{cases} x_{i\mu} = q_{i\mu} & \mu \in \{0\} \cup \left(\mathbb{N} \cap [1,3]\right) \\ \boldsymbol{x}_i = x_{ij}\boldsymbol{e}_j & i,j \in \mathbb{N} \cap [1,3], \begin{cases} \boldsymbol{e}_1 = \mathrm{i} \\ \boldsymbol{e}_2 = \mathrm{j} \\ \boldsymbol{e}_3 = \mathrm{k} \end{cases} \end{cases} \\
=& x_{10}x_{20} + x_{10}\boldsymbol{x}_2 + x_{20}\boldsymbol{x}_1 + \boldsymbol{x}_1\boldsymbol{x}_2 \\
=& x_{10}x_{20} + x_{10}\boldsymbol{x}_2 + x_{20}\boldsymbol{x}_1 - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 + \boldsymbol{x}_1 \times \boldsymbol{x}_2 \\
=& \left(x_{10}x_{20} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2\right) + \left(x_{10}\boldsymbol{x}_2 + x_{20}\boldsymbol{x}_1 + \boldsymbol{x}_1 \times \boldsymbol{x}_2\right)
\end{aligned}
$$

$$
\begin{aligned}
x_{10}x_{20} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 =& \begin{pmatrix} q_{10} & q_{11} & q_{12} & q_{13} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = q_1^\mu \eta_{\mu\nu} q_2^\nu \\
=& \begin{pmatrix} q_{10} & q_{11} & q_{12} & q_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = \boldsymbol{q}_1^\intercal H \boldsymbol{q}_2, H = \left[\eta_{\mu\nu}\right]_{4\times 4} = \eta_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
q_1q_2 =& \left(x_{10}x_{20} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2\right) + \left(x_{10}\boldsymbol{x}_2 + x_{20}\boldsymbol{x}_1 + \boldsymbol{x}_1 \times \boldsymbol{x}_2\right) \\
QP =& \left(Q_0P_0 - \boldsymbol{Q} \cdot \boldsymbol{P}\right) + \left(Q_0\boldsymbol{P} + P_0\boldsymbol{Q} + \boldsymbol{Q} \times \boldsymbol{P}\right)
\end{aligned}
$$

$$ab = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}) + (a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b})$$

Minkowski metric tensor

$$\eta = H = [\eta_{\mu\nu}]_{4\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta_{\mu\nu}$$

and quaternions as 4-vectors or four-vectors

$$\boldsymbol{q}_1^\intercal = \begin{pmatrix} q_{10} & q_{11} & q_{12} & q_{13} \end{pmatrix}$$

$$\boldsymbol{q}_2 = \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix}$$

#### 32.1.3.2 commutativity vs. anticommutativity

##### 32.1.3.2.1 commutativity 交換律 = 交換性 = 對易性

$$AB = BA \Leftrightarrow AB - BA = 0$$

$$AB = BA \Rightarrow AB \equiv BA$$

##### 32.1.3.2.2 anticommutativity 反交換律 = 反交換性 = 反對易性

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

$$AB = -BA \Rightarrow AB \not\equiv BA$$

#### 32.1.3.3 bracket

##### 32.1.3.3.1 self-invented bracket 自創括號 = 自創括

###### 32.1.3.3.1.1 commutative bracket 交換括號 = 對易式

$$[X, Y] = \frac{XY - YX}{2}$$

###### 32.1.3.3.1.2 anticommutative bracket 反交換括號 = 反對易式

$$\{X, Y\} = \frac{XY + YX}{2}$$

##### 32.1.3.3.2 Poisson bracket https://en.wikipedia.org/wiki/Poisson_bracket

##### 32.1.3.3.3 Lagrange bracket

##### 32.1.3.3.4 Lie bracket

#### 32.1.3.4 triple product

product = double product = Grassmann product

$$ab = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}) + (a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b})$$

pure imaginary

$$\begin{aligned} ab &= (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}) + (a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}) \\ \begin{cases} a_0 = 0 \\ b_0 = 0 \end{cases} &= \quad (00 - \boldsymbol{a} \cdot \boldsymbol{b}) + (0\boldsymbol{b} + 0\boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}) \\ &= -\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b} \end{aligned}$$

pure imaginary product can get both ( real & imaginary ) / ( scalar & vector ) parts

$$ab = \boldsymbol{ab} = -\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}, \begin{cases} a = 0 + \boldsymbol{a} = \boldsymbol{a} \\ b = 0 + \boldsymbol{b} = \boldsymbol{b} \end{cases}$$

---

triple product

https://en.wikipedia.org/wiki/Triple_product

pure imaginary

$$\begin{cases} a = 0 + \boldsymbol{a} = \boldsymbol{a} \\ b = 0 + \boldsymbol{b} = \boldsymbol{b} \\ c = 0 + \boldsymbol{c} = \boldsymbol{c} \end{cases}$$

$$\begin{aligned} abc = \boldsymbol{abc} &= (\boldsymbol{ab})\,\boldsymbol{c} = (-\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b})\,\boldsymbol{c} \\ &= \boldsymbol{a}\,(\boldsymbol{bc}) = \boldsymbol{a}\,(-\boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}) \end{aligned}$$

$$\begin{aligned} \boldsymbol{abc} = (\boldsymbol{ab})\,\boldsymbol{c} &= (-\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b})\,\boldsymbol{c} \\ &= -(\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} + (\boldsymbol{a} \times \boldsymbol{b})\,\boldsymbol{c} \\ &= -(\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} + (-(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} + (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}) \\ &= [-(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}] + [(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}] \\ = \boldsymbol{a}\,(\boldsymbol{bc}) &= \boldsymbol{a}\,(-\boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}) \\ &= -\boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c}) + \boldsymbol{a}\,(\boldsymbol{b} \times \boldsymbol{c}) \\ &= -\boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c}) + (-\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})) \\ &= [-\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})] + [\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c})] \end{aligned}$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

$$\begin{aligned} &\begin{cases} -(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = -\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \\ (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c}) \end{cases} \\ \Rightarrow &\begin{cases} (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) & (s) \\ (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{c}) & (v) \end{cases} \end{aligned}$$

---

permutation

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots \\ \sigma(x_1) & \sigma(x_2) & \cdots \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

---

$$(s)\Rightarrow\begin{cases}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) & \begin{pmatrix}a&b&c\\a&b&c\end{pmatrix},s_1\\ (\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) & \begin{pmatrix}a&b&c\\b&c&a\end{pmatrix},s_2\\ (\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) & \begin{pmatrix}a&b&c\\c&a&b\end{pmatrix},s_3\\ (\boldsymbol{b}\times\boldsymbol{a})\cdot\boldsymbol{c}=\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c}) & \begin{pmatrix}a&b&c\\b&a&c\end{pmatrix},s_4\\ (\boldsymbol{a}\times\boldsymbol{c})\cdot\boldsymbol{b}=\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b}) & \begin{pmatrix}a&b&c\\a&c&b\end{pmatrix},s_5\\ (\boldsymbol{c}\times\boldsymbol{b})\cdot\boldsymbol{a}=\boldsymbol{c}\cdot(\boldsymbol{b}\times\boldsymbol{a}) & \begin{pmatrix}a&b&c\\c&b&a\end{pmatrix},s_6\end{cases}$$

$$\begin{aligned}&\overset{\cdot\text{ commutative}}{\Rightarrow}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\overset{s_1}{=}(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}\overset{s_2}{=}(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}\overset{s_3}{=}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\\ &\overset{\times\text{ anticommutative}}{=}-(\boldsymbol{b}\times\boldsymbol{a})\cdot\boldsymbol{c}\overset{s_6}{=}-(\boldsymbol{c}\times\boldsymbol{b})\cdot\boldsymbol{a}\overset{s_5}{=}-(\boldsymbol{a}\times\boldsymbol{c})\cdot\boldsymbol{b}\overset{s_4}{=}-(\boldsymbol{b}\times\boldsymbol{a})\cdot\boldsymbol{c}\\ &\Leftrightarrow\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})\\ &=-\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b})=-\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c})=-\boldsymbol{c}\cdot(\boldsymbol{b}\times\boldsymbol{a})\end{aligned}$$

---

$$(v)\Rightarrow\begin{cases}(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}-(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})-\boldsymbol{a}\,(\boldsymbol{b}\cdot\boldsymbol{c}) & \begin{pmatrix}a&b&c\\a&b&c\end{pmatrix},v_1\\ (\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}-(\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}=\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})-\boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{a}) & \begin{pmatrix}a&b&c\\b&c&a\end{pmatrix},v_2\\ (\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}-(\boldsymbol{c}\cdot\boldsymbol{a})\,\boldsymbol{b}=\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})-\boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b}) & \begin{pmatrix}a&b&c\\c&a&b\end{pmatrix},v_3\\ (\boldsymbol{b}\times\boldsymbol{a})\times\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{a})\,\boldsymbol{c}=\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})-\boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c}) & \begin{pmatrix}a&b&c\\b&a&c\end{pmatrix},v_4\\ (\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}=\boldsymbol{a}\times(\boldsymbol{c}\times\boldsymbol{b})-\boldsymbol{a}\,(\boldsymbol{c}\cdot\boldsymbol{b}) & \begin{pmatrix}a&b&c\\a&c&b\end{pmatrix},v_5\\ (\boldsymbol{c}\times\boldsymbol{b})\times\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{b})\,\boldsymbol{a}=\boldsymbol{c}\times(\boldsymbol{b}\times\boldsymbol{a})-\boldsymbol{c}\,(\boldsymbol{b}\cdot\boldsymbol{a}) & \begin{pmatrix}a&b&c\\c&b&a\end{pmatrix},v_6\end{cases}$$

$$\Rightarrow\begin{cases}-Z-C=X-A & v_1\\-X-A=Y-B & v_2\\-Y-B=Z-C & v_3\\Z-C=-Y-B & v_4\\Y-B=-X-A & v_5\\X-A=-Z-C & v_6\end{cases},\begin{cases}\cdot\text{ and scalar-vector product} & \text{commutative}\\ \times & \text{anticommutative}\end{cases},$$

$$\begin{cases}X=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=-(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}=-\boldsymbol{a}\times(\boldsymbol{c}\times\boldsymbol{b})=(\boldsymbol{c}\times\boldsymbol{b})\times\boldsymbol{a}\\ Y=\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})=-(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=-\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b}\\ Z=\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=-(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=-\boldsymbol{c}\times(\boldsymbol{b}\times\boldsymbol{a})=(\boldsymbol{b}\times\boldsymbol{a})\times\boldsymbol{c}\\ A=\boldsymbol{a}\,(\boldsymbol{b}\cdot\boldsymbol{c})=\boldsymbol{a}\,(\boldsymbol{c}\cdot\boldsymbol{b})=(\boldsymbol{c}\cdot\boldsymbol{b})\,\boldsymbol{a}=(\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}\\ B=\boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{a})=\boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}=(\boldsymbol{c}\cdot\boldsymbol{a})\,\boldsymbol{b}\\ C=\boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})=\boldsymbol{c}\,(\boldsymbol{b}\cdot\boldsymbol{a})=(\boldsymbol{b}\cdot\boldsymbol{a})\,\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}\end{cases}$$

$$\Rightarrow\begin{cases}-Z-C=X-A & v_1=v_6\\-X-A=Y-B & v_2=v_5\\-Y-B=Z-C & v_3=v_4\end{cases}\Leftrightarrow\begin{cases}Z+X=A-C\\X+Y=B-A\\Y+Z=C-B\end{cases}\Leftrightarrow\begin{cases}X+Y=B-A\\Y+Z=C-B\\Z+X=A-C\end{cases}$$

$$\Leftrightarrow\begin{cases}2\,(X+Y+Z)=0 & \Rightarrow X+Y+Z=0\\ Y+Z=C-B & \Rightarrow X=B-C\Leftrightarrow\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{a})-\boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})\ \text{"back cab"}\\ Z+X=A-C & \Rightarrow Y=C-A\Leftrightarrow\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})-\boldsymbol{a}\,(\boldsymbol{b}\cdot\boldsymbol{c})\\ X+Y=B-A & \Rightarrow Z=A-B\Leftrightarrow\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{a}\,(\boldsymbol{b}\cdot\boldsymbol{c})-\boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{a})\end{cases}$$

#### 32.1.3.5 differential operator

https://en.wikipedia.org/wiki/Differential_operator

**32.1.3.5.1 4-differential operator** 4-differential operator / four-differential operator = d'Alembert operator

$$\begin{aligned}\mathrm{D} &= \frac{\partial}{\partial t} + \mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z} = \partial_t + \mathrm{i}\partial_x + \mathrm{j}\partial_y + \mathrm{k}\partial_z \\ &= \frac{\partial}{\partial t} + \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \partial_t + \boldsymbol{i}\partial_x + \boldsymbol{j}\partial_y + \boldsymbol{k}\partial_z \\ &= \frac{\partial}{\partial x_0} + \boldsymbol{e}_1\frac{\partial}{\partial x_1} + \boldsymbol{e}_2\frac{\partial}{\partial x_2} + \boldsymbol{e}_3\frac{\partial}{\partial x_3} = \partial_0 + \boldsymbol{e}_i\partial_i = \partial_0 + \boldsymbol{\nabla}\end{aligned}$$

$$\mathrm{D} = \partial_0 + \mathrm{i}\partial_1 + \mathrm{j}\partial_2 + \mathrm{k}\partial_3 = \partial_0 + \boldsymbol{\nabla} = \mathrm{D}_0 + \mathbf{D}$$

**32.1.3.5.2 nabla** nabla = spatial differential operator = 3-differential operator / three-differential operator

$$\boldsymbol{\nabla} = \boldsymbol{e}_i\partial_i = \sum_{i=1}^{3}\boldsymbol{e}_i\partial_i = \sum_{i=1}^{3}\boldsymbol{e}_i\frac{\partial}{\partial x_i} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}^{\mathsf{T}} = \begin{pmatrix}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z}\end{pmatrix}$$

**32.1.3.5.3 Laplace operator** Laplace operator = Laplacian

$$\triangle = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

**32.1.3.5.4 d'Alembert operator**

$$\Box = \Box_c = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

$$\Box_1 = \Box_{c=1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \triangle = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

#### 32.1.3.6 electromagnetism

Maxwell

**32.1.3.6.1 4-potential** electromagnetic 4-potential / four-potential

$$A = A_0 + +\mathrm{i}A_1 + \mathrm{j}A_2 + \mathrm{k}A_3 = A_0 + \boldsymbol{A}$$

4-differential operator[32.1.3.5.1]

$$\mathrm{D} = \partial_0 + \mathrm{i}\partial_1 + \mathrm{j}\partial_2 + \mathrm{k}\partial_3 = \partial_0 + \boldsymbol{\nabla} = \mathrm{D}_0 + \mathbf{D}$$

$$QP = (Q_0P_0 - \boldsymbol{Q}\cdot\boldsymbol{P}) + (Q_0\boldsymbol{P} + P_0\boldsymbol{Q} + \boldsymbol{Q}\times\boldsymbol{P})$$

commutative bracket[32.1.3.3.1.1]

$$
\begin{aligned}
[\mathrm{D}, A] =& \frac{\mathrm{D}A - A\mathrm{D}}{2} \\
2\,[\mathrm{D}, A] =& \mathrm{D}A - A\mathrm{D} \\
=& \left(\partial_0 + \mathrm{i}\partial_1 + \mathrm{j}\partial_2 + \mathrm{k}\partial_3\right)\left(A_0 + +\mathrm{i}A_1 + \mathrm{j}A_2 + \mathrm{k}A_3\right) \\
& - \left(A_0 + +\mathrm{i}A_1 + \mathrm{j}A_2 + \mathrm{k}A_3\right)\left(\partial_0 + \mathrm{i}\partial_1 + \mathrm{j}\partial_2 + \mathrm{k}\partial_3\right) \\
\mathrm{D}A =& \left(\mathrm{D}_0 A_0 - \mathbf{D}\cdot\boldsymbol{A}\right) + \left(\mathrm{D}_0\boldsymbol{A} + A_0\mathbf{D} + \mathbf{D}\times\boldsymbol{A}\right) \\
=& \left(\mathrm{D}_0 A_0 - \mathbf{D}\cdot\boldsymbol{A}\right) + \left(\mathrm{D}_0\boldsymbol{A} + \mathbf{D}A_0 + \mathbf{D}\times\boldsymbol{A}\right) \\
=& \mathrm{D}_0\left(A_0 + \boldsymbol{A}\right) - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 + \mathbf{D}\times\boldsymbol{A} \\
=& \mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 + \mathbf{D}\times\boldsymbol{A} \\
=& \partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0 + \boldsymbol{\nabla}\times\boldsymbol{A} \\
A\mathrm{D} =& \left(A_0\mathrm{D}_0 - \boldsymbol{A}\cdot\mathbf{D}\right) + \left(A_0\mathbf{D} + \mathrm{D}_0\boldsymbol{A} + \boldsymbol{A}\times\mathbf{D}\right) \\
=& \left(\mathrm{D}_0 A_0 - \mathbf{D}\cdot\boldsymbol{A}\right) + \left(\mathbf{D}A_0 + \mathrm{D}_0\boldsymbol{A} - \mathbf{D}\times\boldsymbol{A}\right) \\
=& \left(\mathrm{D}_0 A_0 - \mathbf{D}\cdot\boldsymbol{A}\right) + \left(\mathrm{D}_0\boldsymbol{A} + \mathbf{D}A_0 - \mathbf{D}\times\boldsymbol{A}\right) \\
=& \mathrm{D}_0\left(A_0 + \boldsymbol{A}\right) - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 - \mathbf{D}\times\boldsymbol{A} \\
=& \mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 - \mathbf{D}\times\boldsymbol{A} \\
=& \partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0 - \boldsymbol{\nabla}\times\boldsymbol{A}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}A =& \mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 + \mathbf{D}\times\boldsymbol{A} = \partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0 + \boldsymbol{\nabla}\times\boldsymbol{A} \\
A\mathrm{D} =& \mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 - \mathbf{D}\times\boldsymbol{A} = \partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0 - \boldsymbol{\nabla}\times\boldsymbol{A}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}A - A\mathrm{D} = 2\mathbf{D}\times\boldsymbol{A} = 2\boldsymbol{\nabla}\times\boldsymbol{A} \\
[\mathrm{D}, A] = \frac{\mathrm{D}A - A\mathrm{D}}{2} = \mathbf{D}\times\boldsymbol{A} = \boldsymbol{\nabla}\times\boldsymbol{A}
\end{aligned}
$$

anticommutative bracket[32.1.3.3.1.2]

$$
\begin{aligned}
\mathrm{D}A + A\mathrm{D} = 2\left(\mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0\right) = 2\left(\partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0\right) \\
\{\mathrm{D}, A\} = \frac{\mathrm{D}A + A\mathrm{D}}{2} = \mathrm{D}_0 A - \mathbf{D}\cdot\boldsymbol{A} + \mathbf{D}A_0 = \partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla}A_0
\end{aligned}
$$

commutation and anticommutation on differential operator and any quaternion

$$
\begin{aligned}
[\mathrm{D}, Q] =& \boldsymbol{\nabla}\times\boldsymbol{Q} \\
\{\mathrm{D}, Q\} =& \partial_0 Q - \boldsymbol{\nabla}\cdot\boldsymbol{Q} + \boldsymbol{\nabla}Q_0
\end{aligned}
$$

or more evident

$$
\begin{aligned}
[\mathrm{D}, Q] =& \boldsymbol{\nabla}\times\boldsymbol{Q} \\
\{\mathrm{D}, Q\} =& \partial_0 Q - \boldsymbol{\nabla}\cdot\boldsymbol{Q} + \boldsymbol{\nabla}Q_0 \\
=& \partial_0\left(Q_0 + \boldsymbol{Q}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{Q} + \boldsymbol{\nabla}Q_0 \\
=& \left(\partial_0 Q_0 - \boldsymbol{\nabla}\cdot\boldsymbol{Q}\right) + \left(\partial_0\boldsymbol{Q} + \boldsymbol{\nabla}Q_0\right) \\
=& \left(\frac{\partial Q_0}{\partial t} - \boldsymbol{\nabla}\cdot\boldsymbol{Q}\right) + \left(\frac{\partial\boldsymbol{Q}}{\partial t} + \boldsymbol{\nabla}Q_0\right)
\end{aligned}
$$

**32.1.3.6.2 Maxwell compromise for both quaternion and 3-vector** electric potential and vector potential

$$
A = A_0 + +\mathrm{i}A_1 + \mathrm{j}A_2 + \mathrm{k}A_3 = A_0 + \boldsymbol{A} = U + \boldsymbol{A}
$$

electric quaternion and electric field

$$
\begin{aligned}
E =& -\{\mathrm{D}, A\} \\
=& -(\partial_0 A - \boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{\nabla} A_0) \\
=& -\partial_0 A + \boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla} A_0 \\
=& -\partial_t (U + \boldsymbol{A}) + \boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla} U \\
=& -\frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla} U - \frac{\partial \boldsymbol{A}}{\partial t} \\
=& E_0 + \boldsymbol{E}, \begin{cases} E_0 = -\dfrac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} \\ \boldsymbol{E} = -\boldsymbol{\nabla} U - \dfrac{\partial \boldsymbol{A}}{\partial t} \qquad \text{electric field 3-vector} \end{cases}
\end{aligned}
$$

magnetic field

$$
B = [\mathrm{D}, A] = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \boldsymbol{B}
$$

Work on time? Yes.

$$
qE = qE_0 + q\boldsymbol{E} = qE_0 + \boldsymbol{F}_E
$$

force equivalent on time

$$
qE_0
$$

---

$$
\begin{cases} E = -\{\mathrm{D}, A\} = E_0 + \boldsymbol{E} \\ B = +[\mathrm{D}, A] = B_0 + \boldsymbol{B} = 0 + \boldsymbol{B} = \boldsymbol{B} \quad B_0 = 0 \end{cases}
$$

---

for any quaternion commutating and anticommutating with differential operator

$$
\begin{aligned}
[\mathrm{D}, Q] =& \boldsymbol{\nabla}\times\boldsymbol{Q} \\
\{\mathrm{D}, Q\} =& \partial_0 Q - \boldsymbol{\nabla}\cdot\boldsymbol{Q} + \boldsymbol{\nabla} Q_0 \\
=& \partial_0 (Q_0 + \boldsymbol{Q}) - \boldsymbol{\nabla}\cdot\boldsymbol{Q} + \boldsymbol{\nabla} Q_0 \\
=& (\partial_0 Q_0 - \boldsymbol{\nabla}\cdot\boldsymbol{Q}) + (\partial_0 \boldsymbol{Q} + \boldsymbol{\nabla} Q_0) \\
=& \left(\frac{\partial Q_0}{\partial t} - \boldsymbol{\nabla}\cdot\boldsymbol{Q}\right) + \left(\frac{\partial \boldsymbol{Q}}{\partial t} + \boldsymbol{\nabla} Q_0\right)
\end{aligned}
$$

---

$$
\begin{cases}
[\mathrm{D}, E] = \boldsymbol{\nabla}\times\boldsymbol{E} = (0) + (\boldsymbol{\nabla}\times\boldsymbol{E}) \\
\{\mathrm{D}, E\} = (\partial_0 E_0 - \boldsymbol{\nabla}\cdot\boldsymbol{E}) + (\partial_0 \boldsymbol{E} + \boldsymbol{\nabla} E_0) = \left(\dfrac{\partial E_0}{\partial t} - \boldsymbol{\nabla}\cdot\boldsymbol{E}\right) + \left(\dfrac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0\right) \\
[\mathrm{D}, B] = \boldsymbol{\nabla}\times\boldsymbol{B} = (0) + (\boldsymbol{\nabla}\times\boldsymbol{B}) \\
\{\mathrm{D}, B\} = (\partial_0 B_0 - \boldsymbol{\nabla}\cdot\boldsymbol{B}) + (\partial_0 \boldsymbol{B} + \boldsymbol{\nabla} B_0) \overset{B_0=0}{=} -\boldsymbol{\nabla}\cdot\boldsymbol{B} + \partial_0 \boldsymbol{B} = (-\boldsymbol{\nabla}\cdot\boldsymbol{B}) + \left(\dfrac{\partial \boldsymbol{B}}{\partial t}\right)
\end{cases}
$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

Maxwell equations without source terms

$$\begin{cases} [\mathrm{D}, B] = +\{\mathrm{D}, E\} & \Leftrightarrow (0) + (\boldsymbol{\nabla} \times \boldsymbol{B}) = \left(\frac{\partial E_0}{\partial t} - \boldsymbol{\nabla} \cdot \boldsymbol{E}\right) + \left(\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0\right) \\ [\mathrm{D}, E] = -\{\mathrm{D}, B\} & \Leftrightarrow (0) + (\boldsymbol{\nabla} \times \boldsymbol{E}) = (-\boldsymbol{\nabla} \cdot \boldsymbol{B}) + \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\partial E_0}{\partial t} - \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 & \Leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\partial E_0}{\partial t} \\ \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0 = \boldsymbol{\nabla} \times \boldsymbol{B} & \Leftrightarrow \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0 \text{ 動電生磁} \\ -\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 & \Leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \\ \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{E} & \Leftrightarrow \boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t} \text{ 動磁生電} \end{cases}$$

#### 32.1.3.7 Joule heat vs. Thomson heat (Kelvin heat?)

The Lord Kelvin = William Thomson

##### 32.1.3.7.1 thermoelectric effect

thermoelectric effect = Seebeck effect = Peltier effect = Thomson effect

#### 32.1.3.8 source term

$$J = J_0 + \mathrm{i}J_1 + \mathrm{j}J_2 + \mathrm{k}J_3 = J_0 + \boldsymbol{J} = \rho + \boldsymbol{J}$$

Maxwell equations with source terms

$$\begin{cases} [\mathrm{D}, B] = J + \{\mathrm{D}, E\} & \Leftrightarrow (0) + (\boldsymbol{\nabla} \times \boldsymbol{B}) = \left(\rho + \frac{\partial E_0}{\partial t} - \boldsymbol{\nabla} \cdot \boldsymbol{E}\right) + \left(\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0\right) \\ [\mathrm{D}, E] = 0 - \{\mathrm{D}, B\} & \Leftrightarrow (0) + (\boldsymbol{\nabla} \times \boldsymbol{E}) = (-\boldsymbol{\nabla} \cdot \boldsymbol{B}) + \left(\frac{\partial \boldsymbol{B}}{\partial t}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} \rho + \frac{\partial E_0}{\partial t} - \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 & \Leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho + \frac{\partial E_0}{\partial t} \\ \boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0 = \boldsymbol{\nabla} \times \boldsymbol{B} & \Leftrightarrow \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} E_0 \text{ 動電生磁} \\ -\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 & \Leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \\ \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{E} & \Leftrightarrow \boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t} \text{ 動磁生電} \end{cases}$$

### 32.1.4 quaternion group

https://en.wikipedia.org/wiki/Quaternion_group

group theory[35]

or please first see quaternion group representation[35.8].

https://www.bilibili.com/video/BV1rj41117VW

#### 32.1.4.1 2D rotation

##### 32.1.4.1.1 matrix

$$\boldsymbol{r} = (x, y) = \langle x, y \rangle = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$$

$$\boldsymbol{r}' = (x', y') = \langle x', y' \rangle = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix}$$

$$
\begin{aligned}
\boldsymbol{r}' =& \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = r\begin{pmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{pmatrix} \\
=& r\begin{pmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \sin\alpha\cos\theta + \cos\alpha\sin\theta \end{pmatrix} = r\begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{pmatrix} \\
=& r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \\
=& R\boldsymbol{r}, \begin{cases} R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R(\theta) = R_\theta \\ \boldsymbol{r} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}, \boldsymbol{r}' = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} \end{cases}
\end{aligned}
$$

orthonormal matrix

$$\boldsymbol{r}' = O\boldsymbol{r}$$

$$
\begin{aligned}
|\boldsymbol{r}'|^2 =& |\boldsymbol{r}|^2 \\
\boldsymbol{r}'\cdot\boldsymbol{r}' =& \boldsymbol{r}\cdot\boldsymbol{r} \\
\boldsymbol{r}'^\intercal\boldsymbol{r}' =& \boldsymbol{r}^\intercal\boldsymbol{r} \\
(O\boldsymbol{r})^\intercal(O\boldsymbol{r}) =& \\
\boldsymbol{r}^\intercal O^\intercal O\boldsymbol{r} =& \\
\boldsymbol{r}^\intercal O^\intercal O\boldsymbol{r} =& \boldsymbol{r}^\intercal\boldsymbol{r} \\
O^\intercal O =& 1 = I = I_2
\end{aligned}
$$

$$
\begin{aligned}
R^\intercal R =& \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^\intercal \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
=& \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
=& \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\
=& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = I \\
R^\intercal R =& 1 \Rightarrow R \in \{O | O^\intercal O = 1\}
\end{aligned}
$$

https://en.wikipedia.org/wiki/Transformation_matrix#Affine_transformations

reflection matrix

$$
\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & P_x^\intercal P_x = P_x^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_x \in \{O|O^\intercal O = 1\} \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & P_y^\intercal P_y = P_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_y \in \{O|O^\intercal O = 1\} \end{cases}
$$

translation matrix?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$O(2)$ group

$$
\begin{aligned}
O(2) =& \{1, R, P_x, P_y\} \\
=& \{I_2, R_\theta, P_x, P_y\} \subseteq \{O|O^\intercal O = 1\}
\end{aligned}
$$

$$\begin{aligned}
1 =& O^{\intercal} O \\
1 =& \det 1 = \det I = \det (I_2) \\
=& \det (O^{\intercal} O) = (\det O^{\intercal}) (\det O) = (\det O) (\det O) = (\det O)^2 \\
1 =& (\det O)^2 \\
\det O =& \pm 1
\end{aligned}$$

$$\det R = \det R_\theta = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - \left(-\sin^2\theta\right) = \cos^2\theta + \sin^2\theta = 1$$

$$\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \det P_x = -1 \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \det P_y = -1 \end{cases}$$

special orthonormal group of degree 2

$$\begin{aligned}
SO(2) = \{1, R\} = \{I_2, R_\theta\} \subseteq& \left\{ O \middle| \begin{cases} O^{\intercal} O = 1 & O \\ \det O = 1 & S \end{cases} \right\} \\
\subset& \{1, R, P_x, P_y\} = O(2) \subseteq \{O | O^{\intercal} O = 1\}
\end{aligned}$$

See also Lie group[32.2.8]

Mathemaniac: Lie group https://www.youtube.com/playlist?list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP https://www.youtube.com/watch?v=erA0jb9dSm0&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=2

#### 32.1.4.1.2 complex

$$z = r(\cos\alpha + \mathrm{i}\sin\alpha) = r\mathrm{e}^{\mathrm{i}\alpha}$$

$$z' = r(\cos(\alpha+\theta) + \mathrm{i}\sin(\alpha+\theta)) = r\mathrm{e}^{\mathrm{i}(\alpha+\theta)}$$

$$\begin{aligned}
z' =& z_\theta z \\
z_\theta =& \frac{z'}{z} = \frac{r' \mathrm{e}^{\mathrm{i}(\alpha+\theta)}}{r\mathrm{e}^{\mathrm{i}\alpha}} = \frac{r'}{r}\mathrm{e}^{\mathrm{i}\theta} = \frac{r'}{r}(\cos\theta + \mathrm{i}\sin\theta)
\end{aligned}$$

$$\begin{aligned}
z_\theta z =& \left[\frac{r'}{r}(\cos\theta + \mathrm{i}\sin\theta)\right] [r(\cos\alpha + \mathrm{i}\sin\alpha)] \\
=& r' [(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + \mathrm{i}(\sin\theta\cos\alpha + \cos\theta\sin\alpha)] \\
=& r' [\cos(\alpha+\theta) + \mathrm{i}\sin(\alpha+\theta)] = z'
\end{aligned}$$

$$\hat{z}_\theta = z_\theta \left(\frac{r'}{r} = 1\right) = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

$$\hat{z}_\theta^* = \overline{\hat{z}_\theta} = \mathrm{e}^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$$

$$\hat{z}_\theta^* \hat{z}_\theta = \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\theta + (-\mathrm{i}\theta)} = \mathrm{e}^{\mathrm{i}0} = \mathrm{e}^0 = 1$$

unitary group of degree 1

$$U(1) = \{1, \hat{z}_\theta\} = \left\{\mathrm{e}^{\mathrm{i}0}, \mathrm{e}^{\mathrm{i}\theta}\right\}$$

#### 32.1.4.1.3 $SO(2) \cong U(1)$ $\mathbb{C} \leftrightarrow \mathcal{M}_{2\times 2}(\mathbb{R}) = \mathcal{M}_2(\mathbb{R})$ complex group representation[35.7]

$$x + y\mathrm{i} \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

$$\begin{aligned}
U(1) &= \{1, \hat{z}_\theta\} = \left\{\mathrm{e}^{\mathrm{i}0}, \mathrm{e}^{\mathrm{i}\theta}\right\} \\
&= \{\cos 0 + \mathrm{i}\sin 0, \cos\theta + \mathrm{i}\sin\theta\} \\
&\leftrightarrow \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos 0 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\theta\right\} \\
&= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}1 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\theta\right\} \\
&= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\} = \{I_2, R_\theta\} = \{1, R\} = SO(2) \\
U(1) &\cong SO(2) \Leftrightarrow SO(2) \cong U(1)
\end{aligned}$$

unitary group of degree 1 and special orthonormal group of degree 2 are isomorphism

### 32.1.4.2 3D rotation

#### 32.1.4.2.1 matrix

##### 32.1.4.2.1.1 construction with 2D rotation matrix

$$\begin{aligned}
\boldsymbol{r}' &= \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \\ z' \end{pmatrix} \overset{z'=z}{=} \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \\ z \end{pmatrix} = \begin{pmatrix} R(\theta) & \\ & 1 \end{pmatrix} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ z \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ z \end{pmatrix} = R_z(\theta)\,\boldsymbol{r}, \begin{cases} R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \boldsymbol{r} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ z \end{pmatrix} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{r}' &= \begin{pmatrix} x' \\ r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} \overset{x'=x}{=} \begin{pmatrix} x \\ r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} 1 & \\ & R(\theta) \end{pmatrix} \begin{pmatrix} x \\ r\cos\alpha \\ r\sin\alpha \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ r\cos\alpha \\ r\sin\alpha \end{pmatrix} = R_x(\theta)\,\boldsymbol{r}, \begin{cases} R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \\ \boldsymbol{r} = \begin{pmatrix} x \\ r\cos\alpha \\ r\sin\alpha \end{pmatrix} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{r}' &= \begin{pmatrix} r\sin(\alpha+\theta) \\ y' \\ r\cos(\alpha+\theta) \end{pmatrix} \overset{y'=y}{=} \begin{pmatrix} r\sin(\alpha+\theta) \\ y \\ r\cos(\alpha+\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} r\sin\alpha \\ y \\ r\cos\alpha \end{pmatrix} = R_y(\theta)\,\boldsymbol{r}, \begin{cases} R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\ \boldsymbol{r} = \begin{pmatrix} r\sin\alpha \\ y \\ r\cos\alpha \end{pmatrix} \end{cases}
\end{aligned}$$

#### 32.1.4.2.1.2 Euler angle $z \to x \to z : \alpha \to \beta \to \gamma$

$$
\begin{aligned}
&R_z(\gamma) R_x(\beta) R_z(\alpha) \\
=&\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
=&\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \cos\beta\sin\alpha & \cos\beta\cos\alpha & -\sin\beta \\ \sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix} \\
=&\begin{pmatrix} \cos\gamma\cos\alpha - \sin\gamma\cos\beta\sin\alpha & -\cos\gamma\sin\alpha - \sin\gamma\cos\beta\cos\alpha & \sin\gamma\sin\beta \\ \sin\gamma\cos\alpha + \cos\gamma\cos\beta\sin\alpha & -\sin\gamma\sin\alpha - \cos\gamma\cos\beta\cos\alpha & -\cos\gamma\sin\beta \\ \sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix}
\end{aligned}
$$

$x \to y \to z : \alpha \to \beta \to \gamma$

$$
\begin{aligned}
&R_z(\gamma) R_y(\beta) R_x(\alpha) \\
=&\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \\
=&\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta\sin\alpha & \sin\beta\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{pmatrix} \\
=&\begin{pmatrix} \cos\gamma\cos\beta & \cos\gamma\sin\beta\sin\alpha - \sin\gamma\cos\alpha & \cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha \\ \sin\gamma\cos\beta & \sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta\cos\alpha - \cos\gamma\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{pmatrix}
\end{aligned}
$$

#### 32.1.4.2.1.3 3D rotation about an arbitrary axis https://math.stackexchange.com/questions/4550704/rotation-around-an-arbitrary-axis

spherical coordinate unit vector

$$
\begin{cases} \hat{x} = r\sin\theta\cos\phi & \overset{r=1}{=} \sin\theta\cos\phi \\ \hat{y} = r\sin\theta\sin\phi & \overset{r=1}{=} \sin\theta\sin\phi \\ \hat{z} = r\cos\theta & \overset{r=1}{=} \cos\theta \end{cases}
$$

although I prefer $\theta$ and $\phi$ switched back to be compatible with 2D coordinate

$$
\begin{cases} \hat{x} = r\sin\phi\cos\theta & \overset{r=1}{=} \sin\phi\cos\theta \\ \hat{y} = r\sin\phi\sin\theta & \overset{r=1}{=} \sin\phi\sin\theta \\ \hat{z} = r\cos\phi & \overset{r=1}{=} \cos\phi \end{cases}
$$

or cos first in $x, y$-plane

$$
\begin{cases} \hat{x} = r\cos\phi\cos\theta & \overset{r=1}{=} \cos\phi\cos\theta \\ \hat{y} = r\cos\phi\sin\theta & \overset{r=1}{=} \cos\phi\sin\theta \\ \hat{z} = r\sin\phi & \overset{r=1}{=} \sin\phi \end{cases}
$$

---

still use the most convention

$$
\hat{\boldsymbol{n}} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}
$$

---

$$\begin{cases} \hat{\boldsymbol{n}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\ \hat{\boldsymbol{u}} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \Leftarrow \cos\theta\sin\theta - \cos\theta\sin\theta = 0 \Rightarrow \qquad \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}} = 0 \Leftrightarrow \hat{\boldsymbol{u}} \perp \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{v}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{u}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \hat{\boldsymbol{v}} \perp \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{v}} \perp \hat{\boldsymbol{u}} \end{cases} \end{cases}$$

$$S = \{\hat{\boldsymbol{n}}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}\} = \{\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{n}}\} \text{ is a basis of the spherical coordinate}$$

$$[\boldsymbol{V}]_S = \begin{pmatrix} u \\ v \\ n \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{V} &= \begin{pmatrix} \hat{\boldsymbol{u}} & \hat{\boldsymbol{v}} & \hat{\boldsymbol{n}} \end{pmatrix} \begin{pmatrix} u \\ v \\ n \end{pmatrix} = u\hat{\boldsymbol{u}} + v\hat{\boldsymbol{v}} + n\hat{\boldsymbol{n}} \\ &= \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ n \end{pmatrix} = S\,[\boldsymbol{V}]_S \\ \begin{pmatrix} u \\ v \\ n \end{pmatrix} &= \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^{-1} \boldsymbol{V} \\ [\boldsymbol{V}]_S &= S^{-1}\boldsymbol{V} \end{aligned}$$

$$\begin{aligned} S^{-1} &= \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^{-1}, S \in \{O | O^\intercal O = 1\} \\ &= \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^\intercal \\ &= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{u}}^\intercal \\ \hat{\boldsymbol{v}}^\intercal \\ \hat{\boldsymbol{n}}^\intercal \end{pmatrix} \\ S^{-1}S &= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \end{aligned}$$

$\hat{\boldsymbol{n}}$ as $z$ direction

$$\begin{aligned} [\boldsymbol{V}]_S &= S^{-1}\boldsymbol{V} \\ [\boldsymbol{V}']_S = R_z(\gamma)\,[\boldsymbol{V}]_S &= R_z(\gamma)\,S^{-1}\boldsymbol{V} \\ \boldsymbol{V}' = S\,[\boldsymbol{V}']_S &= SR_z(\gamma)\,[\boldsymbol{V}]_S = SR_z(\gamma)\,S^{-1}\boldsymbol{V} \\ \boldsymbol{V}' &= SR_z(\gamma)\,S^{-1}\boldsymbol{V} \end{aligned}$$

https://www.symbolab.com/

$$
\begin{aligned}
& SR_z(\gamma)S^{-1} \\
={}& \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \\
={}& \begin{pmatrix} c_1c_2 & -s_2 & s_1c_2 \\ c_1s_2 & c_2 & s_1s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1c_2 & c_1s_2 & -s_1 \\ -s_2 & c_2 & 0 \\ s_1c_2 & s_1s_2 & c_1 \end{pmatrix}, \begin{cases} c_1 = \cos\theta & s_1 = \sin\theta \\ c_2 = \cos\phi & s_2 = \sin\phi \\ c_3 = \cos\gamma & s_3 = \sin\gamma \end{cases} \\
={}& \begin{pmatrix} c_1c_2 & -s_2 & s_1c_2 \\ c_1s_2 & c_2 & s_1s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3c_1c_2 + s_3s_2 & c_3c_1s_2 - c_2s_3 & -c_3s_1 \\ c_1c_2s_3 - c_3s_2 & c_3c_2 + c_1s_3s_2 & -s_3s_1 \\ c_2s_1 & s_2s_1 & c_1 \end{pmatrix} \\
={}& \left( \begin{matrix} c_1^2c_2^2c_3 + c_3s_2^2 + c_2^2s_1^2 & c_1s_2\left(c_1c_2c_3 - s_2s_3\right) + c_2\left(-c_1c_2s_3 - c_3s_2\right) + c_2s_2s_1^2 & c_1c_2s_1 - s_1\left(c_1c_2c_3 - s_2 \right. \\ c_1c_2\left(c_1c_3s_2 + c_2s_3\right) - s_2\left(c_2c_3 - c_1s_2s_3\right) + c_2s_2s_1^2 & c_1^2c_3s_2^2 + s_2^2s_1^2 + c_2^2c_3 & c_1s_2s_1 - s_1\left(c_1c_3s_2 + c_2 \right. \\ -c_1c_2c_3s_1 - s_2s_1s_3 + c_1c_2s_1 & -c_1c_3s_2s_1 + c_2s_1s_3 + c_1s_2s_1 & c_3s_1^2 + c_1^2 \end{matrix} \right.
\end{aligned}
$$

#### 32.1.4.2.2 quaternion https://math.stackexchange.com/questions/328117/how-does-one-derive-this-rotation-quaternion-formula

$$
\begin{aligned}
q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, & \begin{cases} q \in \mathbb{H} \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ a, b, c, d \in \mathbb{R} & \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \end{cases} \\
= w = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z, & \begin{cases} w \in \mathbb{H} \\ \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 = \mathrm{ijk} & \Rightarrow \mathrm{ij} = \mathrm{k} \\ t, x, y, z \in \mathbb{R} & \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \end{cases} \\
= t + \begin{pmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + & \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + \boldsymbol{x}, \begin{cases} \boldsymbol{e}_1 = \boldsymbol{i} = \mathrm{i} \\ \boldsymbol{e}_2 = \boldsymbol{j} = \mathrm{j} \\ \boldsymbol{e}_3 = \boldsymbol{k} = \mathrm{k} \end{cases}
\end{aligned}
$$

$$q_1q_2 = \left(x_{10}x_{20} - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2\right) + \left(x_{10}\boldsymbol{x}_2 + x_{20}\boldsymbol{x}_1 + \boldsymbol{x}_1 \times \boldsymbol{x}_2\right)$$

$$QP = \left(Q_0P_0 - \boldsymbol{Q} \cdot \boldsymbol{P}\right) + \left(Q_0\boldsymbol{P} + P_0\boldsymbol{Q} + \boldsymbol{Q} \times \boldsymbol{P}\right)$$

$$q = q_0 + \boldsymbol{q} = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$
\begin{aligned}
q^* = \overline{q} ={}& \overline{q_0 + \boldsymbol{q}} = q_0 - \boldsymbol{q} \\
={}& \overline{q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}} = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k} \\
={}& q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}
\end{aligned}
$$

$$v = v_0 + \boldsymbol{v} = v_0 + v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} = v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

$$\boldsymbol{v} = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

$$
\begin{aligned}
qv ={}& \left(q_0v_0 - \boldsymbol{q} \cdot \boldsymbol{v}\right) + \left(q_0\boldsymbol{v} + v_0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{v}\right) \\
\overset{v_0=0}{=}{}& \left(q_0 0 - \boldsymbol{q} \cdot \boldsymbol{v}\right) + \left(q_0\boldsymbol{v} + 0\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{v}\right) \\
={}& \left(-\boldsymbol{q} \cdot \boldsymbol{v}\right) + \left(q_0\boldsymbol{v} + \boldsymbol{q} \times \boldsymbol{v}\right) \\
q\boldsymbol{v} ={}& \left(-\boldsymbol{q} \cdot \boldsymbol{v}\right) + \left(q_0\boldsymbol{v} + \boldsymbol{q} \times \boldsymbol{v}\right)
\end{aligned}
$$

$$\begin{aligned} v\overline{q} &= (v_0 q_0 - \boldsymbol{v}\cdot\overline{\boldsymbol{q}}) + (v_0\overline{\boldsymbol{q}} + q_0\boldsymbol{v} + \boldsymbol{v}\times\overline{\boldsymbol{q}}) \\ &\overset{v_0=0}{=} (0q_0 - \boldsymbol{v}\cdot\overline{\boldsymbol{q}}) + (0\overline{\boldsymbol{q}} + q_0\boldsymbol{v} + \boldsymbol{v}\times\overline{\boldsymbol{q}}) \\ &= (-\boldsymbol{v}\cdot\overline{\boldsymbol{q}}) + (q_0\boldsymbol{v} + \boldsymbol{v}\times\overline{\boldsymbol{q}}) \\ &= (-\boldsymbol{v}\cdot(-\boldsymbol{q})) + (q_0\boldsymbol{v} + \boldsymbol{v}\times(-\boldsymbol{q})) \\ &= (\boldsymbol{v}\cdot\boldsymbol{q}) + (q_0\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}) \\ \boldsymbol{v}\overline{q} &= (\boldsymbol{v}\cdot\boldsymbol{q}) + (q_0\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{q}) \end{aligned}$$

https://math.stackexchange.com/questions/41574/can-eulers-identity-be-extended-to-quaternions

$$\begin{aligned} q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ =& a + r\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}, r^2 = b^2 + c^2 + d^2 \\ =& a + \theta\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{\theta}, \theta^2 = b^2 + c^2 + d^2 \end{aligned}$$

$$\begin{aligned} \left(\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\right)^2 =& \frac{-b^2 - c^2 - d^2 + bc(\mathrm{ij} + \mathrm{ji}) + cd(\mathrm{jk} + \mathrm{kj}) + db(\mathrm{ki} + \mathrm{ik})}{r^2} \\ =& \frac{-b^2 - c^2 - d^2 + bc(\mathrm{k} - \mathrm{k}) + cd(\mathrm{i} - \mathrm{i}) + db(\mathrm{j} - \mathrm{j})}{r^2} \\ =& \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\ =& \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\ \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r} =& \pm\sqrt{-1} \end{aligned}$$

$$\begin{aligned} \mathrm{e}^q =& \mathrm{e}^{a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d} \\ =& \mathrm{e}^{a + r\sqrt{-1}} = \mathrm{e}^a\mathrm{e}^{r\sqrt{-1}} = \mathrm{e}^{a + \theta\sqrt{-1}} = \mathrm{e}^a\mathrm{e}^{\theta\sqrt{-1}} \\ =& \mathrm{e}^a\left(\cos r + \sqrt{-1}\sin r\right) = \mathrm{e}^a\left(\cos\theta + \sqrt{-1}\sin\theta\right) \\ =& \mathrm{e}^a\left(\cos r + \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\sin r\right) = \mathrm{e}^a\left[\cos r + (\mathrm{i}b + \mathrm{j}c + \mathrm{k}d)\frac{\sin r}{r}\right] \\ =& \mathrm{e}^a\left(\cos\theta + \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{\theta}\sin\theta\right) = \mathrm{e}^a\left[\cos\theta + (\mathrm{i}b + \mathrm{j}c + \mathrm{k}d)\frac{\sin\theta}{\theta}\right] \end{aligned}$$

$$\begin{aligned} q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ =& a + r\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}, r^2 = b^2 + c^2 + d^2 \\ =& \sqrt{a^2 + r^2}\left(\frac{a}{\sqrt{a^2 + r^2}} + \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\frac{r}{\sqrt{a^2 + r^2}}\right) \\ =& \rho\left(\cos\phi + \sqrt{-1}\sin\phi\right), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan\phi = \dfrac{r}{a} \Leftrightarrow \phi = \arctan\dfrac{r}{a} \end{cases} \\ =& \rho\mathrm{e}^{\phi\sqrt{-1}} \\ q =& \rho\mathrm{e}^{\phi\sqrt{-1}}, \begin{cases} q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ \rho = \sqrt{a^2 + r^2} = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan\phi = \dfrac{r}{a} \Leftrightarrow \phi = \arctan\dfrac{r}{a} = \arctan\dfrac{\pm\sqrt{b^2 + c^2 + d^2}}{a} \end{cases} \end{aligned}$$

$$\begin{aligned} \rho\mathrm{e}^{-\phi\sqrt{-1}} =& \rho\left[\cos(-\phi) + \sqrt{-1}\sin(-\phi)\right] \\ =& \rho\left[\cos\phi - \sqrt{-1}\sin\phi\right] \\ =& \sqrt{a^2 + r^2}\left[\frac{a}{\sqrt{a^2 + r^2}} - \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\frac{r}{\sqrt{a^2 + r^2}}\right] \\ =& a - (\mathrm{i}b + \mathrm{j}c + \mathrm{k}d) = a - \mathrm{i}b - \mathrm{j}c - \mathrm{k}d = \overline{q} = q^* \end{aligned}$$

### 32.1.5 octonion

## 32.2 Krasjet

https://github.com/Krasjet/quaternion

https://krasjet.github.io/quaternion/

https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf

### 32.2.1 Rodrigues rotation

$$\boldsymbol{v} \to \boldsymbol{v}'$$

$$\boldsymbol{v} \xrightarrow{\text{rotate about } \boldsymbol{u}} \boldsymbol{v}'$$

$$\boldsymbol{v} \xrightarrow{\text{rotate about } \boldsymbol{n}} \boldsymbol{v}'$$

$$\begin{cases} \boldsymbol{v} = \boldsymbol{v}_{\parallel \boldsymbol{n}} + \boldsymbol{v}_{\perp \boldsymbol{n}} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp} & \Rightarrow \boldsymbol{v}_{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\parallel} \\ \boldsymbol{v}' = \boldsymbol{v}'_{\parallel \boldsymbol{n}} + \boldsymbol{v}'_{\perp \boldsymbol{n}} = \boldsymbol{v}'_{\parallel} + \boldsymbol{v}'_{\perp} & \Rightarrow \boldsymbol{v}'_{\perp} = \boldsymbol{v}' - \boldsymbol{v}'_{\parallel} \end{cases}$$

$$\begin{aligned} \boldsymbol{v}_{\parallel \boldsymbol{n}} = \boldsymbol{v}_{\parallel} =& \mathrm{proj}_{\boldsymbol{n}} \boldsymbol{v} = \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|} \hat{\boldsymbol{n}} \\ =& \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|} \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|^2} \boldsymbol{n} \end{aligned}$$

$$\begin{aligned} \boldsymbol{v}_{\parallel \hat{\boldsymbol{n}}} = \boldsymbol{v}_{\parallel} =& \mathrm{proj}_{\hat{\boldsymbol{n}}} \boldsymbol{v} = \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{n}}}{\|\hat{\boldsymbol{n}}\|} \hat{\boldsymbol{n}} \overset{\|\hat{\boldsymbol{n}}\|=1}{=} (\boldsymbol{v} \cdot \hat{\boldsymbol{n}}) \hat{\boldsymbol{n}} = \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \hat{\boldsymbol{n}} \\ =& \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{n}}}{\|\hat{\boldsymbol{n}}\|} \frac{\hat{\boldsymbol{n}}}{\|\hat{\boldsymbol{n}}\|} = \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{n}}}{\|\hat{\boldsymbol{n}}\|^2} \hat{\boldsymbol{n}} \overset{\|\hat{\boldsymbol{n}}\|=1}{=} (\boldsymbol{v} \cdot \hat{\boldsymbol{n}}) \hat{\boldsymbol{n}} = \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \hat{\boldsymbol{n}} \end{aligned}$$

$$\boldsymbol{n} = \hat{\boldsymbol{n}}$$

$$\begin{aligned} \boldsymbol{v}_{\parallel \boldsymbol{n}} = \boldsymbol{v}_{\parallel} =& \mathrm{proj}_{\boldsymbol{n}} \boldsymbol{v} = \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|} \hat{\boldsymbol{n}} \overset{\|\boldsymbol{n}\|=1}{=} (\boldsymbol{v} \cdot \hat{\boldsymbol{n}}) \hat{\boldsymbol{n}} = \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \hat{\boldsymbol{n}} \\ =& \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|} \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|^2} \boldsymbol{n} \overset{\|\boldsymbol{n}\|=1}{=} (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n} \boldsymbol{n} \end{aligned}$$

$$\boldsymbol{v}_{\parallel} = (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n}$$

$$\boldsymbol{v}_{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\parallel} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n}$$

$$\boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp} = \boldsymbol{v} \xrightarrow{\text{rotate about } \boldsymbol{n}} \boldsymbol{v}' = \boldsymbol{v}'_{\parallel} + \boldsymbol{v}'_{\perp}$$

$$\boldsymbol{v}_{\parallel} \xrightarrow{\text{rotate about } \boldsymbol{n}} \boldsymbol{v}'_{\parallel}$$

$$\boldsymbol{v}_{\parallel} = \boldsymbol{v}_{\parallel \boldsymbol{n}} = \boldsymbol{v}'_{\parallel \boldsymbol{n}} = \boldsymbol{v}'_{\parallel}$$

$$\boldsymbol{v}_{\parallel} = \boldsymbol{v}'_{\parallel}$$

$$\boldsymbol{v}'_{\parallel} = \boldsymbol{v}_{\parallel}$$

$$\begin{cases} \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{v}_{\perp} \\ \|\boldsymbol{u}\| = \|\boldsymbol{n} \times \boldsymbol{v}_{\perp}\| = \|\boldsymbol{n}\| \, \|\boldsymbol{v}_{\perp}\| \sin \frac{\pi}{2} = \|\boldsymbol{v}_{\perp}\| \end{cases} \quad \begin{cases} \boldsymbol{u} \perp \boldsymbol{n} \\ \boldsymbol{u} \perp \boldsymbol{v}_{\perp} \\ \boldsymbol{n} \perp \boldsymbol{v}_{\perp} \\ \|\boldsymbol{n}\| = 1 \end{cases}$$

$$\begin{aligned} \boldsymbol{v}'_{\perp} =& \boldsymbol{v}'_{\|\boldsymbol{v}_{\perp}} + \boldsymbol{v}'_{\|\boldsymbol{u}} \\ &= (\cos\theta)\, \boldsymbol{v}_{\perp} + (\sin\theta)\, \boldsymbol{u}, \theta = \angle \boldsymbol{v}'_{\perp} \boldsymbol{v}_{\perp} \\ &= \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \boldsymbol{u} \\ &= \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \\ \boldsymbol{v}'_{\perp} &= \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \end{aligned}$$

$$\boldsymbol{v}'_{\perp} = \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \, , \theta = \angle \boldsymbol{v}'_{\perp} \boldsymbol{v}_{\perp}$$

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$$\boldsymbol{v}'_{\perp} = \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp})$$

---

$$\begin{aligned} \boldsymbol{v}' =& \boldsymbol{v}'_{\|} + \boldsymbol{v}'_{\perp}, \begin{cases} \boldsymbol{v}'_{\|} = \boldsymbol{v}_{\|} \\ \boldsymbol{v}'_{\perp} = \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \, , \theta = \angle \boldsymbol{v}'_{\perp} \boldsymbol{v}_{\perp} \end{cases} \\ =& \boldsymbol{v}_{\|} + \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \, , \theta = \angle \boldsymbol{v}'_{\perp} \boldsymbol{v}_{\perp} \\ =& \boldsymbol{v}_{\|} + \cos\theta \boldsymbol{v}_{\perp} + \sin\theta \, (\boldsymbol{n} \times \boldsymbol{v}_{\perp}) \, , \begin{cases} \boldsymbol{v}_{\|} = (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} \\ \boldsymbol{v}_{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\|} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} \end{cases} \\ =& (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \cos\theta \, [\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}] + \sin\theta \, [\boldsymbol{n} \times (\boldsymbol{v} - \boldsymbol{v}_{\|})] \\ =& (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \cos\theta \boldsymbol{v} - \cos\theta \, (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \sin\theta \, [\boldsymbol{n} \times \boldsymbol{v} - \boldsymbol{n} \times \boldsymbol{v}_{\|}] \\ =& \cos\theta \boldsymbol{v} + (1 - \cos\theta)\, (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \sin\theta \, [\boldsymbol{n} \times \boldsymbol{v} - \boldsymbol{0}] \\ =& \cos\theta \boldsymbol{v} + (1 - \cos\theta)\, (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \sin\theta \boldsymbol{n} \times \boldsymbol{v} \end{aligned}$$

---

$$\boldsymbol{v}' = \cos\theta \boldsymbol{v} + (1 - \cos\theta)\, (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \sin\theta \boldsymbol{n} \times \boldsymbol{v}$$

---

$$\begin{aligned} \boldsymbol{v}' &= \cos\theta \boldsymbol{v} + (1 - \cos\theta)\, (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \sin\theta \boldsymbol{n} \times \boldsymbol{v} \\ &= \cos\theta \boldsymbol{v} + \left[1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right] (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] \boldsymbol{n} \times \boldsymbol{v} \\ &= \cos\theta \boldsymbol{v} + \left[2\sin^2\frac{\theta}{2}\right] (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] \boldsymbol{n} \times \boldsymbol{v} \\ &= \cos\theta \boldsymbol{v} + 2\sin\frac{\theta}{2} \left[\left(\sin\frac{\theta}{2}\right) (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n} + \left(\cos\frac{\theta}{2}\right) \boldsymbol{n} \times \boldsymbol{v}\right] \\ &= \cos\theta \boldsymbol{v} + 2\sin\frac{\theta}{2} \left[\left(\cos\frac{\theta}{2}\right) \boldsymbol{n} \times \boldsymbol{v} + \left(\sin\frac{\theta}{2}\right) (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}\right] \\ &= \left[2\cos^2\frac{\theta}{2} - 1\right] \boldsymbol{v} + 2\sin\frac{\theta}{2} \left[\left(\cos\frac{\theta}{2}\right) \boldsymbol{n} \times \boldsymbol{v} + \left(\sin\frac{\theta}{2}\right) (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}\right] \end{aligned}$$

### 32.2.2 quaternion operation

$$\begin{aligned} q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \\ \varrho =& \alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \end{aligned}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \leftarrow q\varrho = (a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})(\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k})$$

$$\begin{aligned}
&= a(\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k}) \\
&+ b\mathrm{i}(\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k}) \\
&+ c\mathrm{j}(\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k}) \\
&+ d\mathrm{k}(\alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k}) \\
&= a\alpha + a\beta\mathrm{i} + a\gamma\mathrm{j} + a\delta\mathrm{k} \\
&+ b\alpha\mathrm{i} - b\beta + b\gamma\mathrm{k} - b\delta\mathrm{j} \\
&+ c\alpha\mathrm{j} - c\beta\mathrm{k} - c\gamma + c\delta\mathrm{i} \\
&+ d\alpha\mathrm{k} + d\beta\mathrm{j} - d\gamma\mathrm{i} - d\delta \\
&= (a\alpha - b\beta - c\gamma - d\delta) \\
&+ (b\alpha + a\beta - d\gamma + c\delta)\,\mathrm{i} \\
&+ (c\alpha + d\beta + a\gamma - b\delta)\,\mathrm{j} \\
&+ (d\alpha - c\beta + b\gamma + a\delta)\,\mathrm{k}
\end{aligned}$$

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \leftarrow$$

$$\begin{aligned}
&= (\alpha a - \beta b - \gamma c - \delta d) \\
&+ (\beta a + \alpha b + \delta c - \gamma d)\,\mathrm{i} \\
&+ (\gamma a - \delta b + \alpha c + \beta d)\,\mathrm{j} \\
&+ (\delta a + \gamma b - \beta c + \alpha d)\,\mathrm{k}
\end{aligned}$$

$$\begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \leftarrow$$

$$qv \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_\iota v$$

concept like quaternion group representation[35.8]

$$\begin{aligned}
L(q) = Q_\iota = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} a + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} b \\
&+ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} c + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} d \\
&\leftrightarrow 1a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d
\end{aligned}$$

$$\mathrm{ij} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \mathrm{k}$$

$$\mathrm{ji} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -\mathrm{k}$$

$$\mathrm{ki} \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow \mathrm{j}$$

$$Q_l = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} a & -\begin{pmatrix} b & c & d \end{pmatrix} \\ \begin{pmatrix} b \\ c \\ d \end{pmatrix} & \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{pmatrix} = \begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i \end{pmatrix}, \begin{cases} a = q_0 \\ \boldsymbol{q} = \begin{pmatrix} b \\ c \\ d \end{pmatrix} \\ Q_i = \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{cases}$$

Grassmann product

$$qv \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_l v = \begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \boldsymbol{q}^\intercal \boldsymbol{v} \\ \boldsymbol{q} v_0 + Q_i \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \boldsymbol{q} \cdot \boldsymbol{v} \\ \boldsymbol{q} v_0 + q_0 \boldsymbol{v} + \boldsymbol{q} \times \boldsymbol{v} \end{pmatrix}$$

$$vq \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_r v$$

$$\begin{aligned} R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} &= a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &+ c \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &\leftrightarrow a1 + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \end{aligned}$$

$$\mathrm{ij} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -\mathrm{k}$$

$$\mathrm{ji} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \mathrm{k}$$

$$\mathrm{ki} \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow -\mathrm{j}$$

$$\begin{aligned} R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} &\leftrightarrow \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} \\ &= \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \begin{cases} \alpha = a+bi & \overline{\alpha} = a-bi \\ \beta = c+di & \overline{\beta} = c-di \end{cases} \end{aligned}$$

$$Q_r=\begin{pmatrix}a&-b&-c&-d\\b&a&d&-c\\c&-d&a&b\\d&c&-b&a\end{pmatrix}=\begin{pmatrix}a&-\begin{pmatrix}b&c&d\end{pmatrix}\\\begin{pmatrix}b\\c\\d\end{pmatrix}&\begin{pmatrix}a&d&-c\\-d&a&b\\c&-b&a\end{pmatrix}\end{pmatrix}=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i^\intercal\end{pmatrix},\begin{cases}a=q_0\\\boldsymbol{q}=\begin{pmatrix}b\\c\\d\end{pmatrix}\\Q_i=\begin{pmatrix}a&-d&c\\d&a&-b\\-c&b&a\end{pmatrix}\end{cases}$$

$$vq\leftrightarrow\begin{pmatrix}a&-b&-c&-d\\b&a&d&-c\\c&-d&a&b\\d&c&-b&a\end{pmatrix}\begin{pmatrix}v_0\\v_1\\v_2\\v_3\end{pmatrix}=Q_rv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}v_0q_0-\boldsymbol{q}^\intercal\boldsymbol{v}\\v_0\boldsymbol{q}+Q_i^\intercal\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}v_0q_0-\boldsymbol{v}\cdot\boldsymbol{q}\\v_0\boldsymbol{q}+q_0\boldsymbol{v}+\boldsymbol{v}\times\boldsymbol{q}\end{pmatrix}=\begin{pmatrix}q_0v_0-\boldsymbol{q}\cdot\boldsymbol{v}\\\boldsymbol{q}v_0+q_0\boldsymbol{v}-\boldsymbol{q}\times\boldsymbol{v}\end{pmatrix}$$

$$\begin{cases}qv\leftrightarrow Q_lv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}q_0v_0-\boldsymbol{q}^\intercal\boldsymbol{v}\\\boldsymbol{q}v_0+Q_i\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}q_0v_0-\boldsymbol{q}\cdot\boldsymbol{v}\\\boldsymbol{q}v_0+q_0\boldsymbol{v}+\boldsymbol{q}\times\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}q_0v_0-\boldsymbol{q}\cdot\boldsymbol{v}\\\boldsymbol{q}v_0+q_0\boldsymbol{v}+\boldsymbol{q}\times\boldsymbol{v}\end{pmatrix}\\vq\leftrightarrow Q_rv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}v_0q_0-\boldsymbol{q}^\intercal\boldsymbol{v}\\v_0\boldsymbol{q}+Q_i^\intercal\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}v_0q_0-\boldsymbol{v}\cdot\boldsymbol{q}\\v_0\boldsymbol{q}+\boldsymbol{v}q_0+\boldsymbol{v}\times\boldsymbol{q}\end{pmatrix}=\begin{pmatrix}q_0v_0-\boldsymbol{q}\cdot\boldsymbol{v}\\\boldsymbol{q}v_0+q_0\boldsymbol{v}-\boldsymbol{q}\times\boldsymbol{v}\end{pmatrix}\end{cases}$$

$$\begin{cases}qv=\begin{pmatrix}q_0\\\boldsymbol{q}\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}=L(q)\,v=Q_lv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}\\vq=\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}\begin{pmatrix}q_0\\\boldsymbol{q}\end{pmatrix}=R(q)\,v=Q_rv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\\boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}v_0\\\boldsymbol{v}\end{pmatrix}\end{cases}$$

$$\begin{cases}\boldsymbol{v}\xrightarrow{\text{rotate about }\boldsymbol{n}}\boldsymbol{v}'\\\begin{cases}\boldsymbol{v}=\boldsymbol{v}_{\parallel\boldsymbol{n}}+\boldsymbol{v}_{\perp\boldsymbol{n}}=\boldsymbol{v}_\parallel+\boldsymbol{v}_\perp=&\Rightarrow\boldsymbol{v}_\perp=\boldsymbol{v}-\boldsymbol{v}_\parallel\\\boldsymbol{v}'=\boldsymbol{v}'_{\parallel\boldsymbol{n}}+\boldsymbol{v}'_{\perp\boldsymbol{n}}=\boldsymbol{v}'_\parallel+\boldsymbol{v}'_\perp&\Rightarrow\boldsymbol{v}'_\perp=\boldsymbol{v}'-\boldsymbol{v}'_\parallel\end{cases}\\\begin{cases}\boldsymbol{u}=\boldsymbol{n}\times\boldsymbol{v}_\perp&\begin{cases}\boldsymbol{u}\perp\boldsymbol{n}\\\boldsymbol{u}\perp\boldsymbol{v}_\perp\end{cases}\\\|\boldsymbol{u}\|=\|\boldsymbol{n}\times\boldsymbol{v}_\perp\|=\|\boldsymbol{n}\|\,\|\boldsymbol{v}_\perp\|\sin\dfrac{\pi}{2}=\|\boldsymbol{v}_\perp\|&\begin{cases}\boldsymbol{n}\perp\boldsymbol{v}_\perp\\\|\boldsymbol{n}\|=1\end{cases}\end{cases}\\\boldsymbol{v}'_\parallel=\boldsymbol{v}_\parallel\\\boldsymbol{v}'_\perp=\cos\theta\boldsymbol{v}_\perp+\sin\theta\,(\boldsymbol{n}\times\boldsymbol{v}_\perp)\,,\theta=\angle\boldsymbol{v}'_\perp\boldsymbol{v}_\perp\end{cases}$$

$$\leftrightarrow\begin{cases}v\xrightarrow{\text{rotate about }n}v'\\\begin{cases}v=\begin{pmatrix}0\\\boldsymbol{v}\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{v}_{\parallel\boldsymbol{n}}\end{pmatrix}+\begin{pmatrix}0\\\boldsymbol{v}_{\perp\boldsymbol{n}}\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{v}_\parallel\end{pmatrix}+\begin{pmatrix}0\\\boldsymbol{v}_\perp\end{pmatrix}=v_\parallel+v_\perp\quad\Rightarrow v_\perp=v-v_\parallel\\v'=\begin{pmatrix}0\\\boldsymbol{v}'\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{v}'_{\parallel\boldsymbol{n}}\end{pmatrix}+\begin{pmatrix}0\\\boldsymbol{v}'_{\perp\boldsymbol{n}}\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{v}'_\parallel\end{pmatrix}+\begin{pmatrix}0\\\boldsymbol{v}'_\perp\end{pmatrix}=v'_\parallel+v'_\perp\quad\Rightarrow v'_\perp=v'-v'_\parallel\end{cases}\\\begin{cases}u=\begin{pmatrix}0\\\boldsymbol{u}\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{n}\times\boldsymbol{v}_\perp\end{pmatrix}=\begin{pmatrix}00-0\\\boldsymbol{n}0+0\boldsymbol{v}_\perp+\boldsymbol{n}\times\boldsymbol{v}_\perp\end{pmatrix}=\begin{pmatrix}n_0v_{\perp0}-\boldsymbol{n}\cdot\boldsymbol{v}_\perp\\\boldsymbol{n}v_{\perp0}+n_0\boldsymbol{v}_\perp+\boldsymbol{n}\times\boldsymbol{v}_\perp\end{pmatrix}=nv_\perp&\begin{cases}\boldsymbol{u}\perp\boldsymbol{n}\\\boldsymbol{u}\perp\boldsymbol{v}_\perp\end{cases}\\\|\boldsymbol{u}\|=\|\boldsymbol{n}\times\boldsymbol{v}_\perp\|=\|\boldsymbol{n}\|\,\|\boldsymbol{v}_\perp\|\sin\dfrac{\pi}{2}=\|\boldsymbol{v}_\perp\|&\begin{cases}\boldsymbol{n}\perp\boldsymbol{v}_\perp\\\|\boldsymbol{n}\|=1\end{cases}\end{cases}\\v'_\parallel=\begin{pmatrix}0\\\boldsymbol{v}'_\parallel\end{pmatrix}=\begin{pmatrix}0\\\boldsymbol{v}_\parallel\end{pmatrix}=v_\parallel\\v'_\perp=\begin{pmatrix}0\\\boldsymbol{v}'_\perp\end{pmatrix}=\cos\theta\begin{pmatrix}0\\\boldsymbol{v}_\perp\end{pmatrix}+\sin\theta\begin{pmatrix}0\\\boldsymbol{n}\times\boldsymbol{v}_\perp\end{pmatrix}=\cos\theta v_\perp+\sin\theta nv_\perp=(\cos\theta+\sin\theta n)\,v_\perp\end{cases}$$

$$v'_\perp=(\cos\theta+\sin\theta n)\,v_\perp$$

$$v'_\perp=pv_\perp,p=\cos\theta+\sin\theta n=\cos\theta\begin{pmatrix}1\\\boldsymbol{0}\end{pmatrix}+\sin\theta\begin{pmatrix}0\\\boldsymbol{n}\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\boldsymbol{n}\end{pmatrix}$$

$$p = \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix} = \cos\theta + \sin\theta n, \|\boldsymbol{n}\| = 1$$

$$|p| = \left\| \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix}} = \sqrt{\cos^2\theta + \sin^2\theta \|\boldsymbol{n}\|^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$|p| = \left\| \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix} \right\| = 1$$

$$p^* = \overline{p} = \begin{pmatrix} p_0 \\ -\boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta\boldsymbol{n} \end{pmatrix} = \cos\theta - \sin\theta n$$

$$\begin{aligned} p^*p = \overline{p}p &= \begin{pmatrix} p_0 \\ -\boldsymbol{p} \end{pmatrix} \begin{pmatrix} p_0 \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} p_0p_0 - (-\boldsymbol{p}) \cdot \boldsymbol{p} \\ -\boldsymbol{p}p_0 + p_0\boldsymbol{p} + (-\boldsymbol{p}) \times \boldsymbol{p} \end{pmatrix} \\ &= \begin{pmatrix} p_0p_0 + \boldsymbol{p} \cdot \boldsymbol{p} \\ \boldsymbol{0} + \boldsymbol{0} \end{pmatrix} = p_0p_0 + \boldsymbol{p} \cdot \boldsymbol{p} = p_0^2 + \|\boldsymbol{p}\|^2 = |p|^2 = 1^2 = 1 \\ &= \begin{pmatrix} p_0p_0 - \boldsymbol{p} \cdot (-\boldsymbol{p}) \\ \boldsymbol{p}p_0 + p_0(-\boldsymbol{p}) + \boldsymbol{p} \times (-\boldsymbol{p}) \end{pmatrix} = p\overline{p} = pp^* \end{aligned}$$

$$p^*p = \overline{p}p = 1 = p\overline{p} = pp^* = |p|^2 = p_0^2 + \|\boldsymbol{p}\|^2 \Rightarrow p^* = \frac{1}{p} = p^{-1}$$

$$\begin{aligned} p^2 = pp &= \begin{pmatrix} p_0p_0 - \boldsymbol{p} \cdot \boldsymbol{p} \\ \boldsymbol{p}p_0 + p_0\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta \|\boldsymbol{n}\|^2 \\ 2(\sin\theta\boldsymbol{n})\cos\theta + \boldsymbol{0} \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta\boldsymbol{n} \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta\boldsymbol{n} \end{pmatrix} \\ \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix}^2 &= \begin{pmatrix} \cos 2\theta \\ \sin 2\theta\boldsymbol{n} \end{pmatrix} \\ q^2 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\boldsymbol{n} \end{pmatrix}^2 &= \begin{pmatrix} \cos\theta \\ \sin\theta\boldsymbol{n} \end{pmatrix} = p \end{aligned}$$

$qv_{\parallel} = v_{\parallel}q$

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\parallel\boldsymbol{n}} &= \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\parallel} = \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} \begin{pmatrix} 0 \\ \boldsymbol{v}_{\parallel} \end{pmatrix} \\ &= \begin{pmatrix} \alpha 0 - \beta\boldsymbol{n} \cdot \boldsymbol{v}_{\parallel} \\ \beta\boldsymbol{n}0 + \alpha\boldsymbol{v}_{\parallel} + \beta\boldsymbol{n} \times \boldsymbol{v}_{\parallel} \end{pmatrix} = \begin{pmatrix} -\beta\boldsymbol{n} \cdot \boldsymbol{v}_{\parallel} \\ \alpha\boldsymbol{v}_{\parallel} + \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} -\beta\boldsymbol{n} \cdot \boldsymbol{v}_{\parallel} \\ \alpha\boldsymbol{v}_{\parallel} \end{pmatrix} \\ &= \begin{pmatrix} 0\alpha - \boldsymbol{v}_{\parallel} \cdot \beta\boldsymbol{n} \\ \boldsymbol{v}_{\parallel}\alpha + 0\beta\boldsymbol{n} + \boldsymbol{v}_{\parallel} \times \beta\boldsymbol{n} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{v}_{\parallel} \cdot \beta\boldsymbol{n} \\ \alpha\boldsymbol{v}_{\parallel} + \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} -\beta\boldsymbol{n} \cdot \boldsymbol{v}_{\parallel} \\ \alpha\boldsymbol{v}_{\parallel} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \boldsymbol{v}_{\parallel} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} = v_{\parallel} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} = v_{\parallel\boldsymbol{n}} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} \\ \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\parallel} &= v_{\parallel} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} \end{aligned}$$

$qv_{\perp} = v_{\perp}q^*$

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\perp\boldsymbol{n}} &= \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\perp} = \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} \begin{pmatrix} 0 \\ \boldsymbol{v}_{\perp} \end{pmatrix} \\ &= \begin{pmatrix} \alpha 0 - \beta\boldsymbol{n} \cdot \boldsymbol{v}_{\perp} \\ \beta\boldsymbol{n}0 + \alpha\boldsymbol{v}_{\perp} + \beta\boldsymbol{n} \times \boldsymbol{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha\boldsymbol{v}_{\perp} + \beta\boldsymbol{n} \times \boldsymbol{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha\boldsymbol{v}_{\perp} + \boldsymbol{v}_{\perp} \times (-\beta\boldsymbol{n}) \end{pmatrix} \\ &= \begin{pmatrix} 0\alpha - \boldsymbol{v}_{\perp} \cdot (-\beta\boldsymbol{n}) \\ \boldsymbol{v}_{\perp}\alpha + 0(-\beta\boldsymbol{n}) + \boldsymbol{v}_{\perp} \times (-\beta\boldsymbol{n}) \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha\boldsymbol{v}_{\perp} + \boldsymbol{v}_{\perp} \times (-\beta\boldsymbol{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha\boldsymbol{v}_{\perp} + \beta\boldsymbol{n} \times \boldsymbol{v}_{\perp} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \boldsymbol{v}_{\perp} \end{pmatrix} \begin{pmatrix} \alpha \\ -\beta\boldsymbol{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ -\beta\boldsymbol{n} \end{pmatrix} = v_{\perp\boldsymbol{n}} \begin{pmatrix} \alpha \\ -\beta\boldsymbol{n} \end{pmatrix} \\ \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix} v_{\perp} &= v_{\perp} \begin{pmatrix} \alpha \\ -\beta\boldsymbol{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix}^* = v_{\perp} \overline{\begin{pmatrix} \alpha \\ \beta\boldsymbol{n} \end{pmatrix}} \end{aligned}$$

$$q=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}n\in\left\{p\,\middle|\,\begin{cases}p=\cos\theta+\sin\theta n\\ n=\begin{pmatrix}0\\ \boldsymbol{n}\end{pmatrix},\|\boldsymbol{n}\|=1\end{cases}\right\}\subset\left\{\begin{pmatrix}\alpha\\ \beta\boldsymbol{n}\end{pmatrix}\,\middle|\,\begin{cases}\alpha,\beta\in\mathbb{R}\\ \|\boldsymbol{n}\|=1\end{cases}\right\}$$

$$\Rightarrow\begin{cases}q^*q=\overline{q}q=1=q\overline{q}=qq^*=|q|^2=q_0^2+\|\boldsymbol{q}\|^2\Rightarrow q^*=\dfrac{1}{q}=q^{-1} & (u)\,\text{unit quaternion}\\ qv_\parallel=v_\parallel q & (c)\,\text{commutativity}\\ qv_\perp=v_\perp q^*=v_\perp\overline{q}\Leftrightarrow\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\boldsymbol{n}\end{pmatrix}v_\perp=v_\perp\begin{pmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\boldsymbol{n}\end{pmatrix} & (a)\,\text{anticommutativity}\end{cases}$$

$$\begin{aligned}v'=&v_\parallel'+v_\perp',\begin{cases}v_\parallel'=v_\parallel & q=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}n,|n|=1\\ v_\perp'=pv_\perp=q^2v_\perp & p=q^2=\cos\theta+\sin\theta n\end{cases}\\ =&v_\parallel+q^2v_\perp\overset{(u)}{=}1v_\parallel+qqv_\perp\\ \overset{(a)}{=}&qq^*v_\parallel+qv_\perp q^*\overset{(c)}{=}qv_\parallel q^*+qv_\perp q^*\\ =&q\left(v_\parallel+v_\perp\right)q^*=q\left(v\right)q^*=qvq^*\end{aligned}$$

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$$\begin{aligned}v'=&qvq^{-1},q=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}n,|n|=1\\ =&qvq^*=q\left(v_\parallel+v_\perp\right)q^*\\ =&qv_\parallel q^*+qv_\perp q^*=qq^*v_\parallel+qqv_\perp=1v_\parallel+q^2v_\perp\\ =&v_\parallel+pv_\perp,p=q^2=\cos\theta+\sin\theta n\end{aligned}$$

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$$\begin{aligned}v'=qvq^*=q\left(vq^*\right)\leftrightarrow&L\left(q\right)R\left(q^*\right)v=Q_lQ_rv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i\end{pmatrix}\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}v_0\\ \boldsymbol{v}\end{pmatrix}\\ =\left(qv\right)q^*\leftrightarrow&R\left(q^*\right)L\left(q\right)v=Q_rQ_lv=\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i\end{pmatrix}\begin{pmatrix}v_0\\ \boldsymbol{v}\end{pmatrix}\end{aligned}$$

$$\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i\end{pmatrix}\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i^\intercal\end{pmatrix}=\begin{pmatrix}q_0^2-\boldsymbol{q}^\intercal\boldsymbol{q}&-q_0\boldsymbol{q}^\intercal-\boldsymbol{q}^\intercal Q_i^\intercal\\ \boldsymbol{q}q_0+Q_i\boldsymbol{q}&-\boldsymbol{q}\boldsymbol{q}^\intercal+Q_iQ_i^\intercal\end{pmatrix}=L\left(q\right)R\left(q^*\right)$$

$$\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i^\intercal\end{pmatrix}\begin{pmatrix}q_0&-\boldsymbol{q}^\intercal\\ \boldsymbol{q}&Q_i\end{pmatrix}=\begin{pmatrix}q_0^2-\boldsymbol{q}^\intercal\boldsymbol{q}&-q_0\boldsymbol{q}^\intercal-\boldsymbol{q}^\intercal Q_i\\ \boldsymbol{q}q_0+Q_i^\intercal\boldsymbol{q}&-\boldsymbol{q}\boldsymbol{q}^\intercal+Q_i^\intercal Q_i\end{pmatrix}=R\left(q^*\right)L\left(q\right)$$

$$q=a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}n=\cos\frac{\theta}{2}+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}$$

$$\begin{aligned}\Rightarrow Q_iQ_i^\intercal=&\begin{pmatrix}a&-d&c\\ d&a&-b\\ -c&b&a\end{pmatrix}\begin{pmatrix}a&d&-c\\ -d&a&b\\ c&-b&a\end{pmatrix}=\begin{pmatrix}a^2+d^2+c^2&-cb&-db\\ -cb&d^2+a^2+b^2&-dc\\ -db&-dc&c^2+b^2+a^2\end{pmatrix}\\ =Q_i^\intercal Q_i=&\begin{pmatrix}a&d&-c\\ -d&a&b\\ c&-b&a\end{pmatrix}\begin{pmatrix}a&-d&c\\ d&a&-b\\ -c&b&a\end{pmatrix}=\begin{pmatrix}a^2+d^2+c^2&-cb&-db\\ -cb&d^2+a^2+b^2&-dc\\ -db&-dc&c^2+b^2+a^2\end{pmatrix}\\ \boldsymbol{q}^\intercal Q_i^\intercal=&\begin{pmatrix}b&c&d\end{pmatrix}\begin{pmatrix}a&d&-c\\ -d&a&b\\ c&-b&a\end{pmatrix}=\begin{pmatrix}ba&ca&da\end{pmatrix}\\ =\boldsymbol{q}^\intercal Q_i=&\begin{pmatrix}b&c&d\end{pmatrix}\begin{pmatrix}a&-d&c\\ d&a&-b\\ -c&b&a\end{pmatrix}=\begin{pmatrix}ba&ca&da\end{pmatrix}\\ Q_i\boldsymbol{q}=&\begin{pmatrix}a&-d&c\\ d&a&-b\\ -c&b&a\end{pmatrix}\begin{pmatrix}b\\ c\\ d\end{pmatrix}=\begin{pmatrix}ab\\ ac\\ ad\end{pmatrix}\\ =Q_i^\intercal\boldsymbol{q}=&\begin{pmatrix}a&d&-c\\ -d&a&b\\ c&-b&a\end{pmatrix}\begin{pmatrix}b\\ c\\ d\end{pmatrix}=\begin{pmatrix}ab\\ ac\\ ad\end{pmatrix}\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i \end{pmatrix}\begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i^\intercal \end{pmatrix} = \begin{pmatrix} q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q} & -q_0\boldsymbol{q}^\intercal - \boldsymbol{q}^\intercal Q_i^\intercal \\ \boldsymbol{q}q_0 + Q_i\boldsymbol{q} & -\boldsymbol{q}\boldsymbol{q}^\intercal + Q_iQ_i^\intercal \end{pmatrix} = L(q)\,R(q^*) \\
=&\begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i^\intercal \end{pmatrix}\begin{pmatrix} q_0 & -\boldsymbol{q}^\intercal \\ \boldsymbol{q} & Q_i \end{pmatrix} = \begin{pmatrix} q_0^2 - \boldsymbol{q}^\intercal\boldsymbol{q} & -q_0\boldsymbol{q}^\intercal - \boldsymbol{q}^\intercal Q_i \\ \boldsymbol{q}q_0 + Q_i^\intercal\boldsymbol{q} & -\boldsymbol{q}\boldsymbol{q}^\intercal + Q_i^\intercal Q_i \end{pmatrix} = R(q^*)\,L(q) \\
&\Downarrow \\
L(q)\,R(q^*) =& R(q^*)\,L(q) \\
&\updownarrow \\
v' = qvq^* = q\,(vq^*) =& (qv)\,q^*
\end{aligned}$$

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$$a = \cos\frac{\theta}{2} \Rightarrow \theta = 2\arccos a = 2\cos^{-1} a$$

$$\sin\frac{\theta}{2}n = \sin\frac{2\arccos a}{2}n = \sin\left(\cos^{-1} a\right) n$$

$$n = \begin{pmatrix} 0 \\ \boldsymbol{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{n}} \end{pmatrix}, \|\boldsymbol{n}\| = 1, \boldsymbol{n} = \begin{pmatrix} \sin\gamma\cos\varphi \\ \sin\gamma\sin\varphi \\ \cos\gamma \end{pmatrix}, \begin{cases} \varphi = \angle\hat{\boldsymbol{n}}_{xy}\hat{\boldsymbol{x}} \\ \gamma = \angle\hat{\boldsymbol{n}}\hat{\boldsymbol{z}} \end{cases}$$

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$$\begin{aligned}
q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \\
=& \cos\frac{\theta}{2} + \sin\frac{\theta}{2}n, n = \begin{pmatrix} 0 \\ \boldsymbol{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{n}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\gamma\cos\varphi \\ \sin\gamma\sin\varphi \\ \cos\gamma \end{pmatrix}, \|\boldsymbol{n}\| = 1, \begin{cases} \varphi = \angle\hat{\boldsymbol{n}}_{xy}\hat{\boldsymbol{x}} \\ \gamma = \angle\hat{\boldsymbol{n}}\hat{\boldsymbol{z}} \end{cases} \\
=& \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{n}, \boldsymbol{n} = (\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k} \\
=& \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right] \\
=& \cos\frac{\theta}{2} + \boldsymbol{n}\sin\frac{\theta}{2} \\
q^{-1} = q^* =& \cos\frac{\theta}{2} - \boldsymbol{n}\sin\frac{\theta}{2} \\
=& \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right] \\
\boldsymbol{v}' = v' =& qvq^{-1} = q\boldsymbol{v}q^{-1}, \begin{cases} v = \boldsymbol{v} = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} \\ v' = \boldsymbol{v}' = v_1'\mathrm{i} + v_2'\mathrm{j} + v_3'\mathrm{k} \end{cases} \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right]\right\}\boldsymbol{v} \\
& \left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right]\right\} \\
=& q\,(\theta,\gamma,\varphi)\,\boldsymbol{v}q^{-1}\,(\theta,\gamma,\varphi) \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}\right]\right\}\boldsymbol{v}\left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}\right]\right\} \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} \pm \sqrt{1 - n_x^2 - n_z^2}\mathrm{j} + n_z\mathrm{k}\right]\right\}\boldsymbol{v} \\
& \left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} \pm \sqrt{1 - n_x^2 - n_z^2}\mathrm{j} + n_z\mathrm{k}\right]\right\} \\
=& q\,(\theta,n_z,n_x)\,\boldsymbol{v}q^{-1}\,(\theta,n_z,n_x)
\end{aligned}$$

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$$\boldsymbol{v}' = v' = qvq^{-1} = q\boldsymbol{v}q^{-1}, \begin{cases} v = \boldsymbol{v} = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} \\ v' = \boldsymbol{v}' = v_1'\mathrm{i} + v_2'\mathrm{j} + v_3'\mathrm{k} \\ q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right] \\ q^{-1} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right] \end{cases}$$

### 32.2.3 matrix form

$$q \overset{|q|=1}{=} a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}, \begin{cases} a = \cos\frac{\theta}{2} \\ b = \sin\frac{\theta}{2}\sin\gamma\cos\varphi \\ c = \sin\frac{\theta}{2}\sin\gamma\sin\varphi \\ d = \sin\frac{\theta}{2}\cos\gamma \end{cases}$$

$$L(q) = Q_l = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

$$R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

$$R(q^*) = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

$$\begin{aligned} v' = qvq^* &= (qv)\,q^* \leftrightarrow R(q^*)\,L(q)\,v \\ &= q\,(vq^*) \leftrightarrow L(q)\,R(q^*)\,v \\ &= \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} v \\ &= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & b^2+a^2-d^2-c^2 & -2ad+2bc & 2ac+2bd \\ 0 & 2ad+2bc & c^2-d^2+a^2-b^2 & 2cd-2ab \\ 0 & -2ac+2bd & 2cd+2ab & d^2-c^2-b^2+a^2 \end{pmatrix} v \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\left(d^2+c^2\right) & 2(bc-ad) & 2(bd+ac) \\ 0 & 2(bc+ad) & 1-2\left(d^2+b^2\right) & 2(cd-ab) \\ 0 & 2(bd-ac) & 2(cd+ab) & 1-2\left(c^2+b^2\right) \end{pmatrix} v \\ \begin{pmatrix} 0 \\ \boldsymbol{v}' \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\left(d^2+c^2\right) & 2(bc-ad) & 2(bd+ac) \\ 0 & 2(bc+ad) & 1-2\left(d^2+b^2\right) & 2(cd-ab) \\ 0 & 2(bd-ac) & 2(cd+ab) & 1-2\left(c^2+b^2\right) \end{pmatrix} \begin{pmatrix} 0 \\ \boldsymbol{v} \end{pmatrix} \end{aligned}$$

$$\boldsymbol{v}' = \begin{pmatrix} 1-2\left(d^2+c^2\right) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 1-2\left(d^2+b^2\right) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 1-2\left(c^2+b^2\right) \end{pmatrix} \boldsymbol{v}$$

$$q \overset{|q|=1}{=} a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}, \begin{cases} a = \cos\dfrac{\theta}{2} \\ b = \sin\dfrac{\theta}{2}\sin\gamma\cos\varphi \\ c = \sin\dfrac{\theta}{2}\sin\gamma\sin\varphi \\ d = \sin\dfrac{\theta}{2}\cos\gamma \end{cases}$$

### 32.2.4 exponential form

$$\begin{aligned} q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ =& a + r\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}, r^2 = b^2 + c^2 + d^2 \\ \overset{|q|=1}{=}& \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{n}, \begin{cases} a = \cos\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} = a \\ r = \sin\dfrac{\theta}{2} & \sin\dfrac{\theta}{2} = r \\ \dfrac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r} = \boldsymbol{n} & \boldsymbol{n} = \dfrac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r} \end{cases} \end{aligned}$$

$$\begin{aligned} \|\boldsymbol{n}\|^2 = \left(\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\right)^2 =& \frac{-b^2 - c^2 - d^2 + bc(\mathrm{ij} + \mathrm{ji}) + cd(\mathrm{jk} + \mathrm{kj}) + db(\mathrm{ki} + \mathrm{ik})}{r^2} \\ =& \frac{-b^2 - c^2 - d^2 + bc(\mathrm{k} - \mathrm{k}) + cd(\mathrm{i} - \mathrm{i}) + db(\mathrm{j} - \mathrm{j})}{r^2} \\ =& \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\ =& \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\ \boldsymbol{n} = \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r} =& \pm\sqrt{-1} \end{aligned}$$

$$\begin{aligned} q =& a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ =& a + r\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}, r^2 = b^2 + c^2 + d^2 \\ =& \sqrt{a^2 + r^2}\left(\frac{a}{\sqrt{a^2 + r^2}} + \frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r}\frac{r}{\sqrt{a^2 + r^2}}\right) \\ =& \rho\left(\cos\phi + \sqrt{-1}\sin\phi\right), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan\phi = \dfrac{r}{a} \Leftrightarrow \phi = \arctan\dfrac{r}{a} = \dfrac{\theta}{2} \end{cases} \\ =& \rho \mathrm{e}^{\phi\sqrt{-1}} \overset{|q|=1}{=} |q|\,\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} = \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} \\ q =& \rho \mathrm{e}^{\phi\sqrt{-1}} \overset{|q|=1}{=} \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}, \begin{cases} q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} = 1 \\ \tan\dfrac{\theta}{2} = \dfrac{r}{a} \Leftrightarrow \dfrac{\theta}{2} = \arctan\dfrac{r}{a} = \arctan\dfrac{\pm\sqrt{b^2 + c^2 + d^2}}{a} \end{cases} \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{v}' = v' =& qvq^{-1} = q\boldsymbol{v}q^{-1}, \begin{cases} v = \boldsymbol{v} = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} \\ v' = \boldsymbol{v}' = v_1'\mathrm{i} + v_2'\mathrm{j} + v_3'\mathrm{k} \end{cases}, \boldsymbol{n} = \begin{pmatrix} \sin\gamma\cos\varphi \\ \sin\gamma\sin\varphi \\ \cos\gamma \end{pmatrix}, \begin{cases} \varphi = \angle\hat{\boldsymbol{n}}_{xy}\hat{\boldsymbol{x}} \\ \gamma = \angle\hat{\boldsymbol{n}}\hat{\boldsymbol{z}} \end{cases} \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right]\right\}\boldsymbol{v} \\
& \left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[(\sin\gamma\cos\varphi)\,\mathrm{i} + (\sin\gamma\sin\varphi)\,\mathrm{j} + (\cos\gamma)\,\mathrm{k}\right]\right\} \\
=& q(\theta,\gamma,\varphi)\,\boldsymbol{v}q^{-1}(\theta,\gamma,\varphi) = q\left(\frac{\theta}{2},\gamma,\varphi\right)\boldsymbol{v}q^{-1}\left(\frac{\theta}{2},\gamma,\varphi\right) \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}\right]\right\}\boldsymbol{v}\left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}\right]\right\} \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} \pm \sqrt{1 - n_x^2 - n_z^2}\mathrm{j} + n_z\mathrm{k}\right]\right\}\boldsymbol{v} \\
& \left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} \pm \sqrt{1 - n_x^2 - n_z^2}\mathrm{j} + n_z\mathrm{k}\right]\right\} \\
=& \left\{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} \pm \sqrt{1 - n_x^2 - n_y^2}\mathrm{k}\right]\right\}\boldsymbol{v} \\
& \left\{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} \pm \sqrt{1 - n_x^2 - n_y^2}\mathrm{k}\right]\right\} \\
=& q(n_x,n_y,\theta)\,\boldsymbol{v}q^{-1}(n_x,n_y,\theta) = q\left(n_x,n_y,\frac{\theta}{2}\right)\boldsymbol{v}q^{-1}\left(n_x,n_y,\frac{\theta}{2}\right) \\
=& \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\boldsymbol{v}\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}
\end{aligned}
$$

---

$$
\begin{aligned}
\boldsymbol{v}' =& q\boldsymbol{v}q^{-1}, \\
& \begin{cases} q = q(n_x,n_y,\theta) & = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} \pm \sqrt{1 - n_x^2 - n_y^2}\mathrm{k}\right] \\ q^{-1} = q^* = \overline{q} & = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x\mathrm{i} + n_y\mathrm{j} \pm \sqrt{1 - n_x^2 - n_y^2}\mathrm{k}\right] \end{cases}, \\
& \begin{cases} n_x & = \sin\gamma\cos\varphi \\ n_y & = \sin\gamma\sin\varphi \end{cases} \Rightarrow \pm\sqrt{1 - n_x^2 - n_y^2} = n_z = \cos\gamma
\end{aligned}
$$

---

$$
\boldsymbol{v}' = \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\boldsymbol{v}\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}
$$

---

general quaternion exponential form

$$
\begin{aligned}
q =& \rho\mathrm{e}^{\phi\sqrt{-1}} = |q|\,\mathrm{e}^{\boldsymbol{n}\phi}, \begin{cases} q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d = a + r\frac{\mathrm{i}b + \mathrm{j}c + \mathrm{k}d}{r} = a + r\boldsymbol{n} \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan\phi = \frac{r}{a} \Leftrightarrow \phi = \arctan\frac{r}{a} = \arctan\frac{\pm\sqrt{b^2 + c^2 + d^2}}{a} \end{cases} \\
=& \mathrm{e}^{\ln|q|}\mathrm{e}^{\boldsymbol{n}\phi} = \mathrm{e}^{\ln|q| + \boldsymbol{n}\phi}
\end{aligned}
$$

### 32.2.5 double cover

https://www.bilibili.com/video/BV1rj41117VW/?t=15m15s

$$
\begin{array}{ccccccccccccccccc}
\theta & 0 & \to & \frac{\pi}{2} & \to & \pi & \to & \frac{3\pi}{2} & \to & 2\pi & \to & \frac{5\pi}{2} & \to & 3\pi & \to & \frac{7\pi}{2} & \to & 4\pi \\
\frac{\theta}{2} & 0 & \to & \frac{\pi}{4} & \to & \frac{\pi}{2} & \to & \frac{3\pi}{4} & \to & \pi & \to & \frac{5\pi}{4} & \to & \frac{3\pi}{2} & \to & \frac{7\pi}{4} & \to & 2\pi \\
\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} & 1 & \to & \mathrm{e}^{\boldsymbol{n}\frac{\pi}{4}} & \to & \boldsymbol{n} & \to & \mathrm{e}^{\boldsymbol{n}\frac{3\pi}{4}} & \to & -1 & \to & \mathrm{e}^{\boldsymbol{n}\frac{5\pi}{4}} & \to & -\boldsymbol{n} & \to & \mathrm{e}^{\boldsymbol{n}\frac{7\pi}{4}} & \to & 1 \\
\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}} & 1 & \to & \mathrm{e}^{\boldsymbol{n}\frac{-\pi}{4}} & \to & -\boldsymbol{n} & \to & \mathrm{e}^{\boldsymbol{n}\frac{-3\pi}{4}} & \to & -1 & \to & \mathrm{e}^{\boldsymbol{n}\frac{-5\pi}{4}} & \to & \boldsymbol{n} & \to & \mathrm{e}^{\boldsymbol{n}\frac{-7\pi}{4}} & \to & 1 \\
\boldsymbol{v}' = \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\boldsymbol{v}\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}} & \boldsymbol{v} & \to & & \to & -\boldsymbol{n}\boldsymbol{v}\boldsymbol{n} & \to & & \to & \boldsymbol{v} & \to & & \to & -\boldsymbol{n}\boldsymbol{v}\boldsymbol{n} & \to & & \to & \boldsymbol{v}
\end{array}
$$

$$SU(2) = \left\{ 1 (\cdot) 1, \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} (\cdot) \mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}} \right\}$$

3D rotation about an arbitrary axis[32.1.4.2.1.3]

$$\boldsymbol{V}' = SR_z(\gamma) S^{-1} \boldsymbol{V}$$

$$\boldsymbol{v}' = SR_z(\theta) S^{-1} \boldsymbol{v}$$

$$SO(3) = \left\{ 1, SR_z(\theta) S^{-1} \right\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, SR_z(\theta) S^{-1} \right\}$$

$$SU(2) \cong 2 \cdot SO(3)$$

Special unitary group of degree 2 $SU(2)$ is the boss of special orthonormal group of degree 3 $SO(3)$, $SU(2)$ double covers $SO(3)$.

https://en.wikipedia.org/wiki/Special_unitary_group#The_group_SU(2)

https://en.wikipedia.org/wiki/Unitary_group

“Particle is group representation, spin is square root of 4-vector.”?

「粒子是群表示，自旋是四維向量開根號。」？

See also spinor[70] and Lie group[32.2.8]

https://zhuanlan.zhihu.com/p/143972449

### 32.2.6 right-hand vs. left-hand coordinates

Krasjet 四元數與三維旋轉 p.63 73

#### 32.2.6.1 clock

$$\boldsymbol{v}' = \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}} \boldsymbol{v} \mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}$$

$$\boldsymbol{v}' = \mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}} \boldsymbol{v} \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}$$

right-hand = anticlockwise

$$[\boldsymbol{v}']_R = \left[\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\right]_R [\boldsymbol{v}]_R \left[\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}\right]_R$$

$$[\boldsymbol{v}']_L = \left[\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}\right]_R [\boldsymbol{v}]_L \left[\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\right]_R$$

$$[\boldsymbol{v}']_L = \left[\mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\right]_L [\boldsymbol{v}]_L \left[\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}\right]_L$$

#### 32.2.6.2 Rodrigues rotation in right-hand vs. left-hand coordinates

$$\boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{v}_\perp$$

$$[\boldsymbol{u}]_R = [\boldsymbol{n}]_R \times_R [\boldsymbol{v}_\perp]_R = [\boldsymbol{n}]_L \times_L [\boldsymbol{v}_\perp]_L = [\boldsymbol{u}]_L$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[[\boldsymbol{u}]_L]_R = -[\boldsymbol{u}]_R$$

$$\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_L\right]_R = -\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}_R$$

$$\det O = \pm 1$$

$\det O = -1$ will change handedness of the coordinate.

### 32.2.7 gimbal lock

https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf

https://www.zhihu.com/question/47736315/answer/3146533323

### 32.2.8 Lie group

https://zhuanlan.zhihu.com/p/684801875

https://www.zhihu.com/question/541029773/answer/3327615524

#### 32.2.8.1 Mathemaniac

https://www.youtube.com/playlist?list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP

https://www.youtube.com/watch?v=ZRca3Ggpy_g&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=3

https://www.youtube.com/watch?v=ZRca3Ggpy_g&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=3&t=7m23s

$$\begin{aligned}
\boldsymbol{v}' =& q\boldsymbol{v}q^{-1},\\
&\begin{cases} q = q\left(n_x, n_y, \theta\right) & = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left[n_x \mathrm{i} + n_y \mathrm{j} \pm \sqrt{1-n_x^2-n_y^2}\mathrm{k}\right]\\ q^{-1} = q^* = \overline{q} & = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left[n_x \mathrm{i} + n_y \mathrm{j} \pm \sqrt{1-n_x^2-n_y^2}\mathrm{k}\right]\end{cases},\\
&\begin{cases} n_x & = \sin\gamma\cos\varphi\\ n_y & = \sin\gamma\sin\varphi\end{cases} \Rightarrow \pm\sqrt{1-n_x^2-n_y^2} = n_z = \cos\gamma
\end{aligned}$$

$q$ or $q\left(\cdot\right)q^{-1} = \mathrm{e}^{\boldsymbol{n}\frac{\theta}{2}}\left(\cdot\right)\mathrm{e}^{-\boldsymbol{n}\frac{\theta}{2}}$ corresponding 3D manifold is a solid ball by coordinate $\left(n_x, n_y, \theta\right)$

$$q = q\left(n_x, n_y, \theta\right) = q\left(\begin{pmatrix}n_x\\n_y\\\theta\end{pmatrix}\right) = q\left(\left\langle\begin{matrix}n_x\\n_y\\\theta\end{matrix}\right\rangle\right) = q\begin{pmatrix}n_x\\n_y\\\theta\end{pmatrix} = q\begin{pmatrix}\sin\gamma\cos\varphi\\\sin\gamma\sin\varphi\\\theta\end{pmatrix}, \begin{cases}\gamma\in[0,\pi]\\\varphi\in[-\pi,\pi]\\\theta\in[-\pi,\pi]\end{cases}$$

https://www.youtube.com/watch?v=9CBS5CAynBE&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=4

See also (by Elliot Schneider) Taylor expansion or Taylor series[43.1] Elliot Schneider section or

https://www.youtube.com/watch?v=HQsZG8Yxb7w

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=12m25s

https://www.youtube.com/watch?v=B2PJh2K-jdU&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=5

See also (by Elliot Schneider) https://www.youtube.com/watch?v=0kY3Wpvutfs Hamiltonian flow and its relation to matrix exponential

https://www.youtube.com/watch?v=0kY3Wpvutfs&t=22m45s

https://www.youtube.com/watch?v=gj4kvpy1eCE&list=PLDcSwjT2BF_WDki-WvmJ___Q0nLIHuNPbP&index=6

See also Shinonome Masaki[70.2] especially

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 與 SO(3) 的夢幻聯動 https://zhuanlan.zhihu.com/p/341625428

##### 32.2.8.2 Liang, Can-Bin

https://www.bilibili.com/video/BV1pV4y1a7y5

https://www.youtube.com/playlist?list=PLstdOGDXMaWLhtcJwsGiw6V9YLGmCmmhO

周彬

https://www.youtube.com/playlist?list=PLstdOGDXMaWK4JpSoKjH8mnfxXwh9dtb6

##### 32.2.8.3 special unitary group

### 32.2.9 geometric algebra and Clifford algebra

https://en.wikipedia.org/wiki/Geometric_algebra

https://en.wikipedia.org/wiki/Clifford_algebra

John Vince_2009_Geometric Algebra_ An Algebraic System for Computer Games and Animation

https://www.amazon.com/Geometric-Algebra-Algebraic-Computer-Animation/dp/1848823789

### 32.2.10 dual quaternion

### 32.2.11 interpolation

##### 32.2.11.1 Lerp

##### 32.2.11.2 Nlerp

##### 32.2.11.3 Slerp

## 32.3 visualization

https://zhuanlan.zhihu.com/p/684379741

https://space.bilibili.com/3546564685466124/video

## 32.4 3Blue1Brown

### 32.4.1 Ben Eater

https://eater.net/quaternions/video/intro

https://eater.net/quaternions

### 32.4.2 3B1B

https://www.youtube.com/watch?v=zjMuIxRvygQ

https://www.youtube.com/watch?v=d4EgbgTm0Bg

### 32.4.3 Sutrabla

https://www.youtube.com/watch?v=syQnn_xuB8U

https://www.youtube.com/watch?v=4mXL751ko0w

https://www.newscientist.com/article/mg20427391-600-alices-adventures-in-algebra-wonderland-solved/

https://threejs.org/

## 32.5 CCJou: LA Revelation

https://ccjou.wordpress.com/2014/04/21/四元數/

# Chapter 33

# DICOM

## 33.1 Innolitics: DICOM Standard Browser

https://dicom.innolitics.com/ciods

https://dicom.innolitics.com/ciods/parametric-map/parametric-map-multi-frame-functional-groups/52009229/0048021a/0040072a

## 33.2 David Clunie: Medical Image Format Site

https://www.dclunie.com/

## 33.3 Hsiao, Chia-Hung

https://www.youtube.com/@user-zp9yy1ln6u/videos

# Chapter 34

# tendon pathophysiology

Mark E. Schweitzer, MD

https://www.youtube.com/watch?v=uIo58pQgxhY

## 34.1 histology

MSk MRI

Lee, Jee Eun

https://www.youtube.com/@MSKMRI/playlists

# Chapter 35

# group theory

https://www.bilibili.com/video/BV1hm4y1v75R

https://en.wikipedia.org/wiki/Group_(mathematics)

## 35.1 matrix group

subset of two-by-two matrices at least excluding zero matrix

$$\mathcal{M} = (\mathcal{M}, \cdot) \subset (\mathcal{M}_{2\times 2}(\mathbb{C}) - \{0\}, \cdot) = \mathcal{M}_{2\times 2}(\mathbb{C}) - \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

matrix multiplication

$$\begin{aligned}
&\forall \langle M_1, M_2 \rangle \in \mathcal{M}^2, \exists M_1M_2 \in \mathcal{M}\, [M_1M_2 = M_1 \cdot M_2] \\
\Leftrightarrow &\cdot : \mathcal{M} \times \mathcal{M} = \mathcal{M}^2 \to \mathcal{M} \\
\Leftrightarrow &\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \\
\Leftrightarrow &\mathcal{M} \times \mathcal{M} \dot{\to} \mathcal{M} \\
\Leftrightarrow &\mathcal{M}^2 \dot{\to} \mathcal{M}
\end{aligned}$$

matrix group

$$\begin{aligned}
&\begin{cases}
\forall \langle M_1, M_2, M_3 \rangle \in \mathcal{M}^3\, [M_1 (M_2M_3) = (M_1M_2) M_3] & \text{associativity} \\
\exists I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}, \forall M \in \mathcal{M}\, [IM = M] & \text{left unit element} \\
\forall M \in \mathcal{M}, \exists M^{-1} \in \mathcal{M}\, \left[M^{-1}M = I\right] & \text{left inverse (element)}
\end{cases} \\
\Rightarrow &\mathcal{M} = (\mathcal{M}, \cdot) \text{ is a matrix group}
\end{aligned}$$

## 35.2 group definition and basic theorem

https://en.wikipedia.org/wiki/Group_(mathematics)#Elementary_consequences_of_the_group_axioms

**Definition 35.1** (group)**.** group definition by a set and a binary operation on the set

$$\begin{aligned}
&\begin{cases}
\circ : G \times G = G^2 \to G & \text{binary operation} \\
\forall \langle g_1, g_2, g_3 \rangle \in G^3\, [g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3] & \text{associativity} \\
\exists e =\in G, \forall g \in G\, [e \circ g = g] & \text{left unit element} \\
\forall g \in G, \exists g^{-1} \in G\, \left[g^{-1} \circ g = e\right] & \text{left inverse (element)}
\end{cases} \\
\Leftrightarrow &G = (G, \circ) \text{ is a group}
\end{aligned}$$

**Theorem 35.1.** *group left inverses equal right inverses*

Proof:

to be proved

□

## 35.3 EpicOrganism = AIRoswell = Pan, Yi-Wen[1]

https://space.bilibili.com/14316464/video

https://space.bilibili.com/14316464/channel/collectiondetail?sid=1768137

https://www.bilibili.com/video/BV1mC4y1Z78k

https://www.bilibili.com/video/BV1Sp4y1w76M

https://www.bilibili.com/video/BV1cc411o7Hc

https://www.bilibili.com/video/BV1d8411v7t9

https://www.bilibili.com/video/BV1Rw411e7km

https://www.bilibili.com/video/BV1Q84y127ZQ

https://www.bilibili.com/video/BV1hm4y1g7A9

https://www.bilibili.com/video/BV1s94y1L7MU

https://www.bilibili.com/video/BV1j84y1d7eY

https://www.bilibili.com/video/BV18z4y1P7oB

https://www.bilibili.com/video/BV1cN4y1m77z

https://www.bilibili.com/video/BV18N4y1a7a3

https://www.bilibili.com/video/BV1Ma4y1r7nT

https://www.bilibili.com/video/BV11N4y187QW

## 35.4 PKU-CFCS

https://www.bilibili.com/video/BV11r421E7kD

## 35.5 Mathemaniac

https://www.youtube.com/playlist?list=PLDcSwjT2BF_VuNbn8HiHZKKy59SgnIAeO

## 35.6 Polya enumeration theorem

https://www.bilibili.com/video/BV17s4y1R7fW

## 35.7 complex group representation

### 35.7.1 complex basis group

$$\begin{aligned} G &= \{1, \mathrm{i}, -1, -\mathrm{i}\} \\ &= \left\{\mathrm{i}^0, \mathrm{i}^1, \mathrm{i}^2, \mathrm{i}^3\right\} \end{aligned}$$

$$\begin{aligned} &\forall \langle g_1, g_2\rangle \in G^2, \exists g_1 g_2 \in G\,[g_1 g_2 = g_1 \cdot g_2] \\ \Leftrightarrow &\cdot : G \times G = G^2 \to G \end{aligned}$$

[1] https://www.linkedin.com/in/yiwen-pan-16a90076

https://tex.stackexchange.com/questions/627708/tikz-how-to-put-tables-within-arbitrarily-placed-nodes

| $\cdot$ | 1 | i | −1 | −i |
|---|---|---|---|---|
| 1 | 1 | i | −1 | −i |
| i | i | −1 | −i | 1 |
| −1 | −1 | −i | 1 | i |
| −i | −i | 1 | i | −1 |

202403281312-group-theory_files/figure-latex/unnamed-chunk-3-1.p

Figure 35.1: complex basis group table

### 35.7.2 $\mathbb{C} \to \mathcal{M}_{2\times 2}(\mathbb{R})$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = I$$

$$\begin{aligned} c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} =& x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ =& xI + yJ, J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}), \langle x, y \rangle \in \mathbb{R}^2 \end{aligned}$$

---

$$J^2 = -I$$

---

$$J^2 = -I$$

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ca + cd & cb + d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} a^2 + bc = -1 & b = 0 \Rightarrow a^2 = -1 \Rightarrow\Leftarrow a \in \mathbb{R} \Rightarrow b \neq 0 \\ ab + bd = 0 & (b = 0) \vee (a = -d) \overset{b\neq 0}{\Rightarrow} a = -d \\ ca + cd = 0 & \\ cb + d^2 = -1 & a^2 = d^2 \Rightarrow (a = d) \vee (a = -d) \end{cases} \quad \text{if } \underset{\Rightarrow}{a=d} \begin{cases} a = d \\ a = -d \end{cases}$$

$$\Rightarrow a = d = 0 \Rightarrow bc = -1$$

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} = J(a, b), b \neq 0$$

$$J(a, b) = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix}, b \neq 0$$

---

$$J(a = 1, b) = \begin{pmatrix} 1 & b \\ \frac{-2}{b} & -1 \end{pmatrix} \Rightarrow J^2(a = 1, b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$xI + yJ(a = 1, b) = \begin{pmatrix} x + y & yb \\ y \cdot \frac{-2}{b} & x - y \end{pmatrix}$$

---

$$J(a = 0, b) = \begin{pmatrix} 0 & b \\ \frac{-1}{b} & 0 \end{pmatrix}$$

---

$$J\left(a=0,b=1\right)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

---

$$\begin{aligned} J\left(a=0,b=-1\right)&=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=-J\left(a=0,b=1\right)\\ \Rightarrow J^2\left(a=0,b=-1\right)&=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=-I\end{aligned}$$

$$\begin{aligned} J=J\left(a=0,b=-1\right)&=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &\Rightarrow\begin{cases}1\leftrightarrow I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ \mathrm{i}\leftrightarrow J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{cases}\\ \Rightarrow x+y\mathrm{i}\leftrightarrow& xI+yJ\\ &=x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+y\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}x & -y\\ y & x\end{pmatrix}\end{aligned}$$

---

$$x+y\mathrm{i}\leftrightarrow\begin{pmatrix}x & -y\\ y & x\end{pmatrix}=xI+yJ$$

realizing

$$\mathbb{C}\to\mathcal{M}_{2\times 2}\left(\mathbb{R}\right)=\mathcal{M}_2\left(\mathbb{R}\right)$$

### 35.7.3 ( determinant of complex group representation ) equivalent to ( squared modulus of complex number )

$$\det\left(xI+yJ\right)=\det\begin{pmatrix}x & -y\\ y & x\end{pmatrix}=\begin{vmatrix}x & -y\\ y & x\end{vmatrix}=x^2+y^2=\left|x+y\mathrm{i}\right|^2$$

#### 35.7.3.1 Lagrange identity

Lagrange identity

generalization of Brahmagupta–Fibonacci identity

specialization of Binet–Cauchy identity

cf. Euler identity[35.8.1.1]

$$\begin{aligned} &\det\left[\left(aI+bJ\right)\left(cI+dJ\right)\right]\\ =&\det\left[\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}c & -d\\ d & c\end{pmatrix}\right]\\ =&\det\begin{pmatrix}ac-bd & -ad-bc\\ ad+bc & ac-bd\end{pmatrix}=\left|\left(ac-bd\right)+\left(ad+bc\right)\mathrm{i}\right|^2=\left(ac-bd\right)^2+\left(ad+bc\right)^2\\ =&\left[\det\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\right]\left[\det\begin{pmatrix}c & -d\\ d & c\end{pmatrix}\right]=\left|a+b\mathrm{i}\right|^2\left|c+d\mathrm{i}\right|^2=\left(a^2+b^2\right)\left(c^2+d^2\right)\end{aligned}$$

$$\left|a+b\mathrm{i}\right|^2\left|c+d\mathrm{i}\right|^2=\left(a^2+b^2\right)\left(c^2+d^2\right)=\left(ac-bd\right)^2+\left(ad+bc\right)^2$$

$$
\begin{aligned}
&\det\left[(x_1 I + y_1 J)(x_2 I + y_2 J)\right] \\
=&\det\left[\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\right] \\
=&\det\begin{pmatrix} x_1x_2 - y_1y_2 & -x_1y_2 - y_1x_2 \\ x_1y_2 + y_1x_2 & x_1x_2 - y_1y_2 \end{pmatrix} = \left|(x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)\,\mathrm{i}\right|^2 = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 \\
=&\left[\det\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\right]\left[\det\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\right] = |x_1 + y_1\mathrm{i}|^2\,|x_2 + y_2\mathrm{i}|^2 = \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right)
\end{aligned}
$$

$$
|x_1 + y_1\mathrm{i}|^2\,|x_2 + y_2\mathrm{i}|^2 = \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2
$$

### 35.7.4 Euler formula proved by complex group representation

https://www.bilibili.com/video/BV1mM4y1J79a

$$
\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \cos n\frac{\theta}{n} & -\sin n\frac{\theta}{n} \\ \sin n\frac{\theta}{n} & \cos n\frac{\theta}{n} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{n} & -\sin\frac{\theta}{n} \\ \sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} R_{\frac{\theta}{n}} = \lim_{n\to\infty}\begin{pmatrix} \cos\frac{\theta}{n} & -\sin\frac{\theta}{n} \\ \sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{pmatrix} &= \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n}\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \\
&= I + \frac{\theta}{n} R_{\frac{\pi}{2}}
\end{aligned}
$$

$$
\lim_{n\to\infty} R_{\frac{\theta}{n}} = \lim_{n\to\infty}\begin{pmatrix} \cos\frac{\theta}{n} & -\sin\frac{\theta}{n} \\ \sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} = I + \frac{\theta}{n} R_{\frac{\pi}{2}}
$$

$$
\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} = \lim_{n\to\infty}\begin{pmatrix} x' \\ y' \end{pmatrix} &= \lim_{n\to\infty} R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix} = \left[\lim_{n\to\infty} R_{\frac{\theta}{n}}^n\right]\left[\lim_{n\to\infty}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \left[\lim_{n\to\infty} R_{\frac{\theta}{n}}^n\right]\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n\to\infty}\left[\lim_{n\to\infty} R_{\frac{\theta}{n}}\right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty}\left[I + \frac{\theta}{n} R_{\frac{\pi}{2}}\right]^n \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n\to\infty}\left[I + \frac{\theta}{n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty}\left[I + \frac{\theta J}{n}\right]^n \begin{pmatrix} x \\ y \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \mathrm{e}^{J\theta}\begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

$$
\begin{cases}
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathrm{e}^{J\theta}\begin{pmatrix} x \\ y \end{pmatrix}
\end{cases}
$$

$$
\Rightarrow \mathrm{e}^{J\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \xRightarrow{x+y\mathrm{i} \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ} \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta
$$

□

## 35.8 quaternion group representation

https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Two-dimensional_irreducible_representation_over_a_splitting_field

$$
\begin{aligned}
q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, &\begin{cases} q \in \mathbb{H} \\ a, b, c, d \in \mathbb{R} \quad \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \end{cases} \\
= w = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z, &\begin{cases} w \in \mathbb{H} \\ t, x, y, z \in \mathbb{R} \quad \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \end{cases} \\
= a1 + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = t1 + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} = x_0 1 + \boldsymbol{e}_i x_i &
\end{aligned}
$$

### 35.8.1 $\mathbb{H} \to \mathcal{M}_{2\times 2}(\mathbb{C})$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_2 = 1$$

$$
\begin{aligned}
\boldsymbol{e} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \frac{-a^2-1}{b} & -a \end{pmatrix} \\
= \begin{cases} \begin{pmatrix} a & b \\ \frac{-a^2-1}{b} & -a \end{pmatrix} = \begin{pmatrix} 0 & b \\ \frac{-1}{b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \Rightarrow \boldsymbol{e}_2 = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathrm{j} & a = 0 \\ \begin{pmatrix} a & b \\ \frac{-a^2-1}{b} & -a \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} & a \neq 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix} \alpha^2+\beta^2 & 0 \\ 0 & \beta^2+\alpha^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = \boldsymbol{e}^2 = -1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\Downarrow \\
\alpha^2+\beta^2 = -1 \Leftrightarrow \beta^2+\alpha^2 = -1
\end{aligned}
$$

$$\alpha^2+\beta^2 = -1 \Rightarrow \langle \alpha, \beta \rangle \notin \mathbb{R}^2 \Rightarrow\Leftarrow \langle \alpha, \beta \rangle \in \mathbb{R}^2 \Rightarrow \alpha^2+\beta^2 \geq 0$$

quaternion group has no irreducible two-dimensional representation over the reals [2]

$$\langle \alpha, \beta \rangle \in \mathbb{C}^2 - \mathbb{R}^2$$

$$\alpha^2+\beta^2 = -1 = \beta^2+\alpha^2$$

$$\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} \beta^2+\alpha^2 & 0 \\ 0 & \alpha^2+\beta^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathrm{i} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \mathrm{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} = \mathrm{k}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

[2] https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field

$$1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\mathrm{i}=\begin{pmatrix}\alpha&\beta\\\beta&-\alpha\end{pmatrix},\mathrm{j}=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\mathrm{k}=\begin{pmatrix}\beta&-\alpha\\-\alpha&-\beta\end{pmatrix}$$

$$\mathrm{jk}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\beta&-\alpha\\-\alpha&-\beta\end{pmatrix}=\begin{pmatrix}\alpha&\beta\\\beta&-\alpha\end{pmatrix}=\mathrm{i}$$

---

| | $\alpha^2+\beta^2=-1$ | $\begin{cases}\alpha=\sqrt{-1}\\\beta=0\end{cases}$ | $\begin{cases}\alpha=\sqrt{-2}\\\beta=1\end{cases}$ | $\beta=\alpha^2,n\in\{1,2,4,5\}$ |
|---|---|---|---|---|
| 1 | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ |
| −1 | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ |
| i | $\begin{pmatrix}\alpha&\beta\\\beta&-\alpha\end{pmatrix}$ | $\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix}$ | $\begin{pmatrix}\sqrt{-2}&1\\1&-\sqrt{-2}\end{pmatrix}$ | $\begin{pmatrix}\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}&\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}\\\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}&-\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}\end{pmatrix}$ |
| −i | $\begin{pmatrix}-\alpha&-\beta\\-\beta&\alpha\end{pmatrix}$ | $\begin{pmatrix}-\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix}$ | $\begin{pmatrix}-\sqrt{-2}&-1\\-1&\sqrt{-2}\end{pmatrix}$ | $\begin{pmatrix}-\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}&-\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}\\-\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}&\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}\end{pmatrix}$ |
| j | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ |
| −j | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ |
| k | $\begin{pmatrix}\beta&-\alpha\\-\alpha&-\beta\end{pmatrix}$ | $\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix}$ | $\begin{pmatrix}1&-\sqrt{-2}\\-\sqrt{-2}&-1\end{pmatrix}$ | $\begin{pmatrix}\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}&-\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}\\-\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}&-\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}\end{pmatrix}$ |
| −k | $\begin{pmatrix}-\beta&\alpha\\\alpha&\beta\end{pmatrix}$ | $\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix}$ | $\begin{pmatrix}-1&\sqrt{-2}\\\sqrt{-2}&1\end{pmatrix}$ | $\begin{pmatrix}-\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}&\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}\\\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}}&\mathrm{e}^{\pi\frac{2n}{3}\sqrt{-1}}\end{pmatrix}$ |

---

$$\begin{aligned}
-1=&\alpha^2+\beta^2\\
\overset{\beta=\alpha^2}{=}&\alpha^2+\alpha^4\\
\alpha^4+\alpha^2+1=&0,\alpha^4+\alpha^2+1=\left(\alpha^2+\alpha+1\right)\left(\alpha^2-\alpha+1\right)\\
\left(\alpha^2-1\right)\left(\alpha^4+\alpha^2+1\right)=&0\\
\alpha^6-1=&0\\
\alpha^6=&1=\mathrm{e}^{2\pi k\sqrt{-1}},k\in\mathbb{Z}\\
\alpha=&\mathrm{e}^{2\pi\frac{n}{6}\sqrt{-1}},n\in\{0,1,2,3,4,5\}-\{0,3\}\\
=&\mathrm{e}^{\pi\frac{n}{3}\sqrt{-1}},n\in\{1,2,4,5\}
\end{aligned}$$

---

| | $\alpha^2+\beta^2=-1$ | $\begin{cases}\alpha=i\\\beta=0\end{cases}$ | $\begin{cases}\alpha=\sqrt{2}i\\\beta=1\end{cases}$ | $\omega=\mathrm{e}^{i\pi\frac{n}{3}},n\in\{1,2,4,5\}$ |
|---|---|---|---|---|
| 1 | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ |
| −1 | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ |
| i | $\begin{pmatrix}\alpha&\beta\\\beta&-\alpha\end{pmatrix}$ | $\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $\begin{pmatrix}\sqrt{2}i&1\\1&-\sqrt{2}i\end{pmatrix}$ | $\begin{pmatrix}\omega&\omega^2\\\omega^2&-\omega\end{pmatrix}$ |
| −i | $\begin{pmatrix}-\alpha&-\beta\\-\beta&\alpha\end{pmatrix}$ | $\begin{pmatrix}-i&0\\0&i\end{pmatrix}$ | $\begin{pmatrix}-\sqrt{2}i&-1\\-1&\sqrt{2}i\end{pmatrix}$ | $\begin{pmatrix}-\omega&-\omega^2\\-\omega^2&\omega\end{pmatrix}$ |
| j | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ | $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ |
| −j | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ | $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ |
| k | $\begin{pmatrix}\beta&-\alpha\\-\alpha&-\beta\end{pmatrix}$ | $\begin{pmatrix}0&-i\\-i&0\end{pmatrix}$ | $\begin{pmatrix}1&-\sqrt{2}i\\-\sqrt{2}i&-1\end{pmatrix}$ | $\begin{pmatrix}\omega^2&-\omega\\-\omega&-\omega^2\end{pmatrix}$ |
| −k | $\begin{pmatrix}-\beta&\alpha\\\alpha&\beta\end{pmatrix}$ | $\begin{pmatrix}0&i\\i&0\end{pmatrix}$ | $\begin{pmatrix}-1&\sqrt{2}i\\\sqrt{2}i&1\end{pmatrix}$ | $\begin{pmatrix}-\omega^2&\omega\\\omega&\omega^2\end{pmatrix}$ |

realizing

$$\mathbb{H} \to \mathcal{M}_{2\times 2}(\mathbb{C}) = \mathcal{M}_2(\mathbb{C})$$

#### 35.8.1.1 Euler identity

cf. Lagrange identity[35.7.3.1]

$$\begin{aligned}
&\det(a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k})\\
&=\det\left[a\begin{pmatrix}1&0\\0&1\end{pmatrix}+b\begin{pmatrix}i&0\\0&-i\end{pmatrix}+c\begin{pmatrix}0&-1\\1&0\end{pmatrix}+d\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]\\
&=\det\begin{pmatrix}a+bi&-c-di\\c-di&a-bi\end{pmatrix}=\begin{vmatrix}a+bi&-c-di\\c-di&a-bi\end{vmatrix}\\
&=\left(a^2+b^2\right)+\left(c^2+d^2\right)=a^2+b^2+c^2+d^2
\end{aligned}$$

$$\det(a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k})=\det\begin{pmatrix}a+bi&-c-di\\c-di&a-bi\end{pmatrix}=a^2+b^2+c^2+d^2$$

$$\begin{aligned}
&\det\left[\left(q_{10}+q_{11}\mathrm{i}+q_{12}\mathrm{j}+q_{13}\mathrm{k}\right)\left(q_{20}+q_{21}\mathrm{i}+q_{22}\mathrm{j}+q_{23}\mathrm{k}\right)\right]=\det\left[\left(a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}\right)\left(\alpha+\beta\mathrm{i}+\gamma\mathrm{j}+\delta\mathrm{k}\right)\right]\\
&=\det\left\{\left[q_{10}\begin{pmatrix}1&0\\0&1\end{pmatrix}+q_{11}\begin{pmatrix}i&0\\0&-i\end{pmatrix}+q_{12}\begin{pmatrix}0&-1\\1&0\end{pmatrix}+q_{13}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]\right.\\
&\quad\left.\left[q_{20}\begin{pmatrix}1&0\\0&1\end{pmatrix}+q_{21}\begin{pmatrix}i&0\\0&-i\end{pmatrix}+q_{22}\begin{pmatrix}0&-1\\1&0\end{pmatrix}+q_{23}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]\right\}\\
&=\det\left\{\left[a\begin{pmatrix}1&0\\0&1\end{pmatrix}+b\begin{pmatrix}i&0\\0&-i\end{pmatrix}+c\begin{pmatrix}0&-1\\1&0\end{pmatrix}+d\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]\right.\\
&\quad\left.\left[\alpha\begin{pmatrix}1&0\\0&1\end{pmatrix}+\beta\begin{pmatrix}i&0\\0&-i\end{pmatrix}+\gamma\begin{pmatrix}0&-1\\1&0\end{pmatrix}+\delta\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]\right\}\\
&=\det\left\{\begin{pmatrix}a+bi&-c-di\\c-di&a-bi\end{pmatrix}\begin{pmatrix}\alpha+\beta i&-\gamma-\delta i\\\gamma-\delta i&\alpha-\beta i\end{pmatrix}\right\}\\
&\quad=\det\begin{pmatrix}a+bi&-c-di\\c-di&a-bi\end{pmatrix}\det\begin{pmatrix}\alpha+\beta i&-\gamma-\delta i\\\gamma-\delta i&\alpha-\beta i\end{pmatrix}=\left(a^2+b^2+c^2+d^2\right)\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)\\
&=\det\left\{\begin{pmatrix}[a+bi][\alpha+\beta i]-[c+di][\gamma-\delta i]&[a+bi][-\gamma-\delta i]-[c+di][\alpha-\beta i]\\ [c-di][\alpha+\beta i]+[a-bi][\gamma-\delta i]&[c-di][-\gamma-\delta i]+[a-bi][\alpha-\beta i]\end{pmatrix}\right\}\\
&=\det\left\{\begin{pmatrix}(a\alpha-b\beta-c\gamma-d\delta)+i(a\beta+b\alpha+c\delta-d\gamma)&-(a\gamma-b\delta+c\alpha+d\beta)-i(a\delta+b\gamma-c\beta+d\alpha)\\(a\gamma-b\delta+c\alpha+d\beta)-i(a\delta+b\gamma-c\beta+d\alpha)&(a\alpha-b\beta-c\gamma-d\delta)-i(a\beta+b\alpha+c\delta-d\gamma)\end{pmatrix}\right\}\\
&=\det\left\{(a\alpha-b\beta-c\gamma-d\delta)+(a\beta+b\alpha+c\delta-d\gamma)\,\mathrm{i}+(a\gamma-b\delta+c\alpha+d\beta)\,\mathrm{j}+(a\delta+b\gamma-c\beta+d\alpha)\,\mathrm{k}\right\}\\
&=(a\alpha-b\beta-c\gamma-d\delta)^2+(a\beta+b\alpha+c\delta-d\gamma)^2+(a\gamma-b\delta+c\alpha+d\beta)^2+(a\delta+b\gamma-c\beta+d\alpha)^2\\
&=\left(q_{10}q_{20}-q_{11}q_{21}-q_{12}q_{22}-q_{13}q_{23}\right)^2+\left(q_{10}q_{21}+q_{11}q_{20}+q_{12}q_{23}-q_{13}q_{22}\right)^2\\
&+\left(q_{10}q_{22}-q_{11}q_{23}+q_{12}q_{20}+q_{13}q_{21}\right)^2+\left(q_{10}q_{23}+q_{11}q_{22}-q_{12}q_{21}+q_{13}q_{20}\right)^2
\end{aligned}$$

$$\begin{aligned}
&\left(a^2+b^2+c^2+d^2\right)\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)\\
&=(a\alpha-b\beta-c\gamma-d\delta)^2+(a\beta+b\alpha+c\delta-d\gamma)^2\\
&\quad+(a\gamma-b\delta+c\alpha+d\beta)^2+(a\delta+b\gamma-c\beta+d\alpha)^2
\end{aligned}$$

---

**Theorem 35.2.** *For any two integers greater than zero, their multiplication can be the summation of squared four integers greater than zero.*

$$\forall\langle m_1,m_2\rangle\in(\mathbb{N}\cup\{0\})^2,\exists\langle k_1,k_2,k_3,k_4\rangle\in(\mathbb{N}\cup\{0\})^4\left[m_1m_2=k_1^2+k_2^2+k_3^2+k_4^2\right]$$

Proof:

Let $\begin{cases}m_1=a^2+b^2+c^2+d^2&\langle a,b,c,d\rangle\in(\mathbb{N}\cup\{0\})^4\\m_2=\alpha^2+\beta^2+\gamma^2+\delta^2&\langle\alpha,\beta,\gamma,\delta\rangle\in(\mathbb{N}\cup\{0\})^4\end{cases}\overset{\text{closure property}}{\Rightarrow}\begin{pmatrix}m_1\\m_2\end{pmatrix}=\begin{pmatrix}a^2+b^2+c^2+d^2\\\alpha^2+\beta^2+\gamma^2+\delta^2\end{pmatrix}\in(\mathbb{N}\cup\{0\})^2$,

$$
\begin{aligned}
m_1 m_2 &= \left(a^2+b^2+c^2+d^2\right)\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)\\
&\overset{\text{Euler identity}}{=} (a\alpha-b\beta-c\gamma-d\delta)^2+(a\beta+b\alpha+c\delta-d\gamma)^2\\
&+(a\gamma-b\delta+c\alpha+d\beta)^2+(a\delta+b\gamma-c\beta+d\alpha)^2\\
&=|a\alpha-b\beta-c\gamma-d\delta|^2\\
&+|a\beta+b\alpha+c\delta-d\gamma|^2\\
&+|a\gamma-b\delta+c\alpha+d\beta|^2\\
&+|a\delta+b\gamma-c\beta+d\alpha|^2\\
&=k_1^2+k_2^2+k_3^2+k_4^2, \begin{pmatrix}k_1\\k_2\\k_3\\k_4\end{pmatrix}=\begin{pmatrix}|a\alpha-b\beta-c\gamma-d\delta|\\|a\beta+b\alpha+c\delta-d\gamma|\\|a\gamma-b\delta+c\alpha+d\beta|\\|a\delta+b\gamma-c\beta+d\alpha|\end{pmatrix}\in(\mathbb{N}\cup\{0\})^4
\end{aligned}
$$

$$
\because\begin{cases}\langle a,b,c,d\rangle\in(\mathbb{N}\cup\{0\})^4\\\langle\alpha,\beta,\gamma,\delta\rangle\in(\mathbb{N}\cup\{0\})^4\end{cases}
$$

$$
\overset{\text{closure property}}{\Rightarrow}\begin{pmatrix}|a\alpha-b\beta-c\gamma-d\delta|\\|a\beta+b\alpha+c\delta-d\gamma|\\|a\gamma-b\delta+c\alpha+d\beta|\\|a\delta+b\gamma-c\beta+d\alpha|\end{pmatrix}\in(\mathbb{N}\cup\{0\})^4
$$

□

### 35.8.2 $\mathbb{H}\to\mathcal{M}_{4\times4}(\mathbb{R})$

https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field

$$
\mathbb{H}\to\mathcal{M}_2(\mathbb{C})\overset{\mathbb{C}\to\mathcal{M}_2(\mathbb{R})}{\to}\mathcal{M}_{4\times4}(\mathbb{R})=\mathcal{M}_4(\mathbb{R})
$$

$$
\mathbb{C}\to\mathcal{M}_2(\mathbb{R})\Leftarrow\begin{cases}1\to\begin{pmatrix}1&0\\0&1\end{pmatrix}\\ i\to\begin{pmatrix}0&-1\\1&0\end{pmatrix}\end{cases}
$$

---

$$
1\to\begin{pmatrix}1&0\\0&1\end{pmatrix}\to\begin{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}
$$

$$
-1\to\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\to\begin{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\end{pmatrix}\to\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}
$$

$$
\mathrm{i}\to\begin{pmatrix}i&0\\0&-i\end{pmatrix}\to\begin{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}0&1\\-1&0\end{pmatrix}\end{pmatrix}\to\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}
$$

$$
-\mathrm{i}\to\begin{pmatrix}-i&0\\0&i\end{pmatrix}\to\begin{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}0&-1\\1&0\end{pmatrix}\end{pmatrix}\to\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}
$$

$$\mathrm{j} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$-\mathrm{j} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathrm{k} \rightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$-\mathrm{k} \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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some examinations

$$\mathrm{ij} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \leftarrow \mathrm{k}$$

$$\mathrm{ji} \rightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftarrow -\mathrm{k}$$

$$\mathrm{i}^2 = \mathrm{ii} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \leftarrow -1$$

---

| | | | | |
|---|---|---|---|---|
| | $\begin{cases}\alpha = i \\ \beta = 0\end{cases}$ | $\begin{cases}1 \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ i \to \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{cases}$ | $\begin{cases}\alpha = \sqrt{2}i \\ \beta = 1\end{cases}$ | $\begin{cases}1 \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ i \to \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{cases}$ |
| 1 | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ |
| −1 | $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ | $\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ | $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ | $\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ |
| i | $\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ | $\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$ | $\begin{pmatrix}\sqrt{2}i & 1\\ 1 & -\sqrt{2}i\end{pmatrix}$ | $\begin{pmatrix}0&-\sqrt{2}&1&0\\\sqrt{2}&0&0&1\\1&0&0&\sqrt{2}\\0&1&-\sqrt{2}&0\end{pmatrix}$ |
| −i | $\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$ | $\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$ | $\begin{pmatrix}-\sqrt{2}i & -1\\ -1 & \sqrt{2}i\end{pmatrix}$ | $\begin{pmatrix}0&-1&-1&0\\1&0&0&-1\\-1&0&0&1\\0&-1&-1&0\end{pmatrix}$ |
| j | $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ | $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ |
| −j | $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}$ | $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}$ |
| k | $\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$ | $\begin{pmatrix}1 & -\sqrt{2}i\\ -\sqrt{2}i & -1\end{pmatrix}$ | $\begin{pmatrix}1&0&0&\sqrt{2}\\0&1&-\sqrt{2}&0\\0&\sqrt{2}&-1&0\\-\sqrt{2}&0&0&-1\end{pmatrix}$ |
| −k | $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$ | $\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}$ | $\begin{pmatrix}-1 & \sqrt{2}i\\ \sqrt{2}i & 1\end{pmatrix}$ | $\begin{pmatrix}-1&0&0&-\sqrt{2}\\0&-1&\sqrt{2}&0\\0&-\sqrt{2}&1&0\\\sqrt{2}&0&0&1\end{pmatrix}$ |

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some examinations

$$\text{ij} \to \begin{pmatrix}0&-\sqrt{2}&1&0\\\sqrt{2}&0&0&1\\1&0&0&\sqrt{2}\\0&1&-\sqrt{2}&0\end{pmatrix}\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&\sqrt{2}\\0&1&-\sqrt{2}&0\\0&\sqrt{2}&-1&0\\-\sqrt{2}&0&0&-1\end{pmatrix} \leftarrow \text{k}$$

# Chapter 36

# tensor

## 36.1 Einstein summation convention

### 36.1.1 dummy index

### 36.1.2 free index

## 36.2 EpicOrganism = AIRoswell = Pan, Yi-Wen[1]

https://space.bilibili.com/14316464/video

https://www.bilibili.com/video/BV1T5411D7mS

https://www.bilibili.com/video/BV1QA4y1X7Xk

https://www.bilibili.com/video/BV1K34y1i75w

https://www.bilibili.com/video/BV1ZU411o7xL

## 36.3 EigenChris

### 36.3.1 tensor for beginner

https://www.youtube.com/playlist?list=PLJHszsWbB6hrkmmq57lX8BV-o-YIOFsiG

### 36.3.2 tensor calculus

https://www.youtube.com/playlist?list=PLJHszsWbB6hpk5h8lSfBkVrpjsqvUGTCx

## 36.4 Elliot Schneider

- Elliot Schneider: Physics with Elliot

### 36.4.1 Fundamentals of Cartesian Tensors

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2176662131

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076341

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076344

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745848

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745896

[1] https://www.linkedin.com/in/yiwen-pan-16a90076

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745882

vector notation

$$\vec{V} = \overrightarrow{V} = \overset{\text{\harpoon}}{V} = \mathbf{V} = \boldsymbol{V}$$

`\boldsymbol{}` is relatively more elegant.

`\del` or `\nabla`

vector del = vector nabla

$$\vec{\nabla} = \overrightarrow{\nabla} = \overset{\text{\harpoon}}{\nabla} = \nabla = \boldsymbol{\nabla}$$

scalar del = scalar nabla

$$\nabla$$

vector field

e.g. [magnetic] vector potential

$$\begin{aligned}\boldsymbol{A}\left(\boldsymbol{r}\right) = \vec{A}\left(\vec{r}\right) =& \boldsymbol{A}\left(\left(x_1, x_2, x_3\right)\right) = \boldsymbol{A}\left(x_1, x_2, x_3\right) = \begin{pmatrix} A_1\left(x_1, x_2, x_3\right) \\ A_2\left(x_1, x_2, x_3\right) \\ A_3\left(x_1, x_2, x_3\right) \end{pmatrix} \\ =& \boldsymbol{A}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \boldsymbol{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_1\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \\ A_2\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \\ A_3\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \end{pmatrix} = \begin{pmatrix} A_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ A_2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ A_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{pmatrix}\end{aligned}$$

scalar field

e.g. temperature

$$\begin{aligned}T\left(\boldsymbol{r}\right) = T\left(\vec{r}\right) =& T\left(\left(x_1, x_2, x_3\right)\right) = T\left(x_1, x_2, x_3\right) \\ =& T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\end{aligned}$$

or generally

$$f\left(\boldsymbol{r}\right) = f\left(\vec{r}\right) = f\left(x_1, x_2, x_3\right) = f\left(\cdots, x_i, \cdots\right)$$

**Definition 36.1.** directional derivative

$$\begin{aligned}\nabla_{\boldsymbol{V}} f\left(\boldsymbol{r}\right) =& \lim_{\epsilon\to 0} \frac{f\left(\boldsymbol{r} + \epsilon \boldsymbol{V}\right) - f\left(\boldsymbol{r}\right)}{\epsilon} \\ = \nabla_{\vec{V}} f\left(\vec{r}\right) =& \lim_{\epsilon\to 0} \frac{f\left(\vec{r} + \epsilon \vec{V}\right) - f\left(\vec{r}\right)}{\epsilon}\end{aligned}$$

1D directional derivative

$$\begin{aligned} f_x(\boldsymbol{r}) = f'(x) = &\begin{cases} \dfrac{\partial f}{\partial x} & f = f(\cdots, x, \cdots) \\ \dfrac{\mathrm{d} f}{\mathrm{d} x} & f = f(x) \end{cases} \\ =&\nabla_{\frac{\vec{x}}{\|\vec{x}\|}} f(\vec{r}) = \nabla_{\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}} f(\boldsymbol{r}) = \nabla_{\frac{\vec{x}}{|\vec{x}|}} f(\vec{r}) = \nabla_{\frac{\boldsymbol{x}}{|\boldsymbol{x}|}} f(\boldsymbol{r}) \\ =&\nabla_{\hat{x}} f(\vec{r}) = \nabla_{\hat{\boldsymbol{x}}} f(\boldsymbol{r}) = \begin{cases} \lim\limits_{\epsilon \to 0} \dfrac{f(\boldsymbol{r} + \epsilon \hat{\boldsymbol{x}}) - f(\boldsymbol{r})}{\epsilon} & f = f(\cdots, x, \cdots) \\ \lim\limits_{\epsilon \to 0} \dfrac{f(x + \epsilon) - f(x)}{\epsilon} & f = f(x) \end{cases} \end{aligned}$$

---

According to $\mathrm{d}f$[39],

$$\mathrm{d}f = f'(x)\,\mathrm{d}x = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

is the differential of $f$, read as "d $f$".

$$\begin{aligned} \mathrm{d}f = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = &\left(\frac{\mathrm{d}f}{\mathrm{d}x}\hat{\boldsymbol{x}}\right) \cdot (\mathrm{d}x\hat{\boldsymbol{x}}) \\ =&\boldsymbol{\nabla}_{\hat{\boldsymbol{x}}} f(\boldsymbol{r}) \cdot \mathrm{d}\hat{\boldsymbol{x}} \\ =&\boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{r} = \vec{\nabla} f \cdot \mathrm{d}\vec{r} \\ \overset{\text{commutative}}{=}&\mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla} f = \mathrm{d}\vec{r} \cdot \vec{\nabla} f \end{aligned}$$

---

$$\mathrm{d}f = \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla} f = \mathrm{d}\vec{r} \cdot \vec{\nabla} f$$

$\boldsymbol{\nabla} f$ is the gradient of $f$, read as "del $f$".

---

According to $\mathrm{d}f$[39],

$$\mathrm{d}f = f'(x)\,\mathrm{d}x = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

is the differential of $f$, read as "d $f$".

---

2D directional derivative

According to $\mathrm{d}f$[39],

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

$$\begin{aligned} \mathrm{d}f =&\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ =&\left(\frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}}\right) \cdot (\mathrm{d}x\hat{\boldsymbol{x}} + \mathrm{d}y\hat{\boldsymbol{y}}) \\ =&\boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{r}, \begin{cases} \boldsymbol{\nabla} f = \dfrac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \dfrac{\partial f}{\partial y}\hat{\boldsymbol{y}} \\ \mathrm{d}\boldsymbol{r} = \mathrm{d}x\hat{\boldsymbol{x}} + \mathrm{d}y\hat{\boldsymbol{y}} \end{cases} \end{aligned}$$

---

example:

$f(\boldsymbol{r}) = f(x_1, x_2) = f(x, y) = \sqrt{x^2 + y^2}$

using Feynman method of differentiation / derivative technique[42.1],

$$\begin{aligned}\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} =& \frac{\partial\sqrt{x^2+y^2}}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\sqrt{x^2+y^2}}{\partial y}\hat{\boldsymbol{y}}\\ \overset{\text{Feynman}}{=}& \frac{\frac{1}{2}\cdot 2x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{\frac{1}{2}\cdot 2y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}\\ =& \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}\\ =& \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{\sqrt{x^2+y^2}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} = \frac{\vec{r}}{\|\vec{r}\|}\end{aligned}$$

---

geometric meaning of a gradient

$\boldsymbol{\nabla} f$ points in the direction of “steepest ascent” of $f$.

directional derivative: from definition to calculation

$$\begin{aligned}\mathrm{d}f =& \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{\nabla} f = \mathrm{d}\vec{r}\cdot\vec{\nabla} f\\ =& \epsilon\boldsymbol{V}\cdot\boldsymbol{\nabla} f = \epsilon\vec{V}\cdot\vec{\nabla} f\end{aligned}$$

$$\mathrm{d}f = f\left(\boldsymbol{r}+\epsilon\boldsymbol{V}\right) - f\left(\boldsymbol{r}\right) = f\left(\vec{r}+\epsilon\vec{V}\right) - f\left(\vec{r}\right)$$

$$\begin{aligned}&\nabla_{\boldsymbol{V}} f\left(\boldsymbol{r}\right) = \nabla_{\vec{V}} f\left(\vec{r}\right)\\ =& \lim_{\epsilon\to 0}\frac{f\left(\boldsymbol{r}+\epsilon\boldsymbol{V}\right) - f\left(\boldsymbol{r}\right)}{\epsilon} = \lim_{\epsilon\to 0}\frac{f\left(\vec{r}+\epsilon\vec{V}\right) - f\left(\vec{r}\right)}{\epsilon}\\ =& \lim_{\epsilon\to 0}\frac{\mathrm{d}f}{\epsilon} = \lim_{\epsilon\to 0}\frac{\mathrm{d}f}{\epsilon}\\ =& \lim_{\epsilon\to 0}\frac{\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{\nabla} f}{\epsilon} = \lim_{\epsilon\to 0}\frac{\mathrm{d}\vec{r}\cdot\vec{\nabla} f}{\epsilon}\\ =& \lim_{\epsilon\to 0}\frac{\epsilon\boldsymbol{V}\cdot\boldsymbol{\nabla} f}{\epsilon} = \lim_{\epsilon\to 0}\frac{\epsilon\vec{V}\cdot\vec{\nabla} f}{\epsilon}\\ =& \lim_{\epsilon\to 0}\frac{\boldsymbol{V}\cdot\boldsymbol{\nabla} f}{1} = \lim_{\epsilon\to 0}\frac{\vec{V}\cdot\vec{\nabla} f}{1}\\ =& \boldsymbol{V}\cdot\boldsymbol{\nabla} f = \vec{V}\cdot\vec{\nabla} f\end{aligned}$$

---

$$\nabla_{\boldsymbol{V}} f = \nabla_{\boldsymbol{V}} f\left(\boldsymbol{r}\right) = \boldsymbol{V}\cdot\boldsymbol{\nabla} f = \nabla_{\vec{V}} f = \nabla_{\vec{V}} f\left(\vec{r}\right) = \vec{V}\cdot\vec{\nabla} f$$

---

$$\begin{aligned}\nabla_{\boldsymbol{V}} f =& \boldsymbol{V}\cdot\boldsymbol{\nabla} f\\ = \nabla_{\vec{V}} f =& \vec{V}\cdot\vec{\nabla} f\end{aligned}$$

---

in the view of operator acting on scalar field(s)

$$\boldsymbol{V} = V_x\hat{\boldsymbol{x}} + V_y\hat{\boldsymbol{y}}$$

grad or gradient = vector del or vector nabla as a vector operator acting on scalar field(s)

$$\begin{aligned}\boldsymbol{\nabla} f = \vec{\nabla} f =& \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} = \nabla_{\hat{x}} f\hat{\boldsymbol{x}} + \nabla_{\hat{y}} f\hat{\boldsymbol{y}}\\ =& \hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} = \hat{\boldsymbol{x}}\nabla_{\hat{x}} f + \hat{\boldsymbol{y}}\nabla_{\hat{y}} f\\ =& \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right) f = \left(\hat{\boldsymbol{x}}\nabla_{\hat{x}} + \hat{\boldsymbol{y}}\nabla_{\hat{y}}\right) f\\ &\Downarrow\\ \boldsymbol{\nabla} = \vec{\nabla} =& \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} = \hat{\boldsymbol{x}}\nabla_{\hat{x}} + \hat{\boldsymbol{y}}\nabla_{\hat{y}}\end{aligned}$$

directional derivative = scalar del or scalar nabla as a scalar operator acting on scalar field(s)

$$\begin{aligned}\nabla_{\boldsymbol{V}} f = \nabla_{\vec{V}} f =& \boldsymbol{V} \cdot \boldsymbol{\nabla} f = \vec{V} \cdot \vec{\nabla} f \\ =& \left(V_x \hat{\boldsymbol{x}} + V_y \hat{\boldsymbol{y}}\right) \cdot \left(\nabla_{\hat{\boldsymbol{x}}} f \hat{\boldsymbol{x}} + \nabla_{\hat{\boldsymbol{y}}} f \hat{\boldsymbol{y}}\right) \\ =& V_x \nabla_{\hat{\boldsymbol{x}}} f + V_y \nabla_{\hat{\boldsymbol{y}}} f = \left(V_x \nabla_{\hat{\boldsymbol{x}}} + V_y \nabla_{\hat{\boldsymbol{y}}}\right) f \\ =& V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} = \left(V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y}\right) f \\ & \Downarrow \\ \nabla_{\boldsymbol{V}} = \nabla_{\vec{V}} =& V_x \nabla_{\hat{\boldsymbol{x}}} + V_y \nabla_{\hat{\boldsymbol{y}}} = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y}\end{aligned}$$

---

product rule

$$\begin{aligned}\nabla_{\boldsymbol{V}} (fg) =& \nabla_{\vec{V}} (fg) = \boldsymbol{V} \cdot \boldsymbol{\nabla} (fg) = \vec{V} \cdot \vec{\nabla} (fg) \\ =& \boldsymbol{V} \cdot \left[(\boldsymbol{\nabla} f) g + f \boldsymbol{\nabla} g\right] = \boldsymbol{V} \cdot \left[g \boldsymbol{\nabla} f + f \boldsymbol{\nabla} g\right] = \boldsymbol{V} \cdot \left[f \boldsymbol{\nabla} g + g \boldsymbol{\nabla} f\right] \\ =& f (\boldsymbol{V} \cdot \boldsymbol{\nabla} g) + g (\boldsymbol{V} \cdot \boldsymbol{\nabla} f) = g (\boldsymbol{V} \cdot \boldsymbol{\nabla} f) + f (\boldsymbol{V} \cdot \boldsymbol{\nabla} g) \\ =& f (\nabla_{\boldsymbol{V}} g) + g (\nabla_{\boldsymbol{V}} f) = g (\nabla_{\boldsymbol{V}} f) + f (\nabla_{\boldsymbol{V}} g) \\ =& f \boldsymbol{V} \cdot \boldsymbol{\nabla} g + g \boldsymbol{V} \cdot \boldsymbol{\nabla} f = g \boldsymbol{V} \cdot \boldsymbol{\nabla} f + f \boldsymbol{V} \cdot \boldsymbol{\nabla} g \\ =& f \nabla_{\boldsymbol{V}} g + g \nabla_{\boldsymbol{V}} f = g \nabla_{\boldsymbol{V}} f + f \nabla_{\boldsymbol{V}} g \\ \nabla_{\boldsymbol{V}} (fg) =& f \nabla_{\boldsymbol{V}} g + g \nabla_{\boldsymbol{V}} f\end{aligned}$$

---

two kinds of linearities or bilinearity of directional derivative operator

$$\begin{aligned}\nabla_{\boldsymbol{V}} (f + \lambda g) =& \nabla_{\vec{V}} (f + \lambda g) = \boldsymbol{V} \cdot \boldsymbol{\nabla} (f + \lambda g) = \vec{V} \cdot \vec{\nabla} (f + \lambda g) \\ =& \boldsymbol{V} \cdot \left[\boldsymbol{\nabla} f + \boldsymbol{\nabla} (\lambda g)\right] = \boldsymbol{V} \cdot \left[\boldsymbol{\nabla} f + \lambda \boldsymbol{\nabla} g\right] \\ =& \boldsymbol{V} \cdot \boldsymbol{\nabla} f + \lambda (\boldsymbol{V} \cdot \boldsymbol{\nabla} g) \\ =& \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{V}} g \\ \nabla_{\boldsymbol{V}} (f + \lambda g) =& \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{V}} g\end{aligned}$$

$$\begin{aligned}\nabla_{\boldsymbol{V} + \lambda \boldsymbol{W}} f =& \nabla_{\vec{V} + \lambda \vec{W}} f = (\boldsymbol{V} + \lambda \boldsymbol{W}) \cdot \boldsymbol{\nabla} f = \left(\vec{V} + \lambda \vec{W}\right) \cdot \vec{\nabla} f \\ =& \boldsymbol{V} \cdot \boldsymbol{\nabla} f + \lambda \boldsymbol{W} \cdot \boldsymbol{\nabla} f = \vec{V} \cdot \vec{\nabla} f + \lambda \vec{W} \cdot \vec{\nabla} f \\ =& \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{W}} f = \nabla_{\vec{V}} f + \lambda \nabla_{\vec{W}} f \\ \nabla_{\boldsymbol{V} + \lambda \boldsymbol{W}} f =& \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{W}} f\end{aligned}$$

---

$$\begin{cases} \nabla_{\boldsymbol{V}} (f + \lambda g) & = \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{V}} g \\ \nabla_{\boldsymbol{V} + \lambda \boldsymbol{W}} f & = \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{W}} f \end{cases}$$

$$\begin{cases} \nabla_{\boldsymbol{V}} (f + \lambda g) = \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{V}} g & \Leftrightarrow \nabla_{\boldsymbol{V}} (\lambda f + \mu g) = \lambda \nabla_{\boldsymbol{V}} f + \mu \nabla_{\boldsymbol{V}} g \\ \nabla_{\boldsymbol{V} + \lambda \boldsymbol{W}} f = \nabla_{\boldsymbol{V}} f + \lambda \nabla_{\boldsymbol{W}} f & \Leftrightarrow \nabla_{\lambda \boldsymbol{V} + \mu \boldsymbol{W}} f = \lambda \nabla_{\boldsymbol{V}} f + \mu \nabla_{\boldsymbol{W}} f \end{cases}$$

$$\begin{aligned}\nabla_{\lambda \boldsymbol{V} + \mu \boldsymbol{W}} f =& \nabla_{\lambda \boldsymbol{V}} f + \nabla_{\mu \boldsymbol{W}} f \\ =& \lambda \nabla_{\boldsymbol{V}} f + \mu \nabla_{\boldsymbol{W}} f \\ =& \lambda \frac{\partial}{\partial V} f + \mu \frac{\partial}{\partial W} f \\ =& \lambda \frac{\partial f}{\partial V} + \mu \frac{\partial f}{\partial W} \\ \nabla_{\boldsymbol{V}} f = \nabla_{V_x \hat{\boldsymbol{x}} + V_y \hat{\boldsymbol{y}}} f =& \nabla_{V_x \hat{\boldsymbol{x}}} f + \nabla_{V_y \hat{\boldsymbol{y}}} f \\ =& V_x \nabla_{\hat{\boldsymbol{x}}} f + V_y \nabla_{\hat{\boldsymbol{y}}} f \\ =& V_x \frac{\partial}{\partial x} f + V_y \frac{\partial}{\partial y} f \\ =& V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y}\end{aligned}$$

---

directional derivative of a 3D vector field

$$
\begin{aligned}
\nabla_{\boldsymbol{V}} \boldsymbol{A} = \nabla_{\vec{V}} \vec{A} =& \boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{A} = \vec{V} \cdot \vec{\nabla} \vec{A} \\
&= \left(\hat{\boldsymbol{x}} V_x + \hat{\boldsymbol{y}} V_y + \hat{\boldsymbol{z}} V_z\right) \cdot \left(\hat{\boldsymbol{x}} \frac{\partial \boldsymbol{A}}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial \boldsymbol{A}}{\partial x} + \hat{\boldsymbol{z}} \frac{\partial \boldsymbol{A}}{\partial x}\right) \\
&= \left(\hat{\boldsymbol{x}} V_x + \hat{\boldsymbol{y}} V_y + \hat{\boldsymbol{z}} V_z\right) \cdot \left(\hat{\boldsymbol{x}} \frac{\partial \vec{A}}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial \vec{A}}{\partial x} + \hat{\boldsymbol{z}} \frac{\partial \vec{A}}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} \boldsymbol{A} = \vec{\nabla} \vec{A} =& \hat{\boldsymbol{x}} \frac{\partial \boldsymbol{A}}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial \boldsymbol{A}}{\partial x} + \hat{\boldsymbol{z}} \frac{\partial \boldsymbol{A}}{\partial x} = \hat{\boldsymbol{x}} \frac{\partial \vec{A}}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial \vec{A}}{\partial x} + \hat{\boldsymbol{z}} \frac{\partial \vec{A}}{\partial x} \\
=& \hat{\boldsymbol{x}} \frac{\partial \left(\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z\right)}{\partial x} \\
&+ \hat{\boldsymbol{y}} \frac{\partial \left(\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z\right)}{\partial x} \\
&+ \hat{\boldsymbol{z}} \frac{\partial \left(\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z\right)}{\partial x} \\
=& \hat{\boldsymbol{x}} \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{x}} \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{x}} \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x} \\
&+ \hat{\boldsymbol{y}} \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{y}} \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{y}} \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x} \\
&+ \hat{\boldsymbol{z}} \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{z}} \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{z}} \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x} \\
=& \hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x} \\
&+ \hat{\boldsymbol{y}} \otimes \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{y}} \otimes \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{y}} \otimes \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x} \\
&+ \hat{\boldsymbol{z}} \otimes \hat{\boldsymbol{x}} \frac{\partial A_x}{\partial x} + \hat{\boldsymbol{z}} \otimes \hat{\boldsymbol{y}} \frac{\partial A_y}{\partial x} + \hat{\boldsymbol{z}} \otimes \hat{\boldsymbol{z}} \frac{\partial A_z}{\partial x}
\end{aligned}
$$

$\boldsymbol{\nabla} \boldsymbol{A} = \vec{\nabla} \vec{A}$ is a rank-2 tensor.

$\otimes$ is tensor product.

$\nabla_{\boldsymbol{V}} \boldsymbol{A} = \nabla_{\vec{V}} \vec{A} = \boldsymbol{V} \cdot \boldsymbol{\nabla} \boldsymbol{A} = \vec{V} \cdot \vec{\nabla} \vec{A}$ yields a vector, instead of a scalar.

Dan Fleisch: What's a Tensor?

https://www.youtube.com/watch?v=f5liqUk0ZTw

UdiProd: tensor

https://www.youtube.com/watch?v=YxXyN2ifK8A&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=1

https://www.youtube.com/watch?v=A95jdIuUUW0&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=2

https://www.youtube.com/watch?v=51ARho2bvQY&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=3

```
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
## (-8.0, 8.0, -8.0, 8.0)
```

202403291050-tensor_files/figure-latex/unnamed-chunk-3-1.pdf

```
import numpy as np
import matplotlib.pyplot as plt

A = np.array([[ 0,-1],
              [ 1, 0]])

B = np.array([[ 1, 0],
              [ 0, 1]])

C = np.array([[ 1,-1],
              [ 1, 1]])

D = np.array([[ 1, 1],
              [ 1, 1]])

E = np.array([[-1, 0],
              [ 0,-1]])

F = np.array([[ 3, 1],
              [-2, 2]])

t = np.arange(0, 2 * np.pi, np.pi/5)

# Creating arrow
X = np.cos(t)
Y = np.sin(t)

# X, Y = np.meshgrid(x, y)

# V = np.array([X, Y])

# u = x
# v = y

# u, v = np.matmul(A, V)
u, v = np.matmul(A, np.array([X, Y]))
dx, dy = np.matmul(B, np.array([X, Y]))
dx2, dy2 = np.matmul(C, np.array([X, Y]))
dx3, dy3 = np.matmul(D, np.array([X, Y]))
dx4, dy4 = np.matmul(E, np.array([X, Y]))
dx5, dy5 = np.matmul(F, np.array([X, Y]))

# n = -2

# Defining color
# color = np.sqrt(((v-n)/2)*2 + ((u-n)/2)*2)
# color = (u**2+v**2)**(1/2)
color = u*v

# Creating plot
fig, ax = plt.subplots(3,3,figsize =(15,15),
↪   gridspec_kw = {'wspace':0.1,
↪   'hspace':0.1})

# ax.quiver(X, Y, u, v, color, alpha = 1)
for i in range(-2,3):
    for j in range(-2,3):
        # ax.quiver(1.0*X+4*i, 1.0*Y+4*j, u,
        ↪   v)#, color, alpha = 1)
        # ax.quiver(1.5*X+4*i, 1.5*Y+4*j, u,
        ↪   v)#, color, alpha = 1)
        # ax.quiver(2.0*X+4*i, 2.0*Y+4*j, u,
        ↪   v)#, color, alpha = 1)

        ax[0,0].quiver(0.5*X+4*i, 0.5*Y+4*j,
↪   u, v, u*v, alpha = 0.50, cmap='spring')
        ax[0,0].quiver(1.0*X+4*i, 1.0*Y+4*j,
```

https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745916

### 36.4.2 Fundamentals of Curvilinear Tensors

### 36.4.3 Fundamentals of Spacetime Tensors

## 36.5 Lin, Hsiu-Hau / Porcupine Lin / Hedgehog Note

# Chapter 37

# dual space

dual space and linear functional

https://ccjou.wordpress.com/2011/06/13/線性泛函與對偶空間/

## 37.1 linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n =& b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n =& b_2 \\
& \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n =& b_m
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

$$
A\boldsymbol{x} = \boldsymbol{b}
$$

$$
\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_i \end{pmatrix} = b_i
$$

$$
\begin{pmatrix} a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_i \end{pmatrix} = b_i
$$

$$
\begin{pmatrix} \cdots & a_{ij} & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix} = b_i
$$

$$
\boldsymbol{a}_i^{\intercal}\boldsymbol{x} = \boldsymbol{a}_i \cdot \boldsymbol{x} = b_i
$$

$$
a_{ij}x_j = \boldsymbol{a}_i^{\intercal}\boldsymbol{x} = \boldsymbol{a}_i \cdot \boldsymbol{x} = b_i
$$

$$\boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^\intercal \boldsymbol{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots$$

if finite,

$$\boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^\intercal \boldsymbol{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

## 37.2 matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

$$AX = B$$

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix} = \begin{pmatrix} b_{ik} \end{pmatrix} = b_{ik}$$

$$\begin{pmatrix} a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_{1k} \\ \vdots \\ x_{jk} \\ \vdots \\ x_{nk} \end{pmatrix} = \begin{pmatrix} b_{ik} \end{pmatrix} = b_{ik}$$

$$\begin{pmatrix} \cdots & a_{ij} & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ x_{jk} \\ \vdots \end{pmatrix} = b_{ik}$$

$$a_{ij}x_j = \boldsymbol{a}_i^\intercal \boldsymbol{x} = \boldsymbol{a}_i \cdot \boldsymbol{x} = b_i$$

$$\boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^\intercal \boldsymbol{x} = a_{ij}x_j$$

`\iddots` in MathJax

https://math.meta.stackexchange.com/questions/23273/mathjax-and-iddots-udots-or-reflectbox

$$b_{ik} = a_{ij}x_{jk} = \boldsymbol{a}_i \cdot \boldsymbol{x}_k = \boldsymbol{a}_i^\intercal \boldsymbol{x}_k$$

$$\operatorname{row}(A)\operatorname{col}(X) = b_{\text{row,col}}$$

$$\boldsymbol{a}_i \cdot \boldsymbol{x}_k = \boldsymbol{a}_i^\intercal \boldsymbol{x}_k = a_{ij}x_{jk}$$

$$\boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^\intercal \boldsymbol{x}_j = a_{ik}x_{kj}$$

---

$$\boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^\intercal \boldsymbol{x}_j = a_{ik}x_{kj} = \cdots + a_{ik}x_{kj} + \cdots$$

if finite,

$$\boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^\intercal \boldsymbol{x}_j = a_{ik}x_{kj} = a_{i1}x_{1j} + \cdots + a_{ik}x_{kj} + \cdots + a_{in}x_{nj}$$

## 37.3 functional

( innder product or dot product ) or linear equations

$$\boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^\intercal \boldsymbol{x} = a_{ij} x_j = \cdots + a_{ij} x_j + \cdots$$

if finite,

$$\boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^\intercal \boldsymbol{x} = a_{ij} x_j = a_{i1} x_1 + \cdots + a_{ij} x_j + \cdots + a_{in} x_n$$

---

actually, several rows

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & \cdots + a_{ij} x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & a_{i1} x_1 + \cdots + a_{ij} x_j + \cdots + a_{in} x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

---

in functional aspect,

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & \cdots + a_{ij} x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & a_{i1} x_1 + \cdots + a_{ij} x_j + \cdots + a_{in} x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

---

$$\begin{array}{ccccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\cdots, x_j, \cdots) & = & f_i(x_j) & = & f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & \cdots + a_{ij} x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \cdots, x_j, \cdots, x_n) & = & f_i(x_j) & = & f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^\intercal \boldsymbol{x} & = & a_{ij} x_j & = & a_{i1} x_1 + \cdots + a_{ij} x_j + \cdots + a_{in} x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

---

or simply

$$f_i(\cdots, x_j, \cdots) = f_i(x_j) = f_i(\boldsymbol{x}) = \boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^\intercal \boldsymbol{x}_j = a_{ik} x_{kj} = \cdots + a_{ij} x_j + \cdots$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^\infty \to \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^{\infty} \to \mathbb{F}$$

or more abstract notation,

$$f_i : F^{\infty} \to F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^{\infty} \to \mathbb{R}$$

---

if finite,

$$f_i(x_1, \cdots, x_j, \cdots, x_n) = f_i(x_j) = f_i(\boldsymbol{x}) = \boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^{\intercal}\boldsymbol{x}_j = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

$$\begin{aligned} f_i(x_1, x_2, \cdots, x_n) =& f_i(x_1, \cdots, x_j, \cdots, x_n) = f_i(x_j) = f_i(\boldsymbol{x}) \\ =& \boldsymbol{a}_i \cdot \boldsymbol{x}_j = \boldsymbol{a}_i^{\intercal}\boldsymbol{x}_j = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \end{aligned}$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^n \to \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^n \to \mathbb{F}$$

or more abstract notation,

$$f_i : F^n \to F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

---

functionals are a set of fuctions mapping $n$-dimensional vectors to scalars

$$f_i : F^n \to F$$

---

$$\begin{array}{ccccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\cdots, x_j, \cdots) & = & f_i(x_j) & = & f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^{\intercal}\boldsymbol{x} & = & a_{ij}x_j & = & \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^{\infty} \to F\} = \left\{ \begin{array}{c} \vdots \\ f_i(\cdots, x_j, \cdots) = f_i(x_j) = f_i(\boldsymbol{x}) = \boldsymbol{a}_i \cdot \boldsymbol{x} = \boldsymbol{a}_i^{\intercal}\boldsymbol{x} = a_{ij}x_j, \\ \vdots \end{array} \right\}$$

if finite,

$$\begin{array}{ccccccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \cdots, x_j, \cdots, x_n) & = & f_i(x_j) & = & f_i(\boldsymbol{x}) & = & \boldsymbol{a}_i \cdot \boldsymbol{x} & = & \boldsymbol{a}_i^{\intercal}\boldsymbol{x} & = & a_{ij}x_j & = & a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^n \to F\} = \left\{ \begin{array}{c} \vdots \\ f_i(x_1, x_2, \cdots, x_n) = f_i(\boldsymbol{x}) = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n, \\ \vdots \end{array} \right\}$$

---

linear functionals are a set of fuctions mapping vectors to scalars linearly

$$f = \{f_i | f_i : F^n \to F\} = \left\{ \begin{array}{c} \vdots \\ f_i(x_1, x_2, \cdots, x_n) = f_i(\boldsymbol{x}) = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n, \\ \vdots \end{array} \right\}$$

## 37.4 definition of linear functional

functionals generalized to general vector space

$$f = \{f_i | f_i : V \to F\}$$

linear functionals generalized to general vector space is a linear transformation

$$f = \left\{ f_i \middle| \begin{cases} f_i : V \to F \\ \forall \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in V^2 [f_i(\boldsymbol{x} + \boldsymbol{y}) = f_i(\boldsymbol{x}) + f_i(\boldsymbol{y})] \\ \forall \boldsymbol{x} \in V, c \in F [f_i(c\boldsymbol{x}) = cf_i(\boldsymbol{x})] \end{cases} \right\}$$

$$f = \left\{ f_i \middle| \begin{cases} f_i : V \to F \\ \forall \boldsymbol{x}, \boldsymbol{y} \in V [f_i(\boldsymbol{x} + \boldsymbol{y}) = f_i(\boldsymbol{x}) + f_i(\boldsymbol{y})] \\ \forall \boldsymbol{x} \in V, c \in F [f_i(c\boldsymbol{x}) = cf_i(\boldsymbol{x})] \end{cases} \right\}$$

if $\begin{cases} F = \mathbb{C} \\ V = \mathbb{C}^n \end{cases}$,

$$f = \left\{ f_i \middle| \begin{cases} f_i : \mathbb{C}^n \to \mathbb{C} \\ \forall \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in (\mathbb{C}^n)^2 [f_i(\boldsymbol{x} + \boldsymbol{y}) = f_i(\boldsymbol{x}) + f_i(\boldsymbol{y})] \\ \forall \boldsymbol{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\boldsymbol{x}) = cf_i(\boldsymbol{x})] \end{cases} \right\}$$

$$f = \left\{ f_i \middle| \begin{cases} f_i : \mathbb{C}^n \to \mathbb{C} \\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n [f_i(\boldsymbol{x} + \boldsymbol{y}) = f_i(\boldsymbol{x}) + f_i(\boldsymbol{y})] \\ \forall \boldsymbol{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\boldsymbol{x}) = cf_i(\boldsymbol{x})] \end{cases} \right\}$$

then

$$f_i(\boldsymbol{x}) = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

satisfying

$$\begin{aligned} f_i(\boldsymbol{x} + \boldsymbol{y}) = & a_{i1}(x + y)_1 + \cdots + a_{ij}(x + y)_j + \cdots + a_{in}(x + y)_n \\ = & a_{i1}(x_1 + y_1) + \cdots + a_{ij}(x_j + y_j) + \cdots + a_{in}(x_n + y_n) \\ = & (a_{i1}x_1 + a_{i1}y_1) + \cdots + (a_{ij}x_j + a_{ij}y_j) + \cdots + (a_{in}x_n + a_{in}y_n) \\ = & (a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n) + (a_{i1}y_1 + \cdots + a_{ij}y_j + \cdots + a_{in}y_n) \\ = & f_i(\boldsymbol{x}) + f_i(\boldsymbol{y}) \end{aligned}$$

$$\begin{aligned} f_i(c\boldsymbol{x}) = & a_{i1}(cx)_1 + \cdots + a_{ij}(cx)_j + \cdots + a_{in}(cx)_n \\ = & a_{i1}(cx_1) + \cdots + a_{ij}(cx_j) + \cdots + a_{in}(cx_n) \\ = & c(a_{i1}x_1) + \cdots + c(a_{ij}x_j) + \cdots + c(a_{in}x_n) \\ = & c(a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n) \\ = & cf_i(\boldsymbol{x}) \end{aligned}$$

---

different functional has different $a_{ij}$

let

$$a_{ij} = f_i(\boldsymbol{e}_j), \boldsymbol{e}_j = \left\langle \overbrace{0, \cdots, 0}^{j-1}, 1, 0, \cdots, 0 \right\rangle = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

if $f_i$ is a linear functional, then

$$\begin{aligned} f_i(\boldsymbol{x}) =& f_i(x_1\boldsymbol{e}_1 + \cdots + x_j\boldsymbol{e}_j + \cdots + x_n\boldsymbol{e}_n) \\ =& f_i(x_1\boldsymbol{e}_1) + \cdots + f_i(x_j\boldsymbol{e}_j) + \cdots + f_i(x_n\boldsymbol{e}_n) \\ =& x_1 f_i(\boldsymbol{e}_1) + \cdots + x_j f_i(\boldsymbol{e}_j) + \cdots + x_n f_i(\boldsymbol{e}_n) \\ =& x_1 a_{i1} + \cdots + x_j a_{ij} + \cdots + x_n a_{in} \\ =& a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ =& f_i(\boldsymbol{x}) \end{aligned}$$

## 37.5 set of all linear transformations is a vector space

https://ccjou.wordpress.com/2011/04/08/線性變換集合構成向量空間/

vector space[38]

https://math.stackexchange.com/questions/2381942/the-set-of-all-linear-maps-tv-w-is-a-vector-space

$$T : V \to W \Leftrightarrow \forall \boldsymbol{v} \in V, \exists! \boldsymbol{w} \in W \left[\boldsymbol{w} = T(\boldsymbol{v})\right]$$

$$\begin{cases} V, W \text{ are vector spaces} \\ T : V \to W \\ \begin{cases} \forall \boldsymbol{u}, \boldsymbol{v} \in V \left[T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})\right] \\ \forall \boldsymbol{v} \in V, c \in F \left[T(c\boldsymbol{v}) = cT(\boldsymbol{v})\right] \end{cases} & \text{linearity} \end{cases}$$
$$\Leftrightarrow T \text{ is a linear tranformation}$$

$$\begin{cases} V, W \text{ are vector spaces, both over } F \\ T, U : V \to W \begin{cases} T : V \to W \\ U : V \to W \end{cases} \\ T, U \text{ are both linear tranformations} \\ \boldsymbol{v} \in V \\ c \in F \end{cases}$$

There is still linearity over linear transformations

$(va)$

$$\begin{aligned} (T + U)(\boldsymbol{u} + \boldsymbol{v}) =& T(\boldsymbol{u} + \boldsymbol{v}) + U(\boldsymbol{u} + \boldsymbol{v}) \\ &= \left[T(\boldsymbol{u}) + T(\boldsymbol{v})\right] + \left[U(\boldsymbol{u}) + U(\boldsymbol{v})\right] \\ &= \left[T(\boldsymbol{u}) + U(\boldsymbol{u})\right] + \left[T(\boldsymbol{v}) + U(\boldsymbol{v})\right] \\ &= (T + U)(\boldsymbol{u}) + (T + U)(\boldsymbol{v}) \end{aligned}$$

$(sm)$

$$\begin{aligned}(T+U)(c\boldsymbol{v}) =& T(c\boldsymbol{v})+U(c\boldsymbol{v})\\ =& cT(\boldsymbol{v})+cU(\boldsymbol{v})\\ =& c\left[T(\boldsymbol{v})+U(\boldsymbol{v})\right]\\ =& c(T+U)(\boldsymbol{v})\end{aligned}$$

so we can define

$$\begin{cases}(T+U)(\boldsymbol{v})=T(\boldsymbol{v})+U(\boldsymbol{v}) & \text{linear transformation addition}\\ (cT)(\boldsymbol{v})=cT(\boldsymbol{v}) & \text{scalar linear transformation multiplication}\end{cases}$$

---

the set of all linear tranformations is a vector space

$\mathcal{T}$ is the set of all linear tranformations

$$\begin{cases}F & (f)\, F \text{ is a field}\\ \mathcal{T}\neq\emptyset & (ne)\,\text{nonempty set}\\ +:\mathcal{T}\times\mathcal{T}=\mathcal{T}^2\xrightarrow{+}\mathcal{T}\Leftrightarrow\forall T,U\in\mathcal{T},\exists S\in\mathcal{T}\left[S=T+U\right] & (va)\,\text{vector addition}\\ \cdot:F\times\mathcal{T}\xrightarrow{\cdot}\mathcal{T}\Leftrightarrow\forall c\in F,\forall T\in\mathcal{T},\exists U\in\mathcal{T}\left[U=cT=c\cdot T\right] & (sm)\,\text{scalar multiplication}\\ \begin{cases}\forall S,T,U\in\mathcal{T}\left[S+(T+U)=(S+T)+U\right] & (a)\\ \forall T,U\in\mathcal{T}\left[T+U=U+T\right] & (c)\\ \exists!O\in\mathcal{T},\forall T\in\mathcal{T}\left[O+T=T\right] & (e)\\ \forall T\in\mathcal{T},\exists!-T\in\mathcal{T}\left[(-T)+T=O\right] & (i)\end{cases} & (va)\,\text{vector addition axioms}\\ \begin{cases}\forall b,c\in F,T\in\mathcal{T}\left[b(cT)=(bc)T\right] & (a)\\ \exists!1\in F,\forall T\in\mathcal{T}\left[1T=T\right] & (e)\\ \forall c\in F,T,U\in\mathcal{T}\left[c(T+U)=cT+cU\right] & (dv)\\ \forall b,c\in F,T\in\mathcal{T}\left[(b+c)T=bT+cT\right] & (ds)\end{cases} & (sm)\,\text{scalar multiplication axioms}\end{cases}$$
$$\Leftrightarrow\mathcal{T}=\mathcal{T}(F,+,\cdot)=(\mathcal{T},F,+,\cdot)\text{ is a vector space over the field } F$$
$$\Leftrightarrow\mathcal{T}\text{ is a vector space}$$

Selected proofs of 8 vector space axioms due to some trivial field and vector space properties:

$(va)\,(a)$

$$\begin{aligned}(S+(T+U))(\boldsymbol{v}) =& S(\boldsymbol{v})+(T+U)(\boldsymbol{v})\\ =& S(\boldsymbol{v})+T(\boldsymbol{v})+U(\boldsymbol{v})\\ =& (S+T)(\boldsymbol{v})+U(\boldsymbol{v})\\ =& ((S+T)+U)(\boldsymbol{v})\end{aligned}$$

$(va)\,(c)$

$$\begin{aligned}(T+U)(\boldsymbol{v}) =& T(\boldsymbol{v})+U(\boldsymbol{v})\\ =& U(\boldsymbol{v})+T(\boldsymbol{v})\\ =& (U+T)(\boldsymbol{v})\end{aligned}$$

$(va)\,(e)$

$$O(\boldsymbol{v})=0\boldsymbol{w}\in W$$

$$\begin{aligned}(O+T)(\boldsymbol{v}) =& O(\boldsymbol{v})+T(\boldsymbol{v})\\ =& 0\boldsymbol{w}+T(\boldsymbol{v})\\ =& T(\boldsymbol{v})\end{aligned}$$

$$O_1(\boldsymbol{v})-O_2(\boldsymbol{v})=0\boldsymbol{w}-0\boldsymbol{w}=0\boldsymbol{w}\Rightarrow O_1(\boldsymbol{v})=O_2(\boldsymbol{v})$$

$(sm)\,(dv)$

$$\begin{aligned}(c\,(T+U))\,(\boldsymbol{v}) =& c\,(T+U)\,(\boldsymbol{v})\\ =& c\,[T\,(\boldsymbol{v})+U\,(\boldsymbol{v})]\\ =& cT\,(\boldsymbol{v})+cU\,(\boldsymbol{v})\\ =& (cT+cU)\,(\boldsymbol{v})\end{aligned}$$

The set of all linear tranformations $\mathcal{T}$ is a vector space.

□

## 37.6 definition of dual space

$$\begin{aligned}&V^* = L\,(V,F)\\ =&f = \left\{ f_i \middle| \begin{cases} f_i : V \to F & \text{functional mapping vector to field scalar} \\ \begin{cases} \forall \boldsymbol{x},\boldsymbol{y}\in V\,[f_i\,(\boldsymbol{x}+\boldsymbol{y}) = f_i\,(\boldsymbol{x}) + f_i\,(\boldsymbol{y})] \\ \forall \boldsymbol{x}\in V, c\in F\,[f_i\,(c\boldsymbol{x}) = cf_i\,(\boldsymbol{x})] \end{cases} & (L)\,\text{linearity} \end{cases} \right\}\\ \Leftrightarrow & V^* \text{ is a dual space, a set of linear functionals } f_i \text{ mapping vectors in the vector space } V \text{ to scalars in the field } F\end{aligned}$$

vector space[38]

https://web.math.sinica.edu.tw/mathmedia/HTMLarticle18.jsp?mID=31304

https://web.math.sinica.edu.tw/mathmedia/author18.jsp?query_filter=%E9%BE%94%E6%98%87

## 37.7 double dual

double dual = second dual

https://ccjou.wordpress.com/2014/04/10/雙重對偶/

https://www.zhihu.com/question/444079322/answer/1749490720

Hough Transform

https://www.cnblogs.com/php-rearch/p/6760683.html

# Chapter 38

# vector space

https://www.bilibili.com/video/BV1ez4y1n7h8

https://ccjou.wordpress.com/2010/04/15/同構的向量空間/

## 38.1 What is a vector?

**What is a vector?** or **What is an element in a vector space?**

**Binary operations defined on a vector space satisfying some properties** is more important than **what is a vector**.

ultimate answer: double dual concept[38.4.1.2]

## 38.2 vector space definition

https://tex.stackexchange.com/a/141489 multiline node

202403311536-vector-space_files/figure-latex/vector-construction

Figure 38.1: vector space construction

$$
\begin{cases}
F \text{ is a field} & (f)\text{ field} \\
V \neq \emptyset & (ne)\text{ nonempty set} \\
+ : V \times V = V^2 \xrightarrow{+} V \Leftrightarrow \forall \boldsymbol{u}, \boldsymbol{v} \in V, \exists! \boldsymbol{w} \in V \left[\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}\right] & (va)\text{ vector addition} \\
\cdot : F \times V \xrightarrow{\cdot} V \Leftrightarrow \forall s \in F, \forall \boldsymbol{v} \in V, \exists! \boldsymbol{u} \in V \left[\boldsymbol{u} = s\boldsymbol{v} = s \cdot \boldsymbol{v}\right] & (sm)\text{ scalar multiplication} \\
\begin{cases}
\exists! \boldsymbol{0} \in V, \forall \boldsymbol{v} \in V \left[\boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v}\right] & (e)\text{ identity} \\
\forall \boldsymbol{v} \in V, \exists! -\boldsymbol{v} \in V \left[(-\boldsymbol{v}) + \boldsymbol{v} = \boldsymbol{0}\right] & (i)\text{ inverse} \\
\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V \left[\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}\right] & (a)\text{ associativity} \\
\forall \boldsymbol{u}, \boldsymbol{v} \in V \left[\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}\right] & (c)\text{ commutativity}
\end{cases} & (va)\text{ axioms} \\
\begin{cases}
\exists! 1 \in F, \forall \boldsymbol{v} \in V \left[1\boldsymbol{v} = \boldsymbol{v}\right] & (e)\text{ identity} \\
\forall s, t \in F, \boldsymbol{v} \in V \left[s(t\boldsymbol{v}) = (st)\boldsymbol{v}\right] & (a)\text{ associativity} \\
\forall s, t \in F, \boldsymbol{v} \in V \left[(s + t)\boldsymbol{v} = s\boldsymbol{v} + t\boldsymbol{v}\right] & (ds)\text{ scalar distributivity} \\
\forall s \in F, \boldsymbol{u}, \boldsymbol{v} \in V \left[s(\boldsymbol{u} + \boldsymbol{v}) = s\boldsymbol{u} + s\boldsymbol{v}\right] & (dv)\text{ vector distributivity}
\end{cases} & (sm)\text{ axioms}
\end{cases}
$$

$$\Leftrightarrow V = V(F, +, \cdot) = (V, F, +, \cdot) \text{ is a vector space over the field } F$$

$$\Leftrightarrow V \text{ is a vector space}$$

### 38.2.1 commutative group structure of vector space

$(va)$ axioms = vector addition axioms

$$V = (V,+) \text{ is a commutative group} \Leftrightarrow V = (V,+) \text{ is an abelian group}$$

$$\Leftrightarrow \begin{cases} V = (V,+) = (V,+_V) \text{ is a group} & (g)\,\text{group} \\ \forall \boldsymbol{u},\boldsymbol{v} \in V\,[\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}] & (c)\,\text{commutativity} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{cases} + : V \times V = V^2 \xrightarrow{+} V \Leftrightarrow \forall \boldsymbol{u},\boldsymbol{v} \in V, \exists! \boldsymbol{w} \in V\,[\boldsymbol{w} = \boldsymbol{u}+\boldsymbol{v}] & (cl)\,\text{closure} \\ \exists! \boldsymbol{0} \in V, \forall \boldsymbol{v} \in V\,[\boldsymbol{0}+\boldsymbol{v} = \boldsymbol{v}] & (e)\,\text{identity} \\ \forall \boldsymbol{v} \in V, \exists! -\boldsymbol{v} \in V\,[(-\boldsymbol{v})+\boldsymbol{v} = \boldsymbol{0}] & (i)\,\text{inverse} \\ \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V\,[\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}] & (a)\,\text{associativity} \end{cases} & (g) \\ \forall \boldsymbol{u},\boldsymbol{v} \in V\,[\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}] & (c) \end{cases}$$

$V$ is a vector space

$$\Leftrightarrow V = V(F,+,\cdot) = (V,F,+,\cdot) \text{ is a vector space over the field } F$$

$$\Leftrightarrow \begin{cases} F \text{ is a field} & (f)\,\text{field} \\ V \neq \emptyset & (ne)\,\text{nonempty set} \\ V = (V,+) \text{ is a commutative group} \Leftrightarrow V = (V,+) \text{ is an abelian group} & (va)\,\text{vector addition} \\ \cdot : F \times V \xrightarrow{\cdot} V \Leftrightarrow \forall s \in F, \forall \boldsymbol{v} \in V, \exists! \boldsymbol{u} \in V\,[\boldsymbol{u} = s\boldsymbol{v} = s \cdot \boldsymbol{v}] & (sm)\,\text{scalar multiplication} \\ \begin{cases} \exists! 1 \in F, \forall \boldsymbol{v} \in V\,[1\boldsymbol{v} = \boldsymbol{v}] & (e)\,\text{identity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[s(t\boldsymbol{v}) = (st)\boldsymbol{v}] & (a)\,\text{associativity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[(s+t)\boldsymbol{v} = s\boldsymbol{v}+t\boldsymbol{v}] & (ds)\,\text{scalar distributivity} \\ \forall s \in F, \boldsymbol{u},\boldsymbol{v} \in V\,[s(\boldsymbol{u}+\boldsymbol{v}) = s\boldsymbol{u}+s\boldsymbol{v}] & (dv)\,\text{vector distributivity} \end{cases} & (sm)\,\text{axioms} \end{cases}$$

$$\Leftrightarrow \begin{cases} F \text{ is a field} & (f)\,\text{field} \\ V \neq \emptyset & (ne)\,\text{nonempty set} \\ \begin{cases} V = (V,+) = (V,+_V) \text{ is a group} & (g)\,\text{group} \\ \forall \boldsymbol{u},\boldsymbol{v} \in V\,[\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}] & (c)\,\text{commutativity} \end{cases} & (va)\,\text{vector addition} \\ \cdot = \cdot_{F\times V} : F \times V \xrightarrow{\cdot} V \Leftrightarrow \forall s \in F, \forall \boldsymbol{v} \in V, \exists! \boldsymbol{u} \in V\,[\boldsymbol{u} = s\boldsymbol{v} = s \cdot \boldsymbol{v}] & (sm)\,\text{scalar multiplication} \\ \begin{cases} \exists! 1 \in F, \forall \boldsymbol{v} \in V\,[1\boldsymbol{v} = \boldsymbol{v}] & (e)\,\text{identity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[s(t\boldsymbol{v}) = (st)\boldsymbol{v}] & (a)\,\text{associativity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[(s+t)\boldsymbol{v} = s\boldsymbol{v}+t\boldsymbol{v}] & (ds)\,\text{scalar distributivity} \\ \forall s \in F, \boldsymbol{u},\boldsymbol{v} \in V\,[s(\boldsymbol{u}+\boldsymbol{v}) = s\boldsymbol{u}+s\boldsymbol{v}] & (dv)\,\text{vector distributivity} \end{cases} & (sm)\,\text{axioms} \end{cases}$$

$$\Leftrightarrow \begin{cases} F = F(+_F,\cdot_F) = (F,+_F,\cdot_F) = (F,+,\cdot) \text{ is a field} & (f) \\ V \neq \emptyset & (ne) \\ \begin{cases} \begin{cases} + : V \times V = V^2 \xrightarrow{+} V \Leftrightarrow \forall \boldsymbol{u},\boldsymbol{v} \in V, \exists! \boldsymbol{w} \in V\,[\boldsymbol{w} = \boldsymbol{u}+\boldsymbol{v}] & (cl)\,\text{closure} \\ \exists! \boldsymbol{0} \in V, \forall \boldsymbol{v} \in V\,[\boldsymbol{0}+\boldsymbol{v} = \boldsymbol{v}] & (e)\,\text{identity} \\ \forall \boldsymbol{v} \in V, \exists! -\boldsymbol{v} \in V\,[(-\boldsymbol{v})+\boldsymbol{v} = \boldsymbol{0}] & (i)\,\text{inverse} \\ \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V\,[\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}] & (a)\,\text{associativity} \end{cases} & (g) \\ \forall \boldsymbol{u},\boldsymbol{v} \in V\,[\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}] & (c) \end{cases} & (va) \\ \begin{cases} \cdot : F \times V \xrightarrow{\cdot} V \Leftrightarrow \forall s \in F, \forall \boldsymbol{v} \in V, \exists! \boldsymbol{u} \in V\,[\boldsymbol{u} = s\boldsymbol{v} = s \cdot \boldsymbol{v}] & (cl)\,\text{closure} \\ \exists! 1 \in F, \forall \boldsymbol{v} \in V\,[1\boldsymbol{v} = \boldsymbol{v}] & (e)\,\text{identity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[s(t\boldsymbol{v}) = s \cdot_{F\times V} (t \cdot_{F\times V} \boldsymbol{v}) = (s \cdot_F t) \cdot_{F\times V} \boldsymbol{v} = (st)\boldsymbol{v}] & (a)\,\text{associativity} \\ \forall s,t \in F, \boldsymbol{v} \in V\,[(s+t)\boldsymbol{v} = (s +_F t)\boldsymbol{v} = s\boldsymbol{v} +_V t\boldsymbol{v} = s\boldsymbol{v}+t\boldsymbol{v}] & (ds)\,\text{scalar distributivity} \\ \forall s \in F, \boldsymbol{u},\boldsymbol{v} \in V\,[s(\boldsymbol{u}+\boldsymbol{v}) = s\boldsymbol{u}+s\boldsymbol{v}] & (dv)\,\text{vector distributivity} \end{cases} & (sm) \end{cases}$$

### 38.2.2 scalar distributivity

$(sm)\,(ds)$

$$\forall s,t \in F, \boldsymbol{v} \in V\,[(s+t)\boldsymbol{v} = s\boldsymbol{v}+t\boldsymbol{v}]$$

$$\forall s,t \in F, \boldsymbol{v} \in V\,[(s +_F t)\boldsymbol{v} = s\boldsymbol{v}+t\boldsymbol{v}]$$

$$\forall s,t \in F, \boldsymbol{v} \in V\,[(s +_F t)\boldsymbol{v} = s\boldsymbol{v} +_V t\boldsymbol{v}]$$

## 38.3 linearity

$$\begin{cases} f(x+y) = f(x) + f(y) & \text{additivity} \\ f(\lambda x) = \lambda f(x) & \text{homogeneity} \end{cases}$$
$$\Leftrightarrow f(\lambda x + y) = \lambda f(x) + f(y)$$
$$\Leftrightarrow f \text{ is linear}$$

### 38.3.1 linear structure of vector space

$$\forall s \in F, \boldsymbol{u}, \boldsymbol{v} \in V \, [\boldsymbol{u} + s\boldsymbol{v} \in V]$$

$$\forall s \in F, \forall \boldsymbol{u}, \boldsymbol{v} \in V \, [\boldsymbol{u} + s\boldsymbol{v} \in V]$$

$$\forall s \in F, \langle \boldsymbol{u}, \boldsymbol{v} \rangle \in V^2 \, [\boldsymbol{u} + s\boldsymbol{v} \in V]$$

$$\begin{cases} \forall \boldsymbol{u}, \boldsymbol{v} \in V \, [\boldsymbol{u} + \boldsymbol{v} \in V] & \text{vector addition closure} \\ \forall s \in F, \boldsymbol{v} \in V \, [s\boldsymbol{v} \in V] & \text{scalar multiplication closure} \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall \boldsymbol{u}, \boldsymbol{v} \in V \, [\boldsymbol{u} + \boldsymbol{v} \in V] & (a) \, \text{additivity} \\ \forall s \in F, \boldsymbol{v} \in V \, [s\boldsymbol{v} \in V] & (h) \, \text{homogeneity} \end{cases}$$
$$\Leftrightarrow \forall s \in F, \boldsymbol{u}, \boldsymbol{v} \in V \, [\boldsymbol{u} + s\boldsymbol{v} \in V] \quad (l) \, \text{linearity}$$

### 38.3.2 linear transformation or linear map

$$\begin{cases} V, W \text{ are vector spaces} \\ T : V \to W \\ \begin{cases} \forall \boldsymbol{u}, \boldsymbol{v} \in V \, [T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})] & (a) \, \text{additivity} \\ \forall \boldsymbol{v} \in V, c \in F \, [T(c\boldsymbol{v}) = cT(\boldsymbol{v})] & (h) \, \text{homogeneity} \end{cases} \end{cases} \quad (L)$$
$$\Leftrightarrow \begin{cases} V, W \text{ are vector spaces} \\ T : V \to W \\ \forall \boldsymbol{u}, \boldsymbol{v} \in V, c \in F \, [T(\boldsymbol{u} + c\boldsymbol{v}) = T(\boldsymbol{u}) + cT(\boldsymbol{v})] \quad (l) \, \text{linearity} \end{cases}$$
$$\Leftrightarrow T \text{ is a linear map from } V \text{ to } W$$
$$\Leftrightarrow T \text{ is a linear tranformation}$$

## 38.4 vector space example

- arrow vector
- number
  - integer
  - real
  - complex
  - quaternion
- function
  - polynomial function
  - continuous function
- matrix
  - real matrix
  - complex matrix
- reciprocal space

https://www.bilibili.com/video/BV1NC4y1J7UL

applications in different disciplines

- math
  - recursive number series
  - Fourier series
- physics

  - electrical circuit: linear response / superposition theorem in linear circuit / linear network
- chemistry
  - balancing chemical equation

### 38.4.1 reciprocal space

reciprocal space = 倒易空間

$$\begin{cases} \boldsymbol{e}_1 = \boldsymbol{a} & \boldsymbol{a}\times\boldsymbol{b}\neq\boldsymbol{0} \\ \boldsymbol{e}_2 = \boldsymbol{b} & \boldsymbol{b}\times\boldsymbol{c}\neq\boldsymbol{0} \\ \boldsymbol{e}_3 = \boldsymbol{c} & \boldsymbol{c}\times\boldsymbol{a}\neq\boldsymbol{0} \end{cases} \Rightarrow \begin{cases} \boldsymbol{e}'_1 = \dfrac{\boldsymbol{b}\times\boldsymbol{c}}{\Omega} \\ \boldsymbol{e}'_2 = \dfrac{\boldsymbol{c}\times\boldsymbol{a}}{\Omega} \\ \boldsymbol{e}'_3 = \dfrac{\boldsymbol{a}\times\boldsymbol{b}}{\Omega} \end{cases},$$

$$\Omega = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

reciprocal space as dual space and contravariant vector

$$\begin{aligned} \operatorname{span}\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\} &= \operatorname{span}\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\} = V \\ = \mathbb{R}^3 = &\operatorname{span}\{\boldsymbol{e}'_1,\boldsymbol{e}'_2,\boldsymbol{e}'_3\} = \operatorname{span}\left\{\frac{\boldsymbol{b}\times\boldsymbol{c}}{\Omega},\frac{\boldsymbol{c}\times\boldsymbol{a}}{\Omega},\frac{\boldsymbol{a}\times\boldsymbol{b}}{\Omega}\right\} \\ = &\operatorname{span}\{\boldsymbol{e}^1,\boldsymbol{e}^2,\boldsymbol{e}^3\} = \operatorname{span}\{\boldsymbol{e}^*\}_{*\in\{1,2,3\}} = V^* \end{aligned}$$

#### 38.4.1.1 Kronecker delta

$$\begin{pmatrix} \boldsymbol{e}_1\cdot\boldsymbol{e}'_1 & \boldsymbol{e}_1\cdot\boldsymbol{e}'_2 & \boldsymbol{e}_1\cdot\boldsymbol{e}'_3 \\ \boldsymbol{e}_2\cdot\boldsymbol{e}'_1 & \boldsymbol{e}_2\cdot\boldsymbol{e}'_2 & \boldsymbol{e}_2\cdot\boldsymbol{e}'_3 \\ \boldsymbol{e}_3\cdot\boldsymbol{e}'_1 & \boldsymbol{e}_3\cdot\boldsymbol{e}'_2 & \boldsymbol{e}_3\cdot\boldsymbol{e}'_3 \end{pmatrix} = [\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}^1\cdot\boldsymbol{e}_1 & \boldsymbol{e}^1\cdot\boldsymbol{e}_2 & \boldsymbol{e}^1\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}^2\cdot\boldsymbol{e}_1 & \boldsymbol{e}^2\cdot\boldsymbol{e}_2 & \boldsymbol{e}^2\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}^3\cdot\boldsymbol{e}_1 & \boldsymbol{e}^3\cdot\boldsymbol{e}_2 & \boldsymbol{e}^3\cdot\boldsymbol{e}_3 \end{pmatrix}$$

Kronecker delta

$$\boldsymbol{e}_i\cdot\boldsymbol{e}'_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Kronecker delta tensor = Kronecker tensor

$$\boldsymbol{e}^i(\boldsymbol{e}_j) = \boldsymbol{e}^i\cdot\boldsymbol{e}_j = \delta^i_j = \delta^i{}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\boldsymbol{v} = v_a\boldsymbol{a} + v_b\boldsymbol{b} + v_c\boldsymbol{c} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$$

$$\boldsymbol{e}^1\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_1 = v_1\boldsymbol{e}_1\cdot\boldsymbol{e}'_1 + v_2\boldsymbol{e}_2\cdot\boldsymbol{e}'_1 + v_3\boldsymbol{e}_3\cdot\boldsymbol{e}'_1 = v_1$$

$$\boldsymbol{e}^2\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_2 = v_1\boldsymbol{e}_1\cdot\boldsymbol{e}'_2 + v_2\boldsymbol{e}_2\cdot\boldsymbol{e}'_2 + v_3\boldsymbol{e}_3\cdot\boldsymbol{e}'_2 = v_2$$

$$\boldsymbol{e}^3\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_3 = v_1\boldsymbol{e}_1\cdot\boldsymbol{e}'_3 + v_2\boldsymbol{e}_2\cdot\boldsymbol{e}'_3 + v_3\boldsymbol{e}_3\cdot\boldsymbol{e}'_3 = v_3$$

$$\begin{aligned} \boldsymbol{v} =& v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \\ &= (\boldsymbol{v}\cdot\boldsymbol{e}'_1)\,\boldsymbol{e}_1 + (\boldsymbol{v}\cdot\boldsymbol{e}'_2)\,\boldsymbol{e}_2 + (\boldsymbol{v}\cdot\boldsymbol{e}'_3)\,\boldsymbol{e}_3 \\ &= (\boldsymbol{e}^1\cdot\boldsymbol{v})\,\boldsymbol{e}_1 + (\boldsymbol{e}^2\cdot\boldsymbol{v})\,\boldsymbol{e}_2 + (\boldsymbol{e}^3\cdot\boldsymbol{v})\,\boldsymbol{e}_3 \\ &= \boldsymbol{e}^1(\boldsymbol{v})\,\boldsymbol{e}_1 + \boldsymbol{e}^2(\boldsymbol{v})\,\boldsymbol{e}_2 + \boldsymbol{e}^3(\boldsymbol{v})\,\boldsymbol{e}_3 \end{aligned}$$

$$\begin{cases} \boldsymbol{e}^1(\boldsymbol{v}) = \boldsymbol{e}^1\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_1 = v_1 \\ \boldsymbol{e}^2(\boldsymbol{v}) = \boldsymbol{e}^2\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_2 = v_2 \\ \boldsymbol{e}^3(\boldsymbol{v}) = \boldsymbol{e}^3\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{e}'_3 = v_3 \end{cases}$$

$$\boldsymbol{e}^i(\boldsymbol{e}_j) = \boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j = \delta^i{}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

reciprocal space is a dual space of its original vector space

$$\begin{aligned} V =& \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = \{v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3\} \\ =& \left\{\sum_{j=1}^{3} v_j\boldsymbol{e}_j\right\} = \left\{v_j\boldsymbol{e}_j \middle| \begin{cases} v_j \in F \\ \boldsymbol{e}_j \in F^3 \end{cases}\right\} = \{\boldsymbol{v}|\boldsymbol{v} \in V\} \\ V^* =& \text{span}\{\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3\} = \{v^{*1}\boldsymbol{e}^1 + v^{*2}\boldsymbol{e}^2 + v^{*3}\boldsymbol{e}^3\} \\ =& \left\{\sum_{i=1}^{3} v^{*i}\boldsymbol{e}^i\right\} = \left\{v^{*i}\boldsymbol{e}^i \middle| \begin{cases} v^{*i} \in F \\ \boldsymbol{e}^i \in F^3 \end{cases}\right\} = \{\boldsymbol{v}^*|\boldsymbol{v}^* \in V^*\} \end{aligned}$$

$$\begin{aligned} \boldsymbol{v}^*(\boldsymbol{v}) &= (v^{*i}\boldsymbol{e}^i)(\boldsymbol{v}), \boldsymbol{v} \in V \\ &= (v^{*1}\boldsymbol{e}^1 + v^{*2}\boldsymbol{e}^2 + v^{*3}\boldsymbol{e}^3)(\boldsymbol{v}) \\ &= v^{*1}\boldsymbol{e}^1(\boldsymbol{v}) + v^{*2}\boldsymbol{e}^2(\boldsymbol{v}) + v^{*3}\boldsymbol{e}^3(\boldsymbol{v}) \\ &= v^{*1}v_1 + v^{*2}v_2 + v^{*3}v_3 \in F \end{aligned}$$

element in dual space is a functional or mapping from its original vector space to the field

$$\boldsymbol{v}^* : V \to F$$

$$V \overset{\boldsymbol{v}^*}{\to} F$$

---

$V^* = \{\boldsymbol{v}^*|\boldsymbol{v}^* : V \to F\}$

$$\begin{array}{cccccccccccc} & & & V & = \{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots \} \\ & \boldsymbol{e}^1 & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ V^* = \{ & \boldsymbol{e}^2 & \} & F & \supseteq \{ & 1 & 0 & 0 & v_1 & \cdots \} \\ & \boldsymbol{e}^3 & & V & = \{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots \} \\ & \boldsymbol{v}^* & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \vdots & & F & \supseteq \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i}v_i & \cdots \} \end{array}$$

---

$$\begin{aligned} V^* = \{\boldsymbol{v}^*|\boldsymbol{v}^* \in V^*\} &= \{\boldsymbol{v}^*|\boldsymbol{v}^* : V \to F\} \\ &= \left\{\boldsymbol{v}^* \middle| V \overset{\boldsymbol{v}^*}{\to} F\right\} \\ &= \{\boldsymbol{\omega}|\boldsymbol{\omega} : V \to F\} \\ &= \left\{\omega^i\boldsymbol{e}^i \middle| \begin{cases} \omega^i \in F \\ \boldsymbol{e}^i \in F^3 \end{cases}\right\} \end{aligned}$$

By defining vector addition and scalar multiplication on the dual space

$$\begin{cases} + : V^* \times V^* \to V^* \Leftrightarrow \forall \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in V^*, \exists! (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \in V^* [(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)(\boldsymbol{v}) = \boldsymbol{\omega}_1(\boldsymbol{v}) + \boldsymbol{\omega}_2(\boldsymbol{v})] \\ \cdot : F \times V^* \to V^* \Leftrightarrow \forall k \in F, \forall \boldsymbol{\omega} \in V^*, \exists! (k\boldsymbol{\omega}) \in V^* [(k\boldsymbol{\omega})(\boldsymbol{v}) = k \cdot \boldsymbol{\omega}(\boldsymbol{v})] \\ \forall \boldsymbol{\omega} \in V^*, \exists! \boldsymbol{0} \in V^* [(\boldsymbol{\omega} + \boldsymbol{0})(\boldsymbol{v}) = \boldsymbol{\omega}(\boldsymbol{v}) + \boldsymbol{0}(\boldsymbol{v}) = \boldsymbol{\omega}(\boldsymbol{v})] \end{cases}$$

the dual space also becomes a vector space.

#### 38.4.1.2 double dual concept

double dual space = second dual space

$$\begin{aligned} V^{**} &= (V^*)^* \\ &= \{\boldsymbol{\omega}^* | \boldsymbol{\omega}^* : V^* \to F\} \\ &= \{\boldsymbol{\omega}^* | \boldsymbol{\omega}^* \in V^{**}\} \end{aligned}$$

$$\begin{aligned} V^{**} = (V^*)^* =& \text{span}\,\{\boldsymbol{e}^\mu\}^*_{\mu\in\{1,2,3\}} \\ =& \text{span}\,\{\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3\}^* \\ =& \text{span}\,\{\boldsymbol{e}^{1*}, \boldsymbol{e}^{2*}, \boldsymbol{e}^{3*}\} \\ =& \text{span}\,\{\boldsymbol{e}^{\nu*}\}_{\nu\in\{1,2,3\}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}^*(\boldsymbol{\omega}) &= (\omega^{*\nu}\boldsymbol{e}^{\nu*})(\boldsymbol{\omega}), \boldsymbol{\omega} \in V^* \\ &= (\omega^{*1}\boldsymbol{e}^{1*} + \omega^{*2}\boldsymbol{e}^{2*} + \omega^{*3}\boldsymbol{e}^{3*})(\boldsymbol{\omega}) \\ &= \omega^{*1}\boldsymbol{e}^{1*}(\boldsymbol{\omega}) + \omega^{*2}\boldsymbol{e}^{2*}(\boldsymbol{\omega}) + \omega^{*3}\boldsymbol{e}^{3*}(\boldsymbol{\omega}) \\ &= \omega^{*1}\omega_1 + \omega^{*2}\omega_2 + \omega^{*3}\omega_3 \in F \end{aligned}$$

---

$V^{**} = \{\boldsymbol{\omega}^* | \boldsymbol{\omega}^* : V^* \to F\}$

$$\begin{array}{cccccccccccc} & & & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{\omega} & \cdots\} \\ & \boldsymbol{e}^{1*} & : & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ V^{**} = \{ & \boldsymbol{e}^{2*} & \} & & F & \supseteq\{ & 1 & 0 & 0 & \omega^1 & \cdots\} \\ & \boldsymbol{e}^{3*} & & & V^* & =\{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots\} \\ & \boldsymbol{\omega}^* & : & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \vdots & & & F & \supseteq\{ & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*}\omega^\mu & \cdots\} \end{array}$$

---

$$\begin{cases} \boldsymbol{e}^{1*}(\boldsymbol{\omega}) = \boldsymbol{e}^{1*} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{e}^{1*} = \boldsymbol{\omega}(\boldsymbol{e}^{1*}) \\ \boldsymbol{e}^{2*}(\boldsymbol{\omega}) = \boldsymbol{e}^{2*} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{e}^{2*} = \boldsymbol{\omega}(\boldsymbol{e}^{2*}) \\ \boldsymbol{e}^{3*}(\boldsymbol{\omega}) = \boldsymbol{e}^{3*} \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{e}^{3*} = \boldsymbol{\omega}(\boldsymbol{e}^{3*}) \end{cases}$$

$$\boldsymbol{\omega}^*(\boldsymbol{\omega}) = \boldsymbol{\omega}^* \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\omega}^* = \boldsymbol{\omega}(\boldsymbol{\omega}^*)$$

i.e. $\boldsymbol{f}$ acts on $\boldsymbol{x}$ equivalent to $\boldsymbol{x}$ acts on $\boldsymbol{f}$

$$\boldsymbol{x}(\boldsymbol{f}) = \boldsymbol{x} \cdot \boldsymbol{f} = \boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$$

$$\boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \boldsymbol{e}^\mu \cdot \boldsymbol{e}_\nu = \delta^\mu_\nu = \delta^\mu{}_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$$\begin{aligned} \boldsymbol{e}^{\nu*}(\boldsymbol{e}^\mu) \stackrel{\text{def.}}{=} \boldsymbol{e}_\nu \cdot \boldsymbol{e}^\mu =& \boldsymbol{e}^\mu \cdot \boldsymbol{e}_\nu = \boldsymbol{e}^\mu(\boldsymbol{e}_\nu) \\ & \Downarrow \\ V^{**} = \text{span}\,\{\boldsymbol{e}^{\nu*}\}_{\nu\in\{1,2,3\}} \cong& \text{span}\,\{\boldsymbol{e}_\nu\}_{\nu\in\{1,2,3\}} = V \\ V^{**} \cong& V \\ & \Downarrow \begin{cases} V^{**} \cong V & V, V^{**} \text{ are isomorphism} \\ & \text{independent of choice of bases} \end{cases} \\ V, V^{**} & \text{ are naturally isomorphism} \end{aligned}$$

---

$V^{**} = \{\boldsymbol{\omega}^* | \boldsymbol{\omega}^* : V^* \to F\} \cong V = \{v | v : V^* \to F\}$

$$
\begin{array}{rcccccccccc}
 & & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{\omega} & \cdots\} \\
 & \boldsymbol{e}^{1*} & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
V^{**}=\{ & \boldsymbol{e}^{2*} & \} & F & \supseteq\{ & 1 & 0 & 0 & \omega^1 & \cdots\} \\
 & \boldsymbol{e}^{3*} & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{\omega} & \cdots\} \\
 & \boldsymbol{\omega}^* & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & & F & \supseteq\{ & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*}\omega^{\mu} & \cdots\} \\
 & & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{v}^* & \cdots\} \\
 & \boldsymbol{e}_1 & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cong V=\{ & \boldsymbol{e}_2 & \} & F & \supseteq\{ & 1 & 0 & 0 & v^{*1} & \cdots\} \\
 & \boldsymbol{e}_3 & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{v}^* & \cdots\} \\
 & \boldsymbol{v} & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & & F & \supseteq\{ & v_1 & v_2 & v_3 & v_\mu v^{*\mu} & \cdots\}
\end{array}
$$

---

$$V \cong V^{**}$$

---

$$V \cong V^{**} = \{\boldsymbol{\omega}^* | \boldsymbol{\omega}^* : V^* \to F\}$$

$$V = \{v | v : V^* \to F\}$$

i.e. **vector space is a set of functionals or mappings from its dual space to the field**, answering What is a vector?[38.1], and satifying Fig: 38.1.

## 38.5 EpicOrganism = AIRoswell = Pan, Yi-Wen[1]

https://space.bilibili.com/14316464/video

https://www.bilibili.com/video/BV1dC4y1171H

https://www.bilibili.com/video/BV14H4y1C7A9

https://www.bilibili.com/video/BV1MW421N7HS

## 38.6 field

https://web.math.sinica.edu.tw/math_media/d312/31202.pdf

## 38.7 module

https://web.math.sinica.edu.tw/mathmedia/HTMLarticle18.jsp?mID=31304

## 38.8 subspace

[1] https://www.linkedin.com/in/yiwen-pan-16a90076

# Chapter 39

# d$f$

## 39.1 d$f$ decomposed with partials as a set of basis in vector space

$$f = \{f_i\} = \{f_1, f_2, \cdots\} = \{f, g, \cdots\}$$

$$\boldsymbol{v} : f \to F$$

$$\boldsymbol{v}(af + bg) = a\boldsymbol{v}(f) + b\boldsymbol{v}(g)$$

$$\boldsymbol{v}(fg) = f|_P \boldsymbol{v}(g) + \boldsymbol{v}(f)\, g|_P$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)\, g(x)]\,|_{x=x_0} = f(x_0)\frac{\mathrm{d}}{\mathrm{d}x}g(x)\,|_{x=x_0} + \frac{\mathrm{d}}{\mathrm{d}x}f(x)\,|_{x=x_0}g(x_0)$$

$$V = \{\boldsymbol{v}|\boldsymbol{v} : f \to F\}$$

$$\begin{aligned} f =& f(\boldsymbol{x}) \\ =& f(x_1, \cdots, x_j, \cdots, x_n) \\ =& f(x^1, \cdots, x^j, \cdots, x^n) \end{aligned}$$

$$\boldsymbol{x} = \langle x^1, \cdots, x^j, \cdots, x^n \rangle$$

$$\boldsymbol{x}(t) = \langle x^1(t), \cdots, x^j(t), \cdots, x^n(t) \rangle$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} =& \frac{\mathrm{d}x^1}{\mathrm{d}t}\frac{\partial f}{\partial x^1} + \cdots + \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\partial f}{\partial x^j} + \cdots + \frac{\mathrm{d}x^n}{\mathrm{d}t}\frac{\partial f}{\partial x^n} \\ =& \cdots + \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\partial f}{\partial x^j} + \cdots = \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\partial f}{\partial x^j} = \frac{\mathrm{d}x^j}{\mathrm{d}t}\partial_j f \end{aligned}$$

$$\begin{aligned} V =& \mathrm{span}\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_j, \cdots, \boldsymbol{e}_n\} \\ =& \mathrm{span}\left\{\frac{\partial}{\partial x^1}|_P, \cdots, \frac{\partial}{\partial x^j}|_P, \cdots, \frac{\partial}{\partial x^n}|_P\right\} \\ =& \mathrm{span}\{\boldsymbol{\partial}_1, \cdots, \boldsymbol{\partial}_j, \cdots, \boldsymbol{\partial}_n\} \\ =& \left\{\boldsymbol{\partial}_t \middle| \boldsymbol{\partial}_t = a_j\boldsymbol{e}_j = a_j\boldsymbol{\partial}_j = a_j\frac{\partial}{\partial x^j}|_P\right\} \\ =& \left\{\frac{\partial}{\partial t}|_P \middle| \frac{\partial}{\partial t}|_P = a_1\frac{\partial}{\partial x^1}|_P + \cdots + a_j\frac{\partial}{\partial x^j}|_P + \cdots + a_n\frac{\partial}{\partial x^n}|_P\right\} \end{aligned}$$

## 39.2 dual space of span of partials

$$V^* = \{\boldsymbol{\omega}_f | \boldsymbol{\omega}_f : V \to F\}$$

$$\boldsymbol{\omega}_f (\boldsymbol{e}_j) = \boldsymbol{\omega}_f (\boldsymbol{\partial}_j) = \boldsymbol{\omega}_f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F$$

$$\begin{aligned} \boldsymbol{\omega}_{fg} (\boldsymbol{\partial}_j) = \frac{\partial fg}{\partial x^j}|_P =& f|_P \frac{\partial g}{\partial x^j}|_P + \frac{\partial f}{\partial x^j}|_P g|_P \\ =& f|_P \boldsymbol{\omega}_g (\boldsymbol{\partial}_j) + \boldsymbol{\omega}_f (\boldsymbol{\partial}_j) g|_P \end{aligned}$$

$$\boldsymbol{\omega}_{x^i} (\boldsymbol{\partial}_j) = \boldsymbol{\omega}_{x^i} \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\begin{aligned} V^* =& \{\boldsymbol{\omega}_f | \boldsymbol{\omega}_f : V \to F\} = \left\{ \boldsymbol{\omega}_f \middle| \begin{cases} \boldsymbol{\omega}_f (\boldsymbol{e}_j) = \boldsymbol{\omega}_f (\boldsymbol{\partial}_j) = \boldsymbol{\omega}_f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ \boldsymbol{\omega}_{fg} (\boldsymbol{\partial}_j) = f|_P \boldsymbol{\omega}_g (\boldsymbol{\partial}_j) + \boldsymbol{\omega}_f (\boldsymbol{\partial}_j) g|_P \\ \boldsymbol{\omega}_{x^i} (\boldsymbol{\partial}_j) = \boldsymbol{\omega}_{x^i} \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{cases} \right\} \\ =& \{\mathrm{d}f | \mathrm{d}f : V \to F\} = \left\{ \mathrm{d}f \middle| \begin{cases} \mathrm{d}f (\boldsymbol{e}_j) = \mathrm{d}f (\boldsymbol{\partial}_j) = \mathrm{d}f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ \mathrm{d}fg (\boldsymbol{\partial}_j) = f|_P (\mathrm{d}g) + (\mathrm{d}f) g|_P \\ \mathrm{d}x^i (\boldsymbol{\partial}_j) = \mathrm{d}x^i \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{cases} \right\} \end{aligned}$$

$$\mathrm{d}x^i \left( \frac{\partial}{\partial x^j}|_P \right) = \delta_{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}_j \Rightarrow \begin{cases} \boldsymbol{e}^i = \mathrm{d}x^i \\ \boldsymbol{e}_j = \frac{\partial}{\partial x^j}|_P \end{cases}$$

$$\begin{aligned} V^* =& \{\mathrm{d}f | \mathrm{d}f : V \to F\} = \left\{ \mathrm{d}f \middle| \begin{cases} \mathrm{d}f (\boldsymbol{e}_j) = \mathrm{d}f (\boldsymbol{\partial}_j) = \mathrm{d}f \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial f}{\partial x^j}|_P \in F \\ \mathrm{d}fg (\boldsymbol{\partial}_j) = f|_P (\mathrm{d}g) + (\mathrm{d}f) g|_P \\ \mathrm{d}x^i (\boldsymbol{\partial}_j) = \mathrm{d}x^i \left( \frac{\partial}{\partial x^j}|_P \right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{cases} \right\} \\ =& \mathrm{span} \{\mathrm{d}x^1, \cdots, \mathrm{d}x^i, \cdots, \mathrm{d}x^n\} = \mathrm{span} \{\boldsymbol{e}^1, \cdots, \boldsymbol{e}^j, \cdots, \boldsymbol{e}^n\} \end{aligned}$$

## 39.3 directional derivative

$$\begin{aligned} \mathrm{d}f (\boldsymbol{v}) =& \mathrm{d}f (v_j \boldsymbol{e}_j) = v_j \mathrm{d}f (\boldsymbol{e}_j) \\ =& v_j \mathrm{d}f (\boldsymbol{\partial}_j) = v_j \frac{\partial f}{\partial x^j}|_P \\ =& v_1 \frac{\partial f}{\partial x^1}|_P + \cdots + v_j \frac{\partial f}{\partial x^j}|_P + \cdots + v_n \frac{\partial f}{\partial x^n}|_P \\ =& \begin{pmatrix} v_1 & \cdots & v_j & \cdots & v_n \end{pmatrix} \boldsymbol{\nabla} f \end{aligned}$$

---

$$\overset{\frown}{PQ} = C(t) - C(0) = Q - P$$

$$\boldsymbol{v} = \frac{\partial}{\partial t}|_P$$

$$\begin{aligned} \mathrm{d}f (s\boldsymbol{v}) =& \mathrm{d}f \left( s \frac{\partial}{\partial t}|_P \right) = s \frac{\partial f}{\partial t}|_P \\ =& s\boldsymbol{v} (f) = s \cdot \lim_{t \to 0} \frac{f(C(t)) - f(C(0))}{t} \\ \approx& s \cdot \frac{f(Q) - f(P)}{s} = f(Q) - f(P) = \Delta f \end{aligned}$$

## 39.4 coefficient of linear combination for vector space and dual space

$$
\begin{aligned}
V =& \{\boldsymbol{v} | \boldsymbol{v} : f \to F\} \\
=& \text{span}\left\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_j, \cdots, \boldsymbol{e}_n\right\} \\
=& \text{span}\left\{\frac{\partial}{\partial x^1}|_P, \cdots, \frac{\partial}{\partial x^j}|_P, \cdots, \frac{\partial}{\partial x^n}|_P\right\} = \text{span}\left\{\boldsymbol{\partial}_1, \cdots, \boldsymbol{\partial}_j, \cdots, \boldsymbol{\partial}_n\right\} \\
=& \left\{\boldsymbol{\partial}_t \middle| \boldsymbol{\partial}_t = a_j \boldsymbol{e}_j = a_j \boldsymbol{\partial}_j = a_j \frac{\partial}{\partial x^j}|_P\right\} \\
=& \left\{\frac{\partial}{\partial t}|_P \middle| \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \cdots + a_j \frac{\partial}{\partial x^j}|_P + \cdots + a_n \frac{\partial}{\partial x^n}|_P\right\} \\
V^* =& \{\mathrm{d}f | \mathrm{d}f : V \to F\} \\
=& \text{span}\left\{\boldsymbol{e}^1, \cdots, \boldsymbol{e}^i, \cdots, \boldsymbol{e}^n\right\} \\
=& \text{span}\left\{\mathrm{d}x^1, \cdots, \mathrm{d}x^i, \cdots, \mathrm{d}x^n\right\} \\
=& \left\{\mathrm{d}f | \mathrm{d}f = b^i \boldsymbol{e}^i = b^i \mathrm{d}x^i\right\} \\
=& \left\{\mathrm{d}f | \mathrm{d}f = b^1 \mathrm{d}x^1 + \cdots + b^i \mathrm{d}x^i + \cdots + b^n \mathrm{d}x^n\right\}
\end{aligned}
$$

---

or more simply to be comparison

$$
\begin{array}{lllllllll}
V & = \text{span}\{ & \boldsymbol{e}_j = & \partial_j & \} = \{ & \boldsymbol{v} = & \partial_t|_P & = a_j\, \boldsymbol{e}_j = & a_j\, \partial_j|_P & : f \to F & \} \\
V^* & = \text{span}\{ & \boldsymbol{e}^i = & \mathrm{d}x^i & \} = \{ & \boldsymbol{\omega} = & \mathrm{d}f & = b^i\, \boldsymbol{e}^i = & b^i\, \mathrm{d}x^i & : V \to F & \}
\end{array}
$$

---

$$
\begin{aligned}
& \begin{cases}
\mathrm{d}x^i\left(\boldsymbol{\partial}_j\right) = \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\
\boldsymbol{\partial}_t = a_j \boldsymbol{e}_j = a_j \boldsymbol{\partial}_j \Leftrightarrow \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \cdots + a_j \frac{\partial}{\partial x^j}|_P + \cdots + a_n \frac{\partial}{\partial x^n}|_P
\end{cases} \\
\Rightarrow & \begin{cases}
\mathrm{d}x^i\left(\boldsymbol{\partial}_t\right) = \mathrm{d}x^i\left(\frac{\partial}{\partial t}|_P\right) = \frac{\partial x^i}{\partial t}|_P \\
\mathrm{d}x^i\left(\boldsymbol{\partial}_t\right) = \mathrm{d}x^i\left(a_j \boldsymbol{\partial}_j\right) = a_j \mathrm{d}x^i\left(\boldsymbol{\partial}_j\right) = a_j \delta_{ij} = a_i
\end{cases} \Rightarrow a_i = \mathrm{d}x^i\left(\boldsymbol{\partial}_t\right) = \frac{\partial x^i}{\partial t}|_P \\
\Rightarrow & a_i = \frac{\partial x^i}{\partial t}|_P \Rightarrow a_j = \frac{\partial x^j}{\partial t}|_P = \partial_t x^j|_P \\
\Rightarrow & \frac{\partial}{\partial t}|_P = a_i \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t}|_P \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}|_P \Rightarrow \frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \\
\Rightarrow & \partial_t|_P = \frac{\partial x^j}{\partial t} \partial_j|_P \Leftrightarrow \partial_t|_P = \partial_t x^j \partial_j|_P
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}f =& b^i \boldsymbol{e}^i = b^i \mathrm{d}x^i \\
\frac{\partial f}{\partial x^j} = \mathrm{d}f\left(\boldsymbol{\partial}_j\right) = \mathrm{d}f\left(\boldsymbol{e}_j\right) =& b^i \boldsymbol{e}^i \cdot \boldsymbol{e}_j = b^i \delta_{ij} = b^j \\
b^j =& \frac{\partial f}{\partial x^j} \\
b^i =& \frac{\partial f}{\partial x^i} = \partial_i f \\
\mathrm{d}f = b^i \boldsymbol{e}^i = b^i \mathrm{d}x^i =& \frac{\partial f}{\partial x^i} \mathrm{d}x^i \\
\mathrm{d}f =& \frac{\partial f}{\partial x^i} \mathrm{d}x^i \\
\mathrm{d}f =& \partial_i f \mathrm{d}x^i
\end{aligned}
$$

---

$$
\begin{array}{lllllllll}
V & = \text{span}\{ & \boldsymbol{e}_j = & \partial_j & \} = \{ & \boldsymbol{v} = & \partial_t|_P & = a_j\, \boldsymbol{e}_j = & \partial_t x^j\, \partial_j|_P & : f \to F & \} \\
V^* & = \text{span}\{ & \boldsymbol{e}^i = & \mathrm{d}x^i & \} = \{ & \boldsymbol{\omega} = & \mathrm{d}f & = b^i\, \boldsymbol{e}^i = & \partial_i f\, \mathrm{d}x^i & : V \to F & \}
\end{array}
$$

## 39.5 change of basis / change of coordinate

$$
\frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \overset{t = x'^j}{\Rightarrow} \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} = \frac{\partial x^1}{\partial x'^j} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^j} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^j} \frac{\partial}{\partial x^3}
$$

$$\begin{aligned} \mathrm{d}f =& \frac{\partial f}{\partial x^i}\mathrm{d}x^i \\ f = x'^j \Downarrow& \\ \mathrm{d}x'^j =& \frac{\partial x'^j}{\partial x^i}\mathrm{d}x^i \end{aligned}$$

---

$$\begin{cases} \dfrac{\partial}{\partial x'^j} = \dfrac{\partial x^i}{\partial x'^j}\dfrac{\partial}{\partial x^i} = \sum_i \dfrac{\partial x^i}{\partial x'^j}\dfrac{\partial}{\partial x^i} \\ \mathrm{d}x'^j = \dfrac{\partial x'^j}{\partial x^i}\mathrm{d}x^i = \sum_i \dfrac{\partial x'^j}{\partial x^i}\mathrm{d}x^i \end{cases}$$

## 39.6 ambiguity with partial notation

https://www.youtube.com/watch?v=mICbKwwHziI

## 39.7 1-form

1-form = one-form

## 39.8 What is Math: differential geometry

https://www.youtube.com/playlist?list=PLXo8Tdaw0czOWyRD-esa6mNajlPZnjHQs

## 39.9 Liang, Can-bin: differential geometry and general relativity

https://www.bilibili.com/video/BV1o4411L72E

https://www.youtube.com/playlist?list=PLstdOGDXMaWIKCWheiNIRumejII0gItYM

https://www.youtube.com/playlist?list=PLstdOGDXMaWICAkLFdCX24pwcWww5YzyQ

# Chapter 40

# determinant

https://www.bilibili.com/video/BV13e411m7Js

https://www.youtube.com/watch?v=Sv7VseMsOQc

## 40.1 induction of determinant axioms

## 40.2 determinant axioms

determinant axioms

$$
\begin{aligned}
&\begin{cases}
\begin{cases}
\det(\boldsymbol{u},\boldsymbol{v}) = \det(\boldsymbol{u}+s\boldsymbol{v},\boldsymbol{v}) \\
\det(\boldsymbol{u},\boldsymbol{v}) = \det(\boldsymbol{u},\boldsymbol{v}+s\boldsymbol{u})
\end{cases} & \text{translation invariance} \\
\begin{cases}
\det(s\boldsymbol{u},\boldsymbol{v}) = s\det(\boldsymbol{u},\boldsymbol{v}) \\
\det(\boldsymbol{u},s\boldsymbol{v}) = s\det(\boldsymbol{u},\boldsymbol{v})
\end{cases} & \text{scaling} \\
\begin{cases}
\det(\boldsymbol{u}_1+\boldsymbol{u}_2,\boldsymbol{v}) = \det(\boldsymbol{u}_1,\boldsymbol{v}) + \det(\boldsymbol{u}_2,\boldsymbol{v}) \\
\det(\boldsymbol{u},\boldsymbol{v}_1+\boldsymbol{v}_2) = \det(\boldsymbol{u},\boldsymbol{v}_1) + \det(\boldsymbol{u},\boldsymbol{v}_2)
\end{cases} & \text{decomposition}
\end{cases} \\
\Leftrightarrow &\begin{cases}
\begin{cases}
\det(\boldsymbol{u},\boldsymbol{v}) = \det(\boldsymbol{u}+s\boldsymbol{v},\boldsymbol{v}) \\
\det(\boldsymbol{u},\boldsymbol{v}) = \det(\boldsymbol{u},\boldsymbol{v}+s\boldsymbol{u})
\end{cases} & \text{translation invariance} \\
\begin{cases}
\det(\boldsymbol{u}_1+s\boldsymbol{u}_2,\boldsymbol{v}) = \det(\boldsymbol{u}_1,\boldsymbol{v}) + s\det(\boldsymbol{u}_2,\boldsymbol{v}) \\
\det(\boldsymbol{u},\boldsymbol{v}_1+s\boldsymbol{v}_2) = \det(\boldsymbol{u},\boldsymbol{v}_1) + s\det(\boldsymbol{u},\boldsymbol{v}_2)
\end{cases} & \text{linearity}
\end{cases}
\end{aligned}
$$

## 40.3 determinant theorems or properties

## 40.4 ellipse area

## 40.5 Cramer rule geometry perspective

# Chapter 41

# hypergeometric function

## 41.1 linear space of function

https://www.bilibili.com/video/BV1PX4y167RS

quantum state[43.2.3]

Taylor vs. Fourier[@ref(taylor-vs.-fourier)]

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

$$f(x) = x_0 x^0 + x_1 x^1 + x_2 x^2 + \cdots = \sum_{k=0}^{\infty} x_k x^k$$

Def: 43.3

$$\langle f|g\rangle = \int_a^b \overline{f(x)} g(x)\,\mathrm{d}x \overset{f,g:\mathbb{R}\to\mathbb{R}}{=} \int_a^b f(x)\,g(x)\,\mathrm{d}x$$

Dirac bracket[43.5]

$$\left\langle x^2 \middle| x\right\rangle \overset{x^2,x:\mathbb{R}\to\mathbb{R}}{=} \int_a^b x^2 x \mathrm{d}x = \int_a^b x^3 \mathrm{d}x = \left[\frac{x^4}{4}\right]_a^b \not\equiv 0$$

$$x^0 \not\perp x^1, x^1 \not\perp x^2, \cdots$$

$$\langle x_m | x^n \rangle = \int_a^b x_m x^n \mathrm{d}x = \delta_{mn}$$

$$\langle 1 | x^n \rangle = \int_a^b x_0 x^n \mathrm{d}x = \delta_{0n} \Rightarrow x_0 = \delta(x) = \delta(x-0)$$

$$\langle x_m | x^n \rangle = \int_a^b x_m x^n \mathrm{d}x = \delta_{mn} \Rightarrow x_m = \frac{(-1)^m}{m!} \delta^{(m)}(x)$$

$$|f\rangle = 1\,|f\rangle = \left(\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right|\right) |f\rangle = \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i \middle| f\right\rangle$$

$$\begin{aligned}
\left|f\right\rangle =&1\left|f\right\rangle = \left(\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right|\right)\left|f\right\rangle = \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\middle|f\right\rangle \\
=&1\left|f\right\rangle = \left(\sum_n \left|x^n\right\rangle \left\langle x^n\right|\right)\left|f\right\rangle = \sum_n \left|x^n\right\rangle \left\langle x^n|f\right\rangle = \sum_n \left\langle x^n|f\right\rangle \left|x^n\right\rangle \\
\left\langle x^n\right|\left|f\right\rangle =& \left\langle x^n|f\right\rangle = \int_a^b x_n f\left(x\right)\mathrm{d}x = \int_a^b \frac{\left(-1\right)^n}{n!}\delta^{(n)}\left(x\right)f\left(x\right)\mathrm{d}x = \frac{f^{(n)}\left(0\right)}{n!} \\
\left|f\right\rangle =& \sum_n \left\langle x^n|f\right\rangle \left|x^n\right\rangle = \sum_n \frac{f^{(n)}\left(0\right)}{n!}\left|x^n\right\rangle \\
\left|f\right\rangle =& \sum_n \frac{f^{(n)}\left(0\right)}{n!}\left|x^n\right\rangle \\
&\Downarrow \\
f\left(x\right) =& \sum_n \frac{f^{(n)}\left(0\right)}{n!}x^n
\end{aligned}$$

## 41.2 beta function

https://www.bilibili.com/video/BV1pa4y1P7Da

$$\begin{aligned}
\binom{n}{k} = \mathrm{C}_k^n =& \frac{n!}{(n-k)!k!} \\
=& \frac{n\left(n-1\right)\cdots\left(n-k+1\right)}{k!}, \begin{cases} n\in\mathbb{N} \\ k\in\left(\{0\}\cup\mathbb{N}\right)\end{cases} \\
\binom{r}{k} =& \begin{cases} \dfrac{r\left(r-1\right)\cdots\left(r-k+1\right)}{k!} & k\geq 0, k\in\mathbb{Z} \\ 0 & k<0, k\in\mathbb{Z}\end{cases}
\end{aligned}$$

$$\sum_{k=0}^{n}\binom{r}{k}\left(\cdot\right)$$

$$\sum_{k=-\infty}^{n}\binom{r}{k}\left(\cdot\right) = \left(0+0+\cdots\right) + \sum_{k=0}^{n}\binom{r}{k}\left(\cdot\right)$$

$$\sum_{k=-\infty}^{\infty}\binom{r}{k}\left(\cdot\right)$$

---

$$n! = \Gamma\left(n+1\right) = \int_0^\infty x^{(n+1)-1}\mathrm{e}^{-x}\mathrm{d}x$$

$$\Gamma\left(z\right) = \int_0^\infty x^{z-1}\mathrm{e}^{-x}\mathrm{d}x$$

$$\Gamma\left(z+1\right) = z\Gamma\left(z\right)$$

$$\Gamma\left(z\right)\Gamma\left(1-z\right) = \frac{\pi}{\sin\left(\pi z\right)}$$

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) =& \frac{\pi}{\sin(\pi z)} \\
[\Gamma(z)\Gamma(1-z)]_{z=-n} =& \left[\frac{\pi}{\sin(\pi z)}\right]_{z=-n}, n\in\mathbb{N} \\
\Gamma(-n)\,n! = \Gamma(n+1) = \Gamma(-n)\Gamma(1-(-n)) =& \frac{\pi}{\sin(\pi(-n))} = \frac{\pi}{-\sin(n\pi)} \\
\Gamma(-n) =& \frac{-\pi}{n!\sin(n\pi)} = \frac{-\pi}{n!0} \to -\infty, n\in\mathbb{N}
\end{aligned}$$

$$\begin{aligned}
\binom{n}{k} = \mathrm{C}_k^n =& \frac{n!}{(n-k)!k!} \\
=& \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}
\end{aligned}$$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}$$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \overset{k\leq 0}{\equiv} \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(1)} = \dfrac{\Gamma(n+1)}{\Gamma(n+1)1} = 1 & k=0 \\ \dfrac{\Gamma(n+1)}{\Gamma(n-k+1)(-\infty)} = 0 & k\leq -1, k\in\mathbb{Z} \end{cases}$$

beta function = $\beta$ function

**Definition 41.1.** beta function = $\beta$ function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\begin{aligned}
\binom{n}{k} =& \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \\
\left[\binom{n}{k}\right]_{\begin{cases} n=a+b \\ k=a \end{cases}} =& \left[\frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}\right]_{\begin{cases} n=a+b \\ k=a \end{cases}} \\
\binom{a+b}{a} =& \frac{\Gamma(a+b+1)}{\Gamma(a+b-a+1)\Gamma(a+1)} \\
=& \frac{\Gamma(a+b+1)}{\Gamma(b+1)\Gamma(a+1)}
\end{aligned}$$

$$\begin{aligned}
B(p,q) =& \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\
[B(p,q)]_{\begin{cases} p=a+1 \\ q=b+1 \end{cases}} =& \left[\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\right]_{\begin{cases} p=a+1 \\ q=b+1 \end{cases}} \\
B(a+1,b+1) =& \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+1+b+1)} \\
=& \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma([a+b+1]+1)} = \frac{\Gamma(a+1)\Gamma(b+1)}{[a+b+1]\Gamma(a+b+1)}
\end{aligned}$$

$$\begin{aligned}
\binom{a+b}{a} =& \frac{\Gamma(a+b+1)}{\Gamma(b+1)\Gamma(a+1)} = \frac{1}{\dfrac{\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+1)}} \\
=& \frac{1}{[a+b+1]\dfrac{\Gamma(b+1)\Gamma(a+1)}{[a+b+1]\Gamma(a+b+1)}} \\
=& \frac{1}{[a+b+1]B(a+1,b+1)}
\end{aligned}$$

https://en.wikipedia.org/wiki/Beta_function

https://en.wikipedia.org/wiki/Beta_function#Other_identities_and_formulas

https://en.wikipedia.org/wiki/Beta_function#Multivariate_beta_function

https://www.bilibili.com/video/BV1pa4y1P7Da/?t=4m

### 41.2.1 Wallis product formula

https://en.wikipedia.org/wiki/Wallis_product

https://www.bilibili.com/video/BV1pa4y1P7Da/?t=4m10s

https://www.math.sinica.edu.tw/mathmedia/journals/4739?keywords%5B%5D=Paul+Dirac

## 41.3 gamma function

https://www.bilibili.com/video/BV1cT411H7Hp

## 41.4 recursion

https://www.bilibili.com/video/BV1FV4y1Z7jm

https://www.bilibili.com/video/BV1Sg4y1L7DF

## 41.5 mean and variance of discrete probability distributions

https://www.bilibili.com/video/BV1Tk4y1n7NX

## 41.6 CORDIC = coordinate rotation digital computer

https://www.bilibili.com/video/BV1ge411e7K7

https://space.bilibili.com/11008987/channel/collectiondetail?sid=2053177

https://www.bilibili.com/video/BV1AW4y1A7HN

https://www.bilibili.com/video/BV115411i7VS

# Chapter 42

# Feynman method

## 42.1 Feynman method of differentiation / derivative technique

https://www.bilibili.com/video/BV1hG411Z7Cb

分式微分不是難而是煩

### 42.1.1 principle

**Theorem 42.1.** *Feynman method of differentiation / derivative*

$$f(t) = k[u(t)]^a [v(t)]^b [w(t)]^c \cdots$$
$$\Downarrow$$
$$f'(t) = f(t)\left[a\frac{u'(t)}{u(t)} + b\frac{v'(t)}{v(t)} + c\frac{w'(t)}{w(t)} + \cdots\right]$$

Proof:

$$f(t) = k[u(t)]^a [v(t)]^b [w(t)]^c \cdots$$

$$f = ku^a v^b w^c \cdots = k \cdot u^a \cdot v^b \cdot w^c \cdot \cdots$$

$$\begin{aligned}
f =& ku^a v^b w^c \cdots = k \cdot u^a \cdot v^b \cdot w^c \cdot \cdots \\
\ln f =& \ln\left(ku^a v^b w^c \cdots\right) = \ln k + \ln u^a + \ln v^b + \ln w^c + \cdots \\
=& \ln k + a\ln u + b\ln v + c\ln w + \cdots \\
\frac{\mathrm{d}}{\mathrm{d}t}\ln f =& \frac{\mathrm{d}}{\mathrm{d}t}(\ln k + a\ln u + b\ln v + c\ln w + \cdots) \\
\frac{\frac{\mathrm{d}}{\mathrm{d}t}f}{f} =& 0 + \frac{\mathrm{d}}{\mathrm{d}t}(a\ln u) + \frac{\mathrm{d}}{\mathrm{d}t}(b\ln v) + \frac{\mathrm{d}}{\mathrm{d}t}(c\ln w) + \cdots \\
=& a\frac{\mathrm{d}}{\mathrm{d}t}\ln u + b\frac{\mathrm{d}}{\mathrm{d}t}\ln v + c\frac{\mathrm{d}}{\mathrm{d}t}\ln w + \cdots \\
=& a\frac{\frac{\mathrm{d}}{\mathrm{d}t}u}{u} + b\frac{\frac{\mathrm{d}}{\mathrm{d}t}v}{v} + c\frac{\frac{\mathrm{d}}{\mathrm{d}t}w}{w} + \cdots \\
\frac{f'}{f} =& a\frac{u'}{u} + b\frac{v'}{v} + c\frac{w'}{w} + \cdots \\
f' =& f\left(a\frac{u'}{u} + b\frac{v'}{v} + c\frac{w'}{w} + \cdots\right) \\
f'(t) =& f(t)\left[a\frac{u'(t)}{u(t)} + b\frac{v'(t)}{v(t)} + c\frac{w'(t)}{w(t)} + \cdots\right]
\end{aligned}$$

□

### 42.1.2 examples

$f(x) = x^x$

$$(x^x)' = x^x + x^x \ln x$$

$$\begin{aligned} f(x) =& x^x \\ \ln f(x) =& x \ln x \\ \frac{\mathrm{d}}{\mathrm{d}x} \ln f(x) =& \frac{\mathrm{d}}{\mathrm{d}x} [x \ln x] \\ \frac{f'(x)}{f(x)} =& [x \ln x]' \\ f'(x) =& f(x) [x \ln x]' = x^x \left[(x)' \ln x + x (\ln x)'\right] \\ =& x^x \left[1 \ln x + x \frac{1}{x}\right] = x^x [\ln x + 1] = x^x [1 + \ln x] \\ =& x^x + x^x \ln x \end{aligned}$$

□

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3D delta function[43.2.2]

$$\Delta\left(\frac{1}{r}\right) = \boldsymbol{\nabla}^2 \left(\frac{1}{\sqrt{\boldsymbol{r}^2}}\right) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left(\frac{1}{\sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}}\right) = \boldsymbol{\nabla} \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\begin{aligned} & \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{\partial}{\partial x} \left[\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}}\right] \\ =& \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} \left[\frac{-1}{2} \frac{2x}{x^2 + y^2 + z^2}\right] \\ =& \frac{-x}{(x^2 + y^2 + z^2)^{\frac{-3}{2}}} \end{aligned}$$

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = \boldsymbol{\nabla}\left(\frac{1}{\sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = \left(\frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\right)$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\left(\frac{1}{r}\right) =& \boldsymbol{\nabla} \cdot \left(\frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\right) \\ =& \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \end{aligned}$$

$$\frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}}$$

$$\begin{aligned} & \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ =& \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \left[1 \cdot \frac{1}{x} + \frac{-3}{2} \frac{2x}{x^2 + y^2 + z^2}\right] \\ =& \frac{-1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ =& \frac{-\left(x^2 + y^2 + z^2\right) + 3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \end{aligned}$$

$$\nabla \cdot \nabla \left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\frac{-x}{\left(x^2+y^2+z^2+\epsilon^2\right)^{\frac{3}{2}}} + \frac{\partial}{\partial y}\frac{-y}{\left(x^2+y^2+z^2+\epsilon^2\right)^{\frac{3}{2}}} + \frac{\partial}{\partial z}\frac{-z}{\left(x^2+y^2+z^2+\epsilon^2\right)^{\frac{3}{2}}}$$

$$= \frac{+2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} + \frac{-x^2+2y^2-z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} + \frac{-x^2-y^2+2z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} = 0$$

$$\Delta\left(\frac{1}{r}\right) = \nabla \cdot \nabla \left(\frac{1}{r}\right) = 0$$

□

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$$\frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} + \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} + \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}}\right]$$

---

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}}\right]$$

$$= \frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} \cdot [$$

$$= \frac{6\left[1+2t^2\right]\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} \cdot [1 \cdot \frac{1}{1+2t^2} \cdot 4t, \left[1+2t^2\right] \to \begin{cases} \text{exponential :} & 1 \\ \text{linear to denominator :} & \dfrac{1}{1+2t^2} \\ \text{differentiation} & 4t \end{cases}$$

$$= \frac{6\left(1+2t^2\right)\left[\left(t^3-t\right)^2\right]}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} \cdot [\frac{4t}{1+2t^2} + 2 \cdot \frac{1}{t^3-t} \cdot \left(3t^2-1\right)$$

$$, \left[\left(t^3-t\right)^2\right] \to \begin{cases} \text{exponential :} & 2 \\ \text{linear to denominator :} & \dfrac{1}{t^3-t} \\ \text{differentiation} & 3t^2-1 \end{cases}$$

$$= \frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\left[\sqrt{t+5t^2}\right](4t)^{\frac{3}{2}}} \cdot [\frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} + \frac{-1}{2} \cdot \frac{1}{t+5t^2} \cdot (1+10t)$$

$$, \left[\sqrt{t+5t^2}\right] \to \begin{cases} \text{exponential :} & \dfrac{-1}{2} \\ \text{linear to denominator :} & \dfrac{1}{t+5t^2} \\ \text{differentiation} & 1+10t \end{cases}$$

$$= \frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\left[(4t)^{\frac{3}{2}}\right]} \cdot \left[\frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} - \frac{1+10t}{2t+10t^2} + \frac{-3}{2} \cdot \frac{1}{4t} \cdot 4\right]$$

$$, \left[(4t)^{\frac{3}{2}}\right] \to \begin{cases} \text{exponential :} & \dfrac{-3}{2} \\ \text{linear to denominator :} & \dfrac{1}{4t} \\ \text{differentiation} & 4 \end{cases}$$

$$= \frac{6\left(1+2t^2\right)\left(t^3-t\right)^2}{\sqrt{t+5t^2}\,(4t)^{\frac{3}{2}}} \left[\frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} - \frac{1+10t}{2t+10t^2} - \frac{3}{2t}\right]$$

---

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}}\right]$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}}\right]\\=&\frac{\sqrt{1+2t}}{t+\left[\sqrt{1+t^2}\right]}\left[\frac{1}{2}\frac{2}{1+2t}+(-1)\frac{1+\left[\frac{1}{2}\frac{2t}{\sqrt{1+t^2}}\right]}{t+\sqrt{1+t^2}}\right]\end{aligned}$$

□

## 42.2 Feynman method of integration / integral technique

### 42.2.1 principle

https://www.bilibili.com/video/BV1Lj411L79X/?t=2m38s

https://www.youtube.com/watch?v=GW86SShcYbM

$$I=\int f(x)\,\mathrm{d}x$$

$$I(t)=\int f(x,t)\,\mathrm{d}x$$

$$\begin{aligned}I&=\int f(x)\,\mathrm{d}x\\I(t)&=\int f(x,t)\,\mathrm{d}x\\\frac{\mathrm{d}}{\mathrm{d}t}I(t)=I'(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\int f(x,t)\,\mathrm{d}x\\&\overset{\text{Lebniz integral rule}}{=}f(x,b(x),t)\frac{\mathrm{d}b(x)}{\mathrm{d}x}-f(x,a(x),t)\frac{\mathrm{d}a(x)}{\mathrm{d}x}+\int_{a(x)}^{b(x)}\frac{\partial}{\partial t}f(x,t)\,\mathrm{d}x\\&\overset{\cdots}{=}\int\frac{\partial}{\partial t}f(x,t)\,\mathrm{d}x=\int\frac{\partial f(x,t)}{\partial t}\mathrm{d}x=\int f_t(x,t)\,\mathrm{d}x\\I(t)&=\int I'(t)\,\mathrm{d}t=\int I'(t)\,\mathrm{d}t+C=\left[\int I'(t)\,\mathrm{d}t\right](t)\\&\Downarrow\text{ if }f(x,t=0)=f(x,0)=f(x)\\I&\overset{f(x,0)=f(x)}{=}I(0)=\left[\int I'(t)\,\mathrm{d}t\right](0)\end{aligned}$$

### 42.2.2 Dirichlet integral

as a example by Feynman method of integration / integral technique

https://en.wikipedia.org/wiki/Dirichlet_integral

https://www.bilibili.com/video/BV1Lj411L79X/?t=4m38s

$\int_0^\infty\frac{\sin x}{x}\mathrm{d}x$

https://www.youtube.com/watch?v=ZZccxuOpb4k

https://blog.csdn.net/zhuoqingjoking97298/article/details/127950915

$\int_{-\infty}^\infty\frac{\sin x}{x}\mathrm{d}x$

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\mathrm{d}x$$

$$f(x) = \frac{\sin x}{x} \text{ is an even function} \Leftrightarrow f(-x) = \frac{\sin(-x)}{(-x)} = \frac{-\sin(x)}{-x} = \frac{\sin x}{x} = f(x)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x &= \int_{-\infty}^{0} \frac{\sin x}{x} \mathrm{d}x + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{x=-\infty}^{x=0} \frac{\sin x}{x} \mathrm{d}x + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{x=-\infty}^{x=0} \frac{\sin x}{x} \mathrm{d}x \,(, x' = -x \Leftrightarrow x = -x') + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{(-x')=-\infty}^{(-x')=0} \frac{\sin(-x')}{(-x')} \mathrm{d}(-x') + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{x'=-(-\infty)}^{x'=-0} \frac{-\sin x'}{-x'} (-\mathrm{d}x') + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{x'=\infty}^{x'=0} \frac{\sin x'}{x'} (-\mathrm{d}x') + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= -\int_{x'=\infty}^{x'=0} \frac{\sin x'}{x'} \mathrm{d}x' + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{x'=0}^{x'=\infty} \frac{\sin x'}{x'} \mathrm{d}x' + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x \\
&= \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x + \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x = 2\int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x
\end{aligned}$$

---

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x = 2\int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x$$

---

$\int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x$

$$
\begin{aligned}
I = &\int_0^\infty \frac{\sin x}{x} \mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x, f(x) = \frac{\sin x}{x} \\
\int_0^\infty &\mathrm{e}^{-tx}\frac{\sin x}{x} \mathrm{d}x = \int_0^\infty \mathrm{e}^{-tx} f(x)\,\mathrm{d}x,\ \text{Laplacian transform of } f(x) \\
&= \int_0^\infty f(x,t)\,\mathrm{d}x, \begin{cases} f(x,t) = \mathrm{e}^{-tx} f(x) \\ f(x,0) = \mathrm{e}^{-0x} f(x) = 1 \cdot f(x) = f(x) \quad \Downarrow \end{cases} \\
I(t) &= \int_0^\infty f(x,t)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-tx} f(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-tx}\frac{\sin x}{x}\mathrm{d}x \Rightarrow I(0) = I \\
\frac{\mathrm{d}}{\mathrm{d}t} I(t) = I'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty f(x,t)\,\mathrm{d}x = \int_0^\infty \frac{\partial f(x,t)}{\partial t}\mathrm{d}x = \int_0^\infty \frac{\partial \mathrm{e}^{-tx}\frac{\sin x}{x}}{\partial t}\mathrm{d}x \\
&= \int_0^\infty \frac{\sin x}{x}\left[\frac{\partial \mathrm{e}^{-tx}}{\partial t}\right]\mathrm{d}x = \int_0^\infty \frac{\sin x}{x}\left[-x\mathrm{e}^{-tx}\right]\mathrm{d}x = \int_0^\infty \sin x\left[-\mathrm{e}^{-tx}\right]\mathrm{d}x \\
&= -\int_0^\infty \mathrm{e}^{-tx}\sin x\mathrm{d}x,\ \text{Laplacian transform of } -\sin x \\
&= \int_0^\infty \mathrm{e}^{-tx}(-\sin x)\,\mathrm{d}x = \int_{x=0}^{x=\infty} \mathrm{e}^{-tx}\mathrm{d}\cos x, I'(t) = \int_0^\infty \mathrm{e}^{-tx}(-\sin x)\,\mathrm{d}x \\
&= \left[\mathrm{e}^{-tx}\cos x\right]_{x=0}^{x=\infty} - \int_{x=0}^{x=\infty}\cos x\mathrm{d}\mathrm{e}^{-tx} \\
&= \left[\mathrm{e}^{-t\infty}\cos\infty\right] - \left[\mathrm{e}^{-t0}\cos 0\right] - \int_{x=0}^{x=\infty}\cos x\left(-t\mathrm{e}^{-tx}\mathrm{d}x\right) \\
&= \left[0\cos\infty\right] - \left[1\cdot 1\right] + \int_0^\infty t\mathrm{e}^{-tx}\cos x\mathrm{d}x \\
&= 0 - 1 + \int_0^\infty t\mathrm{e}^{-tx}\cos x\mathrm{d}x = -1 + \int_{x=0}^{x=\infty} t\mathrm{e}^{-tx}\mathrm{d}\sin x \\
&= -1 + \left[t\mathrm{e}^{-tx}\sin x\right]_{x=0}^{x=\infty} - \int_{x=0}^{x=\infty}\sin x\mathrm{d}\left(t\mathrm{e}^{-tx}\right) \\
&= -1 + [0-0] - \int_0^\infty \sin x\left(t\mathrm{e}^{-tx}(-t)\,\mathrm{d}x\right) = -1 - \int_0^\infty t^2\mathrm{e}^{-tx}(-\sin x)\,\mathrm{d}x \\
&= -1 - t^2\int_0^\infty \mathrm{e}^{-tx}(-\sin x)\,\mathrm{d}x = -1 - t^2 I'(t) \\
I'(t) &= -1 - t^2 I'(t) \\
I'(t) &= \frac{-1}{1+t^2} \\
I(t) &= \int \frac{-1}{1+t^2}\mathrm{d}t \overset{t=\tan\theta}{=} \int \frac{-1}{1+\tan^2\theta}\mathrm{d}\tan\theta = -\int \frac{1}{\sec^2\theta}\sec^2\theta\mathrm{d}\theta = -\int \mathrm{d}\theta = -\theta + C \\
&= -\arctan t + C = -\tan^{-1} t + C \\
&\int_0^\infty \mathrm{e}^{-tx}\frac{\sin x}{x}\mathrm{d}x = I(t) = -\arctan t + C \\
&\Downarrow \Rightarrow I = \int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \int_0^\infty \mathrm{e}^{-0x}\frac{\sin x}{x}\mathrm{d}x = I(0) = -\arctan 0 + C = -0 + C \Rightarrow C = I \\
&\Rightarrow 0 = \int_0^\infty 0\frac{\sin x}{x}\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\infty x}\frac{\sin x}{x}\mathrm{d}x = I(\infty) = -\arctan\infty + C = \frac{-\pi}{2} + C \\
&0 = \frac{-\pi}{2} + C \Rightarrow C = \frac{\pi}{2} \\
I &= I(0) = C = \frac{\pi}{2}, I = \int_0^\infty \frac{\sin x}{x}\mathrm{d}x \\
\int_0^\infty &\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}
\end{aligned}
$$

---

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$$

---

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x = 2 \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x = 2 \cdot \frac{\pi}{2} = \pi$$

---

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x = \pi$$

□

#### 42.2.2.1 residue method

https://www.bilibili.com/video/BV1Lj411L79X/?t=7m26s

#### 42.2.2.2 sinc function

https://en.wikipedia.org/wiki/Sinc_function

https://en.wikipedia.org/wiki/Anti-aliasing_filter

https://en.wikipedia.org/wiki/Borwein_integral

https://blog.udn.com/paraquat/22455342

`202404150929-Feynman-method_files/figure-latex/unnamed-chunk-2-1`

Figure 42.1: $\sin(x)$

https://tex.stackexchange.com/questions/235006/declaring-sinc-in-tikz

`202404150929-Feynman-method_files/figure-latex/unnamed-chunk-3-1`

Figure 42.2: $\mathrm{sinc}(x)$

### 42.2.3 Gaussian integral

https://www.youtube.com/watch?v=jP-6j6mEpRg

$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x$$

$$f(x) = \mathrm{e}^{-x^2} \text{ is an even function} \Leftrightarrow f(-x) = \mathrm{e}^{-(-x)^2} = \mathrm{e}^{-x^2} = f(x)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x &= \int_{-\infty}^{0} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{x=-\infty}^{x=0} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{x=-\infty}^{x=0} \mathrm{e}^{-x^2} \mathrm{d}x \left(, x' = -x \Leftrightarrow x = -x'\right) + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{(-x')=-\infty}^{(-x')=0} \mathrm{e}^{-(-x')^2} \mathrm{d}\left(-x'\right) + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{x'=-(-\infty)}^{x'=-0} \mathrm{e}^{-(x')^2} \left(-\mathrm{d}x'\right) + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{x'=\infty}^{x'=0} \mathrm{e}^{-(x')^2} \mathrm{d}x' + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= -\int_{x'=\infty}^{x'=0} \mathrm{e}^{-(x')^2} \mathrm{d}x' + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{x'=0}^{x'=\infty} \mathrm{e}^{-(x')^2} \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\
&= \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = 2\int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x
\end{aligned}$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = 2\int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x$$

$\int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x$

$$I(t) = \left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]^2$$

$$\begin{aligned}
I'(t) = \frac{\mathrm{d}I(t)}{\mathrm{d}t} =& \frac{\mathrm{d}\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]^2}{\mathrm{d}t} = \frac{\mathrm{d}\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]^2}{\mathrm{d}\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]} \frac{\mathrm{d}\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]}{\mathrm{d}t} = 2\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right] \\
=& 2\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right] \overset{\text{FToC}}{=} 2\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]\left[\mathrm{e}^{-t^2}\right] \\
=& 2\int_0^t \mathrm{e}^{-x^2}\mathrm{e}^{-t^2} \mathrm{d}x = 2\int_0^t \mathrm{e}^{-\left(x^2+t^2\right)} \mathrm{d}x = 2\int_{x=0}^{x=t} \mathrm{e}^{-\left(x^2+t^2\right)} \mathrm{d}x, x' = \frac{x}{t} \Leftrightarrow x = tx' \\
=& 2\int_{tx'=0}^{tx'=t} \mathrm{e}^{-\left((tx')^2+t^2\right)} \mathrm{d}tx' = 2\int_{x'=0}^{x'=1} \mathrm{e}^{-t^2\left((x')^2+1\right)} t\mathrm{d}x' = 2\int_0^1 t\mathrm{e}^{-t^2\left(x^2+1\right)} \mathrm{d}x \\
=& -\int_0^1 (-2t)\,\mathrm{e}^{-t^2\left(x^2+1\right)} \mathrm{d}x = -\int_0^1 \frac{\partial}{\partial t}\frac{\mathrm{e}^{-t^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{\mathrm{e}^{-t^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x \\
\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]^2 =& I(t) = -\int_0^1 \frac{\mathrm{e}^{-t^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x + C \\
0 = \left[\int_0^0 \mathrm{e}^{-x^2} \mathrm{d}x\right]^2 =& I(0) = \int_0^1 \frac{1}{x^2+1} \mathrm{d}x + C \overset{x=\tan\theta}{=} \left[-\arctan x\right]_0^1 + C = \frac{-\pi}{4} + C \Rightarrow C = \frac{\pi}{4} \\
\left[\int_0^t \mathrm{e}^{-x^2} \mathrm{d}x\right]^2 =& I(t) = -\int_0^1 \frac{\mathrm{e}^{-t^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x + C = -\int_0^1 \frac{\mathrm{e}^{-t^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x + \frac{\pi}{4} \\
\left[\int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x\right]^2 =& \lim_{t\to\infty} I(t) = I(\infty) = -\int_0^1 \frac{\mathrm{e}^{-\infty^2\left(x^2+1\right)}}{x^2+1} \mathrm{d}x + \frac{\pi}{4} = -\int_0^1 \frac{0}{x^2+1} \mathrm{d}x + \frac{\pi}{4} = \frac{\pi}{4} \\
\left[\int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x\right]^2 =& \frac{\pi}{4} \\
\int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x =& \frac{\sqrt{\pi}}{2}
\end{aligned}$$

---

$$\int_0^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

---

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = 2\int_0^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

---

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

□

### 42.2.4 other examples

https://www.youtube.com/watch?v=GW86SShcYbM

$\int_0^1 \ln x \mathrm{d}x$

$$I = \int_0^1 \ln x \mathrm{d}x = \int_0^1 \ln(x)\, \mathrm{d}x, f(x) = \ln x$$

$$I(t) = \int_0^1 \ln tx \mathrm{d}x = \int_0^1 f(tx)\, \mathrm{d}x = \int_0^1 f(x,t)\, \mathrm{d}x, \begin{cases} f(x,t) = f(tx) = \ln tx \\ f(x,1) = f(x) = \ln x \end{cases}$$

$$\begin{aligned}
I(t) &= \int_0^1 f(x,t)\, \mathrm{d}x = \int_0^1 f(tx)\, \mathrm{d}x = \int_0^1 \ln tx \mathrm{d}x \Rightarrow I(1) = I \\
\frac{\mathrm{d}}{\mathrm{d}t} I(t) = I'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 f(x,t)\, \mathrm{d}x = \int_0^1 \frac{\partial f(x,t)}{\partial t} \mathrm{d}x = \int_0^1 \frac{\partial \ln tx}{\partial t} \mathrm{d}x \\
&= \int_0^1 \frac{1}{tx} x \mathrm{d}x = \int_0^1 \frac{1}{t} \mathrm{d}x = \frac{1}{t} \int_0^1 \mathrm{d}x = \frac{1}{t} [x]_0^1 = \frac{1}{t} \\
I(t) &= \int I'(t)\, \mathrm{d}t = \int \frac{1}{t} \mathrm{d}t = \ln|t| + C \\
&\Downarrow \Rightarrow I = \int_0^1 \ln x \mathrm{d}x = \int_0^1 \ln 1x \mathrm{d}x = I(1) = \ln|1| + C = 0 + C \Rightarrow C = I
\end{aligned}$$

no more known boundary condition

$$\begin{aligned}
\Im(t) &= \int_0^1 x^t \mathrm{d}x = \left[\frac{x^{t+1}}{t+1}\right]_{x=0}^1 = \frac{1}{t+1} \\
\frac{-1}{(t+1)^2} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{t+1} = \frac{\mathrm{d}}{\mathrm{d}t} \Im(t) = \Im'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 x^t \mathrm{d}x = \int_0^1 \frac{\partial x^t}{\partial t} \mathrm{d}x = \int_0^1 x^t \ln x \mathrm{d}x \\
-1 = \left[\frac{-1}{(t+1)^2}\right]_{t=0} &= \Im'(0) = \int_0^1 x^0 \ln x \mathrm{d}x = \int_0^1 1 \ln x \mathrm{d}x = \int_0^1 \ln x \mathrm{d}x \\
\int_0^1 \ln x \mathrm{d}x &= -1
\end{aligned}$$

---

https://zhuanlan.zhihu.com/p/687355703

https://www.zhihu.com/question/646881575/answer/3417318090

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TaylorCatAlice: Feynman method of integration and residue theorem

https://www.bilibili.com/video/BV1Lj411L79X

MatheManiac: complex integral

https://www.youtube.com/watch?v=EyBDtUtyshk&list=PLDcSwjT2BF_UDdkQ3KQjX5SRQ2DLLwv0R&index=11

https://www.youtube.com/watch?v=EyBDtUtyshk

## 42.3 Feynman method of path integral

Elliot Schneider: Physics with Elliot

https://www.youtube.com/watch?v=W8QZ-yxebFA

https://www.youtube.com/watch?v=Se-CpexiJLQ

Feynman method of path integral in quantum dynamics

https://www.youtube.com/watch?v=Sp5SvdDh2u8

history of path integral

https://en.wikipedia.org/wiki/Path_integral_formulation

https://www.youtube.com/watch?v=7yjv-gLHFqg

# Chapter 43

# Hilbert space

```r
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

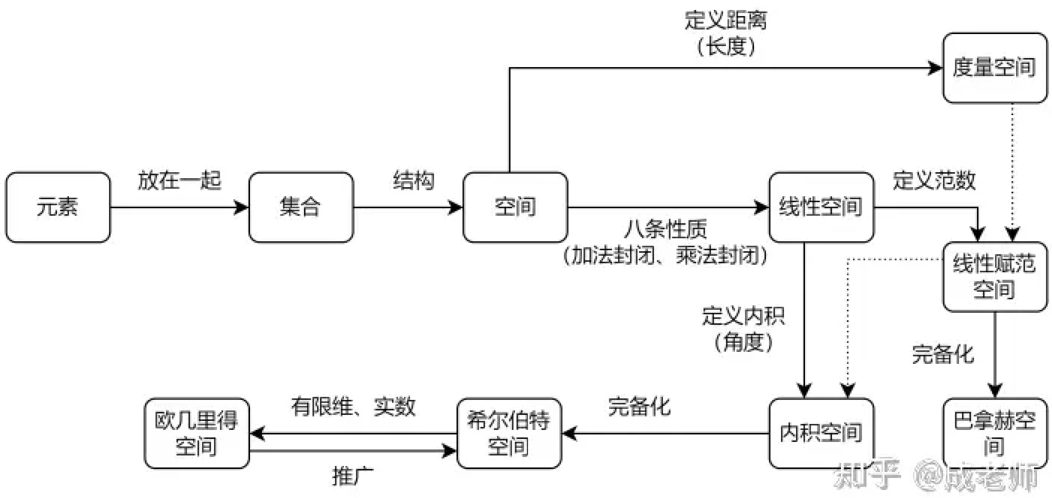


Figure 43.1: Euclid space construction

[1]

## 43.1 Taylor expansion or Taylor series

https://www.bilibili.com/video/BV15s4y1g7HQ

**Lemma 43.1.** *Newton-Leibniz formula = N-LF, equivalent to first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1*

牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF, equivalent to 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 **??**

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a) \tag{43.1}$$

[1] https://zhuanlan.zhihu.com/p/85867887

$$\begin{array}{l}\left\{\begin{array}{l}\left\{\begin{array}{ll}f:[a,b]\to\mathbb{R} & (0)\\ f\text{ continuous on }[a,b] & (1)\\ f\text{ differentiable on }(a,b) & (2)\end{array}\right.\\ \left\{\begin{array}{ll}f':[a,b]\to\mathbb{R}\ (4) & a<b\ (3)\\ f'\text{ continuous on }[a,b] & (5)\end{array}\right.\Rightarrow\exists c\in(a,b)\left(f'(c)=\dfrac{\int_a^b f'(x)\,\mathrm{d}x}{b-a}\right)\quad\text{MVTi1 ??}\end{array}\right.\\ \Rightarrow\begin{array}{c}\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}\ 43.1\\ \Updownarrow\\ \int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\ 43.1\end{array}\end{array}$$

(⇓):

$$\begin{array}{l}\left\{\begin{array}{l}\left\{\begin{array}{ll}f:[a,b]\to\mathbb{R} & (0)\\ f\text{ continuous on }[a,b] & (1)\\ f\text{ differentiable on }(a,b) & (2)\end{array}\right.\overset{??}{\Rightarrow}\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}\quad\text{MVTd1 ??}\\ {[a,b]}=[x_0,x_n]=[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n]\quad 43.2\end{array}\right.\\ \overset{??}{\underset{??}{\Rightarrow}}\displaystyle\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\end{array}$$

(⇑):

$$\begin{array}{l}\left\{\begin{array}{l}\left\{\begin{array}{ll}f:[a,b]\to\mathbb{R} & (0)\\ f\text{ continuous on }[a,b] & (1)\\ f\text{ differentiable on }(a,b) & (2)\end{array}\right.\overset{??}{\underset{??}{\Rightarrow}}\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\quad\text{N-LF } 43.1\\ \left\{\begin{array}{ll}f':[a,b]\to\mathbb{R}\ (4) & a<b\ (3)\\ f'\text{ continuous on }[a,b] & (5)\end{array}\right.\Rightarrow\exists c\in(a,b)\left(f'(c)=\dfrac{\int_a^b f'(x)\,\mathrm{d}x}{b-a}\right)\quad\text{MVTi1 ??}\end{array}\right.\\ \Rightarrow\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}\end{array}$$

Proof: (⇓)

重疊端點分割 $[a,b]$ 成 $n$ 部分聯集 $(n\in\mathbb{N})$

$$[a,b]=[x_0,x_n]=[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n] \tag{43.2}$$

$$\begin{cases}\begin{cases}f:[a,b]\to\mathbb{R} & (0)\\ f\text{ continuous on }[a,b] & (1)\\ f\text{ differentiable on }(a,b) & (2)\end{cases}\overset{??}{\Rightarrow}\exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}\quad\text{MVTd1 }\mathbf{??}\\ [a,b]=[x_0,x_n]=[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n]\qquad 43.2\end{cases}$$

$$\Rightarrow\begin{cases}(f:[x_0,x_n]\to\mathbb{R})=(f:[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n]\to\mathbb{R}) & \Leftarrow(0)\\ f\text{ continuous on }[x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n] & \Leftarrow(1)\\ f\text{ differentiable on }(x_0,x_1]\cup[x_0,x_2]\cup\cdots\cup[x_{n-1},x_n) & \Leftarrow(2)\end{cases}$$

$$\Rightarrow\begin{cases}f:[x_0,x_1]\to\mathbb{R},f:[x_0,x_2]\to\mathbb{R},\cdots,f:[x_{n-1},x_n]\to\mathbb{R} & \Leftarrow(0)\\ f\text{ continuous on }[x_0,x_1],[x_0,x_2],\cdots,[x_{n-1},x_n] & \Leftarrow(1)\\ f\text{ differentiable on }(x_0,x_1],[x_0,x_2],\cdots,[x_{n-1},x_n)\Rightarrow f\text{ differentiable on }(x_0,x_1),(x_0,x_2),\cdots,(x_{n-1},x_n) & \Leftarrow(2)\end{cases}$$

$$\overset{??}{\overset{??}{\Rightarrow}}\begin{cases}\exists c_1\in(x_0,x_1)\left\{f'(c_1)(x_1-x_0)=f(x_1)-f(x_0)\right\}\\ \exists c_2\in(x_1,x_2)\left\{f'(c_2)(x_2-x_1)=f(x_2)-f(x_1)\right\}\\ \vdots\\ \exists c_k\in(x_{k-1},x_k)\left\{f'(c_k)(x_k-x_{k-1})=f(x_k)-f(x_{k-1})\right\}\quad\forall k\in\mathbb{N}\cap[1,n]\\ \vdots\\ \exists c_n\in(x_{n-1},x_n)\left\{f'(c_n)(x_n-x_{n-1})=f(x_n)-f(x_{n-1})\right\}\end{cases}$$

$$\Rightarrow f'(c_k)(x_k-x_{k-1})=f(x_k)-f(x_{k-1})$$

$$\Rightarrow\sum_k f'(c_k)(x_k-x_{k-1})=\sum_k f(x_k)-f(x_{k-1})$$

$$\Rightarrow\sum_{k=1}^n f'(c_k)(x_k-x_{k-1})=\sum_{k=1}^n f(x_k)-f(x_{k-1})$$

$$=[f(x_1)-f(x_0)]+[f(x_2)-f(x_1)]+\cdots+[f(x_n)-f(x_{n-1})]$$

$$=f(x_n)-f(x_0)=f(b)-f(a)$$

$$\Rightarrow\sum_{k=1}^n f'(c_k)(x_k-x_{k-1})=f(b)-f(a)\overset{\Delta x_k=x_k-x_{k-1}}{\Leftrightarrow}\sum_{k=1}^n f'(c_k)\Delta x_k=f(b)-f(a)$$

$$\Rightarrow\lim_{n\to\infty}\lim_{\Delta x_k=\frac{x_n-x_0}{n}}\sum_{k=1}^n f'(c_k)\Delta x_k=\lim_{n\to\infty}\lim_{\Delta x_k=\frac{b-a}{n}\to0}f(b)-f(a)=f(b)-f(a)$$

$$\Rightarrow\int_a^b f'(x)\,\mathrm{d}x=\int_{x_0}^{x_n}f'(x)\,\mathrm{d}x=f(b)-f(a)$$

$$\Rightarrow\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)$$

$$\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)$$

得到 牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF [eq:N-LF]

□

Proof: (⇑)

$$
\left\{\begin{array}{l}
\left\{\begin{array}{ll} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{array}\right. \overset{??}{\overset{??}{\Rightarrow}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \qquad \text{N-LF } 43.1\\
\left\{\begin{array}{ll} f:[a,b]\to\mathbb{R} & a<b\ (3)\\ f \text{ continuous on } [a,b] & (1)\end{array}\right. \Rightarrow \exists c\in(a,b)\left(f(c)=\dfrac{\int_a^b f(x)\,\mathrm{d}x}{b-a}\right) \qquad \text{MVTi1 } \mathbf{??}
\end{array}\right.
$$

$$
\Rightarrow\left\{\begin{array}{l}
\left\{\begin{array}{ll} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{array}\right. \overset{??}{\overset{??}{\Rightarrow}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \qquad \text{N-LF } 43.1\\
\left\{\begin{array}{ll} f:[a,b]\to\mathbb{R} & a<b\ (3)\\ f \text{ continuous on } [a,b] & (1)\end{array}\right. \Rightarrow \exists c\in(a,b)\left\{f(c)(b-a)=\int_a^b f(x)\,\mathrm{d}x\right\} \qquad \text{MVTi1 } \mathbf{??}
\end{array}\right.
$$

$$
\Rightarrow\left\{\begin{array}{l}
\left\{\begin{array}{ll} f:[a,b]\to\mathbb{R} & (0)\\ f \text{ continuous on } [a,b] & (1)\\ f \text{ differentiable on } (a,b) & (2)\end{array}\right. \overset{??}{\overset{??}{\Rightarrow}} \int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a) \qquad \text{N-LF } 43.1\\
\left\{\begin{array}{ll} \text{if } f':[a,b]\to\mathbb{R} & (4)\\ \text{if } f' \text{ continuous on } [a,b] & (5)\end{array}\right. \Rightarrow \exists c\in(a,b)\left\{f'(c)(b-a)=\int_a^b f'(x)\,\mathrm{d}x\right\} \qquad \text{MVTi1 } \mathbf{??}
\end{array}\right.
$$

$$
\begin{aligned}
&\Rightarrow \exists c\in(a,b)\left\{f'(c)(b-a)=\int_a^b f'(x)\,\mathrm{d}x=f(b)-f(a)\right\}\\
&\Rightarrow \exists c\in(a,b)\left\{f'(c)(b-a)=f(b)-f(a)\right\}
\end{aligned}
$$

$$\exists c\in(a,b)\left[f'(c)(b-a)=f(b)-f(a)\right]$$

得到 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 [thm:MVTd1]

□

$$
\begin{aligned}
\int_a^b f'(x)\,\mathrm{d}x =& f(b)-f(a)\\
\int_{x_0}^x f'(t)\,\mathrm{d}t =& f(x)-f(x_0)\\
f(x) =& f(x_0)+\int_{x_0}^x f'(t)\,\mathrm{d}t
\end{aligned}
$$

---

$$f(x)=f(x_0)+\int_{x_0}^x f'(t)\,\mathrm{d}t$$

---

**Lemma 43.2.** *univariable product rule*

$$\mathrm{d}(uv)=(\mathrm{d}u)v+u\mathrm{d}v=v\mathrm{d}u+u\mathrm{d}v$$

$$\frac{\mathrm{d}(uv)}{\mathrm{d}t}=v\frac{\mathrm{d}u}{\mathrm{d}t}+u\frac{\mathrm{d}v}{\mathrm{d}t}$$

**Lemma 43.3.** *integration by parts*

$$\begin{aligned}
\int \frac{\mathrm{d}\left(uv\right)}{\mathrm{d}t}\mathrm{d}t &= \int v\frac{\mathrm{d}u}{\mathrm{d}t} + u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t \\
\int \mathrm{d}\left(uv\right) = & = \int v\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t + \int u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t \\
uv = & = \int v\mathrm{d}u + \int u\mathrm{d}v \\
\int u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \text{ "switching derivatives"} \qquad (43.3) \\
\int u\mathrm{d}v &= uv - \int v\mathrm{d}u \text{ "switching differentials"} \qquad (43.4)
\end{aligned}$$

**Theorem 43.1.** *univariable Taylor theorem with the remainder in integral form*

$$f\left(x\right) = \sum_{k=0}^{n} \frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^{k} + \int_{a}^{x} f^{(n+1)}\left(t\right)\frac{\left(t-x\right)^{n}}{n!}\mathrm{d}t$$

$$f\left(x\right) = \sum_{k=0}^{n} \frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^{k} + \int_{a}^{x} f^{(n+1)}\left(t\right)\frac{\left(x-t\right)^{n}}{n!}\mathrm{d}t$$

Proof: relatively weird $(t-x)$

$$\begin{aligned}
& f(x) - f(a) && (43.5)\\
\overset{43.1}{=}\ & \int_a^x f'(t)\,\mathrm{d}t = \int_a^x f'(t)\frac{\mathrm{d}(t-x)}{\mathrm{d}t}\mathrm{d}t \overset{43.3}{=} \left[f'(t)(t-x)\right]_{t=a}^{x} - \int_a^x (t-x)\frac{\mathrm{d}f'(t)}{\mathrm{d}t}\mathrm{d}t && (43.6)\\
=\ & \left[-f'(a)(a-x)\right] - \int_a^x f''(t)(t-x)\,\mathrm{d}t = f'(a)(x-a) - \int_a^x f''(t)\frac{\mathrm{d}\frac{(t-x)^2}{2}}{\mathrm{d}t}\mathrm{d}t && (43.7)\\
=\ & f'(a)(x-a) - \left(\left[f''(t)\frac{(t-x)^2}{2}\right]_{t=a}^{x} - \int_a^x \frac{(t-x)^2}{2}\frac{\mathrm{d}f''(t)}{\mathrm{d}t}\mathrm{d}t\right) && (43.8)\\
=\ & f'(a)(x-a) - \left(\left[-f''(a)\frac{(x-a)^2}{2}\right] - \int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t\right) && (43.9)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t && (43.10)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \int_a^x f'''(t)\frac{\mathrm{d}\frac{(t-x)^3}{2\cdot 3}}{\mathrm{d}t}\mathrm{d}t && (43.11)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \left(\left[f'''(t)\frac{(t-x)^3}{2\cdot 3}\right]_{t=a}^{x} - \int_a^x \frac{(t-x)^3}{2\cdot 3}\frac{\mathrm{d}f'''(t)}{\mathrm{d}t}\mathrm{d}t\right) && (43.12)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \left(\left[f'''(a)\frac{(x-a)^3}{2\cdot 3}\right] - \int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t\right) && (43.13)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(t)\frac{(x-a)^3}{2\cdot 3} - \int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t && (43.14)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(t)\frac{(x-a)^3}{2\cdot 3} - \int_a^x f^{(4)}(t)\frac{\mathrm{d}\frac{(t-x)^4}{2\cdot 3\cdot 4}}{\mathrm{d}t}\mathrm{d}t && (43.15)\\
=\ & f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(t)\frac{(x-a)^3}{2\cdot 3} && (43.16)\\
& - \left(\left[f^{(4)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\right]_{t=a}^{x} - \int_a^x \frac{(t-x)^4}{2\cdot 3\cdot 4}\frac{\mathrm{d}f^{(4)}(t)}{\mathrm{d}t}\mathrm{d}t\right) && (43.17)\\
=\ & f'(t)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(t)\frac{(x-a)^3}{2\cdot 3} && (43.18)\\
& - \left(\left[-f^{(4)}(a)\frac{(x-a)^4}{2\cdot 3\cdot 4}\right] - \int_a^x f^{(5)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\mathrm{d}t\right) && (43.19)\\
=\ & f'(t)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{2\cdot 3} + f^{(4)}(a)\frac{(x-a)^4}{2\cdot 3\cdot 4} && (43.20)\\
\vdots\ & + \int_a^x f^{(5)}(t)\frac{(t-x)^4}{2\cdot 3\cdot 4}\mathrm{d}t && (43.21)\\
=\ & \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\mathrm{d}t \text{ remainder in integral form} && (43.22)
\end{aligned}$$

□

Proof:

$$
\begin{aligned}
& f(x) - f(a) \\
& \overset{43.1}{=} \int_a^x f'(t)\,\mathrm{d}t = \int_a^x f'(t) \frac{\mathrm{d}(t-x)}{\mathrm{d}t}\mathrm{d}t \overset{43.3}{=} \left[f'(t)(t-x)\right]_{t=a}^{x} - \int_a^x (t-x)\frac{\mathrm{d}f'(t)}{\mathrm{d}t}\mathrm{d}t \\
& = \left[-f'(a)(a-x)\right] - \int_a^x f''(t)(t-x)\,\mathrm{d}t = f'(a)(x-a) + \int_a^x f''(t)(x-t)\,\mathrm{d}t \\
& = f'(a)(x-a) - \int_a^x f''(t)(t-x)\,\mathrm{d}t = \sum_{k=1}^{1}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f''(t)(x-t)\,\mathrm{d}t \\
& = f'(a)(x-a) - \int_a^x f''(t)\frac{\mathrm{d}\frac{(t-x)^2}{2}}{\mathrm{d}t}\mathrm{d}t \\
& = f'(a)(x-a) - \left(\left[f''(t)\frac{(t-x)^2}{2}\right]_{t=a}^{x} - \int_a^x \frac{(t-x)^2}{2}\frac{\mathrm{d}f''(t)}{\mathrm{d}t}\mathrm{d}t\right) \\
& = f'(a)(x-a) - \left(\left[-f''(a)\frac{(x-a)^2}{2}\right] - \int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t\right) \\
& = f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t \\
& = \sum_{k=1}^{2}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f'''(t)\frac{(t-x)^2}{2}\mathrm{d}t = \sum_{k=1}^{2}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f'''(t)\frac{(x-t)^2}{2!}\mathrm{d}t \\
& = f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \int_a^x f'''(t)\frac{\mathrm{d}\frac{(t-x)^3}{2\cdot 3}}{\mathrm{d}t}\mathrm{d}t \\
& = f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \left(\left[f'''(t)\frac{(t-x)^3}{2\cdot 3}\right]_{t=a}^{x} - \int_a^x \frac{(t-x)^3}{2\cdot 3}\frac{\mathrm{d}f'''(t)}{\mathrm{d}t}\mathrm{d}t\right) \\
& = f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \left(\left[f'''(a)\frac{(x-a)^3}{2\cdot 3}\right] - \int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t\right) \\
& = f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(t)\frac{(x-a)^3}{2\cdot 3} - \int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t \\
& = \sum_{k=1}^{3}\frac{f^{(k)}(a)}{k!}(x-a)^k - \int_a^x f^{(4)}(t)\frac{(t-x)^3}{2\cdot 3}\mathrm{d}t = \sum_{k=1}^{3}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(4)}(t)\frac{(x-t)^3}{3!}\mathrm{d}t \\
& \overset{\vdots}{=} \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
f(x) - f(a) & = \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
f(x) & = f(a) + \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
& = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
& = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t
\end{aligned}
$$

□

**Theorem 43.2.** *univariable Taylor theorem with the remainder in Lagrange form*

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Proof:

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT **??** and 連續函數 介值定理 / 中間值定理 continuous function intermediate value theorem = CFIVT / intermediate value theorem = IVT **??**

$$\text{let } f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + r_n(x) \tag{43.23}$$

$$r_n(x) = \int_a^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\mathrm{d}t \tag{43.24}$$

$$\text{let } n \in \{2k-1 | k \in \mathbb{N}\} \tag{43.25}$$

$$\text{let } f(t) \overset{??}{\in} [m, M] \subseteq f((a,x)) \text{ when } a < x \tag{43.26}$$

$$\int_a^x m\frac{(t-x)^n}{n!}\mathrm{d}t \le r_n(x) \le \int_a^x M\frac{(t-x)^n}{n!}\mathrm{d}t \tag{43.27}$$

$$m\int_a^x \frac{(t-x)^n}{n!}\mathrm{d}t = \quad = M\int_a^x \frac{(t-x)^n}{n!}\mathrm{d}t \tag{43.28}$$

$$m\left[\frac{(t-x)^{n+1}}{(n+1)!}\right]_{t=a}^{x} = \quad = M\left[\frac{(t-x)^{n+1}}{(n+1)!}\right]_{t=a}^{x} \tag{43.29}$$

$$m\frac{(x-a)^{n+1}}{(n+1)!} = \quad = M\frac{(x-a)^{n+1}}{(n+1)!} \tag{43.30}$$

$$\Downarrow \quad \mathbf{??} \tag{43.31}$$

$$r_n(x) \overset{\exists \xi \in (a,x)}{=} f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \tag{43.32}$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!} \tag{43.33}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{43.34}$$

□

**Theorem 43.3.** *univariable Taylor theorem with the remainder in big O form*

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \mathrm{O}\left(h^{n+1}\right)$$

Proof:

$$\text{if } \left|f^{(n+1)}(t)\right| \le K \,\forall t \in (a,x) \tag{43.35}$$

$$|r_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{K}{(n+1)!}|x-a|^{n+1} \tag{43.36}$$

$$r_n(x) \in \mathrm{O}\left((x-a)^{n+1}\right) \tag{43.37}$$

$$\text{let } R_n(h) = r_n(a+h) \tag{43.38}$$

$$R_n(h) \in \mathrm{O}\left(h^{n+1}\right) \tag{43.39}$$

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} \tag{43.40}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \mathrm{O}\left(h^{n+1}\right) \tag{43.41}$$

□

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$f(x) = \sin(x)$

$$\sin(x) = \sum_{k=0}^{n} \frac{\sin^{(k)}(a)}{k!}(x-a)^k + \int_a^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$a = 0$

$$\sin(x) = \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!}(x-0)^k + \int_0^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$$\begin{aligned}
\sin(x) =& \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!}(x-0)^k + \int_0^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t\\
=& \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!}x^k + \int_0^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t\\
=& \frac{\sin^{(0)}(0)}{0!}x^0 + \frac{\sin^{(1)}(0)}{1!}x^1 + \frac{\sin^{(2)}(0)}{2!}x^2 + \frac{\sin^{(3)}(0)}{3!}x^3\\
&+ \int_0^x \sin^{(3+1)}(t)\frac{(x-t)^3}{3!}\mathrm{d}t\\
=& \frac{0}{0!}x^0 + \frac{\cos(0)}{1!}x^1 + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3\\
&+ \int_0^x \sin^{(4)}(t)\frac{(x-t)^3}{3!}\mathrm{d}t\\
=& 0 + \frac{1}{1}x + 0 - \frac{1}{6}x^3 + \int_0^x \sin(t)\frac{(x-t)^3}{6}\mathrm{d}t\\
=& x - \frac{1}{6}x^3 + \frac{1}{6}\int_0^x (x-t)^3\sin(t)\,\mathrm{d}t\\
\sin(x) =& x - \frac{1}{6}x^3 + \frac{1}{6}\int_0^x (x-t)^3\sin(t)\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\sin(x) =& \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!}(x-0)^k + \int_0^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t\\
=& \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!}x^k + \int_0^x \sin^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t\\
=& \frac{\sin^{(0)}(0)}{0!}x^0 + \frac{\sin^{(1)}(0)}{1!}x^1 + \frac{\sin^{(2)}(0)}{2!}x^2 + \frac{\sin^{(3)}(0)}{3!}x^3 + \frac{\sin^{(4)}(0)}{4!}x^4\\
&+ \int_0^x \sin^{(4+1)}(t)\frac{(x-t)^4}{4!}\mathrm{d}t\\
=& \frac{0}{0!}x^0 + \frac{\cos(0)}{1!}x^1 + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 + \frac{\sin(0)}{4!}x^4\\
&+ \int_0^x \sin^{(5)}(t)\frac{(x-t)^4}{4!}\mathrm{d}t\\
=& 0 + \frac{1}{1}x + 0 - \frac{1}{6}x^3 + 0 + \int_0^x \cos(t)\frac{(x-t)^4}{24}\mathrm{d}t\\
=& x - \frac{1}{6}x^3 + \frac{1}{24}\int_0^x (x-t)^4\cos(t)\,\mathrm{d}t\\
\sin(x) =& x - \frac{1}{6}x^3 + \frac{1}{24}\int_0^x (x-t)^4\cos(t)\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\sin(x) =& x - \frac{1}{6}x^3 + \frac{1}{6}\int_0^x (x-t)^3 \sin(t)\,\mathrm{d}t \\
=& x - \frac{1}{6}x^3 + \frac{1}{24}\int_0^x (x-t)^4 \cos(t)\,\mathrm{d}t \\
& \Downarrow \\
\frac{1}{6}\int_0^x (x-t)^3 \sin(t)\,\mathrm{d}t =& \frac{1}{24}\int_0^x (x-t)^4 \cos(t)\,\mathrm{d}t \\
& \Downarrow \\
\int_0^x (x-t)^4 \cos(t)\,\mathrm{d}t =& 4\int_0^x (x-t)^3 \sin(t)\,\mathrm{d}t
\end{aligned}$$

---

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$f(x) = \mathrm{e}^x = \exp(x)$

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{\exp^{(k)}(a)}{k!}(x-a)^k + \int_a^x \exp^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$a = 0$

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{\exp^{(k)}(0)}{k!}(x-0)^k + \int_0^x \exp^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t$$

$$\begin{aligned}
\mathrm{e}^x =& \sum_{k=0}^{n} \frac{\exp^{(k)}(0)}{k!}(x-0)^k + \int_0^x \exp^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
=& \sum_{k=0}^{n} \frac{\exp^{(k)}(0)}{k!}x^k + \int_0^x \exp^{(n+1)}(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
=& \sum_{k=0}^{n} \frac{\exp(0)}{k!}x^k + \int_0^x \exp(t)\frac{(x-t)^n}{n!}\mathrm{d}t \\
=& \sum_{k=0}^{n} \frac{\mathrm{e}^0}{k!}x^k + \int_0^x \mathrm{e}^t\frac{(x-t)^n}{n!}\mathrm{d}t = \sum_{k=0}^{n} \frac{1}{k!}x^k + \int_0^x \mathrm{e}^t\frac{(x-t)^n}{n!}\mathrm{d}t \\
=& \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{1}{n!}\int_0^x (x-t)^n \mathrm{e}^t\mathrm{d}t \\
\mathrm{e}^x =& \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{1}{n!}\int_0^x (x-t)^n \mathrm{e}^t\mathrm{d}t
\end{aligned}$$

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{1}{n!}\int_0^x (x-t)^n \mathrm{e}^t\mathrm{d}t$$

---

$$\lim_{n\to\infty} \frac{1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}}{\mathrm{e}^n}$$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1+n+\dfrac{n^2}{2}+\cdots+\dfrac{n^n}{n!}}{\mathrm{e}^n},\wedge\mathrm{e}^x=\sum_{k=0}^{n}\frac{x^k}{k!}+\frac{1}{n!}\int_0^x(x-t)^n\,\mathrm{e}^t\mathrm{d}t\\
=&\lim_{n\to\infty}\frac{1+n+\dfrac{n^2}{2}+\cdots+\dfrac{n^n}{n!}}{\left[\sum\limits_{k=0}^{n}\frac{x^k}{k!}+\frac{1}{n!}\int_0^x(x-t)^n\,\mathrm{e}^t\mathrm{d}t\right]_{x=n}}\\
=&\lim_{n\to\infty}\frac{\sum\limits_{k=0}^{n}\frac{n^k}{k!}}{\sum\limits_{k=0}^{n}\frac{n^k}{k!}+\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^t\mathrm{d}t}=\lim_{n\to\infty}\frac{1}{1+\dfrac{\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^t\mathrm{d}t}{\sum\limits_{k=0}^{n}\frac{n^k}{k!}}}
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1+n+\dfrac{n^2}{2}+\cdots+\dfrac{n^n}{n!}}{\mathrm{e}^n},\wedge\mathrm{e}^x=\sum_{k=0}^{n}\frac{x^k}{k!}+\frac{1}{n!}\int_0^x(x-t)^n\,\mathrm{e}^t\mathrm{d}t\\
=&\lim_{n\to\infty}\frac{\sum\limits_{k=0}^{n}\frac{n^k}{k!}}{\mathrm{e}^n},\wedge\mathrm{e}^n=\sum_{k=0}^{n}\frac{n^k}{k!}+\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^t\mathrm{d}t\Rightarrow\sum_{k=0}^{n}\frac{n^k}{k!}=\mathrm{e}^n-\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^t\mathrm{d}t\\
=&\lim_{n\to\infty}\frac{\mathrm{e}^n-\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^t\mathrm{d}t}{\mathrm{e}^n}=\lim_{n\to\infty}\left[1-\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t\right],\ \text{if}\ \lim_{n\to\infty}\frac{1}{n!}\int_0^n(t-n)^n\,\mathrm{e}^{t-n}\mathrm{d}t\in\mathbb{R}\\
=&\lim_{n\to\infty}1-\lim_{n\to\infty}\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t=1-\lim_{n\to\infty}\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t
\end{aligned}$$

$\lim\limits_{n\to\infty}\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n!}\int_0^n(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t,x=n-t\\
=&\lim_{n\to\infty}\frac{1}{n!}\int_{t=0}^{t=n}(n-t)^n\,\mathrm{e}^{t-n}\mathrm{d}t,t=n-x\\
=&\lim_{n\to\infty}\frac{1}{n!}\int_{n-x=0}^{n-x=n}(n-(n-x))^n\,\mathrm{e}^{(n-x)-n}\mathrm{d}\,(n-x)\\
=&\lim_{n\to\infty}\frac{1}{n!}\left(-\int_{x=n}^{x=0}x^n\mathrm{e}^{-x}\mathrm{d}x\right)=\lim_{n\to\infty}\frac{1}{n!}\left(\int_{x=0}^{x=n}x^n\mathrm{e}^{-x}\mathrm{d}x\right)\\
=&\lim_{n\to\infty}\frac{1}{n!}\int_0^n x^n\mathrm{e}^{-x}\mathrm{d}x
\end{aligned}$$

$\int x^n\mathrm{e}^{-x}\mathrm{d}x$

difficult process

$$\begin{aligned}
\int x^n\mathrm{e}^{-x}\mathrm{d}x&=-\int x^n\mathrm{d}\mathrm{e}^{-x}\\
&=-\left[x^n\mathrm{e}^{-x}-\int\mathrm{e}^{-x}\mathrm{d}x^n\right]\\
&=-\left[x^n\mathrm{e}^{-x}-\int\mathrm{e}^{-x}nx^{n-1}\mathrm{d}x\right]\\
&=-x^n\mathrm{e}^{-x}+n\int x^{n-1}\mathrm{e}^{-x}\mathrm{d}x
\end{aligned}$$

$$\int x^n\mathrm{e}^{-x}\mathrm{d}x=-x^n\mathrm{e}^{-x}+n\int x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$$

$$\begin{aligned}\int_0^n x^n \mathrm{e}^{-x}\mathrm{d}x &= \left[-x^n \mathrm{e}^{-x}\right]_{x=0}^n + n\int_0^n x^{n-1}\mathrm{e}^{-x}\mathrm{d}x \\ &= \left[-n^n \mathrm{e}^{-n} - \left(-0^n \mathrm{e}^{-0}\right)\right] + n\int_0^n x^{n-1}\mathrm{e}^{-x}\mathrm{d}x \\ &= -\, n^n \mathrm{e}^{-n} + n\int_0^n x^{n-1}\mathrm{e}^{-x}\mathrm{d}x\end{aligned}$$

$$\int_0^n x^n \mathrm{e}^{-x}\mathrm{d}x = -n^n \mathrm{e}^{-n} + n\int_0^n x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$$

more formal process with gamma function

https://www.wolframalpha.com/input?i2d=true&i=Divide%5BIntegrate%5BPower%5Bx%2Cn%5DPower%5Be%2C-x%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D

https://www.wolframalpha.com/input?i2d=true&i=Limit%5BDivide%5BIntegrate%5BPower%5Bx%2Cn%5DPower%5Be%2C-x%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D%2Cn-%3E%E2%88%9E%5D

to be proved

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https://www.bilibili.com/video/BV1du411a7qB

### 43.1.1 Elliot Schneider: Taylor series

https://www.youtube.com/watch?v=HQsZG8Yxb7w

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=12m25s

$$f\left(x\right) = \sum_{k=0}^{n} \frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k + \frac{f^{(n+1)}\left(\xi\right)}{(n+1)!}\left(x-x_0\right)^{n+1}$$

$$f\left(x_0+\epsilon\right) = \sum_{k=0}^{n} \frac{f^{(k)}\left(x_0\right)}{k!}\epsilon^k + \mathrm{O}\left(\epsilon^{n+1}\right), \epsilon = x - x_0$$

$$f\left(x+\epsilon\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(x\right)}{n!}\epsilon^n$$

$$\begin{aligned}f\left(x+\epsilon\right) &= \sum_{n=0}^{\infty} \frac{f^{(n)}\left(x\right)}{n!}\epsilon^n = \sum_{n=0}^{\infty} \frac{\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\left(x\right)}{n!}\epsilon^n = \sum_{n=0}^{\infty} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f\left(x\right)}{n!}\epsilon^n \\ &= \sum_{n=0}^{\infty} \frac{\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f\left(x\right)}{n!} = \left(\sum_{n=0}^{\infty} \frac{\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}x}\right)^n}{n!}\right) f\left(x\right) = \mathrm{e}^{\epsilon\frac{\mathrm{d}}{\mathrm{d}x}} f\left(x\right)\end{aligned}$$

---

univariable Taylor operator

$$f\left(x+\epsilon\right) = \mathrm{e}^{\epsilon\frac{\mathrm{d}}{\mathrm{d}x}} f\left(x\right)$$

$$\mathrm{e}^{\epsilon\frac{\mathrm{d}}{\mathrm{d}x}} = \sum_{n=0}^{\infty} \frac{\left(\epsilon\frac{\mathrm{d}}{\mathrm{d}x}\right)^n}{n!} = 1 + \epsilon\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\epsilon^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \cdots$$

---

$f(x) = mx + b$,

$$\begin{aligned}
e^{\epsilon\frac{d}{dx}} f(x) = & \left(1 + \epsilon\frac{d}{dx} + \frac{1}{2}\epsilon^2\frac{d^2}{dx^2} + \cdots\right)(mx+b) \\
= & 1(mx+b) + \epsilon\frac{d}{dx}(mx+b) + \frac{1}{2}\epsilon^2\frac{d^2}{dx^2}(mx+b) + \cdots \\
= & (mx+b) + (\epsilon m) + \left(\frac{1}{2}\epsilon^2 0\right) + (0+0+\cdots) \\
= & (mx+b) + (\epsilon m) + (0+0+\cdots) \\
= & (mx+b) + (\epsilon m) + 0 \\
= & mx + b + m\epsilon = m(x+\epsilon) + b = f(x+\epsilon)
\end{aligned}$$

---

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=15m34s

multivariable ...

$$\begin{aligned}
f(x+\epsilon) = & e^{\epsilon\frac{d}{dx}} f(x) \\
f(x+\epsilon_x, y+\epsilon_y, z+\epsilon_z) = & f(x, y+\epsilon_y, z+\epsilon_z) \\
& + \epsilon\frac{\partial}{\partial x} f(x, y+\epsilon_y, z+\epsilon_z) \\
& + \frac{1}{2}\epsilon^2\frac{\partial^2}{\partial x^2} f(x, y+\epsilon_y, z+\epsilon_z) \\
& + \cdots
\end{aligned}$$

$$f(x+\epsilon) = e^{\epsilon\frac{d}{dx}} f(x)$$

$$f(x+\epsilon_x, y+\epsilon_y, z+\epsilon_z) = f(\boldsymbol{r}+\boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}} f(\boldsymbol{r}), \begin{cases} \boldsymbol{r} = (x, y, z) \\ \boldsymbol{\epsilon} = (\epsilon_x, \epsilon_y, \epsilon_z) \\ \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \end{cases}$$

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla} = (\epsilon_x, \epsilon_y, \epsilon_z)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \epsilon_x\frac{\partial}{\partial x} + \epsilon_y\frac{\partial}{\partial y} + \epsilon_z\frac{\partial}{\partial z}$$

---

multivariable Taylor operator

$$f(\boldsymbol{r}+\boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}} f(\boldsymbol{r})$$

---

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=17m36s

physics

#### 43.1.1.1 making complicated equastion simple

linearize

simple pendulum

sin(x) to x linearization

potential energy

$$f(x_0+\epsilon) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\epsilon^k + O\left(\epsilon^{n+1}\right)$$

$$U(x_0+\epsilon) = \sum_{k=0}^{n}\frac{U^{(k)}(x_0)}{k!}\epsilon^k + O\left(\epsilon^{n+1}\right) = U(x_0) + U'(x_0)x + \frac{1}{2}U''(x_0)x^2 + \cdots$$

$$F = \frac{-dU}{dx} = -U'(x_0) - U''(x_0)x - \cdots \overset{\text{if } U'(x_0)=0}{=} -U''(x_0)x = -kx, k = U''(x_0)$$

#### 43.1.1.2 non-relativistic limit

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=22m32s

$$
\begin{aligned}
E =& \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2} = \sqrt{m^2c^4 + p^2c^2} \\
=& mc^2\sqrt{1 + \left(\frac{pc}{mc^2}\right)^2} = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2} = mc^2\left(1 + \left(\frac{p}{mc}\right)^2\right)^{\frac{1}{2}} \\
=& mc^2\left[\frac{1}{0!} + \frac{1}{1!}\left(\frac{1}{2}\right)\left(\left(\frac{p}{mc}\right)^2\right)^1 + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\left(\frac{p}{mc}\right)^2\right)^2 + \cdots\right] \\
=& mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \cdots\right] = mc^2\left[1 + \frac{1}{2}\frac{p^2}{m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4} + \cdots\right] \\
=& mc^2 + \frac{1}{2}\frac{p^2}{m} - \frac{1}{8}\frac{p^4}{m^3c^2} + \cdots = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots \\
=& E_0 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots, \begin{cases} E_0 = mc^2 \\ \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{mv^2}{2} = \frac{1}{2}mv^2 \end{cases}
\end{aligned}
$$

sluppy here $p = \gamma mv$

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=24m50s

leading relativistic correction

$$
-\frac{p^4}{8m^3c^2}
$$

binding energy of hydrogen atom

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=25m40s

fine-structure constant

#### 43.1.1.3 quantum momentum operator

https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=29m13s

## 43.2 Dirac delta function

https://www.bilibili.com/video/BV1qu411578c

Dirac function = Dirac delta function

https://tikz.net/delta_function/

https://tikz.dev/tikz-arrows

$$
\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1
$$

Figure 43.2: Dirac function = Dirac delta function

$$
\operatorname{supp}(f) = \left\{ x \middle| \begin{cases} x \in \mathcal{D} \\ f(x) \neq 0 \end{cases} \right\}
$$

$$
\operatorname{supp}(\delta) = \left\{ x \middle| \begin{cases} x \in \mathcal{D} = \mathbb{R} \\ \delta(x) \neq 0 \end{cases} \right\} = \{0\}
$$

$$
f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}
$$

Figure 43.3: $f(x) = 1$ if $x = 0$ else 0

https://tex.stackexchange.com/questions/45275/tikz-get-values-for-predefined-dash-patterns

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| \le \frac{1}{2n} \end{cases}, \forall n \in \mathbb{N}$$

Figure 43.4: $\delta_n(x)$

$$\delta(x) = \{\delta_n(x) | n \in \mathbb{N}\} = \{\delta_1(x), \delta_2(x), \cdots\} = \lim_{n\to\infty} \delta_n(x)$$

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \Leftrightarrow \begin{cases} x > \frac{1}{2n} \\ x < \frac{-1}{2n} \end{cases} \\ n & |x| \le \frac{1}{2n} \Leftrightarrow \frac{-1}{2n} \le x \le \frac{1}{2n} \Leftrightarrow x \in \left[\frac{-1}{2n}, \frac{1}{2n}\right] \end{cases}, \forall n \in \mathbb{N}$$

$$\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{d}x = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\mathrm{d}x = n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \mathrm{d}x = n\,[x]_{\frac{-1}{2n}}^{\frac{1}{2n}} = n\left[\frac{1}{2n} - \frac{-1}{2n}\right] = n \cdot \frac{1}{n} = 1$$

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x, \begin{cases} f(x) \in [m, M] \subseteq f\left(\left[\frac{-1}{2n}, \frac{1}{2n}\right]\right) \\ \Downarrow \\ m \le f(x) \le M \end{cases}\end{aligned}$$

$$\int_{-\frac{1}{2n}}^{\frac{1}{2n}} nm\mathrm{d}x \le \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x \le \int_{-\frac{1}{2n}}^{\frac{1}{2n}} nM\mathrm{d}x$$

$$m = m \cdot 1 = m\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\mathrm{d}x \le \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x \le M\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\mathrm{d}x = M \cdot 1 = M$$

$$\Downarrow \mathbf{??}$$

$$\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x \overset{\exists \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right)}{=} f(\xi_n) \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\mathrm{d}x = f(\xi_n) \cdot 1 = f(\xi_n)$$

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x &= \lim_{n\to\infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\ &= f(0)\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right)\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-0) f(x)\,\mathrm{d}x = f(0)$$

Figure 43.5: $\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$

---

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\int_{-\infty}^{\infty}\delta\left(x-0\right)f\left(x\right)\mathrm{d}x=f\left(0\right)$$

https://math.stackexchange.com/questions/73010/proof-of-dirac-deltas-sifting-property

https://www.youtube.com/watch?v=2QaRZ7u-BgM

SIFTing property

SIFT = scale-invariant feature transform

https://en.wikipedia.org/wiki/Scale-invariant_feature_transform

Proof:

Similarly to:

$$\begin{aligned}\int_{-\infty}^{\infty}\delta\left(x\right)f\left(x\right)\mathrm{d}x&=\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_{n}\left(x\right)f\left(x\right)\mathrm{d}x\\&=\lim_{n\to\infty}\int_{-\frac{1}{2n}}^{\frac{1}{2n}}\delta_{n}\left(x\right)f\left(x\right)\mathrm{d}x\\&=\lim_{n\to\infty}\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nf\left(x\right)\mathrm{d}x,\begin{cases}f\left(x\right)\in\left[m,M\right]\subseteq f\left(\left[\frac{-1}{2n},\frac{1}{2n}\right]\right)\\\Downarrow\\m\le f\left(x\right)\le M\end{cases}\end{aligned}$$

$$\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nm\mathrm{d}x\le\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\le\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nM\mathrm{d}x$$

$$m=m\cdot1=m\int_{-\frac{1}{2n}}^{\frac{1}{2n}}n\mathrm{d}x\le\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\le M\int_{-\frac{1}{2n}}^{\frac{1}{2n}}n\mathrm{d}x=M\cdot1=M$$

$$\Downarrow\mathbf{??}$$

$$\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\overset{\exists\xi_{n}\in\left(\frac{-1}{2n},\frac{1}{2n}\right)}{=}f\left(\xi_{n}\right)\int_{-\frac{1}{2n}}^{\frac{1}{2n}}n\mathrm{d}x=f\left(\xi_{n}\right)\cdot1=f\left(\xi_{n}\right)$$

$$\begin{aligned}\int_{-\infty}^{\infty}\delta\left(x\right)f\left(x\right)\mathrm{d}x=\lim_{n\to\infty}\int_{-\frac{1}{2n}}^{\frac{1}{2n}}nf\left(x\right)\mathrm{d}x&=\lim_{n\to\infty}f\left(\xi_{n}\right),\xi_{n}\in\left(\frac{-1}{2n},\frac{1}{2n}\right)\\&=f\left(0\right)\end{aligned}$$

$$\int_{-\infty}^{\infty}\delta\left(x\right)f\left(x\right)\mathrm{d}x=f\left(0\right)$$

$f\left(x\right)\in\left[m,M\right]\subseteq f\left(\left[x_{0}-\frac{1}{2n},x_{0}+\frac{1}{2n}\right]\right)$,

$$\int_{-\infty}^{\infty}\delta\left(x-x_0\right)f\left(x\right)\mathrm{d}x=\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n\left(x-x_0\right)f\left(x\right)\mathrm{d}x$$
$$=\lim_{n\to\infty}\left[\int_{-\infty}^{x_0-\frac{1}{2n}}\delta_n\left(x-x_0\right)f\left(x\right)\mathrm{d}x+\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}\delta_n\left(x-x_0\right)f\left(x\right)\mathrm{d}x+\lim_{n\to\infty}\int_{x_0+\frac{1}{2n}}^{\infty}\delta_n\left(x-x_0\right)f\left(x\right)\mathrm{d}x\right]$$
$$=\lim_{n\to\infty}\left[\int_{-\infty}^{x_0-\frac{1}{2n}}0f\left(x\right)\mathrm{d}x+\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x+\lim_{n\to\infty}\int_{x_0+\frac{1}{2n}}^{\infty}0f\left(x\right)\mathrm{d}x\right]$$
$$=\lim_{n\to\infty}\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x,\begin{cases}f\left(x\right)\in\left[m,M\right]\subseteq f\left(\left[x_0-\frac{1}{2n},x_0+\frac{1}{2n}\right]\right)\\\Downarrow\\m\leq f\left(x\right)\leq M\end{cases}$$
$$\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nm\mathrm{d}x\leq\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\leq\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nM\mathrm{d}x$$
$$m=m\cdot1=m\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}n\mathrm{d}x\leq\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\leq M\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}n\mathrm{d}x=M\cdot1=M$$
$$\Downarrow??$$
$$\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x\overset{\exists\xi_n\in\left(x_0-\frac{1}{2n},x_0+\frac{1}{2n}\right)}{=}f\left(\xi_n\right)\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}n\mathrm{d}x=f\left(\xi_n\right)\cdot1=f\left(\xi_n\right)$$
$$\int_{-\infty}^{\infty}\delta\left(x\right)f\left(x\right)\mathrm{d}x=\lim_{n\to\infty}\int_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}nf\left(x\right)\mathrm{d}x=\lim_{n\to\infty}f\left(\xi_n\right),\xi_n\in\left(x_0-\frac{1}{2n},x_0+\frac{1}{2n}\right)$$
$$=f\left(x_0\right)$$
$$\int_{-\infty}^{\infty}\delta\left(x-x_0\right)f\left(x\right)\mathrm{d}x=f\left(x_0\right)$$

$$\int_{-\infty}^{\infty}\delta\left(x-x_0\right)f\left(x\right)\mathrm{d}x=f\left(x_0\right)$$

□

$$\int_{-\infty}^{\infty}\delta\left(x-x_0\right)f\left(x\right)\mathrm{d}x=f\left(x_0\right)$$

$$\int_{-\infty}^{\infty}\delta\left(x-x'\right)f\left(x\right)\mathrm{d}x=f\left(x'\right)$$

---

https://www.youtube.com/watch?v=nDa3cqFk80o

$$\left\{\left\{f_n\left(x\right)|n\in\mathbb{N}\right\}\middle|\begin{cases}f_n:\mathbb{R}\to\mathbb{R}\\\int_{-\infty}^{\infty}f_n\left(x\right)\mathrm{d}x=1\end{cases}\right\}$$
$$=\left\{\left\{\frac{n}{2}\mathrm{e}^{-n|x|}\middle|n\in\mathbb{N}\right\},\right.$$
$$\left\{\frac{1}{\pi}\frac{n}{n^2x^2+1}\middle|n\in\mathbb{N}\right\},$$
$$\left\{\frac{n}{\sqrt{\pi}}\mathrm{e}^{-n^2x^2}\middle|n\in\mathbb{N}\right\},$$
$$\left.\left\{\frac{1}{\pi}\frac{\sin\left(nx\right)}{x}\middle|n\in\mathbb{N}\right\},\cdots\right\}$$

$$\left\{\delta\left(x\right)\middle|\begin{cases}\delta:\mathbb{R}\to\mathbb{R}\\\int_{-\infty}^{\infty}\delta\left(x\right)\cdot1\mathrm{d}x=1\end{cases}\right\}$$

$$\left\{\delta(x) \middle| \begin{cases} \delta : \mathbb{R} \to \mathbb{R} \\ \int_{-\infty}^{\infty} \delta(x-0) \cdot f(x)\,\mathrm{d}x = f(0) \end{cases} \right\}$$

In measure theory, we can define the distance of two functions by

$$d(f,g) = \sqrt{\int_{-\infty}^{\infty} [f(x) - g(x)]^2\,\mathrm{d}x}$$

for real distance of two square delta function approximations,

$$\begin{aligned}
d(\delta_m, \delta_n) =& \sqrt{\int_{-\infty}^{\infty} [\delta_m(x) - \delta_n(x)]^2\,\mathrm{d}x} \\
=& \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2 - 2\delta_m(x)\,\delta_n(x) + [\delta_n(x)]^2\,\mathrm{d}x} \\
=& \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2\,\mathrm{d}x - 2\int_{-\infty}^{\infty} \delta_m(x)\,\delta_n(x)\,\mathrm{d}x + \int_{-\infty}^{\infty} [\delta_n(x)]^2\,\mathrm{d}x} \\
=& \sqrt{\int_{-\infty}^{\infty} \delta_m(x)\,\delta_m(x)\,\mathrm{d}x - 2\int_{-\infty}^{\infty} \delta_m(x)\,\delta_n(x)\,\mathrm{d}x + \int_{-\infty}^{\infty} \delta_n(x)\,\delta_n(x)\,\mathrm{d}x} \\
=& \sqrt{\delta_m(\xi_m) - 2\int_{-\infty}^{\infty} \delta_m(x)\,\delta_n(x)\,\mathrm{d}x + \delta_n(\xi_n)}, \begin{cases} \xi_m \in \left(\frac{-1}{2m}, \frac{1}{2m}\right) \\ \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \end{cases} \\
=& \sqrt{m - 2\int_{-\infty}^{\infty} \delta_m(x)\,\delta_n(x)\,\mathrm{d}x + n}, \text{ if } m > n \\
=& \sqrt{m - 2\delta_n(\xi_n) + n}, \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\
=& \sqrt{m - 2n + n} = \sqrt{m-n} \in \mathbb{R} \\
d(\delta_m, \delta_n) \overset{m \geq n}{=}& \sqrt{m-n} \in \mathbb{R}
\end{aligned}$$

$$\langle d(\delta_m, \delta_n)\rangle_{n\in\mathbb{N}} = \left\langle \sqrt{m-n} \right\rangle_{n\in\mathbb{N}} \text{ is not a Cauchy series, not even mentioned convergence}$$

Def: 43.2

### 43.2.1 complex delta function

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$$

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\\
&=\lim_{k'\to 0}\left[\int_{-\infty}^{k'-}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k+\int_{k'-}^{k'+}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k+\int_{k'+}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[\int_{-\infty}^{0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k+\int_{k'-}^{k'+}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}0x}\mathrm{d}x\right)f\left(0,x\right)\mathrm{d}k+\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[\int_{-\infty}^{0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k+\int_{k'-}^{k'+}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}1\mathrm{d}x\right)f\left(0,x\right)\mathrm{d}k+\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[-\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k+\int_{k'-}^{k'+}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}1\mathrm{d}x\right)f\left(0,x\right)\mathrm{d}k+\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[-\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\right)f\left(x\right)\mathrm{d}k+\int_{k'-}^{k'+}\left(\frac{1}{2\pi}\infty\right)f\left(0,x\right)\mathrm{d}k+\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[\int_{k'-}^{k'+}\frac{1}{\mathrm{d}k}f\left(0,x\right)\mathrm{d}k+\int_{0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}-\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\right]\\
&\approx\lim_{k'\to 0}\left[f\left(0,x\right)+\int_{0}^{\infty}\left(\frac{1}{2\pi}0\right)f\left(k,x\right)\mathrm{d}k\right]\approx f\left(0,x\right)\\
&f\left(0,x\right)=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k=\int_{-\infty}^{\infty}\delta\left(k\right)f\left(k,x\right)\mathrm{d}k=\int_{-\infty}^{\infty}\delta\left(k-0\right)f\left(k,x\right)\mathrm{d}k
\end{aligned}$$

https://tex.stackexchange.com/questions/150138/how-can-i-create-a-polar-plot-on-a-cartesian-grid

Figure 43.6: $\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k$

$$\begin{aligned}
f\left(0,x\right)&=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x\right)f\left(k,x\right)\mathrm{d}k\\
&=\int_{-\infty}^{\infty}\delta\left(k-0\right)f\left(k,x\right)\mathrm{d}k
\end{aligned}$$

---

[13] p.18~20

$$\delta\left(x\right)=\begin{cases}0 & x\neq 0\\ \int_{-\infty}^{\infty}\delta\left(x\right)\mathrm{d}x=1 & \end{cases}$$

According the SIFTing property above,

$$\int_{-\infty}^{\infty}\delta\left(x-x'\right)f\left(x\right)\mathrm{d}x=f\left(x'\right)$$

And according to Fourier transform and inverse transform, Fourier analysis[48] with $2\pi$ in the powers,

$$f\left(x\right)=\int_{-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}xk}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}kx'}f\left(x'\right)\mathrm{d}x'\right)\mathrm{d}k$$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x k} \left( \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} k x'} f(x')\,\mathrm{d}x' \right) \mathrm{d}k \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x k} f(x')\,\mathrm{d}x' \left( \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} k x'} \right) \mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(x')\,\mathrm{d}x' \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x k} \mathrm{e}^{-2\pi \mathrm{i} k x'} \right) \mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(x')\,\mathrm{d}x' \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \right) \mathrm{d}k \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \right) f(x')\,\mathrm{d}x' \mathrm{d}k \\
&\overset{\text{Fubini}}{=} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \right) f(x')\,\mathrm{d}k \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \mathrm{d}k \right) f(x')\,\mathrm{d}x' \\
f(x) &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \mathrm{d}k \right) f(x')\,\mathrm{d}x'
\end{aligned}$$

comparing with SIFTing definition of Dirac delta function,

$$\int_{-\infty}^{\infty} \delta(x - x') f(x)\,\mathrm{d}x = f(x')$$

thus

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \mathrm{d}k = \delta(x - x')$$

i.e.

$$\delta(x - x') = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} k \left(x - x'\right)} \mathrm{d}k$$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk\right) f(x')\,dx' \\
&= \int_{-\infty}^{\infty} \delta(x-x') f(x')\,dx', \begin{cases} k' = 2\pi k & \Leftrightarrow k = \dfrac{k'}{2\pi} \\ x - x' = \varepsilon & \Leftrightarrow x' = x - \varepsilon \end{cases} \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\varepsilon k'} dk'\right) f(x')\,dx' = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{2\pi}\int_{-K}^{K} e^{i\varepsilon k'} dk'\right) f(x')\,dx', K > 0 \\
&= \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{2\pi}\left[\frac{e^{i\varepsilon k'}}{i\varepsilon}\right]_{k'=-K}^{K}\right) f(x')\,dx' = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{2\pi}\left[\frac{e^{i\varepsilon K} - e^{i\varepsilon(-K)}}{i\varepsilon}\right]\right) f(x')\,dx' \\
&= \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi\varepsilon}\left[\frac{e^{i\varepsilon K} - e^{-i\varepsilon K}}{2i}\right]\right) f(x')\,dx' = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi\varepsilon}\left[\sin(\varepsilon K)\right]\right) f(x')\,dx' \\
&= \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(\varepsilon K)}{\varepsilon}\right) f(x')\,dx' = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(K\varepsilon)}{\varepsilon}\right) f(x')\,dx', \begin{cases} x' - x = -\varepsilon & \Leftrightarrow \varepsilon = x - x' \\ & \Downarrow x = 0 \\ x' = -\varepsilon & \Leftrightarrow \varepsilon = -x' \\ Kx' = u & \Leftrightarrow x' = \dfrac{u}{K} \end{cases} \\
&\overset{x=0}{=} \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(K[-x'])}{-x'}\right) f(x')\,dx' = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{-\sin(Kx')}{-x'}\right) f(x')\,dx' \\
&= \int_{x'=-\infty}^{x'=\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(Kx')}{x'}\right) f(x')\,dx', \begin{cases} x' = -\varepsilon & \Leftrightarrow \varepsilon = -x' \\ Kx' = u & \Leftrightarrow x' = \dfrac{u}{K}, \wedge K > 0 \end{cases} \\
&= \int_{\frac{u}{K}=-\infty}^{\frac{u}{K}=\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(u)}{\frac{u}{K}}\right) f\left(\frac{u}{K}\right) d\frac{u}{K} = \int_{u=-\infty}^{u=\infty}\lim_{K\to\infty}\left(\frac{K}{\pi}\frac{\sin(u)}{u}\right) f\left(\frac{u}{K}\right)\frac{1}{K}du \\
&= \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{K}{\pi}\frac{\sin(u)}{u}\right) f\left(\frac{u}{K}\right)\frac{1}{K}du = \int_{-\infty}^{\infty}\lim_{K\to\infty}\left(\frac{1}{\pi}\frac{\sin(u)}{u}\right) f\left(\frac{u}{K}\right)du \\
&= \frac{1}{\pi}\lim_{K\to\infty}\int_{-\infty}^{\infty}\frac{\sin(u)}{u} f\left(\frac{u}{K}\right)du \overset{\text{MVT}}{=} \frac{1}{\pi}\lim_{K\to\infty} f\left(\frac{\xi}{K}\right)\int_{-\infty}^{\infty}\frac{\sin(u)}{u}du, \exists\xi \in (-\epsilon, +\epsilon), \epsilon \in \mathbb{R}_{>0} \\
&= \frac{1}{\pi} f(0)\int_{-\infty}^{\infty}\frac{\sin(u)}{u}du \overset{\text{Feynman method or residue method}}{=} \frac{1}{\pi} f(0)\,\pi = f(0) = [f(x)]_{x=0}
\end{aligned}$$

$\int_{-\infty}^{\infty}\frac{\sin(u)}{u}du = \pi$ see Feynman method of integration / integral technique[42.2]

---

According to Fourier transform and inverse transform, Fourier analysis[48] without $2\pi$ in the powers,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\cdot\omega}\left(\int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s)\,ds\right) d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}\left(\int_{-\infty}^{\infty} e^{-ikx'} f(x')\,dx'\right) dk$$

or

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\cdot\omega}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s)\,ds\right) d\omega$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixk}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'} f(x')\,dx'\right) dk$$

---

complex function if analytic always with better properties than real function

$$\left\{ \left\{ \delta\left(k\right) | k\in\mathbb{R} \right\} \middle| \begin{cases} \delta:\mathbb{R}\to\mathbb{C} \\ \int_{-\infty}^{\infty}\delta\left(k-0\right)f\left(k,x\right)\mathrm{d}k=f\left(0,x\right) \end{cases} \right\}$$
$$=\left\{ \left\{ K\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\left(\cdot\right)\mathrm{d}k \middle| k\in\mathbb{R} \right\}, \cdots \right\}$$

---

$$\delta\left(k\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\left(\cdot\right)\mathrm{d}x$$

$$\delta\left(x\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\left(\cdot\right)\mathrm{d}k$$

$$\delta\left(x'\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx'}\left(\cdot\right)\mathrm{d}k$$

---

$\psi\left(x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\psi\left(k\right)\mathrm{d}k$

$$\psi\left(k\right)=\frac{1}{\sqrt{2\pi}}\int\mathrm{e}^{-\mathrm{i}kx}\psi\left(x\right)\mathrm{d}x$$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\psi\left(k\right)\mathrm{d}k \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx'}\psi\left(x'\right)\mathrm{d}x'\right)\mathrm{d}k \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}kx'}\psi\left(x'\right)\mathrm{d}x'\right)\mathrm{d}k \\
\overset{\text{Fubini}}{=}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\psi\left(x'\right)\mathrm{d}k\right)\mathrm{d}x' \\
=&\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\psi\left(x'\right)\mathrm{d}k\right)\mathrm{d}x' \\
=&\int_{-\infty}^{\infty}\psi\left(x'\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\mathrm{d}k\right)\mathrm{d}x' \\
=&\int_{-\infty}^{\infty}\psi\left(x'\right)\delta\left(x-x'\right)\mathrm{d}x'=\int_{-\infty}^{\infty}\delta\left(x-x'\right)\psi\left(x'\right)\mathrm{d}x'=\psi\left(x\right)
\end{aligned}$$

$$\psi\left(x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\psi\left(k\right)\mathrm{d}k$$

---

$\int_{-\infty}^{\infty}\left|\psi\left(k\right)\right|^2\mathrm{d}k=\int_{-\infty}^{\infty}\left|\psi\left(x\right)\right|^2\mathrm{d}x$

$$\begin{cases} \psi\left(x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\psi\left(k\right)\mathrm{d}k \\ \psi\left(k\right)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\psi\left(x\right)\mathrm{d}x \end{cases}$$

$$\begin{aligned}
\psi^*\left(k\right)=\overline{\psi}\left(k\right)=&\overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\psi\left(x\right)\mathrm{d}x} \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\mathrm{e}^{-\mathrm{i}kx}\psi\left(x\right)}\mathrm{d}x \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\mathrm{e}^{-\mathrm{i}kx}}\,\overline{\psi\left(x\right)}\mathrm{d}x \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\psi^*\left(x\right)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |\psi(k)|^2 \,\mathrm{d}k = \int_{-\infty}^{\infty} \psi^*(k)\,\psi(k)\,\mathrm{d}k \\
=&\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\psi^*(x)\,\mathrm{d}x\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx'}\psi(x')\,\mathrm{d}x'\right)\mathrm{d}k \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\psi^*(x)\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx'}\psi(x')\,\mathrm{d}x'\right)\mathrm{d}k \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}kx'}\psi(x')\,\mathrm{d}x'\mathrm{d}x\mathrm{d}k \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\psi(x')\,\mathrm{d}x'\mathrm{d}x\mathrm{d}k \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\psi(x')\,\mathrm{d}x'\mathrm{d}x\mathrm{d}k \\
\overset{\text{Fubini}}{=}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\psi(x')\,\mathrm{d}k\mathrm{d}x'\mathrm{d}x \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x')\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\mathrm{d}k\mathrm{d}x'\mathrm{d}x \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x')\,\delta(x-x')\,\mathrm{d}x'\mathrm{d}x \\
=&\int_{-\infty}^{\infty} \psi^*(x)\int_{-\infty}^{\infty} \psi(x')\,\delta(x-x')\,\mathrm{d}x'\mathrm{d}x \\
=&\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} |\psi(k)|^2\,\mathrm{d}k = \int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x
$$

---

convolution

$\psi(x) = \int_{-\infty}^{\infty} \psi_1(y)\,\psi_2(x-y)\,\mathrm{d}y \Rightarrow \psi(k) = \sqrt{2\pi}\psi_1(k)\,\psi_2(k)$

$$
\psi(x) = \int_{-\infty}^{\infty} \psi_1(y)\,\psi_2(x-y)\,\mathrm{d}y
$$

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1y}\psi_1(k_1)\,\mathrm{d}k_1
$$

$$
\psi_2(x-y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_2(x-y)}\psi_2(k_2)\,\mathrm{d}k_2
$$

$$
\begin{cases}
\psi(x) = & \int_{-\infty}^{\infty} \psi_1(y)\,\psi_2(x-y)\,\mathrm{d}y \\
\psi_1(y) = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1y}\psi_1(k_1)\,\mathrm{d}k_1 \\
\psi_2(x-y) = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_2(x-y)}\psi_2(k_2)\,\mathrm{d}k_2
\end{cases}
$$

$$
\begin{aligned}
&\psi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\psi(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty} \psi_1(y)\,\psi_2(x-y)\,\mathrm{d}y\mathrm{d}x \\
=&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1y}\psi_1(k_1)\,\mathrm{d}k_1\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_2(x-y)}\psi_2(k_2)\,\mathrm{d}k_2\right)\mathrm{d}y\mathrm{d}x \\
=&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1y}\mathrm{e}^{-\mathrm{i}k_2y}\psi_1(k_1)\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k_2x}\psi_2(k_2)\,\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}y\mathrm{d}x \\
=&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(k_1-k_2)y}\psi_1(k_1)\,\mathrm{e}^{\mathrm{i}(k_2-k)x}\psi_2(k_2)\,\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}y\mathrm{d}x \\
\overset{\text{Fubini}}{=}&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(k_1-k_2)y}\psi_1(k_1)\,\mathrm{e}^{\mathrm{i}(k_2-k)x}\psi_2(k_2)\,\mathrm{d}y\mathrm{d}x\mathrm{d}k_1\mathrm{d}k_2 \\
=&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(k_1-k_2)y}\psi_1(k_1)\,\mathrm{d}y\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(k_2-k)x}\psi_2(k_2)\,\mathrm{d}x\right)\mathrm{d}k_1\mathrm{d}k_2
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_1-k_2)y}\psi_1\left(k_1\right)\mathrm{d}y\right)\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_2-k)x}\psi_2\left(k_2\right)\mathrm{d}x\right)\mathrm{d}k_1\mathrm{d}k_2\\
=&\sqrt{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_1-k_2)y}\psi_1\left(k_1\right)\mathrm{d}y\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_2-k)x}\psi_2\left(k_2\right)\mathrm{d}x\right]\mathrm{d}k_1\mathrm{d}k_2\\
=&\sqrt{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\psi_1\left(k_1\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_1-k_2)y}\mathrm{d}y\right]\left[\psi_2\left(k_2\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_2-k)x}\mathrm{d}x\right]\mathrm{d}k_1\mathrm{d}k_2\\
=&\sqrt{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\psi_1\left(k_1\right)\delta\left(k_1-k_2\right)\right]\left[\psi_2\left(k_2\right)\delta\left(k_2-k\right)\right]\mathrm{d}k_1\mathrm{d}k_2\\
\overset{\text{Fubini}}{=}&\sqrt{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\psi_1\left(k_1\right)\delta\left(k_1-k_2\right)\right]\left[\psi_2\left(k_2\right)\delta\left(k_2-k\right)\right]\mathrm{d}k_2\mathrm{d}k_1\\
=&\sqrt{2\pi}\int_{-\infty}^{\infty}\left[\psi_1\left(k_1\right)\right]\left[\int_{-\infty}^{\infty}\delta\left(k_1-k_2\right)\psi_2\left(k_2\right)\delta\left(k_2-k\right)\mathrm{d}k_2\right]\mathrm{d}k_1\\
=&\sqrt{2\pi}\int_{-\infty}^{\infty}\left[\psi_1\left(k_1\right)\right]\left[\delta\left(k_1-k\right)\psi_2\left(k\right)\right]\mathrm{d}k_1=\sqrt{2\pi}\psi_2\left(k\right)\int_{-\infty}^{\infty}\psi_1\left(k_1\right)\delta\left(k_1-k\right)\mathrm{d}k_1\\
=&\sqrt{2\pi}\psi_2\left(k\right)\psi_1\left(k\right)=\sqrt{2\pi}\psi_1\left(k\right)\psi_2\left(k\right)
\end{aligned}
$$

$$
\begin{gathered}
\begin{cases}
\psi\left(x\right)= & \int_{-\infty}^{\infty}\psi_1\left(y\right)\psi_2\left(x-y\right)\mathrm{d}y\\
\psi_1\left(y\right)= & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_1y}\psi_1\left(k_1\right)\mathrm{d}k_1\\
\psi_2\left(x-y\right)= & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_2(x-y)}\psi_2\left(k_2\right)\mathrm{d}k_2
\end{cases}\\
\Downarrow\\
\psi\left(k\right)=\sqrt{2\pi}\psi_1\left(k\right)\psi_2\left(k\right)
\end{gathered}
$$

### 43.2.2 3D delta function

https://www.youtube.com/watch?v=Y8y965ZAmQE

$$
\delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)=\delta\left(x-x'\right)\delta\left(y-y'\right)\delta\left(z-z'\right)
$$

Poisson equation

Feynman method of differentiation / derivative technique[42.1]

$$
\begin{aligned}
\Delta\left(\frac{1}{r}\right)=&\boldsymbol{\nabla}^2\left(\frac{1}{\sqrt{\boldsymbol{r}^2}}\right)=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}}\right)=\boldsymbol{\nabla}\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\\
=&\boldsymbol{\nabla}\cdot\left(\frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}},\frac{-y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}},\frac{-z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)=\boldsymbol{\nabla}\cdot\frac{-\boldsymbol{r}}{r^3}\\
=&\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}},\frac{-y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}},\frac{-z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)\\
=&\frac{\partial}{\partial x}\frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{\partial}{\partial y}\frac{-y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{\partial}{\partial z}\frac{-z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\\
=&\frac{+2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}+\frac{-x^2+2y^2-z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}+\frac{-x^2-y^2+2z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}=0
\end{aligned}
$$

$$
\Delta\left(\frac{1}{r}\right)=-4\pi\delta\left(\boldsymbol{r}\right)
$$

https://math.stackexchange.com/questions/3774483/derivatives-of-frac1r-and-dirac-delta-function

$$\begin{aligned}
\Delta\frac{1}{r(\epsilon)} =&\boldsymbol{\nabla}^2\frac{1}{\sqrt{\boldsymbol{r}^2+\epsilon^2}} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2+\epsilon^2}}\\
=&\boldsymbol{\nabla}\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2+\epsilon^2}}\right)\\
=&\boldsymbol{\nabla}\cdot\left(\frac{-x}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}},\frac{-y}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}},\frac{-z}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}}\right)\\
=&\frac{\partial}{\partial x}\frac{-x}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}}+\frac{\partial}{\partial y}\frac{-y}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}}+\frac{\partial}{\partial z}\frac{-z}{(x^2+y^2+z^2+\epsilon^2)^{\frac{3}{2}}}\\
=&\frac{+2x^2-y^2-z^2-\epsilon^2}{(x^2+y^2+z^2+\epsilon^2)^{\frac{5}{2}}}+\frac{-x^2+2y^2-z^2-\epsilon^2}{(x^2+y^2+z^2+\epsilon^2)^{\frac{5}{2}}}+\frac{-x^2-y^2+2z^2-\epsilon^2}{(x^2+y^2+z^2+\epsilon^2)^{\frac{5}{2}}}\\
=&\frac{-3\epsilon^2}{(x^2+y^2+z^2+\epsilon^2)^{\frac{5}{2}}} = \frac{-3\epsilon^2}{(\boldsymbol{r}^2+\epsilon^2)^{\frac{5}{2}}}
\end{aligned}$$

$$\Delta\left(\frac{1}{r}\right) = \boldsymbol{\nabla}^2\left(\frac{1}{r}\right) = \boldsymbol{\nabla}^2\frac{1}{|\boldsymbol{r}|} = \boldsymbol{\nabla}^2\frac{1}{|\boldsymbol{r}-\boldsymbol{0}|} = \Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{0}|} = \Delta(\boldsymbol{r}-\boldsymbol{0}) = \Delta(\boldsymbol{r})$$

$$\Delta\frac{1}{r(\epsilon)} = \Delta_\epsilon(r) = \Delta_\epsilon(\boldsymbol{r}) = \Delta_\epsilon(\boldsymbol{r}-\boldsymbol{0})$$

$$\lim_{\epsilon\to 0}\Delta_\epsilon(\boldsymbol{r}) = \lim_{\epsilon\to 0}\Delta\frac{1}{r(\epsilon)} = \Delta\left(\frac{1}{r}\right) = \Delta(\boldsymbol{r})$$

$$K_3\int_{\mathbb{R}^3} f(\boldsymbol{r})\,\Delta(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \lim_{\epsilon\to 0}K_3\int_{\mathbb{R}^3} f(\boldsymbol{r})\,\Delta_\epsilon(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \lim_{\epsilon\to 0}K_3\int_{\mathbb{R}^3} f(\boldsymbol{r})\,\Delta_\epsilon(\boldsymbol{r}-\boldsymbol{0})\,\mathrm{d}\boldsymbol{r} = f(\boldsymbol{0})$$

$$K_1\int_{\mathbb{R}^1} f(x)\,\delta(x)\,\mathrm{d}x = \lim_{n\to\infty}K_1\int_{\mathbb{R}^1} f(x)\,\delta_n(x)\,\mathrm{d}x = K_1\int_{\mathbb{R}^1} f(x)\,\delta_n(x-0)\,\mathrm{d}x = f(0)$$

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(\boldsymbol{r})\,\Delta_\epsilon(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \int_{\mathbb{R}^3} f(\boldsymbol{r})\,\frac{-3\epsilon^2}{(\boldsymbol{r}^2+\epsilon^2)^{\frac{5}{2}}}\mathrm{d}\boldsymbol{r},\ \begin{cases}\boldsymbol{r} = r\boldsymbol{n}_{\mathbb{S}}\\ \boldsymbol{n}_{\mathbb{S}} = \begin{pmatrix}\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}\end{cases}\\
=&\int_0^\infty\int_{\mathbb{S}^2} f(r\boldsymbol{n}_{\mathbb{S}})\,\frac{-3\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}}r^2\sigma(\mathrm{d}\boldsymbol{n}_S)\,\mathrm{d}r,\ \begin{cases}\mathbb{S}^2 = \text{unit sphere centered at the origin}\\ \sigma(\mathrm{d}\boldsymbol{n}_S)\ \text{ is the surface measure of }\mathbb{S}^2\\ r^2\sigma(\mathrm{d}\boldsymbol{n}_S) = r^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\\ r = \epsilon s\end{cases}\\
=&\int_0^\infty\int_{\mathbb{S}^2} f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{-3\epsilon^2}{\left((\epsilon s)^2+\epsilon^2\right)^{\frac{5}{2}}}(\epsilon s)^2\,\sigma(\mathrm{d}\boldsymbol{n}_S)\,\mathrm{d}\epsilon s\\
=&\int_0^\infty\int_{\mathbb{S}^2} f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{-3\epsilon^5}{\epsilon^5(s^2+1)^{\frac{5}{2}}}s^2\sigma(\mathrm{d}\boldsymbol{n}_S)\,\mathrm{d}s = -\int_0^\infty\int_{\mathbb{S}^2} f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{3s^2}{(s^2+1)^{\frac{5}{2}}}\sigma(\mathrm{d}\boldsymbol{n}_{\mathbb{S}})\,\mathrm{d}s\\
=&-\int_0^\infty\int_0^{2\pi}\int_0^{2\pi} f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{3s^2}{(s^2+1)^{\frac{5}{2}}}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}s = -\int_0^\infty 4\pi f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{3s^2}{(s^2+1)^{\frac{5}{2}}}\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} f(\boldsymbol{r})\,\Delta_\epsilon(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\\
=&\lim_{\epsilon\to 0}\left[-\int_0^\infty 4\pi f(\epsilon s\boldsymbol{n}_{\mathbb{S}})\,\frac{3s^2}{(s^2+1)^{\frac{5}{2}}}\mathrm{d}s\right]\\
=&-4\pi f(\boldsymbol{0})\int_0^\infty\frac{3s^2}{(s^2+1)^{\frac{5}{2}}}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\int \frac{3s^2}{\left(s^2+1\right)^{\frac{5}{2}}}\mathrm{d}s, s=\tan u\\
=&\int \frac{3\tan^2 u}{\left(\tan^2 u+1\right)^{\frac{5}{2}}}\mathrm{d}\tan u=\int \frac{3\tan^2 u}{\left(\tan^2 u+1\right)^{\frac{5}{2}}}\sec^2 u\mathrm{d}u\\
=&\int \frac{3\tan^2 u}{\left(\sec^2 u\right)^{\frac{5}{2}}}\sec^2 u\mathrm{d}u=\int \frac{3\tan^2 u}{\sec^5 u}\sec^2 u\mathrm{d}u=\int \frac{3\tan^2 u}{\sec^3 u}\mathrm{d}u\\
=&\int \frac{3\left(\frac{\sin u}{\cos u}\right)^2}{\left(\frac{1}{\cos u}\right)^3}\mathrm{d}u=3\int \sin^2 u\cos u\mathrm{d}u=3\int \sin^2 u\mathrm{d}\sin u=3\frac{\sin^3 u}{3}+C\\
=&\sin^3 u+C=\left(\frac{s}{\sqrt{s^2+1}}\right)^3+C=\frac{s^3}{\left(s^2+1\right)^{\frac{3}{2}}}+C\overset{s\geq 0}{=}\frac{1}{\left(1+s^{-2}\right)^{\frac{3}{2}}}+C
\end{aligned}
$$

$$
\int_0^{\infty}\frac{3s^2}{\left(s^2+1\right)^{\frac{5}{2}}}\mathrm{d}s=\left[\frac{s^3}{\left(s^2+1\right)^{\frac{3}{2}}}\right]_0^{\infty}=\left[\frac{1}{\left(1+s^{-2}\right)^{\frac{3}{2}}}\right]_{s=\infty}-\left[\frac{s^3}{\left(s^2+1\right)^{\frac{3}{2}}}\right]_{s=0}=1-0=1
$$

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} f\left(\boldsymbol{r}\right)\Delta_{\epsilon}\left(\boldsymbol{r}\right)\mathrm{d}\boldsymbol{r}\\
=&\lim_{\epsilon\to 0}\left[-\int_0^{\infty}4\pi f\left(\epsilon s\boldsymbol{n}_{\mathbb{S}}\right)\frac{3s^2}{\left(s^2+1\right)^{\frac{5}{2}}}\mathrm{d}s\right]\\
=&-4\pi f\left(\boldsymbol{0}\right)\int_0^{\infty}\frac{3s^2}{\left(s^2+1\right)^{\frac{5}{2}}}\mathrm{d}s=-4\pi f\left(\boldsymbol{0}\right)1=-4\pi f\left(\boldsymbol{0}\right)
\end{aligned}
$$

$$
\int_{\mathbb{R}^3} f\left(\boldsymbol{r}\right)\Delta\left(\boldsymbol{r}\right)\mathrm{d}\boldsymbol{r}=\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} f\left(\boldsymbol{r}\right)\Delta_{\epsilon}\left(\boldsymbol{r}\right)\mathrm{d}\boldsymbol{r}=-4\pi f\left(\boldsymbol{0}\right)=-4\pi\int_{\mathbb{R}^3} f\left(\boldsymbol{r}\right)\delta\left(\boldsymbol{r}\right)\mathrm{d}\boldsymbol{r}
$$

$$
\Delta\left(\frac{1}{r}\right)=\Delta\left(\boldsymbol{r}\right)=-4\pi\delta\left(\boldsymbol{r}\right)
$$

Big Delta reciprocal r is minus 4 pi delta r.

□

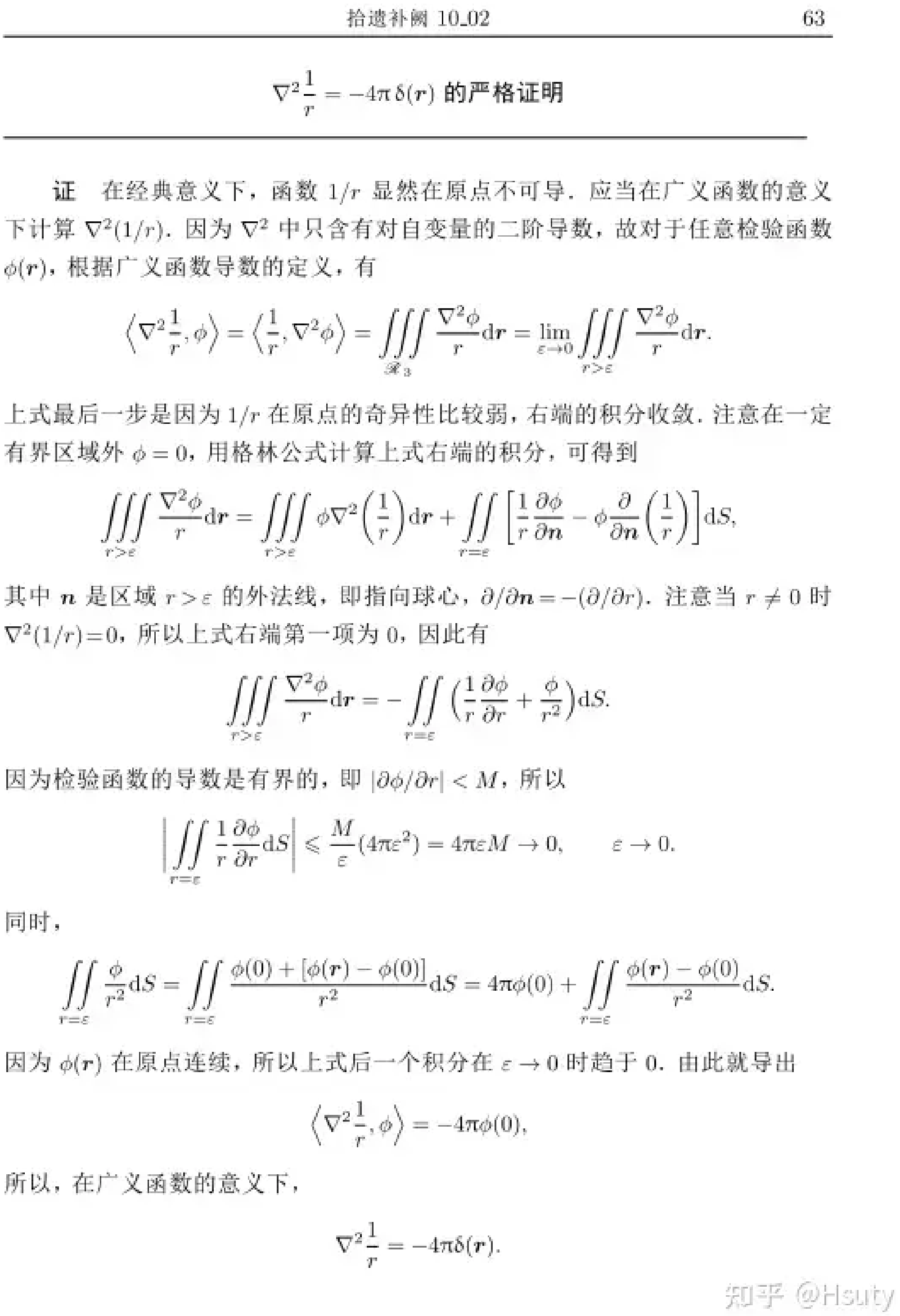
---

$$
\Delta\left(\frac{1}{r}\right)=-4\pi\delta\left(\boldsymbol{r}\right)
$$

---

Poisson equation rigorous proof by general function and Green theorem

https://zhuanlan.zhihu.com/p/154094937



拾遗补阙 10.02 63

$$\nabla^2\frac{1}{r}=-4\pi\,\delta(\boldsymbol{r})$$ 的严格证明

**证** 在经典意义下，函数 $1/r$ 显然在原点不可导．应当在广义函数的意义下计算 $\nabla^2(1/r)$．因为 $\nabla^2$ 中只含有对自变量的二阶导数，故对于任意检验函数 $\phi(\boldsymbol{r})$，根据广义函数导数的定义，有

$$\left\langle\nabla^2\frac{1}{r},\phi\right\rangle=\left\langle\frac{1}{r},\nabla^2\phi\right\rangle=\iiint\limits_{\mathscr{R}_3}\frac{\nabla^2\phi}{r}\mathrm{d}\boldsymbol{r}=\lim_{\varepsilon\to0}\iiint\limits_{r>\varepsilon}\frac{\nabla^2\phi}{r}\mathrm{d}\boldsymbol{r}.$$

上式最后一步是因为 $1/r$ 在原点的奇异性比较弱，右端的积分收敛．注意在一定有界区域外 $\phi=0$，用格林公式计算上式右端的积分，可得到

$$\iiint\limits_{r>\varepsilon}\frac{\nabla^2\phi}{r}\mathrm{d}\boldsymbol{r}=\iiint\limits_{r>\varepsilon}\phi\nabla^2\left(\frac{1}{r}\right)\mathrm{d}\boldsymbol{r}+\iint\limits_{r=\varepsilon}\left[\frac{1}{r}\frac{\partial\phi}{\partial\boldsymbol{n}}-\phi\frac{\partial}{\partial\boldsymbol{n}}\left(\frac{1}{r}\right)\right]\mathrm{d}S,$$

其中 $\boldsymbol{n}$ 是区域 $r>\varepsilon$ 的外法线，即指向球心，$\partial/\partial\boldsymbol{n}=-(\partial/\partial r)$．注意当 $r\neq0$ 时 $\nabla^2(1/r)=0$，所以上式右端第一项为 0，因此有

$$\iiint\limits_{r>\varepsilon}\frac{\nabla^2\phi}{r}\mathrm{d}\boldsymbol{r}=-\iint\limits_{r=\varepsilon}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\phi}{r^2}\right)\mathrm{d}S.$$

因为检验函数的导数是有界的，即 $|\partial\phi/\partial r|<M$，所以

$$\left|\iint\limits_{r=\varepsilon}\frac{1}{r}\frac{\partial\phi}{\partial r}\mathrm{d}S\right|\leqslant\frac{M}{\varepsilon}(4\pi\varepsilon^2)=4\pi\varepsilon M\to0,\qquad\varepsilon\to0.$$

同时，

$$\iint\limits_{r=\varepsilon}\frac{\phi}{r^2}\mathrm{d}S=\iint\limits_{r=\varepsilon}\frac{\phi(0)+[\phi(\boldsymbol{r})-\phi(0)]}{r^2}\mathrm{d}S=4\pi\phi(0)+\iint\limits_{r=\varepsilon}\frac{\phi(\boldsymbol{r})-\phi(0)}{r^2}\mathrm{d}S.$$

因为 $\phi(\boldsymbol{r})$ 在原点连续，所以上式后一个积分在 $\varepsilon\to0$ 时趋于 0．由此就导出

$$\left\langle\nabla^2\frac{1}{r},\phi\right\rangle=-4\pi\phi(0),$$

所以，在广义函数的意义下，

$$\nabla^2\frac{1}{r}=-4\pi\delta(\boldsymbol{r}).$$



Figure 43.7: Poisson equation rigorous proof by general function and Green theorem

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https://math.stackexchange.com/questions/368155/where-does-the-relation-nabla21-r-4-pi-delta3-bf-r-between-laplacian

---

$$\left\{\{\delta(k)|k\in\mathbb{R}\}\left|\begin{cases}\delta:\mathbb{R}\to\mathbb{C}\\\int_{-\infty}^{\infty}\delta(k-0)\,f(k,x)\,\mathrm{d}k=f(0,x)\end{cases}\right.\right\}$$
$$=\left\{\left\{K\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\,(\cdot)\,\mathrm{d}k\middle|k\in\mathbb{R}\right\},\cdots\right\}$$

[32.1.3]

$$\left\{\left\{\delta\left(\boldsymbol{k}\right)\middle|\boldsymbol{k}\in\mathbb{R}^3\right\}\middle|\begin{cases}\delta:\mathbb{R}^3\to\mathbb{H}\\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(\boldsymbol{k}-\boldsymbol{0}\right)f\left(\boldsymbol{k},\boldsymbol{r}\right)\mathrm{d}^3\boldsymbol{r}=f\left(\boldsymbol{0},\boldsymbol{r}\right)\end{cases}\right\}$$
$$=\left\{\left\{K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\left(\cdot\right)\mathrm{d}^3\boldsymbol{k}\middle|\boldsymbol{k}\in\mathbb{R}^3\right\},\cdots\right\}$$

---

Columb law

$$F_{\mathrm{e}}=k\frac{Qq}{r^2}$$

$$\boldsymbol{F}_{\mathrm{e}}=k\frac{Qq}{r^2}\hat{\boldsymbol{r}}$$

electric field from point electric charge

$$\begin{aligned}\boldsymbol{E}\left(\boldsymbol{r}\right)=\boldsymbol{E}\left(r\right)=\boldsymbol{E}=&k\frac{q}{r^2}\hat{\boldsymbol{r}}=k_{\mathrm{e}}\frac{q}{\left|\boldsymbol{r}\right|^2}\frac{\boldsymbol{r}}{\left|\boldsymbol{r}\right|}=k_{\mathrm{e}}q\frac{\boldsymbol{r}}{\left|\boldsymbol{r}\right|^3}=k_{\mathrm{e}}q\frac{\boldsymbol{r}}{\left(\boldsymbol{r}^2\right)^{\frac{3}{2}}}\\ =&k_{\mathrm{e}}q\frac{\boldsymbol{r}}{\left(\boldsymbol{r}^2\right)^{\frac{3}{2}}}=k_{\mathrm{e}}q\left(-\boldsymbol{\nabla}\frac{1}{\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}}\right)=-k_{\mathrm{e}}q\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}\end{aligned}$$

$$\boldsymbol{E}\left(\boldsymbol{r}\right)=-k_{\mathrm{e}}q\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}$$

electric field from tiny point electric charge

$$\begin{aligned}\mathrm{d}\boldsymbol{E}\left(\boldsymbol{r}\right)=\mathrm{d}\boldsymbol{E}\left(r\right)=\mathrm{d}\boldsymbol{E}=&k\frac{\mathrm{d}q}{r^2}\hat{\boldsymbol{r}}=k\frac{\mathrm{d}q\left(r\right)}{r^2}\hat{\boldsymbol{r}}=k_{\mathrm{e}}\mathrm{d}q\left(\boldsymbol{r}\right)\frac{\boldsymbol{r}}{\left(\boldsymbol{r}^2\right)^{\frac{3}{2}}}\\ =&k_{\mathrm{e}}\mathrm{d}q\left(\boldsymbol{r}\right)\frac{\boldsymbol{r}}{\left(\boldsymbol{r}^2\right)^{\frac{3}{2}}}=k_{\mathrm{e}}\mathrm{d}q\left(\boldsymbol{r}\right)\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}\end{aligned}$$

$$\mathrm{d}\boldsymbol{E}\left(\boldsymbol{r}\right)=-k_{\mathrm{e}}\mathrm{d}q\left(\boldsymbol{r}\right)\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}$$

$$\mathrm{d}q=\rho\mathrm{d}V\Leftrightarrow\rho=\frac{\mathrm{d}q}{\mathrm{d}V}$$

$$\mathrm{d}q\left(\boldsymbol{r}\right)=\rho\left(\boldsymbol{r}\right)\mathrm{d}V\Leftrightarrow\rho\left(\boldsymbol{r}\right)=\frac{\mathrm{d}q\left(\boldsymbol{r}\right)}{\mathrm{d}V}$$

$$q=\int_q\mathrm{d}q\left(\boldsymbol{r}\right)=\int_V\rho\left(\boldsymbol{r}\right)\mathrm{d}V=\iiint_V\rho\left(\boldsymbol{r}\right)\mathrm{d}^3\boldsymbol{r}$$

$$\begin{aligned}\boldsymbol{E}=&\iiint_V\boldsymbol{E}\left(\boldsymbol{r}\right)\mathrm{d}^3\boldsymbol{r}=\iiint_V\left(-k_{\mathrm{e}}\rho\left(\boldsymbol{r}\right)\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}\right)\mathrm{d}^3\boldsymbol{r}\\ =&-k_{\mathrm{e}}\iiint_V\left(\rho\left(\boldsymbol{r}\right)\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}\right\|}\right)\mathrm{d}^3\boldsymbol{r}\\ =&-k_{\mathrm{e}}\iiint_V\left(\rho\left(\boldsymbol{r}\right)\boldsymbol{\nabla}\frac{1}{\left\|\boldsymbol{r}-\boldsymbol{0}\right\|}\right)\mathrm{d}^3\boldsymbol{r}=\boldsymbol{E}\left(\boldsymbol{0}\right)\end{aligned}$$

$$
\begin{aligned}
&\Downarrow \begin{cases} Q \text{ at } \boldsymbol{r} \\ \mathrm{d}q \text{ or } \rho \text{ at } \boldsymbol{r}',\ \text{totally } V \end{cases} \\
\boldsymbol{E}(\boldsymbol{r}) =& \iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\boldsymbol{\nabla}\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' = \iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}}\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' \\
\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\boldsymbol{E}(\boldsymbol{r}) =& \boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}}\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' = \iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' \\
=& \iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}}^2\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' = \iiint_V \left(-k_{\mathrm{e}}\rho(\boldsymbol{r}')\Delta_{\boldsymbol{r}}\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)\mathrm{d}^3\boldsymbol{r}' \\
=& \iiint_V f(\boldsymbol{r}')\Delta(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}', f(\boldsymbol{r}') = -k_{\mathrm{e}}\rho(\boldsymbol{r}') \\
=& \iiint_V f(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}', f(\boldsymbol{r}') = -k_{\mathrm{e}}\rho(\boldsymbol{r}') \\
=& -4\pi f(\boldsymbol{r}'=\boldsymbol{r}) = -4\pi\left[-k_{\mathrm{e}}\rho(\boldsymbol{r}'=\boldsymbol{r})\right] \\
=& -4\pi f(\boldsymbol{r}) = -4\pi\left[-k_{\mathrm{e}}\rho(\boldsymbol{r})\right] = 4\pi k_{\mathrm{e}}\rho(\boldsymbol{r}) \\
\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r}) =& 4\pi k_{\mathrm{e}}\rho(\boldsymbol{r})
\end{aligned}
$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r}) = 4\pi k_{\mathrm{e}}\rho(\boldsymbol{r})$$

---

gravity

https://zhuanlan.zhihu.com/p/695160620

---

https://www.bilibili.com/video/BV1qu411578c/?t=7m26s

https://en.wikipedia.org/wiki/Helmholtz_equation

Helmholtz equation

$$\left(\Delta + k^2\right)\frac{\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{r} = -4\pi\delta(\boldsymbol{r})$$

$$\left(\boldsymbol{\nabla}_{\boldsymbol{r}}^2 + \boldsymbol{k}^2\right)\frac{\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{\|\boldsymbol{r}\|} = -4\pi\delta(\boldsymbol{r})$$

### 43.2.3 quantum state

https://www.bilibili.com/video/BV1qu411578c/?t=7m45s

https://www.bilibili.com/video/BV1xu4y1S7W2

### 43.2.4 rigorous definition of delta function

$$\int_{-\infty}^{\infty}\delta(x-0)\cdot f(x)\,\mathrm{d}x = f(0)$$

$$\int \delta(x) f(x)\,\mathrm{d}x = f(0)$$

$$\int \delta(x)(\cdot)\,\mathrm{d}x : f(x) \to f(0)$$

$$f(x) \overset{\int\delta(x)(\cdot)\mathrm{d}x}{\to} f(0)$$

$f$ must be with good properties

$$\text{continuous linear functional}\begin{cases}\text{function-like} & \eta_\varphi(f)=\int\varphi(x)\,f(x)\,\mathrm{d}x\\ \text{non-function-like, e.g.} & \delta_{x'}(f)=f(x')\end{cases}$$

general function with “formally” integral notation

$$\delta_{x'}(f)=\int\delta(x-x')\,f(x)\,\mathrm{d}x=f(x')$$

https://www.bilibili.com/video/BV1qu411578c/?t=11m36s

Quantum mechanics is founded on Hilbert space; Hoever, Dirac delta function is not a linear functional on Hilbert space.?

One of the reason is that we cannot find 1-1 of Dirac delta function, but many-to-one or one-to-many functions.

量子力學本不需 Dirac delta function, 其 卻仍大行其道, 實 廣義函數 為其負重前行.

狄拉克 函數 不是 希爾伯特空間 的 線性泛函. Dirac delta function is not a linear functional on Hilbert space.

狄拉克 括號 卻是 希爾伯特空間 的 向量表示. Dirac bra-ket brackets are vector representation on Hilbert space.

## 43.3 Fourier expansion or Fourier series

vector space definition[38.2]

Fig: 38.1

linear space of function[41.1]

https://www.bilibili.com/video/BV1PX4y167RS

https://www.bilibili.com/video/BV1m24y1A74K

https://www.bilibili.com/video/BV1m24y1A74K/?=3m2s

### 43.3.1 Taylor vs. Fourier

$$f(x)=x_0x^0+x_1x^1+x_2x^2+\cdots=\sum_{k=0}^{\infty}x_kx^k,\,x_k=\frac{f^{(k)}(0)}{k!}$$

$$f(r)=r_0r^0+r_1r^1+r_2r^2+\cdots=\sum_{k=0}^{\infty}r_kr^k,\,r_k=\frac{f^{(k)}(0)}{k!}$$

$$f(\theta)=a_0\cos(0\theta)+a_1\cos(1\theta)+a_2\cos(2\theta)+\cdots=\sum_{k=0}^{\infty}a_k\cos(k\theta)$$

$$a_k=\int_{-\infty}^{\infty}\cos(k\theta)\,f(\theta)\,\mathrm{d}\theta$$

$$\begin{aligned}f(\theta)&=a_0\cos(0\theta)+a_1\cos(1\theta)+a_2\cos(2\theta)+\cdots=\sum_{k=0}^{\infty}a_k\cos(k\theta)\\&=a_0\Re\left[\mathrm{e}^{\mathrm{i}0\theta}\right]+a_1\Re\left[\mathrm{e}^{\mathrm{i}1\theta}\right]+a_2\Re\left[\mathrm{e}^{\mathrm{i}2\theta}\right]+\cdots=\sum_{k=0}^{\infty}a_k\Re\left[\mathrm{e}^{\mathrm{i}k\theta}\right]=\sum_{k=0}^{\infty}a_k\Re\left[\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k\right]\\&=\Re\left[a_0\mathrm{e}^{\mathrm{i}0\theta}+a_1\mathrm{e}^{\mathrm{i}1\theta}+a_2\mathrm{e}^{\mathrm{i}2\theta}+\cdots\right]=\Re\left[\sum_{k=0}^{\infty}a_k\mathrm{e}^{\mathrm{i}k\theta}\right]=\Re\left[\sum_{k=0}^{\infty}a_k\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k\right]\end{aligned}$$

$$f(x)=x_0x^0+x_1x^1+x_2x^2+\cdots=\sum_{k=0}^{\infty}x_kx^k$$

$$f(z)=z_0z^0+z_1z^1+z_2z^2+\cdots=\sum_{k=0}^{\infty}z_kz^k$$

$$z = r\mathrm{e}^{\mathrm{i}\theta}$$

$$\begin{aligned}
f(z) =& z_0 z^0 + z_1 z^1 + z_2 z^2 + \cdots = \sum_{k=0}^{\infty} z_k z^k \\
f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) =& z_0\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^0 + z_1\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^1 + z_2\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^2 + \cdots = \sum_{k=0}^{\infty} z_k\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^k \\
= f(r) =& z_0\left(\mathrm{e}^{\mathrm{i}\theta}\right)^0 r^0 + z_1\left(\mathrm{e}^{\mathrm{i}\theta}\right)^1 r^1 + z_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2 r^2 + \cdots = \sum_{k=0}^{\infty} z_k\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k r^k \\
=& r_0 r^0 + r_1 r^1 + r_2 r^2 + \cdots = \sum_{k=0}^{\infty} r_k r^k, r_k = z_k\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k \\
= f\left(\mathrm{e}^{\mathrm{i}\theta}\right) =& z_0 r^0\left(\mathrm{e}^{\mathrm{i}\theta}\right)^0 + z_1 r^1\left(\mathrm{e}^{\mathrm{i}\theta}\right)^1 + z_2 r^2\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2 + \cdots = \sum_{k=0}^{\infty} z_k r^k\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k \\
=& \left(\mathrm{e}^{\mathrm{i}\theta}\right)_0\left(\mathrm{e}^{\mathrm{i}\theta}\right)^0 + \left(\mathrm{e}^{\mathrm{i}\theta}\right)_1\left(\mathrm{e}^{\mathrm{i}\theta}\right)^1 + \left(\mathrm{e}^{\mathrm{i}\theta}\right)_2\left(\mathrm{e}^{\mathrm{i}\theta}\right)^2 + \cdots = \sum_{k=0}^{\infty}\left(\mathrm{e}^{\mathrm{i}\theta}\right)_k\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k \\
&, \left(\mathrm{e}^{\mathrm{i}\theta}\right)_k = z_k r^k
\end{aligned}$$

https://www.bilibili.com/video/BV1nV411F7es

## 43.4 Hilbert space construction

https://www.zhihu.com/question/19967778/answer/2572975608

https://www.bilibili.com/video/BV1cV411T7hb

**Definition 43.1.** sequence limit

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}\left(n > N \Rightarrow |a_n - a| < \epsilon\right)$$

$$x_1, x_2, \cdots, x_n, x_{n+1}, \cdots$$

- to find convergence
  - monotone convergence theorem
  - split into known sequence
  - fraction: Stolz theorem
  - sequence itself
    * ratio $\dfrac{x_m}{x_n}$, but not good for some terms being zero
    * difference $x_m - x_n$

**Theorem 43.4.** *Cauchy convergence theorem*

$$\begin{gathered}
\{x_n\}_{n\in\mathbb{N}} \text{ is convergent} \\
\Updownarrow \\
\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \in \mathbb{N}\left(m, n > N \Rightarrow |x_m - x_n| < \epsilon\right) \Leftrightarrow \{x_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence}
\end{gathered}$$

**Definition 43.2.** Cauchy sequence

$$\begin{gathered}
\{x_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence} \\
\Updownarrow \\
\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \in \mathbb{N}\left(m, n > N \Rightarrow |x_m - x_n| < \epsilon\right)
\end{gathered}$$

$$\text{convergence}\begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

$$\text{convergence}\begin{cases}|x_n - x| < \epsilon & \text{convergent sequence}\\ \Downarrow \text{ easy proof} & \Uparrow \text{ hard proof, weak condition needs all real number as supporting base}\\ |x_m - x_n| < \epsilon & \text{Cauchy sequence}\end{cases}$$

$$\text{convergence}\begin{cases}|x_n - x| < \epsilon & \text{convergent sequence}\\ \Downarrow & \Uparrow x_n, x_m, x \in \mathbb{R} \text{ with real completeness}\\ |x_m - x_n| < \epsilon & \text{Cauchy sequence}\end{cases}$$

### 43.4.1 inner product space

vector space definition[38.2]

Fig: 38.1

Figure 43.8: inner product space construction

$$\langle\cdot|\cdot\rangle : \begin{cases}\langle f|g+h\rangle = \langle f|g\rangle + \langle f|h\rangle\\ \langle f|cg\rangle = c\langle f|g\rangle\\ \langle f|g\rangle = \overline{\langle g|f\rangle} = \langle g|f\rangle^*\\ \langle f|f\rangle \underset{f=0\Leftrightarrow\langle f|f\rangle=0}{\geq} 0\end{cases}$$

definition of distance

**Definition 43.3.** distance

$$d(f,g) = d\langle f|g\rangle = \sqrt{\langle f-g|f-g\rangle}$$

For real number,

$$\begin{cases}d(f,g) = d\langle f|g\rangle = \sqrt{\langle f-g|f-g\rangle}\\ f,g \in \mathbb{R}\\ \langle f|g\rangle = fg = f\cdot g\end{cases}$$
$$\Downarrow$$
$$d\langle f|g\rangle = \sqrt{(f-g)^2} = |f-g|$$

For $n$-dimensional vector

$$\begin{cases}d(f,g) = d\langle f|g\rangle = \sqrt{\langle f-g|f-g\rangle}\\ f,g \in F^n\\ \langle f|g\rangle = \sum_{k=1}^{n} f_k g_k\end{cases}$$
$$\Downarrow$$
$$d\langle f|g\rangle = \sqrt{\sum_{k=1}^{n}(f_k - g_k)^2}$$

For continuous complex function on $[a,b]$,

$$\begin{cases}d(f,g) = d\langle f|g\rangle = \sqrt{\langle f-g|f-g\rangle}\\ f,g \in C_{\mathbb{C}}([a,b]) & \text{continuous complex function}\\ \langle f|g\rangle = \int_a^b \overline{f(x)}g(x)\,\mathrm{d}x\end{cases}$$
$$\Downarrow$$
$$\underline{d\langle f|g\rangle = \sqrt{\int_a^b \overline{[f-g](x)}\,[f-g](x)\,\mathrm{d}x} = \sqrt{\int_a^b |[f-g](x)|^2\,\mathrm{d}x}}$$

For real number sequence,

$$\begin{cases} d\left(f,g\right) = d\left\langle f|g\right\rangle = \sqrt{\left\langle f-g|f-g\right\rangle} \\ f,g \in \mathbb{R} \\ \left\langle f|g\right\rangle = fg = f\cdot g \end{cases}$$
$$\Downarrow$$
$$d\left\langle f|g\right\rangle = \sqrt{\left(f-g\right)^2} = |f-g|$$

$$\text{convergence}\begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

$$\text{convergence}\begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ \Downarrow & \Uparrow x_n, x_m, x \in \mathbb{R} \text{ with real completeness} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

For complex function sequence,

$$\begin{cases} d\left(f,g\right) = d\left\langle f|g\right\rangle = \sqrt{\left\langle f-g|f-g\right\rangle} \\ f,g \in C_{\mathbb{C}}\left([a,b]\right) & \text{continuous complex function} \\ \left\langle f|g\right\rangle = \int_a^b \overline{f\left(x\right)} g\left(x\right) \mathrm{d}x \end{cases}$$
$$\Downarrow$$
$$d\left\langle f|g\right\rangle = \sqrt{\int_a^b \overline{\left[f-g\right]\left(x\right)} \left[f-g\right]\left(x\right) \mathrm{d}x} = \sqrt{\int_a^b \left|\left[f-g\right]\left(x\right)\right|^2 \mathrm{d}x}$$

$$\text{function convergence}\begin{cases} d\left\langle f_n|f\right\rangle < \epsilon & \text{convergent function sequence} \\ d\left\langle f_m|f_n\right\rangle < \epsilon & \text{Cauchy function sequence} \end{cases}$$

$$\text{function convergence}\begin{cases} d\left\langle f_n|f\right\rangle < \epsilon & \text{convergent function sequence} \\ \Downarrow & \not\Uparrow \text{ not always continuous} \\ d\left\langle f_m|f_n\right\rangle < \epsilon & \text{Cauchy function sequence} \end{cases}$$

Figure 43.9: Hilbert space construction

https://www.zhihu.com/question/19967778/answer/2572975608

| 度量 | | 范数 | | 内积 | |
|---|---|---|---|---|---|
| d(x,y)>=0 | 非负性 | $\|\|x\|\| \geq 0$ | 非负性 | $\langle x, x\rangle \geq 0$ | 正定性 |
| d(x,y)=0, only if x=y | 非退化性 | $\|\|x\|\| = 0$, only if x=0 | 非退化性 | <x,x>=0, only if x=0 | 非退化性 |
| d(x,y)=d(y,x) | 对称性 | | | $\langle x, y\rangle = \overline{\langle y, x\rangle}$ | 共轭对称性 |
| | | $\|\|ax\|\| = \|a\| \cdot \|\|x\|\|$ | 齐次性 | $\langle ax + by, z\rangle$ $= a\langle x, z\rangle + b\langle y, z\rangle$ | 第一变元线性 |
| | | | | $\langle z, ax + by\rangle$ $= \bar{a}\langle z, x\rangle + \bar{b}\langle z, y\rangle$ | 第二变元共轭线性 |
| d(x,y)<=d(x,z)+d(z,y) | 三角不等式 | $\|x + y\| \leq \|x\| + \|y\|$ | 三角不等式 | | |

Figure 43.10: metric vs. inner product

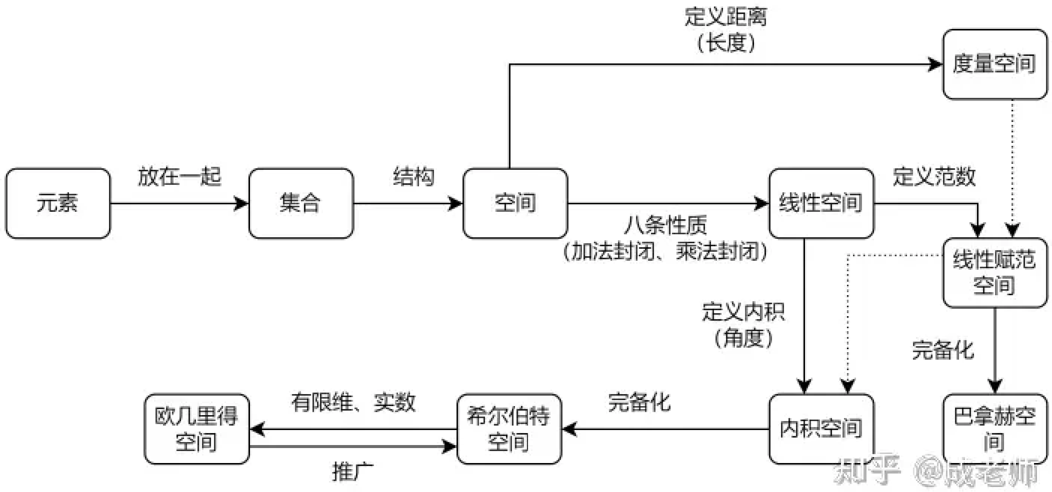


Figure 43.11: Euclid space construction

https://zhuanlan.zhihu.com/p/85867887

### 43.4.2 Riesz representation theorem

https://zhuanlan.zhihu.com/p/308879557

## 43.5 Dirac bracket

https://www.bilibili.com/video/BV1GN411m7A9

reciprocal space[38.4.1]

double dual concept[38.4.1.2]

$$
\begin{array}{cccccccccc}
 & & & V & =\{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots\} \\
 & \boldsymbol{e}^1 & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
V^* = \{ & \boldsymbol{e}^2 & \} & F & \supseteq \{ & 1 & 0 & 0 & v_1 & \cdots\} \\
 & \boldsymbol{e}^3 & & V & =\{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots\} \\
 & \boldsymbol{v}^* & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & & F & \supseteq \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i}v_i & \cdots\}
\end{array}
$$

---

$$
\begin{array}{cccccccccc}
 & & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{\omega} & \cdots\} \\
 & \boldsymbol{e}^{1*} & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
V^{**} = \{ & \boldsymbol{e}^{2*} & \} & F & \supseteq \{ & 1 & 0 & 0 & \omega^1 & \cdots\} \\
 & \boldsymbol{e}^{3*} & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{\omega} & \cdots\} \\
 & \boldsymbol{\omega}^* & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & & F & \supseteq \{ & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*}\omega^{\mu} & \cdots\} \\
 & & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{v}^* & \cdots\} \\
 & \boldsymbol{e}_1 & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cong V = \{ & \boldsymbol{e}_2 & \} & F & \supseteq \{ & 1 & 0 & 0 & v^{*1} & \cdots\} \\
 & \boldsymbol{e}_3 & & V^* & =\{ & \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 & \boldsymbol{v}^* & \cdots\} \\
 & \boldsymbol{v} & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & & F & \supseteq \{ & v_1 & v_2 & v_3 & v_{\mu}v^{*\mu} & \cdots\}
\end{array}
$$

---

dual space of span of partials[39.2]

coefficient of linear combination for vector space and dual space[39.4]

$$V^* = \left\{ \begin{array}{c} \boldsymbol{e}^1 \\ \boldsymbol{e}^2 \\ \boldsymbol{e}^3 \\ \boldsymbol{v}^* \\ \vdots \end{array} \right\} \begin{array}{cccccccc} & V & = \{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & 1 & 0 & 0 & v_1 & \cdots \} \\ & V & = \{ & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i} v_i & \cdots \} \end{array}$$

inner product space[43.4.1]

$$\begin{cases} d(f,g) = d\langle f|g\rangle = \sqrt{\langle f-g|f-g\rangle} \\ f,g \in C_{\mathbb{C}}([a,b]) & \text{continuous complex function} \\ (f,g) = \langle f|g\rangle = \int_a^b \overline{f(x)} g(x)\,\mathrm{d}x = \int_a^b f^*(x) g(x)\,\mathrm{d}x \end{cases}$$

$$\Downarrow$$

$$d\langle f|g\rangle = \sqrt{\int_a^b \overline{[f-g](x)}\,[f-g](x)\,\mathrm{d}x} = \sqrt{\int_a^b |[f-g](x)|^2\,\mathrm{d}x}$$

$$v^*(f) = \eta_f(\cdot) = (f,\cdot) = \langle f|\cdot\rangle = \int_a^b \overline{f(x)}\,(\cdot)\,\mathrm{d}x = \int_a^b f^*(x)\,(\cdot)\,\mathrm{d}x$$

$$\eta_f(g) = (f,g) = \langle f|g\rangle = \int_a^b \overline{f(x)} g(x)\,\mathrm{d}x = \int_a^b f^*(x) g(x)\,\mathrm{d}x$$

https://en.wikipedia.org/wiki/Antilinear_map

$$\begin{cases} v^*(f+g) = v^*(f) + v^*(g) & \text{addivity} = \text{superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \overline{\lambda} v^*(f) & \text{conjugate homogeneity} = \text{complex conjugate} \end{cases}$$

$$\begin{cases} v^*(f+g) = v^*(f) + v^*(g) & \text{addivity} = \text{superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \overline{\lambda} v^*(f) & \text{conjugate homogeneity} = \text{complex conjugate} \end{cases}$$

$$\Leftrightarrow \begin{cases} v^*(f+g) = \lambda^* v^*(f) + v^*(g) = \overline{\lambda} v^*(f) + v^*(g) & \text{antilinearity} \end{cases}$$

$$V^* = \left\{ \begin{array}{c} \hat{f}^1 = \left(\hat{f}_1,\cdot\right) \\ \hat{f}^2 = \left(\hat{f}_2,\cdot\right) \\ \hat{f}^3 = \left(\hat{f}_3,\cdot\right) \\ f^* = (f,\cdot) \\ \vdots \end{array} \right\} \begin{array}{cccccccc} & V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & \left(\hat{f}_1,\hat{f}_1\right) = 1 & \left(\hat{f}_1,\hat{f}_2\right) = 0 & \left(\hat{f}_1,\hat{f}_3\right) = 0 & \left(\hat{f}_1,f\right) & \cdots \} \\ & V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & \left(f,\hat{f}_1\right) & \left(f,\hat{f}_2\right) & \left(f,\hat{f}_3\right) & (f,f) & \cdots \} \end{array}$$

$$V^* = \left\{ \begin{array}{c} \hat{f}^1 = \left\langle \hat{f}_1 \middle| \cdot \right\rangle \\ \hat{f}^2 = \left\langle \hat{f}_2 \middle| \cdot \right\rangle \\ \hat{f}^3 = \left\langle \hat{f}_3 \middle| \cdot \right\rangle \\ f^* = \langle f|\cdot\rangle \\ \vdots \end{array} \right\} \begin{array}{cccccccc} & V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & \left\langle \hat{f}_1 \middle| \hat{f}_1 \right\rangle = 1 & \left\langle \hat{f}_1 \middle| \hat{f}_2 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| \hat{f}_3 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| f \right\rangle & \cdots \} \\ & V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \cdots \} \\ : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & F & \supseteq \{ & \left\langle f \middle| \hat{f}_1 \right\rangle & \left\langle f \middle| \hat{f}_2 \right\rangle & \left\langle f \middle| \hat{f}_3 \right\rangle & \langle f|f\rangle & \cdots \} \end{array}$$

row vector is dual vector of column vector

$$f = \begin{pmatrix} a \\ b \end{pmatrix} \to v^*(f) = \eta_f = \begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix} = \overline{\begin{pmatrix} a \\ b \end{pmatrix}^{\mathsf{T}}} = \overline{\begin{pmatrix} a \\ b \end{pmatrix}}^{\mathsf{T}} = \begin{pmatrix} a \\ b \end{pmatrix}^{\dagger}$$

$$\eta_f \cdot f = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^* a + b^* b$$

https://www.bilibili.com/video/BV1GN411m7A9/?t=2m40s

https://en.wikipedia.org/wiki/Riesz_representation_theorem

**Theorem 43.5.** *Riesz representation theorem*

to be proved

https://www.bilibili.com/video/BV1GN411m7A9/?t=6m20s

$$\text{Hilbert space } V \begin{cases} B = \left\{\hat{f}_i\right\}_{i\in I} = \left\{\cdots, \hat{f}_i, \cdots\right\} \\ \begin{cases} \sum_i c_i \hat{f}_i = 0 \Rightarrow c_i = 0 & \text{bases are linear independent} \\ f = \sum_i c_i \hat{f}_i, c_i \in \mathbb{C} & \text{expansion over bases} \end{cases} \\ \begin{cases} \left(\hat{f}_i, \hat{f}_j\right) = \left\langle \hat{f}_i \middle| \hat{f}_j \right\rangle = \delta_{ij} \\ \text{completeness} : \forall \hat{f}_i \in B, \nexists f \in V - B \left[\left(\hat{f}_i, f\right) = \left\langle \hat{f}_i \middle| f \right\rangle = 0\right] \end{cases} \end{cases}$$

$$f = \sum_i \left(\hat{f}_i \cdot f\right) \hat{f}_i = \sum_i \left(\hat{f}_i, \hat{f}_j\right) \hat{f}_i = \sum_i \left\langle \hat{f}_i \middle| \hat{f}_j \right\rangle \hat{f}_i$$

https://www.bilibili.com/video/BV1GN411m7A9/?t=7m55s

### 43.5.1 Dirac bracket symmetry

$$f = \sum_i \left(\hat{f}_i, f\right) \hat{f}_i = \sum_i \hat{f}_i \left(\hat{f}_i, f\right)$$

$$\begin{aligned} f^* = (f, \cdot) &= \left(\sum_i \left(\hat{f}_i, f\right) \hat{f}_i, \cdot\right) = \left(\sum_i \hat{f}_i \left(\hat{f}_i, f\right), \cdot\right) \\ &\overset{\text{antilinearity}}{=} \overline{\sum_i \left(\hat{f}_i, f\right)} \left(\hat{f}_i, \cdot\right) = \sum_i \overline{\left(\hat{f}_i, f\right)} \left(\hat{f}_i, \cdot\right) \\ &= \sum_i \left(f, \hat{f}_i\right) \left(\hat{f}_i, \cdot\right) \\ &\overset{\text{Dirac bracket}}{\rightarrow} \sum_i \left\langle f \middle| \hat{f}_i \right\rangle \left\langle \hat{f}_i \right| = \langle f| \\ &\Updownarrow \\ f &= \sum_i \hat{f}_i \left(\hat{f}_i, f\right) \\ &\overset{\text{Dirac bracket}}{\rightarrow} \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i \middle| f \right\rangle = |f\rangle \end{aligned}$$

---

$$\begin{cases} |f\rangle = \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i \middle| f \right\rangle = \sum_i \left\langle \hat{f}_i \middle| f \right\rangle \left|\hat{f}_i\right\rangle \\ \langle f| = \sum_i \left\langle f \middle| \hat{f}_i \right\rangle \left\langle \hat{f}_i \right| \\ \left\langle \hat{f}_i \middle| \hat{f}_j \right\rangle = \delta_{ij} \end{cases}$$

---

$$\begin{array}{cccccccccc} & & & V & = \{ & \left|\hat{f}_1\right\rangle & \left|\hat{f}_2\right\rangle & \left|\hat{f}_3\right\rangle & |f\rangle & \cdots\} \\ & \left\langle \hat{f}_1 \right| & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ V^* = \{ & \left\langle \hat{f}_2 \right| & \} & F & \supseteq \{ & \left\langle \hat{f}_1 \middle| \hat{f}_1 \right\rangle = 1 & \left\langle \hat{f}_1 \middle| \hat{f}_2 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| \hat{f}_3 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| f \right\rangle & \cdots\} \\ & \left\langle \hat{f}_3 \right| & & V & = \{ & \left|\hat{f}_1\right\rangle & \left|\hat{f}_2\right\rangle & \left|\hat{f}_3\right\rangle & |f\rangle & \cdots\} \\ & \langle f| & : & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \vdots & & F & \supseteq \{ & \left\langle f \middle| \hat{f}_1 \right\rangle & \left\langle f \middle| \hat{f}_2 \right\rangle & \left\langle f \middle| \hat{f}_3 \right\rangle & \langle f|f\rangle & \cdots\} \end{array}$$

$$\begin{cases} v^*(f+g) = v^*(f) + v^*(g) & \text{addivity = superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \overline{\lambda} v^*(f) & \text{conjugate homogeneity = complex conjugate} \end{cases}$$

$$\Leftrightarrow \begin{cases} v^*(f+g) = \lambda^* v^*(f) + v^*(g) = \overline{\lambda} v^*(f) + v^*(g) & \text{antilinearity} \end{cases}$$

$$\begin{cases} (c\psi, \cdot) \to \langle c\psi| = c^* \langle\psi| & c\psi \to |c\psi\rangle = c|\psi\rangle \\ \langle\psi| \leftrightarrow |\psi\rangle & \\ \langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle} & \langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle} \\ \langle c\psi|\phi\rangle = c^* \langle\psi|\phi\rangle & \langle\phi|c\psi\rangle = c\langle\phi|\psi\rangle \end{cases}$$

---

$|f\rangle\langle g| = ?$

$$|f\rangle\langle g| |h\rangle = |f\rangle\langle g|h\rangle = \langle g|h\rangle |f\rangle$$
$$\Downarrow$$
$$|h\rangle \overset{|f\rangle\langle g|}{\to} |f\rangle\langle g| |h\rangle = \langle g|h\rangle |f\rangle$$

$$\langle h| |f\rangle\langle g| = \langle h|f\rangle\langle g|$$
$$\Downarrow$$
$$\langle h| \overset{|f\rangle\langle g|}{\to} \langle h| |f\rangle\langle g| = \langle h|f\rangle\langle g|$$

$$|f\rangle\langle g| \text{ is a linear transform or rank-2 tensor}$$

$$\begin{aligned} |f\rangle &= \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i \middle| f\right\rangle \\ &= \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right| |f\rangle \\ &= \left(\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right|\right) |f\rangle = 1|f\rangle = |f\rangle \\ 1 &= \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right| \end{aligned}$$

$$\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right| = 1$$

---

$$\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right| = 1$$

the above is completeness relationship.

---

$$\begin{cases} |f\rangle = 1|f\rangle = \left(\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right|\right) |f\rangle = \sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i \middle| f\right\rangle \\ \langle f| = \langle f| 1 = \langle f| \left(\sum_i \left|\hat{f}_i\right\rangle \left\langle \hat{f}_i\right|\right) = \sum_i \left\langle f \middle| \hat{f}_i\right\rangle \left\langle \hat{f}_i\right| \end{cases}$$

---

> The sculpture is already complete within the marble block, before I start my work. It is already there, I just have to chisel away the superfluous material. Michelangelo

https://www.goodreads.com/quotes/1191114-the-sculpture-is-already-complete-within-the-marble-block-before

https://www.bilibili.com/video/BV1y84y1U7Ps

## 43.6 quantum operator

https://www.bilibili.com/video/BV1za4y1F7xo

## 43.7 Gram-Schmidt algorithm to find orthonormal basis

https://www.bilibili.com/video/BV1eu4y1S77C

## 43.8 Hermite polynomials and Legendre polynomials

https://www.bilibili.com/video/BV1Lu411V7Rp

# Chapter 44

# Lagrange inversion

## 44.1 Newton

https://www.bilibili.com/video/BV1ge411e7K7

## 44.2 Lagrange

Lagrange inversion theorem

拉格朗日反演

https://www.bilibili.com/video/BV13V411L7XL

https://www.bilibili.com/video/BV1AM41167y2

# Chapter 45

# quantum color dynamics

https://www.youtube.com/watch?v=FL3ImtGcHqQ

# Chapter 46

# Lagrangian

- Elliot Schneider: Physics with Elliot
  - Lagrangian fundamentals
  - Lagrangian mechanics

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458875

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458876

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458877

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458878

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458879

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458880

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458881

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458882

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458883

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458884

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458885

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458886

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458887

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458888

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458889

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458890

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2158458891 https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912436

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912460

https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912464

https://www.youtube.com/watch?v=0DHNGtsmmH8

https://www.youtube.com/watch?v=sUk9y23FPHk

https://www.youtube.com/watch?v=O0NYaO_OnH4

https://www.youtube.com/watch?v=KVk1QNTWBx

https://www.youtube.com/watch?v=h2SEK6Jjv3Y

https://www.youtube.com/watch?v=KpIaWiWvuRs

https://www.youtube.com/watch?v=13hCkUiu_mI

# Chapter 47

# Hamiltonian

- Elliot Schneider: Physics with Elliot
  - Hamiltonian mechanics

https://www.youtube.com/watch?v=Nd4b0_vJZUk

https://www.youtube.com/watch?v=uncm8DChdhc

https://www.youtube.com/watch?v=_lz1VfI6Wxk

# Chapter 48

# Fourier analysis

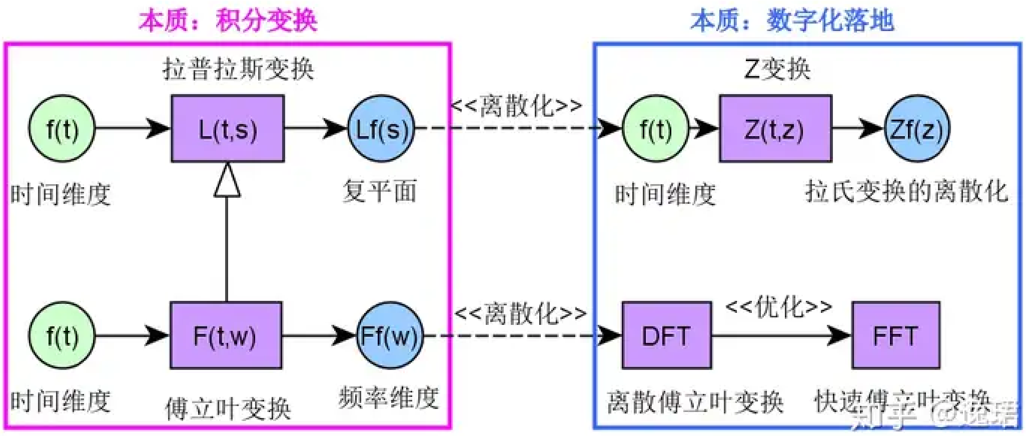


Figure 48.1: transform relationhsip: Fourier vs Laplace vs Z

https://zhuanlan.zhihu.com/p/342952028

https://zhuanlan.zhihu.com/p/19763358

## 48.1 basic calculation

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(n\frac{2\pi}{\tau}t+\phi\right)\mathrm{d}t = \left[\frac{\sin\left(n\frac{2\pi}{\tau}t+\phi\right)}{n\frac{2\pi}{\tau}}\right]_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} = 0$$

$$\begin{aligned}
\cos\alpha\cos\beta &= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \\
\sin\alpha\sin\beta &= \frac{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \\
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}
\end{aligned}$$

$$
\begin{aligned}
&\int \cos\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t\\
&=\begin{cases}\int \cos\phi_0 \mathrm{d}t & m=n=0\\ \int \frac{\cos\left((m+n)\frac{2\pi}{\tau}t+\phi_n\right)+\cos\left((m-n)\frac{2\pi}{\tau}t-\phi_n\right)\mathrm{d}t}{2} & mn\neq 0\end{cases}\\
&=\begin{cases}t\cos\phi_0 & m=n=0\\ \begin{cases}\int \frac{\cos\left(2n\frac{2\pi}{\tau}t+\phi_n\right)+\cos\left(-\phi_n\right)}{2}\mathrm{d}t & m=n\\ \int \frac{\cos\left((m+n)\frac{2\pi}{\tau}t+\phi_n\right)+\cos\left((m-n)\frac{2\pi}{\tau}t-\phi_n\right)}{2}\mathrm{d}t & m\neq n\end{cases} & mn\neq 0\end{cases}\\
&=\begin{cases}t\cos\phi_0 & m=n=0\\ \begin{cases}\int \frac{\cos\left(2n\frac{2\pi}{\tau}t+\phi_n\right)+\cos\phi_n}{2}\mathrm{d}t & m=n\\ \int \frac{\cos\left((m+n)\frac{2\pi}{\tau}t+\phi_n\right)+\cos\left((m-n)\frac{2\pi}{\tau}t-\phi_n\right)}{2}\mathrm{d}t & m\neq n\end{cases} & mn\neq 0\end{cases}\\
&=\begin{cases}t\cos\phi_0 & m=n=0\\ \begin{cases}\int \frac{\cos\left(2n\frac{2\pi}{\tau}t+\phi_n\right)}{2}\mathrm{d}t+\frac{t\cos\phi_n}{2} & m=n\\ \int \frac{\cos\left((m+n)\frac{2\pi}{\tau}t+\phi_n\right)+\cos\left((m-n)\frac{2\pi}{\tau}t-\phi_n\right)}{2}\mathrm{d}t & m\neq n\end{cases} & mn\neq 0\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\cos\alpha\cos\beta &= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}\\
\sin\alpha\sin\beta &= \frac{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}\\
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}\\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t\\
&=\begin{cases}\tau\cos\phi_0 & m=n=0\\ \begin{cases}\frac{\tau\cos\phi_n}{2} & m=n\\ 0 & m\neq n\end{cases} & mn\neq 0\end{cases}\\
&=\begin{cases}\begin{cases}\tau\cos\phi_0 & n=0\\ \frac{\tau\cos\phi_n}{2} & n\neq 0\end{cases} & m=n\\ 0 & m\neq n\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2}{\tau\cos\phi_n}\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t\\
&=\begin{cases}\begin{cases}2 & n=0\\ 1 & n\neq 0\end{cases} & m=n\\ 0 & m\neq n\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \sin\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t\\
&=\begin{cases}0 & m=n\\ 0 & m\neq n\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right)\sin\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t\\
&=\begin{cases}0 & m=n\\ 0 & m\neq n\end{cases}
\end{aligned}
$$

---

$$\frac{2}{\tau\cos\phi_n}\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi}\cos\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t=\begin{cases}\begin{cases}2 & n=0\\ 1 & n\neq 0\end{cases} & m=n\\ 0 & m\neq n\end{cases}$$

$$\frac{2}{\tau\cos\phi_n}\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi}\sin\left(m\frac{2\pi}{\tau}t\right)\sin\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t=\begin{cases}1 & m=n\neq 0\\ 0 & \neg(m=n\neq 0)\end{cases}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi}\sin\left(m\frac{2\pi}{\tau}t\right)\cos\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t=\begin{cases}0 & m=n\\ 0 & m\neq n\end{cases}$$

$$\int_{\frac{-\tau}{2}+\psi}^{\frac{\tau}{2}+\psi}\cos\left(m\frac{2\pi}{\tau}t\right)\sin\left(n\frac{2\pi}{\tau}t+\phi_n\right)\mathrm{d}t=\begin{cases}0 & m=n\\ 0 & m\neq n\end{cases}$$

## 48.2 TaylorCatAlice: phase velocity vs. group velocity

https://www.bilibili.com/video/BV1nV411F7es

## 48.3 quantum mechanics or wave mechanics

Elliot Schneider: Physics with Elliot

https://www.youtube.com/watch?v=W8QZ-yxebFA

### 48.3.1 uncertainty principle

https://www.youtube.com/watch?v=p7bzE1E5PMY

## 48.4 Simon Xu

### 48.4.1 DFT = discrete Fourier transform

https://www.youtube.com/watch?v=mkGsMWi_j4Q

Reducible

https://www.youtube.com/watch?v=yYEMxqreA10

### 48.4.2 FFT = fast Fourier transform

https://www.youtube.com/watch?v=htCj9exbGo0

Reducible

https://www.youtube.com/playlist?list=PLpXOY-RxVRTNBfxhIuqoZcWtg-JZKCktX

https://www.youtube.com/watch?v=h7apO7q16V0

https://www.youtube.com/watch?v=Ty0JcR6Dvis

### 48.4.3 wavelet

https://www.youtube.com/watch?v=ZnmvUCtUAEE

Artem Kirsanov

https://www.youtube.com/watch?v=jnxqHcObNK4

#### 48.4.3.1 compressed sensing MRI

Michael (Miki) Lustig

https://www.youtube.com/watch?v=AP6JczMW8C8

Tseng, Wen-Yih

https://www.youtube.com/watch?v=P6tzQ9KQ9JQ&list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU&index=14

https://www.youtube.com/watch?v=hxdZdUQ6y2k&list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU&index=16 (Michael Lustig)

## 48.5 sampling and reconstruction

https://www.youtube.com/playlist?list=PLTp0eSi9MdkPtCLf1VxMWvUSI5JI8kAtz

https://www.youtube.com/watch?v=8CPPyE1rvMU

https://www.youtube.com/watch?v=Qd8fLSDwbQo

https://www.youtube.com/watch?v=0255KLvu75g

## 48.6 signal and system

https://www.youtube.com/playlist?list=PLX6FA3vfNTfChkbNQGxVPrIsvkC_DwNV6

## 48.7 image compression / data compression

Reducible

https://www.youtube.com/playlist?list=PLpXOY-RxVRTOR1PAtQUwoZN2tSs1ICSk7

## 48.8 Lin, Chi–Kun

https://www.youtube.com/playlist?list=PLj6E8qlqmkFuX5N1O3FKoDfoySC6Hku-2

# Chapter 49

# axiom of choice

https://www.youtube.com/watch?v=szfsGJ_PGQ0

## 49.1 Zorn lemma

https://zhuanlan.zhihu.com/p/670990411

# Chapter 50

# linear algebra

## 50.1 The Art of Linear Algeba

https://github.com/kenjihiranabe/The-Art-of-Linear-Algebra

https://stackoverflow.com/questions/39173714/r-markdown-can-i-insert-a-pdf-to-the-r-markdown-file-as-an-image

https://stackoverflow.com/a/39177166

<br>

# The Art of Linear Algebra

– Graphic Notes on “Linear Algebra for Everyone” –

Kenji Hiranabe *
with the kindest help of Gilbert Strang †
translator: Kefang Liu ‡
traditionalization and rearrangement: Joey Yu Hsu, MD §

September 1, 2021/updated May 11, 2024

**Abstract**

我嘗試爲Gilbert Strang 在書籍“Linear Algebra for Everyone” 中介紹的矩陣的重要概念進形可視化圖釋, 以促進從矩陣分解的角度對向量、矩陣計算和算法的理解. [1] 它們包括矩陣分解(Column-Row, $\boldsymbol{CR}$)、高斯消去法(Gaussian Elimination, $\boldsymbol{LU}$)、格拉姆-施密特正交化(Gram-Schmidt Orthogonalization, $\boldsymbol{QR}$)、特徵値和對角化(Eigenvalues and Diagonalization, $\boldsymbol{Q\Lambda Q^T}$)、和奇異値分解(Singular Value Decomposition, $\boldsymbol{U\Sigma V^T}$).

# 序言

我很高興能看到Kenji Hiranabe 的線性代數中的矩陣運算的圖片! 這樣的圖片是展示代數的絕佳方式. 我們當然可以通過列·行的點乘來想象矩陣乘法, 但那絕非全部—— 它是“線性組合”與“秩1矩陣”組成的代數與藝術. 我很感激能看到日文翻譯的書籍和Kenji的圖片中的想法.

– Gilbert Strang
麻省理工學院數學教授

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†Massachusetts Institute of Technology, http://www-math.mit.edu/~gs/
‡twitter: @kfchliu, 微博用户: 5717297833
§redwing1304@gmail.com

[1] “Linear Algebra for Everyone”: http://math.mit.edu/everyone/.

<br>

## 50.2 CCJou

https://www.youtube.com/playlist?list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju

### 50.2.1 coordinate

https://www.youtube.com/watch?v=eMUFexQsKXA&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=20

$$
\begin{aligned}
\boldsymbol{v} =& v^1\boldsymbol{v}_1 + v^2\boldsymbol{v}_2 + \cdots + v^n\boldsymbol{v}_n \\
=& v^1\boldsymbol{v}_1 + \cdots + v^n\boldsymbol{v}_n \\
=& c^1\boldsymbol{\beta}_1 + \cdots + c^n\boldsymbol{\beta}_n = c_1\boldsymbol{\beta}_1 + \cdots + c_n\boldsymbol{\beta}_n = B\boldsymbol{c} = B\left[\boldsymbol{v}\right]_B \\
=& e^1\hat{\boldsymbol{\beta}}_1 + \cdots + e^n\hat{\boldsymbol{\beta}}_n = e_1\hat{\boldsymbol{\beta}}_1 + \cdots + e_n\hat{\boldsymbol{\beta}}_n \\
=& e^1\hat{\boldsymbol{e}}_1 + \cdots + e^n\hat{\boldsymbol{e}}_n = e_1\hat{\boldsymbol{e}}_1 + \cdots + e_n\hat{\boldsymbol{e}}_n \\
\overset{\text{e.g.}}{=}& e^1\begin{pmatrix}1\\ \vdots\\ 0\\ \vdots\\ 0\end{pmatrix}_{n\times 1} + \cdots + e^n\begin{pmatrix}0\\ \vdots\\ 0\\ \vdots\\ 1\end{pmatrix}_{n\times 1} = e_1\begin{pmatrix}1\\ \vdots\\ 0\\ \vdots\\ 0\end{pmatrix} + \cdots + e_n\begin{pmatrix}0\\ \vdots\\ 0\\ \vdots\\ 1\end{pmatrix} \\
=& \begin{pmatrix}1 & \cdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ 0 & \cdots & 1 & \cdots & 0\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \cdots & 1\end{pmatrix}_{n\times n}\begin{pmatrix}e^1\\ \vdots\\ e^j\\ \vdots\\ e^n\end{pmatrix}_{n\times 1} = I_{n\times n}\boldsymbol{e} = I\boldsymbol{e}
\end{aligned}
$$

$$\begin{aligned}
B\boldsymbol{c} =& c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^j \begin{pmatrix} | \\ \boldsymbol{\beta}_j \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^j \\ \vdots \\ c^n \end{pmatrix} \\
=& c^1 \begin{bmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{bmatrix} + \cdots + c^j \begin{bmatrix} | \\ \boldsymbol{\beta}_j \\ | \end{bmatrix} + \cdots + c^n \begin{bmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{bmatrix} \begin{bmatrix} c^1 \\ \vdots \\ c^j \\ \vdots \\ c^n \end{bmatrix} \\
=& c^1\boldsymbol{\beta}_1 + c^2\boldsymbol{\beta}_2 + \cdots + c^n\boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + c^2 \begin{pmatrix} | \\ \boldsymbol{\beta}_2 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^n \end{pmatrix} \\
=& c^1\boldsymbol{\beta}_1 + c^2\boldsymbol{\beta}_2 + \cdots + c^n\boldsymbol{\beta}_n = c^1 \begin{bmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{bmatrix} + c^2 \begin{bmatrix} | \\ \boldsymbol{\beta}_2 \\ | \end{bmatrix} + \cdots + c^n \begin{bmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c^1 \\ c^2 \\ \vdots \\ c^n \end{bmatrix} \\
=& c^1\boldsymbol{\beta}_1 + \cdots + c^n\boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix} = B\left[\boldsymbol{v}\right]_B \\
=& c^1\boldsymbol{\beta}_1 + \cdots + c^n\boldsymbol{\beta}_n = c^1 \begin{bmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{bmatrix} + \cdots + c^n \begin{bmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\
B =& \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} = \begin{bmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{bmatrix} \\
\left[\boldsymbol{v}\right]_B =& \boldsymbol{c} = \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in F^n = \mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\}
\end{aligned}$$

---

$$\boldsymbol{v} = B\left[\boldsymbol{v}\right]_B$$

---

$$\begin{aligned}
\boldsymbol{v} =& B\left[\boldsymbol{v}\right]_B \\
B^{-1}\boldsymbol{v} =& B^{-1}B\left[\boldsymbol{v}\right]_B = I\left[\boldsymbol{v}\right]_B = \left[\boldsymbol{v}\right]_B \\
\left[\boldsymbol{v}\right]_B =& B^{-1}\boldsymbol{v}
\end{aligned}$$

---

$$\left[\boldsymbol{v}\right]_B = B^{-1}\boldsymbol{v}$$

$$\left[\boldsymbol{v}\right]_B = B^{-1}\boldsymbol{v} \in\in F^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\}$$

---

$$\boldsymbol{v} = B\left[\boldsymbol{v}\right]_B = B'\left[\boldsymbol{v}\right]_{B'}$$

---

$$\begin{aligned} B[\boldsymbol{v}]_B &= B'[\boldsymbol{v}]_{B'} \\ [\boldsymbol{v}]_B = I[\boldsymbol{v}]_B = B^{-1}B[\boldsymbol{v}]_B &= B^{-1}B'[\boldsymbol{v}]_{B'} \\ [\boldsymbol{v}]_B &= B^{-1}B'[\boldsymbol{v}]_{B'} \\ B'[\boldsymbol{v}]_{B'} &= B[\boldsymbol{v}]_B \\ [\boldsymbol{v}]_{B'} = I[\boldsymbol{v}]_{B'} = B'^{-1}B'[\boldsymbol{v}]_{B'} &= B'^{-1}B[\boldsymbol{v}]_B \\ [\boldsymbol{v}]_{B'} &= B'^{-1}B[\boldsymbol{v}]_B \end{aligned}$$

### 50.2.2 linear transformation

$$\begin{aligned} \boldsymbol{v} &= v^1\boldsymbol{v}_1 + \cdots + v^n\boldsymbol{v}_n = v^j\boldsymbol{v}_j = \sum_{j=1}^{n} v^j\boldsymbol{v}_j \\ &= v^1\begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} + \cdots + v^n\begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = V[\boldsymbol{v}]_V \\ &, \begin{cases} V = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix}, \mathfrak{V} = \{\boldsymbol{v}_j\}_{j=1}^{n} = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\} \\ [\boldsymbol{v}]_V = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in F^n \in \{\mathbb{R}^n, \mathbb{C}^n, \cdots\} \end{cases} \quad \overset{\text{if } V \text{ invertible}}{\Longrightarrow} [\boldsymbol{v}]_V = V^{-1}\boldsymbol{v} \end{aligned}$$

$$\text{\mathfrak}V = \mathfrak{V} \text{ vs.} \text{\mathfrak}B = \mathfrak{B}$$

$$\begin{aligned} &\boldsymbol{v} \in \mathcal{V} \xrightarrow{T} \mathcal{W} \ni \boldsymbol{w} = T(\boldsymbol{v}) \\ &, T : \mathcal{V} \to \mathcal{W} \begin{cases} T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}) & (a) \text{ additivity} \\ T(\lambda\boldsymbol{v}) = \lambda T(\boldsymbol{v}) & (h) \text{ homogeneity} \end{cases} \end{aligned}$$

$$\begin{aligned} \boldsymbol{w} = T(\boldsymbol{v}) &= T(v^1\boldsymbol{v}_1 + \cdots + v^n\boldsymbol{v}_n) \overset{(a)}{=} T(v^1\boldsymbol{v}_1) + \cdots + T(v^n\boldsymbol{v}_n) \overset{(h)}{=} v^1T(\boldsymbol{v}_1) + \cdots + v^nT(\boldsymbol{v}_n) \\ &= v^1\begin{bmatrix} | \\ T(\boldsymbol{v}_1) \\ | \end{bmatrix} + \cdots + v^n\begin{bmatrix} | \\ T(\boldsymbol{v}_n) \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix}\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix}[\boldsymbol{v}]_V \end{aligned}$$

$$\boldsymbol{w} = T(\boldsymbol{v}) = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix}[\boldsymbol{v}]_V = T(V)[\boldsymbol{v}]_V, T(V) = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix}$$

---

$$\boldsymbol{w} = T(\boldsymbol{v}) \overset{\boldsymbol{v}=V[\boldsymbol{v}]_V}{=} T(V[\boldsymbol{v}]_V) \overset{T \text{ linear}}{=} T(V)[\boldsymbol{v}]_V$$

---

$$\begin{aligned} T(\boldsymbol{v}) = \boldsymbol{w} &= w^1\boldsymbol{w}_1 + \cdots + w^m\boldsymbol{w}_m = w^j\boldsymbol{w}_j = \sum_{j=1}^{n} w^j\boldsymbol{w}_j \\ &= w^1\begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix} + \cdots + w^m\begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = W[\boldsymbol{w}]_W \overset{\boldsymbol{w}=T(\boldsymbol{v})}{=} W[T(\boldsymbol{v})]_W \\ &, \begin{cases} W = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}, \mathfrak{W} = \{\boldsymbol{w}_j\}_{j=1}^{m} = \{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m\} \\ [\boldsymbol{w}]_W = \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} \in F^m \in \{\mathbb{R}^m, \mathbb{C}^m, \cdots\} \end{cases} \quad \overset{\text{if } W \text{ invertible}}{\Longrightarrow} [\boldsymbol{w}]_W = W^{-1}\boldsymbol{w} \end{aligned}$$

$$\backslash\text{mathfrak}W = \mathfrak{W} \text{ vs.} \backslash\text{mathfrak}V = \mathfrak{V}$$

$$\begin{aligned}
T(\boldsymbol{v}_1) =& t_1^1 \boldsymbol{w}_1 + \cdots + t_1^m \boldsymbol{w}_m = t_1^1 \begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix} + \cdots + t_1^m \begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix} \\
=& \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\boldsymbol{v}_1)]_W \\ | \end{bmatrix} = W [T(\boldsymbol{v}_1)]_W \\
\vdots & \\
T(\boldsymbol{v}_n) =& t_n^1 \boldsymbol{w}_1 + \cdots + t_n^m \boldsymbol{w}_m = t_n^1 \begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix} + \cdots + t_n^m \begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix} \\
=& \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\boldsymbol{v}_n)]_W \\ | \end{bmatrix} = W [T(\boldsymbol{v}_n)]_W
\end{aligned}$$

$$\begin{aligned}
T(V) = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} &= \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} & \cdots & \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} = WT \\
&, \begin{cases} W = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \\ T = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} \end{cases}
\end{aligned}$$

$$T(V) = \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} = WT, \begin{cases} W = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix} \\ T = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} \end{cases}$$

$$T = \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} \overset{V = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix}}{=} [T(V)]_W$$

---

$$\begin{aligned} T = [T(V)]_W &= W^{-1}T(V) = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}^{-1} T(V), \text{if } W \text{ invertible} \\ &= \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}^{-1} T\left(\begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix}\right) \\ &= \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}^{-1} \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} \\ &= W^{-1} \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ W^{-1}T(\boldsymbol{v}_1) & \cdots & W^{-1}T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} \end{aligned}$$

---

$$T = [T(V)]_W = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}^{-1} \begin{bmatrix} | & & | \\ T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \\ | & & | \end{bmatrix} = W^{-1}T(V)$$

---

$$\begin{aligned} T &= \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix} = [T(V)]_W \\ &= \begin{bmatrix} | & & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\boldsymbol{v}_1)]_W & \cdots & [T(\boldsymbol{v}_n)]_W \\ | & & | \end{bmatrix}, \boldsymbol{t}_j = [T(\boldsymbol{v}_j)]_W \end{aligned}$$

$$\begin{aligned} W[\boldsymbol{w}]_W = \boldsymbol{w} = T(\boldsymbol{v}) &= T(V)[\boldsymbol{v}]_V \overset{T(V)=WT}{=} WT[\boldsymbol{v}]_V = W(T[\boldsymbol{v}]_V) \\ W[\boldsymbol{w}]_W &= W(T[\boldsymbol{v}]_V) \\ [\boldsymbol{w}]_W = I[\boldsymbol{w}]_W = W^{-1}W[\boldsymbol{w}]_W &= W^{-1}W(T[\boldsymbol{v}]_V) = IT[\boldsymbol{v}]_V = T[\boldsymbol{v}]_V, \text{if } W \text{ invertible} \\ [\boldsymbol{w}]_W &= T[\boldsymbol{v}]_V \\ [T(\boldsymbol{v})]_W \overset{\boldsymbol{w}=T(\boldsymbol{v})}{=} [\boldsymbol{w}]_W &= T[\boldsymbol{v}]_V \\ [T(\boldsymbol{v})]_W &= T[\boldsymbol{v}]_V \overset{T=[T(V)]_W}{=} [T(V)]_W[\boldsymbol{v}]_V \end{aligned}$$

---

$$\begin{aligned} [\boldsymbol{w}]_W &= T[\boldsymbol{v}]_V \\ &= [T(\boldsymbol{v})]_W = [T(V)]_W[\boldsymbol{v}]_V \end{aligned}$$

---

$$\begin{aligned} T &= \begin{bmatrix} | & & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} & \cdots & \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix}, \quad \begin{matrix} [\boldsymbol{w}]_W = T[\boldsymbol{v}]_V \\ \Updownarrow \\ w^i = t_j^i v^j = t^i{}_j v^j \end{matrix} \\ &= \begin{bmatrix} \begin{bmatrix} t^1{}_1 \\ \vdots \\ t^m{}_1 \end{bmatrix} & \cdots & \begin{bmatrix} t^1{}_n \\ \vdots \\ t^m{}_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} t^1{}_1 & \cdots & t^1{}_n \\ \vdots & \ddots & \vdots \\ t^m{}_1 & \cdots & t^m{}_n \end{bmatrix} = [t^i{}_j]_{m\times n} = [t^i{}_j] = t^i{}_j \\ &= \begin{bmatrix} \begin{bmatrix} t_{11} \\ \vdots \\ t_{m1} \end{bmatrix} & \cdots & \begin{bmatrix} t_{1n} \\ \vdots \\ t_{mn} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix} = [t_{ij}]_{m\times n} = [t_{ij}] = t_{ij} \end{aligned}$$

---

$$T=\begin{bmatrix} | & & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & & | \end{bmatrix}=\begin{bmatrix} t^1{}_1 & \cdots & t^1{}_n \\ \vdots & \ddots & \vdots \\ t^m{}_1 & \cdots & t^m{}_n \end{bmatrix}=\left[t^i{}_j\right]_{m\times n}=t^i{}_j$$

---

$$\begin{aligned}
T\left(\boldsymbol{v}_1\right)=&t_1{}^1\boldsymbol{w}_1+\cdots+t_1{}^m\boldsymbol{w}_m=t_1{}^1\begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix}+\cdots+t_1{}^m\begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix}=\begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} t_1{}^1 \\ \vdots \\ t_1{}^m \end{bmatrix}\\
=&W\left[T\left(\boldsymbol{v}_1\right)\right]_W\\
&\vdots\\
T\left(\boldsymbol{v}_n\right)=&t_n{}^1\boldsymbol{w}_1+\cdots+t_n{}^m\boldsymbol{w}_m=t_n{}^1\begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix}+\cdots+t_n{}^m\begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix}=\begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} t_n{}^1 \\ \vdots \\ t_n{}^m \end{bmatrix}\\
=&W\left[T\left(\boldsymbol{v}_n\right)\right]_W\\
&\Downarrow\\
\begin{bmatrix} | & & | \\ T\left(\boldsymbol{v}_1\right) & \cdots & T\left(\boldsymbol{v}_n\right) \\ | & & | \end{bmatrix}=&\begin{matrix} t_1{}^1 \\ \vdots \\ t_n{}^1 \end{matrix}\begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix}\begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix}\begin{matrix} t_1{}^m \\ \vdots \\ t_n{}^m \end{matrix}\begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix}\\
=&\begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} t_1{}^1 & \cdots & t_n{}^1 \\ \vdots & \ddots & \vdots \\ t_1{}^m & \cdots & t_n{}^m \end{bmatrix}=\begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} | & & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & & | \end{bmatrix}=WT\\
=&\begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} | & & | \\ \left[T\left(\boldsymbol{v}_1\right)\right]_W & \cdots & \left[T\left(\boldsymbol{v}_n\right)\right]_W \\ | & & | \end{bmatrix}=W\left[T\left(V\right)\right]_W\\
T\left(V\right)=&WT=W\left[T\left(V\right)\right]_W
\end{aligned}$$

---

$$\begin{aligned}
WT=&T\left(V\right)\\
&\Downarrow\\
T=&W^{-1}T\left(V\right)
\end{aligned}$$

---

$$T=\begin{bmatrix} t_1{}^1 & \cdots & t_n{}^1 \\ \vdots & \ddots & \vdots \\ t_1{}^m & \cdots & t_n{}^m \end{bmatrix}=\begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix}=\begin{bmatrix} | & & | \\ \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ | & & | \end{bmatrix}=\left[t_j^i\right]_{m\times n}=\left[t_j{}^i\right]_{m\times n}=t_j{}^i$$

---

$$\left[t_j{}^i\right]_{m\times n}=\left[t_j^i\right]_{m\times n}=\left[t^i{}_j\right]_{m\times n},\begin{cases} w^i=t_j^iv^j=t^i{}_jv^j \\ T\left(\boldsymbol{v}_j\right)=t_j^i\boldsymbol{w}_i=t_j{}^i\boldsymbol{w}_i \end{cases}$$

---

$T=\left[t^i{}_j\right]_{m\times n}=t^i{}_j$ is the matrix representation of the linear transformation $T\left(\cdot\right):\mathcal{V}\to\mathcal{W}$

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Figure 50.1: coordinate under linear transformation

### 50.2.3 change of basis

https://www.youtube.com/watch?v=WAtLPk55ljM&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=22

$$\boldsymbol{v} = B\left[\boldsymbol{v}\right]_B = B'\left[\boldsymbol{v}\right]_{B'}$$

---

$$\begin{aligned}
B\left[\boldsymbol{v}\right]_B &= B'\left[\boldsymbol{v}\right]_{B'}\\
\left[\boldsymbol{v}\right]_B = I\left[\boldsymbol{v}\right]_B = B^{-1}B\left[\boldsymbol{v}\right]_B &= B^{-1}B'\left[\boldsymbol{v}\right]_{B'}\\
\left[\boldsymbol{v}\right]_B &= B^{-1}B'\left[\boldsymbol{v}\right]_{B'}\\
B'\left[\boldsymbol{v}\right]_{B'} &= B\left[\boldsymbol{v}\right]_B\\
\left[\boldsymbol{v}\right]_{B'} = I\left[\boldsymbol{v}\right]_{B'} = B'^{-1}B'\left[\boldsymbol{v}\right]_{B'} &= B'^{-1}B\left[\boldsymbol{v}\right]_B\\
\left[\boldsymbol{v}\right]_{B'} &= B'^{-1}B\left[\boldsymbol{v}\right]_B
\end{aligned}$$

---

$$\begin{aligned}
\boldsymbol{v} &= v^1\boldsymbol{v}_1 + \cdots + v^n\boldsymbol{v}_n = v^j\boldsymbol{v}_j = \sum_{j=1}^{n} v^j\boldsymbol{v}_j = v^1\begin{bmatrix} | \\ \boldsymbol{v}_1 \\ | \end{bmatrix} + \cdots + v^n\begin{bmatrix} | \\ \boldsymbol{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = V\left[\boldsymbol{v}\right]_V\\
&= v'^1\boldsymbol{v}'_1 + \cdots + v'^n\boldsymbol{v}'_n = v'^j\boldsymbol{v}'_j = \sum_{j=1}^{n} v'^j\boldsymbol{v}'_j = v'^1\begin{bmatrix} | \\ \boldsymbol{v}'_1 \\ | \end{bmatrix} + \cdots + v'^n\begin{bmatrix} | \\ \boldsymbol{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{v}'_n \\ | & & | \end{bmatrix}\begin{bmatrix} v'^1 \\ \vdots \\ v'^n \end{bmatrix} = V'\left[\boldsymbol{v}\right]_{V'}\\
\boldsymbol{v} &= V\left[\boldsymbol{v}\right]_V = V'\left[\boldsymbol{v}\right]_{V'}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{w} &= w^1\boldsymbol{w}_1 + \cdots + w^m\boldsymbol{w}_m = w^j\boldsymbol{w}_j = \sum_{j=1}^{m} w^j\boldsymbol{w}_j = w^1\begin{bmatrix} | \\ \boldsymbol{w}_1 \\ | \end{bmatrix} + \cdots + w^m\begin{bmatrix} | \\ \boldsymbol{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \\ | & & | \end{bmatrix}\begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = W\left[\boldsymbol{w}\right]_W\\
&= w'^1\boldsymbol{w}'_1 + \cdots + w'^m\boldsymbol{w}'_m = w'^j\boldsymbol{w}'_j = \sum_{j=1}^{m} w'^j\boldsymbol{w}'_j = w'^1\begin{bmatrix} | \\ \boldsymbol{w}'_1 \\ | \end{bmatrix} + \cdots + w'^m\begin{bmatrix} | \\ \boldsymbol{w}'_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{w}'_m \\ | & & | \end{bmatrix}\begin{bmatrix} w'^1 \\ \vdots \\ w'^m \end{bmatrix} = W'\left[\boldsymbol{w}\right]_{W'}\\
\boldsymbol{w} &= W\left[\boldsymbol{w}\right]_W = W'\left[\boldsymbol{w}\right]_{W'}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{v}_1 &= v'^{\,1}_1\boldsymbol{v}'_1 + \cdots + v'^{\,n}_1\boldsymbol{v}'_n = v'^{\,1}_1\begin{bmatrix} | \\ \boldsymbol{v}'_1 \\ | \end{bmatrix} + \cdots + v'^{\,n}_1\begin{bmatrix} | \\ \boldsymbol{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{v}'_n \\ | & & | \end{bmatrix}\begin{bmatrix} v'^{\,1}_1 \\ \vdots \\ v'^{\,n}_1 \end{bmatrix} = V'\left[\boldsymbol{v}_1\right]_{V'}\\
&\vdots\\
\boldsymbol{v}_n &= v'^{\,1}_n\boldsymbol{v}'_1 + \cdots + v'^{\,n}_n\boldsymbol{v}'_n = v'^{\,1}_n\begin{bmatrix} | \\ \boldsymbol{v}'_1 \\ | \end{bmatrix} + \cdots + v'^{\,n}_n\begin{bmatrix} | \\ \boldsymbol{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{v}'_n \\ | & & | \end{bmatrix}\begin{bmatrix} v'^{\,1}_n \\ \vdots \\ v'^{\,n}_n \end{bmatrix} = V'\left[\boldsymbol{v}_n\right]_{V'}\\
&\Downarrow\\
\begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ | & & | \end{bmatrix} &= \begin{matrix} v'^{\,1}_1 \\ \vdots \\ v'^{\,1}_n \end{matrix}\begin{bmatrix} | \\ \boldsymbol{v}'_1 \\ | \end{bmatrix} \begin{matrix} + & \cdots & + \\ & \curvearrowright & \\ + & \cdots & + \end{matrix} \begin{matrix} v'^{\,n}_1 \\ \vdots \\ v'^{\,n}_n \end{matrix}\begin{bmatrix} | \\ \boldsymbol{v}'_n \\ | \end{bmatrix}\\
&= \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{v}'_n \\ | & & | \end{bmatrix}\begin{bmatrix} v'^{\,1}_1 & \cdots & v'^{\,1}_n \\ \vdots & \ddots & \vdots \\ v'^{\,n}_1 & \cdots & v'^{\,n}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{v}'_1 & \cdots & \boldsymbol{v}'_n \\ | & & | \end{bmatrix}\begin{bmatrix} | & & | \\ \left[\boldsymbol{v}_1\right]_{V'} & \cdots & \left[\boldsymbol{v}_n\right]_{V'} \\ | & & | \end{bmatrix} = V'\left[V\right]_{V'}\\
V &= V'\left[V\right]_{V'} \overset{\text{if } V' \text{ invertible}}{\iff} \left[V\right]_{V'} = V'^{-1}V
\end{aligned}$$

---

$$\left[V\right]_{V'} = V'^{-1}V : [V'|V] \xrightarrow{\text{Gauss-Jordan delimination}} [I|V'^{-1}V] = [I|[V]_{V'}]$$

$$[V']_V = V^{-1}V' : [V|V'] \xrightarrow{\text{Gauss-Jordan delimination}} [I|V^{-1}V'] = [I|[V']_V]$$

---

$$\begin{cases} [V]_{V'} = V'^{-1}V : & [V'|V] \xrightarrow{\text{Gauss-Jordan delimination}} [I|V'^{-1}V] = [I|[V]_{V'}] \\ [V']_V = V^{-1}V' : & [V|V'] \xrightarrow{\text{Gauss-Jordan delimination}} [I|V^{-1}V'] = [I|[V']_V] \end{cases}$$

---

$$\begin{aligned} \boldsymbol{v} =& V\,[\boldsymbol{v}]_V = V'\,[\boldsymbol{v}]_{V'} \\ =& V\,[\boldsymbol{v}]_V\,, \wedge V = V'\,[V]_{V'} \\ =& V'\,[V]_{V'}\,[\boldsymbol{v}]_V \\ V'\,[\boldsymbol{v}]_{V'} =& V'\,[V]_{V'}\,[\boldsymbol{v}]_V \\ [\boldsymbol{v}]_{V'} =& [V]_{V'}\,[\boldsymbol{v}]_V \end{aligned}$$

---

$$[\boldsymbol{v}]_{V'} = [V]_{V'}\,[\boldsymbol{v}]_V$$

---

symmetrically,

$$[\boldsymbol{v}]_V = [V']_V\,[\boldsymbol{v}]_{V'}$$

---

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Figure 50.2: change of coordinate basis under linear transformation

$[\boldsymbol{v}]_V \rightleftharpoons [\boldsymbol{v}]_{V'}$

$$\begin{aligned} [\boldsymbol{v}]_V \overset{[V]_{V'}}{\to} [\boldsymbol{v}]_{V'} &= [V]_{V'}\,[\boldsymbol{v}]_V \\ [\boldsymbol{v}]_V \overset{[V']_V}{\leftarrow} [\boldsymbol{v}]_{V'} &= [V']_V^{-1}\,[\boldsymbol{v}]_V \\ &\Downarrow \\ [V]_{V'} &= [V']_V^{-1} \Leftrightarrow [V']_V = [V]_{V'}^{-1} \end{aligned}$$

$[\boldsymbol{v}]_V \to [\boldsymbol{v}]_{V'}$

$$\begin{aligned} [\boldsymbol{v}]_V \overset{[V]_{V'}}{\to} [\boldsymbol{v}]_{V'} &= [V]_{V'}\,[\boldsymbol{v}]_V \\ [\boldsymbol{v}]_V \overset{V}{\to} \boldsymbol{v} \overset{V'^{-1}}{\to} [\boldsymbol{v}]_{V'} &= V'^{-1}V\,[\boldsymbol{v}]_V \\ &\Downarrow \\ [V]_{V'} &= V'^{-1}V \overset{[V']_V=[V]_{V'}^{-1}}{\Longleftrightarrow} [V']_V = (V'^{-1}V)^{-1} = V^{-1}V' \end{aligned}$$

$[\boldsymbol{v}]_{V'} \to [\boldsymbol{w}]_{W'}$

$$\begin{aligned}
&[\boldsymbol{v}]_{V'} \overset{T'=[T(V')]_{W'}}{\rightarrow} [\boldsymbol{w}]_{W'} = [T(V')]_{W'}[\boldsymbol{v}]_{V'} \\
[\boldsymbol{v}]_{V'} \overset{[V']_V}{\rightarrow} [\boldsymbol{v}]_V \overset{T=[T(V)]_W}{\rightarrow} [\boldsymbol{w}]_W \overset{[W]_{W'}}{\rightarrow} &[\boldsymbol{w}]_{W'} = [W]_{W'}[T(V)]_W [V']_V [\boldsymbol{v}]_{V'} \\
&\Downarrow \\
[T(V')]_{W'} &= [W]_{W'}[T(V)]_W [V']_V \\
&= W'^{-1}W[T(V)]_W V^{-1}V' \\
&= W'^{-1}WW^{-1}T(V)V^{-1}V' \\
&= W'^{-1}IT(V)V^{-1}V' \\
&= W'^{-1}T(V)V^{-1}V'
\end{aligned}$$

---

$[T(V')]_{W'} \rightleftharpoons [T(V)]_W$

$$[T(V')]_{W'} = [W]_{W'}[T(V)]_W [V']_V$$

---

$$[T(V')]_{W'} = W'^{-1}T(V)V^{-1}V'$$

---

$$\begin{aligned}
W'[T(V)]_{W'} &= T(V) \\
&\Downarrow \\
[T(V)]_{W'} &= W'^{-1}T(V)
\end{aligned}$$

---

$$\begin{aligned}
[T(V')]_{W'} &= W'^{-1}T(V)V^{-1}V' \\
&= \{W'^{-1}T(V)\}\{V^{-1}V'\} \\
&= \{[T(V)]_{W'}\}\{V^{-1}V'\} \\
&= [T(V)]_{W'}V^{-1}V' \\
&, \begin{cases} [T(V)]_{W'} = W'^{-1}T(V) & : W'[T(V)]_{W'} = T(V) \\ V^{-1}V' & : [V|V'] \overset{\text{Gauss-Jordan delimination}}{\longrightarrow} [I|V^{-1}V'] \end{cases}
\end{aligned}$$

---

$T' \rightleftharpoons T$

$$\begin{aligned}
[T(V')]_{W'} &= [W]_{W'}[T(V)]_W [V']_V \\
T' = [T(V')]_{W'} &\Updownarrow T = [T(V)]_W \\
T' &= [W]_{W'} T [V']_V \\
&= [W'^{-1}W] T [V^{-1}V'] \\
T' &= W'^{-1}WTV^{-1}V'
\end{aligned}$$

---

$$T' = W'^{-1}WTV^{-1}V'$$

#### 50.2.3.1 differential polynomial

##### 50.2.3.1.1 Chebyshev polynomial

##### 50.2.3.1.2 Hermite polynomial

### 50.2.4 singular value decomposition

https://www.youtube.com/watch?v=oPHM-ZWWvlg&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=42

https://www.youtube.com/watch?v=zagHmMPZyoo&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=43

## 50.3 Chi, Chen-Yu

https://www.youtube.com/playlist?list=PLJWAeYEa8SXBej3kuQMz8vV41VabZUILb

# Chapter 51

# analysis

## 51.1 real analysis

https://zhuanlan.zhihu.com/p/665222184

## 51.2 Chi, Chen-Yu

https://www.youtube.com/playlist?list=PLVJXJebpO4PhAc21JW-cYbzT3sq4s7Qg8

https://www.youtube.com/playlist?list=PLil-R4o6jmGihq7XzdNzb0d5hHqEJbr6L

https://www.youtube.com/playlist?list=PLil-R4o6jmGhUqtKbZf0LIFKd-xN__g_M

https://www.youtube.com/playlist?list=PLil-R4o6jmGhkuZPmKL_A5Y7N4HOsa1nX

## 51.3 Chen, Jin-Tzu

https://www.youtube.com/playlist?list=PLil-R4o6jmGjoxAWZurHXAY0q9yxwXv5F

## 51.4 $\sigma$-algebra

https://www.zhihu.com/question/68135067/answer/259982491

## 51.5 complex anaylsis

### 51.5.1 Mathemaniac

https://www.youtube.com/playlist?list=PLDcSwjT2BF_UDdkQ3KQjX5SRQ2DLLwv0R

# Chapter 52

# matrix calculus

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https://www.youtube.com/playlist?list=PLhcN-s3_Z7-YS6ltpJhjwqvHO1TYDbiZv

$$\boldsymbol{x} = \langle x_1, x_2, \cdots, x_n \rangle = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\intercal = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(x_1, x_2, \cdots, x_n) = f(\langle x_1, x_2, \cdots, x_n \rangle) = f(\boldsymbol{x})$$

$$\boldsymbol{y} = \langle y_1, y_2, \cdots, y_m \rangle = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^\intercal = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

## 52.1 vector-by-scalar

$$\frac{\partial \boldsymbol{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$

## 52.2 scalar-by-vector

$$\boldsymbol{\nabla} f = \frac{\partial}{\partial \boldsymbol{x}} f = \frac{\partial f}{\partial \boldsymbol{x}} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^\intercal = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$f_i = f_i(x_1, x_2, \cdots, x_n) = f_i(\boldsymbol{x})$$

$$\boldsymbol{f} = \langle f_1, f_2, \cdots, f_m \rangle = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix}^\intercal = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

## 52.3 vector-by-vector

### 52.3.1 numerator-layout notation

分子布局

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \boldsymbol{x}} \\ \frac{\partial y_2}{\partial \boldsymbol{x}} \\ \vdots \\ \frac{\partial y_p}{\partial \boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{y}}{\partial x_1} & \frac{\partial \boldsymbol{y}}{\partial x_2} & \cdots & \frac{\partial \boldsymbol{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \left[\frac{\partial y_i}{\partial x_j}\right]_{m\times n}$$

### 52.3.2 denominator-layout notation

分母布局

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial \boldsymbol{y}}{\partial x_1} \\ \frac{\partial \boldsymbol{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \boldsymbol{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial \boldsymbol{x}} & \frac{\partial y_2}{\partial \boldsymbol{x}} & \cdots & \frac{\partial y_m}{\partial \boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \left[\frac{\partial y_j}{\partial x_i}\right]_{n\times m}$$

https://zhuanlan.zhihu.com/p/692195114

# Chapter 53

# genetics

楊欣洲: genetics https://www.youtube.com/playlist?list=PLTp0eSi9MdkPp0swo8-VVplaG8bateq7q

## 53.1 term

- chromosome
- locus (sl.) loci (pl.)
- marker
- allele
- haplotype
- genotype
- phenotype / trait
  - endophenotype

## 53.2 marker

- large marker: STRP = short tandem repeat polymorphism, STRs = short tandem repeats
  - linkage analysis
  - paternity testing
  - taxonomy
- small marker: SNP = single-nucleotide polymorphism
  - association analysis
  - disease diagnosis
  - pharmacodynamics? drug design / pharmacogenomics = custom drug
- RFLP = restriction fragment length polymorphism
- platform
  - customized: MALDI-TOF MS(= mass spectrometry)
  - whole-genome gene chip
    - Affymetrix
      - Taiwan BioBank: TWB2.0
    - Illumina
  - NGS = next-generation sequencing

## 53.3 genome project

- HGP = Human Genome Project
  - 20~25K genes
  - 3 billion bps(base pairs)
  - ELSI = ethical, legal, and social issues
  - 99.9% bps are exactly the same in all people
  - germline mutation
    - male : female = 2 : 1
- HapMap = International HapMap Project
  - SNP, haplotype, tag SNP
    - haplotypes are combination of SNPs
    - tag SNPs can identify unique haplotypes

    - HapMap 1 & 2
        * between-ancestry SNP: 1 common SNP per 5 Kb to 1 common SNP per 1 Kb
            · African
            · European
            · East Asian
                · Han Chinese
                · Japanese
    - HapMap 3: more ancestries
- 1000 Genomes Project
    - NGS
    - identify 95% genetic variants with frequencies at least 1%
    - final phase 77M SNPs
    - browser
        * Ensembl GRCh37
        * Ensembl GRCh38 http://asia.ensembl.org/Homo_sapiens/Info/Index
- TWB = Taiwan BioBank
    - browser: TaiwanView https://taiwanview.twbiobank.org.tw/index
    - pricing: https://www.biobank.org.tw/about_price.php
- TPMI = Taiwan Precision Medicine Initiative https://tpmi.ibms.sinica.edu.tw/www/precision-medicine/

## 53.4 linkage analysis

- Mendel $1^{st}$ & $2^{nd}$ laws
    - law of segregation $\sim 3 : 1$
    - law of assortment $\sim 9 : 3 : 3 : 1$
- phenotypic model by R.A. Fisher
    - $$P = G + E$$
        * $G$ is the genotypic component
        * $E$ is the environmental component
    - $$P = G + E + G \cdot E$$
        * $G \cdot E$ is the interaction between the genotypic component and environmental component
- linkage = co-segregation = cosegregation
    - $\theta$ = recombination fraction: 1% recombination = 1 crossover per 100 meioses = 1 cM(centiMorgan) on genetic map
- statistical hypothesis testing for in linkage mapping
    - statistical hypothesis testing for categorical trait in linkage mapping: PLA = parametric linkage analysis
        * $H_0$: no linkage $\theta = 0.5$
        * $H_1$: linkage $\theta < 0.5$
    - statistical hypothesis testing for quantitative trait in linkage mapping: VCLA = variance component linkage analysis
        * $H_0$: no linkage $\sigma_q^2 = 0$
        * $H_1$: linkage $\sigma_q^2 > 0$
- study design
    - case control
    - trio
    - affected / discordant sib-pair
    - extended pedigree
- data format: linkage format
    - family-based
        * PID = pedigree Id
        * IID = individual Id
        * FID = father Id
        * MID = mother Id
        * gender
        * affection
        * marker
            · M1 = marker 1
            · M2 = marker 2
            · ...
- single major locus model
    - a two-allele $A$ and $a$ locus influences a dichotomous trait

  - allele frequency
    * $p = \mathrm{P}(A)$
    * $q = \mathrm{P}(a) = 1 - p$
  - penetrance
    * $f_{AA} = \mathrm{P}(\text{affected} \mid AA)$
    * $f_{Aa} = \mathrm{P}(\text{affected} \mid Aa) = f_{aA}$
    * $f_{aa} = \mathrm{P}(\text{affected} \mid aa)$
  - disease mode of inheritance
    * dominant model
      · $\begin{cases} f_{AA} = 1 \\ f_{Aa} = 1 \\ f_{aa} = 0 \end{cases}$
    * recessive model
      · $\begin{cases} f_{AA} = 0 \\ f_{Aa} = 0 \\ f_{aa} = 1 \end{cases}$
    * additive model
      · $\begin{cases} f_{AA} = 1 \\ f_{Aa} = \frac{1}{2} \\ f_{aa} = 0 \end{cases}$
    * phenocopy model
      · $f_{aa} > 0$ perhaps due to environmental cause
    * liability model
      · e.g. $f_{AA}, f_{Aa}, f_{aa}$ are age-dependent

### 53.4.1 PLA = parametric linkage analysis

- LOD score

### 53.4.2 VCLA = variance component linkage analysis

- allele-sharing: IBS and IBD
  - IBS = identity-by-state
  - IBD = identity-by-descent

## 53.5 association analysis

- LD = linkage disequilibrium
- genotype & allele frequency
  - diallelic marker
    * $p_{AA}$
    * $p_{Aa}$
    * $p_{aA}$
    * $p_{aa}$
    * $p_A$
    * $p_a$
- HWC = Hardy-Weinberg condition
  - HWE = Hardy-Weinberg equilibrium
  - HWD = Hardy-Weinberg disequilibrium
- gametic or haplotype frequency
  - LE = linkage equilibrium
  - LD = linkage disequilibrium

# Chapter 54

# multivariate normal distribution

**Definition 54.1.** probability density function (PDF) of normal distribution (= Gaussian distribution)

$$\mathcal{N}\left(x \mid \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

If a continuous random variable $X$ follows a normal distribution with mean 0 and variance $\sigma^2$

$$\begin{aligned} X &\sim \mathrm{n}\left(\mu, \sigma^2\right) = \mathcal{N}\left(\mu, \sigma^2\right) \\ \Leftrightarrow f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{\mathrm{e}^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = \mathrm{n}\left(x \mid \mu, \sigma^2\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \mathcal{N}\left(x \mid \mu, \sigma^2\right) \end{aligned}$$

A random variable $X$ can be standardized by subtracting the mean $\mu$ and dividing by the standard deviation $\sigma$, resulting in the standardized random variable $Z$

$$Z = \frac{X-\mu}{\sigma} \text{ or } z = \frac{x-\mu}{\sigma}$$

The standardized random variable $Z$ follows the standard normal distribution

$$\begin{aligned} Z &\sim \mathrm{n}\left(0, 1^2\right) = \mathcal{N}\left(0, 1^2\right) \\ \Leftrightarrow f_Z(z) &= \frac{\mathrm{e}^{\frac{-1}{2}\left(\frac{x-0}{1}\right)^2}}{1 \cdot \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \\ &= \mathrm{n}\left(z \mid 0, 1^2\right) = \mathcal{N}\left(z \mid 0, 1^2\right) \end{aligned}$$

To generalize from univariate random variables to multivariate random vectors, a random vector[15]

$$\boldsymbol{Z} = \langle Z_1, Z_2, \cdots, Z_p \rangle = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_p \end{bmatrix}^\mathsf{T} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}$$

with $p$ random variable components is said to follow the standard multivariate normal distribution if and only if its joint PDF is given by

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{\boldsymbol{z}^\mathsf{T}\boldsymbol{z}}{2}\right\} = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{\boldsymbol{z} \cdot \boldsymbol{z}}{2}\right\} \tag{54.1}$$

(54.1) can be rewritten as the following

$$\begin{aligned}
f_{\boldsymbol{Z}}(\boldsymbol{z}) &= \underbrace{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\cdots\frac{1}{\sqrt{2\pi}}}_{p \text{ times}}\exp\left\{-\frac{z_1^2}{2}-\frac{z_2^2}{2}-\cdots-\frac{z_p^2}{2}\right\}\\
&=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^2}{2}\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_2^2}{2}\right\}\cdots\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_p^2}{2}\right\}\\
&=f(z_1)\,f(z_2)\cdots f(z_p)
\end{aligned}$$

where

$$f_{Z_i}(z_i)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_i^2}{2}\right\}=f(z_i)\Rightarrow Z_i\sim\mathcal{N}\left(0,1^2\right)=\mathrm{n}\left(0,1^2\right) \tag{54.2}$$

$$\begin{aligned}
f_{Z_i}(z_i) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\boldsymbol{Z}}(z_1,\ldots,z_{i-1},z_i,z_{i+1},\ldots,z_p)\,\mathrm{d}z_1\cdots\mathrm{d}z_{i-1}\mathrm{d}z_{i+1}\cdots\mathrm{d}z_p\\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(z_1)\cdots f(z_{i-1})\,f(z_i)\,f(z_{i+1})\cdots f(z_p)\,\mathrm{d}z_1\cdots\mathrm{d}z_{i-1}\mathrm{d}z_{i+1}\cdots\mathrm{d}z_p\\
&=f(z_i)\int_{-\infty}^{\infty}f(z_1)\,\mathrm{d}z_1\cdots\int_{-\infty}^{\infty}f(z_{i-1})\,\mathrm{d}z_{i-1}\int_{-\infty}^{\infty}f(z_{i+1})\,\mathrm{d}z_{i+1}\cdots\int_{-\infty}^{\infty}f(z_p)\,\mathrm{d}z_p\\
&=f(z_i)\overset{(54.2)}{=}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_i^2}{2}\right\}
\end{aligned}$$

covariance matrix[15]

**Definition 54.2.** covariance matrix of a random vector[8]

$$\mathrm{C}[\boldsymbol{X}]=\mathrm{Cov}[\boldsymbol{X}]=\mathrm{V}[\boldsymbol{X}]=\mathrm{E}\left[[\boldsymbol{X}-\mathrm{E}(\boldsymbol{X})][\boldsymbol{X}-\mathrm{E}(\boldsymbol{X})]^{\mathsf{T}}\right]$$

$$\boldsymbol{X}=\langle X_1,X_2,\cdots,X_p\rangle=\begin{bmatrix}X_1 & X_2 & \cdots & X_p\end{bmatrix}^{\mathsf{T}}=\begin{bmatrix}X_1\\X_2\\\vdots\\X_p\end{bmatrix}$$

$$\mathrm{E}[\boldsymbol{X}]=\langle\mathrm{E}[X_1],\mathrm{E}[X_2],\cdots,\mathrm{E}[X_p]\rangle=\begin{bmatrix}\mathrm{E}[X_1] & \mathrm{E}[X_2] & \cdots & \mathrm{E}[X_p]\end{bmatrix}^{\mathsf{T}}=\begin{bmatrix}\mathrm{E}[X_1]\\\mathrm{E}[X_2]\\\vdots\\\mathrm{E}[X_p]\end{bmatrix}$$

$$\boldsymbol{X}-\mathrm{E}[\boldsymbol{X}]=\begin{bmatrix}X_1-\mathrm{E}[X_1]\\X_2-\mathrm{E}[X_2]\\\vdots\\X_p-\mathrm{E}[X_p]\end{bmatrix}$$

$$\begin{aligned}
&[\boldsymbol{X}-\mathrm{E}(\boldsymbol{X})][\boldsymbol{X}-\mathrm{E}(\boldsymbol{X})]^{\mathsf{T}}\\
&=\begin{bmatrix}X_1-\mathrm{E}[X_1]\\X_2-\mathrm{E}[X_2]\\\vdots\\X_p-\mathrm{E}[X_p]\end{bmatrix}\begin{bmatrix}X_1-\mathrm{E}[X_1] & X_2-\mathrm{E}[X_2] & \cdots & X_p-\mathrm{E}[X_p]\end{bmatrix}\\
&=\begin{bmatrix}(X_1-\mathrm{E}[X_1])(X_1-\mathrm{E}[X_1]) & (X_1-\mathrm{E}[X_1])(X_2-\mathrm{E}[X_2]) & \cdots & (X_1-\mathrm{E}[X_1])(X_p-\mathrm{E}[X_p])\\(X_2-\mathrm{E}[X_2])(X_1-\mathrm{E}[X_1]) & (X_2-\mathrm{E}[X_2])(X_2-\mathrm{E}[X_2]) & \cdots & (X_2-\mathrm{E}[X_2])(X_p-\mathrm{E}[X_p])\\\vdots & \vdots & \ddots & \vdots\\(X_p-\mathrm{E}[X_p])(X_1-\mathrm{E}[X_1]) & (X_p-\mathrm{E}[X_p])(X_2-\mathrm{E}[X_2]) & \cdots & (X_p-\mathrm{E}[X_p])(X_p-\mathrm{E}[X_p])\end{bmatrix}\\
&=\begin{bmatrix}(X_1-\mathrm{E}[X_1])^2 & \cdots & (X_1-\mathrm{E}[X_1])(X_p-\mathrm{E}[X_p])\\\vdots & \ddots & \vdots\\(X_p-\mathrm{E}[X_p])(X_1-\mathrm{E}[X_1]) & \cdots & (X_p-\mathrm{E}[X_p])^2\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{E}\left[\left[\boldsymbol{X}-\mathrm{E}\left(\boldsymbol{X}\right)\right]\left[\boldsymbol{X}-\mathrm{E}\left(\boldsymbol{X}\right)\right]^{\intercal}\right] && (54.3)\\
=&\mathrm{E}\begin{bmatrix}\left(X_1-\mathrm{E}\left[X_1\right]\right)^2 & \cdots & \left(X_1-\mathrm{E}\left[X_1\right]\right)\left(X_p-\mathrm{E}\left[X_p\right]\right)\\ \vdots & \ddots & \vdots\\ \left(X_p-\mathrm{E}\left[X_p\right]\right)\left(X_1-\mathrm{E}\left[X_1\right]\right) & \cdots & \left(X_p-\mathrm{E}\left[X_p\right]\right)^2\end{bmatrix} && (54.4)\\
=&\begin{bmatrix}\mathrm{E}\left[\left(X_1-\mathrm{E}\left[X_1\right]\right)^2\right] & \cdots & \mathrm{E}\left[\left(X_1-\mathrm{E}\left[X_1\right]\right)\left(X_p-\mathrm{E}\left[X_p\right]\right)\right]\\ \vdots & \ddots & \vdots\\ \mathrm{E}\left[\left(X_p-\mathrm{E}\left[X_p\right]\right)\left(X_1-\mathrm{E}\left[X_1\right]\right)\right] & \cdots & \mathrm{E}\left[\left(X_p-\mathrm{E}\left[X_p\right]\right)^2\right]\end{bmatrix} && (54.5)\\
=&\begin{bmatrix}\mathrm{V}\left(X_1,X_1\right) & \cdots & \mathrm{V}\left(X_1,X_p\right)\\ \vdots & \ddots & \vdots\\ \mathrm{V}\left(X_p,X_1\right) & \cdots & \mathrm{V}\left(X_p,X_p\right)\end{bmatrix}=\begin{bmatrix}\mathrm{V}\left(X_1,X_1\right) & \mathrm{V}\left(X_1,X_2\right) & \cdots & \mathrm{V}\left(X_1,X_p\right)\\ \mathrm{V}\left(X_2,X_1\right) & \mathrm{V}\left(X_2,X_2\right) & \cdots & \mathrm{V}\left(X_2,X_p\right)\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{V}\left(X_p,X_1\right) & \mathrm{V}\left(X_p,X_2\right) & \cdots & \mathrm{V}\left(X_p,X_p\right)\end{bmatrix} && (54.6)\\
=&\begin{bmatrix}\mathrm{V}\left(X_1\right) & \cdots & \mathrm{V}\left(X_1,X_p\right)\\ \vdots & \ddots & \vdots\\ \mathrm{V}\left(X_p,X_1\right) & \cdots & \mathrm{V}\left(X_p\right)\end{bmatrix}=\begin{bmatrix}\mathrm{V}\left(X_1\right) & \mathrm{V}\left(X_1,X_2\right) & \cdots & \mathrm{V}\left(X_1,X_p\right)\\ \mathrm{V}\left(X_2,X_1\right) & \mathrm{V}\left(X_2\right) & \cdots & \mathrm{V}\left(X_2,X_p\right)\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{V}\left(X_p,X_1\right) & \mathrm{V}\left(X_p,X_2\right) & \cdots & \mathrm{V}\left(X_p\right)\end{bmatrix} && (54.7)\\
=&\begin{bmatrix}\mathrm{V}\left(X_1\right) & \cdots & \mathrm{C}\left(X_1,X_p\right)\\ \vdots & \ddots & \vdots\\ \mathrm{C}\left(X_p,X_1\right) & \cdots & \mathrm{V}\left(X_p\right)\end{bmatrix}=\begin{bmatrix}\mathrm{V}\left(X_1\right) & \mathrm{C}\left(X_1,X_2\right) & \cdots & \mathrm{C}\left(X_1,X_p\right)\\ \mathrm{C}\left(X_2,X_1\right) & \mathrm{V}\left(X_2\right) & \cdots & \mathrm{C}\left(X_2,X_p\right)\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{C}\left(X_p,X_1\right) & \mathrm{C}\left(X_p,X_2\right) & \cdots & \mathrm{V}\left(X_p\right)\end{bmatrix} && (54.8)\\
=&\begin{bmatrix}\sigma_1^2 & \cdots & \sigma_{1p}\\ \vdots & \ddots & \vdots\\ \sigma_{p1} & \cdots & \sigma_p^2\end{bmatrix}=\begin{bmatrix}\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp}\end{bmatrix}=\left[\sigma_{ij}\right]_{p\times p}=\mathit{\Sigma} && (54.9)
\end{aligned}
$$

$$\boldsymbol{X}\sim\mathcal{D}\left(\boldsymbol{\mu},\mathit{\Sigma}\right)=\mathrm{d}\left(\boldsymbol{\mu}_{\boldsymbol{X}},\mathit{\Sigma}_{\boldsymbol{X}}\right)=\mathrm{d}\left(\mathrm{E}\left[\boldsymbol{X}\right],\mathrm{C}\left[\boldsymbol{X}\right]\right)=\mathrm{d}\left(\mathrm{E}\left[\boldsymbol{X}\right],\mathrm{V}\left[\boldsymbol{X}\right]\right)$$

$$\boldsymbol{Z}\sim\mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{Z}},\mathit{\Sigma}_{\boldsymbol{Z}}\right)=\mathrm{n}\left(\mathrm{E}\left[\boldsymbol{Z}\right],\mathrm{V}\left[\boldsymbol{Z}\right]\right)$$

$$
\begin{aligned}
&\mathrm{E}\left[\boldsymbol{Z}\right]=\begin{bmatrix}\mathrm{E}\left[Z_1\right]\\ \mathrm{E}\left[Z_2\right]\\ \vdots\\ \mathrm{E}\left[Z_p\right]\end{bmatrix}=\left[\mathrm{E}\left[Z_i\right]\right]_{p\times 1}\\
\Rightarrow&\mathrm{E}\left[Z_i\right]=\int_{-\infty}^{\infty}z_i f_{Z_i}\left(z_i\right)\mathrm{d}z_i\overset{(54.2)}{=}\int_{-\infty}^{\infty}z_i\frac{\mathrm{e}^{\frac{-1}{2}z_i^2}}{\sqrt{2\pi}}\mathrm{d}z_i=0\\
\Rightarrow&\mathrm{E}\left[\boldsymbol{Z}\right]=\mathbf{0}\\
\Rightarrow&\boldsymbol{Z}\sim\mathcal{N}\left(\boldsymbol{\mu}_{\boldsymbol{Z}}=\mathbf{0},\mathit{\Sigma}_{\boldsymbol{Z}}\right)=\mathrm{n}\left(\mathbf{0},\mathrm{V}\left[\boldsymbol{Z}\right]\right)
\end{aligned}
$$

$$\mathrm{V}\left(Z_i\right)=\int_{-\infty}^{\infty}\left(z_i-\mu_{Z_i}\right)^2 f_{Z_i}\left(z_i\right)\mathrm{d}z_i\overset{(54.2)}{=}\int_{-\infty}^{\infty}\left(z_i-0\right)^2\frac{\mathrm{e}^{\frac{-1}{2}z_i^2}}{\sqrt{2\pi}}\mathrm{d}z_i=1$$

$$\mathrm{V}\left(Z_i,Z_j\right)\overset{i\neq j\Rightarrow Z_i,Z_j\text{are independent}}{=}0 \tag{54.10}$$

$$
\begin{aligned}
\mathrm{V}[\boldsymbol{Z}] &= \begin{bmatrix} \mathrm{V}(Z_1) & \mathrm{V}(Z_1, Z_2) & \cdots & \mathrm{V}(Z_1, Z_p) \\ \mathrm{V}(Z_2, Z_1) & \mathrm{V}(Z_2) & \cdots & \mathrm{V}(Z_2, Z_p) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{V}(Z_p, Z_1) & \mathrm{V}(Z_p, Z_2) & \cdots & \mathrm{V}(Z_p) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix} \overset{54.10}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_{p\times p} = I_p = I
\end{aligned}
$$

$$
\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Z}}, \varSigma_{\boldsymbol{Z}}) = \mathrm{n}(\mathrm{E}[\boldsymbol{Z}], \mathrm{V}[\boldsymbol{Z}]) = \mathcal{N}(\boldsymbol{0}, I) \Leftrightarrow \begin{cases} \boldsymbol{\mu}_{\boldsymbol{Z}} = \mathrm{E}[\boldsymbol{Z}] = \boldsymbol{0} = [0]_p = [0]_{p\times 1} \\ \varSigma_{\boldsymbol{Z}} = \mathrm{V}[\boldsymbol{Z}] = I = I_p = I_{p\times p} \end{cases}
$$

$$
\begin{aligned}
\boldsymbol{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} &= \begin{bmatrix} \dfrac{X_1-\mu_1}{\sigma_1} \\ \dfrac{X_2-\mu_2}{\sigma_2} \\ \vdots \\ \dfrac{X_p-\mu_p}{\sigma_p} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \begin{bmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ \vdots \\ X_p-\mu_p \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \left( \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \right) = B^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \\
\Rightarrow \boldsymbol{X} &= B\boldsymbol{Z} + \boldsymbol{\mu}
\end{aligned}
$$

$$
\boldsymbol{X} = B\boldsymbol{Z} + \boldsymbol{\mu} = T(\boldsymbol{Z})
$$

by $\mathrm{V}[A\boldsymbol{X} + \boldsymbol{b}] = A\mathrm{V}[\boldsymbol{X}]A^\intercal$

$$
\varSigma = \varSigma_{\boldsymbol{X}} = \mathrm{V}[\boldsymbol{X}] = \mathrm{V}[B\boldsymbol{Z} + \boldsymbol{\mu}] = B\mathrm{V}[\boldsymbol{Z}]B^\intercal = BIB^\intercal = BB^\intercal \tag{54.11}
$$

Consider two infinitesimal volumes of $p$-dimensional parallelepipeds in the different $\mathbb{R}^p$ spaces[16]

$$
V_{\boldsymbol{x}} = [x_1, x_1 + \mathrm{d}x_1] \times [x_2, x_2 + \mathrm{d}x_2] \times \cdots \times [x_p, x_p + \mathrm{d}x_p]
$$

and

$$
V_{\boldsymbol{z}} = [z_1, z_1 + \mathrm{d}z_1] \times [z_2, z_2 + \mathrm{d}z_2] \times \cdots \times [z_p, z_p + \mathrm{d}z_p]
$$

Their relationship under linear transformation is

$$
\begin{aligned}
V_{\boldsymbol{x}} = T(V_{\boldsymbol{z}}) = &[T(z_1), T(z_1) + T(\mathrm{d}z_1)] \\
&\times [T(z_2), T(z_2) + T(\mathrm{d}z_2)] \\
&\times \cdots \\
&\times [T(z_p), T(z_p) + T(\mathrm{d}z_p)]
\end{aligned}
$$

and

$$
\mathrm{d}x_i = \sum_j \frac{\partial x_i}{\partial z_j} \mathrm{d}z_j
$$

For examples in 2 dimension,

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial z_1} & \dfrac{\partial x_1}{\partial z_2} \\ \dfrac{\partial x_2}{\partial z_1} & \dfrac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \end{bmatrix}$$

Two element infinitesimal one-directional vectors of $\boldsymbol{Z}$ transformed into another space of $\boldsymbol{X}$ are

$$T\left(\mathrm{d}\boldsymbol{z}_1\right) = \begin{bmatrix} \dfrac{\partial x_1}{\partial z_1} & \dfrac{\partial x_1}{\partial z_2} \\ \dfrac{\partial x_2}{\partial z_1} & \dfrac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \mathrm{d}z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial z_1}\mathrm{d}z_1 \\ \dfrac{\partial x_2}{\partial z_1}\mathrm{d}z_1 \end{bmatrix}$$

and

$$T\left(\mathrm{d}\boldsymbol{z}_2\right) = \begin{bmatrix} \dfrac{\partial x_1}{\partial z_1} & \dfrac{\partial x_1}{\partial z_2} \\ \dfrac{\partial x_2}{\partial z_1} & \dfrac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{d}z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial z_2}\mathrm{d}z_2 \\ \dfrac{\partial x_2}{\partial z_2}\mathrm{d}z_2 \end{bmatrix}$$

Their corresponding area(volume) in the space of $\boldsymbol{X}$ is

$$\begin{aligned} &\int_{A_{\boldsymbol{x}}} \mathrm{d}A_{\boldsymbol{x}} = \int_{A_{\boldsymbol{x}}} \mathrm{d}x_1\mathrm{d}x_2 = \int_{T(A_{\boldsymbol{z}})} \mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_{A_{\boldsymbol{z}}} \left|\left[T\left(\mathrm{d}\boldsymbol{z}_1\right) \quad T\left(\mathrm{d}\boldsymbol{z}_2\right)\right]\right| = \int_{A_{\boldsymbol{z}}} \begin{vmatrix} \dfrac{\partial x_1}{\partial z_1}\mathrm{d}z_1 & \dfrac{\partial x_1}{\partial z_2}\mathrm{d}z_2 \\ \dfrac{\partial x_2}{\partial z_1}\mathrm{d}z_1 & \dfrac{\partial x_2}{\partial z_2}\mathrm{d}z_2 \end{vmatrix} \\ &= \int_{A_{\boldsymbol{z}}} \begin{vmatrix} \dfrac{\partial x_1}{\partial z_1} & \dfrac{\partial x_1}{\partial z_2} \\ \dfrac{\partial x_2}{\partial z_1} & \dfrac{\partial x_2}{\partial z_2} \end{vmatrix} \mathrm{d}z_1\mathrm{d}z_2 = \int_{A_{\boldsymbol{z}}} |J|\,\mathrm{d}A_{\boldsymbol{z}} \end{aligned}$$

To generalize for volumes in $p$ dimension,

$$\begin{aligned} &\int_{V_{\boldsymbol{x}}} \mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{x}}} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_p = \int_{T(V_{\boldsymbol{z}})} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_p \\ &= \int_{A_{\boldsymbol{z}}} \left|\left[T\left(\mathrm{d}\boldsymbol{z}_1\right) \quad T\left(\mathrm{d}\boldsymbol{z}_2\right) \quad \cdots \quad T\left(\mathrm{d}\boldsymbol{z}_p\right)\right]\right| = \int_{V_{\boldsymbol{z}}} \left|\left[\frac{\partial x_i}{\partial z_j}\mathrm{d}z_j\right]_{p\times p}\right| \\ &= \int_{V_{\boldsymbol{z}}} \left|\left[\frac{\partial x_i}{\partial z_j}\right]_{p\times p}\right| \mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_p = \int_{V_{\boldsymbol{z}}} |J|\,\mathrm{d}V_{\boldsymbol{z}} \end{aligned}$$

i.e.

$$\int_{V_{\boldsymbol{x}}} \mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{z}}} |J|\,\mathrm{d}V_{\boldsymbol{z}} \tag{54.12}$$

where $J$ is a Jacobian matrix

$$J = \left[\frac{\partial x_i}{\partial z_j}\right]_{p\times p} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}$$

or $|J|$ is a Jacobian determinant or simply Jacobian

$$|J| = \left|\frac{\partial x_i}{\partial z_j}\right|_{p\times p} = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}\right|$$

The probability of the same event should be invariant under transformation.

$$\begin{aligned}
&\int_{V_{\boldsymbol{x}}} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{x}}} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_p \\
=&\int_{T(V_{\boldsymbol{z}})} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_p \\
=&\int_{V_{\boldsymbol{z}}} f_{\boldsymbol{Z}}(\boldsymbol{z})\,\mathrm{d}V_{\boldsymbol{z}} = \int_{V_{\boldsymbol{z}}} f_{\boldsymbol{Z}}(\boldsymbol{z})\,\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_p
\end{aligned}$$

i.e.

$$\int_{V_{\boldsymbol{x}}} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{z}}} f_{\boldsymbol{Z}}(\boldsymbol{z})\,\mathrm{d}V_{\boldsymbol{z}} \tag{54.13}$$

$$\begin{cases}
\int_{V_{\boldsymbol{x}}} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{z}}} f_{\boldsymbol{Z}}(\boldsymbol{z})\,\mathrm{d}V_{\boldsymbol{z}} & 54.13 \\
\int_{V_{\boldsymbol{x}}} \mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{z}}} |J|\,\mathrm{d}V_{\boldsymbol{z}} & 54.12
\end{cases}$$

$$\begin{aligned}
\boldsymbol{Z} =& B^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \\
\boldsymbol{z} =& B^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \\
\boldsymbol{X} =& B\boldsymbol{Z}+\boldsymbol{\mu} \\
\boldsymbol{x} =& B\boldsymbol{z}+\boldsymbol{\mu}
\end{aligned} \tag{54.14}$$

$$J = \left[\frac{\partial x_i}{\partial z_j}\right]_{p\times p} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}} = B \tag{54.15}$$

$$|J| = \left|\frac{\partial x_i}{\partial z_j}\right|_{p\times p} = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{z}}\right| = |B|$$

$$\begin{aligned}
&\int_{V_{\boldsymbol{z}}} f_{\boldsymbol{Z}}(\boldsymbol{z})\,\mathrm{d}V_{\boldsymbol{z}} \overset{54.13}{=} \int_{V_{\boldsymbol{x}}} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}V_{\boldsymbol{x}} = \int_{V_{\boldsymbol{x}}} \mathrm{d}V_{\boldsymbol{x}} f_{\boldsymbol{X}}(\boldsymbol{x}) \\
\overset{54.12}{=}& \int_{V_{\boldsymbol{z}}} |J|\,\mathrm{d}V_{\boldsymbol{z}} f_{\boldsymbol{X}}(\boldsymbol{x}(\boldsymbol{z})) = \int_{V_{\boldsymbol{z}}} f_{\boldsymbol{X}}(\boldsymbol{x}(\boldsymbol{z}))\,|J|\,\mathrm{d}V_{\boldsymbol{z}} \\
f_{\boldsymbol{Z}}(\boldsymbol{z}) \overset{\Downarrow}{=}& f_{\boldsymbol{X}}(\boldsymbol{x}(\boldsymbol{z}))\,|J| \\
f_{\boldsymbol{X}}(\boldsymbol{x}(\boldsymbol{z})) \overset{\Downarrow}{=}& |J|^{-1} f_{\boldsymbol{Z}}(\boldsymbol{z}) \overset{54.1}{=} |J|^{-1}\frac{1}{(2\pi)^{p/2}}\exp\left\{\frac{-\boldsymbol{z}^\intercal\boldsymbol{z}}{2}\right\} \\
f_{\boldsymbol{X}}(\boldsymbol{x}) \overset{\Downarrow}{=}& |J|^{-1} f_{\boldsymbol{Z}}(\boldsymbol{z}(\boldsymbol{x})) \overset{54.15,54.14}{=} |B|^{-1} f_{\boldsymbol{Z}}\left(B^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right) \\
=& |B|^{-1}(2\pi)^{-p/2}\exp\left\{\frac{-1}{2}\left[B^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]^\intercal\left[B^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]\right\} \\
=& |B|^{-1/2}|B|^{-1/2}(2\pi)^{-p/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\left(B^{-1}\right)^\intercal B^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\
=& |B|^{-1/2}|B^\intercal|^{-1/2}(2\pi)^{-p/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\left(B^\intercal\right)^{-1} B^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\
=& |BB^\intercal|^{-1/2}(2\pi)^{-p/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\left(BB^\intercal\right)^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\
\overset{54.11}{=}& |\Sigma|^{-1/2}(2\pi)^{-p/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\
=& \left(|\Sigma|(2\pi)^p\right)^{-1/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}
\end{aligned}$$

**Definition 54.3.** probability density function (PDF) of multivariate normal distribution (= multivariate Gaussian distribution)

$$\mathcal{N}(\boldsymbol{x}\mid\boldsymbol{\mu},\Sigma) = f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(|\Sigma|(2\pi)^p\right)^{-1/2}\exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

**Definition 54.4.** correlation coefficient

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2}\sqrt{\sigma_j^2}} = \frac{\sigma_{ij}}{\sigma_i\sigma_j} = \frac{\mathrm{V}(X_i, X_j)}{\sqrt{\mathrm{V}(X_i)}\sqrt{\mathrm{V}(X_j)}} = \mathrm{R}(X_i, X_j)$$

## 54.1 bivariate normal distribution

$p = 2$ is the case of bivariate normal distribution

$$\mathit{\Sigma} = [\sigma_{ij}]_{2\times 2} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}$$

$$\rho_{12} = \rho = \rho_{21}$$

$$|\mathit{\Sigma}| = \begin{vmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2(1 - \rho_{12}\rho_{21}) = \sigma_1^2\sigma_2^2\left(1 - \rho^2\right)$$

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$$

$$\begin{aligned} \mathit{\Sigma}^{-1} &= \frac{1}{|\mathit{\Sigma}|}\left[\begin{array}{cc} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{array}\right] \\ &= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\begin{array}{cc} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{array}\right] \\ &= \frac{1}{(1-\rho^2)}\left[\begin{array}{cc} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{array}\right] \end{aligned}$$

$$\begin{aligned} &\mathcal{N}\left(\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathit{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}\right) \\ &= \left(|\mathit{\Sigma}|\,(2\pi)^{p=2}\right)^{-1/2} \exp\left\{\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal\,\mathit{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\ &= \left(\sigma_1^2\sigma_2^2\left(1-\rho^2\right)(2\pi)^2\right)^{-1/2} \exp\left\{\frac{-1}{2}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^\intercal \mathit{\Sigma}^{-1}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right\} \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^\intercal \frac{1}{(1-\rho^2)}\left[\begin{array}{cc} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{array}\right]\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right\} \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^\intercal \left[\begin{array}{cc} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{array}\right]\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right\} \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\} \end{aligned}$$

**Definition 54.5.** probability density function (PDF) of bivariate normal distribution (= bivariate Gaussian distribution)

$$\begin{aligned} &\mathcal{N}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}\right) \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\} \end{aligned}$$

# Chapter 55

# logic

## 55.1 Shai Ben-David

https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1p0eE5LEtv3N

LEAN[77]

# Chapter 56

# algorithm

## 56.1 Chen, Vivian

https://www.youtube.com/playlist?list=PLTpF-A8hKVUMcBaEuGyWbBXBn21seO3vA

## 56.2 The Bubble Sort Curve

https://www.youtube.com/watch?v=Gm8v_MR7TGk

https://linesthatconnect.github.io/blog/a-rigorous-derivation-of-the-bubble-sort-curve/

# Chapter 57

# computer graphics

## 57.1 Cem Yuksel

### 57.1.1 introduction

https://www.youtube.com/playlist?list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh

#### 57.1.1.1 2D transformation

https://www.youtube.com/watch?v=EKN7dTJ4ep8&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh&index=6

##### 57.1.1.1.1 translation

$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{t} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \end{bmatrix}$$

##### 57.1.1.1.2 scale

$$\boldsymbol{p}' = s\boldsymbol{p} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = s \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} sp_x \\ sp_y \end{bmatrix}$$

##### 57.1.1.1.3 non-uniform scale

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} s_x p_x \\ s_y p_y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = S\boldsymbol{p}$$

##### 57.1.1.1.4 rotation

$$\boldsymbol{p} = p_x \hat{\boldsymbol{x}} + p_y \hat{\boldsymbol{y}} = p_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = I\boldsymbol{p}$$

$$\boldsymbol{p}' = p_x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + p_y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_\theta \boldsymbol{p}$$

$$\boldsymbol{p}' = p_x \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} + p_y \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_{-\theta} \boldsymbol{p}$$

##### 57.1.1.1.5 skew = rotation + non-uniform scale + rotation

$$back\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_{-\theta} S R_\theta \boldsymbol{p}$$

##### 57.1.1.1.6 any $2 \times 2$ matrix

$$\boldsymbol{p}' = M\boldsymbol{p}$$

SVD = singular value decomposition

$$M = U\Sigma V^\intercal \overset{\text{e.g.}}{=} RSR^\intercal = R_{-\theta} S R_\theta$$

any $2 \times 2$ matrix + translation

$$\boldsymbol{p}' = M\boldsymbol{p} + \boldsymbol{t}$$

$$\boldsymbol{p}' = M_2\left(M_1\boldsymbol{p} + \boldsymbol{t}_1\right) + \boldsymbol{t}_2$$

**57.1.1.1.7 homogeneous coordinate**

$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{t}$$

$$\boldsymbol{p}' = T\boldsymbol{p} = \boldsymbol{p} + \boldsymbol{t}$$

$$\begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = T\boldsymbol{p}$$

$$\boldsymbol{p}' = M\boldsymbol{p}\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\boldsymbol{p}$$

**57.1.1.1.8 position vs. direction** position vector: transformation affected by rotation and translation

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\boldsymbol{p}$$

direction vector: transformation affected by only rotation but not translation

$$\boldsymbol{d}' = \begin{bmatrix} p'_x \\ p'_y \\ 0 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} = M\boldsymbol{d}$$

**57.1.1.2 3D transformation**

https://www.youtube.com/watch?v=1z1S2kQKXDs&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh&index=7

**57.1.1.2.1 homogeneous coordinate** affine transformation

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\boldsymbol{p}$$

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = M\boldsymbol{p}$$

**57.1.1.2.2 scale**

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = S\boldsymbol{p}$$

**57.1.1.2.3 translation**

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = T\boldsymbol{p}$$

### 57.1.2 interactive

https://www.youtube.com/playlist?list=PLplnkTzzqsZS3R5DjmCQsqupu43oS9CFN

# Chapter 58

# autoregression in time series

張翔老師. 2015. "ARMA Part1." https://www.youtube.com/watch?v=G-0dR57W-fo.

張翔老師. 2015. "ARMA Part2." https://www.youtube.com/watch?v=fQaZzO7E6FE.

張翔老師. 2015. "ARMA Part3." https://www.youtube.com/watch?v=Ocw4NXoO8Xo.

time series [data]

$$\cdots, Y_{t-2}, Y_{t-1}, Y_t, Y_{t+1}, Y_{t+2}, \cdots$$

- lag
  - $1^{\text{st}}$ lag = lag 1

  $$Y_{t-1}$$

  - $k^{\text{th}}$ lag = lag $k$

  $$Y_{t-k}$$

$1^{\text{st}}$ difference

$$\Delta Y_t = Y_t - Y_{t-1}$$

approximation for RoR = rate of return

$$\begin{aligned}\Delta \ln Y_t = \ln Y_t - \ln Y_{t-1} = &\ln \frac{Y_t}{Y_{t-1}} = \ln\left(1 + \frac{Y_t - Y_{t-1}}{Y_{t-1}}\right)\\ =&\frac{Y_t - Y_{t-1}}{Y_{t-1}} + \mathrm{O}\left(\left(\frac{Y_t - Y_{t-1}}{Y_{t-1}}\right)^2\right)\\ \approx&\frac{Y_t - Y_{t-1}}{Y_{t-1}} = \mathrm{RoR}\end{aligned}$$

$$\begin{cases}\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\end{cases}$$

$$\begin{aligned}\frac{1}{1+t} =&1 - t + t^2 - t^3 + \cdots\\ \ln(1+x) =& \int_0^x \frac{1}{1+t} dt\\ =&x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

**Definition 58.1.** autocorrelation = serial correlation

$$\exists t_1 \neq t_2 \left[\mathrm{V}\left(Y_{t_1}, Y_{t_2}\right) \neq 0\right]$$

$$\begin{cases} \mathrm{E}\left(Y_t\right) \triangleq \mu_t \\ \mathrm{V}\left(Y_t\right) \triangleq \sigma_t^2 \end{cases}$$

**Definition 58.2.** $k^{\text{th}}$ order autocovariance

$$\mathrm{V}\left(Y_t, Y_{t-k}\right) \triangleq \gamma\left(t, k\right)$$

$$\sigma_t^2 \triangleq \mathrm{V}\left(Y_t\right) = \mathrm{V}\left(Y_t, Y_t\right) = \mathrm{V}\left(Y_t, Y_{t-0}\right) = \gamma\left(t, k=0\right) = \gamma\left(t, 0\right)$$

**Definition 58.3.** $k^{\text{th}}$ order autocorrelation

$$\rho_{t,t-k} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_{tt}}\sqrt{\sigma_{t-k,t-k}}} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_t^2}\sqrt{\sigma_{t-k}^2}} = \frac{\sigma_{t,t-k}}{\sigma_t\sigma_{t-k}} = \frac{\mathrm{V}\left(Y_t, Y_{t-k}\right)}{\sqrt{\mathrm{V}\left(Y_t\right)}\sqrt{\mathrm{V}\left(Y_{t-k}\right)}} \triangleq \mathrm{R}\left(Y_t, Y_{t-k}\right) \triangleq \rho\left(t, k\right)$$

**Definition 58.4.** stationary time series

$$\begin{cases} \mathrm{E}\left(Y_t\right) \triangleq \mu_t = \mu & (1)\,\text{independent of } t \\ \mathrm{V}\left(Y_t\right) \triangleq \sigma_t^2 = \sigma^2 < \infty & (2)\,\text{independent of } t \\ \mathrm{V}\left(Y_t, Y_{t-k}\right) \triangleq \gamma\left(t, k\right) = \gamma\left(k\right) & (3)\,\text{independent of } t \end{cases}$$

properties

$$\begin{cases} \mathrm{V}\left(Y_t\right) \triangleq \sigma_t^2 = \gamma\left(t, k=0\right) = \gamma\left(k=0\right) \overset{(3)}{=} \gamma\left(0\right) \triangleq \gamma_0 \overset{(2)}{=} \sigma^2 & (4) \Rightarrow \mathrm{V}\left(Y_{t-k}\right) = \gamma\left(0\right) \\ \mathrm{R}\left(Y_t, Y_{t-k}\right) \triangleq \rho\left(t, k\right) \triangleq \dfrac{\mathrm{V}\left(Y_t, Y_{t-k}\right)}{\sqrt{\mathrm{V}\left(Y_t\right)}\sqrt{\mathrm{V}\left(Y_{t-k}\right)}} & \\ = \dfrac{\gamma\left(t, k\right)}{\sqrt{\gamma\left(0\right)}\sqrt{\gamma\left(0\right)}} \overset{(3)}{\underset{(4)}{=}} \dfrac{\gamma\left(k\right)}{\gamma\left(0\right)} \triangleq \dfrac{\gamma_k}{\gamma_0} \triangleq \rho\left(k\right) \triangleq \rho_k & (5) \\ \gamma\left(k\right) = \gamma\left(-k\right) & (6) \Rightarrow \rho\left(k\right) = \rho\left(-k\right) \end{cases}$$

$$\gamma\left(k\right) \overset{(3)}{=} \mathrm{V}\left(Y_t, Y_{t-k}\right) = \mathrm{V}\left(Y_{t-k}, Y_t\right) = \mathrm{V}\left(Y_{t'}, Y_{t'+k}\right) = \mathrm{V}\left(Y_{t'}, Y_{t'-(-k)}\right) \overset{(3)}{=} \gamma\left(-k\right) \Rightarrow (6)$$

point estimation

$$\begin{cases} \widehat{\mathrm{E}}\left(Y_t\right) \triangleq \overline{Y} \triangleq \widehat{\mu} = \dfrac{1}{T}\sum\limits_{t=1}^{T} Y_t & \to \mathrm{E}\left(Y_t\right) = \mu \\ \widehat{\mathrm{V}}\left(Y_t\right) \triangleq \widehat{\gamma}_0 = \dfrac{1}{T}\sum\limits_{t=1}^{T}\left(Y_t - \overline{Y}\right)^2 & \to \mathrm{V}\left(Y_t\right) = \sigma^2 = \gamma_0 \\ \widehat{\mathrm{V}}\left(Y_t, Y_{t-k}\right) \triangleq \widehat{\gamma}_k & \\ = \dfrac{1}{T}\sum\limits_{t=k+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-k} - \overline{Y}\right) = \dfrac{1}{T}\sum\limits_{t=1}^{T-k}\left(Y_t - \overline{Y}\right)\left(Y_{t+k} - \overline{Y}\right) & \to \mathrm{V}\left(Y_t, Y_{t-k}\right) = \gamma_k \\ \dfrac{\widehat{\mathrm{V}}\left(Y_t, Y_{t-k}\right)}{\sqrt{\widehat{\mathrm{V}}\left(Y_t\right)}\sqrt{\widehat{\mathrm{V}}\left(Y_{t-k}\right)}} \triangleq \widehat{\rho}_k = \dfrac{\widehat{\gamma}_k}{\widehat{\gamma}_0} = \dfrac{\dfrac{1}{T}\sum\limits_{t=k+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-k} - \overline{Y}\right)}{\dfrac{1}{T}\sum\limits_{t=1}^{T}\left(Y_t - \overline{Y}\right)^2} & \to \mathrm{R}\left(Y_t, Y_{t-k}\right) = \rho_k \end{cases}$$

$$Y_1Y_{1+k} + Y_2Y_{2+k} + \cdots + Y_tY_{t+k} + \cdots + Y_{T-k}Y_T$$

$1^{\text{st}}$-order autocorrelation estimation

$$\widehat{\rho}_1 = \frac{\widehat{\gamma}_1}{\widehat{\gamma}_0} = \frac{\dfrac{1}{T}\sum\limits_{t=1+1=2}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-1} - \overline{Y}\right)}{\dfrac{1}{T}\sum\limits_{t=1}^{T}\left(Y_t - \overline{Y}\right)^2}$$

## 58.1 AR(1) = $1^{\text{st}}$-order autoregressive model = first-order autoregressive model

張翔老師. 2015. "ARMA Part3." https://www.youtube.com/watch?v=Ocw4NXoO8Xo.

**Definition 58.5.** AR(1) = $1^{\text{st}}$-order autoregressive model$

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \mathcal{E}_t$$

$$\mathcal{E}_t \sim \mathcal{N}\left(0, \sigma_{\mathcal{E}}^2\right)$$

$$\begin{aligned}
Y_t =& \beta_0 + \beta_1 Y_{t-1} + \mathcal{E}_t \\
=& \beta_0 + \beta_1 \left(\beta_0 + \beta_1 Y_{t-2} + \mathcal{E}_{t-1}\right) + \mathcal{E}_t \\
=& \beta_0 \left(1 + \beta_1\right) + \beta_1^2 Y_{t-2} + \mathcal{E}_t + \beta_1 \mathcal{E}_{t-1} \\
=& \beta_0 \left(1 + \beta_1\right) + \beta_1^2 \left(\beta_0 + \beta_1 Y_{t-3} + \mathcal{E}_{t-2}\right) + \mathcal{E}_t + \beta_1 \mathcal{E}_{t-1} \\
=& \beta_0 \left(1 + \beta_1 + \beta_1^2\right) + \beta_1^3 Y_{t-3} + \mathcal{E}_t + \beta_1 \mathcal{E}_{t-1} + \beta_1^2 \mathcal{E}_{t-2} \\
&\vdots \\
=& \beta_0 \left(1 + \beta_1 + \beta_1^2 + \cdots\right) + \mathcal{E}_t + \beta_1 \mathcal{E}_{t-1} + \beta_1^2 \mathcal{E}_{t-2} + \cdots \\
\overset{|\beta_1|<1}{=}& \frac{\beta_0}{1-\beta_1} + \sum_{k=0}^{\infty} \beta_1^k \mathcal{E}_{t-k}
\end{aligned}$$

$$\begin{cases}
\mu = \mathrm{E}\left(Y_t\right) = \mathrm{E}\left(\frac{\beta_0}{1-\beta_1} + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) & \\
= \frac{\beta_0}{1-\beta_1} + \sum_{i=0}^{\infty} \beta_1^i \mathrm{E}\left(\mathcal{E}_{t-i}\right) \overset{\mathcal{E}_t \sim \mathcal{N}\left(0,\sigma_{\mathcal{E}}^2\right)}{=} \frac{\beta_0}{1-\beta_1} & \mu = \frac{\beta_0}{1-\beta_1} \\
\gamma_0 = \sigma^2 = \mathrm{V}\left(Y_t\right) = \mathrm{V}\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) = \mathrm{V}\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}\right) & \\
= \sum_{i=0}^{\infty} \left(\beta_1^i\right)^2 \mathrm{V}\left(\mathcal{E}_{t-i}\right) \overset{\mathcal{E}_t \sim \mathcal{N}\left(0,\sigma_{\mathcal{E}}^2\right)}{=} \sum_{i=0}^{\infty} \left(\beta_1^2\right)^i \sigma_{\mathcal{E}}^2 = \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \gamma_0 = \sigma^2 = \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\
\gamma_k = \mathrm{V}\left(Y_t, Y_{t-k}\right) = \mathrm{V}\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) & \\
= \mathrm{V}\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) = \cdots & \gamma_k = \frac{\beta_1^k \sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\
\rho_k = \mathrm{R}\left(Y_t, Y_{t-k}\right) = \frac{\gamma_k}{\gamma_0} = \frac{\frac{\beta_1^k \sigma_{\mathcal{E}}^2}{1-\beta_1^2}}{\frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2}} = \beta_1^k & \rho_k = \frac{\gamma_k}{\gamma_0} = \beta_1^k
\end{cases}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \beta_1^1 = \beta_1$$

$$\begin{aligned}
\gamma_k = \mathrm{V}\left(Y_t, Y_{t-k}\right) =& \mathrm{V}\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) \\
=& \mathrm{V}\left(\sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \mathcal{E}_{t-k-i}\right) = \cdots
\end{aligned}$$

$$\boldsymbol{Y} = \langle Y_1, Y_2, \cdots, Y_T \rangle = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_T \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix}$$

AR(1) covariance matrix

by (54.3)

$$
\begin{aligned}
\mathrm{V}(\boldsymbol{Y}) =& \mathrm{E}\left[[\boldsymbol{Y}-\mathrm{E}(\boldsymbol{Y})][\boldsymbol{Y}-\mathrm{E}(\boldsymbol{Y})]^{\mathrm{T}}\right] \\
=& \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_T^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_{TT} \end{bmatrix} = [\sigma_{ij}]_{T\times T} = \varSigma \\
=& \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \frac{\beta_1^1\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-1}\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\ \frac{\beta_1^1\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-2}\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_1^{T-1}\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \frac{\beta_1^{T-2}\sigma_{\mathcal{E}}^2}{1-\beta_1^2} & \cdots & \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \end{bmatrix} \\
=& \frac{\sigma_{\mathcal{E}}^2}{1-\beta_1^2} \begin{bmatrix} 1 & \beta_1 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \cdots & \beta_1^{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \beta_1 & \cdots & \beta_1^{T-2} \\ \beta_1^2 & \beta_1 & 1 & \cdots & \beta_1^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \beta_1^{T-3} & \cdots & 1 \end{bmatrix} \\
=& \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{T-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{T-1} & \rho_{T-2} & \rho_{T-3} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}
\end{aligned}
$$

$$\boldsymbol{Y} \sim \mathcal{D}(\boldsymbol{\mu}, \varSigma) = \mathrm{d}(\boldsymbol{\mu}_{\boldsymbol{Y}}, \varSigma_{\boldsymbol{Y}}) = \mathrm{d}(\mathrm{E}[\boldsymbol{Y}], \mathrm{V}[\boldsymbol{Y}])$$

where

$$\varSigma = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}$$

## 58.2 AR(2) = $2^{\text{nd}}$-order autoregressive model = second-order autoregressive model

**Definition 58.6.** AR(2) = $2^{\text{nd}}$-order autoregressive model

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \mathcal{E}_t$$

$$\mathcal{E}_t \sim \mathcal{N}\left(0, \sigma_{\mathcal{E}}^2\right)$$

## 58.3 AR($p$) = $p^{\text{th}}$-order autoregressive model

$$Y_t = \beta_0 + \sum_{i=1}^{p} \beta_i Y_{t-i} + \mathcal{E}_t$$

## 58.4 MA($q$) = $q^{\text{th}}$-order moving-average model

$$Y_t = \mu + \mathcal{E}_t + \sum_{i=1}^{q} \alpha_i \mathcal{E}_{t-i}$$

## 58.5 ARMA($p$,$q$) = $p^{\mathbf{th}}, q^{\mathbf{th}}$-order autoregressive-moving-average model

$$Y_t = \mathcal{E}_t + \sum_{i=1}^{q} \alpha_i \mathcal{E}_{t-i} + \sum_{i=1}^{p} \beta_i Y_{t-i}$$

# Chapter 59

# conductivity

## 59.1 resistance

https://www.bilibili.com/video/BV1tN41147SK

## 59.2 semiconductor

https://www.youtube.com/watch?v=-lHXZk5M6cI

# Chapter 60

# LaTeX annotation by TikZ

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

Figure 60.1: test

```
\begin{tikzpicture}
  \begin{axis}[
        %axis x line = center,
        %axis y line = center,
        %xlabel = {$ x $}, xlabel style =
        ↪  {right},
        %ylabel = {$ y=f\left(x\right) $},
        ↪  ylabel style = {above},
        xmin = -1, xmax = 1,
        ymin = -0.5, ymax = 0.5,
        %hide obscured x ticks=false, % for
        ↪  origin x tick label i.e. xtick =
        ↪  {0}
        xtick= \empty,
        xticklabels= \empty,
        %extra x ticks={0},
        ytick = \empty,
        yticklabels= \empty,
        x = 5cm, y = 5cm, % x y scaling
        %grid = major,
        domain = -1:1,
        %samples = 1000
    ]
    \node at (axis cs: 0,0) {
      $\begin{aligned}
      f\left(x\right)= & x^{x}\\
      \ln f\left(x\right)= & x\ln x\\
      \dfrac{\mathrm{d}}{\mathrm{d}x}\ln
↪   f\left(x\right)= &
↪   \dfrac{\mathrm{d}}{\mathrm{d}x}\left[x\ln
↪   x\right]\\

↪   \dfrac{f^{\prime}\left(x\right)}{f\left(x\right)}=
↪   & \left[x\ln x\right]^{\prime}\\
      f^{\prime}\left(x\right)= &
↪   f\left(x\right)\left[x\ln
↪   x\right]^{\prime}=x^{x}\left[\left(x\right)^{\prime}\ln
↪   x+x\left(\ln x\right)^{\prime}\right]\\
      = & x^{x}\left[1\ln
↪   x+x\dfrac{1}{x}\right]=x^{x}\left[\ln
↪   x+1\right]=x^{x}\left[1+\ln x\right]\\
      = & x^{x}+x^{x}\ln x
      \end{aligned}$
      };
  \end{axis}
\end{tikzpicture}
```

Figure 60.2: test

https://tex.stackexchange.com/questions/670153/how-to-annotate-calculations

Figure 60.3: test

```
\begin{tikzpicture}
  \begin{axis}[
        %axis x line = center,
        %axis y line = center,
        %xlabel = {$ x $}, xlabel style =
        ↪  {right},
        %ylabel = {$ y=f\left(x\right) $},
        ↪  ylabel style = {above},
        xmin = -0.7, xmax = 0.9,
        ymin = -0.3, ymax = 0.3,
        %hide obscured x ticks=false, % for
        ↪  origin x tick label i.e. xtick =
        ↪  {0}
        xtick= \empty,
        xticklabels= \empty,
        %extra x ticks={0},
        ytick = \empty,
        yticklabels= \empty,
        x = 5cm, y = 5cm, % x y scaling
        %grid = major,
        domain = -1:1,
        %samples = 1000
    ]
    \node at (axis cs:0,0) {

      ↪  $\tikzmarknode{a1}{\frac{4}{2}}+\tikzmarknode{a2}{\frac{6}{3}}=
      \frac{\tikzmarknode{b1}{(3\times
↪   4)}+\tikzmarknode{b2}{(2\times
↪   6)}}{\tikzmarknode{b3}{2\times
↪   3}}=\tikzmarknode{c1}{\frac{12+12}{6}}=\tikzmarknode{c2}{\frac{24}{6}}=\tikzmarknode{d1}{4}$};

    \draw[red, thick, ->, shorten <=1em, ,
    ↪  shorten >=1em](a2.south
    ↪  east)--node[below, pos=.45,
    ↪  scale=.4]{1}(a1.north west);
    \node[draw=red, thick, ellipse, fit=(b1),
    ↪  inner ysep=0, inner xsep=-1mm](ell){};
    \draw[thick, red]
    ↪  (ell)--++(0,.5)node[above, red,
    ↪  scale=.4]{This is important!};
    \draw[red, thick] (b2.south
    ↪  west)--node[above=2mm, red,
    ↪  scale=.5]{\bfseries WRONG!}(b2.north
    ↪  east);
    \draw[green!70!black, thick]
    ↪  (b3.east)--node[below]{$\uparrow$}node[below=5mm,
    ↪  align=left, green!70!black,
    ↪  scale=.5]{This multiplication is
    ↪  not\\necessary Try to
    ↪  simplify.}(b3.west)
    ↪  ([yshift=2pt]b3.east)--([yshift=2pt]b3.west);
    \node[draw=blue, thick, fit=(c1), inner
    ↪  xsep=1pt](box){};
    \draw[blue, thick, ->, shorten >=3pt]
    ↪  (box) to[out=90, in=90] (c2.north);
    \node[draw=yellow!70!orange, thick,
    ↪  circle, fit=(d1), inner
    ↪  sep=1pt](cir){};
    \node[draw=yellow!70!orange, thick,
    ↪  text=yellow!70!orange, single arrow,
    ↪  shape border rotate=180, anchor=west,
    ↪  scale=.4, single arrow tip angle=40,
    ↪  minimum height=30mm] at
    ↪  (cir.east){\quad RESULT};
  \end{axis}
\end{tikzpicture}
```

Figure 60.4: test

https://tex.stackexchange.com/questions/494884/anchor-alignment-in-tikzmarknode

Figure 60.5: test

```
\tikzset{every tikzmarknode/.append
↪  style={inner sep=3pt,rounded
corners}}

\begin{tikzpicture}
  \begin{axis}[
        %axis x line = center,
        %axis y line = center,
        %xlabel = {$ x $}, xlabel style =
        ↪  {right},
        %ylabel = {$ y=f\left(x\right) $},
        ↪  ylabel style = {above},
        xmin = -1.5, xmax = 1.5,
        ymin = -0.4, ymax = 0.4,
        %hide obscured x ticks=false, % for
        ↪  origin x tick label i.e. xtick =
        ↪  {0}
        xtick= \empty,
        xticklabels= \empty,
        %extra x ticks={0},
        ytick = \empty,
        yticklabels= \empty,
        x = 5cm, y = 5cm, % x y scaling
        %grid = major,
        domain = -1:1,
        %samples = 1000
    ]
    \node at (axis cs: 0,0) {
      $\begin{aligned}
  \sum\limits_{i=1}^{3}\sum\limits_{j=1}^{4}
↪  a_{ij}&=

↪  \tikzmarknode[fill=red!20]{red1}{\sum\limits_{j=1}^{4}a_{1j}}
↪  +

↪  \tikzmarknode[fill=green!20]{green1}{\sum\limits_{j=1}^{4}a_{2j}}
↪  +

↪  \tikzmarknode[fill=blue!20]{blue1}{\sum\limits_{j=1}^{4}a_{3j}}\\
  \\
  &=
  \tikzmarknode[fill=red!20]{red2}{a_{11} +
↪  a_{12} + a_{13} + a_{14}} +
  \tikzmarknode[fill=green!20]{green2}{a_{21}
↪  + a_{22} + a_{23} + a_{24}} +
  \tikzmarknode[fill=blue!20]{blue2}{a_{31} +
↪  a_{32} + a_{33} + a_{34}}
\end{aligned}$
      };
    \draw[->,red!20] (red1.south)
    ↪  to[out=-90,in=120,looseness=0.3]
    ↪  (red2.north);
    \draw[->,green!20] (green1)
    ↪  to[out=-90,in=120,looseness=0.3]
    ↪  (green2);
    \draw[->,blue!20] (blue1)
    ↪  to[out=-90,in=135,looseness=0.3]
    ↪  (blue2);
  \end{axis}
\end{tikzpicture}
```

Figure 60.6: test

https://zhuanlan.zhihu.com/p/694787336

LaTeX macro: annotate-equations

```
\usepackage{annotate-equations}
```

---

https://github.com/synercys/annotated_latex_equations

https://github.com/synercys/annotated_latex_equations/blob/main/example_prob.tex

# Chapter 61

# Shannon sampling

https://www.youtube.com/watch?v=ePGDQpJAvjE

# Chapter 62

# brachistochrone

https://www.youtube.com/watch?v=2c_bdVC9KS8

# Chapter 63

# magnetic resonance

## 63.1 NMR

### 63.1.1 Bloch equation

https://en.wikipedia.org/wiki/Bloch_equations

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \gamma \boldsymbol{B} - \frac{\boldsymbol{M}_{xy}}{T_2} - \frac{\boldsymbol{M}_z - \boldsymbol{M}_{z0}}{T_1}$$

$$\begin{aligned}
\dot{M}_x &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_x - \frac{M_x}{T_2} \\
\dot{M}_y &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_y - \frac{M_y}{T_2} \\
\dot{M}_z &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_z - \frac{M_z - M_{z0}}{T_1}
\end{aligned}$$

### 63.1.2 Bloch-Torrey equation

Bloch-Torrey equation = Bloch equation + diffusion term

https://journals.aps.org/pr/abstract/10.1103/PhysRev.104.563 https://journals.aps.org/pr/pdf/10.1103/PhysRev.104.563

https://en.wikipedia.org/wiki/Diffusion_MRI#Magnetization_dynamics

https://arxiv.org/abs/1608.02859 https://arxiv.org/pdf/1608.02859

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \gamma \boldsymbol{B} - \frac{\boldsymbol{M}_{xy}}{T_2} - \frac{\boldsymbol{M}_z - \boldsymbol{M}_{z0}}{T_1} + \boldsymbol{\nabla} \cdot D \boldsymbol{\nabla} \boldsymbol{M}$$

$$\begin{aligned}
\dot{M}_x &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_x - \frac{M_x}{T_2} + \boldsymbol{\nabla} \cdot D \boldsymbol{\nabla} (M_x - M_{x0}) \\
\dot{M}_y &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_y - \frac{M_y}{T_2} + \boldsymbol{\nabla} \cdot D \boldsymbol{\nabla} (M_y - M_{y0}) \\
\dot{M}_z &= (\boldsymbol{M} \times \gamma \boldsymbol{B})_z - \frac{M_z - M_{z0}}{T_1} + \boldsymbol{\nabla} \cdot D \boldsymbol{\nabla} (M_z - M_{z0})
\end{aligned}$$

## 63.2 MRI

### 63.2.1 Lin, Hsiu-Hau

Lin, Hsiu-Hau / Porcupine Lin / Hedgehog Note

https://www.youtube.com/watch?v=Y-z-1XCu7fE&list=PLS0SUwlYe8czNqxfQq2XWeAHDqT8vYjmC&index=60

### 63.2.2 MRI: Physics and Image Creation

https://www.youtube.com/playlist?list=PLRaiBmx1XQs5mrhyhzCmZWGHAG4bMlLXj

### 63.2.3 Tseng, Wen-Yih Isaac

https://www.youtube.com/playlist?list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU

# Chapter 64

# OpenMMLab

MMLab

## 64.1 Tongji TommyZihao

https://github.com/TommyZihao

https://space.bilibili.com/1900783/channel/series

## 64.2 official

https://github.com/open-mmlab

https://space.bilibili.com/1293512903/channel/series

# Chapter 65

# linear programming

https://www.youtube.com/watch?v=E72DWgKP_1Y

https://www.bilibili.com/video/BV1KC411J7kU

## 65.1 simplex method

## 65.2 duality method

https://www.bilibili.com/video/BV1kT411b7Ph

## 65.3 integer linear programming

# Chapter 66

# TaylorCatAlice physics

## 66.1 dimensional analysis

dimensional analysis = 量綱分析 = 因次分析

https://www.bilibili.com/video/BV1314y1B7Hx[17] https://www.books.com.tw/products/CN11371614

## 66.2 classic mechanics

### 66.2.1 Galileo

https://www.bilibili.com/video/BV1C14y1B7Bv

### 66.2.2 Newton

https://www.bilibili.com/video/BV1mh4y1c7QD

### 66.2.3 Lagrange

光力類比 = light-force analogy

https://zhuanlan.zhihu.com/p/666330436

https://www.zhihu.com/question/26435474/answer/2723574018

https://www.youtube.com/watch?v=Uvx5OA605x4&t=5m14s

高崇文: 墓碑不留白

https://www.youtube.com/watch?v=9mmUpWw6x-w

### 66.2.4 Hamilton

## 66.3 electromagnetism

UdiProd: tensor

https://www.youtube.com/watch?v=YxXyN2ifK8A&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=1

https://www.youtube.com/watch?v=A95jdIuUUW0&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=2

https://www.youtube.com/watch?v=51ARho2bvQY&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=3

https://zhuanlan.zhihu.com/p/680649946

https://www.bilibili.com/video/BV1PV411P7w4

quaternion[32]

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643029

https://www.bilibili.com/video/BV1SN411b7cn

https://www.bilibili.com/video/BV1pp4y137e4 https://www.bilibili.com/video/BV19P411x7Nv

https://www.bilibili.com/video/BV1Nu4y1E78j

## 66.4 relativity

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643042

https://www.bilibili.com/video/BV1sV4y1Y7fX

https://www.bilibili.com/video/BV1gP411s7PY

## 66.5 quantum theory

### 66.5.1 old quantum theory

https://www.youtube.com/watch?v=D2gS-nURAjE

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643062

https://www.bilibili.com/video/BV1U84y1o7zk

https://www.bilibili.com/video/BV1Zh4y1S7AW

### 66.5.2 black-body radiation

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643054

https://www.bilibili.com/video/BV1G14y1X7o4

https://www.bilibili.com/video/BV17u411G76U

https://www.bilibili.com/video/BV16h4y1k7qj

https://www.bilibili.com/video/BV1YM4y1W7XY

### 66.5.3 quantum mechanics

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643065

#### 66.5.3.1 Heisenberg: matrix mechanics

https://www.bilibili.com/video/BV1Rm4y1M7oL

https://www.bilibili.com/video/BV1mN4y197pY

#### 66.5.3.2 Schrodinger: wave mechanics

https://www.bilibili.com/video/BV1nV411F7es

https://www.bilibili.com/video/BV1gj411B7wz

https://www.bilibili.com/video/BV1kF41117cp

https://www.bilibili.com/video/BV1KK4y1c7S8

https://www.bilibili.com/video/BV18H4y1U7VN

https://www.bilibili.com/video/BV1uF4112785

https://www.bilibili.com/video/BV1zN411t7x1

https://www.bilibili.com/video/BV1Yc411r7sr

https://www.bilibili.com/video/BV1q5411v7p3

https://www.bilibili.com/video/BV1Xr421g7ki

**66.5.3.2.0.1 matrix mechanics = wave mechanics** https://www.bilibili.com/video/BV1vy4y1c7cs

https://www.bilibili.com/video/BV1c5411z7Nc

#### 66.5.3.3 Dirac: relativistic quantum mechanics

https://www.bilibili.com/video/BV1oB4y1d7JV

https://www.bilibili.com/video/BV1GC4y137Nh

https://www.bilibili.com/video/BV1y84y1U7Ps

##### 66.5.3.3.1 spin https://www.bilibili.com/video/BV1mv421i7v7

#### 66.5.3.4 other

https://www.bilibili.com/video/BV1qm4y1g7Km

https://www.bilibili.com/video/BV1ac411o7NJ

### 66.5.4 resistance and conductivity

https://space.bilibili.com/11008987/channel/collectiondetail?sid=1786884

https://www.bilibili.com/video/BV1tN41147SK

https://www.bilibili.com/video/BV1Cw411A75x

https://www.bilibili.com/video/BV1Hj411x7rg

https://www.bilibili.com/video/BV1BM411S7pM

https://www.bilibili.com/video/BV1mN4y1U7b2

https://www.bilibili.com/video/BV1Mw411P7Vt

# Chapter 67

# fundamental theorem of algebra

## 67.1 Gauss

### 67.1.1 FToA

FToA = fundamental theorem of algebra

cf. FToLA = fundamental theorem of linear algebra

https://ccjou.wordpress.com/2009/03/23/線性代數基本定理-一

cf. FToPF = fundamental theorem of prime factorization = fundamental theorem of arithmetic = prime factorization theorem = unique factorization theorem

https://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic

---

https://www.bilibili.com/video/BV1pW4y1c7Zi

https://www.bilibili.com/video/BV1U7421K7gX

### 67.1.2 trial and error

https://www.bilibili.com/video/BV1g94y1w7sy

### 67.1.3 heptadecagon

https://www.bilibili.com/video/BV18V411Q7nD

## 67.2 equation

### 67.2.1 cubic equation

https://en.wikipedia.org/wiki/Cubic_equation

https://www.bilibili.com/video/BV1nd4y1f7t8

https://www.bilibili.com/video/BV1Wh411c7jf

### 67.2.2 qunitic equation

https://en.wikipedia.org/wiki/Quintic_function

https://www.bilibili.com/video/BV1ya4y1G79P

# Chapter 68

# Basel problem

https://en.wikipedia.org/wiki/Basel_problem

## 68.1 TaylorCatAlice

https://www.bilibili.com/video/BV1wW421A75k

## 68.2 3B1B

https://www.youtube.com/watch?v=d-o3eB9sfls

## 68.3 Riemann hypothesis

https://www.youtube.com/playlist?list=PL300nJOfVNKZH3H-2J_g1bZRYOx-dQUp3

https://zhuanlan.zhihu.com/p/672704574

# Chapter 69

# number theory

## 69.1 4 important theorems

https://www.bilibili.com/video/BV1Nw4117757

## 69.2 congruence modulo

https://www.bilibili.com/video/BV18W4y1F7FU

# Chapter 70

# spinor

## 70.1 EigenChris

spinor for beginner

https://www.youtube.com/playlist?list=PLJHszsWbB6hoOo_wMb0b6T44KM_ABZtBs

https://www.zhihu.com/question/554324716/answer/3101029714

https://zhuanlan.zhihu.com/p/341625428

https://www.cnblogs.com/quantum-condensed-matter-physics/category/1911234.html

https://www.cnblogs.com/quantum-condensed-matter-physics/p/14299801.html

https://www.cnblogs.com/quantum-condensed-matter-physics/p/14299854.html

https://arxiv.org/abs/1312.3824

https://arxiv.org/pdf/1312.3824

## 70.2 Shinonome Masaki

東雲正樹：群論 (Group Theory) 終極速成 / 物理系零基礎火箭級 notes https://zhuanlan.zhihu.com/p/294221308

東雲正樹：群論 (Group Theory) 終極速成 / 群表示理論 https://zhuanlan.zhihu.com/p/314567658

東雲正樹：群論 (Group Theory) 終極速成 / 群表示論下的正交性與完備性 https://zhuanlan.zhihu.com/p/337650698

東雲正樹：群論 (Group Theory) 終極速成 / 李群 (Lie group) 的定義與常見李群 https://zhuanlan.zhihu.com/p/340870993

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 與 SO(3) 的夢幻聯動 https://zhuanlan.zhihu.com/p/341625428

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 的全體不可約表示與李群上的積分 https://zhuanlan.zhihu.com/p/342592239

東雲正樹：群論 (Group Theory) 終極速成 / 李群對應的李代數與可愛又迷人的伴隨表示 https://zhuanlan.zhihu.com/p/348680652

東雲正樹：群論 (Group Theory) 終極速成 / 淺談洛倫茲群 (Lorentz group) 與洛倫茲代數 https://zhuanlan.zhihu.com/p/370828021

東雲正樹：群論 (Group Theory) 終極速成 / 旋量空間與哈人的 Clifford 代數 https://zhuanlan.zhihu.com/p/373020066

https://zhuanlan.zhihu.com/p/332873718

# Chapter 71

# singular value decomposition

## 71.1 Visual Kernel

https://www.youtube.com/playlist?list=PLWhu9osGd2dB9uMG5gKBARmk73oHUUQZS

https://www.youtube.com/watch?v=vSczTbgc8Rc&list=PLWhu9osGd2dB9uMG5gKBARmk73oHUUQZS&index=4

https://www.youtube.com/watch?v=vSczTbgc8Rc

## 71.2 CCJou: LA Revelation

https://ccjou.wordpress.com/專題探究/奇異值分解專題/

# Chapter 72

# support vector machine

SVM = support vector machine

## 72.1 Visually Explained

https://www.youtube.com/playlist?list=PLqwozWPBo-FvuHWx3_aYwG2WVdbb-wC6q

https://www.youtube.com/watch?v=_YPScrckx28

# Chapter 73

# principal component analysis

PCA = principal component analysis

## 73.1 Visually Explained

https://www.youtube.com/watch?v=FD4DeN81ODY&list=PLqwozWPBo-FtNyPKLDPTVDOHwK12QbVsM&index=4

https://www.youtube.com/watch?v=FD4DeN81ODY

## 73.2 dimension reduction: PCA, tSNE, UMAP

# Chapter 74

# transformer

## 74.1 Lee, Hung-Yi

https://www.youtube.com/watch?v=uhNsUCb2fJI&list=PLJV_el3uVTsPz6CTopeRp2L2t4aL_KgiI&index=11

https://www.youtube.com/watch?v=uhNsUCb2fJI

## 74.2 3B1B

https://www.youtube.com/playlist?list=PLZHQObOWTQDNU6R1_67000Dx_ZCJB-3pi

https://www.youtube.com/watch?v=wjZofJX0v4M&list=PLZHQObOWTQDNU6R1_67000Dx_ZCJB-3pi&index=5

https://www.youtube.com/watch?v=wjZofJX0v4M

https://www.youtube.com/watch?v=eMlx5fFNoYc&list=PLZHQObOWTQDNU6R1_67000Dx_ZCJB-3pi&index=6

https://www.youtube.com/watch?v=eMlx5fFNoYc

# Chapter 75

# differential equation

DE = differential equation

https://www.youtube.com/watch?v=0kY3Wpvutfs

- guess: ansatz
- energy conservation
- series expansion
- Laplace transform
- Hamiltonian flow
- matrix exponential
- linear operator

## 75.1 ODE = ordinary DE = ordinary differential equation

https://www.zhihu.com/question/33177784/answer/237992320

https://www.zhihu.com/question/572977895/answer/2814177886

https://zhuanlan.zhihu.com/p/666958247

### 75.1.1 Sturm-Liouville theory

https://zhuanlan.zhihu.com/p/156941491

https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory

## 75.2 PDE = partial DE = partial differential equation

# Chapter 76

# Shannon

## 76.1 entropy

https://www.zhihu.com/question/27068465/answer/3318941468

# Chapter 77

# LEAN

L∃∀N = LEAN

## 77.1 LEAN 4

https://en.wikipedia.org/wiki/Lean_(proof_assistant)

https://leanprover-community.github.io/

https://lean-lang.org/

### 77.1.1 type theory

https://www.bilibili.com/video/BV12m4y1x7CA

### 77.1.2 installation

https://space.bilibili.com/520713026/channel/collectiondetail?sid=2671954

## 77.2 ELAN

LEAN version control

## 77.3 MathLib = mathlib

library of LEAN theorems

https://leanprover-community.github.io/mathlib-overview.html

https://leanprover-community.github.io/mathlib4_docs/

# Chapter 78

# recreational math

## 78.1 solved

### 78.1.1 Monty Hall problem

https://en.wikipedia.org/wiki/Monty_Hall_problem

https://www.youtube.com/watch?v=uNYVuk3JgQc

### 78.1.2 goat grazing problem

https://en.wikipedia.org/wiki/Goat_grazing_problem

## 78.2 not yet solved

### 78.2.1 moving sofa problem

https://en.wikipedia.org/wiki/Moving_sofa_problem

# Chapter 79

# causal inference

- Brady Neal: causal inference
  - full lecture
  - short lecture

# Chapter 80

# Einstein-de Hass effect

https://zh.wikipedia.org/wiki/愛因斯坦-德哈斯效應

https://en.wikipedia.org/wiki/Einstein

https://www.youtube.com/watch?v=BbEHApXFTQc

# Chapter 81

# pythagorean theorem

## 81.1 2023 new proof with trigonometry

https://www.youtube.com/watch?v=eFSvQHfrHQc

# Chapter 82

# general relativity

## 82.1 EpicOrganism = AIRoswell = Pan, Yi-Wen[1]

https://space.bilibili.com/14316464/video

https://www.bilibili.com/video/BV1T5411D7mS

https://www.bilibili.com/video/BV1QA4y1X7Xk

https://www.bilibili.com/video/BV1K34y1i75w

https://www.bilibili.com/video/BV1ZU411o7xL

## 82.2 Elliot Schneider

- Elliot Schneider: Physics with Elliot

[1] https://www.linkedin.com/in/yiwen-pan-16a90076

# Chapter 83

# LaTeX RegEx

https://tex.stackexchange.com/questions/534581/tex-compilation-after-regex-replace

https://www.overleaf.com/project/66512b6948077364474110de

```
\documentclass[a4paper,11pt]{article}

\usepackage{expl3,xparse}
\usepackage{textcomp}

\ExplSyntaxOn
\NewDocumentCommand{\midarrow}{m}
{
    \tl_set:Nn \l_tmpa_tl { (#1) }
    \regex_replace_all:nnN { \, } { \c{textrightarrow} } \l_tmpa_tl
    \regex_replace_all:nnN { \; } { \)\; \c{hspace}\cB\{ 1cm \cE\} \( } \l_tmpa_tl
    \tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff

\begin{document}
    \midarrow{a,A;b,B;c,C}
\end{document}
```

# references


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