math

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math on bookdown started on 2024/01/28

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A Minimal Book Example

6.1 About

This is a *sample* book written in **Markdown**. You can use anything that Pandoc's Markdown supports; for example, a math equation $a^2 + b^2 = c^2$.

6.1.1 Usage

Each **bookdown** chapter is an .Rmd file, and each .Rmd file can contain one (and only one) chapter. A chapter *must* start with a first-level heading: `# A good chapter`, and can contain one (and only one) first-level heading.

Use second-level and higher headings within chapters like: `## A short section` or `### An even shorter section`.

The `index.Rmd` file is required, and is also your first book chapter. It will be the homepage when you render the book.

6.1.2 Render book

You can render the HTML version of this example book without changing anything:

1. Find the **Build** pane in the RStudio IDE, and
2. Click on **Build Book**, then select your output format, or select "All formats" if you'd like to use multiple formats from the same book source files.

Or build the book from the R console:

```
bookdown::render_book()
```

To render this example to PDF as a `bookdown::pdf_book`, you'll need to install XeLaTeX. You are recommended to install TinyTeX (which includes XeLaTeX): https://yihui.org/tinytex/.

6.1.3 Preview book

As you work, you may start a local server to live preview this HTML book. This preview will update as you edit the book when you save individual .Rmd files. You can start the server in a work session by using the RStudio add-in "Preview book", or from the R console:

```
bookdown::serve_book()
```

6.2 Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (`#`) per .Rmd file.

6.2.1 A section

All chapter sections start with a second-level (`##`) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

6.3 Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

6.3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
   - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world` = `# Hello world {#hello-world}`.
   - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 6.3.
   - If you prefer text as the link instead of a numbered reference use: any text you want can go here.

6.3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 6.1.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

Don't miss Table 6.1.

```
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
  booktabs = TRUE
)
```

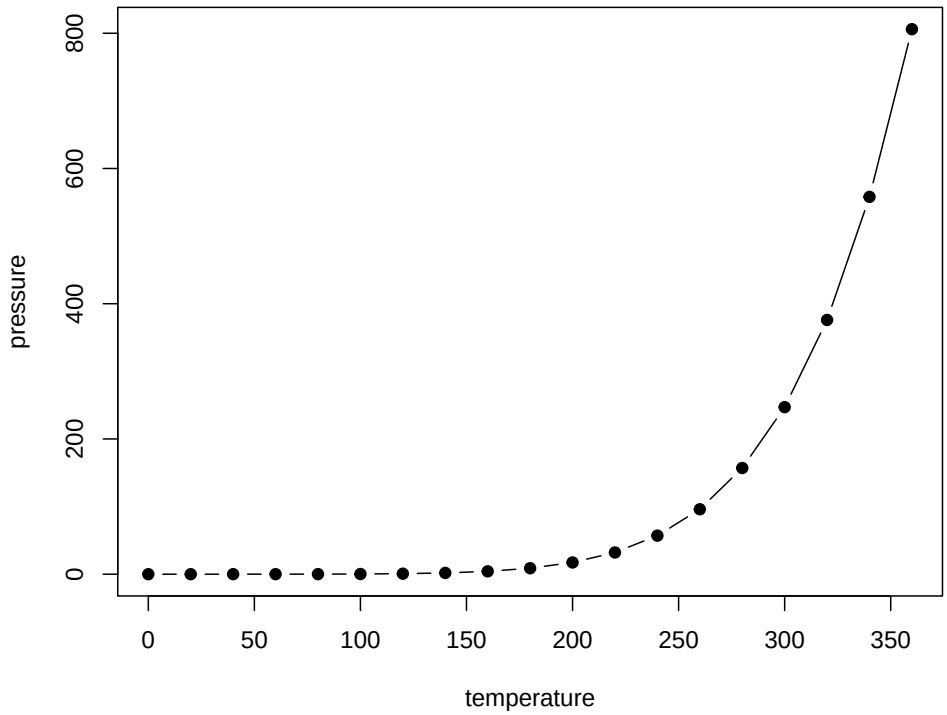


Figure 6.1: Here is a nice figure!

Table 6.1: Here is a nice table!

temperature	pressure
0	0.0002
20	0.0012
40	0.0060
60	0.0300
80	0.0900
100	0.2700
120	0.7500
140	1.8500
160	4.2000
180	8.8000

6.4 Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

6.5 Footnotes and citations

6.5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

6.5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package[1] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr**[2] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

6.6 Blocks

6.6.1 Equations

Here is an equation.

$$f\left(k\right) = \binom{n}{k} p^k \left(1-p\right)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

6.6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

Theorem 6.1. *For a right triangle, if c denotes the* length *of the hypotenuse and a and b denote the lengths of the* ***other*** *two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

[1] This is a footnote.

6.6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

6.7 Sharing your book

6.7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

6.7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

6.7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book- all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```

Chapter 7

test

https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html

7.1 RStudio

7.1.1 Rtools

Rtools43 for Windows https://cran.r-project.org/bin/windows/Rtools/rtools43/rtools.html

7.1.2 addins

https://github.com/rstudio/addinexamples

```
if (!requireNamespace("devtools", quietly = TRUE))
  install.packages("devtools")

devtools::install_github("rstudio/htmltools")
devtools::install_github("rstudio/shiny")
devtools::install_github("rstudio/miniUI")
```

7.2 RMarkdown

https://www.rstudio.com/wp-content/uploads/2015/02/rmarkdown-cheatsheet.pdf

https://slides.yihui.org/2020-taipei-satrday-rmarkdown.html#1

7.2.1 URL

https://stackoverflow.com/questions/29787850/how-do-i-add-a-url-to-r-markdown

7.2.2 superscript and subscript

script$^{superscript}_{subscript}$

```
script^superscript^
```

script^{superscript}

```
~subscript~
```

script$_{\text{subscript}}$

7.2.2.1 LaTeX

https://tex.stackexchange.com/questions/580824/subscript-not-distinguished-enough

https://tex.stackexchange.com/questions/262295/make-subscript-size-smaller-always

7.2.3 equation

https://stackoverflow.com/questions/26049762/erroneous-nesting-of-equation-structures-in-using-beginalign-in-a-multi-l

7.2.4 image

https://stackoverflow.com/questions/25166624/insert-picture-table-in-r-markdown

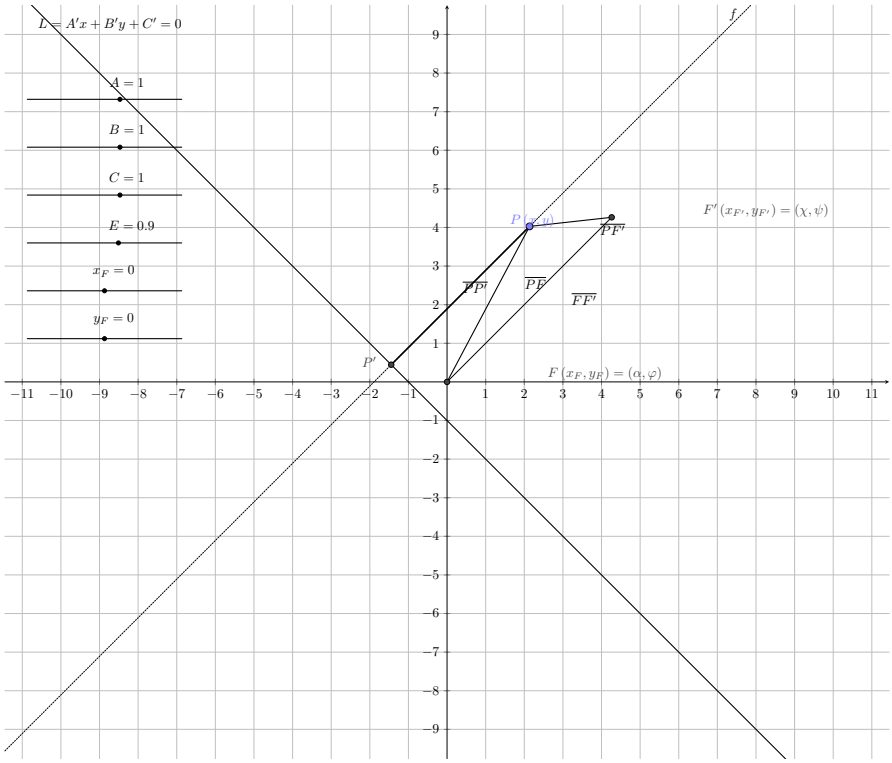


Figure 7.1: conic sections

7.2.4.1 SVG

https://stackoverflow.com/questions/50165404/how-to-make-a-pdf-using-bookdown-including-svg-images

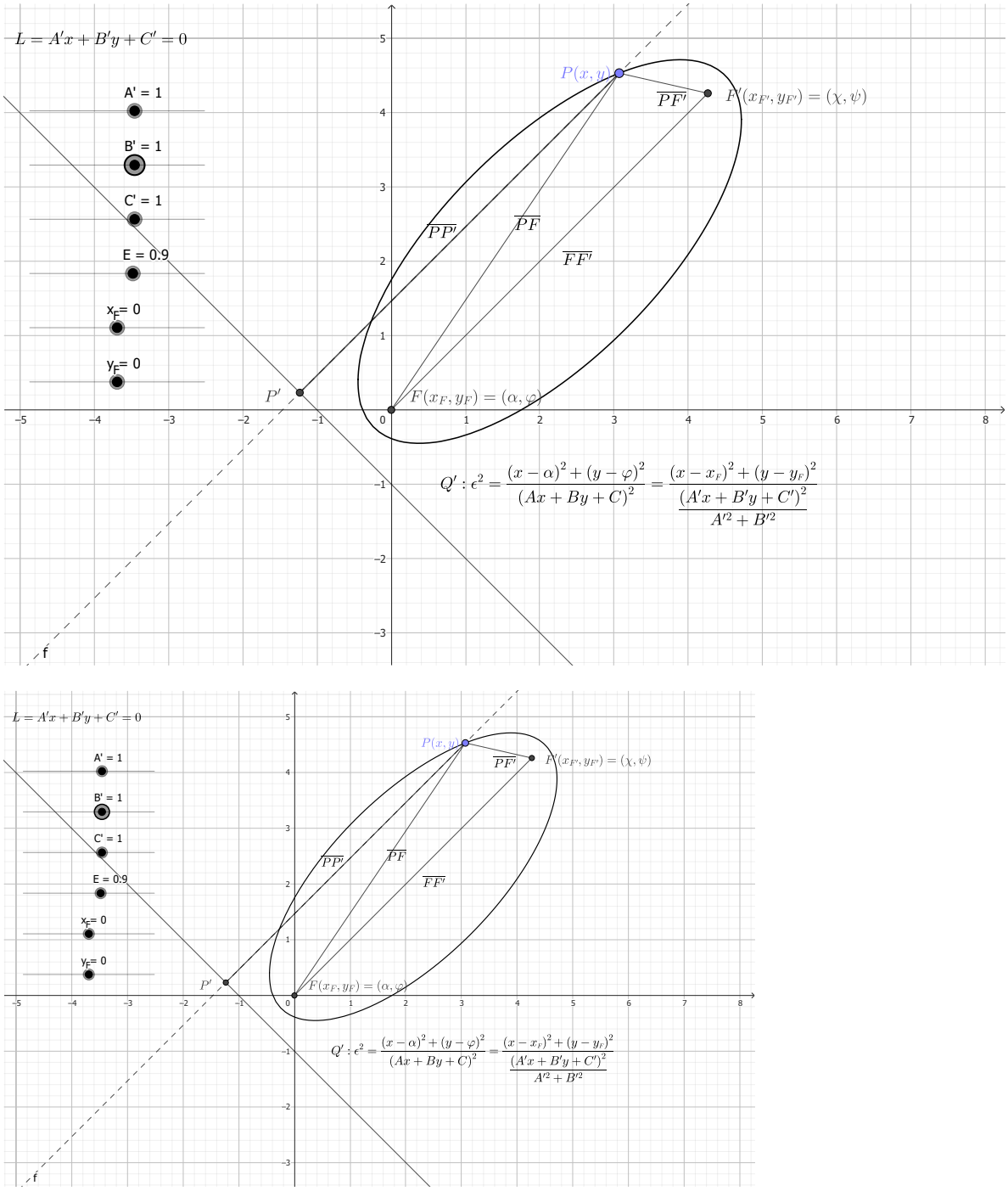


Figure 7.2: conic sections

https://stackoverflow.com/questions/34064292/is-it-possible-to-include-svg-image-in-pdf-document-rendered-by-rmarkdown

***

horizontal rule (or slide break)

```r
dim(iris)
```

```
## [1] 150   5
```

7.2.5 footnote

7.2.6 hyperlink

PDF pandoc internal link will lose focus

equivalence relation [11] equivalence relation[1] equivalence relation[11]

equivalence class [10] equivalence class[2] equivalence class[10]

partition [9] partition[3] partition[9]

- LaTeX
  - TikZ[13]
    - TikZ-3Dplot
    - PGFplots
  - xypic = xy-pic[4]
- OverLeaf
- MathCha
- GeoGebra
- Python
  - MatPlotLib
  - Seaborn
  - Plotly

7.2.7 xaringan

slide realtime preview with RStudio addin Infinite Moon Reader in RStudio viewer

https://github.com/yihui/xaringan

https://www.youtube.com/watch?v=3n9nASHg9gc

7.3 Bookdown

7.3.1 system locale

https://bookdown.org/tpemartin/ntpu-programming-for-data-science/appendix-d-.html

`Sys.getlocale()`

Windows

```
Sys.setlocale(category = "LC_ALL", locale = "UTF-8")
```

MacOS

```
Sys.setlocale(category = "LC_ALL", locale = "en_US.UTF-8")
```

[1] {11} equivalence relation
[2] {10} equivalence class
[3] {9} partition
[4] {14} xy-pic

https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html

7.3.2 render_book()

https://bookdown.org/yihui/bookdown/build-the-book.html

```
render_book(input = ".", output_format = NULL, ..., clean = TRUE,
  envir = parent.frame(), clean_envir = !interactive(),
  output_dir = NULL, new_session = NA, preview = FALSE,
  config_file = "_bookdown.yml")
```

7.3.3 serve_book()

https://bookdown.org/yihui/bookdown/serve-the-book.html

```
serve_book(dir = ".", output_dir = "_book", preview = TRUE,
  in_session = TRUE, quiet = FALSE, ...)
```

7.3.4 LaTeX

7.3.4.1 hyperlink, URL, href

https://www.baeldung.com/cs/latex-hyperref-url-hyperlinks

https://www.omdte.com/小技巧讓-facebook和-line顯示中文網址，網址不再變亂碼/

7.3.4.2 ugly mathptmx $\sum$

PDF LaTeX `\usepackage{fdsymbol}` to have `\overrightharpoon` vector; however, there are too many side effects, including ugly mathptmx $\sum$, ...

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}
```

https://tex.stackexchange.com/questions/315102/different-sum-signs

https://tex.stackexchange.com/questions/275038/how-to-replace-mathptmx-sum-with-cm-sum

https://tex.stackexchange.com/questions/391410/calligraphic-symbols-are-too-fancy-with-mathptmx-package

https://blog.csdn.net/kongtaoxing/article/details/131005044

In `preamble.tex`, add

```
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}

\DeclareMathAlphabet{\mathcal}{OMS}{cmsy}{m}{n}
\DeclareSymbolFont{largesymbols}{OMX}{cmex}{m}{n}
```

7.3.4.3 LaTeX package in HTML document

https://github.com/rstudio/rmarkdown/issues/1829

```
---
title: "assignment"
author: "author"
output: html_document
---

$$
  \require{cancel}
  \cancel{x}
$$
```

$$\cancel{x}$$

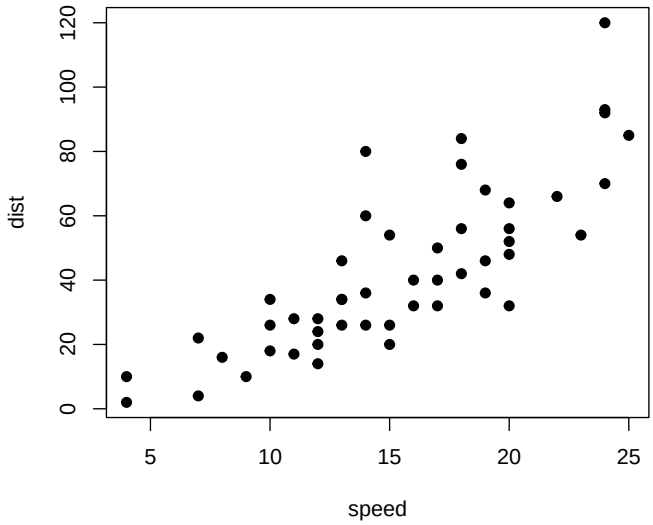
https://stackoverflow.com/questions/18189175/how-to-use-textup-with-mathjax

`\textup` is not available in MathJax. You can replace it with `\mathrm`, **but `\mathrm` does not interpret spaces**.

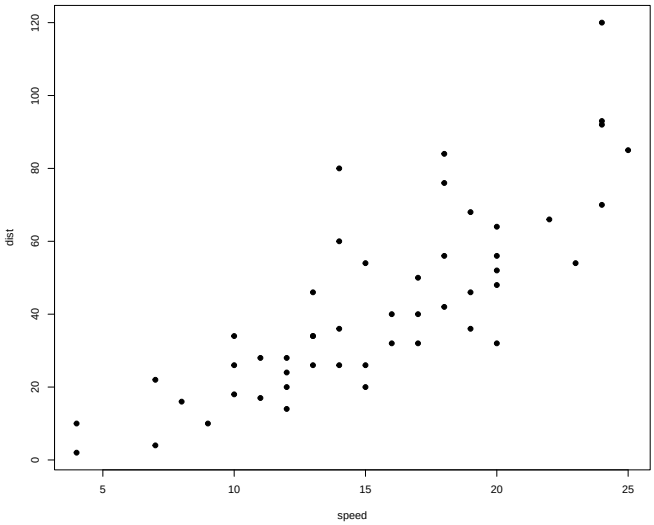
7.3.5 Two columns

Below is a Div containing three child Divs side by side. The Div in the middle is empty, just to add more space between the left and right Divs.

```
::::::: {.cols data-latex=""}

::: {.col data-latex="{0.55\textwidth}"}
![](202401280001-test_files/figure-latex/unnamed-chunk-10-1.pdf)<!-- -->
:::

::: {.col data-latex="{0.05\textwidth}"}
\
<!-- an empty Div (with a white space), serving as
a column separator -->
:::

::: {.col data-latex="{0.4\textwidth}"}
The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do
eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut
enim ad minim veniam, quis nostrud exercitation ullamco laboris
nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor
in reprehenderit in voluptate velit esse cillum dolore eu fugiat
nulla pariatur.
:::
:::::::
```



The figure on the left-hand side shows the `cars` data.
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.



The figure on the left-hand side shows the `cars` data.
Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

7.4 conditional block/chunk for either HTML or PDF, and Chinese issue

https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html

等價關係 equivalence relation

$$
\begin{aligned}
&R \text{ is an equivalence relation over } A \times B \\
\Leftrightarrow &\begin{cases} R = \sim = \{\langle x, y\rangle | x \sim y\} \subseteq A \times B & \text{(e) equivalence 等價} \\ \vdots & \vdots \end{cases} \\
\Leftrightarrow &\begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & (R) \text{ relation} \\ \forall \langle x, y\rangle \in R \left(xRx\right) & (r) \text{ reflexive} \\ \forall \langle x, y\rangle \in R \left(xRy \Rightarrow yRx\right) & (s) \text{ symmetric} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R \left(\begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (t) \text{ transitive} \end{cases} \\
\Leftrightarrow &\begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & \text{關係} \\ \forall \langle x, y\rangle \in R \left(\langle x, x\rangle \in R\right) & \text{自反} \\ \forall \langle x, y\rangle \in R \left(\langle y, x\rangle \in R\right) & \text{對稱} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R \left(\langle x, z\rangle \in R\right) & \text{遞移} \end{cases}
\end{aligned}
$$

7.5 video embedding

https://stackoverflow.com/questions/42543206/r-markdown-compile-error

always_allow_html: true

```
install.packages("webshot")
webshot::install_phantomjs()
```

however webshot not work

Error: cannot find bilibili.com

https://cran.r-project.org/web/packages/vembedr/vignettes/embed.html

```
## embed_youtube("qeMqtt7NFDM")
```

7.5.1 timestamp

- YouTube: https://www.youtube.com/embed/%7BvideoID%7D?start=%7Bsecond%7D
- BiliBili: https://player.bilibili.com/player.html?bvid=%7BvideoID%7D&autoplay=0&t=%7Bsecond%7D

7.6 equation term coloring

7.6.1 font color

RegEx replacement in RStudio for `{\\color{(\w+)}` in LyX to be replaced with `\\color{$1}{` in HTML document, and remain the same for PDF document

In HTML document, if no `{}` for text range, only the first following term will take effect

```
\color{orange}x=y
```

$$\color{orange}x=y$$

`\color{orange}` and `\color{cyan}` are better color for HTML GitBook `White` and `Night` themes and PDF

```
\color{cyan}{x=y}
```

$$\color{cyan}{x=y}$$

```
\color{cyan}{x=y}
```

$$\color{cyan}{x=y}$$

```
::: {show-in="html"}

$$
\begin{aligned}
\color{orange}{\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{2}}=\left(\bold
\left(L\right)=\color{orange}{\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{
=\left(R\right)\\
\left(L\right)=\color{orange}{\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{
\overset{\left(e_{1}\right)}{=} & \left(\lambda_{1}\boldsymbol{x}_{1}\right)^{\intercal}\b
\left(R\right)=\color{orange}{\boldsymbol{x}_{1}^{\intercal}\left(A\boldsymbol{x}_{2}\righ
\lambda_{1}\boldsymbol{x}_{1}^{\intercal}\boldsymbol{x}_{2}=\color{orange}{\left(A\boldsym
\lambda_{1}\boldsymbol{x}_{1}^{\intercal}\boldsymbol{x}_{2}= & \lambda_{2}\boldsymbol{x}_{
\end{aligned}
$$

:::

::: {show-in="pdf"}

$$
\begin{aligned}
{\color{orange}\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{2}}=\left(\bold
\left(L\right)={\color{orange}\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{
\left(L\right)={\color{orange}\left(A\boldsymbol{x}_{1}\right)^{\intercal}\boldsymbol{x}_{
\left(R\right)={\color{orange}\boldsymbol{x}_{1}^{\intercal}\left(A\boldsymbol{x}_{2}\righ
\lambda_{1}\boldsymbol{x}_{1}^{\intercal}\boldsymbol{x}_{2}={\color{orange}\left(A\boldsym
\lambda_{1}\boldsymbol{x}_{1}^{\intercal}\boldsymbol{x}_{2}= & \lambda_{2}\boldsymbol{x}_{
\end{aligned}
$$

:::
```

7.6.2 background color

https://bookdown.org/yihui/rmarkdown-cookbook/font-color.html

LaTex color

https://latexcolor.com/

https://www.overleaf.com/learn/latex/Using_colors_in_LaTeX

https://latex-tutorial.com/color-latex/#:~:text=To%20summarize%2C%20pyellow!50efined%20colors%20in,w

LaTex color methods

color frame

https://tex.stackexchange.com/questions/582748/highlight-equation-with-boxes-and-arrows

color box

https://tex.stackexchange.com/questions/567739/how-to-move-and-size-colorbox

color box with round corners

https://tex.stackexchange.com/questions/568880/color-box-with-rounded-corners

highlighting

https://tex.stackexchange.com/questions/318991/highlighting-math

https://forum.remnote.io/t/highlighting-latex-formulas/149

LyX

https://tex.stackexchange.com/questions/250069/create-a-color-box https://latexlyx.blogspot.com/2013/12/lyx.html

https://tex.stackexchange.com/questions/635486/prevent-lyx-from-escaping-math-in-color-box-title

Bookdown - conditional display of text and code blocks (LaTeX/PDF vs. HTML) https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html

$$F = ma$$

https://community.rstudio.com/t/highlighting-text-inline-in-rmarkdown-or-bookdown-pdf/35118/4

$$F = ma$$

$$F = F$$

$$F = ma \tag{7.1}$$

$$F = ma$$

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_n X_n$$

7.7 link and reference

https://stackoverflow.com/questions/57469501/cross-referencing-bookdownhtml-document2-not-working

$$E = mc^2 \tag{7.2}$$

`\@ref(nice-label)` 1

`[link to partition][partition]` link to partition

`[partition] \@ref(partition)`

partition [#partition] (9) @ref(#partition)

`[equivalence class] \@ref(equivalence class)`

equivalence class [#equivalence class] (@ref(equivalence class)) @ref(#equivalence class)

[equivalence-class] [#equivalence-class] (10) @ref(#equivalence-class)

[equivalence-class.html] [equivalence-class.html#equivalence-class] (@ref(equivalence-class.html)) @ref(equivalence-class.html#equivalence-class)

equivalence relation [#equivalence relation] (@ref(equivalence relation)) @ref(#equivalence relation)

[equivalence-relation] [#equivalence-relation] (11) @ref(#equivalence-relation)

[equivalence-relation.html] [equivalence-relation.html#equivalence-relation] (@ref(equivalence-relation.html)) @ref(equivalence-relation.html#equivalence-relation)

7.8 number and reference equations

https://stackoverflow.com/questions/71595882/rstudio-error-in-windows-running-pdflatex-exe-on-file-name-tex-exit-code-10

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#equations

`\#eq:emc \@ref(eq:emc)`

https://stackoverflow.com/questions/55923290/consistent-math-equation-numbering-in-bookdown-across-pdf-docx-html-output

$$
\begin{aligned}
& C \text{ is an equivalence class of } a \text{ on } A \\
\Leftrightarrow [a]_\sim = C = & \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{cases} \right\} \subseteq A \neq \emptyset \\
\Leftrightarrow [a] = [a]_\sim = & \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{cases} \right\} \subseteq A \neq \emptyset \\
\Rightarrow [a]_\sim = & \{x | x \sim a\} \subseteq A \neq \emptyset
\end{aligned}
\tag{7.3}
$$

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#cross-referencing

This cross reference is the Fig. 7.3

https://stackoverflow.com/questions/51595939/bookdown-cross-reference-figure-in-another-file

I ran into the same issue and came up with this solution if you aim at compiling 2 different pdfs. It relies on LaTeX's xr package for cross references: https://stackoverflow.com/a/52532269/576684

7.9 footnote

noun[5]

7.10 citation

https://stackoverflow.com/questions/48965247/use-csl-file-for-pdf-output-in-bookdown/49145699#49145699

citation 1[3] citation 2[3]

citation 3[4] citation 4[4]

7.10.1 backreference

https://community.rstudio.com/t/how-to-create-a-backreference-to-place-of-citation-in-rmarkdown/84866

https://blog.csdn.net/RobertChenGuangzhi/article/details/50455429

https://latex.org/forum/viewtopic.php?t=3722

7.11 bookdown environment for definition, theorem, proof

https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html

https://github.com/rstudio/rstudio/issues/5264

`@howthebodyworks` Ideally, previews of such equations should also work inside a theorem, although I could survive without that.

https://github.com/rstudio/rstudio/issues/8773

Theorem 7.1 (Theorem Name)**.** *Here is my theorem.*

Proof Name. Here is my proof. □

Theorem 7.2 (Pythagorean theorem)**.** *For a right triangle, if c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, we have*

$$a^2 + b^2 \overset{7.2}{=} c^2$$

Definition 7.1 (Definition Name)**.** Here is my definition.

number and reference equations

(7.3)

(7.2)

7.2

[5]This is a footnote.

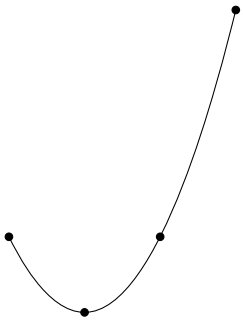


Figure 7.3: parabola arc with points

Chapter 8

test2

8.1 verbatim

https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html

````
```r
1 + 1
```

```
## [1] 2
```
````

````
We can output arbitrary content **verbatim**.


```r
1 + 1
```

```
## [1] 2
```

The content can contain inline code like
78.5398163, too.
````

Chapter 9

partition

$$\{A_i\}_{i\in I} = \{A_i | i \in I\} \text{ is a partition of a set } A$$
$$\Leftrightarrow \begin{cases} \forall i \in I \, (A_i \neq \emptyset) \\ A = \bigcup\limits_{i\in I} A_i \\ \forall i, j \in I \, (i \neq j \Rightarrow A_i \cap A_j = \emptyset) \end{cases}$$

https://proofwiki.org/wiki/Definition:Set_Partition

Chapter 10

equivalence class

$$
\begin{aligned}
&C \text{ is an equivalence class of } a \text{ on } A \\
\Leftrightarrow &[a]_\sim = C = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{cases} \right\} \subseteq A \neq \emptyset \\
\Leftrightarrow &[a] = [a]_\sim = \left\{ x \middle| \begin{cases} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{cases} \right\} \subseteq A \neq \emptyset \\
\Rightarrow &[a]_\sim = \{x | x \sim a\} \subseteq A \neq \emptyset
\end{aligned}
$$

where the definition of equivalence relation can be found in 11.

Chapter 11

equivalence relation

等價關係 equivalence relation

R is an equivalence relation over $A \times B$

$$\Leftrightarrow \begin{cases} R = \sim = \{\langle x, y\rangle | x \sim y\} \subseteq A \times B & (\mathrm{e})\,\text{equivalence 等價} \\ \vdots & \vdots \end{cases}$$

$$\Leftrightarrow \begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & (R)\,\text{relation} \\ \forall \langle x, y\rangle \in R\,(xRx) & (r)\,\text{reflexive} \\ \forall \langle x, y\rangle \in R\,(xRy \Rightarrow yRx) & (s)\,\text{symmetric} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R \left(\begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (t)\,\text{transitive} \end{cases} \Leftrightarrow \begin{cases} R = \{\langle x, y\rangle | xRy\} \subseteq A \times B & \text{關係} \\ \forall \langle x, y\rangle \in R\,(\langle x, x\rangle \in R) & \text{自反} \\ \forall \langle x, y\rangle \in R\,(\langle y, x\rangle \in R) & \text{對稱} \\ \forall \langle x, y\rangle, \langle y, z\rangle \in R\,(\langle x, z\rangle \in R) & \text{遞移} \end{cases}$$

Chapter 12

Python

https://bookdown.org/yihui/rmarkdown/language-engines.html

```
names(knitr::knit_engines$get())
```

```
##  [1] "awk"         "bash"        "coffee"      "gawk"        "groovy"
##  [6] "haskell"     "lein"        "mysql"       "node"        "octave"
## [11] "perl"        "php"         "psql"        "Rscript"     "ruby"
## [16] "sas"         "scala"       "sed"         "sh"          "stata"
## [21] "zsh"         "asis"        "asy"         "block"       "block2"
## [26] "bslib"       "c"           "cat"         "cc"          "comment"
## [31] "css"         "ditaa"       "dot"         "embed"       "eviews"
## [36] "exec"        "fortran"     "fortran95"   "go"          "highlight"
## [41] "js"          "julia"       "python"      "R"           "Rcpp"
## [46] "sass"        "scss"        "sql"         "stan"        "targets"
## [51] "tikz"        "verbatim"    "theorem"     "lemma"       "corollary"
## [56] "proposition" "conjecture"  "definition"  "example"     "exercise"
## [61] "hypothesis"  "proof"       "remark"      "solution"
```

https://rstudio.github.io/reticulate/articles/python_packages.html

```
x = 'hello, python world!'
print(x.split(' '))
```

```
## ['hello,', 'python', 'world!']
```

```
library(reticulate)
virtualenv_python()
```

```
library(reticulate)
# conda_list()
```

```
library(reticulate)
virtualenv_list()
```

https://rstudio.github.io/reticulate/reference/install_python.html

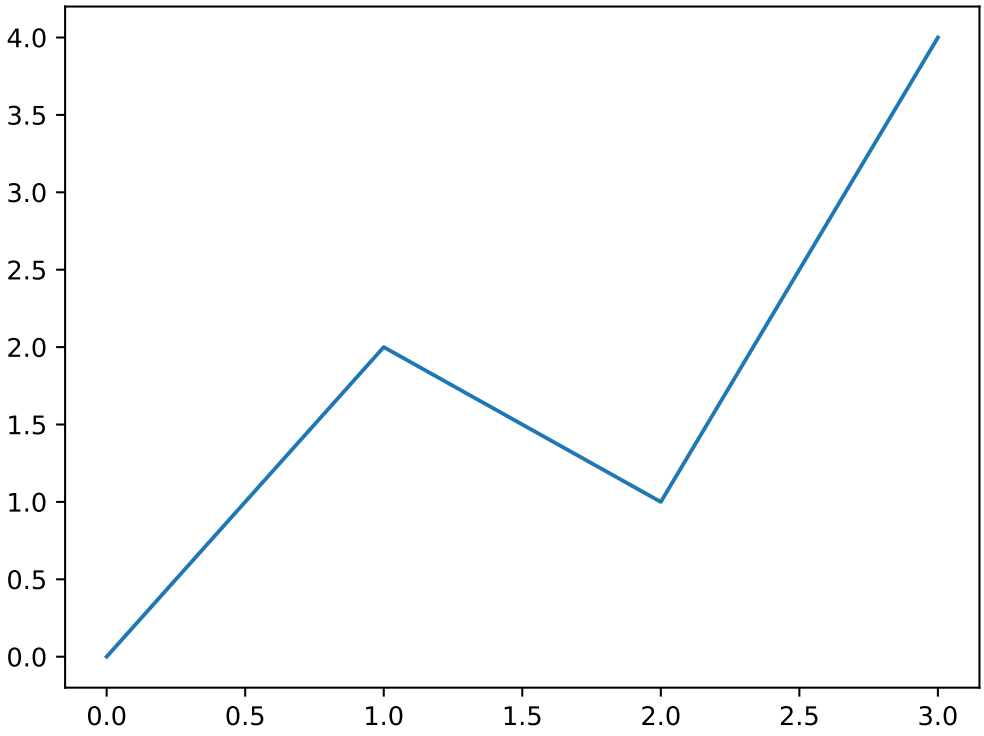
```
library(reticulate)
version <- "3.9.12"
```

```
# install_python(version)

# create a new environment
# virtualenv_create("r-reticulate", version = version)

# use_virtualenv("r-reticulate")

# install MatPlotLib
# virtualenv_install("r-reticulate", "matplotlib")

# import MatPlotLib (it will be automatically discovered in "r-reticulate")
matplotlib <- import("matplotlib")
```

copy `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tc`
and `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl`
two folders to the folder `C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\ver`

```
# library(reticulate)
# use_virtualenv("r-reticulate")
# # matplotlib <- import("matplotlib")
# matplotlib$use("Agg", force = TRUE)
```

```
import matplotlib.pyplot as plt
plt.plot([0, 2, 1, 4])
plt.show()
```

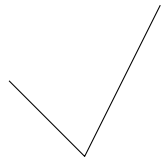

4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Chapter 13

TikZ

two columns [7.3.5]

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



How to speed up bookdown generation?

https://stackoverflow.com/questions/56541371/how-to-speed-up-bookdown-generation

TikZ and PGFplots

What's the relation between packages PGFplots and TikZ?

https://tex.stackexchange.com/questions/285925/whats-the-relation-between-packages-pgfplots-and-tikz

https://www.youtube.com/watch?v=bQugbYq0BVA

https://www.youtube.com/watch?v=ft4Kg9emK1k&list=PLg5nrpKdkk2DWcg3scb75AknF7DJXs8lk&index=18

```
\begin{tikzpicture}
  \def\a{1.5} % amplitude
  \def\b{2}   % frequency
  \draw[->] (-0.2,0)--(4.2,0) node[right, font=\small] {$x$};
  \draw[->] (0,-4)--(0,0.5) node[above] {$y$};
  \draw[domain=0:4,smooth,variable=\t,blue,thick]
    plot ({\a * (\b*\t - sin(deg(\b*\t)))},{-\a * (1 - cos(deg(\b*\t)))});
  % \node[above] at (2, 0.5) {Brachistochrone Curve};
  \node[above, font=\footnotesize] at (2, 1) {Brachistochrone Curve};
  \node[above, font=\footnotesize] at (2, 0) {$\begin{aligned}
& x=r(t-\sin t) \\
& y=r(1-\cos t)
\end{aligned}$};
\end{tikzpicture}
```

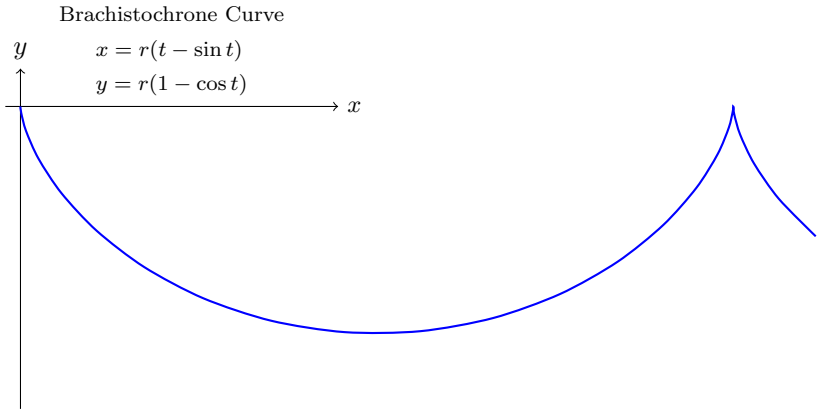


Figure 13.1: Brachistochrone Curve

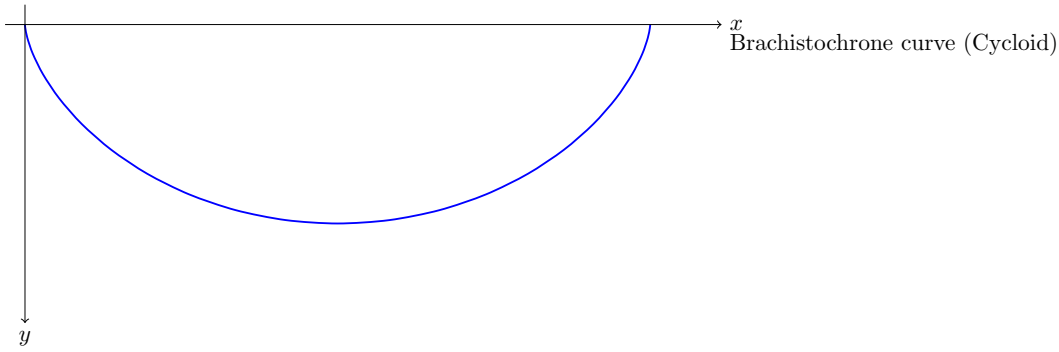
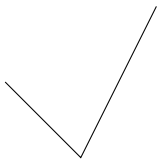


Figure 13.2: Brachistochrone Curve

13.1 2D

https://zhuanlan.zhihu.com/p/127155579?utm_psn=1741479950987960320

```
\begin{tikzpicture}
  \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



2

3

```
\begin{tikzpicture}
  \draw[rounded corners] (-1,1)--(0,0)--(1,2)--(-1,1);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw[rounded corners] (-1,1)--(0,0)--(1,2)--cycle;
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) rectangle (4,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) circle (1);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) circle (1);
  \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (1,1) ellipse (2 and 1);
\end{tikzpicture}
```

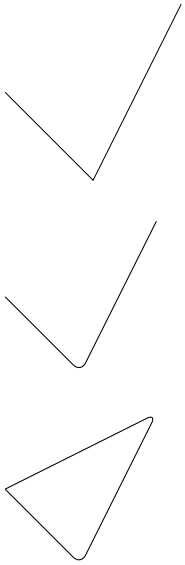


Figure 13.3: rounded corner pseudo-closed triangle

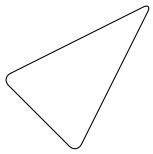


Figure 13.4: rounded corner triangle

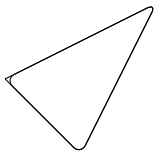


Figure 13.5: triangle vs. pseudo-closed triangle

Figure 13.6: rectangle

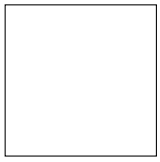


Figure 13.7: square

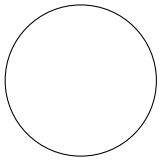


Figure 13.8: circle

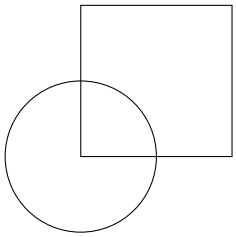


Figure 13.9: circle and square

```
\begin{tikzpicture}
  \draw (1 ,1) arc (0:270:1);
  \draw (6 ,1) arc (0:270:2 and 1);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (-1,1) parabola bend (0,0) (2,4);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (-1,1) parabola bend (0,0) (2,4);
  \filldraw
    (-1,1) circle (.05)
    ( 0,0) circle (.05)
    ( 1,1) circle (.05)
    ( 2,4) circle (.05);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw[step=20pt] (0,0) grid (3,2);
  \draw[help lines ,step=20pt] (4,0) grid (7,2);
\end{tikzpicture}
```

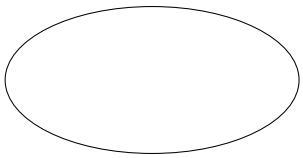


Figure 13.10: ellipse

Figure 13.11: circle and ellipse arcs

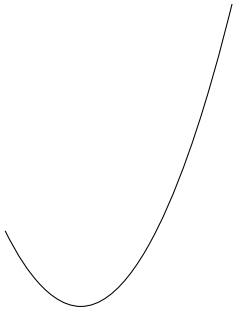


Figure 13.12: parabola arc

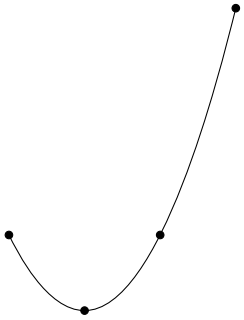
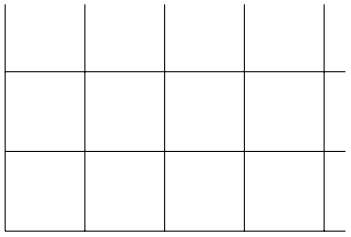


Figure 13.13: parabola arc with points



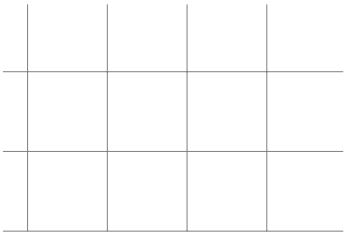
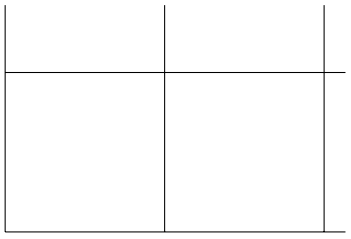


Figure 13.14: grid and help lines



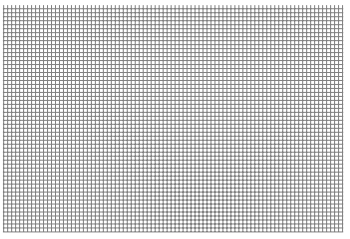


Figure 13.15: grid and help lines

```
\begin{tikzpicture}[scale=0.25]
  \draw[->] (0,0)--(9,0);
  \draw[<-] (0,1)--(9,1);
  \draw[<->] (0,2)--(9,2);
  \draw[>->>] (0,3)--(9,3);
  \draw[|<->|] (0,4)--(9,4);
\end{tikzpicture}
```

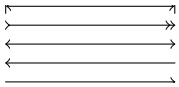


Figure 13.16: arrows

```
\begin{tikzpicture}
  \draw[line width =2pt] (0,6)--(9,6);
  \draw[dotted]          (0,5)--(9,5);
  \draw[densely dotted]  (0,4)--(9,4);
  \draw[loosely dotted]  (0,3)--(9,3);
  \draw[dashed]          (0,2)--(9,2);
  \draw[densely dashed]  (0,1)--(9,1);
  \draw[loosely dashed]  (0,0)--(9,0);
\end{tikzpicture}
```

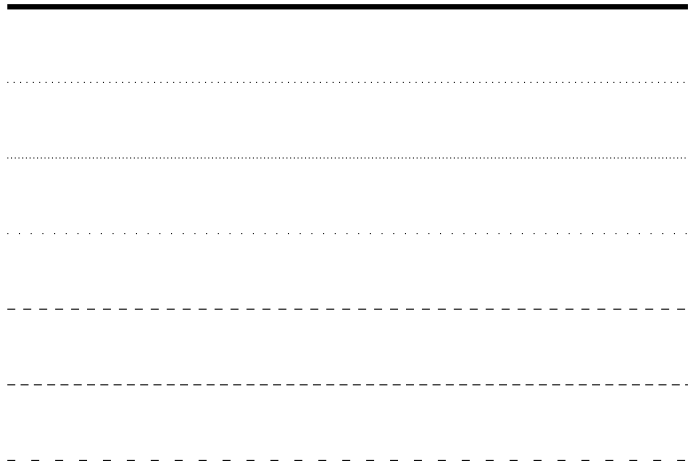


Figure 13.17: arrows

```
\begin{tikzpicture}[dline/.style={color= blue, line width=2pt}]
  \draw[dline] (0,0)--(9,0);
\end{tikzpicture}
```


Figure 13.18: head styling

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
```

```
  \draw[shift={( 3, 0)}] (0,0) rectangle (2,2);
  \draw[shift={( 0, 3)}] (0,0) rectangle (2,2);
  \draw[shift={( 0,-3)}] (0,0) rectangle (2,2);
  \draw[shift={(-3, 0)}] (0,0) rectangle (2,2);
  \draw[shift={( 3, 3)}] (0,0) rectangle (2,2);
  \draw[shift={(-3, 3)}] (0,0) rectangle (2,2);
  \draw[shift={( 3,-3)}] (0,0) rectangle (2,2);
  \draw[shift={(-3,-3)}] (0,0) rectangle (2,2);
\end{tikzpicture}
```

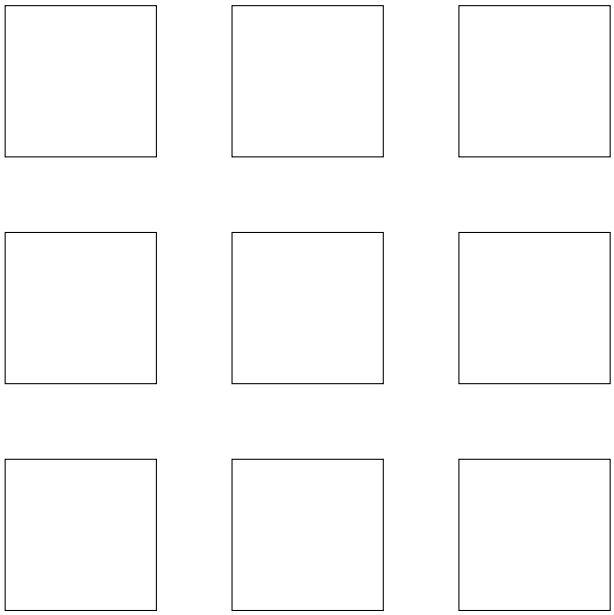


Figure 13.19: transform: shift

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt] (0,0) rectangle (2,2);
  \draw[xshift=-100pt] (0,0) rectangle (2,2);
  \draw[yshift= 100pt] (0,0) rectangle (2,2);
  \draw[yshift=-100pt] (0,0) rectangle (2,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt, xscale=1.5] (0,0) rectangle (2,2);
  \draw[yshift= 100pt, xscale=0.5] (0,0) rectangle (2,2);
  \draw[xshift=-100pt, yscale=1.5] (0,0) rectangle (2,2);
  \draw[yshift=-100pt, yscale=0.5] (0,0) rectangle (2,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift= 100pt, xscale=1.5] (0,0) rectangle (2,2);
```

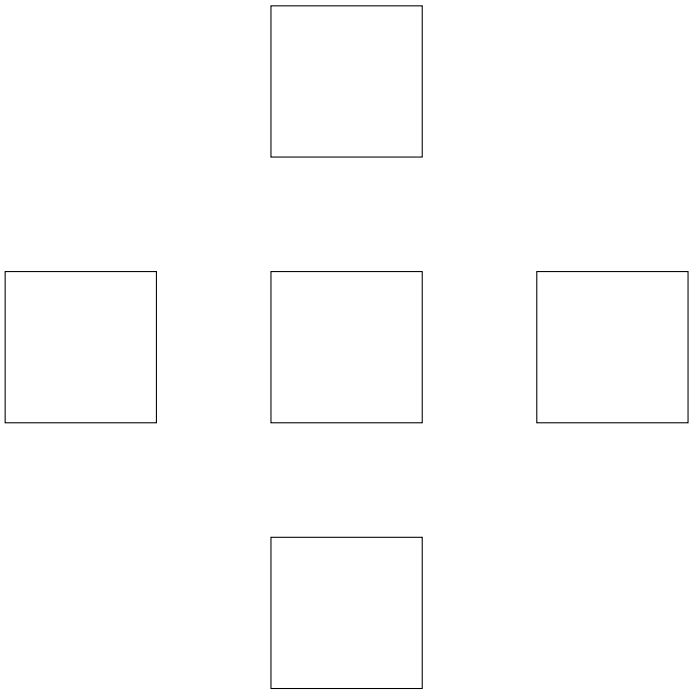


Figure 13.20: transform: shift x, y

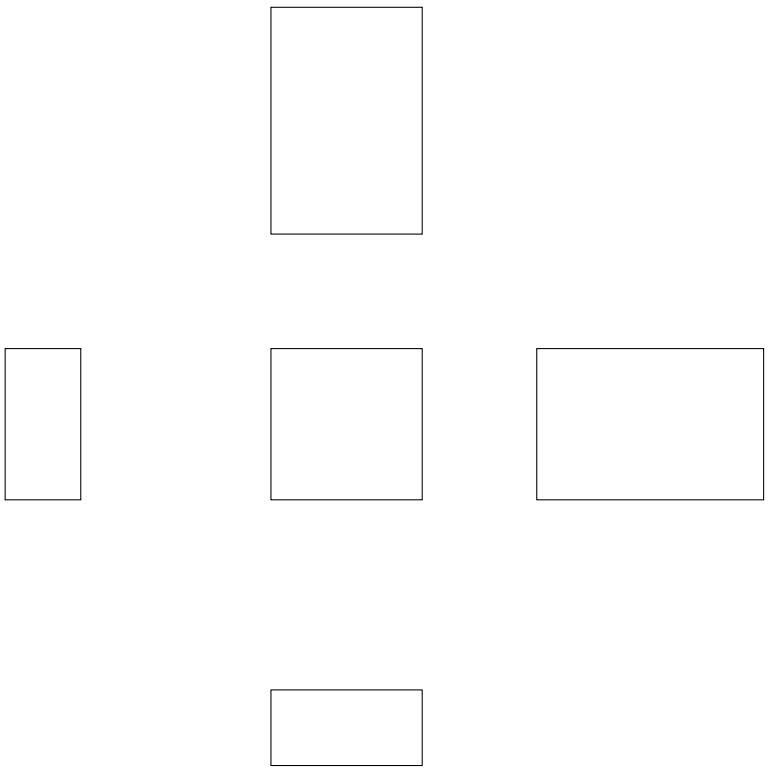


Figure 13.21: transform: scale x, y

```
  \draw[yshift= 100pt, yscale=1.5] (0,0) rectangle (2,2);
  \draw[xshift=-100pt, xscale=0.5] (0,0) rectangle (2,2);
  \draw[yshift=-100pt, yscale=0.5] (0,0) rectangle (2,2);
\end{tikzpicture}
```

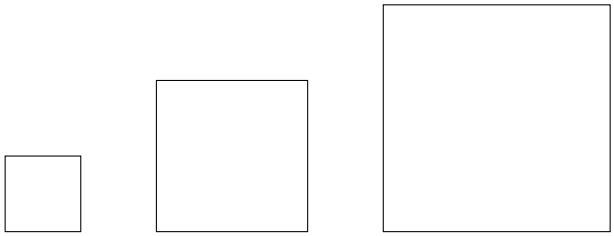


Figure 13.22: transform: scale

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift=125pt,rotate=45] (0,0) rectangle (2,2);
  \draw[xshift=175pt,rotate around={45:(2 ,2)}] (0,0) rectangle (2,2);
\end{tikzpicture}
```

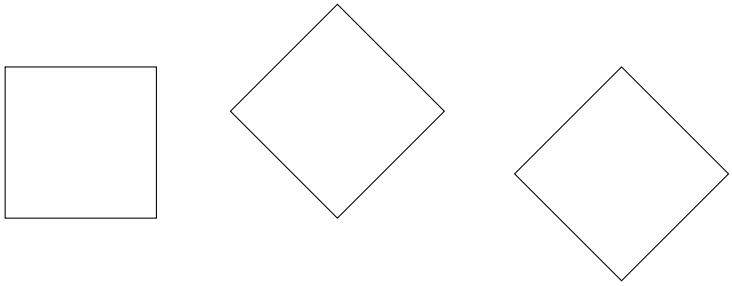


Figure 13.23: transform: rotate

```
\begin{tikzpicture}
  \draw (0,0) rectangle (2,2);
  \draw[xshift=70pt,xslant=1] (0,0) rectangle (2,2);
  \draw[yshift=70pt,yslant=1] (0,0) rectangle (2,2);
\end{tikzpicture}
```

```
\tikzset{
  box/.style={
    draw=blue,
    rectangle,
    rounded corners=5pt,
    minimum width=50pt,
    minimum height=20pt,
    inner sep=5pt
  }
}
\begin{tikzpicture}
  \node[box] (1) at (0,0) {1};
```

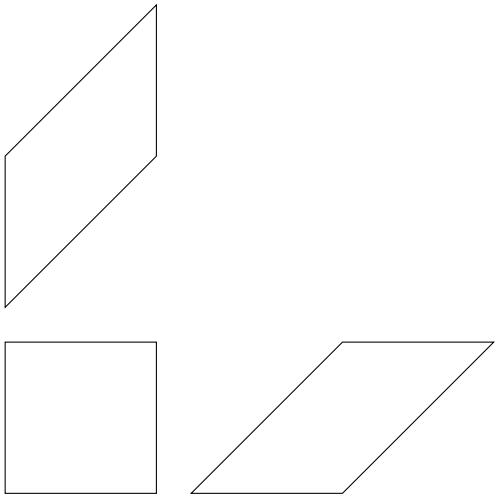


Figure 13.24: transform: slant

```
  \node[box] (2) at (4,0) {2};
  \node[box] (3) at (8,0) {3};
  \draw[->] (1)--(2);
  \draw[->] (2)--(3);
  \node at (2,1) {a};
  \node at (6,1) {b};
\end{tikzpicture}
```

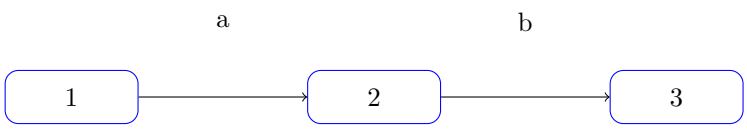


Figure 13.25: flowchart

```
\tikzset{
  box/.style={
    draw=blue,
    fill=blue!20,
    rectangle,
    rounded corners=5pt,
    minimum height=20pt,
    inner sep=5pt
  }
}
\begin{tikzpicture}
  \node[box] {1}
      child {node[box] {2}}
      child {node[box] {3}
          child {node[box] {4}}
          child {node[box] {5}}
          child {node[box] {6}}
```

```
      };
\end{tikzpicture}
```

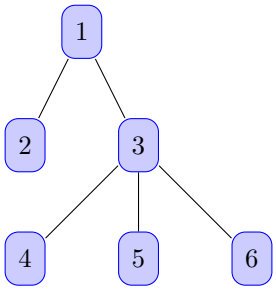


Figure 13.26: tree

```
\begin{tikzpicture}
  \draw[->] (-0.2,0)--(6,0) node[right] {$x$};
  \draw[->] (0,-0.2)--(0,6) node[above] {$f(x)$};
  \draw[domain=0:4] plot (\x ,{0.1* exp(\x)}) node[right] {$f(x)=\frac{1}{10}e^x$};
\end{tikzpicture}
```

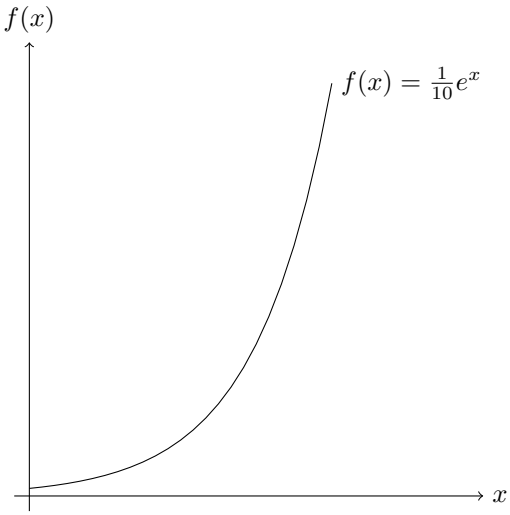


Figure 13.27: tree

https://stackoverflow.com/questions/64897575/tikz-libraries-in-bookdown

It turns out that you can simply put the `\usetikzlibrary{...}` command directly before the `\begin{tikzpicture}` and everything works fine :)

https://stackoverflow.com/questions/56211210/r-markdown-document-with-html-docx-output-using-latex-package-bbm

https://tex.stackexchange.com/questions/171711/how-to-include-latex-package-in-r-markdown

13.2 3D

https://zhuanlan.zhihu.com/p/431732330?utm_psn=1741857547550638080

https://github.com/RRWWW/Stereometry

```
\begin{tikzpicture}
  \coordinate (A) at ( 1, 1, 1);
  \coordinate (B) at ( 1, 1,-1);
  \coordinate (C) at ( 1,-1,-1);
  \coordinate (D) at ( 1,-1, 1);
  \coordinate (E) at (-1,-1, 1);
  \coordinate (F) at (-1,-1,-1);
  \coordinate (G) at (-1, 1,-1);
  \coordinate (H) at (-1, 1, 1);
  \draw (A) node[right=1pt] {$A$}--
        (B) node[right=1pt] {$B$}--
        (C) node[right=1pt] {$C$}--
        (D) node[right=1pt] {$D$}--
        (E) node[left= 1pt] {$E$}--
        (F) node[right=1pt] {$F$}--
        (G) node[right=1pt] {$G$}--
        (H) node[left= 1pt] {$H$}--
        (A) node[right=1pt] {$A$};
\end{tikzpicture}
```

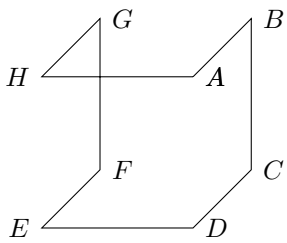


Figure 13.28: cube

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
\tdplotsetmaincoords{45}{45}
\begin{tikzpicture}[tdplot_main_coords]
  \coordinate (A) at ( 1, 1, 1);
  \coordinate (B) at ( 1, 1,-1);
  \coordinate (C) at ( 1,-1,-1);
  \coordinate (D) at ( 1,-1, 1);
  \coordinate (E) at (-1,-1, 1);
  \coordinate (F) at (-1,-1,-1);
  \coordinate (G) at (-1, 1,-1);
  \coordinate (H) at (-1, 1, 1);
  \draw (A) node[right=1pt] {$A$}--
        (B) node[right=1pt] {$B$}--
        (C) node[right=1pt] {$C$}--
        (D) node[right=1pt] {$D$}--
```

```
        (E) node[left= 1pt] {$E$}--
        (F) node[right=1pt] {$F$}--
        (G) node[right=1pt] {$G$}--
        (H) node[left= 1pt] {$H$}--
        (A) node[right=1pt] {$A$};
\end{tikzpicture}
```

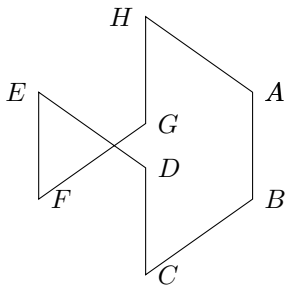


Figure 13.29: cube rotate

https://tex.stackexchange.com/questions/388621/optimizing-perspective-tikz-graphic

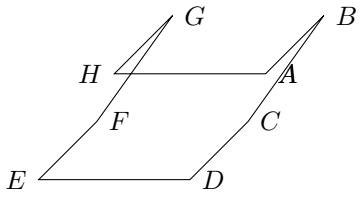


Figure 13.30: cube rotate

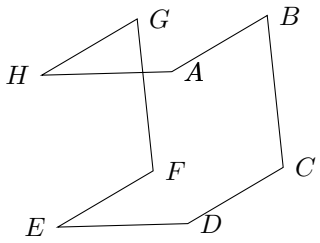


Figure 13.31: cube rotate

https://github.com/XiangyunHuang/bookdown-broken/blob/master/index.Rmd

```
\smartdiagramset{planet color=gray!40!white, uniform color list=gray!40!white for 10
↪  items}
\smartdiagram[bubble diagram]{Basic skills,
  Edit~/\\ (RStudio), Organize~/\\ (bookdown), Cooperate~/\\ (Git), Typeset~/\\
↪  (LaTeX/Pandoc), Compile~/\\ (GitHub Action)}
```

Figure 13.32: modern statistics plot skills

13.3 animation

https://zhuanlan.zhihu.com/p/338402487

Chapter 14

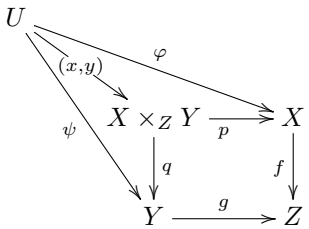
xy-pic

https://bookdown.org/yihui/rmarkdown-cookbook/install-latex-pkgs.html

`tinytex::install_tinytex()`

the following xymatrix from LaTeX package xy for xy-pic is not shown or rendered in HTML:

`$\LaTeX$` can only be used in HTML, not PDF



Chapter 15

statistics

15.1 covariance matrix

[5]

15.1.1 calculation

$$\begin{aligned}
\mathrm{C}[\boldsymbol{X}] = \mathrm{Cov}[\boldsymbol{X}] = \mathrm{V}[\boldsymbol{X}] =& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]\left[\boldsymbol{X}^{\mathrm{T}} - \mathrm{E}(\boldsymbol{X})^{\mathrm{T}}\right]\right] \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} - \mathrm{E}(\boldsymbol{X})\boldsymbol{X}^{\mathrm{T}} - \boldsymbol{X}\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}\left[\mathrm{E}(\boldsymbol{X})\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}\left[\boldsymbol{X}\mathrm{E}(\boldsymbol{X})^{\mathrm{T}}\right] + \mathrm{E}\left[\mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}(\boldsymbol{X})\mathrm{E}\left[\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}[\boldsymbol{X}]\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} + \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} \\
=& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{X} = [X]_{1\times 1} = X \Rightarrow C(X) = \mathrm{C}[\boldsymbol{X}] =& \mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right] - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{X})^{\mathrm{T}} \\
=& \mathrm{E}[XX] - \mathrm{E}(X)\mathrm{E}(X) \\
=& \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = \mathrm{V}(X)
\end{aligned}$$

15.1.2 $\mathrm{V}[\boldsymbol{X} + \boldsymbol{b}] = \mathrm{V}[\boldsymbol{X}]$

$$\begin{aligned}
\mathrm{V}[\boldsymbol{X} + \boldsymbol{b}] =& \mathrm{E}\left[[(\boldsymbol{X} + \boldsymbol{b}) - \mathrm{E}(\boldsymbol{X} + \boldsymbol{b})][(\boldsymbol{X} + \boldsymbol{b}) - \mathrm{E}(\boldsymbol{X} + \boldsymbol{b})]^{\mathrm{T}}\right] \\
\overset{\mathrm{E}(\boldsymbol{X}+\boldsymbol{b})=\mathrm{E}(\boldsymbol{X})+\boldsymbol{b}}{=}& \mathrm{E}\left[[\boldsymbol{X} + \boldsymbol{b} - \mathrm{E}(\boldsymbol{X}) - \boldsymbol{b}][\boldsymbol{X} + \boldsymbol{b} - \mathrm{E}(\boldsymbol{X}) - \boldsymbol{b}]^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^{\mathrm{T}}\right] = \mathrm{V}[\boldsymbol{X}]
\end{aligned}$$

15.1.3 $\mathrm{V}[A\boldsymbol{X}] = A\mathrm{V}[\boldsymbol{X}]A^{\mathrm{T}}$

$$\begin{aligned}
\mathrm{V}[A\boldsymbol{X}] =& \mathrm{E}\left[[(A\boldsymbol{X}) - \mathrm{E}(A\boldsymbol{X})][(A\boldsymbol{X}) - \mathrm{E}(A\boldsymbol{X})]^{\mathrm{T}}\right] \\
\overset{\mathrm{E}(A\boldsymbol{X})=A\mathrm{E}(\boldsymbol{X})}{=}& \mathrm{E}\left[[A\boldsymbol{X} - A\mathrm{E}(\boldsymbol{X})][A\boldsymbol{X} - A\mathrm{E}(\boldsymbol{X})]^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]]^{\mathrm{T}}\right] \\
=& \mathrm{E}\left[A[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^{\mathrm{T}}A^{\mathrm{T}}\right] \\
=& A\mathrm{E}\left[[\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})][\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})]^{\mathrm{T}}\right]A^{\mathrm{T}} = A\mathrm{V}[\boldsymbol{X}]A^{\mathrm{T}}
\end{aligned}$$

15.1.4 $\mathrm{V}[A\boldsymbol{X} + \boldsymbol{b}] = A\mathrm{V}[\boldsymbol{X}]A^{\mathrm{T}}$

$$\mathrm{V}[A\boldsymbol{X} + \boldsymbol{b}] = \mathrm{V}[A\boldsymbol{X}] = A\mathrm{V}[\boldsymbol{X}]A^{\mathrm{T}}$$

Chapter 16

Gosper algorithm

Chapter 17

Lorentz transformation

17.1 Einstein

https://wap.hillpublisher.com/UpFile/202204/20220414165340.pdf

17.2 Bondi k-calculus

https://en.wikipedia.org/wiki/Bondi_k-calculus

17.3 wordline in Minkowski space

17.3.1 Wick rotation

https://ncatlab.org/nlab/show/Wick+rotation

17.3.1.1 Osterwalder-Schrader reconstruction theorem

https://ncatlab.org/nlab/show/Osterwalder-Schrader+theorem

Chapter 18

R

https://bookdown.org/tonykuoyj/eloquentr/getting-started.html

Chapter 19

Laplace transform

Chapter 20

conic section

conic section 圓錐曲線 / 圓錐截痕

https://en.wikipedia.org/wiki/Conic_section

https://tex.stackexchange.com/questions/222882/drawing-minimal-xy-axis

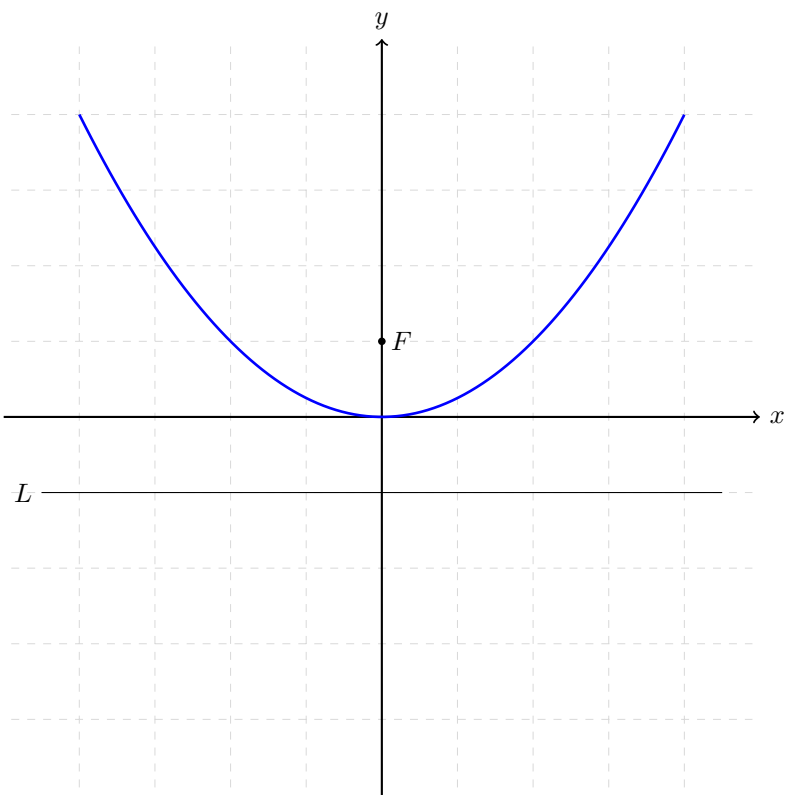


Figure 20.1: parabola defined by focus, directrix, eccentricity

20.1 Cartesian coordinate: focus, directrix, eccentricity

focus, directrix, eccentricity 焦點, 準線, 離心率

$$\begin{cases} F=(0,y_F) & F:\text{focus} \\ L=y-y_L=0 & L:\text{directrix} \\ \epsilon=\dfrac{\overline{PF}}{d(P,L)}=\dfrac{\|(x,y)-(0,y_F)\|}{\|y-y_L\|} & \begin{cases} P=(x,y) \\ \epsilon:\text{eccentricity} \end{cases} \end{cases}$$

$$0\le\epsilon=\frac{\overline{PF}}{d(P,L)}=\frac{\overline{PF}}{\overline{PP'}}=\frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|}=\frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|}=\frac{\sqrt{x^2+(y-y_F)^2}}{\sqrt{(y-y_L)^2}} \tag{20.1}$$

$$\epsilon^2=\frac{x^2+(y-y_F)^2}{(y-y_L)^2}=\frac{x^2+y^2-2y_Fy+y_F^2}{y^2-2y_Ly+y_L^2} \tag{20.2}$$

$$0=x^2+\left(1-\epsilon^2\right)y^2-2\left(y_F-\epsilon^2y_L\right)y+\left(y_F^2-\epsilon^2y_L^2\right) \tag{20.3}$$

$$\overset{\epsilon\neq1}{=}x^2+\left(1-\epsilon^2\right)\left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right] \tag{20.4}$$

$$=x^2+\left(1-\epsilon^2\right) \tag{20.5}$$

$$\left[y^2-\frac{2\left(y_F-\epsilon^2y_L\right)}{1-\epsilon^2}y+\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2-\left(\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{y_F^2-\epsilon^2y_L^2}{1-\epsilon^2}\right] \tag{20.6}$$

$$=x^2+\left(1-\epsilon^2\right)\left[\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)-\left(y_F-\epsilon^2y_L\right)^2}{\left(1-\epsilon^2\right)^2}\right] \tag{20.7}$$

$$=x^2+\left(1-\epsilon^2\right)\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2+\frac{\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)\text{-}\left(y_F-\epsilon^2y_L\right)^2}{1-\epsilon^2} \tag{20.8}$$

$$\begin{aligned}&\left(y_F^2-\epsilon^2y_L^2\right)\left(1-\epsilon^2\right)\text{-}\left(y_F-\epsilon^2y_L\right)^2 \\ &=\left(1-\epsilon^2\right)y_F^2-\left(\epsilon^2-\epsilon^4\right)y_L^2-y_F^2+2\epsilon^2y_Fy_L-\epsilon^4y_L^2 \\ &=-\epsilon^2y_F^2-\epsilon^2y_L^2+2\epsilon^2y_Fy_L=-\epsilon^2\left(y_F-y_L\right)^2\end{aligned}$$

$$\frac{\epsilon^2\left(y_F-y_L\right)^2}{1-\epsilon^2}\overset{\epsilon\neq1}{=}x^2+\left(1-\epsilon^2\right)\left(y-\frac{y_F-\epsilon^2y_L}{1-\epsilon^2}\right)^2$$

$$1\overset{\epsilon\neq0,1}{=}\begin{cases}\left(\dfrac{x-0}{\dfrac{\epsilon\left(y_F-y_L\right)}{\sqrt{1-\epsilon^2}}}\right)^2+\left(\dfrac{y-\dfrac{y_F-\epsilon^2y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2>0\overset{\epsilon\geq0}{\Rightarrow}0<\epsilon<1 \\ -\left(\dfrac{x-0}{\dfrac{\epsilon\left(y_F-y_L\right)}{\sqrt{\epsilon^2-1}}}\right)^2+\left(\dfrac{y-\dfrac{y_F-\epsilon^2y_L}{1-\epsilon^2}}{\dfrac{\epsilon\left(y_F-y_L\right)}{1-\epsilon^2}}\right)^2 & 1-\epsilon^2<0\overset{\epsilon\geq0}{\Rightarrow}\epsilon>1\end{cases}$$

$\epsilon = 0$ or $\lim_{|y_L|\to\infty} \epsilon = 0$

$$r = \overline{PF} = \|(x,y) - (0,y_F)\| = \|(x, y - y_F)\| = \sqrt{x^2 + (y-y_F)^2}$$

$$\epsilon = \frac{r}{d(P,L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x,y) - (0,y_F)\|}{\|(x,y) - (x,y_L)\|} = \frac{\|(x, y-y_F)\|}{\|(0, y-y_L)\|} = \frac{\sqrt{x^2 + (y-y_F)^2}}{|y - y_L|}$$

$$\lim_{|y_L|\to\infty} \epsilon = \lim_{|y_L|\to\infty} \frac{r}{d(P,L)} = \lim_{|y_L|\to\infty} \frac{\sqrt{x^2 + (y-y_F)^2}}{|y-y_L|} = 0$$

$\epsilon = 1$

$$\begin{aligned}
0 =& x^2 + \left(1-\epsilon^2\right) y^2 - 2\left(y_F - \epsilon^2 y_L\right) y + \left(y_F^2 - \epsilon^2 y_L^2\right) \\
\overset{\epsilon=1}{=}& x^2 + \left(1-1^2\right) y^2 - 2\left(y_F - 1^2 y_L\right) y + \left(y_F^2 - 1^2 y_L^2\right) \\
=& x^2 - 2(y_F - y_L) y + \left(y_F^2 - y_L^2\right) \\
=& x^2 - 2(y_F - y_L) y + (y_F + y_L)(y_F - y_L) \\
x^2 =& 2(y_F - y_L)\left(y - \frac{y_F + y_L}{2}\right)
\end{aligned}$$

Let one curve vertex $P = V = (0,0)$ on the curve, and fix the directrix L or y_L,

$\epsilon \neq 1$

$$\begin{aligned}
1 &\overset{P(x,y)=V(0,0)}{=} 0 + \left(\frac{0 - \frac{y_F - \epsilon^2 y_L}{1-\epsilon^2}}{\frac{\epsilon(y_F - y_L)}{1-\epsilon^2}}\right)^2 \\
\Rightarrow & y_F - \epsilon^2 y_L = \pm\epsilon(y_F - y_L) \\
\Rightarrow & \begin{cases} (1-\epsilon) y_F = \epsilon(\epsilon - 1) y_L & + \\ (1+\epsilon) y_F = \epsilon(\epsilon + 1) y_L & - \end{cases} \\
\Rightarrow & y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases}
\end{aligned}$$

$\epsilon = 1$

$$\begin{aligned}
x^2 &= 2(y_F - y_L)\left(y - \frac{y_F + y_L}{2}\right) \\
&\overset{P(x,y)=V(0,0)}{\Rightarrow} 0^2 = 2(y_F - y_L)\left(0 - \frac{y_F + y_L}{2}\right) \\
&\Rightarrow 0 = (y_F - y_L)(y_F + y_L) \\
&\Rightarrow y_F = \mp y_L
\end{aligned}$$

or by definition of eccentricity (20.1)

$$0 \le \epsilon = \frac{\overline{PF}}{d(P,L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x,y)-(0,y_F)\|}{\|(x,y)-(x,y_L)\|} = \frac{\|(x,y-y_F)\|}{\|(0,y-y_L)\|} = \frac{\sqrt{x^2+(y-y_F)^2}}{\sqrt{(y-y_L)^2}}$$

$$\overset{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2+(0-y_F)^2}}{\sqrt{(0-y_L)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2}$$

$$\epsilon^2 = \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp \epsilon y_L$$

actually,

$$y_F = -\epsilon y_L$$

20.2 two-definition equivalence for ellipse and hyperbola

https://math.stackexchange.com/questions/1833973/prove-that-the-directrix-focus-and-focus-focus-definitions-are-equivalent

https://www.geogebra.org/calculator/zkppuxwp

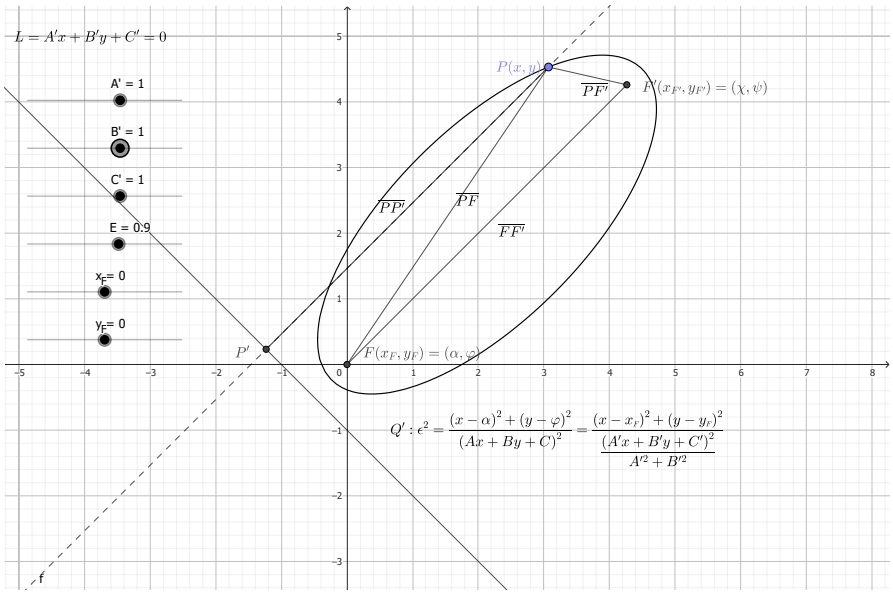


Figure 20.2: conic sections

$$\begin{cases} P = (x,y) \\ F = (x_F, y_F) = (\alpha, \varphi) \qquad F' = (x_{F'}, y_{F'}) = (\chi, \psi) \\ L = A'x + B'y + C' = 0 \end{cases}$$

20.2.1 first definition for conic sections including ellipses and hyperbolas

distance from a point to a line[21]

$$0 \leq \epsilon = \frac{\overline{PF}}{d\left(P,L\right)} = \frac{\sqrt{\left(x-x_{F}\right)^{2}+\left(y-y_{F}\right)^{2}}}{\frac{\left|A'x+B'y+C'\right|}{\sqrt{A'^{2}+B'^{2}}}} = \frac{\sqrt{\left(x-\alpha\right)^{2}+\left(y-\varphi\right)^{2}}}{\left|Ax+By+C\right|}, \begin{cases} A = \frac{A'}{\sqrt{A'^{2}+B'^{2}}} \\ B = \frac{B'}{\sqrt{A'^{2}+B'^{2}}} \\ C = \frac{C'}{\sqrt{A'^{2}+B'^{2}}} \end{cases}$$

$$A^{2}+B^{2} = \left(\frac{A'}{\sqrt{A'^{2}+B'^{2}}}\right)^{2} + \left(\frac{B'}{\sqrt{A'^{2}+B'^{2}}}\right)^{2} = 1$$

or allowing $\epsilon < 0$ by squaring the definition

$$\epsilon^{2} = \frac{\left(x-\alpha\right)^{2}+\left(y-\varphi\right)^{2}}{\left(Ax+By+C\right)^{2}} = \frac{\left(x-x_{F}\right)^{2}+\left(y-y_{F}\right)^{2}}{\frac{\left(A'x+B'y+C'\right)^{2}}{A'^{2}+B'^{2}}}$$

$$\left(x-\alpha\right)^{2}+\left(y-\varphi\right)^{2} = \left[\epsilon\left(Ax+By+C\right)\right]^{2}$$

20.2.2 second definition for ellipses and hyperbolas

$$\begin{aligned} 2c = \overline{FF'} &= \left\|\left(x_{F},y_{F}\right)-\left(x_{F'},y_{F'}\right)\right\| = \left\|\left(\alpha,\varphi\right)-\left(\chi,\psi\right)\right\| \\ &= \sqrt{\left(\alpha-\chi\right)^{2}+\left(\chi-\psi\right)^{2}} \end{aligned}$$

$$\begin{aligned} D &= \begin{cases} \sqrt{\left(x-x_{F}\right)^{2}+\left(y-y_{F}\right)^{2}} + \sqrt{\left(x-x_{F'}\right)^{2}+\left(y-y_{F'}\right)^{2}} & \text{ellipse} \\ \sqrt{\left(x-x_{F}\right)^{2}+\left(y-y_{F}\right)^{2}} - \sqrt{\left(x-x_{F'}\right)^{2}+\left(y-y_{F'}\right)^{2}} & \text{hyperbola} \end{cases} \\ &= \sqrt{\left(x-x_{F}\right)^{2}+\left(y-y_{F}\right)^{2}} \pm \sqrt{\left(x-x_{F'}\right)^{2}+\left(y-y_{F'}\right)^{2}} \\ &= \sqrt{\left(x-\alpha\right)^{2}+\left(y-\varphi\right)^{2}} \pm \sqrt{\left(x-\chi\right)^{2}+\left(y-\psi\right)^{2}} \end{aligned}$$

$$\begin{aligned} \left(x-\alpha\right)^{2}+\left(y-\varphi\right)^{2} &= \left(D \mp \sqrt{\left(x-\chi\right)^{2}+\left(y-\psi\right)^{2}}\right)^{2} \\ &= D^{2} \mp 2D\sqrt{\left(x-\chi\right)^{2}+\left(y-\psi\right)^{2}} \\ &\quad + \left(x-\chi\right)^{2}+\left(y-\psi\right)^{2} \end{aligned}$$

$$
\begin{aligned}
D^2 =& (x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2 \\
& \pm 2\sqrt{\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right]} \\
& (x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2 - D^2 \\
=& \mp 2\sqrt{\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right]} \\
& \left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right]^2 + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
=& 4\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] \\
& \left[(x-\alpha)^2 + (y-\varphi)^2\right]^2 + \left[(x-\chi)^2 + (y-\psi)^2\right]^2 \\
& + 2\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
=& 4\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] \\
0 =& \left[(x-\alpha)^2 + (y-\varphi)^2\right]^2 + \left[(x-\chi)^2 + (y-\psi)^2\right]^2 \\
& - 2\left[(x-\alpha)^2 + (y-\varphi)^2\right]\left[(x-\chi)^2 + (y-\psi)^2\right] + D^4 \\
& - 2D^2\left[(x-\alpha)^2 + (y-\varphi)^2 + (x-\chi)^2 + (y-\psi)^2\right] \\
0 =& \left\{\left[(x-\alpha)^2 + (y-\varphi)^2\right] - \left[(x-\chi)^2 + (y-\psi)^2\right]\right\}^2 + D^4 \\
& - 2D^2\left\{\left[(x-\alpha)^2 + (y-\varphi)^2\right] + \left[(x-\chi)^2 + (y-\psi)^2\right]\right\} \\
0 =& \left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right]\right\}^2 + D^4 \\
& - 2D^2\left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right]\right\} \\
& - 4D^2\left[(x-\alpha)^2 + (y-\varphi)^2\right] \\
& (2D)^2\left[(x-\alpha)^2 + (y-\varphi)^2\right] \\
=& \left\{\left[(x-\chi)^2 + (y-\psi)^2\right] - \left[(x-\alpha)^2 + (y-\varphi)^2\right] - D^2\right\}^2 \\
=& \left\{\left[(x-\chi)^2 - (x-\alpha)^2\right] + \left[(y-\psi)^2 - (y-\varphi)^2\right] - D^2\right\}^2 \\
=& \left\{(2x-\chi-\alpha)(\alpha-\chi) + (2y-\psi-\varphi)(\varphi-\psi) - D^2\right\}^2 \\
=& \left\{2(\alpha-\chi)x - \left(\alpha^2-\chi^2\right) + 2(\varphi-\psi)y - \left(\varphi^2-\psi^2\right) - D^2\right\}^2 \\
=& \left\{2(\alpha-\chi)x + 2(\varphi-\psi)y - \left[\left(\alpha^2-\chi^2\right) + \left(\varphi^2-\psi^2\right) + D^2\right]\right\}^2
\end{aligned}
$$

$D \neq 0$

$$
\begin{aligned}
& (x-\alpha)^2 + (y-\varphi)^2 \\
=& \left[\frac{\alpha-\chi}{D}x + \frac{\varphi-\psi}{D}y - \left(\frac{\alpha^2-\chi^2}{2D} + \frac{\varphi^2-\psi^2}{2D} + \frac{D}{2}\right)\right]^2
\end{aligned}
$$

$$\begin{cases} (x-\alpha)^2+(y-\varphi)^2=[\epsilon\,(Ax+By+C)]^2 \\ (x-\alpha)^2+(y-\varphi)^2=\left[\dfrac{\alpha-\chi}{D}x+\dfrac{\varphi-\psi}{D}y-\left(\dfrac{\alpha^2-\chi^2}{2D}+\dfrac{\varphi^2-\psi^2}{2D}+\dfrac{D}{2}\right)\right]^2 \end{cases}$$

$$(A,B,C)\rightleftarrows(\chi,\psi,D)$$

$$\begin{cases} \epsilon A=\pm\dfrac{\alpha-\chi}{D} & \chi\pm\epsilon AD=\alpha \\ \epsilon B=\pm\dfrac{\varphi-\psi}{D} & \psi\pm\epsilon BD=\varphi \\ \epsilon C=\mp\left(\dfrac{\alpha^2-\chi^2}{2D}+\dfrac{\varphi^2-\psi^2}{2D}+\dfrac{D}{2}\right) & \end{cases}$$

$$\begin{aligned} 2\epsilon C&=\mp\left(\frac{\alpha-\chi}{D}(\alpha+\chi)+\frac{\varphi-\psi}{D}(\varphi+\psi)+D\right) \\ &=\mp\left(\pm\epsilon A(\alpha+\chi)\pm\epsilon B(\varphi+\psi)+D\right) \\ \mp\epsilon(A\alpha+B\varphi+2C)&=\pm\epsilon A\chi\pm\epsilon B\psi+D \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A \\ 0 & 1 & \pm\epsilon B \\ \pm\epsilon A & \pm\epsilon B & 1 \end{pmatrix}\begin{pmatrix}\chi\\ \psi\\ D\end{pmatrix}=\begin{pmatrix}\alpha\\ \varphi\\ \mp\epsilon(A\alpha+B\varphi+2C)\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & \pm\epsilon B & 1\mp\epsilon^2A^2 & \mp\epsilon(2A\alpha+B\varphi+2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1\mp\epsilon^2A^2\mp\epsilon^2B^2 & \mp\epsilon(2A\alpha+2B\varphi+2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm\epsilon A & \alpha \\ 0 & 1 & \pm\epsilon B & \varphi \\ 0 & 0 & 1 & \dfrac{\mp2\epsilon(A\alpha+B\varphi+C)}{1\mp\epsilon^2(A^2+B^2)} \end{pmatrix}$$

$$A^2+B^2=\left(\frac{A'}{\sqrt{A'^2+B'^2}}\right)^2+\left(\frac{B'}{\sqrt{A'^2+B'^2}}\right)^2=1$$

$$\begin{cases} \chi=\alpha\mp\epsilon AD=\alpha\mp\epsilon\dfrac{A'}{\sqrt{A'^2+B'^2}}D \\ \psi=\varphi\mp\epsilon BD=\varphi\mp\epsilon\dfrac{B'}{\sqrt{A'^2+B'^2}}D \\ D=\dfrac{\mp2\epsilon(A\alpha+B\varphi+C)}{1\mp\epsilon^2(A^2+B^2)}=\dfrac{\mp2\epsilon}{1\mp\epsilon^2}\dfrac{A'\alpha+B'\varphi+C'}{\sqrt{A'^2+B'^2}} & A^2+B^2=1 \end{cases}$$

actually, only one of two solutions is true

$$\begin{cases} \chi = \alpha - \epsilon AD = \alpha - \epsilon \dfrac{A'}{\sqrt{A'^2 + B'^2}} D = \alpha - \dfrac{2\epsilon^2}{\epsilon^2 - 1} \dfrac{A'^2\alpha + A'B'\varphi + A'C'}{A'^2 + B'^2} \\ \psi = \varphi - \epsilon BD = \varphi - \epsilon \dfrac{B'}{\sqrt{A'^2 + B'^2}} D = \varphi - \dfrac{2\epsilon^2}{\epsilon^2 - 1} \dfrac{A'B'\alpha + B'^2\varphi + B'C'}{A'^2 + B'^2} \\ D = \dfrac{-2\epsilon\left(A\alpha + B\varphi + C\right)}{1 - \epsilon^2\left(A^2 + B^2\right)} = \dfrac{-2\epsilon}{1 - \epsilon^2} \dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} = \dfrac{2\epsilon}{\epsilon^2 - 1} \dfrac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

$$\begin{cases} \chi = \dfrac{\left(\epsilon^2 - 1\right)\left(A'^2 + B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right)}{\left(\epsilon^2 - 1\right)\left(A'^2 + B'^2\right)} \\ \psi = \dfrac{\left(\epsilon^2 - 1\right)\left(A'^2 + B'^2\right)\varphi - 2\epsilon^2\left(A'B'\alpha + B'^2\varphi + B'C'\right)}{\left(\epsilon^2 - 1\right)\left(A'^2 + B'^2\right)} \\ \left|\dfrac{D}{d\left(F, L\right)}\right| = \left|\dfrac{2\epsilon}{1 - \epsilon^2}\right| \Rightarrow \left(\dfrac{D}{d\left(F, L\right)}\right)^2 = \left(\dfrac{2\epsilon}{1 - \epsilon^2}\right)^2 \end{cases}$$

$$\begin{aligned} &\left(\epsilon^2 - 1\right)\left(A'^2 + B'^2\right)\alpha - 2\epsilon^2\left(A'^2\alpha + A'B'\varphi + A'C'\right) \\ =& \left(-\left(\epsilon^2 + 1\right)A'^2 + \left(\epsilon^2 - 1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \\ =& \left(-\left(\epsilon^2 + 1\right)A'^2 + \left(\epsilon^2 - 1\right)B'^2\right)\alpha - 2\epsilon^2\left(A'B'\varphi + A'C'\right) \end{aligned}$$

Can the above be more simplified?

$$\begin{aligned} \overline{FF'}^2 &= \left(\alpha - \chi\right)^2 + \left(\varphi - \psi\right)^2 \\ &= \left(\alpha - \left(\alpha - \epsilon AD\right)\right)^2 + \left(\varphi - \left(\varphi - \epsilon BD\right)\right)^2 \\ &= \left(\epsilon D\right)^2\left(A^2 + B^2\right) \\ &= \left(\epsilon D\right)^2 \end{aligned}$$

20.2.3 eccentricity and its equivalent representation

$$\left(\frac{c}{a}\right)^2 = \left(\frac{\overline{PF}}{d\left(P, L\right)}\right)^2 = \epsilon^2 = \left(\frac{\overline{FF'}}{D}\right)^2 = \left(\frac{2c}{D}\right)^2 \Rightarrow D = 2a$$

$$\left(\frac{D}{d\left(F, L\right)}\right)^2 = \left(\frac{2\epsilon}{1 - \epsilon^2}\right)^2$$

20.3 Cartesian coordinate: standard form / standard equation

circle	$\left(\dfrac{y-k}{a}\right)^2 + \left(\dfrac{x-h}{a}\right)^2 = 1$			$b = a$
ellipse	$\left(\dfrac{y-k}{b}\right)^2 + \left(\dfrac{x-h}{a}\right)^2 = 1$	vertical		$b > a$
	$\left(\dfrac{y-k}{b}\right)^2 + \left(\dfrac{x-h}{a}\right)^2 = 1$	horizontal		$a > b$
parabola	$(y-k) - 4c\left(x-h\right)^2 = 0$	vertical		
	$-4c\left(y-k\right)^2 + (x-h) = 0$	horizontal		
hyperbola	$\left(\dfrac{y-k}{b}\right)^2 - \left(\dfrac{x-h}{a}\right)^2 = 1$	vertical	$\dfrac{x-h}{a} = 0 \Rightarrow \dfrac{y-k}{b} = \pm 1$	
	$-\left(\dfrac{y-k}{b}\right)^2 + \left(\dfrac{x-h}{a}\right)^2 = 1$	horizontal	$\dfrac{y-k}{b} = 0 \Rightarrow \dfrac{x-h}{a} = \pm 1$	

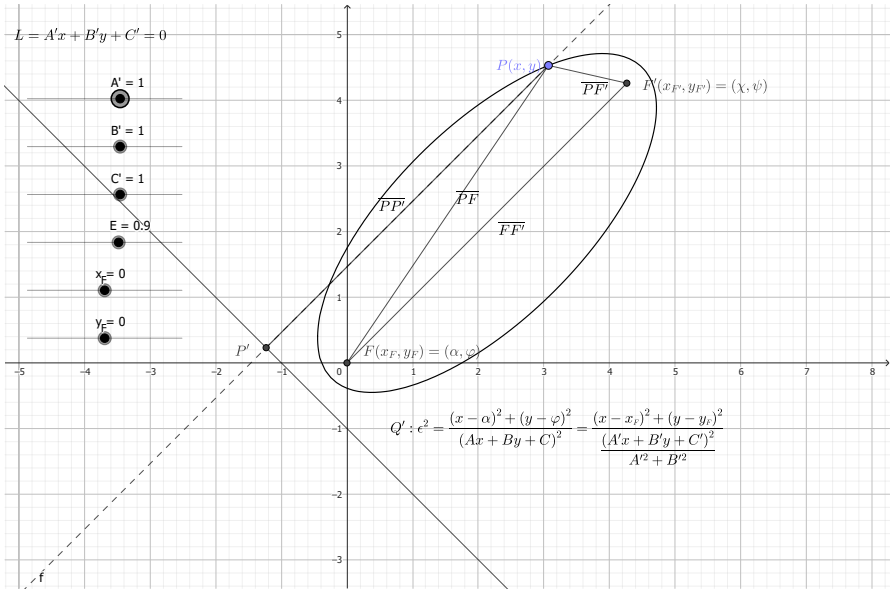


Figure 20.3: conic sections: ellipse

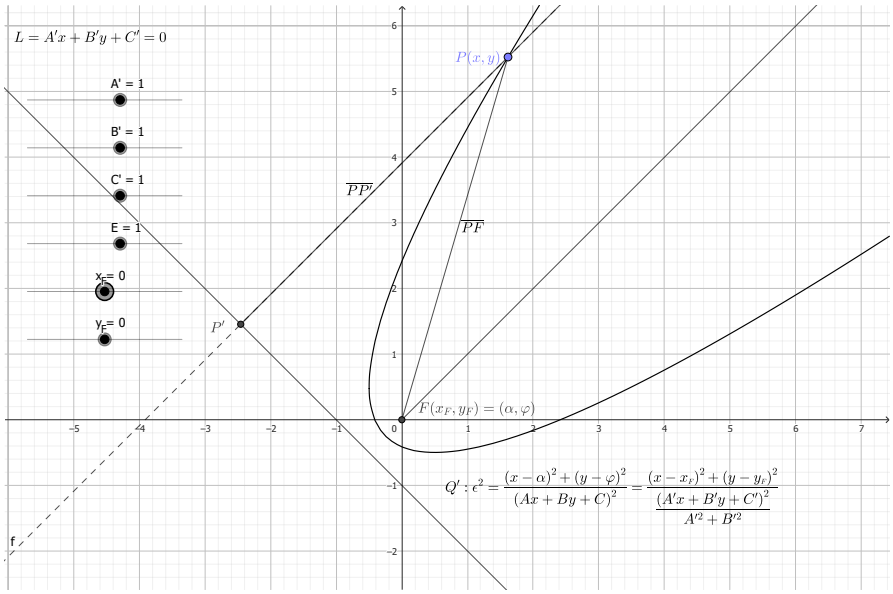


Figure 20.4: conic sections: parabola

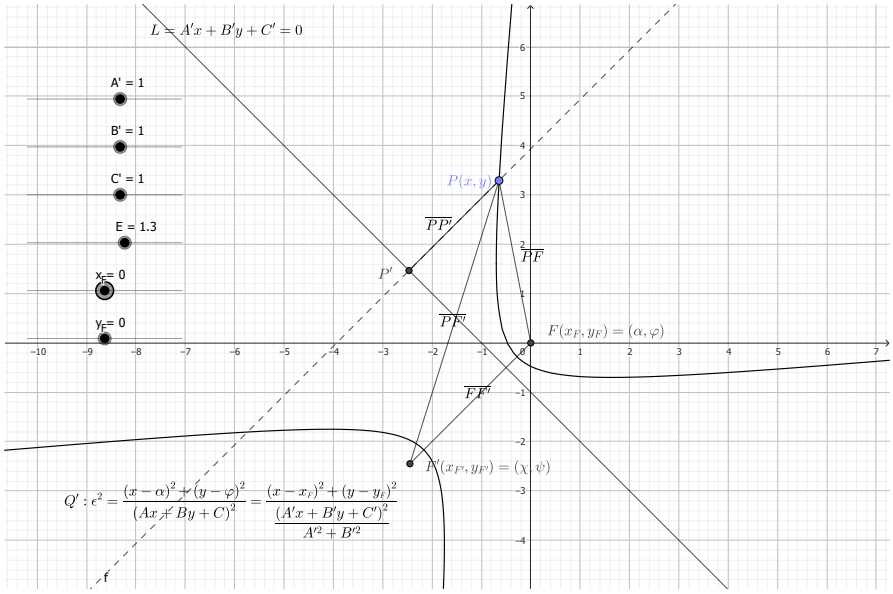


Figure 20.5: conic sections: hyperbola

20.4 parametric equation

circle	$\left(\frac{y-k}{a}\right)^2 + \left(\frac{x-h}{a}\right)^2$	$= 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} a & 0 & h \\ 0 & a & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix}$
ellipse	$\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2$	$= 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix}$
parabola	$(y-k) - 4c\,(x-h)^2$	$= 0$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} 1 & 0 & h \\ 0 & 4c & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} t & 0 & h \\ 0 & t^2 & k \\ 0 & 0 & 1 \end{pmatrix}$
	$-4c\,(y-k)^2 + (x-h)$	$= 0$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} 4c & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} t^2 & 0 & h \\ 0 & t & k \\ 0 & 0 & 1 \end{pmatrix}$
hyperbola	$\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2$	$= 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \pm\cosh t \\ \sinh t \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} \tan t & 0 & h \\ 0 & \sec t & k \\ 0 & 0 & 1 \end{pmatrix}$
	$-\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2$	$= 1$	$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \sinh t \\ \pm\cosh t \\ 1 \end{pmatrix}$	$=$	$\begin{pmatrix} \sec t & 0 & h \\ 0 & \tan t & k \\ 0 & 0 & 1 \end{pmatrix}$

20.5 polar coordinate

$$(x-\alpha)^2 + (y-\varphi)^2 = [\epsilon\,(Ax + By + C)]^2$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(r\cos\theta - \alpha)^2 + (r\sin\theta - \varphi)^2 = [\epsilon\,(Ar\cos\theta + Br\sin\theta + C)]^2$$

If $\begin{cases} F = (x_F, y_F) = (\alpha, \varphi) = (0, 0) \\ L = Ax + By + C = x + p = 0 \end{cases}$

$$\begin{aligned} (r\cos\theta)^2 + (r\sin\theta)^2 &= [\epsilon\,(r\cos\theta + p)]^2 \\ r^2 &= \\ r &= \pm\,\epsilon\,(r\cos\theta + p) \\ &= \pm\,(r\epsilon\cos\theta + \epsilon p) \\ r\,(1 \mp \epsilon\cos\theta) &= \epsilon p \\ r &= \frac{\epsilon p}{1 \mp \epsilon\cos\theta} \end{aligned}$$

https://www.geogebra.org/calculator/azksjxbq

$r = \dfrac{\epsilon p}{1 - \epsilon\cos\theta}$ will not cross $L = x + p = 0$ on graphs, so maybe it is a more correct solution

$$r = \frac{\epsilon p}{1 - \epsilon\cos\theta}$$

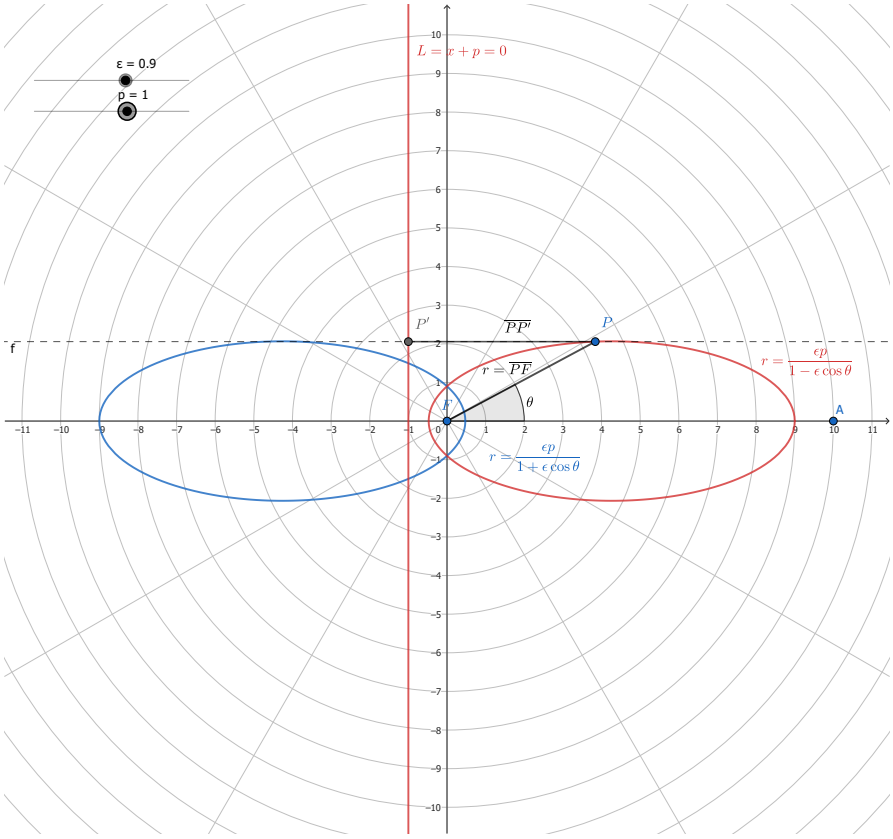


Figure 20.6: polar conic sections: ellipse

https://ccjou.wordpress.com/2013/05/24/%E5%9C%93%E9%8C%90%E6%9B%B2%E7%B7%9A/

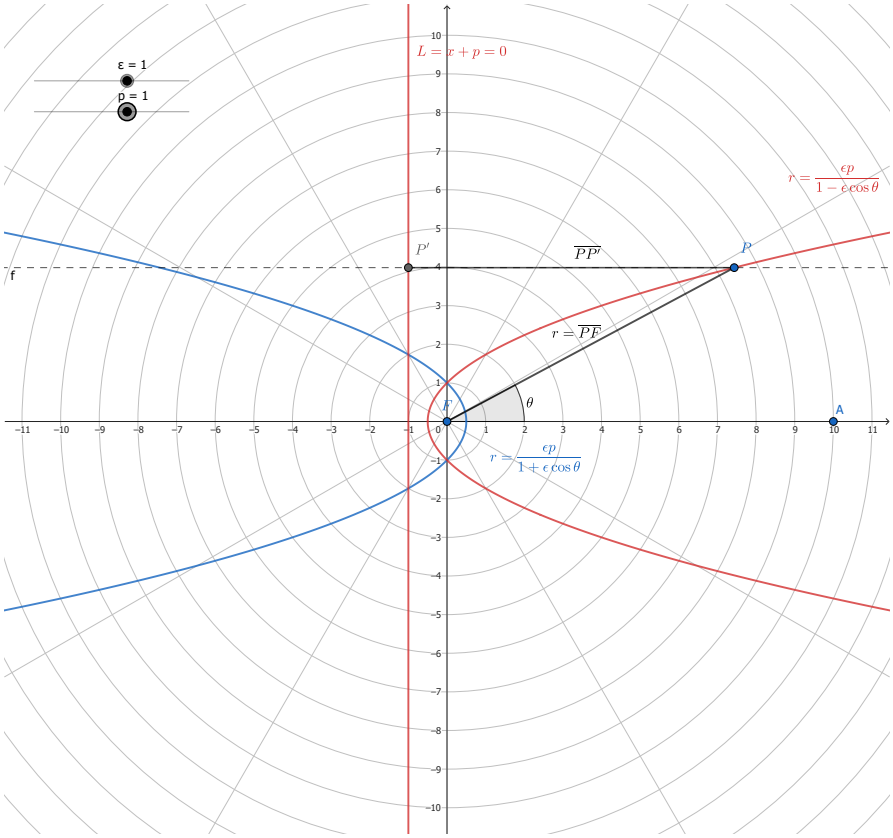


Figure 20.7: polar conic sections: parabola

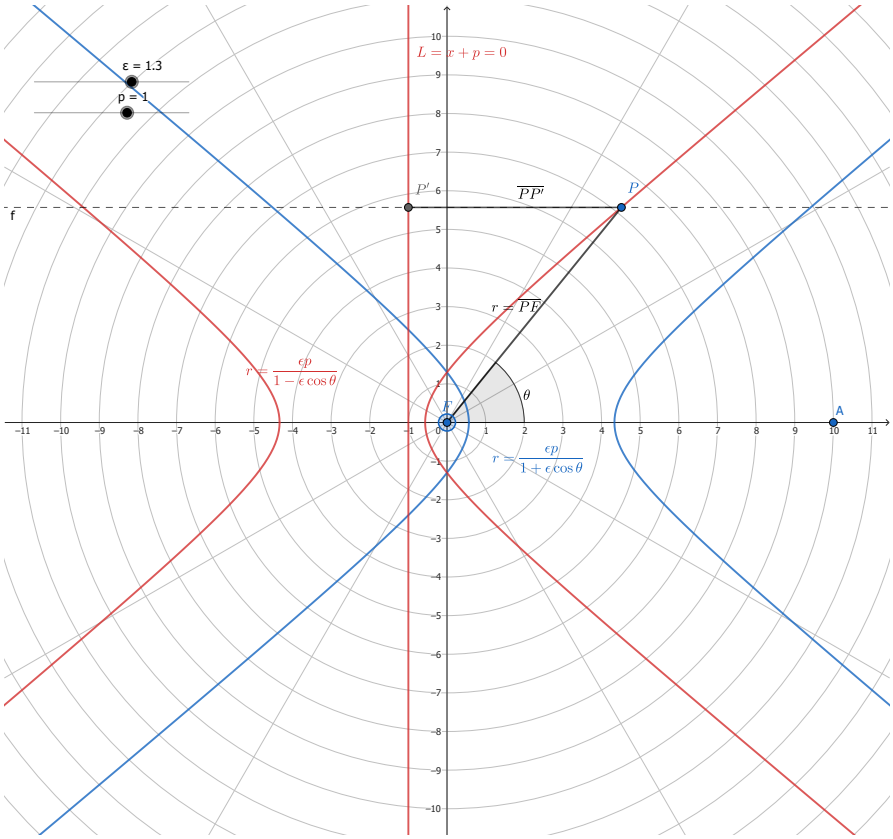


Figure 20.8: polar conic sections: hyperbola

20.6 Cartesian coordinate: general form / quadratic equation

https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections

20.7 homogeneous coordinate

Chapter 21

distance from a point to a line

點到直線距離

Theorem 21.1.

$$\begin{cases} P = P\left(x_0, y_0\right) \\ L = L\left(x, y\right) = Ax + By + C = 0, A^2 + B^2 \neq 0 \end{cases}$$
$$\Downarrow$$
$$d\left(P, L\right) = \frac{\left|Ax_0 + By_0 + C\right|}{\sqrt{A^2 + B^2}}$$

https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line

https://highscope.ch.ntu.edu.tw/wordpress/?p=47407

https://web.math.sinica.edu.tw/mathmedia/HTMLarticle18.jsp?mID=40312

Proofs:

21.1 by shortest $\overline{PP'}$

$$\begin{aligned} P' = P'\left(x, y\right) \in L = Ax + By + C = 0 \\ \Rightarrow y = \frac{-1}{B}\left(Ax + C\right) \end{aligned}$$

$$\begin{aligned} \overline{PP'}^2\left(x, y\right) &= \left(x_0 - x\right)^2 + \left(y_0 - y\right)^2 \\ &= \left(x_0 - x\right)^2 + \left(y_0 - \frac{-1}{B}\left(Ax + C\right)\right)^2 \\ &= \left(x - x_0\right)^2 + \left(\frac{A}{B}x + \frac{C}{B} + y_0\right)^2 = \overline{PP'}^2\left(x\right) \end{aligned}$$

$$\begin{aligned}
0 = \frac{\partial}{\partial x}\overline{PP'}^2(x) =& 2(x-x_0) + 2\left(\frac{A}{B}x + \frac{C}{B} + y_0\right)\frac{A}{B} \\
=& \frac{2}{B^2}\left(B^2(x-x_0) + A^2x + AC + ABy_0\right) \\
=& \frac{2}{B^2}\left[\left(A^2+B^2\right)x - \left(B^2x_0 - ABy_0 - AC\right)\right] \\
x =& \frac{B^2x_0 - ABy_0 - AC}{A^2+B^2}
\end{aligned}$$

or by completing the square to find x

$$\begin{aligned}
&\overline{PP'}^2\left(x = \frac{B^2x_0 - ABy_0 - AC}{A^2+B^2}\right) \\
=& \left(\frac{B^2x_0 - ABy_0 - AC}{A^2+B^2} - x_0\right)^2 + \left(\frac{A}{B}\frac{B^2x_0 - ABy_0 - AC}{A^2+B^2} + \frac{C}{B} + y_0\right)^2 \\
=& \left(\frac{-A^2x_0 - ABy_0 - AC}{A^2+B^2}\right)^2 + \left(\frac{A\left(B^2x_0 - ABy_0 - AC\right) + C\left(A^2+B^2\right) + B\left(A^2+B^2\right)y_0}{B\left(A^2+B^2\right)}\right)^2 \\
=& \left(\frac{-A\left(Ax_0 + By_0 + C\right)}{A^2+B^2}\right)^2 + \left(\frac{AB^2x_0 + B^3y_0 + B^2C}{B\left(A^2+B^2\right)}\right)^2 \\
=& \frac{A^2\left(Ax_0 + By_0 + C\right)^2}{\left(A^2+B^2\right)^2} + \frac{B^2\left(Ax_0 + By_0 + C\right)^2}{\left(A^2+B^2\right)^2} \\
=& \frac{\left(Ax_0 + By_0 + C\right)^2}{A^2+B^2} \\
\overline{PP'} =& \overline{PP'}\left(x = \frac{B^2x_0 - ABy_0 - AC}{A^2+B^2}\right) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}
\end{aligned}$$

21.2 by perpendicular foot

$$y = \frac{-A}{B}x - \frac{C}{B} = \frac{-1}{B}(Ax + C), \text{ if } B \neq 0$$

$$L_\perp : \left(y = \frac{B}{A}x + K\right) \perp \left(y = \frac{-A}{B}x - \frac{C}{B}\right) : L$$

$$L_\perp = L_\perp(x, y) = Bx - Ay + K = 0$$

$$P = P(x_0, y_0) \in L_\perp = B(x - x_0) - A(y - y_0) = 0$$

$$L_\perp = Bx - Ay - (Bx_0 - Ay_0) = 0$$

perpendicular foot = foot of the perpendicular P'

$$
\begin{aligned}
P' \in (L_{\perp} \cap L) &= \begin{cases} L = Ax + By + C = 0 \\ L_{\perp} = Bx - Ay - (Bx_0 - Ay_0) = 0 \end{cases} \\
&= \begin{cases} Ax + By = -C \\ Bx - Ay = Bx_0 - Ay_0 \end{cases} \\
P' = P'(x, y) &= \left(\frac{\begin{vmatrix} -C & B \\ Bx_0 - Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}}, \frac{\begin{vmatrix} A & -C \\ B & Bx_0 - Ay_0 \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}} \right) \\
&= \left(\frac{\begin{vmatrix} C & B \\ -Bx_0 + Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}}, \frac{\begin{vmatrix} A & C \\ B & -Bx_0 + Ay_0 \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}} \right) \\
&= \left(\frac{B^2 x_0 - ABy_0 - AC}{A^2 + B^2}, \frac{-ABx_0 + A^2 y_0 - BC}{A^2 + B^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
& d(P, L) = \overline{PP'} \\
=& \left\| (x_0, y_0) - \left(\frac{B^2 x_0 - ABy_0 - AC}{A^2 + B^2}, \frac{-ABx_0 + A^2 y_0 - BC}{A^2 + B^2} \right) \right\| \\
=& \sqrt{\left(x_0 - \frac{B^2 x_0 - ABy_0 - AC}{A^2 + B^2} \right)^2 + \left(y_0 - \frac{-ABx_0 + A^2 y_0 - BC}{A^2 + B^2} \right)^2} \\
=& \sqrt{\left(\frac{A^2 x_0 + ABy_0 + AC}{A^2 + B^2} \right)^2 + \left(\frac{ABx_0 + B^2 y_0 + BC}{A^2 + B^2} \right)^2} \\
=& \sqrt{\frac{A^2 (Ax_0 + By_0 + C)^2 + B^2 (Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2}} = \sqrt{\frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}} \\
=& \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}
$$

21.3 by normal vector

$$
\begin{cases} \vec{n} = (A, B) \perp L = Ax + By + C = 0 \\ \vec{PP'} = P' - P = (x - x_0, y - y_0) \end{cases}
$$

$$
\begin{aligned}
& \left|\vec{PP'}\cdot\vec{n}\right| \\
=& \left\|\vec{PP'}\right\| \left\|\vec{n}\right\| |\cos\theta| \\
\left\|\vec{PP'}\right\| |\cos\theta| =& \left|\vec{PP'}\cdot\hat{\boldsymbol{n}}\right| \\
=& \frac{\left|\vec{PP'}\cdot\vec{n}\right|}{\left\|\vec{n}\right\|} \\
=& \frac{|(x-x_0,y-y_0)\cdot(A,B)|}{\|(A,B)\|} = \frac{|A(x-x_0)+B(y-y_0)|}{\sqrt{A^2+B^2}} \\
=& \frac{|-Ax_0-By_0+Ax+By|}{\sqrt{A^2+B^2}} \overset{\substack{Ax+By+C=0 \\ Ax+By=-C}}{=} \frac{|-Ax_0-By_0-C|}{\sqrt{A^2+B^2}} \\
=& \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

PDF LaTeX `\usepackage{fdsymbol}` to have `\overrightharpoon` vector; however, there are too many side effects, including ugly mathptmx $\sum$, ...

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}
```

21.4 by Cauchy inequality

$$
\begin{aligned}
Ax+By+C =& 0 \\
Ax+By =& -C \\
(Ax+By)-(Ax_0+By_0) =& -C-(Ax_0+By_0) \\
A(x-x_0)+B(y-y_0) =& -(Ax_0+By_0+C)
\end{aligned}
$$

$$
\begin{aligned}
\overline{PP'}^2 =& (x_0-x)^2+(y_0-y)^2 \\
\left[A^2+B^2\right]\overline{PP'}^2 =& \left[A^2+B^2\right]\left[(x_0-x)^2+(y_0-y)^2\right] \\
\geq& \left[A(x-x_0)+B(y-y_0)\right]^2 \\
=& \left[-(Ax_0+By_0+C)\right]^2 = (Ax_0+By_0+C)^2 \\
\overline{PP'}^2 \geq& \frac{(Ax_0+By_0+C)^2}{A^2+B^2} \\
\overline{PP'} \geq& \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}
\end{aligned}
$$

Chapter 22

real symmetric matrix diagonalizable

https://ccjou.wordpress.com/2011/02/09/實對稱矩陣可正交對角化的證明/

https://tex.stackexchange.com/questions/30619/what-is-the-best-symbol-for-vector-matrix-transpose

Theorem 22.1.

實對稱矩陣的特徵值皆是實數，且對應特徵向量是實向量。

$$\begin{cases} \begin{cases} A \in \mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix} \\ A^{\intercal} = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\ A\boldsymbol{x} = \lambda\boldsymbol{x} & \begin{cases} \lambda \in \mathbb{C} & \text{complex eigenvalue} \\ \boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{C}^n & \text{complex eigenvector} \end{cases} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \lambda \in \mathbb{R} & \text{real eigenvalue (1)} \\ \boldsymbol{x} \in \mathbb{R}^n & \text{real eigenvector (2)} \end{cases}$$

Proof. (1)

$$\begin{aligned} A\boldsymbol{x} &= \lambda\boldsymbol{x} \\ \overline{A\boldsymbol{x}} = \overline{A}\overline{\boldsymbol{x}} &= \overline{\lambda\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}} \\ \overline{\boldsymbol{x}}^{\intercal}\overline{A}^{\intercal} = \left(\overline{A}\overline{\boldsymbol{x}}\right)^{\intercal} &= \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^{\intercal} = \overline{\lambda}\overline{\boldsymbol{x}}^{\intercal} \\ \overline{\boldsymbol{x}}^{\intercal}A \overset{\text{symmetric}}{=} \overline{\boldsymbol{x}}^{\intercal}A^{\intercal} &\overset{\text{real}}{=} \\ \overline{\boldsymbol{x}}^{\intercal}A &= \overline{\lambda}\overline{\boldsymbol{x}}^{\intercal} \\ \lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} = \overline{\boldsymbol{x}}^{\intercal}(\lambda\boldsymbol{x}) \underset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{\overset{\boldsymbol{x}}{=}} \overline{\boldsymbol{x}}^{\intercal}A\boldsymbol{x} &= \overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} \\ \lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} &= \overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} \\ \left(\lambda - \overline{\lambda}\right)\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} = 0 \wedge \begin{cases} \overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} = \sum_{i=1}^{n} |x_i|^2 \\ \boldsymbol{x} \neq \boldsymbol{0} \end{cases} &\Rightarrow \overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x} \neq 0 \\ \lambda - \overline{\lambda} = 0 & \\ \lambda = \overline{\lambda} \Leftrightarrow \lambda \in \mathbb{R} & \end{aligned}$$

□

Proof. (1) fast concept

$$\left(\overline{A}\overline{\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\left(\overline{\boldsymbol{x}}^{\intercal}\overline{A}^{\intercal}\right)\boldsymbol{x}\overset{\text{symmetric}}{=}\left(\overline{\boldsymbol{x}}^{\intercal}\overline{A}\right)\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)$$

$$(L)=\left(\overline{A}\overline{\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)=(R)$$

$$(L)=\left(\overline{A}\overline{\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}\overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=}\left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$

$$(R)=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)\overset{\text{real}}{=}\overline{\boldsymbol{x}}^{\intercal}\left(A\boldsymbol{x}\right)\overset{A\boldsymbol{x}=\lambda\boldsymbol{x}}{=}\overline{\boldsymbol{x}}^{\intercal}\left(\lambda\boldsymbol{x}\right)=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$

$$\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}=\left(\overline{A}\overline{\boldsymbol{x}}\right)^{\intercal}\boldsymbol{x}=\overline{\boldsymbol{x}}^{\intercal}\left(\overline{A}\boldsymbol{x}\right)=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$

$$\overline{\lambda}\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}=\lambda\overline{\boldsymbol{x}}^{\intercal}\boldsymbol{x}$$

□

Proof. (2)

???

推論特徵空間 $N(A-\lambda I)$ ($A-\lambda I$ 的零空間) 為 $\mathbb{R}^n$ 的子空間，故 $\boldsymbol{x}\in N(A-\lambda I)$ 是一個非零實向量。

□

Theorem 22.2.

實對稱矩陣對應相異特徵值的特徵向量互為正交。

$$\begin{cases}\begin{cases}A\in\mathcal{M}_{n\times n}(\mathbb{R}) & \text{real matrix}\\ A^{\intercal}=A & \text{symmetric matrix}\end{cases} & \text{real symmetric matrix}\\ A\boldsymbol{x}=\lambda\boldsymbol{x} & 22.1\begin{cases}\lambda\in\mathbb{R} & \text{real eigenvalue}\\ \boldsymbol{x}\in\mathbb{R}^n & \text{real eigenvector}\end{cases}\\ \begin{cases}A\boldsymbol{x}_1=\lambda_1\boldsymbol{x}_1 & (e_1)\\ A\boldsymbol{x}_2=\lambda_2\boldsymbol{x}_2 & (e_2)\end{cases} & \lambda_1\neq\lambda_2\end{cases}$$

$$\Downarrow$$

$$\boldsymbol{x}_1^{\intercal}\boldsymbol{x}_2=0\Leftrightarrow\boldsymbol{x}_1\perp\boldsymbol{x}_2$$

Proof. (1)

$$
\begin{aligned}
A\boldsymbol{x}_2 =& \lambda_2 \boldsymbol{x}_2 \\
\boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \overset{\boldsymbol{x}_1^\intercal \cdot}{=}& \boldsymbol{x}_1^\intercal \lambda_2 \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (1) \\
A\boldsymbol{x}_1 =& \lambda_1 \boldsymbol{x}_1 \\
\boldsymbol{x}_1^\intercal A^\intercal = (A\boldsymbol{x}_1)^\intercal =& (\lambda_1 \boldsymbol{x}_1)^\intercal = \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A^\intercal =& \lambda_1 \boldsymbol{x}_1^\intercal \\
\boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \overset{\text{symmetric}}{=} \boldsymbol{x}_1^\intercal A^\intercal \boldsymbol{x}_2 \overset{\cdot \boldsymbol{x}_2}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (2) \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \overset{(1)}{=} \boldsymbol{x}_1^\intercal A \boldsymbol{x}_2 \overset{(2)}{=}& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(\lambda_2 - \lambda_1)\, \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0 \wedge \lambda_1 \neq \lambda_2 \\
\boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& 0
\end{aligned}
$$

□

Proof. (1) fast concept

$$
\begin{aligned}
(A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = (\boldsymbol{x}_1^\intercal A^\intercal)\, \boldsymbol{x}_2 \overset{\text{symmetric}}{=}& (\boldsymbol{x}_1^\intercal A)\, \boldsymbol{x}_2 = \boldsymbol{x}_1^\intercal\, (A\boldsymbol{x}_2) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 =& \boldsymbol{x}_1^\intercal\, (A\boldsymbol{x}_2) = (R) \\
(L) = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 \overset{(e_1)}{=}& (\lambda_1 \boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 = \lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
(R) = \boldsymbol{x}_1^\intercal\, (A\boldsymbol{x}_2) \overset{(e_2)}{=}& \boldsymbol{x}_1^\intercal\, (\lambda_2 \boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 = (A\boldsymbol{x}_1)^\intercal \boldsymbol{x}_2 =& \boldsymbol{x}_1^\intercal\, (A\boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 \\
\lambda_1 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2 =& \lambda_2 \boldsymbol{x}_1^\intercal \boldsymbol{x}_2
\end{aligned}
$$

□

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