inspriation of operator theory

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June 2, 2024

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editing since 2024/06/02

Contents

1 algebra

$2+2^2+2^3+\cdots+2^n$

$$1+2+2^2+2^3+\cdots+2^n$$

$$S=2^0+2^1+2^2+2^3+\cdots+2^n=\sum_{\nu=0}^{n}2^\nu$$

$$
\begin{aligned}
S =& 2^0+2^1+2^2+2^3+\cdots+2^n && =\sum_{\nu=0}^{n}2^\nu \\
2S =& \quad 2\cdot 2^0+2\cdot 2^1+2\cdot 2^2+2\cdot 2^3+\cdots+2\cdot 2^n && =2\sum_{\nu=0}^{n}2^\nu \\
=& \quad 2^1+2^2+2^3+\cdots+2^n+2^{n+1} && =\sum_{\nu=0}^{n}2^{\nu+1} \\
2S =& \quad 2^1+2^2+2^3+\cdots+2^n+2^{n+1} && =2\sum_{\nu=0}^{n}2^\nu \\
S =& 2^0+2^1+2^2+2^3+\cdots+2^n && =\sum_{\nu=0}^{n}2^\nu \\
2S-S =& \left(0-2^0\right)+\left(2^1-2^1\right)+\left(2^2-2^2\right)+\left(2^3-2^3\right)+\cdots+\left(2^n-2^n\right)+\left(2^{n+1}-0\right) && =2\sum_{\nu=0}^{n}2^\nu-\sum_{\nu=0}^{n}2^\nu \\
=& \left(-2^0\right)+(0)+(0)+(0)+\cdots+(0)+\left(2^{n+1}\right) \\
=& \left(-2^0\right)+\left(2^{n+1}\right) \\
(2-1)\,S =& 2^{n+1}-2^0 && =(2-1)\sum_{\nu=0}^{n}2^\nu \\
S =& 2^{n+1}-1=\sum_{\nu=0}^{n}2^\nu && \forall n\in\mathbb{N}
\end{aligned}
$$

$a + ax + ax^2 + ax^3 + \cdots + ax^n$

$$
\begin{aligned}
\sum_{\nu=0}^{n} ax^{\nu} =& a + ax^1 + ax^2 + ax^3 + \cdots + ax^n \\
x\sum_{\nu=0}^{n} ax^{\nu} =& \qquad x\cdot ax^0 + x\cdot ax^1 + x\cdot ax^2 + x\cdot ax^3 + \cdots + x\cdot ax^n \\
\sum_{\nu=0}^{n} ax^{\nu+1} =& \qquad ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1} \\
x\sum_{\nu=0}^{n} ax^{\nu} =& \qquad ax^1 + ax^2 + ax^3 + \cdots + ax^n + ax^{n+1} \\
\sum_{\nu=0}^{n} ax^{\nu} =& ax^0 + ax^1 + ax^2 + ax^3 + \cdots + ax^n \\
x\sum_{\nu=0}^{n} ax^{\nu} - \sum_{\nu=0}^{n} ax^{\nu} =& \left(0 - ax^0\right) + \left(ax^1 - ax^1\right) + \left(ax^2 - ax^2\right) + \left(ax^3 - ax^3\right) + \cdots + \left(ax^n - ax^n\right) + \left(ax^{n+1} - 0\right) \\
=& \left(0 - ax^0\right) + (0) + (0) + (0) + \cdots + (0) + \left(ax^{n+1}\right) \\
=& \left(-ax^0\right) + \left(ax^{n+1}\right) \\
(x-1)\sum_{\nu=0}^{n} ax^{\nu} =& ax^{n+1} - ax^0 \\
a\sum_{\nu=0}^{n} x^{\nu} = \sum_{\nu=0}^{n} ax^{\nu} =& \frac{a\left(x^{n+1}-1\right)}{x-1} = \frac{(-1)\,a\left(x^{n+1}-1\right)}{(-1)\,(x-1)} = \frac{a\left(1-x^{n+1}\right)}{1-x} = \frac{a\left(1-x^{n+1}\right)}{1-x} \qquad \forall n \in \mathbb{N} \\
a\sum_{\nu=0}^{\infty} x^{\nu} = \sum_{\nu=0}^{\infty} ax^{\nu} =& \lim_{n\to\infty} \frac{a\left(1-x^{n+1}\right)}{1-x} = \frac{ax\,(1-0)}{1-x} = \frac{a}{1-x} = (1-x)^{-1}\,a \qquad n \to \infty, \forall\,|x| < 1 \\
(1-x)^{-1}\,a =& x^0a + x^1a + x^2a + x^3a + \cdots \qquad \forall\,|x| < 1
\end{aligned}
$$

2 linear algebra

$A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots$

$$
\begin{aligned}
(1-x)^{-1}\,a =& x^0a + x^1a + x^2a + x^3a + \cdots && \forall\,|x| < 1 \\
x^0a + x^1a + x^2a + x^3a + \cdots =& (1-x)^{-1}\,a && \forall\,|x| < 1 \\
A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots =& (I-A)^{-1}\,\boldsymbol{x} && \forall\,||A|| = |\det A| < 1
\end{aligned}
$$

$$
\begin{cases}
A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad |\det A| = \dfrac{1}{2} < 1 \\
\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{cases}
$$

$$
\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x & -x \\ y & y \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$(1-x)^{-1}\,a = x^0a + x^1a + x^2a + x^3a + \cdots \qquad \forall\,|x|<1$$

$$x^0a + x^1a + x^2a + x^3a + \cdots = (1-x)^{-1}\,a \qquad \forall\,|x|<1$$

$$A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots = (I-A)^{-1}\,\boldsymbol{x} \qquad \forall\,|\det A|<1$$

$$\Downarrow$$

$$\begin{aligned}
\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^0\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^1\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix} + \cdots &= \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\
&= \begin{bmatrix}1\\1\end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^0\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^1\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix} + \cdots = \begin{bmatrix}1\\1\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$A^0\boldsymbol{x} + A^1\boldsymbol{x} + A^2\boldsymbol{x} + A^3\boldsymbol{x} + \cdots = (I-A)^{-1}\,\boldsymbol{x}$$

$$\boldsymbol{x} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\left(A^0 + A^1 + A^2 + A^3 + \cdots\right)\boldsymbol{x} =$$

$$\left(\sum_{\nu=0}^{\infty} A^{\nu}\right)\boldsymbol{x} =$$

3 calculus

$\exp(x) = \mathrm{e}^x$

$$
\begin{aligned}
\mathrm{e}^x &\overset{\text{def.}}{=} \mathrm{D}\mathrm{e}^x = \mathrm{D}_x\mathrm{e}^x = \mathrm{d}_x\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} && \forall x \in \mathbb{R}\\
&\Downarrow F(x) = \int_a^x \mathrm{D}F(t) && \text{FToC1}\\
\mathrm{e}^x &= \int \mathrm{D}\mathrm{e}^x && = \text{fundamental theorem of calculus first part}\\
&\Downarrow \mathrm{e}^x \overset{\text{def.}}{=} \mathrm{D}\mathrm{e}^x\\
\mathrm{e}^x &= \int \mathrm{e}^x\\
\mathrm{e}^x - \int \mathrm{e}^x &= 0\\
\left(1 - \int\right)\mathrm{e}^x &= 0\\
\mathrm{e}^x &= \left(1 - \int\right)^{-1} 0 = \left(\sum_{\nu=0}^{\infty}\int^{\nu}\right) 0 && \forall\,|\det A| < 1 \left[(I-A)^{-1}\boldsymbol{x}\right.\\
&= \left(\int^0 + \int^1 + \int^2 + \int^3 + \cdots\right) 0 && \left.= \left(\sum_{\nu=0}^{\infty} A^{\nu}\right)\boldsymbol{x}\right]\\
&= \int^0 0 + \int^1 0 + \int^2 0 + \int^3 0 + \cdots\\
&= 0 + \int \mathrm{d}x 0 + \iint \mathrm{d}x^2 0 + \iiint \mathrm{d}x^3 0 + \cdots\\
&= \quad + C + \int \mathrm{d}x C + \iint \mathrm{d}x^2 C + \cdots\\
&= \quad + C + Cx + \int \mathrm{d}x Cx + \cdots\\
&= \quad + C + \frac{Cx}{1} + \frac{Cx^2}{2\cdot 1} + \cdots && = C\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}\\
1 = \mathrm{e}^0 = \mathrm{e}^{x=0} &= C + \frac{C0}{1} + \frac{C0^2}{2\cdot 1} + \cdots = C + 0 + 0 + \cdots = C && \Rightarrow C = 1\\
\mathrm{e}^x &= 1 + \frac{1x}{1} + \frac{1x^2}{2\cdot 1} + \cdots && = 1\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}\\
&= \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots && = \sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!}
\end{aligned}
$$

4 complex analysis

$$\mathrm{e}^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu!} \quad \forall x \in \mathbb{C}, |x| < 1$$

5 matrix calculus

$$\mathrm{e}^{Ax} = \frac{1}{0!} + \frac{Ax}{1!} + \frac{A^2x^2}{2!} + \cdots = \sum_{\nu=0}^{\infty}\frac{A^{\nu}x^{\nu}}{\nu!} \quad \forall Ax \in \mathcal{M}_{n\times n}, ||Ax|| = |(\det A)\,x| < 1$$

6 Lie algebra

$$\mathrm{Ad}_{\exp X}Y = \exp(\mathrm{Ad}X)\,Y = \frac{Y}{0!} + \frac{[X,Y]}{1!} + \frac{[X,[X,Y]]}{2!} + \frac{[X,[X,[X,Y]]]}{3!}\cdots$$

「Mathematics is the art of giving the same name to different things.」

Henri Poincré (1854~1912)

「Algebra's like sheet music, the important thing isn't can you read music, it's can you hear it. Can you hear the music, Robert?」

Niels Bohr (1885˜1962) $\sim$ Oppenheimer movie (2023)