

math

Joey Yu Hsu

2024-06-26

Contents

| | |
|--|-----------|
| index | 17 |
| I by discipline | 19 |
| 1 mathematics | 21 |
| 1.1 tool | 21 |
| 1.2 discipline | 21 |
| 2 physics | 23 |
| 2.1 discipline | 23 |
| 3 plot | 25 |
| 4 programming language | 27 |
| 4.1 discipline | 27 |
| 4.2 learning map | 27 |
| 5 machine learning | 29 |
| 5.1 Shai Ben-David | 29 |
| 5.2 deep learning | 29 |
| II by date | 37 |
| 6 A Minimal Book Example | 39 |
| 6.1 About | 39 |
| 6.1.1 Usage | 39 |
| 6.1.2 Render book | 39 |
| 6.1.3 Preview book | 39 |
| 6.2 Hello bookdown | 39 |
| 6.2.1 A section | 39 |
| An unnumbered section | 40 |
| 6.3 Cross-references | 40 |
| 6.3.1 Chapters and sub-chapters | 40 |
| 6.3.2 Captioned figures and tables | 40 |
| 6.4 Parts | 41 |
| 6.5 Footnotes and citations | 41 |
| 6.5.1 Footnotes | 41 |
| 6.5.2 Citations | 41 |
| 6.6 Blocks | 41 |
| 6.6.1 Equations | 41 |
| 6.6.2 Theorems and proofs | 42 |
| 6.6.3 Callout blocks | 42 |
| 6.7 Sharing your book | 42 |
| 6.7.1 Publishing | 42 |
| 6.7.2 404 pages | 42 |
| 6.7.3 Metadata for sharing | 42 |
| 7 test | 43 |
| 7.1 RStudio | 43 |
| 7.1.1 RMarkdown preview LaTeX equation options | 43 |

| | | |
|-----------|---|----|
| 7.1.2 | writer options | 43 |
| 7.1.2.1 | line wrapping | 43 |
| 7.1.2.2 | ensuring the same markdown between source / visual mode | 43 |
| 7.1.3 | Rtools | 43 |
| 7.1.4 | addins | 43 |
| 7.1.5 | Git | 44 |
| 7.2 | RMarkdown | 44 |
| 7.2.1 | verbatim | 44 |
| 7.2.2 | Pandoc link | 44 |
| 7.2.3 | URL | 44 |
| 7.2.4 | arrow | 44 |
| 7.2.5 | superscript and subscript | 45 |
| 7.2.5.1 | LaTeX | 45 |
| 7.2.5.2 | equation | 45 |
| 7.2.5.3 | aligned | 45 |
| 7.2.5.4 | proof QED | 45 |
| 7.2.6 | image | 45 |
| 7.2.6.1 | DiagrammeR / mermaid flowchart | 45 |
| 7.2.6.2 | multiple images / figures in the same line | 46 |
| 7.2.6.3 | figure size | 47 |
| 7.2.6.3.1 | knitr | 49 |
| 7.2.6.3.2 | out.width vs. fig.width | 50 |
| 7.2.6.4 | dynamic knitr plot width and height | 52 |
| 7.2.6.5 | web image in PDF | 52 |
| 7.2.6.6 | SVG | 52 |
| 7.2.7 | horizontal rule | 54 |
| 7.2.8 | footnote | 54 |
| 7.2.9 | hyperlink | 54 |
| 7.2.10 | code chunk | 55 |
| 7.2.10.1 | code folding | 55 |
| 7.2.11 | xaringan | 55 |
| 7.2.12 | embed HTML or web page | 55 |
| 7.2.12.1 | iframe | 55 |
| 7.2.12.2 | include URL | 55 |
| 7.2.12.3 | htmltools | 57 |
| 7.2.13 | embed PDF | 57 |
| 7.3 | Bookdown | 57 |
| 7.3.1 | system locale | 57 |
| 7.3.2 | render_book() | 57 |
| 7.3.3 | serve_book() | 57 |
| 7.3.4 | LaTeX | 57 |
| 7.3.4.1 | hyperlink, URL, href | 57 |
| 7.3.4.2 | ugly mathptmx \sum | 57 |
| 7.3.4.3 | LaTeX package in HTML document | 58 |
| 7.3.5 | depth of table of contents toc_depth | 58 |
| 7.3.6 | multi-column layout / two columns | 58 |
| 7.3.6.1 | for both HTML and PDF | 58 |
| 7.3.6.2 | multi-column fig.cap must use fig.pos="H" | 60 |
| 7.3.6.3 | caption above figure | 60 |
| 7.3.6.4 | for only HTML | 60 |
| 7.3.6.4.1 | CSS flex | 60 |
| 7.3.6.4.2 | CSS grid | 61 |
| 7.4 | conditional block/chunk for either HTML or PDF, and Chinese issue | 62 |
| 7.5 | video embedding | 62 |
| 7.5.1 | timestamp | 63 |
| 7.6 | equation term coloring | 63 |
| 7.6.1 | font color | 63 |
| 7.6.2 | background color | 63 |
| 7.7 | link and reference | 64 |
| 7.8 | number and reference equations | 65 |
| 7.9 | footnote | 65 |
| 7.10 | citation | 65 |
| 7.10.1 | citation in fig.cap | 66 |

| | | |
|------------|--|-----|
| 7.10.2 | backreference | 66 |
| 7.11 | environment for definition, theorem, proof | 66 |
| 7.11.1 | collapsible panel or HTML component | 67 |
| 7.11.1.1 | nested collapsible panel or HTML component | 67 |
| 7.12 | slide or presentation | 67 |
| 7.12.1 | Xaringan and Infinite Moon Reader | 67 |
| 7.12.2 | ioslides | 67 |
| 7.12.3 | PowerPoint | 69 |
| 8 | test2 | 71 |
| 9 | partition | 73 |
| 10 | equivalence class | 75 |
| 11 | equivalence relation | 77 |
| 12 | Python | 79 |
| 12.1 | using Python in R / RMarkdown | 79 |
| 12.2 | SoloLearn | 80 |
| 12.3 | list comprehension | 80 |
| 12.4 | functional programming | 81 |
| 12.5 | object-oriented programming = OOP | 81 |
| 13 | TikZ | 83 |
| 13.1 | 2D | 84 |
| 13.2 | 3D | 91 |
| 13.3 | plots of functions | 93 |
| 13.4 | commutative diagram | 94 |
| 13.4.1 | arrows | 100 |
| 13.4.1.1 | predefined arrow tip in TikZ | 100 |
| 13.4.1.1.1 | stealth arrow tip | 100 |
| 13.4.1.1.2 | reversed stealth arrow tip | 100 |
| 13.4.1.1.3 | to arrow tip | 100 |
| 13.4.1.1.4 | latex arrow tip | 100 |
| 13.4.1.1.5 | arrow tip | 100 |
| 13.4.1.2 | arrows.meta | 101 |
| 13.4.1.2.1 | reversing, halving, swapping, opening | 101 |
| 13.4.1.2.2 | mathematical arrows | 101 |
| 13.4.1.2.3 | ray arrow tip | 101 |
| 13.4.1.2.4 | arrow head scale | 101 |
| 13.4.1.2.5 | arrow head size | 102 |
| 13.4.1.2.6 | arrow head slant | 102 |
| 13.4.1.2.7 | arrow head color | 102 |
| 13.4.1.2.8 | multiple arrow head | 102 |
| 13.4.1.3 | default stealth arrow | 103 |
| 13.4.1.4 | harpoon | 103 |
| 13.4.1.5 | bend | 103 |
| 13.4.1.6 | rounded corners | 104 |
| 13.4.1.7 | crossing over | 104 |
| 13.4.1.8 | tweaking to paths | 104 |
| 13.4.2 | label | 105 |
| 13.4.2.1 | single quotation mark to swap | 105 |
| 13.4.2.2 | swap | 106 |
| 13.4.2.3 | every arrow swap | 106 |
| 13.4.3 | box in TikZ-CD | 106 |
| 13.4.4 | examples | 106 |
| 13.4.5 | put a commutative diagram in a nodepart in a tikzpicture | 107 |
| 13.5 | PGFplots | 108 |
| 13.5.1 | the axis environments | 108 |
| 13.5.2 | axis descriptions | 109 |
| 13.5.2.1 | placement of axis descriptions | 109 |
| 13.5.2.1.1 | coordinate system <code>axis description cs</code> | 109 |
| 13.5.2.1.2 | legend | 110 |

| | | |
|------------|---|------------|
| 13.5.2.1.3 | tick option | 110 |
| 13.5.3 | declare function | 110 |
| 13.5.3.1 | pgfmathparse | 110 |
| 13.5.3.2 | pgfmathdeclarefunction | 111 |
| 13.5.4 | - and - in TikZ | 112 |
| 13.5.5 | pgfplotsinvokeforeach | 112 |
| 13.5.6 | interpolation dashed lines | 113 |
| 13.5.7 | Zewbie | 113 |
| 13.5.7.1 | coordinate axis/axes fine-tuing | 114 |
| 13.5.7.1.1 | range | 114 |
| 13.5.7.1.2 | scaling | 114 |
| 13.5.7.1.3 | direction vector | 115 |
| 13.5.7.1.4 | axis style | 116 |
| 13.5.7.1.5 | axis discontinuity | 117 |
| 13.5.7.1.6 | tick | 117 |
| 13.5.7.2 | addplot+ | 119 |
| 13.5.7.3 | point | 120 |
| 13.5.7.4 | line | 122 |
| 13.5.7.5 | plane | 124 |
| 13.5.7.6 | polar coordinate | 125 |
| 13.5.8 | Arnav Bandekar: Using pgfplots to make economic graphs in LaTeX | 125 |
| 13.5.9 | PGFplots gallery | 125 |
| 13.6 | TikZplotLib / tikzplotlib | 126 |
| 13.7 | animation | 128 |
| 14 | xy-pic | 129 |
| 14.1 | pure LaTeX equation environment not rendering xy-pic in HTML | 129 |
| 14.2 | R figure knit with TikZ engine | 129 |
| 15 | statistics | 133 |
| 15.1 | Hung Hung | 133 |
| 15.1.1 | probability theory | 135 |
| 15.1.1.1 | probability function | 135 |
| 15.1.1.2 | conditional probability | 137 |
| 15.1.2 | univariable distribution | 137 |
| 15.1.2.1 | cumulative distribution function | 138 |
| 15.1.2.2 | probability density function | 138 |
| 15.1.2.3 | range vs. support | 139 |
| 15.1.2.4 | continuous monotone transformation | 140 |
| 15.1.2.5 | discrete monotone transformation | 143 |
| 15.1.2.6 | expected value | 144 |
| 15.1.2.7 | moment | 151 |
| 15.1.2.7.1 | moment generating function | 154 |
| 15.1.2.7.2 | characteristic function | 155 |
| 15.1.2.8 | common families of distributions | 156 |
| 15.1.2.8.1 | discrete distribution | 156 |
| 15.1.2.8.2 | continuous distribution | 161 |
| 15.1.2.9 | exponential family | 162 |
| 15.1.2.9.1 | binomial distribution with known n | 163 |
| 15.1.2.9.2 | continuous uniform distribution not in exponential family | 163 |
| 15.1.2.9.3 | normal distribution is in exponential family | 164 |
| 15.1.2.9.4 | normal distribution with unknown equal mean and standard deviation in curved exponential family | 164 |
| 15.1.2.9.5 | properties of exponential family | 165 |
| 15.1.3 | multivariable distribution | 171 |
| 15.2 | Chen, Lin-An | 176 |
| 16 | covariance matrix | 177 |
| 16.1 | vector direct product | 177 |
| 16.1.1 | vector direct product: increasing rank | 177 |
| 16.1.2 | vector inner product: decreasing rank | 177 |
| 16.1.3 | tensor direct product: increasing rank | 178 |
| 16.2 | covariance matrix and its properties | 178 |

| | |
|--|------------|
| 16.2.1 $V[\mathbf{X} + \mathbf{b}] = V[\mathbf{X}]$ | 178 |
| 16.2.2 $V[\mathbf{AX}] = AV[\mathbf{X}] A^\top$ | 178 |
| 16.2.3 $V[\mathbf{AX} + \mathbf{b}] = AV[\mathbf{X}] A^\top$ | 178 |
| 17 Gosper algorithm | 179 |
| 17.1 Zhuli | 179 |
| 17.2 TaylorCatAlice | 179 |
| 18 Lorentz transformation | 181 |
| 18.1 Einstein | 181 |
| 18.2 Bondi k -calculus | 181 |
| 18.3 wordline in Minkowski space | 181 |
| 18.3.1 Wick rotation | 181 |
| 18.3.1.1 Osterwalder-Schrader reconstruction theorem | 181 |
| 19 R | 183 |
| 19.1 TonyKuoYJ | 183 |
| 19.1.1 quick intro | 183 |
| 19.1.1.1 function | 184 |
| 19.1.2 R style | 184 |
| 19.1.3 data workflow or forward pipe | 184 |
| 19.1.3.1 <code>%>%</code> operator | 184 |
| 19.1.4 data processing with <code>dplyr</code> | 185 |
| 19.1.5 visualization statically with <code>ggplot2</code> | 187 |
| 19.1.6 loop | 192 |
| 19.1.7 variable type | 192 |
| 19.1.7.1 date | 193 |
| 19.1.7.2 time | 193 |
| 19.1.7.3 quotient <code>%%</code> operator | 193 |
| 19.1.8 data type | 194 |
| 19.1.8.1 vector | 194 |
| 19.1.8.1.1 logic | 195 |
| 19.1.8.1.2 <code>rep</code> repeat | 195 |
| 19.1.8.1.3 <code>seq</code> sequence | 195 |
| 19.1.8.2 factor | 195 |
| 19.1.8.3 matrix | 195 |
| 19.1.8.4 data frame | 196 |
| 19.1.8.4.1 selecting variable or column | 197 |
| 19.1.8.4.2 filtering observation or row | 197 |
| 19.1.8.4.3 check mixed data type | 197 |
| 19.2 W3School | 198 |
| 19.2.1 same multiple variable | 198 |
| 19.2.2 legal variable name | 198 |
| 19.2.3 complex number | 198 |
| 19.2.4 escape character | 198 |
| 19.2.5 global assignment <code><-></code> | 198 |
| 19.2.6 data type | 199 |
| 19.2.6.1 array | 199 |
| 19.2.6.2 list | 199 |
| 19.3 Apan Liao | 199 |
| 19.3.1 data input | 200 |
| 19.3.2 descriptive statistics | 200 |
| 20 Laplace transform | 201 |
| 21 conic section | 203 |
| 21.1 Cartesian coordinate: focus, directrix, eccentricity | 203 |
| 21.2 two-definition equivalence for ellipse and hyperbola | 205 |
| 21.2.1 first definition for conic sections including ellipses and hyperbolas | 205 |
| 21.2.2 second definition for ellipses and hyperbolas | 206 |
| 21.2.3 eccentricity and its equivalent representation | 209 |
| 21.3 Cartesian coordinate: standard form / standard equation | 209 |
| 21.4 parametric equation | 211 |
| 21.5 polar coordinate | 211 |

| | | |
|------------|---|------------|
| 21.6 | Cartesian coordinate: general form / quadratic equation | 211 |
| 21.7 | homogeneous coordinate | 215 |
| 21.8 | TalyorCatAlice: projective geometry | 216 |
| 21.9 | double cone | 216 |
| 21.10 | theorem proof without analytic method | 216 |
| 22 | distance from a point to a line | 217 |
| 22.1 | by shortest $\overline{PP'}$ | 217 |
| 22.2 | by perpendicular foot | 218 |
| 22.3 | by normal vector | 219 |
| 22.4 | by Cauchy inequality | 219 |
| 23 | real symmetric matrix diagonalizable | 221 |
| 24 | tangent half-angle formula | 225 |
| 25 | homogeneous coordinate | 227 |
| 25.1 | Cem Yuksel | 227 |
| 26 | Archimedean property | 229 |
| 26.1 | integer Archimedean property | 229 |
| 26.2 | rational Archimedean property | 229 |
| 26.3 | real Archimedean property | 229 |
| 27 | MatPlotLib / matplotlib | 231 |
| 27.1 | Timothy H. Wu | 231 |
| 27.1.1 | funcitonal API | 233 |
| 27.1.2 | object-oriented API | 235 |
| 27.1.2.1 | Configure the figure size and DPI | 236 |
| 27.1.2.2 | subplot | 237 |
| 27.1.2.3 | color and linestyle | 238 |
| 27.1.2.4 | other inputs | 238 |
| 27.1.2.5 | legend | 238 |
| 27.1.2.6 | customize style | 239 |
| 27.1.2.7 | save to file | 239 |
| 27.1.3 | Seaborn | 239 |
| 27.1.3.1 | basic | 241 |
| 27.1.3.2 | data frame | 241 |
| 27.1.3.3 | axis-level plot and figure-level plot | 243 |
| 27.1.3.4 | accessing figure and axes objects | 245 |
| 27.1.3.5 | distribution plot | 246 |
| 27.1.3.5.1 | histogram | 246 |
| 27.1.3.5.2 | ECDF = empirical cumulative distrutive function | 247 |
| 27.1.3.5.3 | KDE = kernel density estimation | 247 |
| 27.1.3.6 | categorical plot | 249 |
| 27.1.3.7 | joint plot | 250 |
| 27.1.3.8 | pair plot | 250 |
| 27.1.3.9 | palette | 251 |
| 27.1.3.10 | volcano plot | 251 |
| 27.1.3.11 | heatmap | 251 |
| 27.2 | export | 251 |
| 27.2.1 | .svg | 251 |
| 27.2.2 | .eps | 252 |
| 28 | survial analysis | 253 |
| 28.1 | Python package <code>tableone</code> | 253 |
| 28.2 | Python package <code>lifelines</code> | 253 |
| 29 | Manim | 255 |
| 29.1 | VSCode extension: Manim Sideview | 255 |
| 29.2 | installation | 255 |
| 29.2.1 | Conda | 255 |
| 29.3 | quickstart | 255 |

| | |
|---|------------|
| 30 ggplot2 | 257 |
| 30.1 geom | 257 |
| 30.1.1 basic plot system decomposition | 261 |
| 30.2 statistic | 263 |
| 30.3 scale | 263 |
| 30.4 coordinate system | 264 |
| 30.5 facet | 265 |
| 30.6 jitter | 266 |
| 30.7 font | 266 |
| 31 notational system for design | 267 |
| 31.1 graphic notation | 267 |
| 31.2 pre-experimental design | 267 |
| 31.2.1 one-shot case study | 267 |
| 31.2.2 one-group pretest-posttest design | 267 |
| 31.2.3 static-group comparison | 268 |
| 31.3 true experimental design | 268 |
| 31.3.1 posttest-only control group design | 268 |
| 31.3.2 pretest-posttest control group design | 271 |
| 31.3.3 Solomon four-group design | 271 |
| 31.4 quasi-experimental design | 272 |
| 31.5 correlational and ex post facto designs | 272 |
| 31.6 graphic notation, advanced | 272 |
| 32 design of experiment | 273 |
| 32.1 notational system for design ^[31] | 273 |
| 32.2 terminology | 273 |
| 32.2.1 replication vs. subsample | 274 |
| 32.2.2 replication vs. repeated measurements | 274 |
| 32.2.3 replication, replicate | 274 |
| 32.2.3.1 technical replicate, biological replicate | 274 |
| 32.2.4 Latin square design | 275 |
| 32.2.5 model assumption and experimental unit, measurement/observational unit | 275 |
| 32.3 experiment structure | 275 |
| 32.3.1 treatment structure | 275 |
| 32.3.2 design structure | 277 |
| 32.3.3 size of experimental unit | 277 |
| 32.4 approach to experimentation | 277 |
| 32.5 sample size estimation | 277 |
| 32.6 statistical analysis plan | 277 |
| 32.7 protocol | 277 |
| 32.8 DoE course with six sigma and Minitab | 278 |
| 32.8.1 evolution | 278 |
| 33 quine | 279 |
| 33.1 %r | 279 |
| 34 quaternion | 281 |
| 34.1 TaylorCatAlice | 281 |
| 34.1.1 complex / dionion / bionion | 281 |
| 34.1.2 trionion / triernion / triplex / ternion | 281 |
| 34.1.3 quaternion | 282 |
| 34.1.3.1 true origin of (dot product & cross product) / (inner product & outer product) | 284 |
| 34.1.3.2 commutativity vs. anticommutativity | 285 |
| 34.1.3.2.1 commutativity | 285 |
| 34.1.3.2.2 anticommutativity | 285 |
| 34.1.3.3 bracket | 285 |
| 34.1.3.3.1 self-invented bracket | 285 |
| 34.1.3.3.2 Poisson bracket | 286 |
| 34.1.3.3.3 Lagrange bracket | 286 |
| 34.1.3.3.4 Lie bracket | 286 |
| 34.1.3.4 triple product | 286 |
| 34.1.3.5 differential operator | 288 |
| 34.1.3.5.1 4-differential operator | 288 |

| | |
|--|------------|
| 34.1.3.5.2 nabla | 288 |
| 34.1.3.5.3 Laplace operator | 289 |
| 34.1.3.5.4 d'Alembert operator | 289 |
| 34.1.3.6 electromagnetism | 289 |
| 34.1.3.6.1 4-potential | 289 |
| 34.1.3.6.2 Maxwell compromise for both quaternion and 3-vector | 290 |
| 34.1.3.7 Joule heat vs. Thomson heat (Kelvin heat?) | 291 |
| 34.1.3.7.1 thermoelectric effect | 291 |
| 34.1.3.8 source term | 291 |
| 34.1.4 quaternion group | 292 |
| 34.1.4.1 2D rotation | 292 |
| 34.1.4.1.1 matrix | 292 |
| 34.1.4.1.2 complex | 293 |
| 34.1.4.1.3 $SO(2) \cong U(1)$ | 294 |
| 34.1.4.2 3D rotation | 294 |
| 34.1.4.2.1 matrix | 294 |
| 34.1.4.2.2 quaternion | 297 |
| 34.1.5 octonion | 299 |
| 34.2 Krasjet | 299 |
| 34.2.1 Rodrigues rotation | 299 |
| 34.2.2 quaternion operation | 301 |
| 34.2.3 matrix form | 307 |
| 34.2.4 exponential form | 308 |
| 34.2.5 double cover | 310 |
| 34.2.6 right-hand vs. left-hand coordinates | 310 |
| 34.2.6.1 clock | 310 |
| 34.2.6.2 Rodrigues rotation in right-hand vs. left-hand coordinates | 311 |
| 34.2.7 gimbal lock | 311 |
| 34.2.8 Lie group | 311 |
| 34.2.8.1 Mathemaniac | 311 |
| 34.2.8.2 Liang, Can-Bin | 312 |
| 34.2.8.3 special unitary group | 312 |
| 34.2.9 geometric algebra and Clifford algebra | 312 |
| 34.2.10 dual quaternion | 312 |
| 34.2.11 interpolation | 312 |
| 34.2.11.1 Lerp | 312 |
| 34.2.11.2 Nlerp | 312 |
| 34.2.11.3 Slerp | 312 |
| 34.3 visualization | 312 |
| 34.4 3Blue1Brown | 312 |
| 34.4.1 Ben Eater | 312 |
| 34.4.2 3B1B | 313 |
| 34.4.3 Sutrabla | 313 |
| 34.5 CCJou: LA Revelation | 313 |
| 35 DICOM | 315 |
| 35.1 Innolitics: DICOM Standard Browser | 315 |
| 35.2 David Clunie: Medical Image Format Site | 315 |
| 35.3 Hsiao, Chia-Hung | 315 |
| 36 tendon pathophysiology | 317 |
| 36.1 histology | 317 |
| 37 group theory | 319 |
| 37.1 matrix group | 319 |
| 37.2 group definition and basic theorem | 319 |
| 37.3 EpicOrganism = AIRoswell = Pan, Yi-Wen | 320 |
| 37.4 PKU-CFCS | 320 |
| 37.5 Mathemaniac | 320 |
| 37.6 Polya enumeration theorem | 320 |
| 37.7 complex group representation | 320 |
| 37.7.1 complex basis group | 320 |
| 37.7.2 $\mathbb{C} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ | 321 |

| | |
|--|------------|
| 37.7.3 (determinant of complex group representation) equivalent to (squared modulus of complex number) | 322 |
| 37.7.3.1 Lagrange identity | 322 |
| 37.7.4 Euler formula proved by complex group representation | 323 |
| 37.8 quaternion group representation | 324 |
| 37.8.1 $\mathbb{H} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C})$ | 324 |
| 37.8.1.1 Euler identity | 326 |
| 37.8.2 $\mathbb{H} \rightarrow \mathcal{M}_{4 \times 4}(\mathbb{R})$ | 327 |
| 38 tensor | 331 |
| 38.1 Einstein summation convention | 331 |
| 38.1.1 dummy index | 331 |
| 38.1.2 free index | 331 |
| 38.2 EpicOrganism = AIRoswell = Pan, Yi-Wen | 331 |
| 38.3 EigenChris | 331 |
| 38.3.1 tensor for beginner | 331 |
| 38.3.2 tensor calculus | 331 |
| 38.4 Elliot Schneider | 331 |
| 38.4.1 Fundamentals of Cartesian Tensors | 331 |
| 38.4.2 Fundamentals of Curvilinear Tensors | 339 |
| 38.4.3 Fundamentals of Spacetime Tensors | 339 |
| 38.5 Lin, Hsiu-Hau / Porcupine Lin / Hedgehog Note | 339 |
| 39 dual space | 341 |
| 39.1 linear equations | 341 |
| 39.2 matrix multiplication | 342 |
| 39.3 functional | 343 |
| 39.4 definition of linear functional | 345 |
| 39.5 set of all linear transformations is a vector space | 346 |
| 39.6 definition of dual space | 348 |
| 39.7 double dual | 348 |
| 40 vector space | 349 |
| 40.1 What is a vector? | 349 |
| 40.2 vector space definition | 349 |
| 40.2.1 commutative group structure of vector space | 349 |
| 40.2.2 scalar distributivity | 350 |
| 40.3 linearity | 351 |
| 40.3.1 linear structure of vector space | 351 |
| 40.3.2 linear transformation or linear map | 351 |
| 40.4 vector space example | 351 |
| 40.4.1 reciprocal space | 352 |
| 40.4.1.1 Kronecker delta | 352 |
| 40.4.1.2 double dual concept | 354 |
| 40.5 EpicOrganism = AIRoswell = Pan, Yi-Wen | 355 |
| 40.6 field | 355 |
| 40.7 module | 355 |
| 40.8 subspace | 355 |
| 41 df | 357 |
| 41.1 df decomposed with partials as a set of basis in vector space | 357 |
| 41.2 dual space of span of partials | 358 |
| 41.3 directional derivative | 358 |
| 41.4 coefficient of linear combination for vector space and dual space | 359 |
| 41.5 change of basis / change of coordinate | 359 |
| 41.6 ambiguity with partial notation | 360 |
| 41.7 1-form | 360 |
| 41.8 What is Math: differential geometry | 360 |
| 41.9 Liang, Can-bin: differential geometry and general relativity | 360 |
| 42 determinant | 361 |
| 42.1 induction of determinant axioms | 361 |
| 42.2 determinant axioms | 361 |
| 42.3 determinant theorems or properties | 361 |
| 42.4 ellipse area | 361 |

| | |
|---|------------|
| 42.5 Cramer rule geometry perspective | 361 |
| 43 hypergeometric function | 363 |
| 43.1 linear space of function | 363 |
| 43.2 beta function | 364 |
| 43.2.1 Wallis product formula | 366 |
| 43.3 gamma function | 366 |
| 43.4 recursion | 366 |
| 43.5 mean and variance of discrete probability distributions | 366 |
| 43.6 CORDIC = coordinate rotation digital computer | 366 |
| 44 Feynman method | 367 |
| 44.1 Feynman method of differentiation / derivative technique | 367 |
| 44.1.1 principle | 367 |
| 44.1.2 examples | 368 |
| 44.2 Feynman method of integration / integral technique | 370 |
| 44.2.1 principle | 370 |
| 44.2.2 Dirichlet integral | 370 |
| 44.2.2.1 residue method | 373 |
| 44.2.2.2 sinc function | 373 |
| 44.2.3 Gaussian integral | 373 |
| 44.2.4 other examples | 376 |
| 44.3 Feynman method of path integral | 377 |
| 45 Hilbert space | 379 |
| 45.1 Taylor expansion or Taylor series | 379 |
| 45.1.1 Elliot Schneider: Taylor series | 390 |
| 45.1.1.1 making complicated equation simple | 391 |
| 45.1.1.2 non-relativistic limit | 392 |
| 45.1.1.3 quantum momentum operator | 392 |
| 45.2 Dirac delta function | 392 |
| 45.2.1 complex delta function | 397 |
| 45.2.2 3D delta function | 403 |
| 45.2.3 quantum state | 408 |
| 45.2.4 rigorous definition of delta function | 408 |
| 45.3 Fourier expansion or Fourier series | 409 |
| 45.3.1 Taylor vs. Fourier | 409 |
| 45.4 Hilbert space construction | 410 |
| 45.4.1 inner product space | 411 |
| 45.4.2 Riesz representation theorem | 413 |
| 45.5 Dirac bracket | 413 |
| 45.5.1 Dirac bracket symmetry | 415 |
| 45.6 quantum operator | 417 |
| 45.7 Gram-Schmidt algorithm to find orthonormal basis | 417 |
| 45.8 Hermite polynomials and Legendre polynomials | 417 |
| 46 Lagrange inversion | 419 |
| 46.1 Newton | 419 |
| 46.2 Lagrange | 419 |
| 47 quantum color dynamics | 421 |
| 48 Lagrangian | 423 |
| 49 Hamiltonian | 425 |
| 50 Fourier analysis | 427 |
| 50.1 basic calculation | 427 |
| 50.2 TaylorCatAlice: phase velocity vs. group velocity | 429 |
| 50.3 quantum mechanics or wave mechanics | 429 |
| 50.3.1 uncertainty principle | 429 |
| 50.4 Simon Xu | 429 |
| 50.4.1 DFT = discrete Fourier transform | 429 |
| 50.4.2 FFT = fast Fourier transform | 429 |

| | |
|---|------------|
| 50.4.3 wavelet | 429 |
| 50.4.3.1 compressed sensing MRI | 429 |
| 50.5 sampling and reconstruction | 430 |
| 50.6 signal and system | 430 |
| 50.7 image compression / data compression | 430 |
| 50.8 Lin, Chi-Kun | 430 |
| 51 axiom of choice | 431 |
| 51.1 Zorn lemma | 431 |
| 52 linear algebra | 433 |
| 52.1 The Art of Linear Algebra | 433 |
| 52.2 CCJou | 435 |
| 52.2.1 coordinate | 435 |
| 52.2.2 linear transformation | 437 |
| 52.2.3 change of basis | 440 |
| 52.2.3.1 differential polynomial | 444 |
| 52.2.3.1.1 Chebyshev polynomial | 444 |
| 52.2.3.1.2 Hermite polynomial | 444 |
| 52.2.4 singular value decomposition | 444 |
| 52.3 Chi, Chen-Yu | 444 |
| 53 analysis | 445 |
| 53.1 real analysis | 445 |
| 53.2 Chi, Chen-Yu | 445 |
| 53.3 Chen, Jin-Tzu | 445 |
| 53.4 σ -algebra | 445 |
| 53.5 complex anaylsis | 445 |
| 53.5.1 Mathemaniac | 445 |
| 54 matrix calculus | 447 |
| 54.1 vector-by-scalar | 447 |
| 54.2 scalar-by-vector | 447 |
| 54.3 vector-by-vector | 448 |
| 54.3.1 numerator-layout notation | 448 |
| 54.3.2 denominator-layout notation | 448 |
| 55 genetics | 449 |
| 55.1 term | 449 |
| 55.2 marker | 449 |
| 55.3 genome project | 449 |
| 55.4 linkage analysis | 450 |
| 55.4.1 PLA = parametric linkage analysis | 451 |
| 55.4.2 VCLA = variance component linkage analysis | 451 |
| 55.5 association analysis | 451 |
| 56 multivariate normal distribution | 453 |
| 56.1 bivariate normal distribution | 459 |
| 57 logic | 461 |
| 57.1 Shai Ben-David | 461 |
| 58 algorithm | 463 |
| 58.1 Chen, Vivian | 463 |
| 58.2 The Bubble Sort Curve | 463 |
| 59 computer graphics | 465 |
| 59.1 Cem Yuksel | 465 |
| 59.1.1 introduction | 465 |
| 59.1.1.1 2D transformation | 465 |
| 59.1.1.1.1 translation | 465 |
| 59.1.1.1.2 scale | 465 |
| 59.1.1.1.3 non-uniform scale | 465 |
| 59.1.1.1.4 rotation | 465 |

| | | |
|------------|---|------------|
| 59.1.1.1.5 | skew = rotation + non-uniform scale + rotation | 465 |
| 59.1.1.6 | any 2×2 matrix | 465 |
| 59.1.1.7 | homogeneous coordinate | 466 |
| 59.1.1.8 | position vs. direction | 466 |
| 59.1.1.2 | 3D transformation | 466 |
| 59.1.1.2.1 | homogeneous coordinate | 466 |
| 59.1.1.2.2 | scale | 466 |
| 59.1.1.2.3 | translation | 466 |
| 59.1.2 | interactive | 467 |
| 60 | autoregression in time series | 469 |
| 60.1 | AR(1) = 1 st -order autoregressive model = first-order autoregressive model | 471 |
| 60.2 | AR(2) = 2 nd -order autoregressive model = second-order autoregressive model | 472 |
| 60.3 | AR(p) = p^{th} -order autoregressive model | 472 |
| 60.4 | MA(q) = q^{th} -order moving-average model | 472 |
| 60.5 | ARMA(p,q) = $p^{\text{th}}, q^{\text{th}}$ -order autoregressive-moving-average model | 473 |
| 61 | conductivity | 475 |
| 61.1 | resistance | 475 |
| 61.2 | semiconductor | 475 |
| 62 | LaTeX annotation by TikZ | 477 |
| 63 | Shannon sampling | 483 |
| 64 | brachistochrone | 485 |
| 65 | magnetic resonance | 487 |
| 65.1 | NMR | 487 |
| 65.1.1 | Bloch equation | 487 |
| 65.1.2 | Bloch-Torrey equation | 487 |
| 65.2 | MRI | 487 |
| 65.2.1 | Lin, Hsiu-Hau | 487 |
| 65.2.2 | MRI: Physics and Image Creation | 487 |
| 65.2.3 | Tseng, Wen-Yih Isaac | 488 |
| 66 | OpenMMLab | 489 |
| 66.1 | Tongji TommyZihao | 489 |
| 66.2 | official | 489 |
| 67 | linear programming | 491 |
| 67.1 | simplex method | 491 |
| 67.2 | duality method | 491 |
| 67.3 | integer linear programming | 491 |
| 68 | TaylorCatAlice physics | 493 |
| 68.1 | dimensional analysis | 493 |
| 68.2 | classic mechanics | 493 |
| 68.2.1 | Galileo | 493 |
| 68.2.2 | Newton | 493 |
| 68.2.3 | Lagrange | 493 |
| 68.2.4 | Hamilton | 493 |
| 68.3 | electromagnetism | 493 |
| 68.4 | relativity | 494 |
| 68.5 | quantum theory | 494 |
| 68.5.1 | old quantum theory | 494 |
| 68.5.2 | black-body radiation | 494 |
| 68.5.3 | quantum mechanics | 494 |
| 68.5.3.1 | Heisenberg: matrix mechanics | 494 |
| 68.5.3.2 | Schrodinger: wave mechanics | 494 |
| 68.5.3.3 | Dirac: relativistic quantum mechanics | 495 |
| 68.5.3.3.1 | spin | 495 |
| 68.5.3.4 | other | 495 |
| 68.5.4 | resistance and conductivity | 495 |

| | |
|---|------------|
| 69 fundamental theorem of algebra | 497 |
| 69.1 Gauss | 497 |
| 69.1.1 FToA | 497 |
| 69.1.2 trial and error | 497 |
| 69.1.3 heptadecagon | 497 |
| 69.2 equation | 497 |
| 69.2.1 cubic equation | 497 |
| 69.2.2 quintic equation | 497 |
| 70 Basel problem | 499 |
| 70.1 TaylorCatAlice | 499 |
| 70.2 3B1B | 499 |
| 70.3 Riemann hypothesis | 499 |
| 71 number theory | 501 |
| 71.1 4 important theorems | 501 |
| 71.2 congruence modulo | 501 |
| 72 spinor | 503 |
| 72.1 EigenChris | 503 |
| 72.2 Shinonome Masaki | 503 |
| 73 singular value decomposition | 505 |
| 73.1 Visual Kernel | 505 |
| 73.2 CCJou: LA Revelation | 505 |
| 74 support vector machine | 507 |
| 74.1 Visually Explained | 507 |
| 75 principal component analysis | 509 |
| 75.1 Visually Explained | 509 |
| 75.2 dimension reduction: PCA, tSNE, UMAP | 509 |
| 76 transformer | 511 |
| 76.1 Lee, Hung-Yi | 511 |
| 76.2 3B1B | 511 |
| 77 differential equation | 513 |
| 77.1 ODE = ordinary DE = ordinary differential equation | 513 |
| 77.1.1 Sturm-Liouville theory | 513 |
| 77.2 PDE = partial DE = partial differential equation | 513 |
| 78 Shannon | 515 |
| 78.1 entropy | 515 |
| 79 LEAN | 517 |
| 79.1 LEAN 4 | 517 |
| 79.1.1 type theory | 517 |
| 79.1.2 installation | 517 |
| 79.2 ELAN | 517 |
| 79.3 MathLib = mathlib | 517 |
| 79.3.1 yugu233 / Crisper : MathLibExplorer | 517 |
| 79.4 LEAN game server | 517 |
| 79.4.1 A LEAN Intro to Logic | 517 |
| 79.4.2 1.2.1.2 Set Theory Game | 517 |
| 79.4.3 Natural Number Game | 518 |
| 79.4.3.1 Natural Number Game in LEAN 4 | 518 |
| 79.5 LEAN 4 proof tutorial | 518 |
| 80 recreational math | 519 |
| 80.1 solved | 519 |
| 80.1.1 Monty Hall problem | 519 |
| 80.1.2 goat grazing problem | 519 |
| 80.2 not yet solved | 519 |
| 80.2.1 moving sofa problem | 519 |

| | |
|---|------------|
| 81 causal inference | 521 |
| 82 Einstein-de Hass effect | 523 |
| 83 pythagorean theorem | 525 |
| 83.1 2023 new proof with trigonometry | 525 |
| 84 general relativity | 527 |
| 84.1 EpicOrganism = AIRoswell = Pan, Yi-Wen | 527 |
| 84.2 Elliot Schneider | 527 |
| 85 LaTeX RegEx | 529 |
| 86 operator theory | 533 |
| 87 physics monsters | 535 |
| 88 List of Theorem | 537 |
| 89 Galois theory | 539 |
| 90 Sage | 541 |
| references | 543 |

index

math on bookdown started on 2024/01/28

Joey Yu Hsu, MD

“If you are doing research, you may not get a bunch of money but \$\$.” by Joey Yu Hsu, MD

Part I

by discipline

Chapter 1

mathematics

1.1 tool

- formula typesetting
 - TeX
 - * LaTeX
 - pdfLaTeX
 - XeLaTeX
 - editor/tool:
 - LyX
 - OverLeaf
 - MathPix Snip
 - Micro\$oft Office Word
 - WordTeX <https://tomwildenhain.com/wordtex/>
 - Pandoc dependent
 - <https://superuser.com/questions/1114697/select-a-different-math-font-in-microsoft-word>
 - https://www.youtube.com/watch?v=jIX_pThh7z8
 - Micro\$oft Office PowerPoint
 - IguanaTeX <https://www.jonathanleroux.org/software/iguanatex/>
 - MathML
 - MathJax: JavaScript
 - plot^[3]
 - proof: type theory
 - LEAN^[79]
 - symbolic computing
 - SageMath: by University of Washington
 - * Sage^[90]
 - Maple: by MapleSoft
 - Mathematica: by Wolfram
 - numeric computing
 - MatLab: by MathWorks

equivalence relation^[11]

equivalence class^[10]

partition^[9]

1.2 discipline

Chapter 2

physics

2.1 discipline

- relativity
 - special relativity
 - * **Lorentz transformation**^[18]
 - general relativity
- analytic mechanics
 - Lagrangian mechanics
 - Hamiltonian mechanics
- electromagnetism
- quantum mechanics
- field theory

Chapter 3

plot

- LaTeX
 - TikZ^[13]
 - * <https://tikz.dev/>
 - * TikZ-3Dplot
 - * PGFplots^[13.5]
 - <https://tikz.dev/pgfplots/>
 - <https://pgfplots.sourceforge.net/gallery.html>
 - <https://pgfplots.net/>
 - * editor / export
 - <https://zhuanlan.zhihu.com/p/660371706>
 - offline
 - [TikzEdt](#): WYSIWYG and live preview
 - [TikZiT](#)
 - online
 - OverLeaf
 - MathCha
 - GeoGebra Classic
 - Python
 - TikZplotLib / tikzplotlib^[13.6]
 - matplotlib export to TikZ .tex
 - [PyPI](#)
 - [GitHub](#)
 - R
 - TikZDevice / tikzDevice
 - r chunk engine='tikz' knitr out.width=if (knitr:::is_html_output()) '100%'
 - [CRAN](#)
 - [reference manual](#)
 - vignette: [TikZDevice - LaTeX Graphics for R](#)
 - [GitHub](#)
 - * [TikZ library](#)
 - xypic = [xy-pic](#)^[14]
- OverLeaf
- MathCha
- GeoGebra
 - GeoGebra Classic: to export TikZ
 - GeoGebra Calculator Suite
- Python
 - Matplotlib / matplotlib^[27]
 - Seaborn / seaborn^[27.1.3]
 - Plotly
 - Manim
- R
 - Modern Statistical Graphics
 - [ggplot2](#)^[30]
 - * Modern Statistical Graphics [section 5.1](#)
 - GraphViz .gv
 - Mermaid .mmd

- * [about](#)
- * JavaScript based diagramming and charting tool that renders Markdown-inspired text definitions to create and modify diagrams dynamically
- Shiny
 - * R Markdown Guide [section 5.1](#)
- tool
 - * Jamovi

neural network plot/draw <https://github.com/ashishpatel26/Tools-to-Design-or-Visualize-Architecture-of-Neural-Network>

Chapter 4

programming language

4.1 discipline

- Python^[12]
- JavaScript
- SQL = structured query language
- R^[19]
 - RMarkdown
 - * Bookdown
 - knitr: engine
 - * TikZ^[13]
 - reticulate: Python
 - Jamovi
- C#
 - web
 - * MVC
 - * .NET
 - desktop
 - * UWP = Universal Windows Platform
 - * WPF = Windows Presentation Foundation
 - * WinForms = Windows Forms
 - 3D/game
 - * Unity

4.2 learning map

- W3School
- SoloLearn
- Codecademy

Chapter 5

machine learning

5.1 Shai Ben-David

<https://www.youtube.com/playlist?list=PLPW2keNyw-usgymR7FTQ3ZRjfLs5jT4BO>

5.2 deep learning

我妻幸長

Esc = Einstein summation convention

$$\begin{aligned}
 W\mathbf{x} &= \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{\nu=0}^n w_{0\nu} x_\nu \\ \sum_{\nu=0}^n w_{1\nu} x_\nu \\ \vdots \\ \sum_{\nu=0}^n w_{m\nu} x_\nu \end{pmatrix} \stackrel{\text{Esc}}{=} \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = w_{\mu\nu} x_\nu \\
 \mathbf{y} &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x} \\
 \mathbf{y}^\top &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = y_\mu^\top = (w_{\mu\nu} x_\nu)^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \left[\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right]^\top = [W\mathbf{x}]^\top \\
 \mathbf{x}_\nu^\top w_{\mu\nu}^\top &= \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \mathbf{x}^\top W^\top \\
 \mathbf{x}^\top W^\top &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = x_\nu^\top w_{\mu\nu}^\top = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top
 \end{aligned}$$

$$\begin{aligned}
W\mathbf{x} &= \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{\nu=0}^n w_{0\nu} x_\nu \\ \sum_{\nu=0}^n w_{1\nu} x_\nu \\ \vdots \\ \sum_{\nu=0}^n w_{m\nu} x_\nu \end{pmatrix} \stackrel{\text{Esc}}{=} \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = w_{\mu\nu} x_\nu, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= W\mathbf{x} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_0 + \sum_{j=1}^n w_{0j} x_j \\ b_1 + \sum_{j=1}^n w_{1j} x_j \\ \vdots \\ b_m + \sum_{j=1}^n w_{mj} x_j \end{pmatrix} \stackrel{\text{Esc}}{=} \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = b_\mu + w_{\mu j} x_j \\
\mathbf{y} &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= \mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu j} x_j + b_\mu = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\mathbf{y}^\top &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = y_\mu^\top = (w_{\mu\nu} x_\nu)^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \left[\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right]^\top = [W\mathbf{x}]^\top \\
&= x_\nu^\top w_{\mu\nu}^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \mathbf{x}^\top W^\top, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= b_\mu^\top + x_j^\top w_{\mu j}^\top = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \mathbf{x}^\top W^\top \\
\mathbf{x}^\top W^\top &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = x_\nu^\top w_{\mu\nu}^\top = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top \\
\mathbf{x}^\top W^\top &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top = b_\mu^\top + x_j^\top w_{\mu j}^\top = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top
\end{aligned}$$

$$\begin{aligned}
W\mathbf{x} &= \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{\nu=0}^n w_{0\nu} x_\nu \\ \sum_{\nu=0}^n w_{1\nu} x_\nu \\ \vdots \\ \sum_{\nu=0}^n w_{m\nu} x_\nu \end{pmatrix} \stackrel{\text{Esc}}{=} \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = w_{\mu\nu} x_\nu, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= W\mathbf{x} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_0 + \sum_{j=1}^n w_{0j} x_j \\ b_1 + \sum_{j=1}^n w_{1j} x_j \\ \vdots \\ b_m + \sum_{j=1}^n w_{mj} x_j \end{pmatrix} \stackrel{\text{Esc}}{=} \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = b_\mu + w_{\mu j} x_j \\
\mathbf{y} &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= \mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu j} x_j + b_\mu = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\mathbf{y}^\top &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = y_\mu^\top = (w_{\mu\nu} x_\nu)^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \left[\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right]^\top = [W\mathbf{x}]^\top \\
&= x_\nu^\top w_{\mu\nu}^\top = x_\nu^\top w_{\nu\mu} = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \mathbf{x}^\top W^\top \\
&= b_\mu^\top + x_j^\top w_{\mu j}^\top = b_\mu^\top + x_j^\top w_{j\mu} = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top \\
\mathbf{x}^\top W^\top &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = x_\nu^\top w_{\nu\mu} = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top \\
\mathbf{x}^\top W^\top &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top = b_\mu^\top + x_j^\top w_{j\mu} = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top
\end{aligned}$$

$$\begin{aligned}
\mathbf{y} &= \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu\nu} x_\nu = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= \mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = y_\mu = w_{\mu j} x_j + b_\mu = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = W\mathbf{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\mathbf{x}^\top W^\top &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top = x_\nu^\top w_{\mu\nu}^\top = x_\nu^\top w_{\nu\mu} = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top \\
&= \mathbf{x}^\top W^\top = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top = b_\mu^\top + x_j^\top w_{j\mu} = y_\mu^\top = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top = \mathbf{y}^\top
\end{aligned}$$

$$\begin{aligned}
\sigma(\mathbf{y}) &= \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma(y_\mu) = \sigma(w_{\mu\nu} x_\nu) = \sigma \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \sigma \left(\begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \sigma(W\mathbf{x}), \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= \sigma(\mathbf{y}) = \sigma \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma(y_\mu) = \sigma(w_{\mu j} x_j + b_\mu) = \sigma \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = \sigma \left(\begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \sigma(W\mathbf{x}), \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
(\mathbf{x}^\top W^\top) \varsigma &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top \varsigma = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top \varsigma = (x_\nu^\top w_{\mu\nu}^\top) \varsigma = (x_\nu^\top w_{\nu\mu}) \varsigma = (y_\mu^\top) \varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top \varsigma = (\mathbf{y}^\top) \varsigma \\
&= (\mathbf{x}^\top W^\top) \varsigma = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top \varsigma = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top \varsigma = (b_\mu^\top + x_j^\top w_{j\mu}) \varsigma = (y_\mu^\top) \varsigma = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top \varsigma = (\mathbf{y}^\top) \varsigma
\end{aligned}$$

$$\begin{aligned}
\mathbf{z} &= \sigma \mathbf{y} = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma_\mu y_\mu = \sigma_\mu w_{\mu\nu} x_\nu = \sigma_\mu \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix} = \sigma \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W \mathbf{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= \mathbf{z} = z_\mu = \sigma_\mu \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_\mu \end{pmatrix} = \sigma_\mu y_\mu = \sigma_\mu (w_{\mu j} x_j + b_\mu) = \sigma_\mu \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix} = \sigma \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \sigma W \mathbf{x}, \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\mathbf{x}^\top W^\top \varsigma &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top \varsigma = \begin{pmatrix} w_{0\nu} x_\nu \\ w_{1\nu} x_\nu \\ \vdots \\ w_{m\nu} x_\nu \end{pmatrix}^\top \varsigma = x_\nu^\top w_{\mu\nu}^\top \varsigma_\mu = x_\nu^\top w_{\nu\mu} \varsigma_\mu = y_\mu^\top \varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top \varsigma_\mu = \mathbf{y}^\top \varsigma = \mathbf{z}^\top \varsigma \\
&= \mathbf{x}^\top W^\top \varsigma = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}^\top \begin{pmatrix} b_0 & w_{01} & \cdots & w_{0n} \\ b_1 & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & w_{m1} & \cdots & w_{mn} \end{pmatrix}^\top \varsigma = \begin{pmatrix} b_0 + w_{0j} x_j \\ b_1 + w_{1j} x_j \\ \vdots \\ b_m + w_{mj} x_j \end{pmatrix}^\top \varsigma = (b_\mu^\top + x_j^\top w_{j\mu}) \varsigma_\mu = y_\mu^\top \varsigma_\mu = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix}^\top \varsigma_\mu = z_\mu^\top = \mathbf{z}^\top
\end{aligned}$$

$$\begin{aligned}
\mathbf{z} &= \sigma \mathbf{y} = \sigma_\mu w_{\mu\nu} x_\nu = \sigma W \mathbf{x}, \begin{cases} x_0 = 1 \\ w_{\mu 0} = b_\mu \end{cases} \\
&= z_\mu = \sigma_\mu y_\mu = \sigma_\mu (w_{\mu j} x_j + b_\mu), \begin{cases} 1 = x_0 \\ b_\mu = w_{\mu 0} \end{cases} \\
\mathbf{x}^\top W^\top \varsigma &= x_\nu^\top w_{\mu\nu}^\top \varsigma_\mu = x_\nu^\top w_{\nu\mu} \varsigma_\mu = \mathbf{y}^\top \varsigma = \mathbf{z}^\top \\
&= (b_\mu^\top + x_j^\top w_{j\mu}) \varsigma_\mu = y_\mu^\top \varsigma_\mu = z_\mu^\top
\end{aligned}$$

matrix calculus^[54]

4-15

wrong or incompatible transpose

$$\begin{aligned}
\mathbf{x}^\top W &= (x_0 \quad x_1 \quad \cdots \quad x_m) \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \\
&= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} \sum_{\mu=1}^m x_\mu w_{\mu 0} \\ \sum_{\mu=1}^m x_\mu w_{\mu 1} \\ \vdots \\ \sum_{\mu=1}^m x_\mu w_{\mu n} \end{pmatrix}^\top \\
\text{Einstein summation convention } &\equiv \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^\top \\
&= x_\mu^\top w_{\mu\nu} = (x_\mu w_{\mu\nu})^\top?
\end{aligned}$$

4-18

wrong or incompatible transpose

$$\begin{aligned}
\mathbf{x}^\top W &= \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^\top, \quad \begin{cases} x_0 = 1 \\ w_{0\nu} = b_\nu \end{cases} \\
&= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} = \begin{pmatrix} x_i w_{i0} + b_0 \\ x_i w_{i1} + b_1 \\ \vdots \\ x_i w_{in} + b_n \end{pmatrix}^\top, \quad \begin{cases} 1 = x_0 \\ b_\nu = w_{0\nu} \end{cases}
\end{aligned}$$

wrong or incompatible transpose

$$\begin{aligned}
\sigma(\mathbf{x}^\top W) &= \sigma \left(\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} x_\mu w_{\mu 0} \\ x_\mu w_{\mu 1} \\ \vdots \\ x_\mu w_{\mu n} \end{pmatrix}^\top \right) = \begin{pmatrix} \sigma_0(x_\mu w_{\mu 0}) \\ \sigma_1(x_\mu w_{\mu 1}) \\ \vdots \\ \sigma_n(x_\mu w_{\mu n}) \end{pmatrix}^\top, \quad \begin{cases} x_0 = 1 \\ w_{0\nu} = b_\nu \end{cases} \\
&= \sigma \left(\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}^\top \begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ w_{10} & w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mn} \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} x_i w_{i0} + b_0 \\ x_i w_{i1} + b_1 \\ \vdots \\ x_i w_{in} + b_n \end{pmatrix}^\top \right) = \sigma_\nu(x_\mu w_{\mu\nu}), \quad \begin{cases} 1 = x_0 \\ b_\nu = w_{0\nu} \end{cases}
\end{aligned}$$

Part II

by date

Chapter 6

A Minimal Book Example

6.1 About

This is a *sample* book written in **Markdown**. You can use anything that Pandoc’s Markdown supports; for example, a math equation $a^2 + b^2 = c^2$.

6.1.1 Usage

Each **bookdown** chapter is an .Rmd file, and each .Rmd file can contain one (and only one) chapter. A chapter *must* start with a first-level heading: `# A good chapter`, and can contain one (and only one) first-level heading.

Use second-level and higher headings within chapters like: `## A short section` or `### An even shorter section`.

The `index.Rmd` file is required, and is also your first book chapter. It will be the homepage when you render the book.

6.1.2 Render book

You can render the HTML version of this example book without changing anything:

1. Find the **Build** pane in the RStudio IDE, and
2. Click on **Build Book**, then select your output format, or select “All formats” if you’d like to use multiple formats from the same book source files.

Or build the book from the R console:

```
bookdown::render_book()
```

To render this example to PDF as a `bookdown::pdf_book`, you’ll need to install XeLaTeX. You are recommended to install TinyTeX (which includes XeLaTeX): <https://yihui.org/tinytex/>.

6.1.3 Preview book

As you work, you may start a local server to live preview this HTML book. This preview will update as you edit the book when you save individual .Rmd files. You can start the server in a work session by using the RStudio add-in “Preview book”, or from the R console:

```
bookdown::serve_book()
```

6.2 Hello bookdown

All chapters start with a first-level heading followed by your chapter title, like the line above. There should be only one first-level heading (#) per .Rmd file.

6.2.1 A section

All chapter sections start with a second-level (##) or higher heading followed by your section title, like the sections above and below here. You can have as many as you want within a chapter.

An unnumbered section

Chapters and sections are numbered by default. To un-number a heading, add a `{.unnumbered}` or the shorter `{-}` at the end of the heading, like in this section.

6.3 Cross-references

Cross-references make it easier for your readers to find and link to elements in your book.

6.3.1 Chapters and sub-chapters

There are two steps to cross-reference any heading:

1. Label the heading: `# Hello world {#nice-label}`.
 - Leave the label off if you like the automated heading generated based on your heading title: for example, `# Hello world = # Hello world {#hello-world}`.
 - To label an un-numbered heading, use: `# Hello world {-#nice-label}` or `{# Hello world .unnumbered}`.
2. Next, reference the labeled heading anywhere in the text using `\@ref(nice-label)`; for example, please see Chapter 6.3.
 - If you prefer text as the link instead of a numbered reference use: `any text you want can go here`.

6.3.2 Captioned figures and tables

Figures and tables *with captions* can also be cross-referenced from elsewhere in your book using `\@ref(fig:chunk-label)` and `\@ref(tab:chunk-label)`, respectively.

See Figure 6.1.

```
par(mar = c(4, 4, .1, .1))
plot(pressure, type = 'b', pch = 19)
```

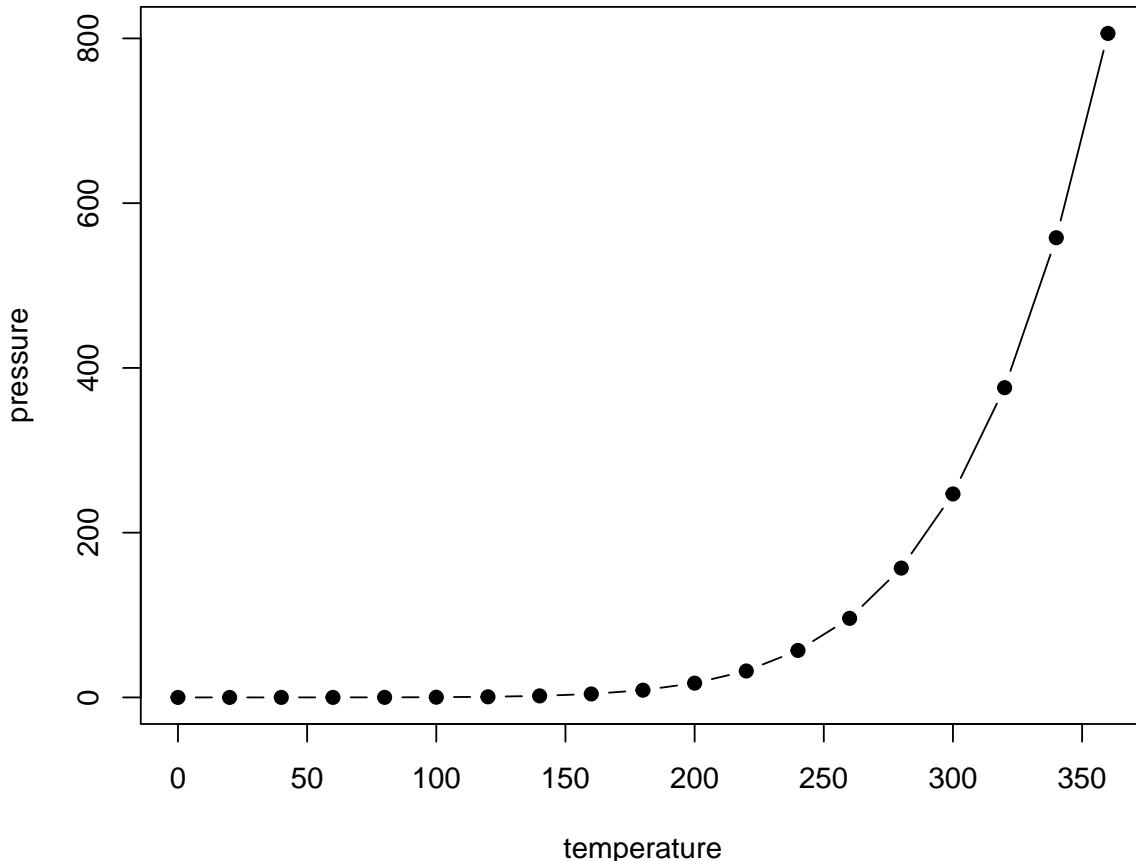


Figure 6.1: Here is a nice figure!

Don't miss Table 6.1.

```
knitr::kable(
  head(pressure, 10), caption = 'Here is a nice table!',
```

Table 6.1: Here is a nice table!

| temperature | pressure |
|-------------|----------|
| 0 | 0.0002 |
| 20 | 0.0012 |
| 40 | 0.0060 |
| 60 | 0.0300 |
| 80 | 0.0900 |
| 100 | 0.2700 |
| 120 | 0.7500 |
| 140 | 1.8500 |
| 160 | 4.2000 |
| 180 | 8.8000 |

```
booktabs = TRUE
)
```

6.4 Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: # (PART) Act one {-} (followed by # A chapter)

Add an unnumbered part: # (PART*) Act one {-} (followed by # A chapter)

Add an appendix as a special kind of un-numbered part: # (APPENDIX) Other stuff {-} (followed by # A chapter). Chapters in an appendix are prepended with letters instead of numbers.

6.5 Footnotes and citations

6.5.1 Footnotes

Footnotes are put inside the square brackets after a caret ^ []. Like this one ¹.

6.5.2 Citations

Reference items in your bibliography file(s) using @key.

For example, we are using the **bookdown** package¹ (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr**² (this citation was added manually in an external file book.bib). Note that the .bib files need to be listed in the index.Rmd with the YAML **bibliography** key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: <https://rstudio.github.io/visual-markdown-editing/#/citations>

6.6 Blocks

6.6.1 Equations

Here is an equation.

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (6.1)$$

You may refer to using \@ref(eq:binom), like see Equation (6.1).

¹This is a footnote.

6.6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

Theorem 6.1. *For a right triangle, if c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here <https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html>.

6.6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: <https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html>

6.7 Sharing your book

6.7.1 Publishing

HTML books can be published online, see: <https://bookdown.org/yihui/bookdown/publishing.html>

6.7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

6.7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book- all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

<https://pkgs.rstudio.com/bookdown/reference/gitbook.html>

Or use:

```
?bookdown::gitbook
```

Chapter 7

test

7.1 RStudio

7.1.1 RMarkdown preview LaTeX equation options

<https://tex.stackexchange.com/questions/474458/r-markdown-preview-latex-by-scrolling-over>

7.1.2 writer options

<https://rstudio.github.io/visual-markdown-editing/markdown.html#writer-options>

7.1.2.1 line wrapping

<https://rstudio.github.io/visual-markdown-editing/markdown.html#line-wrapping>

7.1.2.2 ensuring the same markdown between source / visual mode

<https://stackoverflow.com/questions/71775027/rstudio-switch-markdown-editing-mode-between-source-and-visual-changes-special>

canonical mode

<https://rstudio.github.io/visual-markdown-editing/markdown.html#canonical-mode>

```
---
```

```
title: "My Document"
editor_options:
  markdown:
    wrap: 72
    references:
      location: block
    canonical: true
---
```

7.1.3 Rtools

Rtools43 for Windows <https://cran.r-project.org/bin/windows/Rtools/rtools43/rtools.html>

7.1.4 addins

<https://github.com/rstudio/addinexamples>

```
if (!requireNamespace("devtools", quietly = TRUE))
  install.packages("devtools")

devtools::install_github("rstudio/htmltools")
devtools::install_github("rstudio/shiny")
devtools::install_github("rstudio/miniUI")
```

7.1.5 Git

commit: filename or extension is too long

<https://stackoverflow.com/questions/22575662/filename-too-long-in-git-for-windows>

<https://stackoverflow.com/questions/55327408/how-to-fix-git-for-windows-error-could-not-lock-config-file-c-file-path-to-g>

7.2 RMarkdown

R Markdown 指南 <https://cosname.github.io/rmarkdown-guide/index.html>

<https://www.rstudio.com/wp-content/uploads/2015/02/rmarkdown-cheatsheet.pdf>

<https://slides.yihui.org/2020-taipei-satrdy-rmarkdown.html#1>

7.2.1 verbatim

<https://community.rstudio.com/t/continued-issues-with-new-verbatim-in-rstudio/139737>

<https://bookdown.org/yihui/rmarkdown-cookbook/verbatim-code-chunks.html>

7.2.2 Pandoc link

<https://pandoc.org/chunkedhtml-demo/8.16-links-1.html>

<https://stackoverflow.com/questions/39281266/use-internal-links-in-rmarkdown-html-output>

<https://community.rstudio.com/t/how-to-hyperlink-between-different-rmd-files-in-rmarkdown/62289>

7.2.3 URL

<https://stackoverflow.com/questions/29787850/how-do-i-add-a-url-to-r-markdown>

[I'm an inline-style link] (<https://www.google.com>)

[I'm an inline-style link with title] (<https://www.google.com> "Google's Homepage")

[I'm a reference-style link] [Arbitrary case-insensitive reference text]

[I'm a relative reference to a repository file] (../blob/master/LICENSE)

[You can use numbers for reference-style link definitions] [1]

Or leave it empty and use the [link text itself]

Some text to show that the reference links can follow later.

[arbitrary case-insensitive reference text]: <https://www.mozilla.org>

[1]: <http://slashdot.org>

[link text itself]: <http://www.reddit.com>

7.2.4 arrow

<https://reimbar.org/dev/arrows/>

Up arrow: ↑

Down arrow: ↓

Left arrow: ←

Right arrow: →

Double headed arrow: ↔

7.2.5 superscript and subscript

```
scriptsuperscriptsubscript
scriptsuperscript
~subscript~
```

script_{subscript}

7.2.5.1 LaTeX

<https://tex.stackexchange.com/questions/580824/subscript-not-distinguished-enough>

<https://tex.stackexchange.com/questions/262295/make-subscript-size-smaller-always>

7.2.5.2 equation

<https://stackoverflow.com/questions/26049762/erroneous-nesting-of-equation-structures-in-using-beginalign-in-a-multi-l>

7.2.5.3 aligned

$$\begin{aligned}
 c^2 &= a^2 + b^2 \\
 &= b^2 + a^2 \quad + \text{commutavie} \\
 &= b^2 + a^2 \quad + \text{commutavie} \\
 &= b^2 + a^2 \quad \square
 \end{aligned}$$

$$\begin{aligned}
 c^2 &= a^2 + b^2 \\
 &= b^2 + a^2 \quad + \text{commutavie} \\
 &= a^2 + b^2 \quad + \text{commutavie} \\
 &= b^2 + a^2 \quad \square
 \end{aligned}$$

$$\begin{array}{ll}
 c^2 = a^2 + b^2 & c^2 = a^2 + b^2 \\
 = b^2 + a^2 + \text{commutavie} + \text{commutavie} & = b^2 + a^2 + \text{commutavie} \\
 = a^2 + b^2 + \text{commutavie} + \text{commutavie} & = a^2 + b^2 + \text{commutavie} \\
 = b^2 + a^2 & = b^2 + a^2 \quad \square
 \end{array}$$

7.2.5.4 proof QED

<https://math.meta.stackexchange.com/questions/3582/qed-for-mathjax-here-on-stackexchange>

\tag*{\$\Box\$}

$$a^2 + b^2 = c^2$$

\tag*{\$\blacksquare\$}

$$a^2 + b^2 = c^2$$

7.2.6 image

<https://stackoverflow.com/questions/25166624/insert-picture-table-in-r-markdown>

7.2.6.1 DiagrammeR / mermaid flowchart

Error: Functions that produce HTML output found in document targeting latex output.
Please change the output type of this document to HTML.

If you're aiming to have some HTML widgets shown in non-HTML format as a screenshot,
please install webshot or webshot2 R package for knitr to do the screenshot.

Alternatively, you can allow HTML output in non-HTML formats
by adding this option to the YAML front-matter of

your rmarkdown file:

```
always_allow_html: true
```

Note however that the HTML output will not be visible in non-HTML formats.

<https://bookdown.org/yihui/rmarkdown-cookbook/diagrams.html#diagrams>

<https://stackoverflow.com/questions/40803017/how-to-include-diagrammer-mermaid-flowchart-in-a-rmarkdown-file>

```
{r}
library(DiagrammeR)
mermaid("
graph LR
A-->B
",
width = 100
)
```

<https://github.com/rich-iannone/DiagrammeR/issues/364>

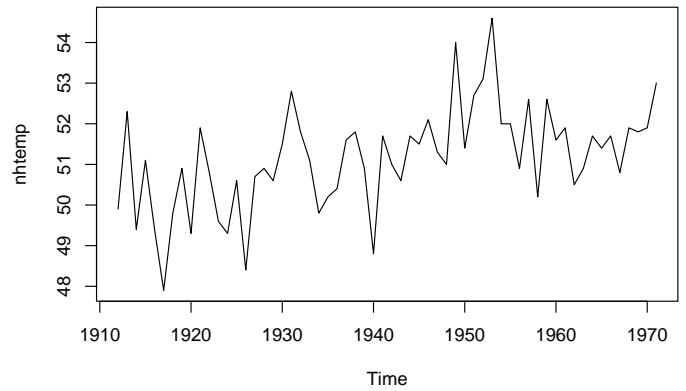
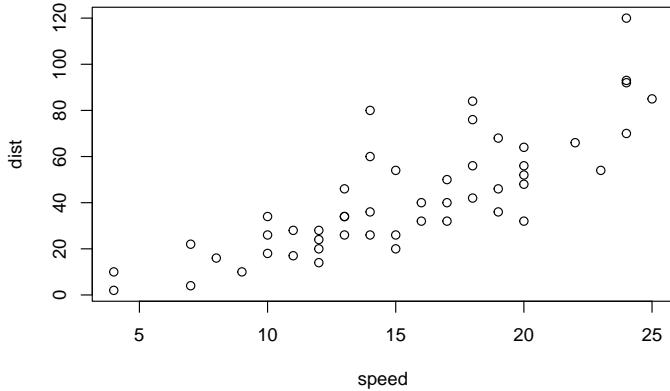
<https://stackoverflow.com/questions/55994210/how-to-solve-diagrammer-waste-of-space-issue-in-rmarkdown>

7.2.6.2 multiple images / figures in the same line

<https://cosname.github.io/rmarkdown-guide/rmarkdown-base.html#element-figure>

```
{r, fig.show = "hold", out.width = "50%"}
plot(cars)
plot(nhtemp)

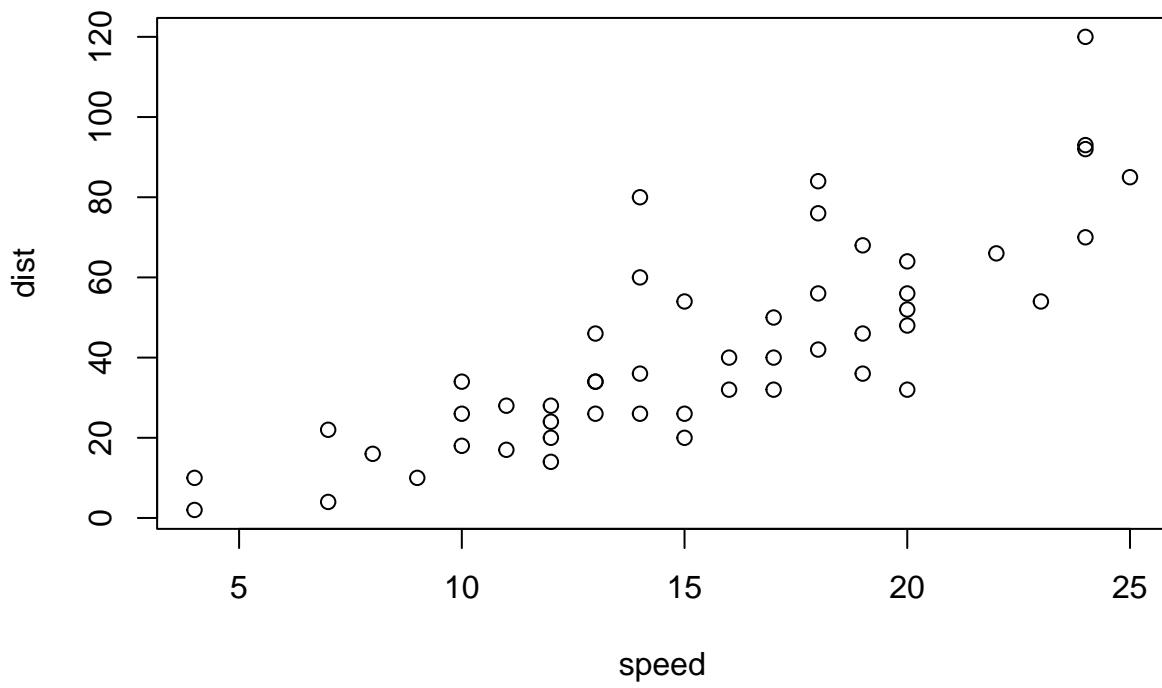
plot(cars)
plot(nhtemp)
```



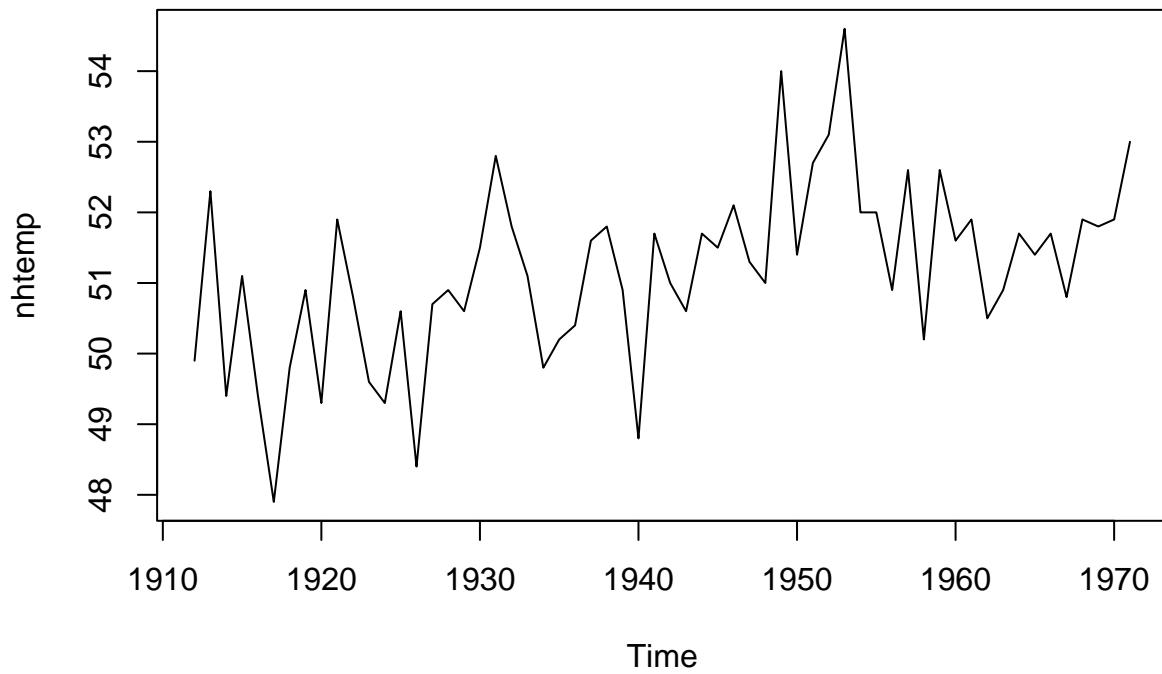
cf.

```
{r}
plot(cars)
plot(nhtemp)

plot(cars)
```



```
plot(nhtemp)
```



7.2.6.3 figure size

https://sebastiansauer.github.io/figure_sizing_knitr/

YAML in index.Rmd

```
--  
title: "My Document"  
output: html_document:  
fig_width: 6  
fig_height: 4  
--
```

first R-chunk in your RMD document

```
knitr::opts_chunk$set(fig.width=12, fig.height=8)
```

width, height and options

```
```{r fig.height = 3, fig.width = 5}
```

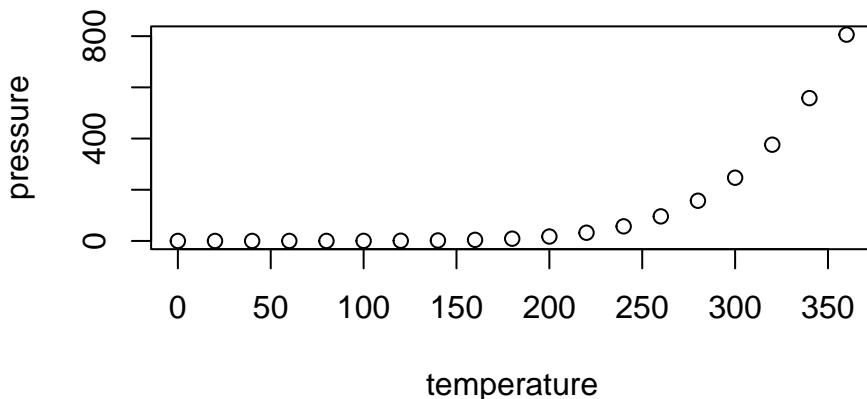
```
plot(pressure)
```

```

```

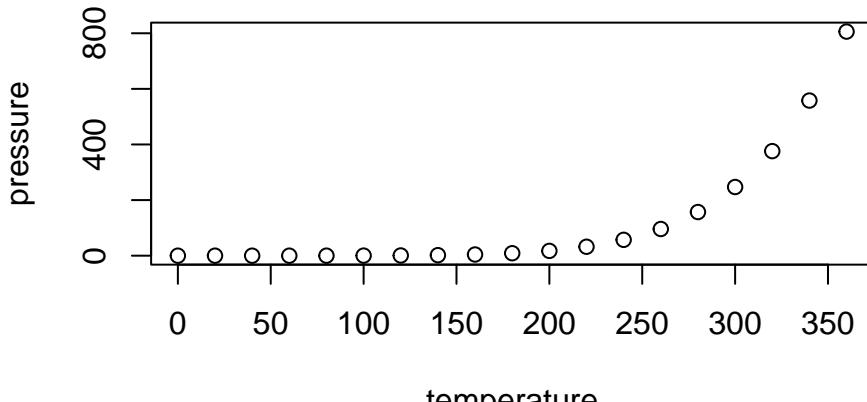
```
{r fig.height = 3, fig.width = 5}
```

```
plot(pressure)
```



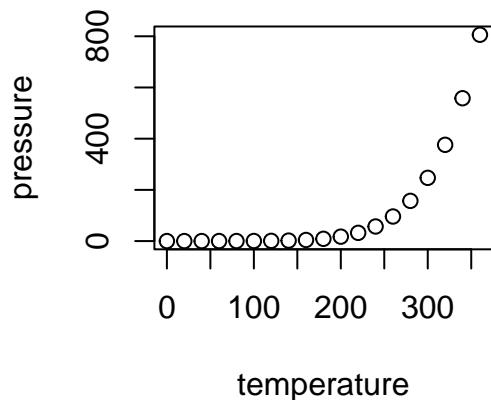
```
{r fig.height = 3, fig.width = 3, fig.align = "center"}
```

```
plot(pressure)
```



```
{r fig.width = 5, fig.asp = .62}
```

```
plot(pressure)
```

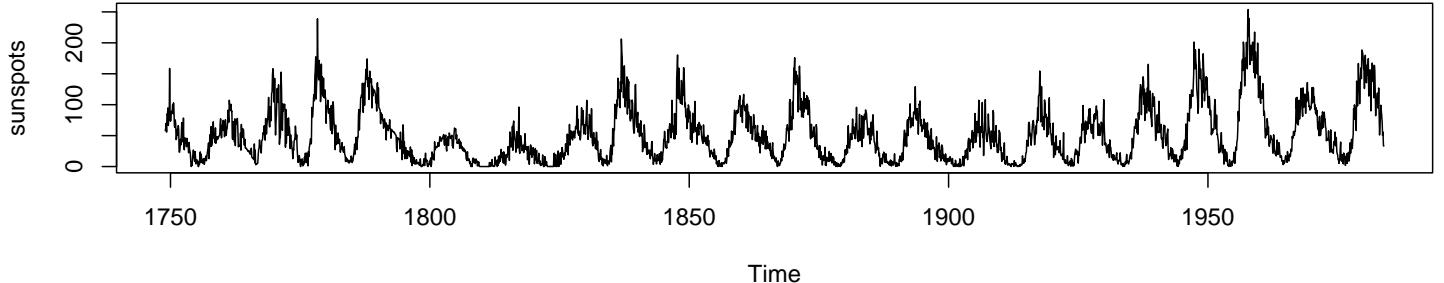


```
<center>
! [] (https://bookdown.org/yihui/rmarkdown-cookbook/images/cover.png){width=20%}
</center>
```

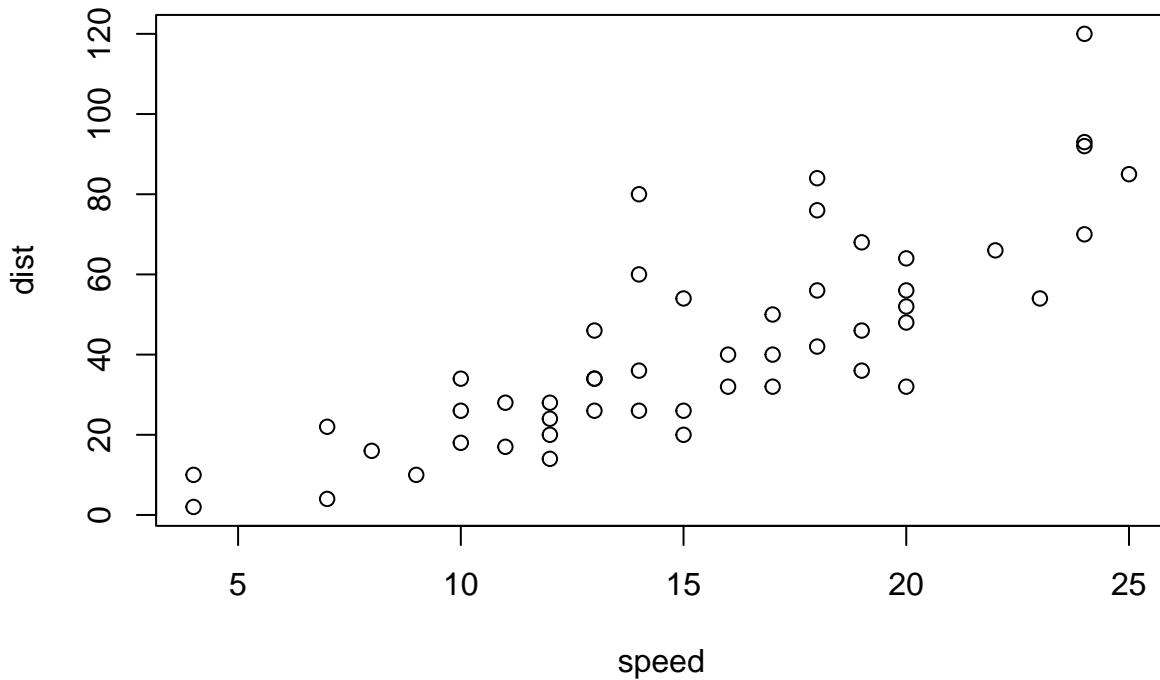
### 7.2.6.3.1 knitr <https://yihui.org/knitr/options/>

<https://bookdown.org/yihui/rmarkdown/tufte-figures.html>

```
par(mar = c(4, 4, .1, .2)); plot(sunspots)
```



```
plot(cars)
```



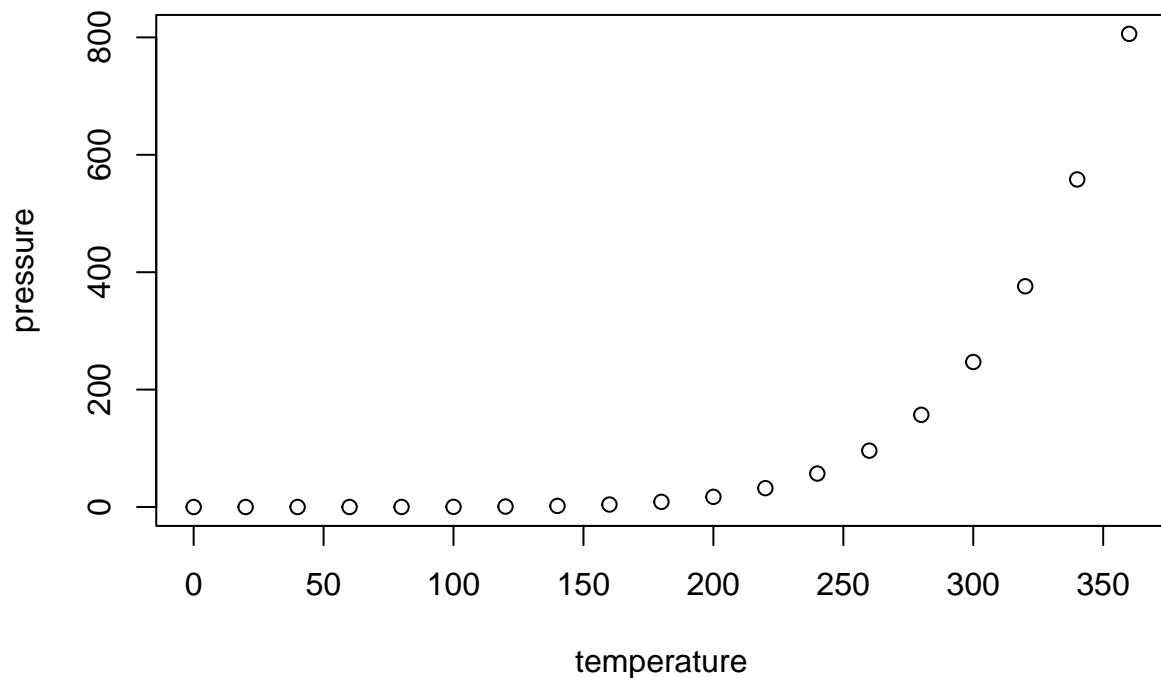
We know from the first fundamental theorem of calculus that  
for  $x$  in  $[a, b]$ :  

$$\frac{d}{dx} \left( \int_a^x f(u) du \right) = f(x).$$

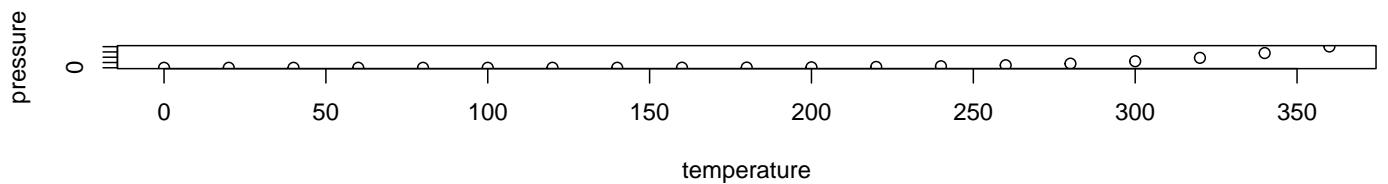
### 7.2.6.3.2 `out.width` vs. `fig.width` <https://stackoverflow.com/questions/29657777/how-to-make-fig-width-and-out-width-consistent-with-knitr>

when chunk option `cache=FALSE` is set, then `out.width` has no effect because no PDF output is created. Hence one has to specify exact measures in inches for `fig.width` and `fig.height` for each chunk

<https://stackoverflow.com/questions/59567235/a-ggmap-too-small-when-rendered-within-a-rmd-file>



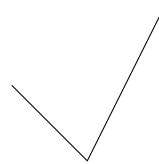
```
plot(pressure)
```



problem: `out.width='100%'` causing LaTeX Error: Not in outer par mode.

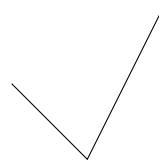
solution: `out.width=if (knitr:::is_html_output()) '100%'`

```
\begin{tikzpicture}
 \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



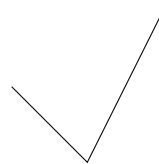
`fig.width=10, fig.height=2`

```
\begin{tikzpicture}
 \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



`out.width=if (knitr:::is_html_output()) '100%'`

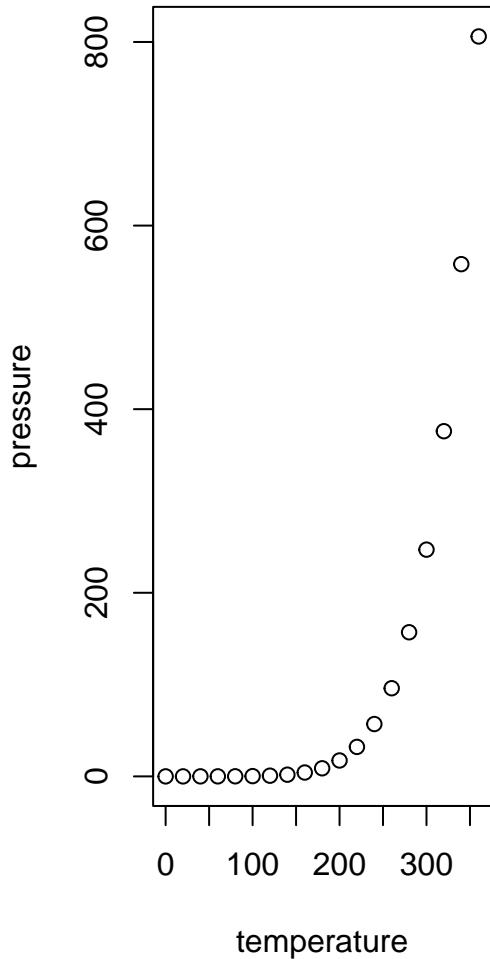
```
\begin{tikzpicture}
 \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



#### 7.2.6.4 dynamic knitr plot width and height

<https://stackoverflow.com/questions/15365829/dynamic-height-and-width-for-knitr-plots>

```
plot(pressure)
```



#### 7.2.6.5 web image in PDF

<https://stackoverflow.com/questions/46331896/how-can-i-insert-an-image-from-internet-to-the-pdf-file-produced-by-r-bookdown-i>

```
cover_url = 'https://bookdown.org/yihui/bookdown/images/cover.jpg'
if (!file.exists(cover_file <- xfun::url_filename(cover_url)))
 xfun::download_file(cover_url)
knitr::include_graphics(if (knitr::pandoc_to('html')) cover_url else cover_file)
```

#### 7.2.6.6 SVG

<https://stackoverflow.com/questions/50165404/how-to-make-a-pdf-using-bookdown-including-svg-images>

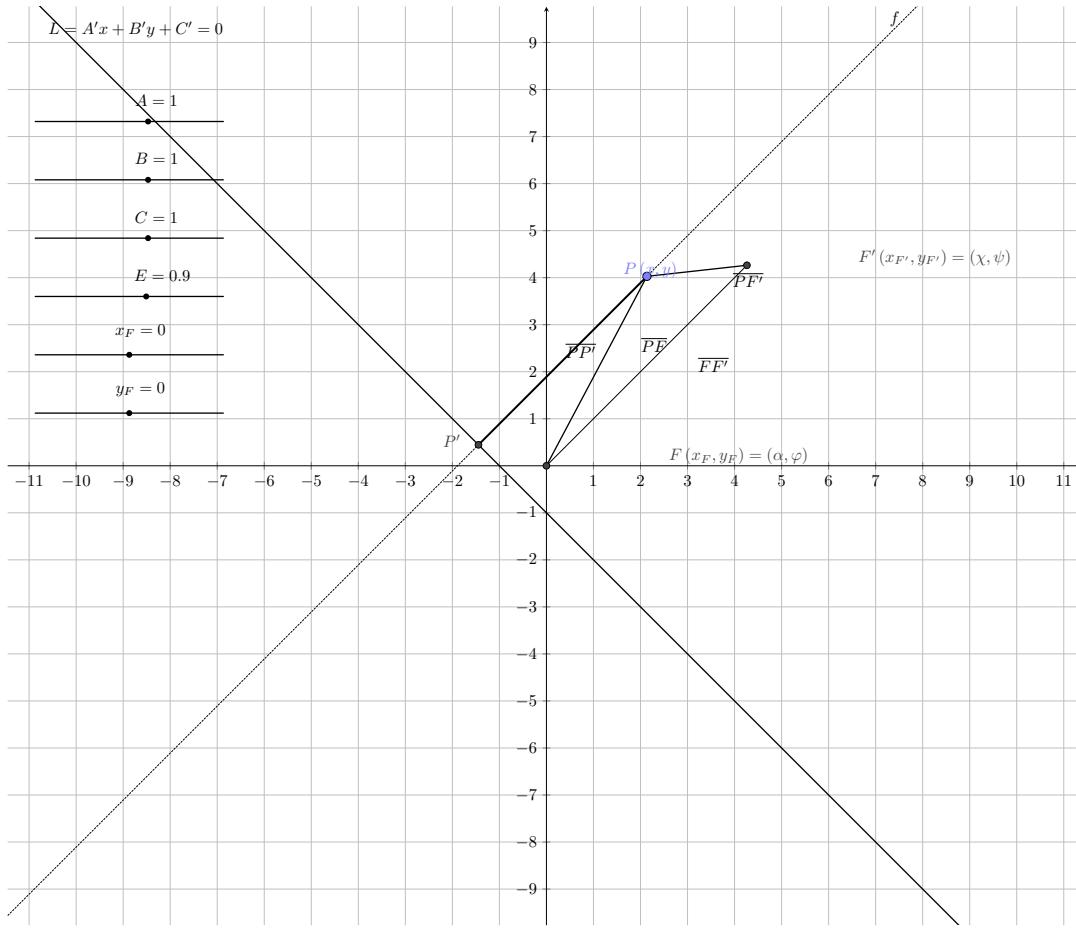
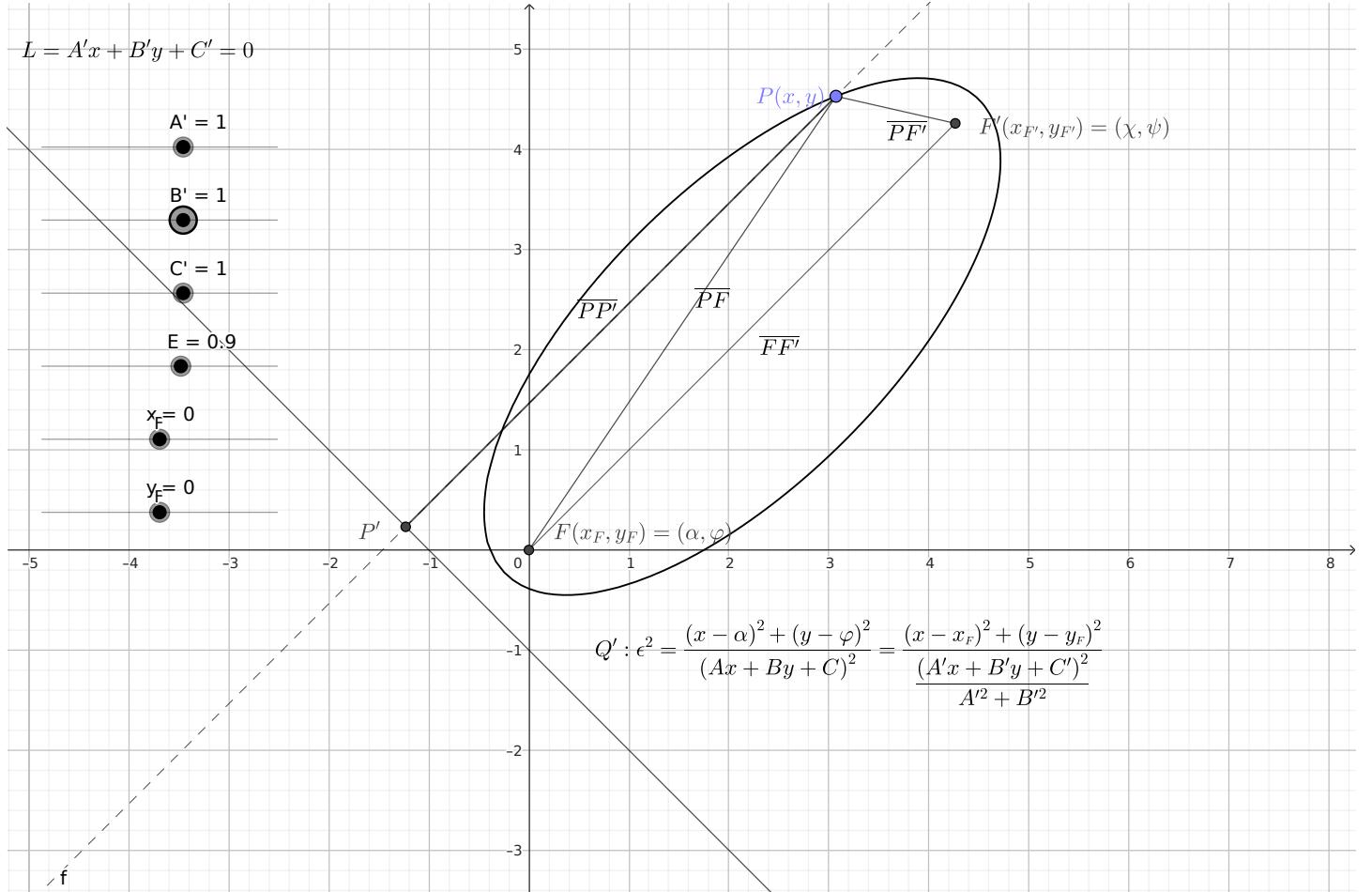


Figure 7.1: conic sections



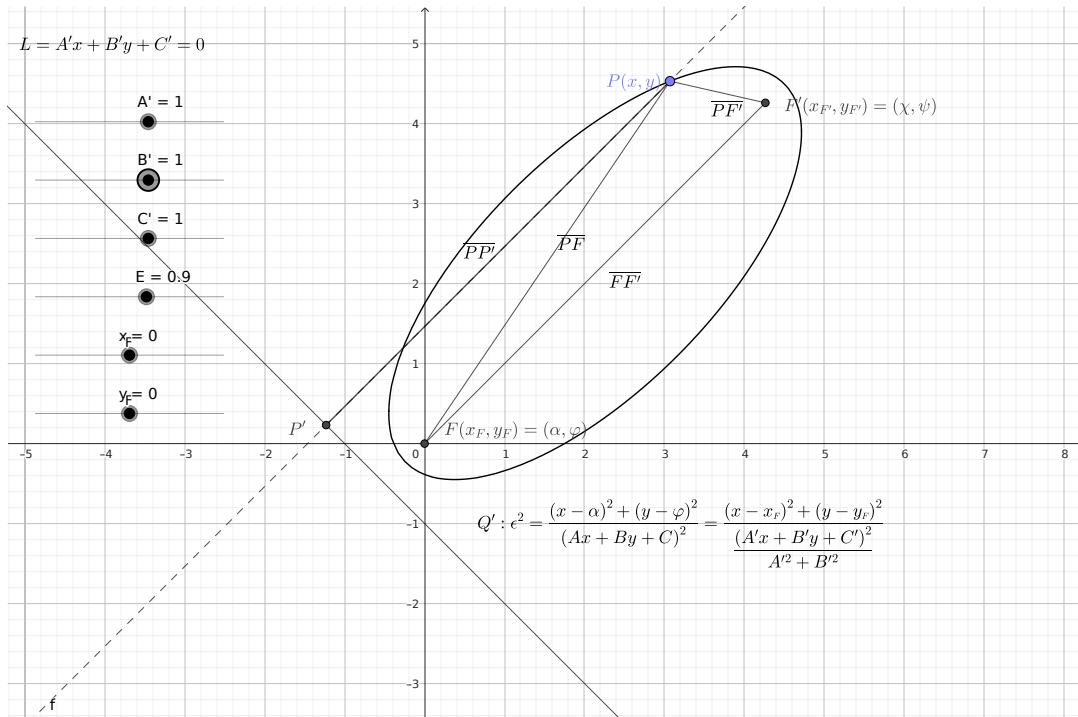


Figure 7.2: conic sections

<https://stackoverflow.com/questions/34064292/is-it-possible-to-include-svg-image-in-pdf-document-rendered-by-markdown>

### 7.2.7 horizontal rule

\*\*\*

horizontal rule (or slide break)

---

```
dim(iris)
```

```
[1] 150 5
```

### 7.2.8 footnote

### 7.2.9 hyperlink

PDF pandoc internal link will lose focus

equivalence relation [11] equivalence relation<sup>1</sup> equivalence relation<sup>[11]</sup>

equivalence class [10] equivalence class<sup>2</sup> equivalence class<sup>[10]</sup>

partition [9] partition<sup>3</sup> partition<sup>[9]</sup>

- LaTeX
  - TikZ<sup>[13]</sup>
    - \* TikZ-3Dplot
    - \* PGFplots
    - xypic = **xy-pic**<sup>4</sup>
- OverLeaf
- MathCha
- GeoGebra
- Python

<sup>1</sup>{11} equivalence relation

<sup>2</sup>{10} equivalence class

<sup>3</sup>{9} partition

<sup>4</sup>{14} **xy-pic**

- Matplotlib
- Seaborn
- Plotly

## 7.2.10 code chunk

### 7.2.10.1 code folding

<https://cosname.github.io/rmarkdown-guide/rmarkdown-document.html#html-code-folding>

## 7.2.11 xaringan

slide realtime preview with RStudio addin Infinite Moon Reader in RStudio viewer

<https://github.com/yihui/xaringan>

<https://www.youtube.com/watch?v=3n9nASHg9gc>

## 7.2.12 embed HTML or web page

### 7.2.12.1 iframe

### 7.2.12.2 include URL

<https://bookdown.org/yihui/rmarkdown-cookbook/include-url.html>

```
knitr::include_url("https://yihui.org")
```

I'm currently a freelancer, and was a software engineer at Posit, PBC (2013-2023). I earned my PhD from the Department of Statistics, Iowa State University. My thesis was *Dynamic Graphics and Reporting for Statistics*, advised by Di Cook and Heike Hofmann. I have developed a few R packages either seriously or for fun (or both), such as knitr, animation, bookdown, blogdown, pagedown, xaringan, and tinytex. I founded a Chinese website called "Capital of Statistics" in 2006, which has grown into a large online community on statistics. I initiated the China R conference in 2008. I'm a big fan of GitHub, LyX and Pandoc. I used to hate IE but no longer care since it has almost died. I fall asleep when I see beamer slides, and I yell at people who use \textbf{} to write \title{}. I know I cannot eat code, so I cook almost every day to stay away from my computer for two hours.

这是谢益辉的主页。2013年底我从 Ames 村办大学统计系毕业，终于解决了人生前三十年被问最多的问题：“你怎么还没毕业？”“目前为自由职业者。我支持开源，喜欢折腾网站和代码，是个高度自我驱动的人。打羽毛球爱勾对角，打乒乓球像太极，网球满场子捡球，篮球容易被撞飞，攀岩一次，腿软。宅，口重、嗜辣、屡教不改。智商中等偏下，对麻将和三国杀有不可逾越的认知障碍，实变函数课上曾被老师叫醒。略好读书与文，对诗词楹联感兴趣，目前较中意的一联是：千秋邈矣独留我；百战归来再读书。最喜欢的一首词是：

深情似海，问相逢初度，是何年纪？依约而今还记取，不是前生夙世。放学花前，题诗石上，春水园亭里。逢君一笑，人间无此欢喜。  
无奈苍狗看云，红羊数劫，惆怅休提起。客气渐多真气少，汨没心灵何已。  
千古声名，百年担负，事事违初意。心头阁住，儿时那种情味。

### 7.2.12.3 htmltools

<https://stackoverflow.com/questions/36524238/include-html-files-in-r-markdown-file>

however `out.height` etc. not works

### 7.2.13 embed PDF

<https://stackoverflow.com/questions/39173714/r-markdown-can-i-insert-a-pdf-to-the-r-markdown-file-as-an-image>

## 7.3 Bookdown

### 7.3.1 system locale

<https://bookdown.org/tpemartin/ntpu-programming-for-data-science/appendix-d-.html>

`Sys.getlocale()`

Windows

`Sys.setlocale(category = "LC_ALL", locale = "UTF-8")`

MacOS

`Sys.setlocale(category = "LC_ALL", locale = "en_US.UTF-8")`

<https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html>

### 7.3.2 render\_book()

<https://bookdown.org/yihui/bookdown/build-the-book.html>

```
render_book(input = ".", output_format = NULL, ..., clean = TRUE,
 envir = parent.frame(), clean_envir = !interactive(),
 output_dir = NULL, new_session = NA, preview = FALSE,
 config_file = "_bookdown.yml")
```

### 7.3.3 serve\_book()

<https://bookdown.org/yihui/bookdown/serve-the-book.html>

```
serve_book(dir = ".", output_dir = "_book", preview = TRUE,
 in_session = TRUE, quiet = FALSE, ...)
```

## 7.3.4 LaTeX

### 7.3.4.1 hyperlink, URL, href

<https://www.baeldung.com/cs/latex-hyperref-url-hyperlinks>

<https://www.omidte.com/小技巧讓facebook和line顯示中文網址，網址不再變亂碼/>

### 7.3.4.2 ugly mathptmx $\sum$

PDF LaTeX `\usepackage{fdsymbol}` to have `\overrightharpoon` vector; however, there are too many side effects, including ugly mathptmx  $\sum$ , ...

```
\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMlargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMlargesymbols}{"50}
```

<https://tex.stackexchange.com/questions/315102/different-sum-signs>

<https://tex.stackexchange.com/questions/275038/how-to-replace-mathptmx-sum-with-cm-sum>

<https://tex.stackexchange.com/questions/391410/calligraphic-symbols-are-too-fancy-with-mathptmx-package>

<https://blog.csdn.net/kongtaoxing/article/details/131005044>

In `preamble.tex`, add

```
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMLargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMLargesymbols}{50}

\DeclareMathAlphabet{\mathcal}{OMS}{cmsy}{m}{n}
\DeclareSymbolFont{largetsymbol}{OMX}{cmex}{m}{n}
```

### 7.3.4.3 LaTeX package in HTML document

<https://github.com/rstudio/rmarkdown/issues/1829>

```

title: "assignment"
author: "author"
output: html_document

$$
\require{cancel}
\cancel{x}
$$
```

$\cancel{x}$

<https://stackoverflow.com/questions/18189175/how-to-use-textup-with-mathjax>

\textup is not available in MathJax. You can replace it with \mathrm, but \mathrm does not interpret spaces.

### 7.3.5 depth of table of contents toc\_depth

<https://stackoverflow.com/questions/49009212/how-to-change-toc-depth-in-r-bookdown-gitbook>

```
bookdown::gitbook:
 toc_depth: 2
```

<https://stackoverflow.com/questions/68537309/how-can-i-specific-the-initial-level-to-have-my-table-of-contents-be-expanded-to>

```
toc:
 collapse: section
```

### 7.3.6 multi-column layout / two columns

<https://bookdown.org/yihui/rmarkdown-cookbook/multi-column.html>

#### 7.3.6.1 for both HTML and PDF

figure size<sup>[7.2.6.3]</sup>

Below is a Div containing three child Divs side by side. The Div in the middle is empty, just to add more space between the left and right Divs.

```
::::::::::: {.cols data-latex=""}

:::: {.col data-latex="{0.55\textwidth}"}
<!-- -->
:::

:::: {.col data-latex="{0.05\textwidth}"}
\
<!-- an empty Div (with a white space), serving as
a column separator -->
:::
```

```
::: {.col data-latex="{0.4\textwidth}"}
```

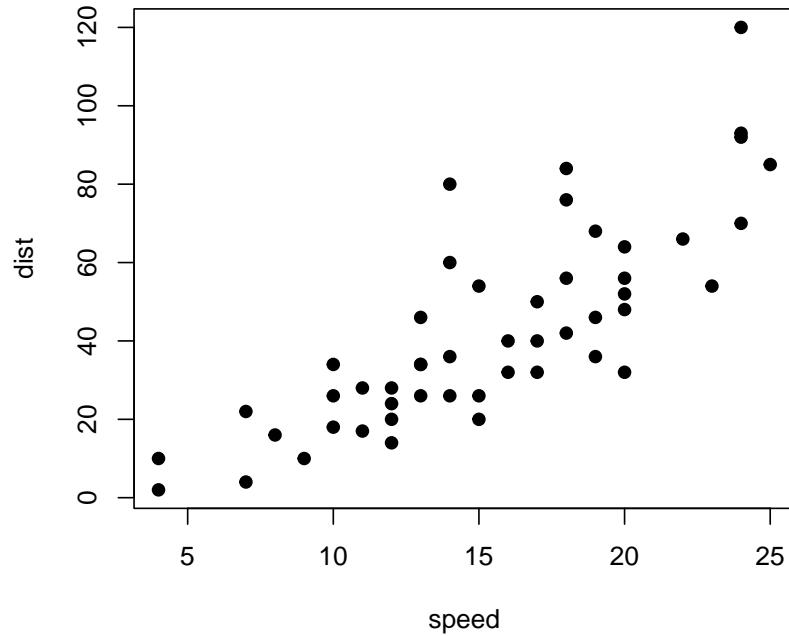
The figure on the left-hand side shows the `cars` data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

```
:::
```

```
:::::::
```

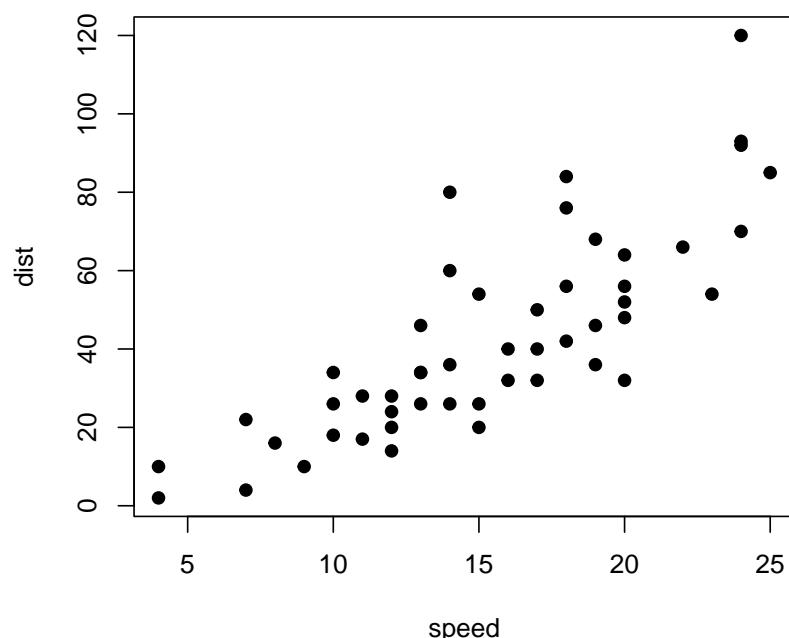
```
{r, echo=FALSE, fig.width=5, fig.height=4}
```



The figure on the left-hand side shows the cars data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

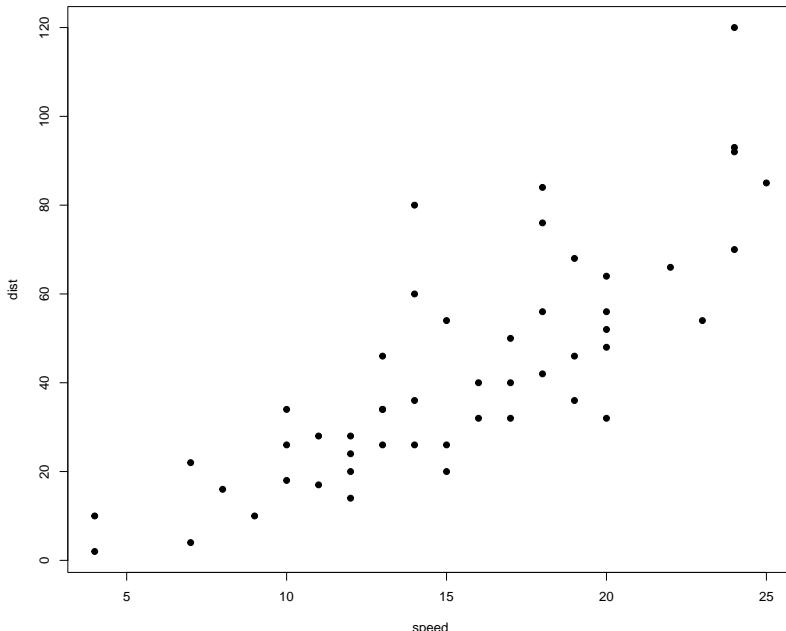
```
{r, echo=FALSE, fig.width=10, fig.height=8, out.width = "100%"}
```



The figure on the left-hand side shows the cars data.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur.

```
{r, echo=FALSE, fig.width=10, fig.height=8}
```



### 7.3.6.2 multi-column fig.cap must use fig.pos="H"

<https://community.rstudio.com/t/adding-fig-cap-caption-text-to-code-chunk-causes-figure-to-print-at-top-of-page-instead-of-where-it-should-be/30297>

<https://bookdown.org/yihui/rmarkdown-cookbook/figure-placement.html>

to avoid LaTeX Error: Not in outer par mode for caption in multi-column LaTeX PDF

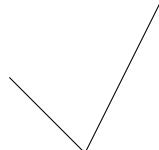
in output.yml add extra\_dependencies: ["float"] under bookdown::pdf\_book:

include first chunk knitr::opts\_chunk\$set(fig.pos = "H", out.extra = "") in .Rmd

add out.width=if (knitr:::is\_html\_output()) '50%' for TikZ chunk

thus complete chunk beginning with {r, echo=FALSE, cache=TRUE, engine='tikz', fig.ext=if (knitr:::is\_latex\_output('pdf') else 'png', fig.width=10, fig.height=2, out.width=if (knitr:::is\_html\_output()) '100%', fig.cap='')

```
\begin{tikzpicture}
 \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



LaTeX package caption

<https://tex.stackexchange.com/questions/128485/how-to-make-a-caption-via-captionof-and-extra-margins-adhere-to-minipage-marg>

What is the different between using \captionof{table}{ABC} and \caption{ABC}?

<https://tex.stackexchange.com/questions/514286/what-is-the-different-between-using-captionoftableabc-and-captionabc-side-by-side-table>

<https://stackoverflow.com/questions/73745714/how-to-print-gt-tbl-tables-side-by-side-with-knitr-kable>

R ternary operator

<https://stackoverflow.com/questions/8790143/does-the-ternary-operator-exist-in-r>

### 7.3.6.3 caption above figure

<https://stackoverflow.com/questions/56979022/caption-above-figure-in-html-rmarkdown>

fig.topcaption=TRUE

### 7.3.6.4 for only HTML

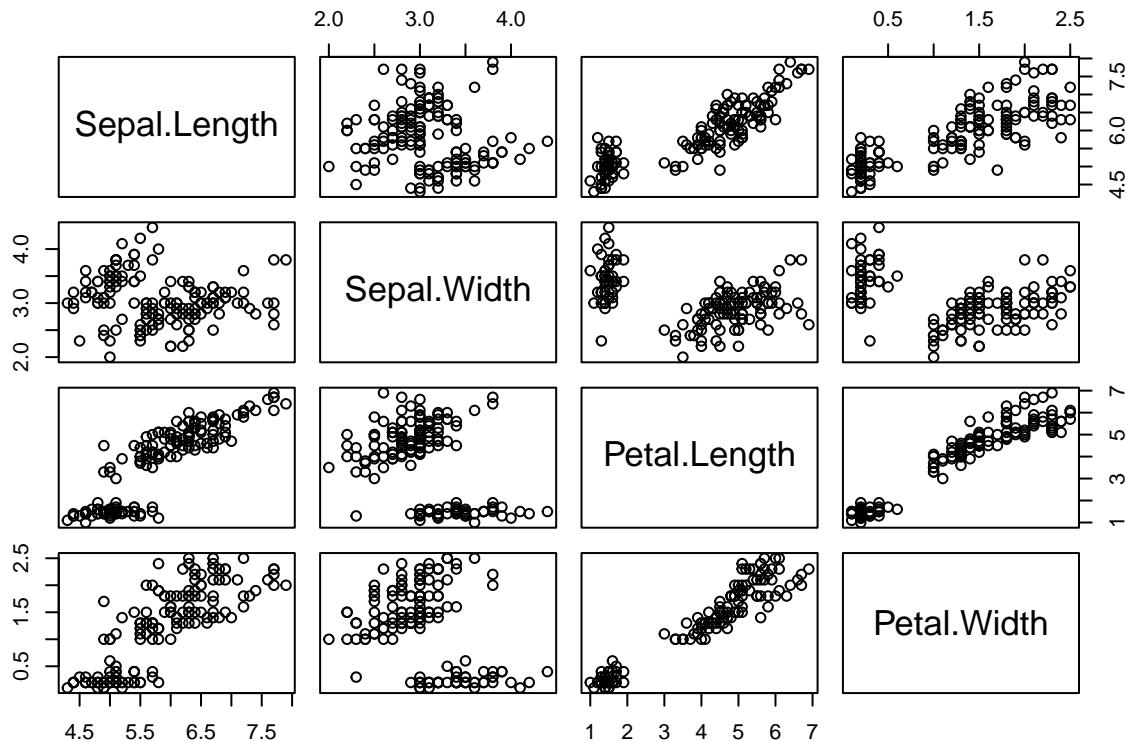
7.3.6.4.1 CSS flex Here is the first Div.

```
str(iris)
```

```
'data.frame': 150 obs. of 5 variables:
$ Sepal.Length: num 5.1 4.9 4.7 4.6 5 5.4 4.6 5 4.4 4.9 ...
$ Sepal.Width : num 3.5 3 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 ...
$ Petal.Length: num 1.4 1.4 1.3 1.5 1.4 1.7 1.4 1.5 1.4 1.5 ...
$ Petal.Width : num 0.2 0.2 0.2 0.2 0.2 0.4 0.3 0.2 0.2 0.1 ...
$ Species : Factor w/ 3 levels "setosa","versicolor",...: 1 1 1 1 1 1 1 1 1 1 ...
```

And this block will be put on the right:

```
plot(iris[, -5])
```



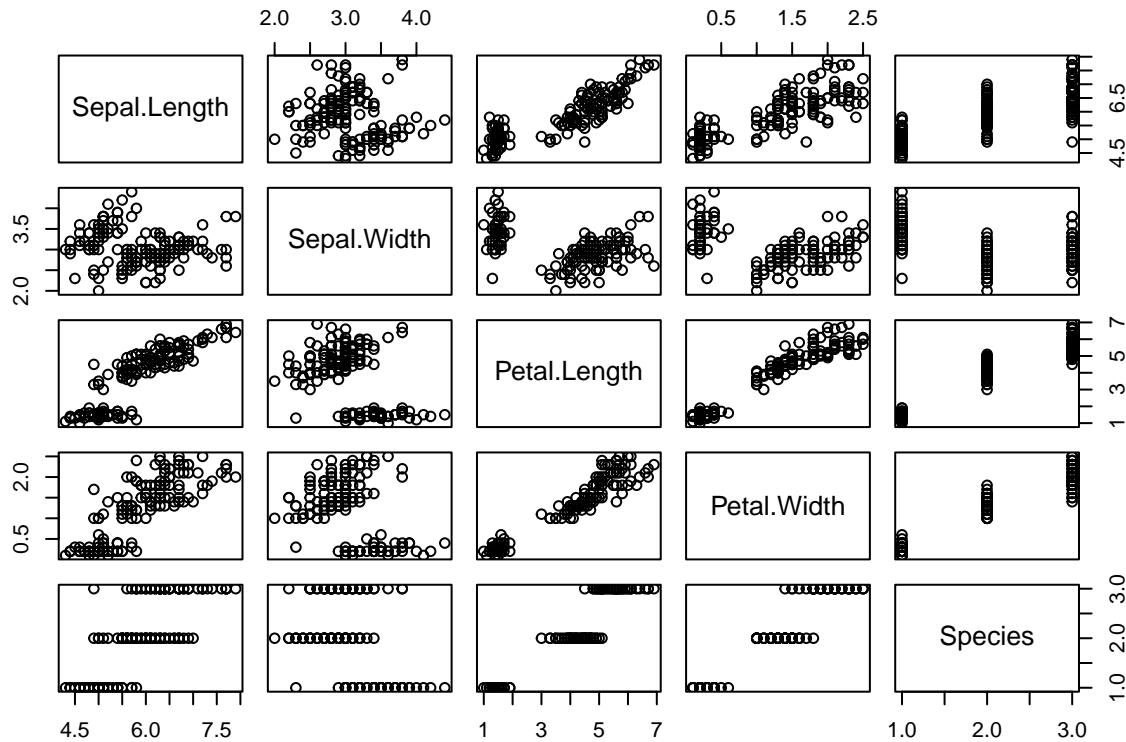
#### 7.3.6.4.2 CSS grid <https://github.com/yihui/knitr/issues/1743>

<https://medium.com/enjoy-life-enjoy-coding/css-所以我说那个版能不能好切一点-grid-基本用法-cd763091cf70>

```
head(iris)
```

```
Sepal.Length Sepal.Width Petal.Length Petal.Width Species
1 5.1 3.5 1.4 0.2 setosa
2 4.9 3.0 1.4 0.2 setosa
3 4.7 3.2 1.3 0.2 setosa
4 4.6 3.1 1.5 0.2 setosa
5 5.0 3.6 1.4 0.2 setosa
6 5.4 3.9 1.7 0.4 setosa
```

```
plot(iris)
```



## 7.4 conditional block/chunk for either HTML or PDF, and Chinese issue

<https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html>  
等價關係 equivalence relation

$R$  is an equivalence relation over  $A \times B$

$$\Leftrightarrow \begin{cases} R = \sim = \{(x, y) | x \sim y\} \subseteq A \times B & (\text{e}) \text{ equivalence 等價} \\ \vdots & \vdots \\ R = \{(x, y) | xRy\} \subseteq A \times B & (\text{R}) \text{ relation} \\ \forall (x, y) \in R (xRx) & (\text{r}) \text{ reflexive} \\ \forall (x, y) \in R (xRy \Rightarrow yRx) & (\text{s}) \text{ symmetric} \\ \forall (x, y), (y, z) \in R \left( \begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (\text{t}) \text{ transitive} \end{cases} \Leftrightarrow \begin{cases} R = \{(x, y) | xRy\} \subseteq A \times B & \text{關係} \\ \forall (x, y) \in R ((x, x) \in R) & \text{自反} \\ \forall (x, y) \in R ((y, x) \in R) & \text{對稱} \\ \forall (x, y), (y, z) \in R ((x, z) \in R) & \text{遞移} \end{cases}$$

## 7.5 video embedding

<https://stackoverflow.com/questions/42543206/r-markdown-compile-error>

```
always_allow_html: true
```

```
install.packages("webshot")
webshot::install_phantomjs()
```

however webshot not work

Error: cannot find bilibili.com

<https://cran.r-project.org/web/packages/vembedr/vignettes/embed.html>

```
Warning: package 'vembedr' was built under R version 4.2.3
```

```
embed_youtube("OLFg5dvP0oc")
```

### 7.5.1 timestamp

- YouTube: <https://www.youtube.com/embed/%7BvideoID%7D?start=%7Bsecond%7D>
- BiliBili: <https://player.bilibili.com/player.html?bvid=%7BvideoID%7D&autoplay=0&t=%7Bsecond%7D>

## 7.6 equation term coloring

LaTeX annotation by TikZ<sup>[62]</sup>

### 7.6.1 font color

RegEx replacement in RStudio for `\color{(\w+)}` in LyX to be replaced with `\color{$1}{}` in HTML document, and remain the same for PDF document

In HTML document, if no `{}` for text range, only the first following term will take effect

`\color{orange}x=y`

*x = y*

`\color{orange}` and `\color{cyan}` are better color for HTML GitBook White and Night themes and PDF

`\color{cyan}{x=y}`

*x = y*

`\color{cyan}{x=y}`

*x = y*

```
:::: {show-in="html"}

$$
\frac{\colorbox{#FFD1DC}{\epsilon^2}}{\left(y_{\{\{\scriptstyle F\}}}-y_{\{\{\scriptstyle L\}}}\right)^2}\{1-\epsilon^2\}
$$

::::

::: {show-in="pdf"}

$$
\frac{\text{\colorbox{red!50}{\epsilon^2}}}{\left(y_{\{\{\scriptstyle F\}}}-y_{\{\{\scriptstyle L\}}}\right)^2}\{1-\epsilon^2\}
$$

::::
```

### 7.6.2 background color

<https://bookdown.org/yihui/rmarkdown-cookbook/font-color.html>

LaTeX color

<https://latexcolor.com/>

[https://www.overleaf.com/learn/latex/Using\\_colors\\_in\\_LaTeX](https://www.overleaf.com/learn/latex/Using_colors_in_LaTeX)

<https://latex-tutorial.com/color-latex/#:~:text=To%20summarize%2C%20pyellow!50efined%20colors%20in,when%20loading%20the>

LaTeX color methods

color frame

<https://tex.stackexchange.com/questions/582748/highlight-equation-with-boxes-and-arrows>

color box

<https://tex.stackexchange.com/questions/567739/how-to-move-and-size-colorbox>

color box with round corners

<https://tex.stackexchange.com/questions/568880/color-box-with-rounded-corners>

highlighting

<https://tex.stackexchange.com/questions/318991/highlighting-math>

<https://forum.remnote.io/t/highlighting-latex-formulas/149>

LyX

<https://tex.stackexchange.com/questions/250069/create-a-color-box> <https://latexlyx.blogspot.com/2013/12/lyx.html>

<https://tex.stackexchange.com/questions/635486/prevent-lyx-from-escaping-math-in-color-box-title>

Bookdown - conditional display of text and code blocks (LaTeX/PDF vs. HTML) <https://stackoverflow.com/questions/76240244/bookdown-conditional-display-of-text-and-code-blocks-latex-pdf-vs-html>

$$F = ma$$

<https://community.rstudio.com/t/highlighting-text-inline-in-rmarkdown-or-bookdown-pdf/35118/4>

$$F = ma$$

$$F = F$$

$$F = ma \quad (7.1)$$

$$F = ma$$

$$Y = \beta_0 + \beta_1 X_1 + \dots + \beta_n X_n$$

## 7.7 link and reference

<https://stackoverflow.com/questions/57469501/cross-referencing-bookdownhtml-document2-not-working>

$$E = mc^2 \quad (7.2)$$

\@ref(nice-label) 7.8

[link to partition] [partition] link to partition

[partition] \@ref(partition)

partition [#partition] (9) @ref(#partition)

[equivalence class] \@ref(equivalence-class)

equivalence class (10)

equivalence class [#equivalence class] (@ref(equivalence class)) @ref(#equivalence class)

[equivalence-class] [#equivalence-class] (10) @ref(#equivalence-class)

X [equivalence-class.html] [equivalence-class.html#equivalence-class] (@ref(equivalence-class.html)) @ref(equivalence-class.html#equivalence-class)

**equivalence relation** [#equivalence relation] (@ref(equivalence relation)) @ref(#equivalence relation)

[equivalence-relation] [#equivalence-relation] (11) @ref(#equivalence-relation)

X [equivalence-relation.html] [equivalence-relation.html#equivalence-relation] (@ref(equivalence-relation.html)) @ref(equivalence-relation.html#equivalence-relation)

## 7.8 number and reference equations

<https://stackoverflow.com/questions/71595882/rstudio-error-in-windows-running-pdflatex-exe-on-file-name-tex-exit-code-10>

<https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#equations>

\#eq:emc \@ref(eq:emc)

<https://stackoverflow.com/questions/55923290/consistent-math-equation-numbering-in-bookdown-across-pdf-docx-html-output>

$C$  is an equivalence class of  $a$  on  $A$

$$\Leftrightarrow [a]_\sim = C = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{array} \right. \right\} \subseteq A \neq \emptyset \quad (7.3)$$

$$\Leftrightarrow [a] = [a]_\sim = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{array} \right. \right\} \subseteq A \neq \emptyset$$

$$\Rightarrow [a]_\sim = \{x | x \sim a\} \subseteq A \neq \emptyset$$

<https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html#cross-referencing>

This cross reference is the Fig. 7.4

<https://stackoverflow.com/questions/51595939/bookdown-cross-reference-figure-in-another-file>

I ran into the same issue and came up with this solution if you aim at compiling 2 different pdfs. It relies on LaTeX's `xr` package for cross references: <https://stackoverflow.com/a/52532269/576684>

## 7.9 footnote

```
noun5 [This is a footnote]

noun[^202401260000-test-cross-link-1]

[^202401260000-test-cross-link-1]: This is a footnote.
```

noun<sup>5</sup>

## 7.10 citation

<https://stackoverflow.com/questions/48965247/use-csl-file-for-pdf-output-in-bookdown/49145699#49145699>

citation 1<sup>3</sup> citation 2<sup>3</sup>

citation 3<sup>4</sup> citation 4<sup>4</sup>

---

<sup>5</sup>This is a footnote.

### 7.10.1 citation in fig.cap

<https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-rmarkdown>

(ref:rudolph) \*nice\* cite: [@Lam94].

(ref:campbell1963) \*nice\* cite: [@campbell1963].

(ref:campbell1963) ([@campbell1963]

(ref:campbell1963) \ [ @campbell1963]

|                                                                                              | Sources of Invalidity |          |         |                 |            |           |           |                                               |                              |                                |                       |                         |
|----------------------------------------------------------------------------------------------|-----------------------|----------|---------|-----------------|------------|-----------|-----------|-----------------------------------------------|------------------------------|--------------------------------|-----------------------|-------------------------|
|                                                                                              | Internal              |          |         |                 | External   |           |           |                                               |                              |                                |                       |                         |
|                                                                                              | History               | Maturity | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| <i>Pre-Experimental Designs:</i>                                                             |                       |          |         |                 |            |           |           |                                               |                              |                                |                       |                         |
| 1. One-Shot Case Study<br><i>X O</i>                                                         | —                     | —        |         |                 | —          | —         |           |                                               | —                            |                                |                       |                         |
| 2. One-Group Pretest-Posttest Design<br><i>O X O</i>                                         | —                     | —        | —       | —               | ?          | +         | +         | —                                             | —                            | —                              | ?                     |                         |
| 3. Static-Group Comparison<br><i>X O</i><br><i>O</i>                                         | +                     | ?        | +       | +               | +          | —         | —         | —                                             | —                            |                                |                       |                         |
| <i>True Experimental Designs:</i>                                                            |                       |          |         |                 |            |           |           |                                               |                              |                                |                       |                         |
| 4. Pretest-Posttest Control Group Design<br><i>R O X O</i><br><i>R O O</i>                   | +                     | +        | +       | +               | +          | +         | +         | +                                             | —                            | ?                              | ?                     |                         |
| 5. Solomon Four-Group Design<br><i>R O X O</i><br><i>R O O</i><br><i>R X O</i><br><i>R O</i> | +                     | +        | +       | +               | +          | +         | +         | +                                             | +                            | ?                              | ?                     |                         |
| 6. Posttest-Only Control Group Design<br><i>R X O</i><br><i>R O</i>                          | +                     | +        | +       | +               | +          | +         | +         | +                                             | +                            | ?                              | ?                     |                         |

Figure 7.3: pre- and true experimental designs ( <sup>5</sup> p.8)

### 7.10.2 backreference

<https://community.rstudio.com/t/how-to-create-a-backreference-to-place-of-citation-in-rmarkdown/84866>

<https://blog.csdn.net/RobertChenGuangzhi/article/details/50455429>

<https://latex.org/forum/viewtopic.php?t=3722>

## 7.11 environment for definition, theorem, proof

<https://bookdown.org/yihui/rmarkdown/bookdown-markdown.html>

<https://github.com/rstudio/rstudio/issues/5264>

@howthebodyworks Ideally, previews of such equations should also work inside a theorem, although I could survive without that.

<https://github.com/rstudio/rstudio/issues/8773>

**Theorem 7.1** (Theorem Name). *Here is my theorem.*

*Proof Name.* Here is my proof. □

**Theorem 7.2** (Pythagorean theorem). *For a right triangle, if c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, we have*

$$a^2 + b^2 \stackrel{7.2}{=} c^2$$

**Definition 7.1** (Definition Name). Here is my definition.

number and reference equations

(7.3)

(7.2)

7.2

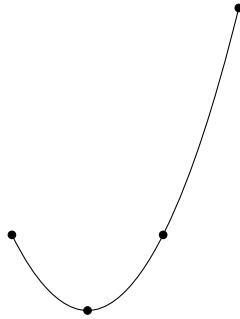


Figure 7.4: parabola arc with points

### 7.11.1 collapsible panel or HTML component

<https://stackoverflow.com/questions/52576626/rmarkdown-collapsible-panel>

**Theorem 7.3.** *This is the description of a theorem*

$$c^2 = a^2 + b^2$$

Proof:

$$c^2 = a^2 + b^2$$

□

#### 7.11.1.1 nested collapsible panel or HTML component

## 7.12 slide or presentation

### 7.12.1 Xaringan and Infinite Moon Reader

<https://rpubs.com/RW1304/xaringan-zh>

<https://slides.yihui.org/xaringan/#1>

<https://slides.yihui.org/xaringan/zh-CN.html#1>

<https://github.com/yihui/xaringan/tree/master>

<https://bookdown.org/yihui/rmarkdown/xaringan.html>

### 7.12.2 ioslides

<https://www.youtube.com/watch?v=gkyjTepCITM>

<https://bookdown.org/yihui/rmarkdown/ioslides-presentation.html>

<https://stackoverflow.com/questions/63749683/how-to-set-up-theorem-environment-in-the-rmarkdown-presentation>

```

```

```
title: "Theorem demo"
output:
 ioslides_presentation:
 css: style.css

```

```
/* theorem environment _ plain */

/*
.theorem {
 display: block;
 font-style: italic;
 font-size: 24px;
 font-family: "Times New Roman";
 color: black;
}
.theorem::before {
 content: "Theorem. ";
 font-weight: bold;
 font-style: normal;
}
.theorem[text]::before {
 content: "Theorem (" attr(text) ") ";
}
.theorem p {
 display: inline;
}
*/
/* theorem environment _ Copenhagen style */

/*
.theorem {
 display: block;
 font-style: italic;
 font-size: 24px;
 font-family: "Times New Roman";
 color: black;
 border-radius: 10px;
 background-color: rgb(222,222,231);
 box-shadow: 5px 10px 8px #888888;
}
.theorem::before {
 content: "Theorem. ";
 font-weight: bold;
 font-style: normal;
 display: inline-block;
 width: -webkit-fill-available;
 color: white;
 border-radius: 10px 10px 0 0;
 padding: 10px 5px 5px 15px;
 background-color: rgb(38, 38, 134);
}
.theorem p {
 padding: 15px 15px 15px 15px;
}
*/

```

### 7.12.3 PowerPoint

<https://bookdown.org/yihui/rmarkdown/powerpoint-presentation.html>



# Chapter 8

## test2

[\newcommand](https://www.overleaf.com/learn/latex/Commands)

[\renewcommand](https://www.overleaf.com/learn/latex/Commands) to replace existing command

<https://www.physicsread.com/latex-newcommand/>

font color<sup>[7.6.1]</sup>

$$\sin(\alpha), \sin^n(\beta), \sin^m(\gamma)$$

$$\cos(2\theta) - \sin(2\theta) = \cos(4\theta)$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

[\regex\\_replace\\_all](https://www.alanshawn.com/latex3-tutorial/#latex3-regular-expression-xxviii)

[\documentclass{article}](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

[\usepackage{tikz,xparse}](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

[\ExplSyntaxOn](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

[\tl\\_new:N \l\\_bob\\_func\\_tl](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

[\NewDocumentCommand \makeline { m }](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

```
{\pgfextra
\tl_set:Nn \l_bob_func_tl { #1 }
\regex_replace_all:nnN { ([0-9])- } { \1)--(} \l_bob_func_tl
\regex_replace_all:nnN { : } { , } \l_bob_func_tl
\exp_last_unbraced:NNV % this is not necessary, but I find it cleaner, correct me if I'm wrong
\endpgfextra
(\l_bob_func_tl)
}
```

[\ExplSyntaxOff](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

[\begin{document}](https://tex.stackexchange.com/questions/422631/expl3-regex-and-missing-semicolon)

```
% \makeline{segment}{1:1--1:2-2:2} % do not use outside `tikz` or `tikzpicture`
```

```
\tikz\draw\makeline{segment}{1:1--1:2-2:2};
```

```
\end{document}
```

<https://stackoverflow.com/questions/41655383/r-markdown-similar-feature-to-newcommand-in-latex>

$\text{Var}(X)$

$$\begin{aligned}\text{Var}[Y] &= x \\ &= 3\end{aligned}$$

$$0 = \frac{\partial}{\partial z_l} (\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda \|h(z_l) \cdot w_{l+1} - z_{l+1}\|)$$

<https://tex.stackexchange.com/questions/353114/latex-equations-colour-all-instances-of-symbol>

$$0 = \frac{\partial}{\partial z_l} (\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda \|h(z_l) \cdot w_{l+1} - z_{l+1}\|)$$

---


$$0 = \frac{\partial}{\partial z_l} (\|h(z_{l-1}) \cdot w_l - z_l\| + \lambda \|h(z_l) \cdot w_{l+1} - z_{l+1}\|)$$


---



---

$$\begin{aligned}& (y_F^2 - \epsilon^2 y_L^2) (1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2 \\&= (1 - \epsilon^2) y_F^2 - (\epsilon^2 - \epsilon^4) y_L^2 - y_F^2 + 2\epsilon^2 y_F y_L - \epsilon^4 y_L^2 \\&= -\epsilon^2 y_F^2 - \epsilon^2 y_L^2 + 2\epsilon^2 y_F y_L = -\epsilon^2 (y_F - y_L)^2\end{aligned}$$

# Chapter 9

## partition

$$\begin{aligned} \{A_i\}_{i \in I} = \{A_i | i \in I\} \text{ is a partition of a set } A \\ \Leftrightarrow \begin{cases} \forall i \in I (A_i \neq \emptyset) \\ A = \bigcup_{i \in I} A_i \\ \forall i, j \in I (i \neq j \Rightarrow A_i \cap A_j = \emptyset) \end{cases} \end{aligned}$$

[https://proofwiki.org/wiki/Definition:Set\\_Partition](https://proofwiki.org/wiki/Definition:Set_Partition)



# Chapter 10

## equivalence class

$C$  is an equivalence class of  $a$  on  $A$

$$\Leftrightarrow [a]_{\sim} = C = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation over } A \times A = A^2 \end{array} \right. \right\} \subseteq A \neq \emptyset$$
$$\Leftrightarrow [a] = [a]_{\sim} = \left\{ x \left| \begin{array}{l} a \in A \\ x \in A \\ x \sim a \\ \sim \text{ is an equivalence relation on } A \end{array} \right. \right\} \subseteq A \neq \emptyset$$
$$\Rightarrow [a]_{\sim} = \{x | x \sim a\} \subseteq A \neq \emptyset$$

where the definition of equivalence relation can be found in 11.



# Chapter 11

## equivalence relation

等價關係 equivalence relation

$R$  is an equivalence relation over  $A \times B$

$$\Leftrightarrow \begin{cases} R = \sim = \{(x, y) | x \sim y\} \subseteq A \times B & (\text{e}) \text{ equivalence 等價} \\ \vdots & \end{cases}$$
$$\Leftrightarrow \begin{cases} R = \{(x, y) | xRy\} \subseteq A \times B & (\text{R}) \text{ relation} \\ \forall (x, y) \in R (xRx) & (\text{r}) \text{ reflexive} \\ \forall (x, y) \in R (xRy \Rightarrow yRx) & (\text{s}) \text{ symmetric} \\ \forall (x, y), (y, z) \in R \left( \begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \right) & (\text{t}) \text{ transitive} \end{cases} \Leftrightarrow \begin{cases} R = \{(x, y) | xRy\} \subseteq A \times B & \text{關係} \\ \forall (x, y) \in R ((x, x) \in R) & \text{自反} \\ \forall (x, y) \in R ((y, x) \in R) & \text{對稱} \\ \forall (x, y), (y, z) \in R ((x, z) \in R) & \text{遞移} \end{cases}$$



# Chapter 12

## Python

### 12.1 using Python in R / RMarkdown

<https://bookdown.org/yihui/rmarkdown/language-engines.html>

```
names(knitr::knit_engines$get())

[1] "awk" "bash" "coffee" "gawk" "groovy"
[6] "haskell" "lein" "mysql" "node" "octave"
[11] "perl" "php" "pgsql" "Rscript" "ruby"
[16] "sas" "scala" "sed" "sh" "stata"
[21] "zsh" "asis" "asy" "block" "block2"
[26] "bslib" "c" "cat" "cc" "comment"
[31] "css" "dittaa" "dot" "embed" "evviews"
[36] "exec" "fortran" "fortran95" "go" "highlight"
[41] "js" "julia" "python" "R" "Rcpp"
[46] "sass" "scss" "sql" "stan" "targets"
[51] "tikz" "verbatim" "theorem" "lemma" "corollary"
[56] "proposition" "conjecture" "definition" "example" "exercise"
[61] "hypothesis" "proof" "remark" "solution" "glue"
[66] "glue_sql" "gluesql"
```

[https://rstudio.github.io/reticulate/articles/python\\_packages.html](https://rstudio.github.io/reticulate/articles/python_packages.html)

```
x = 'hello, python world!'
print(x.split(' '))

['hello,', 'python', 'world!']

library(reticulate)
virtualenv_python()

library(reticulate)
conda_list()

library(reticulate)
virtualenv_list()
```

[https://rstudio.github.io/reticulate/reference/install\\_python.html](https://rstudio.github.io/reticulate/reference/install_python.html)

```
library(reticulate)
version <- "3.9.12"
install_python(version)

create a new environment
virtualenv_create("r-reticulate", version = version)

use_virtualenv("r-reticulate")

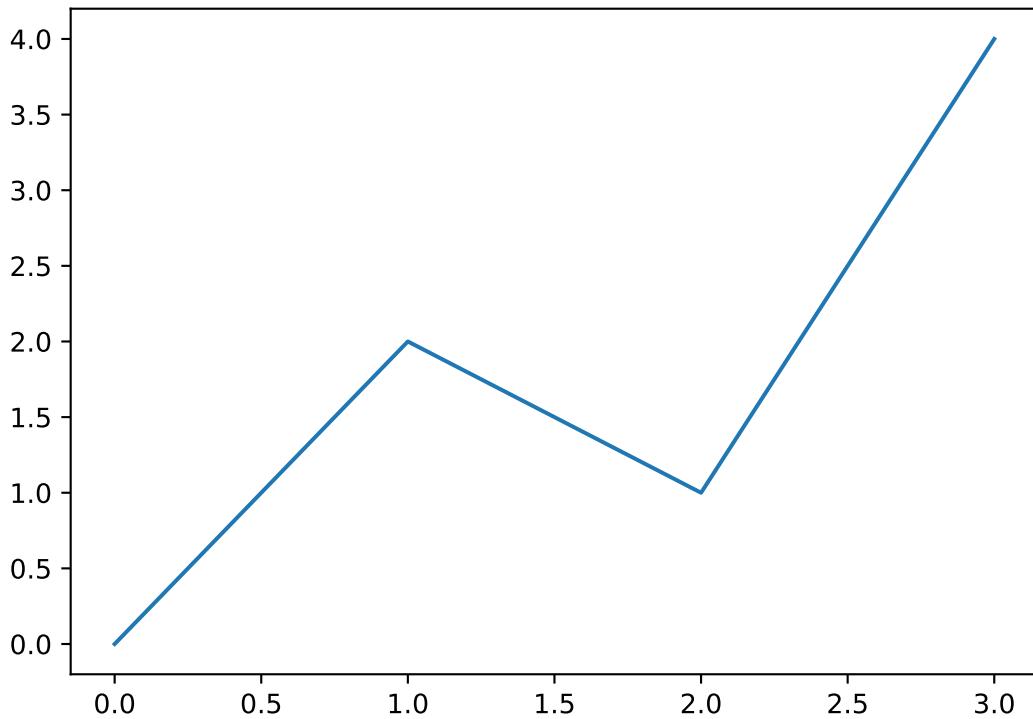
install Matplotlib
```

```
virtualenv_install("r-reticulate", "matplotlib")
import Matplotlib (it will be automatically discovered in "r-reticulate")
matplotlib <- import("matplotlib")
```

```
copy C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl\tcl8.6 and
C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\tcl\tk8.6 two folders
to the folder C:\Users\RW\AppData\Local\r-reticulate\r-reticulate\pyenv\pyenv-win\versions\3.9.12\Lib
```

```
library(reticulate)
use_virtualenv("r-reticulate")
matplotlib <- import("matplotlib")
matplotlib$use("Agg", force = TRUE)
```

```
import matplotlib.pyplot as plt
plt.plot([0, 2, 1, 4])
plt.show()
```



## 12.2 SoloLearn

<https://www.sololearn.com/>

<https://www.sololearn.com/en/learn/courses/python-intermediate>

## 12.3 list comprehension

<https://www.sololearn.com/en/learn/courses/python-intermediate/lesson/1188906590?p=1>

```
cubes = [i**3 for i in range(5)]
print(cubes) ## [0, 1, 8, 27, 64]
```

## 12.4 functional programming

- pure function
- lambda
- map
- filter
- generator
- decorator
- recursion
- \*args
- \*\*kwargs

## 12.5 object-oriented programming = OOP

- class
- inheritance
- magic method
- operator overloading
- data hiding
- static method
- property



# Chapter 13

## TikZ

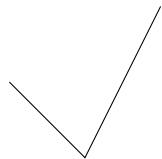
- TikZ
  - PGFplots<sup>[13.5]</sup>
  - tikzplotlib<sup>[13.6]</sup>: Python<sup>[12]</sup> matplotlib<sup>[27]</sup> export to TikZ<sup>[13]</sup> .tex

multi-column 7.3.6

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

<https://bookdown.org/yihui/rmarkdown-cookbook/html-scroll.html>

```
\begin{tikzpicture}
 \draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



How to speed up bookdown generation?

<https://stackoverflow.com/questions/56541371/how-to-speed-up-bookdown-generation>

TikZ and PGFplots

What's the relation between packages PGFplots and TikZ?

<https://tex.stackexchange.com/questions/285925/whats-the-relation-between-packages-pgfplots-and-tikz>

<https://www.youtube.com/watch?v=bQugbYq0BVA>

<https://www.youtube.com/watch?v=ft4Kg9emK1k&list=PLg5nrpKdkk2DWcg3scb75AknF7DJXs8lk&index=18>

```
\begin{tikzpicture}
\def\aa{1.5} % amplitude
\def\bb{2} % frequency
\draw[->] (-0.2,0)--(4.2,0) node[right,
 font=\small] {x};
\draw[->] (0,-4)--(0,0.5) node[above] {y};
\draw[domain=0:4,smooth,variable=\t,blue,thick]
plot ({\aa * (\bb*\t -
sin(deg(\bb*\t)))},{-\aa * (1 -
cos(deg(\bb*\t)))});
% \node[above] at (2, 0.5) {Brachistochrone
% Curve};
\node[above, font=\footnotesize] at (2, 1)
 {Brachistochrone Curve};
\node[above, font=\footnotesize] at (2, 0)
 {\$\\begin{aligned}
&x=r(t-\\sin t) \\
&y=r(1-\\cos t)
\\end{aligned}\$};
\end{tikzpicture}
```

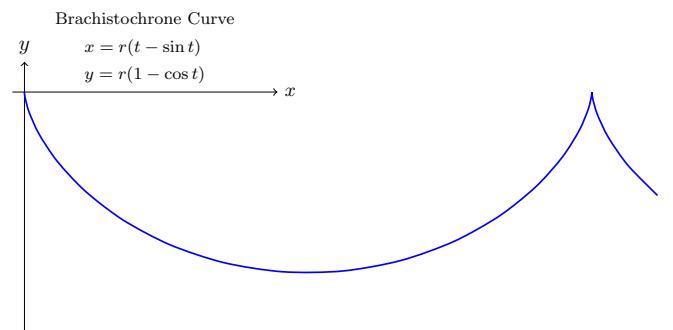


Figure 13.1: Brachistochrone Curve

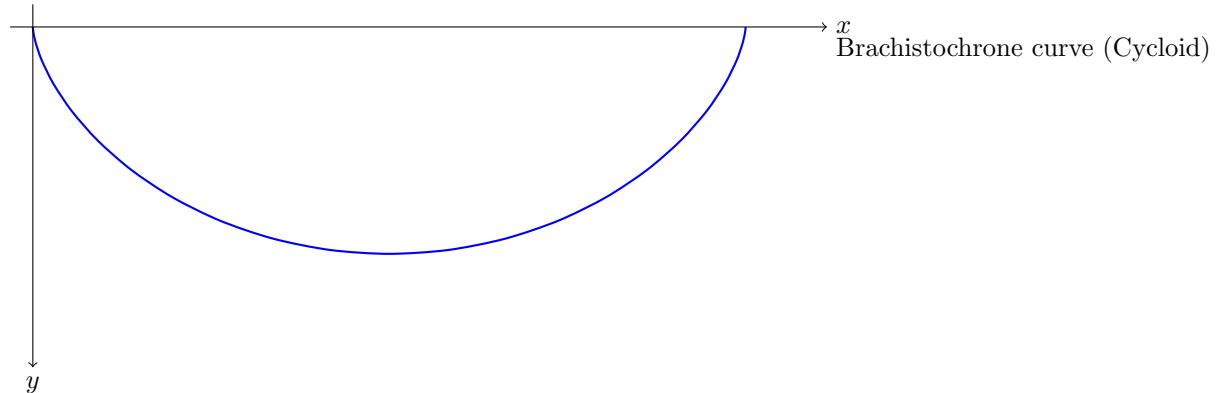


Figure 13.2: Brachistochrone Curve

## 13.1 2D

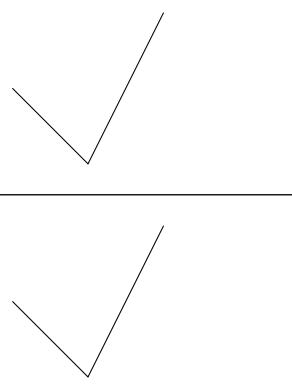
[https://zhuanlan.zhihu.com/p/127155579?utm\\_psn=1741479950987960320](https://zhuanlan.zhihu.com/p/127155579?utm_psn=1741479950987960320)

```
\begin{tikzpicture}
\draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

```
\begin{tikzpicture}
\draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```

```
out.width;if (knitr:::is_html_output()) '20%'

\begin{tikzpicture}
\draw (-1,1)--(0,0)--(1,2);
\end{tikzpicture}
```



```
\begin{tikzpicture}
 \draw[rounded corners]
 -> (-1,1)--(0,0)--(1,2)--(-1,1);
\end{tikzpicture}
```

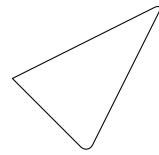


Figure 13.3: rounded corner pseudo-closed triangle

```
\begin{tikzpicture}
 \draw[rounded corners]
 -> (-1,1)--(0,0)--(1,2)--cycle;
\end{tikzpicture}
```

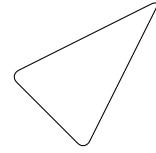


Figure 13.4: rounded corner triangle

```
\begin{tikzpicture}
 \draw[rounded corners]
 -> (-1,1)--(0,0)--(1,2)--cycle;
 \draw[rounded corners]
 -> (-1,1)--(0,0)--(1,2)--(-1,1);
\end{tikzpicture}
```

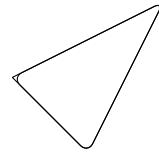


Figure 13.5: triangle vs. pseudo-closed triangle

```
\begin{tikzpicture}
 \draw (0,0) rectangle (4,2);
\end{tikzpicture}
```



Figure 13.6: rectangle

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

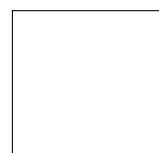


Figure 13.7: square

```
\begin{tikzpicture}
 \draw (0,0) circle (1);
\end{tikzpicture}
```

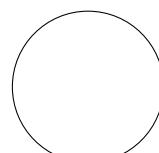


Figure 13.8: circle

```
\begin{tikzpicture}
 \draw (0,0) circle (1);
 \draw (0,0) rectangle (2,2);
\end{tikzpicture}
```

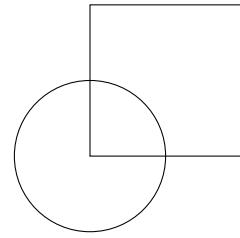


Figure 13.9: circle and square

```
\begin{tikzpicture}
 \draw (1,1) ellipse (2 and 1);
\end{tikzpicture}
```

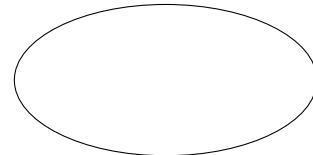


Figure 13.10: ellipse

```
\begin{tikzpicture}
 \draw (1 ,1) arc (0:270:1);
 \draw (6 ,1) arc (0:270:2 and 1);
\end{tikzpicture}
```

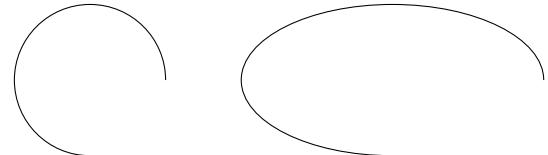


Figure 13.11: circle and ellipse arcs

```
\begin{tikzpicture}
 \draw (-1,1) parabola bend (0,0) (2,4);
\end{tikzpicture}
```

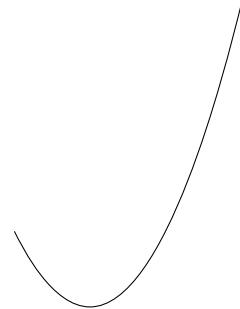


Figure 13.12: parabola arc

```
\begin{tikzpicture}
 \draw (-1,1) parabola bend (0,0) (2,4);
 \filldraw
 (-1,1) circle (.05)
 (0,0) circle (.05)
 (1,1) circle (.05)
 (2,4) circle (.05);
\end{tikzpicture}
```

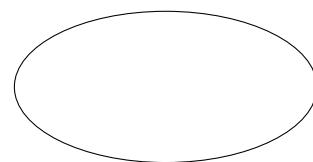


Figure 13.13: parabola arc with points

```
\begin{tikzpicture}
 \draw[step=20pt] (0,0) grid (3,2);
 \draw[help lines ,step=20pt] (4,0) grid
 -> (7,2);
\end{tikzpicture}
```

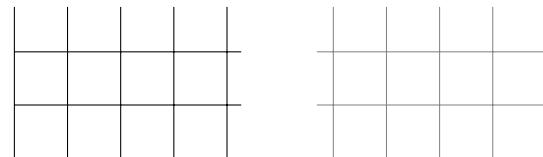


Figure 13.14: grid and help lines

```
\begin{tikzpicture}[scale=0.25]
\draw[->] (0,0)--(9,0);
\draw[<-] (0,1)--(9,1);
\draw[<->] (0,2)--(9,2);
\draw[>->>] (0,3)--(9,3);
\draw[|<-||] (0,4)--(9,4);
\end{tikzpicture}
```

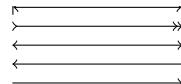


Figure 13.15: arrows

```
\begin{tikzpicture}
\draw[line width =2pt] (0,6)--(9,6);
\draw[dotted] (0,5)--(9,5);
\draw[densely dotted] (0,4)--(9,4);
\draw[loosely dotted] (0,3)--(9,3);
\draw[dashed] (0,2)--(9,2);
\draw[densely dashed] (0,1)--(9,1);
\draw[loosely dashed] (0,0)--(9,0);
\end{tikzpicture}
```



Figure 13.16: lines

```
\begin{tikzpicture}[dline/.style={color= blue,
→ line width=2pt}]
\draw[dline] (0,0)--(9,0);
\end{tikzpicture}
```



Figure 13.17: head styling

```
\begin{tikzpicture}
\draw (0,0) rectangle (2,2);
\draw[shift={(3, 0)}] (0,0) rectangle
→ (2,2);
\draw[shift={(0, 3)}] (0,0) rectangle
→ (2,2);
\draw[shift={(0,-3)}] (0,0) rectangle
→ (2,2);
\draw[shift={(-3, 0)}] (0,0) rectangle
→ (2,2);
\draw[shift={(3, 3)}] (0,0) rectangle
→ (2,2);
\draw[shift={(-3, 3)}] (0,0) rectangle
→ (2,2);
\draw[shift={(3,-3)}] (0,0) rectangle
→ (2,2);
\draw[shift={(-3,-3)}] (0,0) rectangle
→ (2,2);
\end{tikzpicture}
```

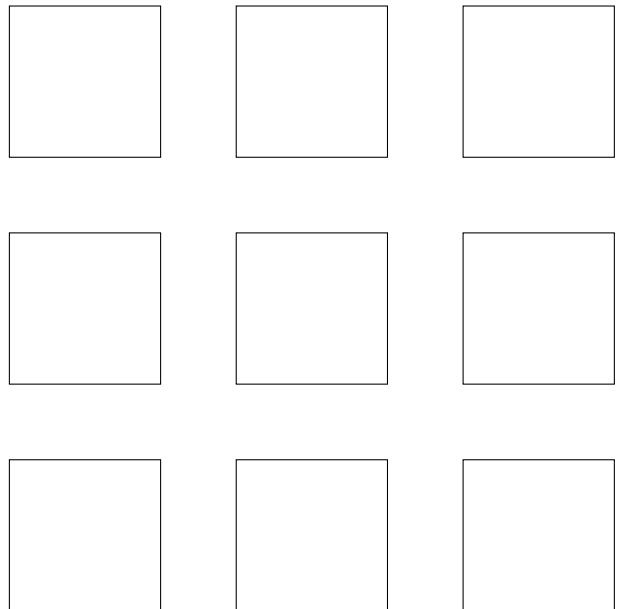


Figure 13.18: transform: shift

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
 \draw[xshift= 100pt] (0,0) rectangle (2,2);
 \draw[xshift=-100pt] (0,0) rectangle (2,2);
 \draw[yshift= 100pt] (0,0) rectangle (2,2);
 \draw[yshift=-100pt] (0,0) rectangle (2,2);
\end{tikzpicture}
```

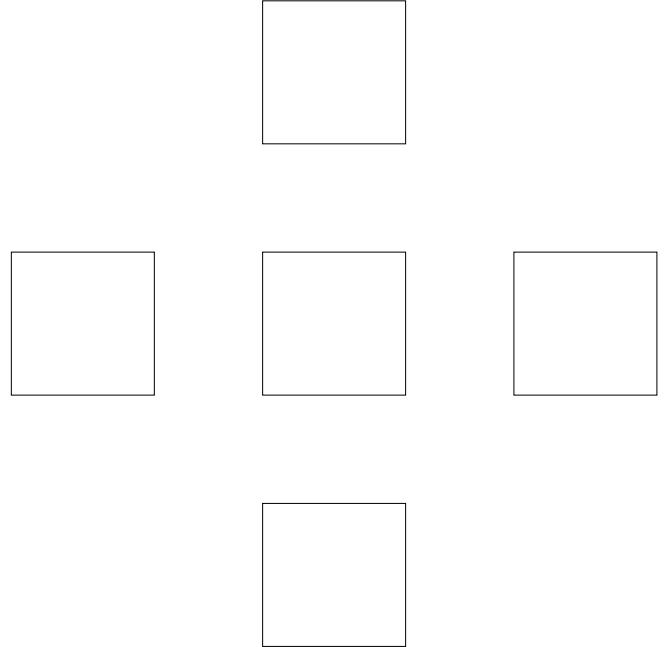


Figure 13.19: transform: shift x, y

---

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
 \draw[xshift= 100pt, xscale=1.5] (0,0)
 -- rectangle (2,2);
 \draw[yshift= 100pt, xscale=0.5] (0,0)
 -- rectangle (2,2);
 \draw[xshift=-100pt, yscale=1.5] (0,0)
 -- rectangle (2,2);
 \draw[yshift=-100pt, yscale=0.5] (0,0)
 -- rectangle (2,2);
\end{tikzpicture}
```

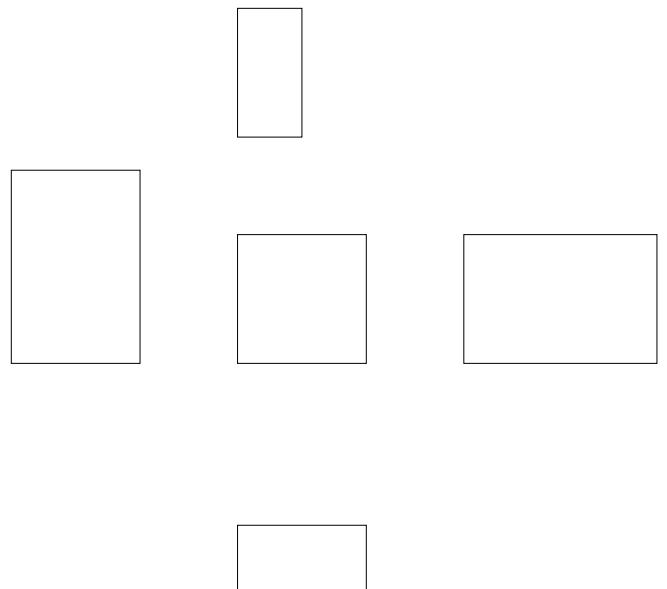


Figure 13.20: transform: scale x, y

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
 \draw[xshift= 100pt, xscale=1.5] (0,0)
 -- rectangle (2,2);
 \draw[yshift= 100pt, yscale=1.5] (0,0)
 -- rectangle (2,2);
 \draw[xshift=-100pt, xscale=0.5] (0,0)
 -- rectangle (2,2);
 \draw[yshift=-100pt, yscale=0.5] (0,0)
 -- rectangle (2,2);
\end{tikzpicture}
```

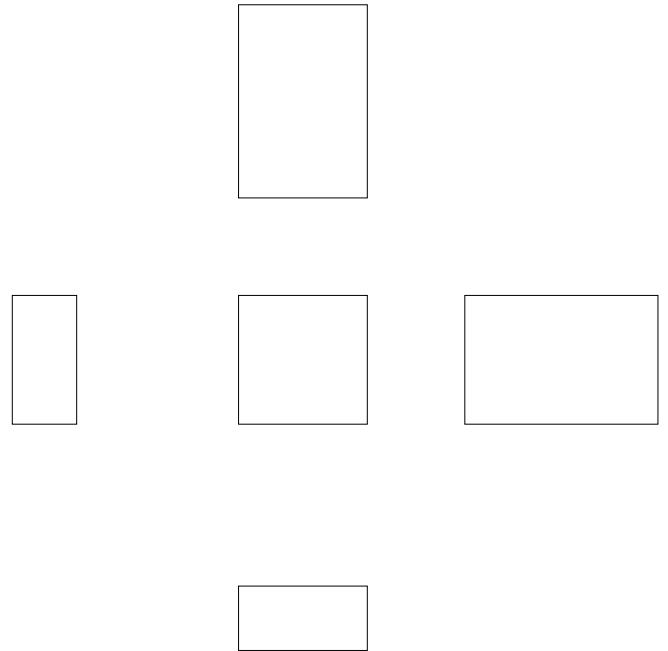


Figure 13.21: transform: scale

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
 \draw[xshift=125pt,rotate=45] (0,0)
 -- rectangle (2,2);
 \draw[xshift=175pt,rotate around={45:(2
 ,2)}] (0,0) rectangle (2,2);
\end{tikzpicture}
```

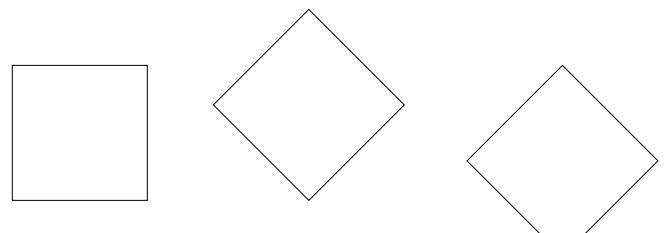


Figure 13.22: transform: rotate

```
\begin{tikzpicture}
 \draw (0,0) rectangle (2,2);
 \draw[xshift=70pt,xslant=1] (0,0) rectangle
 (2,2);
 \draw[yshift=70pt,yslant=1] (0,0) rectangle
 (2,2);
\end{tikzpicture}
```

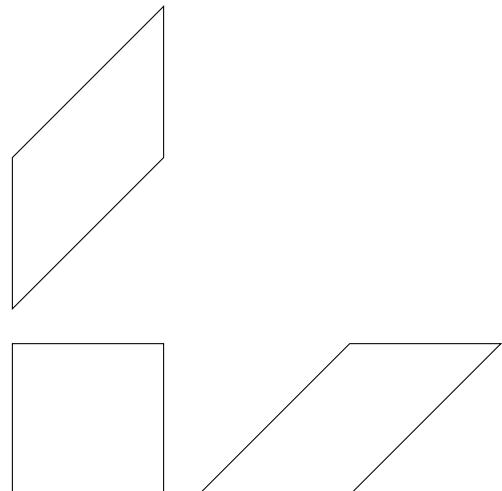


Figure 13.23: transform: slant

```
\tikzset{
 box/.style={
 draw=blue,
 rectangle,
 rounded corners=5pt,
 minimum width=50pt,
 minimum height=20pt,
 inner sep=5pt
 }
}
\begin{tikzpicture}
 \node[box] (1) at (0,0) {1};
 \node[box] (2) at (4,0) {2};
 \node[box] (3) at (8,0) {3};
 \draw[->] (1)--(2);
 \draw[->] (2)--(3);
 \node at (2,1) {a};
 \node at (6,1) {b};
\end{tikzpicture}
```

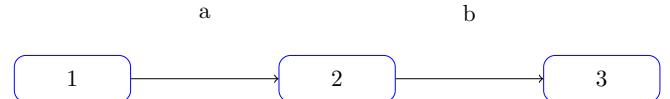


Figure 13.24: flowchart

```
\tikzset{
 box/.style={
 draw=blue,
 fill=blue!20,
 rectangle,
 rounded corners=5pt,
 minimum height=20pt,
 inner sep=5pt
 }
}
\begin{tikzpicture}
 \node[box] {1}
 child {node[box] {2}}
 child {node[box] {3}
 child {node[box] {4}}
 child {node[box] {5}}
 child {node[box] {6}}
 };
\end{tikzpicture}
```

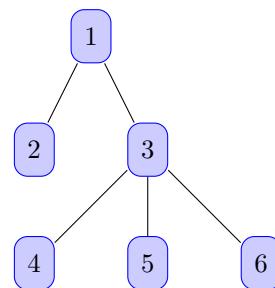


Figure 13.25: tree

```
\begin{tikzpicture}
 \draw[->] (-0.2,0)--(6,0) node[right] {x};
 \draw[->] (0,-0.2)--(0,6) node[above] {$f(x)$};
 \draw[domain=0:4] plot (\x ,{0.1* exp(\x)}) node[right] {$f(x)=\frac{1}{10}e^x$};
\end{tikzpicture}
```

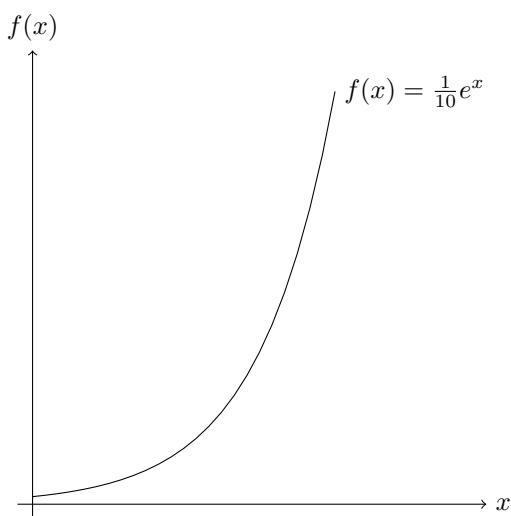


Figure 13.26: function plot

<https://stackoverflow.com/questions/64897575/tikz-libraries-in-bookdown>

It turns out that you can simply put the `\usetikzlibrary{...}` command directly before the `\begin{tikzpicture}` and

everything works fine :)

<https://stackoverflow.com/questions/56211210/r-markdown-document-with-html-docx-output-using-latex-package-bbm>

<https://tex.stackexchange.com/questions/171711/how-to-include-latex-package-in-r-markdown>

## 13.2 3D

[https://zhuanlan.zhihu.com/p/431732330?utm\\_psn=1741857547550638080](https://zhuanlan.zhihu.com/p/431732330?utm_psn=1741857547550638080)

<https://github.com/RRWWW/Stereometry>

```
\begin{tikzpicture}
\coordinate (A) at (1, 1, 1);
\coordinate (B) at (1, 1,-1);
\coordinate (C) at (1,-1,-1);
\coordinate (D) at (1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {A}--(B) node[right=1pt] {B}--(C) node[right=1pt] {C}--(D) node[right=1pt] {D}--(E) node[left= 1pt] {E}--(F) node[right=1pt] {F}--(G) node[right=1pt] {G}--(H) node[left= 1pt] {H}--(A);
\end{tikzpicture}
```

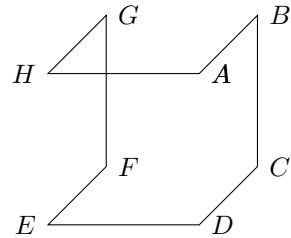


Figure 13.27: cube

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
\tdplotsetmaincoords{45}{45}
\begin{tikzpicture}[tdplot_main_coords]
\coordinate (A) at (1, 1, 1);
\coordinate (B) at (1, 1,-1);
\coordinate (C) at (1,-1,-1);
\coordinate (D) at (1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {A}--(B) node[right=1pt] {B}--(C) node[right=1pt] {C}--(D) node[right=1pt] {D}--(E) node[left= 1pt] {E}--(F) node[right=1pt] {F}--(G) node[right=1pt] {G}--(H) node[left= 1pt] {H}--(A);
\end{tikzpicture}
```

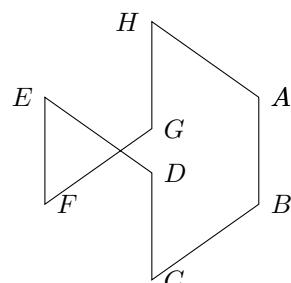


Figure 13.28: cube rotate

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
%\tdplotsetmaincoords{70}{110}
\begin{tikzpicture}[rotate around y=-15,
→ rotate around z=7]
\coordinate (A) at (1, 1, 1);
\coordinate (B) at (1, 1,-1);
\coordinate (C) at (1,-1,-1);
\coordinate (D) at (1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {A}--(B) node[right=1pt] {B}--(C) node[right=1pt] {C}--(D) node[right=1pt] {D}--(E) node[left= 1pt] {E}--(F) node[right=1pt] {F}--(G) node[right=1pt] {G}--(H) node[left= 1pt] {H}--(A) node[right=1pt] {A};
\end{tikzpicture}
```

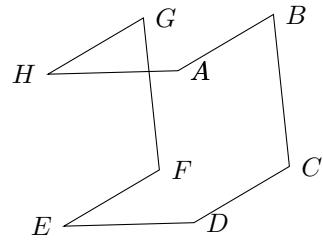


Figure 13.29: cube rotate around

<https://tex.stackexchange.com/questions/388621/optimizing-perspective-tikz-graphic>

```
\usetikzlibrary{patterns}
\usetikzlibrary{3d,calc}
\begin{tikzpicture}[y={(.5cm,.7cm)}]
\coordinate (A) at (1, 1, 1);
\coordinate (B) at (1, 1,-1);
\coordinate (C) at (1,-1,-1);
\coordinate (D) at (1,-1, 1);
\coordinate (E) at (-1,-1, 1);
\coordinate (F) at (-1,-1,-1);
\coordinate (G) at (-1, 1,-1);
\coordinate (H) at (-1, 1, 1);
\draw (A) node[right=1pt] {A}--(B) node[right=1pt] {B}--(C) node[right=1pt] {C}--(D) node[right=1pt] {D}--(E) node[left= 1pt] {E}--(F) node[right=1pt] {F}--(G) node[right=1pt] {G}--(H) node[left= 1pt] {H}--(A) node[right=1pt] {A};
\end{tikzpicture}
```

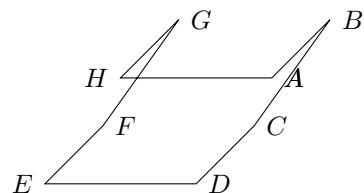


Figure 13.30: cube perspective slant

<https://github.com/XiangyunHuang/bookdown-broken/blob/master/index.Rmd>

```
\smartdiagramset{planet color=gray!40!white,
uniform color list=gray!40!white for 10 items}
\smartdiagram[bubble diagram]{Basic skills,
Edit~/\\" (RStudio),
Organize~/\\" (bookdown),
Cooperate~/\\" (Git),
Typeset~/\\" (LaTeX/Pandoc),
Compile~/\\" (GitHub Action)}
```



Figure 13.31: modern statistics plot skills

### 13.3 plots of functions

<https://tikz.dev/tikz-plots>

A warning before we get started: If you are looking for an easy way to create a normal plot of a function with scientific axes, ignore this section and instead look at the `pgfplots` package or at the `datavisualization` command from Part VI.

<https://tikz.dev/tikz-plots#sec-22.5>

```
\begin{tikzpicture}[domain=0:4]
\draw[very thin,color=gray] (-0.1,-1.1) grid
 (3.9,3.9);
\draw[->] (-0.2,0) -- (4.2,0) node[right]
 {x};
\draw[->] (0,-1.2) -- (0,4.2) node[above]
 {$f(x)$};

\draw[color=red] plot (\x,\x)
 node[right] {$f(x) = x$};
\draw[orange] plot (\x,{0.05*exp(\x)})
 node[right] {$f(x) = \frac{1}{20}e^x$};
\draw[blue] plot (\x,{sin(\x r)})
 node[right] {$f(x) = \sin x$};

\end{tikzpicture}
```

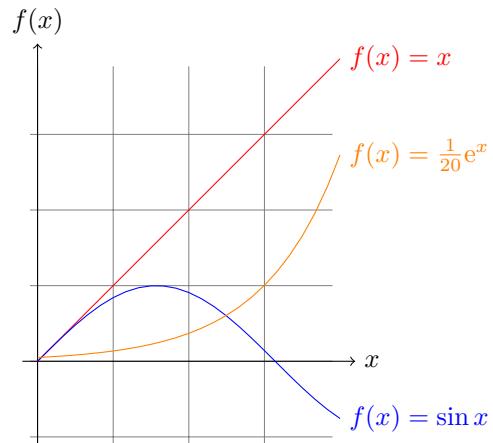


Figure 13.32: plots of functions

```
\tikz
\draw[scale=0.5,domain=-3.141:3.141,smooth,variable=\t]
plot ({\t*sin(\t r)},{\t*cos(\t r)});
```



Figure 13.33: 2D parametric function

```
\tikz \draw [domain=0:360,
 smooth,
 variable=\t]
plot ({sin(\t)},{\t/360},{cos(\t)});
```



Figure 13.34: 3D parametric function

## 13.4 commutative diagram

`cd` = `CD` = commutative diagram

`tikz-cd` = TikZ-CD

`\usetikzlibrary{cd}` vs. `usepackage{tikz-cd}`

<https://tex.stackexchange.com/questions/546392/usepackage{tikz-cd}-vs-\usetikzlibrary{cd}>

No (practical) difference. The code (excluding copyright comment lines) in `tikz-cd.sty` is

```
\ProvidesPackage{tikz-cd}[2014/10/30 v0.9e Commutative diagrams with tikz]
\RequirePackage{tikz}[2013/12/13] % pgf version 3.0.0 required
\usetikzlibrary{cd}

\endinput
```

Never put `\begin{tikzcd}` inside `\begin{tikzpicture}`

<https://tex.stackexchange.com/questions/425296/half-of-tikzcd-diagram-is-missing>

---

```
\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}
11 & 12 & 13 \\
21 & 22 & 23 \\
31 & 32 & 33
\end{tikzcd}
```

|    |    |    |
|----|----|----|
| 11 | 12 | 13 |
| 21 | 22 | 23 |
| 31 | 32 | 33 |

Figure 13.35: learn TikZ-CD or tikz-cd in one picture

---

```
\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}
11 & & 13 \\
& 12 & \\
21 & & 23 \\
& 22 & \\
31 & & 33
\end{tikzcd}
```

|    |    |    |
|----|----|----|
| 11 | 12 | 13 |
| 21 | 22 | 23 |
| 31 | 32 | 33 |

Figure 13.36: TikZ-CD or tikz-cd matrix nodes

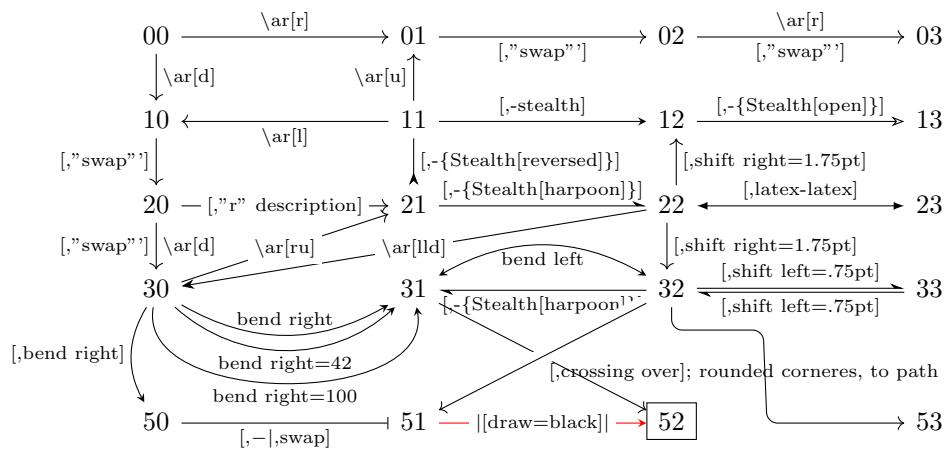


Figure 13.37: TikZ-CD or tikz-cd matrix nodes expanded longitudinally

```

\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}[column
← sep=2.75cm] %small, large, huge
00
\ar[r,"\\backslash \text{ar[r]}"]
\ar[d,"\\backslash \text{ar[d]}"]
&
01
\ar[r,"\\text{[,"swap"]}]"
&
02
\ar[r,"\\backslash
→ \text{ar[r]}","\\text{[,"swap"]}]"
&
03
\\
10
\ar[d,"\\text{[,"swap"]}]"
&
11
\ar[u,"\\backslash \text{ar[u]}"]
\ar[l,"\\backslash \text{ar[l]}"]
\ar[r,-stealth,"\\text{[,-}\text{stealth}\text{]\text{}}"]
\ar[d,-{Stealth[reversed]},"\\text{[,-}\text{\text{Stealth[reversed]}}\text{]\text{}}"]
&
12
\ar[r,-{Stealth[open]},"\\text{[,-}\text{\text{Stealth[open]}}\text{]\text{}}"]
&
13
\\
20
\ar[r,"\\text{[,"r" description}]" description]
\ar[d,"\\backslash
→ \text{ar[d]}","\\text{[,"swap"]}]"
&
21
\ar[r,-{Stealth[harploon]},"\\text{[,-}\text{\text{Stealth[harploon]}}\text{]\text{}}"]
&
22
\ar[u,shift right=1.75pt,"\\text{[,shift
→ right=1.75pt}]"]
\ar[lld,-Stealth,"\\backslash \text{ar[lld]}"
→ description]
\ar[r,latex-latex,"\\text{[,latex-latex}]"]
\ar[d,shift right=1.75pt,"\\text{[,shift
→ right=1.75pt}]"]
&
23
\\
30
\ar[ru,"\\backslash \text{ar[ru]}" description]
\ar[r,bend right,-stealth,"\\text{bend right}"]
\ar[r,bend right=42,-stealth,"\\text{bend
→ right=42}]"]
\ar[r,bend right=100,-stealth,"\\text{bend
→ right=100}]"]
\ar[dd,bend right,-stealth,"\\text{[,,bend
→ right]}"]
&
31
\ar[r,bend left,stealth-stealth,"\\text{bend
→ left}"]
\ar[ddr]
&
32
\ar[l,-{Stealth[harploon]},"\\text{[,-}\text{\text{Stealth[harploon]}}\text{]\text{}}"]
\ar[r,-{Stealth[harploon]} shift

```

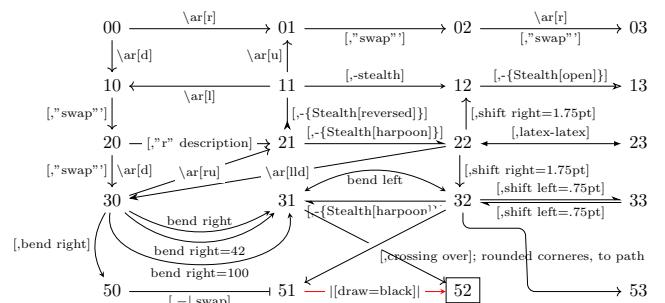


Figure 13.38: learn TikZ-CD or tikz-cd in one picture

<https://tex.stackexchange.com/questions/515267/how-can-i-make-arrows-touch-boxed-expressions-in-tikzcd>

```
cells={nodes={draw}}
```

```
in
```

```
\begin{tikzcd}[column sep=2.75cm, %small,large,huge
 cells={nodes={draw}}
]
```

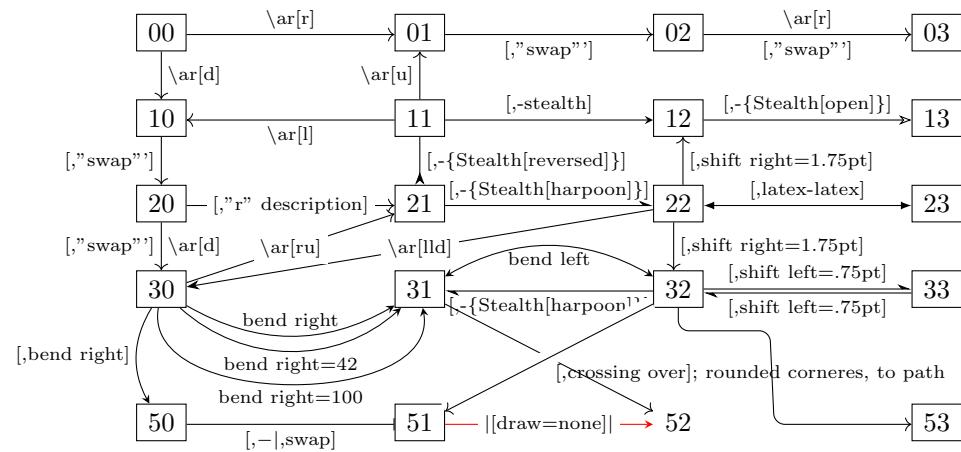


Figure 13.39: learn TikZ-CD or tikz-cd in one picture 2

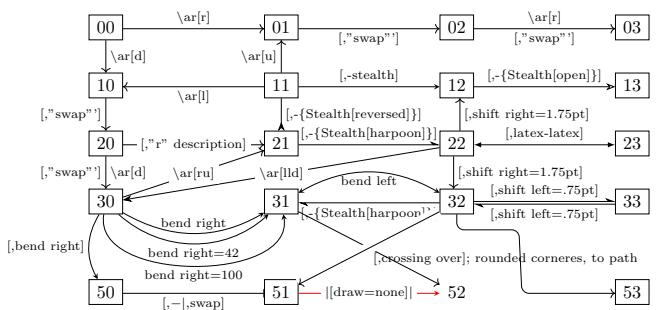


Figure 13.40: learn TikZ-CD or tikz-cd in one picture 2

<https://tex.stackexchange.com/questions/484743/format-single-node-in-tikzcd>

```
\usetikzlibrary{cd,arrows.meta,shapes}
\begin{tikzcd}[cells={nodes={draw=black,ellipse}}]
0 \ar[r] & 1 \ar[r] & \dots \ar[r] & n-1 \ar[r] & n
\end{tikzcd}
```

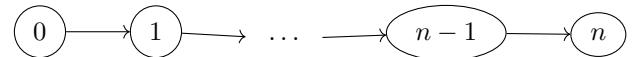


Figure 13.41: individual node without drawing margin

<https://tex.stackexchange.com/questions/586160/drawing-commutative-diagram-with-encircled-nodes-in-tikz-cd-package>

```
\usetikzlibrary{cd,fit,shapes.geometric}
\tikzset{E/.style = {ellipse, draw=blue,
 dashed,
 inner xsep=-2mm,inner
 ysep=-4mm,
 rotate=-30,fit=#1
 }
}
\begin{tikzcd}[arrows = dash,
 execute at end picture = {
 \node[E = (tikz@f@1-2-2)
 \cup (tikz@f@1-3-3)] {};
 \node[E = (tikz@f@1-3-1)
 \cup (tikz@f@1-4-2)] {};
 }% end of execute at
 end picture
]
& R \ar[d] & \\
& A+B \ar[dl] \ar[dr] & \\
A \ar[dr] & & B \ar[dl] \\
& A \cap B &
\end{tikzcd}
```

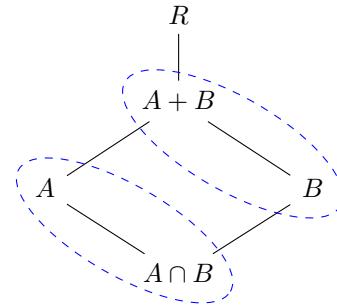


Figure 13.42: enclosed nodes

box in TikZ-CD<sup>[13.4.3]</sup>

```
\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}
A \ar[r,-stealth,"f","g"]
\ar[d,-{Stealth[harpoon]},shift
 left=0.5pt,"\varphi"] \ar[rr,-stealth,bend
 left] & B \ar[d] \\
C \ar[r,-latex,"g"]
\ar[u,-{Stealth[harpoon]},shift
 left=0.5pt,"\varphi^{\{-1\}}"]
\ar[d,harpoon,shift
 left=0.5pt,"\varphi"] & D \ar[r,g]
\ar[d,harpoon,shift
 left=0.5pt,"\varphi"] & E \\
G \ar[r,"f"]
\ar[u,harpoon,shift
 left=0.5pt,"\varphi^{-1}"]
\ar[uur,-stealth,"\phi" description] & H \ar[r,f]
\ar[u,harpoon,shift
 left=0.5pt,"\varphi"] & F
\end{tikzcd}
```

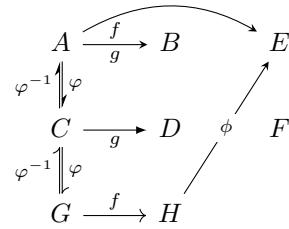


Figure 13.43: TikZ-CD or tikz-cd + arrows.meta

```
\usetikzlibrary{cd,arrows.meta}
\begin{tikzcd}
A \ar[r,-{Stealth[harpoon,swap,open]}] & B
\end{tikzcd}
```

$$A \longrightarrow B$$

Figure 13.44: TikZ-CD or tikz-cd + arrows.meta

TikZcd editor

<https://github.com/yishn/tikzcd-editor>

<https://tikzcd.yichuanshen.de/>

<https://darknmt.github.io/res/tikzcd-editor/>

tikz-cd @ CTAN(Comprehensive TEX Archive Network)

<https://ctan.org/tex-archive/graphics/pgf/contrib/tikz-cd>

tikzcd manual

<https://ctan.mirror.twds.com.tw/tex-archive/graphics/pgf/contrib/tikz-cd/tikz-cd-doc.pdf>

### 13.4.1 arrows

<https://latexdraw.com/exploring-tikz-arrows/>

#### 13.4.1.1 predefined arrow tip in TikZ

##### 13.4.1.1.1 stealth arrow tip $-\text{stealth}$ without capital S = $-\{\text{Stealth}\}$ in `arrows.meta`

```
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth] (0, 0) -- (1, 0);
\draw [stealth-] (0,-1) -- (1,-1);
\draw [stealth-stealth] (0,-2) -- (1,-2);
\node at (0,-2) {};
\end{tikzpicture}
```



Figure 13.45: stealth arrow tip

##### 13.4.1.1.2 reversed stealth arrow tip $-\text{stealth reversed} = -\{\text{Stealth}[reversed]\}$ in `arrows.meta`

```
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth reversed] (0, 0) -- (1, 0);
\draw [stealth reversed-] (0,-1) -- (1,-1);
\draw [stealth reversed-stealth reversed] (0,-2) -- (1,-2);
\node at (0,-2) {};
\end{tikzpicture}
```

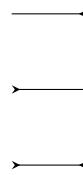


Figure 13.46: reversed stealth arrow tip

##### 13.4.1.1.3 to arrow tip $-\text{to}$

```
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-to] (0, 0) -- (1, 0);
\draw [to-] (0,-1) -- (1,-1);
\draw [to-to] (0,-2) -- (1,-2);
\node at (0,-2) {};
\end{tikzpicture}
```

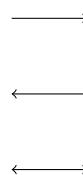


Figure 13.47: to arrow tip

##### 13.4.1.1.4 latex arrow tip $-\text{latex}$ with flat-tailed arrow head, differing from $-\text{stealth}$

```
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-latex] (0, 0) -- (1, 0);
\draw [latex-] (0,-1) -- (1,-1);
\draw [latex-latex] (0,-2) -- (1,-2);
\node at (0,-2) {};
\end{tikzpicture}
```



Figure 13.48: to arrow tip

##### 13.4.1.1.5 | arrow tip $-\mid$ like inhibition in biological pathway

```
\begin{tikzpicture}
 \node at (0, 0) {};
 \draw [-] (0, 0) -- (1, 0);
 \draw [|-] (0,-1) -- (1,-1);
 \draw [|-|] (0,-2) -- (1,-2);
 \node at (0,-2) {};
\end{tikzpicture}
```

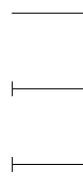


Figure 13.49: to arrow tip

### 13.4.1.2 arrows.meta

customizing arrow heads

#### 13.4.1.2.1 reversing, halving, swapping, opening <https://tikz.dev/tikz-arrows#sec-16.3.5>

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
 \node at (0, 0) {};
 \draw [-{Stealth[reversed]}] (0, 0)
 -- (1, 0);
 \draw [-{Stealth[harpoon]}] (0,-1)
 -- (1,-1);
 \draw [-{Stealth[harpoon,swap]}] (0,-2)
 -- (1,-2);
 \draw [-{Stealth[open]}] (0,-3) --
 (1,-3);
 \node at (0,-3) {};
\end{tikzpicture}
```



Figure 13.50: reversing, halving, swapping, opening

#### 13.4.1.2.2 mathematical arrows

- Classical TikZ Rightarrow
- Computer Modern Rightarrow
- Implies
- To

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
 \node at (0, 0) {};
 \draw [-{Classical TikZ Rightarrow}] (0,
 0) -- (1, 0);
 \draw [-{Computer Modern Rightarrow}]
 (0,-1) -- (1,-1);
 \draw [-{Implies},double]
 (0,-2) -- (1,-2);
 \draw [-To]
 (0,-3) -- (1,-3);
 \node at (0,-3) {};
\end{tikzpicture}
```



Figure 13.51: mathematical arrows

#### 13.4.1.2.3 ray arrow tip -Rays

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
 \node at (0, 0) {};
 \draw [-Rays] (0, 0) -- (1, 0);
 \draw [-{Rays[n=6]}] (0,-1) -- (1,-1);
 \draw [-{Rays[n=8]}] (0,-2) -- (1,-2);
 \node at (0,-2) {};
\end{tikzpicture}
```



Figure 13.52: Ray arrow tip

#### 13.4.1.2.4 arrow head scale e.g. -{Stealth[scale=2]}

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth] (0, 0) -- (1, 0);
\draw [-{Stealth[scale=2]}] (0,-1) -- (1,-1);
\node at (0,-1) {};
\end{tikzpicture}
```



Figure 13.53: arrow head scale

#### 13.4.1.2.5 arrow head size <https://tikz.dev/tikz-arrows#sec-16.3.1>

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth]
(0, 0) -- (1, 0);
\draw [-{Stealth[length=3mm, width=2mm]}]
(0,-1) -- (1,-1);
\node at (0,-1) {};
\end{tikzpicture}
```



Figure 13.54: arrow head size

#### 13.4.1.2.6 arrow head slant e.g. -{Stealth[slant=2]}

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-Stealth]
(0, 0) --
(1, 0);
\draw [-{Stealth[slant=-0.5]}]
(0,-1) --
(1,-1);
\draw [-{Stealth[slant= 1]}]
(0,-2) --
(1,-2);
\node at (0,-2) {};
\end{tikzpicture}
```



Figure 13.55: arrow head slant

#### 13.4.1.2.7 arrow head color e.g.

- -{Stealth[color=cyan]}
- -{Stealth[fill=red]}
- red,-{Stealth}

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth]
(0, 0) --
(1, 0);
\draw [-{Stealth[color=cyan]}]
(0,-1) --
(1,-1);
\draw [-{Stealth[fill=red]}]
(0,-2) --
(1,-2);
\draw [red,-{Stealth}]
(0,-3) --
(1,-3);
\node at (0,-3) {};
\end{tikzpicture}
```



Figure 13.56: arrow and arrow head color

#### 13.4.1.2.8 multiple arrow head e.g. -{Stealth}{Stealth}{Stealth}

```
\usetikzlibrary{arrows.meta}
\begin{tikzpicture}
\node at (0, 0) {};
\draw [-stealth]
 (0, 0) -- (1, 0);
\draw [-{Stealth}{Stealth}[Stealth]]
 (0,-1) -- (1,-1);
\node at (0,-1) {};
\end{tikzpicture}
```



Figure 13.57: multiple arrow head

### 13.4.1.3 default stealth arrow

<https://tex.stackexchange.com/questions/113437/stealth-arrow-in-math>

```
\usetikzlibrary{cd,arrows}
\tikzset{
 commutative diagrams/.cd,
 arrow style=tikz,
 diagrams={>=stealth}
}
\begin{tikzcd}
A \ar[r,"f"] & B \\
C \ar[r] & D \\
G \ar[r, -stealth] & H
\end{tikzcd}
```

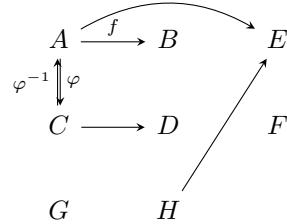


Figure 13.58: default stealth arrow

### 13.4.1.4 harpoon

harpoon ~ half arrow

<https://tex.stackexchange.com/questions/715462/rightleftharpoons-in-commutative-diagram>

```
\usetikzlibrary{cd}
\begin{tikzcd}
cc \ar[r, shift left=1pt, harpoon, "a"]
\hookrightarrow \ar[d, shift left=1pt, harpoon, "r"] &
ct \ar[l, shift left=1pt, harpoon, "b"]
\hookrightarrow \ar[d, shift left=1pt, harpoon, "r"] \\
tc \ar[r, shift left=1pt, harpoon, "a"]
\hookrightarrow \ar[u, shift left=1pt, harpoon, "l"] &
tt \ar[l, shift left=1pt, harpoon, "b"]
\hookrightarrow \ar[u, shift left=1pt, harpoon, "l"]
\end{tikzcd}
```

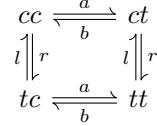


Figure 13.59: harpoon in cd

### 13.4.1.5 bend

<https://tex.stackexchange.com/questions/354480/very-curved-arrow-with-tikzcd>

```
\usetikzlibrary{cd}
\begin{tikzcd}
A
\ar[r]
\ar[r, bend right]
\ar[r, bend left, blue]
\ar[r, bend left=10]
\ar[r, bend left=20]
\ar[r, bend left=50]
\ar[r, bend left=100]
& B
\end{tikzcd}
```

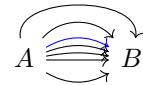


Figure 13.60: arrow bend

### 13.4.1.6 rounded corners

<https://tex.stackexchange.com/questions/295428/rounded-arrow-in-tikzcd-with-text-on-it>

```
\usetikzlibrary{cd,positioning}
\begin{tikzcd}[sep=large, execute at end
 \hookrightarrow picture={\node [below = 1mm of
 tikz@f01-1-2] {${\scriptstyle g'} \circ
 \textcolor{brown}{g\$}$};}]
B \ar[r] [swap] {\textcolor{brown}{g} } \ar[to
 \hookrightarrow path={--([yshift=-4ex]\tikztostart.south)
 -|(\tikztotarget)}, rounded
 corners=12pt] {rr}
 & B' \ar[r] [swap] {g'} & B'' \\
\end{tikzcd}
```

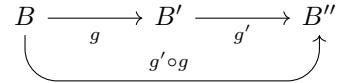


Figure 13.61: rounded corners

### 13.4.1.7 crossing over

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[dr] & B \ar[dl, crossing over] \\
C & D \\
\end{tikzcd}
```

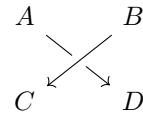


Figure 13.62: crossing over

```
\usetikzlibrary{cd}
\begin{tikzcd}[row sep=scriptsize, column
 \hookrightarrow sep=scriptsize]
& f^* E_V \ar[dl] \ar[rr] \ar[dd] & & E_V \\
f^* E \ar[rr, crossing over] \ar[dd] & & E \ar[dd] \\
& U \ar[dl] \ar[rr] & & V \ar[dl] \\
M \ar[rr] & & N \ar[from=uu, crossing over] \\
\end{tikzcd}
```

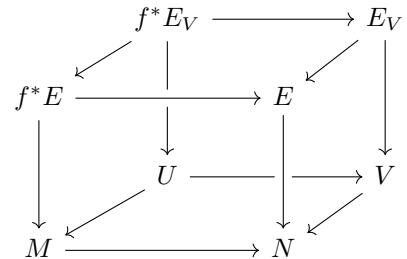


Figure 13.63: pseudo 3D diagram

### 13.4.1.8 tweaking to paths

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[dr, controls={(1.5,0.5) and
 +(-1,0.8)}] \ar[dr, dashed, to path=|-(
 \tikztotarget)] \\
& B \ar[loop right] \\
\end{tikzcd}
```

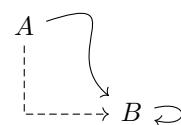


Figure 13.64: crossing over

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r]
& B \ar[r]
& C \ar[dll,
 "\delta",
 rounded corners,
 to
 path={--([xshift=2ex]\tikztostart.east)
 |-(Z)[near
 end]\tikztonodes
 -|([xshift=-2ex]\tikztotarget.west)
 --(\tikztotarget)}] \\
D \ar[r]
& E \ar[r]
& F
\end{tikzcd}
```

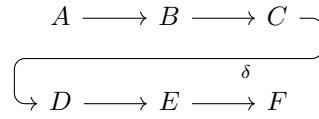


Figure 13.65: crossing over

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r]
& B \ar[r]
& C \ar[dll,
 "\delta",
 rounded corners,
 to
 path={--([xshift=2ex]\tikztostart.east)
 --([xshift=-2ex]\tikztotarget.west)
 --(\tikztotarget)}] \\
D \ar[r]
& E \ar[r]
& F
\end{tikzcd}
```

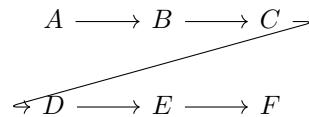


Figure 13.66: crossing over

### 13.4.2 label

<https://tex.stackexchange.com/questions/477733/two-labels-up-and-down-for-same-arrow>

```
\usetikzlibrary{cd}
\begin{tikzcd} & V \\
A \ar[r, "\alpha", "\beta"'] & B
\end{tikzcd}
```

$$A \xrightarrow{\alpha} B$$

Figure 13.67: labels over and under the arrow

#### 13.4.2.1 single quotation mark to swap

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r, "\alpha"] & B
\end{tikzcd}
```

$$A \xrightarrow{\alpha} B$$

Figure 13.68: label over the arrow without single quotation

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r, '\alpha'] & B
\end{tikzcd}
```

$$A \xrightarrow{-\alpha} B$$

Figure 13.69: label under the arrow with single quotation

### 13.4.2.2 swap

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r, "\alpha"] & B
\end{tikzcd}
```

$$A \xrightarrow{\alpha} B$$

Figure 13.70: label over the arrow or no swap

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r, "\alpha", swap] & B
\end{tikzcd}
```

$$A \xleftarrow{\alpha} B$$

Figure 13.71: label swap under the arrow

### 13.4.2.3 every arrow swap

<https://tex.stackexchange.com/questions/515697/how-can-we-change-the-positions-of-the-labels-on-a-tikz-cd-commutative-diagram>

```
every arrow/.append style={swap}
```

### 13.4.3 box in TikZ-CD

<https://stackoverflow.com/questions/50392582/box-in-commutative-diagram>

<https://tex.stackexchange.com/questions/360083/filling-of-diagrams-using-tikzcd/360152#360152>

### 13.4.4 examples

<https://tex.stackexchange.com/questions/218274/how-can-i-draw-commutative-diagrams-in-latex>

```
\usetikzlibrary{cd}
\begin{tikzcd}
A \ar[r, "\varphi"] \ar[swap]{d}{\varrho_f} & B \\
\hookrightarrow \ar[d, "\varrho_g"'] & \\
A_f \ar[r, "\varphi_f"] & B_g
\end{tikzcd}
```

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \varrho_f \downarrow & & \downarrow \varrho_g \\ A_f & \xrightarrow{\varphi_f} & B_g \end{array}$$

Figure 13.72: commutative diagram example 0

<https://tex.stackexchange.com/questions/402572/how-to-make-a-commutative-diagram-using-tikz-cd>

```
\usetikzlibrary{cd}
\begin{tikzcd}[row sep=huge]
A \ar[r, "\phi_n"] \ar[swap]{d}{\pi} & A_n \ar[swap]{r}{\epsilon_{i,n}} & M_{n_k}(\mathbb{C}) \\
A_n \ar[swap]{d}{\psi_n} & \ar[swap]{r}{\pi_n} & \\
B(H) \ar[ll, dashed]{r}{\eta_{i,n}} & B(H_n)
\end{tikzcd}
```

$$\begin{array}{ccccc} A & \xrightarrow{\phi_n} & A_n & \xrightarrow{\epsilon_{i,n}} & M_{n_k}(\mathbb{C}) \\ \pi \downarrow & \swarrow \psi_n & & \searrow \pi_n & \downarrow \eta_{i,n} \\ B(H) & \dashrightarrow & B(H_n) & & \end{array}$$

Figure 13.73: commutative diagram example 1

<https://tex.stackexchange.com/questions/115783/how-to-draw-commutative-diagrams>

```
\usetikzlibrary{cd,arrows}
\begin{tikzcd}
A \ar[r, "f"] \ar[swap]{d}{g \circ f} & B \\
& C
\end{tikzcd}
```

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \circ f & \downarrow g \\ & C & \end{array}$$

Figure 13.74: commutative diagram example 2

<https://tex.stackexchange.com/questions/414292/tikz-commutative-diagrams-compiling-and-best-practice>

```
\usetikzlibrary{cd}
\begin{tikzcd}
C \ar[d, "\langle f_i \rangle_{i \in I} \text{ in } I", dotted] \ar[rd, "f_i"'] & \\
\prod_{i \in I} A_i \ar[r, "\pi_i"] & A_i
\end{tikzcd}
```

<https://tex.stackexchange.com/questions/483801/mapping-arrows-in-commutative-diagrams>

```
\usetikzlibrary{cd}
\begin{tikzcd}
a \ar[r, maps to] & \phi(a) \\
R \ar[r, hook, "\phi"] \ar[dr, hook, "\iotaota"] & K \\
& Q(R) \ar[u, dotted, "\exists! \Phi"] \ar[dr, "\frac{a}{1}"'] \\
a \ar[r, maps to] & \phi(a) \\
& \frac{a}{1}
\end{tikzcd}
```

if not using tikz-cd,

```
\usetikzlibrary{arrows,positioning}
\begin{tikzpicture}
\node (r) at (0,0) {R};
\node (k) at (3,0) {K};
\node (q) at (3,-3) {$Q(R)$};
\node (ra) at (0,.8) {a};
\node (ka) at (3,.8) {$\phi(a)$};
\node (kb) at (4,0) {$\phi(k)$};
\node (qb) at (4,-3) {$\frac{a}{1}$};
\path (q) node[below left=1em and 1em] (qc) {\scriptsize $\exists! \Phi$};
\path (r) node[below left=1em and 1em] (rc) {\scriptsize $\exists! \Phi$};
\draw[right hook->] (r)--(k);
\draw[right hook->] (r)--(q) node[midway,above]{\scriptsize ϕ};
\draw[right hook->] (r)--(a) node[midway,above]{\scriptsize ι};
\draw[dotted,->] (q)--(k) node[midway,right]{\scriptsize $\exists! \Phi$};
\draw[|->] (ra) edge (ka) (qb) edge (kb) (rc)
edge (qc);
\end{tikzpicture}
```

separated diagrams with tikz-cd,

```
\usetikzlibrary{cd}
\begin{tikzcd}
R \ar[r, hook, "\phi"] \ar[dr, hook, "\iotaota"] & K \\
a \ar[r, maps to, "\phi"] \ar[dr, maps to, "\iotaota"] & Q(R) \\
& \frac{a}{1}
\end{tikzcd}
```

register map

<https://tex.stackexchange.com/questions/510088/tikz-register-map>

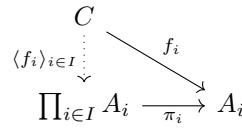


Figure 13.75: commutative diagram example 3

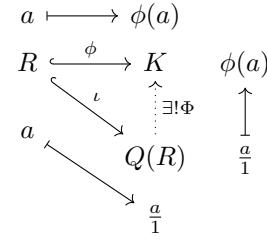


Figure 13.76: commutative diagram example 4

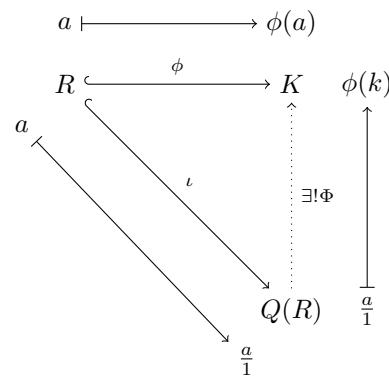


Figure 13.77: commutative diagram example 4 not using tikz-cd

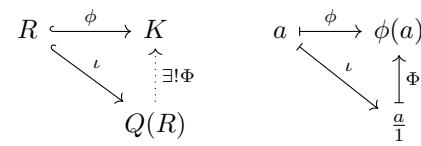


Figure 13.78: commutative diagram example 4

### 13.4.5 put a commutative diagram in a nodepart in a tikzpicture

<https://tex.stackexchange.com/questions/524955/put-a-commutative-diagram-in-a-nodepart-in-a-tikzpicture/525046#525046>

## 13.5 PGFplots

axis similar to matplotlib figure anatomy, see Fig: 27.1

<https://tikz.dev/pgfplots/>

<https://tikz.dev/pgfplots/tutorial1>

Not so common is `\pgfplotsset{compat=1.5}`. A statement like this should always be used in order to (a) benefit from a more or less recent feature set and (b) avoid changes to your picture if you recompile it with a later version of pgfplots.

```
\pgfplotsset{width=7cm,compat=1.18}
\begin{tikzpicture}
\begin{axis}[
]
 % density of Normal distribution:
 \addplot [
 red,
 domain=-3e-3:3e-3,
 samples=201,
]
 {exp(-x^2 / (2e-3^2)) / (1e-3 *
 \sqrt(2*pi))};
\end{axis}
\end{tikzpicture}
```

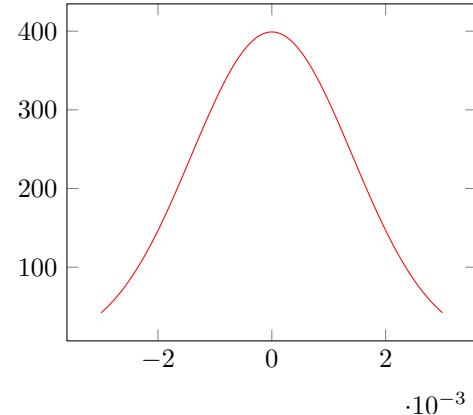


Figure 13.79: PGFplots: normal distribution

### 13.5.1 the axis environments

<https://tikz.dev/pgfplots/reference-axis>

```
\pgfplotsset{every linear axis/.append style={...}}
```

```
\begin{tikzpicture}
\begin{axis}[
 no markers,
 axis x line = center,
 axis y line = center,
 xlabel = {x}, xlabel style =
 {right},
 ylabel = {y}, ylabel style =
 {above},
 xmin = -8, xmax = 8,
 ymin = 0, ymax = 0.45,
 hide obscured x ticks=false, % for
 origin x tick label i.e. xtick = {0}
 xtick={-4, 0, 4},
 xticklabels={\$%
 \mu_{\scriptscriptstyle 1}\$%
 \$%
 \mu_{\scriptscriptstyle 0}\$%
 \$%
 \mu_{\scriptscriptstyle 1}\$},
 %extra x ticks={0},
 ytick = \emptyset,
 x = 1cm, y = 5cm, % x y scaling
 grid = major,
 domain = -10:10,
 samples = 1000
]
\end{axis}
\end{tikzpicture}
```

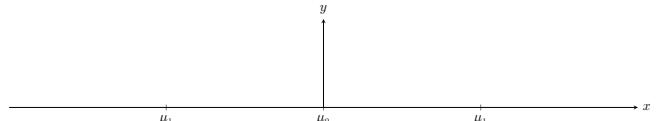


Figure 13.80: beginaxis

<https://tex.stackexchange.com/questions/134959/axis-lines-middle-and-axis-lines-center>

No, there is no difference.

## 13.5.2 axis descriptions

<https://tikz.dev/pgfplots/reference-axisdescription>

### 13.5.2.1 placement of axis descriptions

#### 13.5.2.1.1 coordinate system axis description cs [https://tikz.dev/pgfplots/reference-axisdescription#pgfp.axis\\_description\\_cs](https://tikz.dev/pgfplots/reference-axisdescription#pgfp.axis_description_cs)

\addplot {x}; can change to \addplot {x^2}; still with auto blue dots

```
small dot style,pin=angle:LaTeX label at PGFplots axis coordinate system ;
```

```
\begin{tikzpicture}
 \tikzset{
 every
 pin/.style={fill=yellow!50!white,rectangle,rounded
 corners=3pt,font=\tiny},
 small
 dot/.style={fill=black,circle,scale=0.3},
 }
 \begin{axis}[
 clip=false,
 title=How \texttt{axis description cs}
 works,
]
 \addplot {x};
 %small dot style,pin=angle:LaTeX label
 % at PGFplots axis coordinate system
 %;
 \node [small dot,pin=120:{$(0,0)$}]%
 at (axis description cs:0,0) {};
 \node [small dot,pin=-30:{$(1,1)$}]%
 at (axis description cs:1,1) {};
 \node [small
 dot,pin=-90:{$(1.03,0.5)$}] at
 (axis description cs:1.03,0.5) {};
 \node [small
 dot,pin=125:{$(0.5,0.5)$}] at
 (axis description cs:0.5,0.5) {};
 \end{axis}
\end{tikzpicture}
```

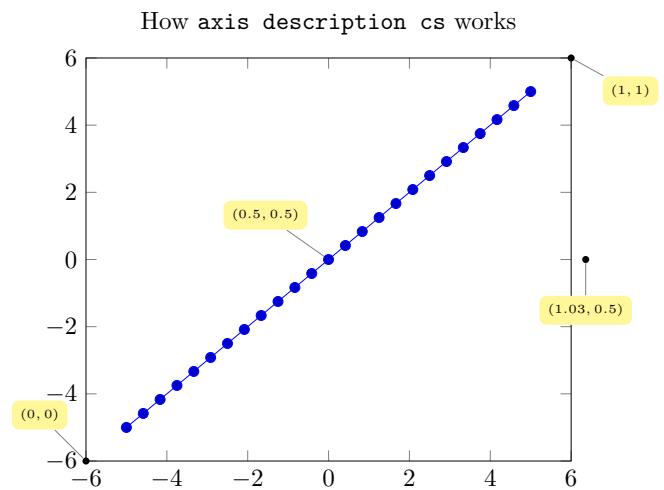
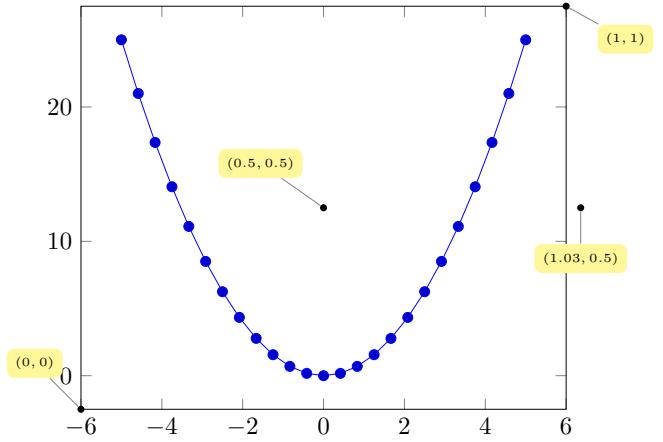


Figure 13.81: PGFplots:  $x$

```
\begin{tikzpicture}
 \tikzset{
 every
 pin/.style={fill=yellow!50!white,rectangle,rounded
 corners=3pt,font=\tiny},
 small
 dot/.style={fill=black,circle,scale=0.3},
 }
 \begin{axis}[
 clip=false,
 title=How \texttt{axis description cs} works,
]
 \addplot {x^2};
 %small dot style,pin=angle:LaTeX label
 % at PGFplots axis coordinate system
 ;
 \node [small dot,pin=120:{\$(0,0)\$}]
 at (axis description cs:0,0)
 {};
 \node [small dot,pin=-30:{\$(1,1)\$}]
 at (axis description cs:1,1)
 {};
 \node [small
 dot,pin=-90:{\$(1.03,0.5)\$}] at
 (axis description cs:1.03,0.5) {};
 \node [small
 dot,pin=125:{\$(0.5,0.5)\$}] at
 (axis description cs:0.5,0.5) {};
 \end{axis}
\end{tikzpicture}
```

How axis description cs works

Figure 13.82: PGFplots:  $x^2$ 

### 13.5.2.1.2 legend <https://tikz.dev/pgfplots/reference-axisdescription#sec-4.9.4>

### 13.5.2.1.3 tick option <https://tikz.dev/pgfplots/reference-tickoptions>

## 13.5.3 declare function

[https://tikz.dev/pgfplots/utility-commands#pgf/declare\\_function](https://tikz.dev/pgfplots/utility-commands#pgf/declare_function)

```
\begin{tikzpicture}
\begin{axis}[
 declare function={
 C=4;
 square(\t)=(\t)^2 + C;
 },
]
\addplot+ [samples=2] {C*x};
\addplot {square(x)};
\end{axis}
\end{tikzpicture}
```

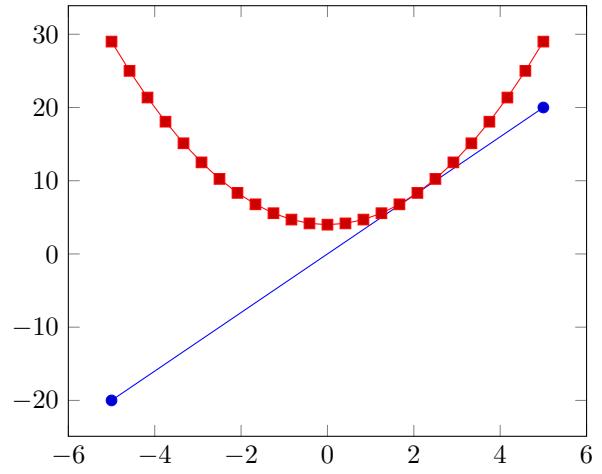


Figure 13.83: declare function

### 13.5.3.1 pgfmathparse

<https://tikz.dev/math-parsing>

<https://tikz.dev/math-parsing#sec-94.1>

This macro parses and returns the result without units in the macro 0.017. Example: `\pgfmathparse{2pt+3.5pt}` will set `\pgfmathresult` to the text 5.5.

```
\pgfmathsqrt{x} = \pgfmathparse{sqrt(x)}
\pgfmathln{x} = \pgfmathparse{ln(x)}
...

```

### 13.5.3.2 pgfmathdeclarefunction

like `pgfplotsinvokeforeach`

replaces any occurrence of #1 inside of (math image)command(math image) once for every element in (math image)list(math image). Thus, it actually assumes that (math image)command(math image) is like a `\newcommand` body.

```
% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
 \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
 \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

\begin{tikzpicture}
\begin{axis}[
 no markers,
 axis x line = center,
 axis y line = center,
 xlabel = {x}, xlabel style = {right},
 ylabel = {y}, ylabel style = {above},
 xmin = -8, xmax = 8,
 ymin = 0, ymax = 0.45,
 hide obscured x ticks=false, % for origin x tick label i.e. xtick = {0}
 xtick={-4, 0, 4},
 xticklabels={\$ \mu_{\scriptscriptstyle 1} \$,
 \$ \mu_{\scriptscriptstyle 0} \$,
 \$ \mu_{\scriptscriptstyle 1} \$},
 %extra x ticks={0},
 ytick = \emptyset,
 x = 1cm, y = 5cm, % x y scaling
 %grid = major,
 domain = -10:10,
 samples = 1000
]
\addplot [fill=cyan!20, draw=none, domain=-10:-2] {gauss(-4, 1)} \closedcycle;
\addplot [fill=cyan!20, draw=none, domain= -2:10] {gauss(4, 1)} \closedcycle;
\addplot [very thick, cyan!50!black] {gauss(-4, 1)};
\addplot [very thick, cyan!50!black] {gauss(0, 1)};
\addplot [very thick, cyan!50!black] {gauss(4, 1)};
%\node [anchor=north] at (axis cs: 0, -0.01) {\$ \mu_1 \$};
%\node at (axis cs: -4, -0.02) {\$ \mu_0 \$};
\draw [dashed, thin] (axis cs: -4, 0) -- (axis cs: -4, 1);
\draw [dashed, thin] (axis cs: 4, 0) -- (axis cs: 4, 1);
\end{axis}
\end{tikzpicture}
```

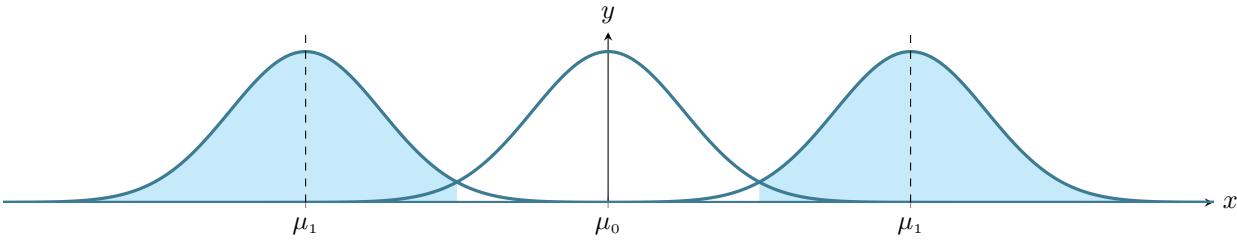


Figure 13.84: PGFmathDeclareFunction: normal distributions hypothesis testing

<https://tex.stackexchange.com/questions/43610/plotting-bell-shaped-curve-in-tikz-pgf>

```
% pgfmathdeclarefunction{name}{num_var}{%
%% #1 = \mu, #2 = \sigma
\pgfmathdeclarefunction{gauss}{2}{%
 \pgfmathparse{1/(\#2*sqrt(2*pi))*exp(-((x-\#1)^2)/(2*\#2^2))}%
}

\begin{tikzpicture}
\begin{axis}[
 no markers, domain=0:10, samples=100,
 axis lines*=left, xlabel=x, ylabel=y,
 every axis y label/.style={at=(current axis.above origin), anchor=south},
 every axis x label/.style={at=(current axis.right of origin), anchor=west},
 height=5cm, width=12cm,
 xtick={4,6.5}, ytick=\emptyset,
 enlargelimits=false, clip=false, axis on top,
 grid = major
]
\addplot [fill=cyan!20, draw=none, domain=0:5.96] {gauss(6.5,1)} \closedcycle;
\addplot [very thick,cyan!50!black] {gauss(4,1)};
\addplot [very thick,cyan!50!black] {gauss(6.5,1)};
\draw [yshift=-0.6cm, latex-latex](axis cs:4,0) -- node [fill=white] {1.96σ} (axis
 cs:5.96,0);
\end{axis}
\end{tikzpicture}
```

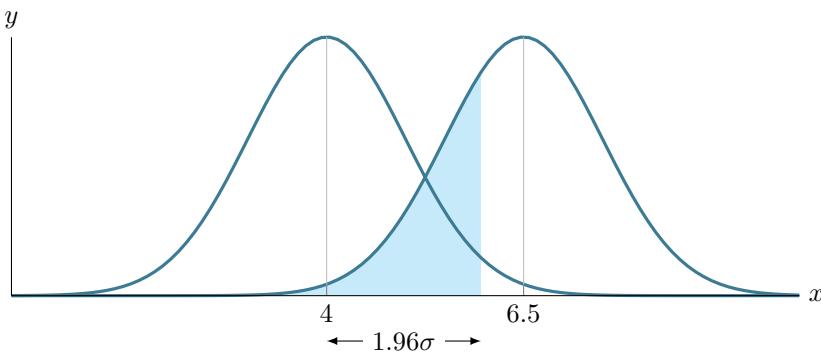


Figure 13.85: PGFmathDeclareFunction: normal distributions

### 13.5.4 |- and -| in TikZ

<https://tex.stackexchange.com/questions/401425/tikz-what-exactly-does-the-the-notation-for-arrows-do>

( $a$   $-|$   $b$ ) where  $a$  and  $b$  are named nodes or coordinates. This means the coordinate that is at the y-coordinate of  $a$ , and x-coordinate of  $b$ . Similarly, ( $a$   $|-$   $b$ ) has the x-coordinate of  $a$  and y-coordinate of  $b$ .

### 13.5.5 pgfplotsinvokeforeach

<https://tikz.dev/pgfplots/pgfplotstable-miscellaneous#/pgfplotsinvokeforeach>

like `\foreach` in TikZ

A variant of `\pgfplotsforeachungrouped` (and such also of `\foreach`) which replaces any occurrence of #1 inside of (math image)command(math image) once for every element in (math image)list(math image). Thus, it actually assumes that (math image)command(math image) is like a `\newcommand` body.

### 13.5.6 interpolation dashed lines

<https://tex.stackexchange.com/questions/193259/what-is-the-easiest-way-to-accomplish-textual-tick-labels-in-tikz>

```
interpa = (10,10), interp = interpolation
```

```
{axis cs:0,0}|-interp#1) = (x of (0,0), y of (interpa)) = (0, 10), ...
```

```
\begin{tikzpicture}
\begin{axis}[
 axis lines=left,
 xmin = 0, xmax = 40,
 ymin = 0, ymax = 40,
 xtick={10,30},
 xticklabels={\$V_i=10\$,\$V_f=30\$},
 ytick={10,30},
 yticklabels={\$P_i=10\$,\$P_f=30\$},
 xlabel={Volume},
 ylabel={Pressure}
]
\addplot[very thick,-latex]
 coordinates{(10,10) (30,30)}
% interpa = (10,10), interp = interpolation
% coordinate[at start](interpa) coordinate[at end](interp);
\pgfplotsinvokeforeach {a,b} {
\draw[ultra thin, dashed]
 % ({axis cs:0,0}|-interp#1) = (x of (0,0), y of (interpa)) = (0, 10),
 % ...
 {axis
 cs:0,0}|-interp#1)--(interp#1)--(interp#1|-{axis
 cs:0,0});
}
\end{axis}
\end{tikzpicture}
```

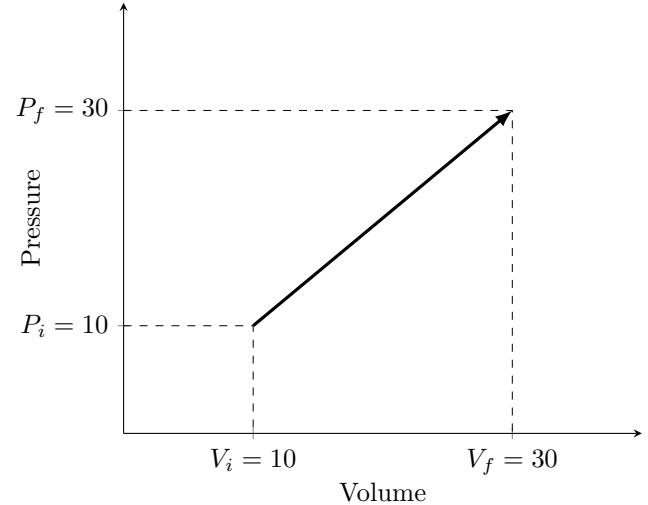


Figure 13.86: tick texts and interpolation dashed lines

### 13.5.7 Zewbie

<https://zhuanlan.zhihu.com/p/551874337>

axis similar to matplotlib figure anatomy, see Fig: 27.1

### 13.5.7.1 coordinate axis/axes fine-tuing

```
\begin{tikzpicture}
 \begin{axis}
 % empty
 \end{axis}
\end{tikzpicture}
```

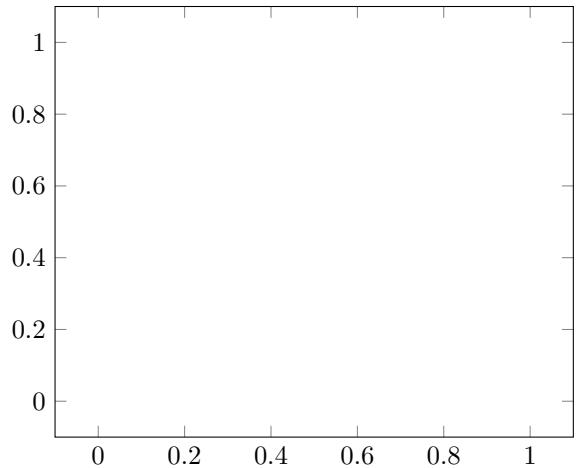


Figure 13.87: PGFplots: 2D axis/axes

#### 13.5.7.1.1 range

---

```
\begin{tikzpicture}
 \begin{axis}[
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
]
 % empty
 \end{axis}
\end{tikzpicture}
```

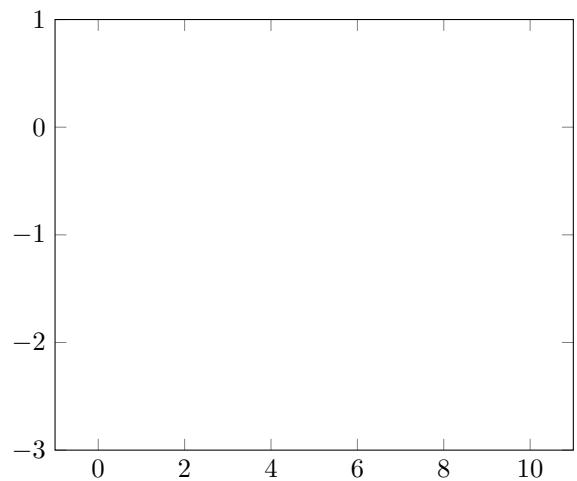


Figure 13.88: PGFplots: axis/axes range

#### 13.5.7.1.2 scaling axis equal image equivalent to unit vector ratio = {1 1 1}

```
\begin{tikzpicture}
 \begin{axis}[
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
 axis equal image, % unit vector ratio
 = {1 1 1},
]
 % empty
 \end{axis}
\end{tikzpicture}
```

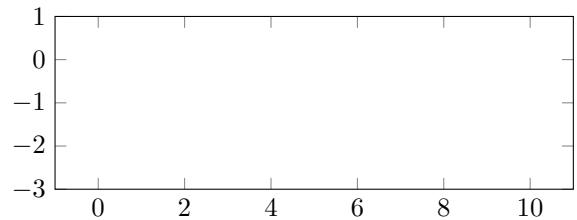


Figure 13.89: PGFplots: axis/axes range

---

scale only axis

width x axis length, height y axis length

```
\begin{tikzpicture}
\begin{axis}[
xmin = -1, xmax = 11,
ymin = -3, ymax = 1,
scale only axis,
width = 5cm, height = 7cm,
]
% empty
\end{axis}
\end{tikzpicture}
```

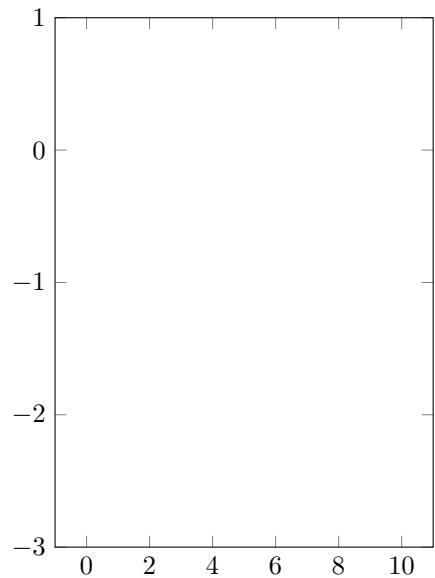


Figure 13.90: PGFplots: axis/axes range

x x unit vector length, y y unit vector length

```
\begin{tikzpicture}
\begin{axis}[
xmin = -1, xmax = 11,
ymin = -3, ymax = 1,
x = 1cm, y = 1cm,
]
% empty
\end{axis}
\end{tikzpicture}
```

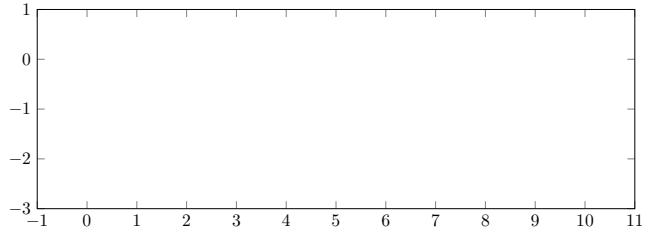


Figure 13.91: PGFplots: axis/axes range

### 13.5.7.1.3 direction vector

```
\begin{tikzpicture}
\begin{axis}[
xmin = -1, xmax = 11,
ymin = -3, ymax = 1,
x={(.2cm,-.1cm)}, y={(-.5cm,.5cm)},
]
% empty
\end{axis}
\end{tikzpicture}
```

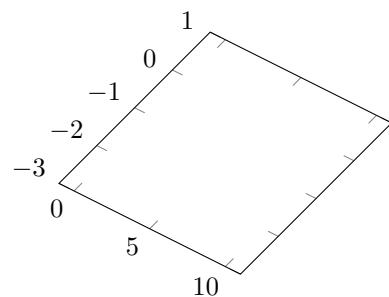


Figure 13.92: PGFplots: axis/axes range

unit vector ratio = {1 1}

```
\begin{tikzpicture}
\begin{axis}[
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
 unit vector ratio = {1 1},
]
% empty
\end{axis}
\end{tikzpicture}
```

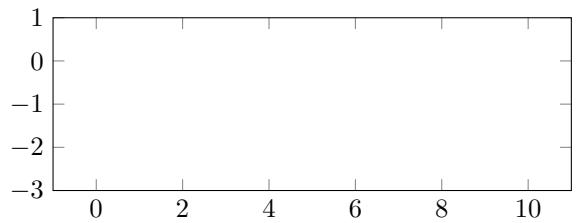


Figure 13.93: PGFplots: axis/axes range

**13.5.7.1.4 axis style** `axis lines` to assign all axes, either `axis lines = box`(default), `axis lines = center`(axis lines with arrows, `x:` bottom, top, `y:` ), or `axis lines = none`(not shown), or even axis lines without arrows `axis lines *= center`

`axis x line`, `axis y line` to assign the respect axis, e.g. `axis x line = center`

`axis lines = center`:

```
\begin{tikzpicture}
\begin{axis}[
 axis lines = center,
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
]
% empty
\end{axis}
\end{tikzpicture}
```

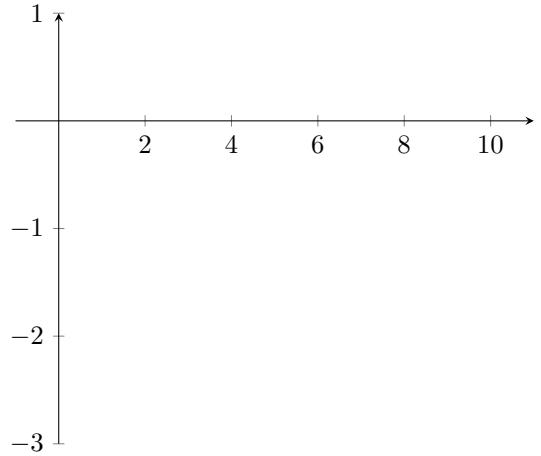


Figure 13.94: PGFplots: axis/axes range

`axis lines *= center`:

x axis line without arrow, y axis box

```
\begin{tikzpicture}
\begin{axis}[
 axis x line*=center, % x axis line
 without arrow, y axis box
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
]
% empty
\end{axis}
\end{tikzpicture}
```

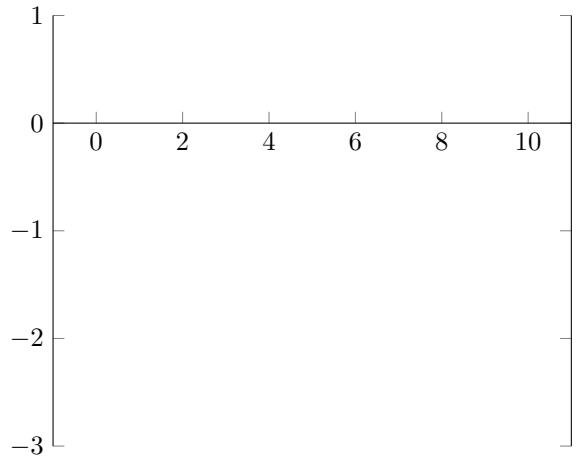


Figure 13.95: PGFplots: axis/axes range

---

x axis line with arrow, y axis line without arrow

```
\begin{tikzpicture}
\begin{axis}[
 axis x line = center, % x axis line
 with arrow
 axis y line* = center, % y axis line
 without arrow
 xmin = -1, xmax = 11,
 ymin = -3, ymax = 1,
]
% empty
\end{axis}
\end{tikzpicture}
```

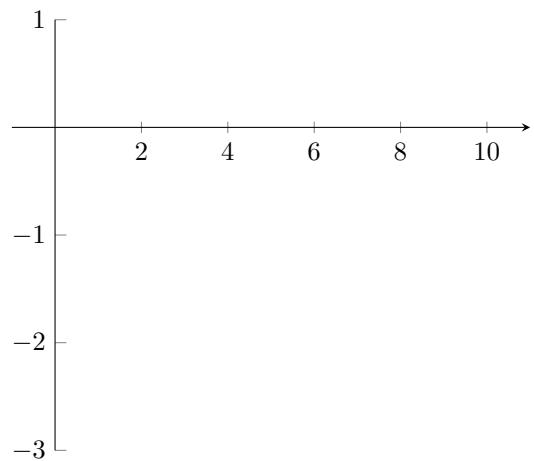


Figure 13.96: PGFplots: axis/axes range

## 13.5.7.1.5 axis discontinuity crunch, parallel, none

crunch

```
\begin{tikzpicture}
\begin{axis}[
 axis x line = bottom,
 axis y line = center,
 xmin = -2, xmax = 10,
 ymin = 0, ymax = 12,
 axis y discontinuity = crunch,
]
% empty
\end{axis}
\end{tikzpicture}
```

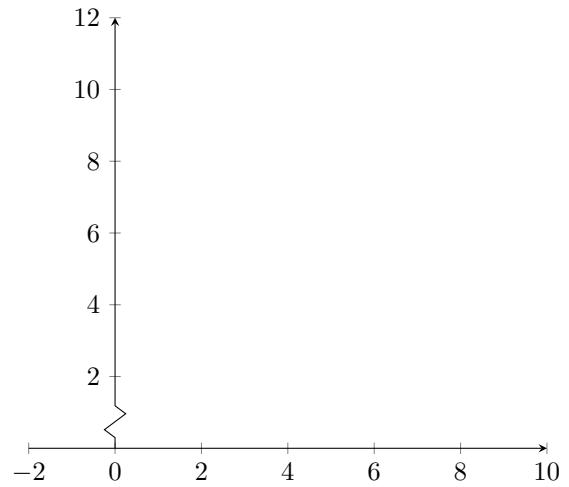


Figure 13.97: PGFplots: axis/axes range

parallel

```
\begin{tikzpicture}
\begin{axis}[
 axis x line = bottom,
 axis y line = center,
 xmin = -2, xmax = 10,
 ymin = 0, ymax = 12,
 axis y discontinuity = parallel,
]
% empty
\end{axis}
\end{tikzpicture}
```

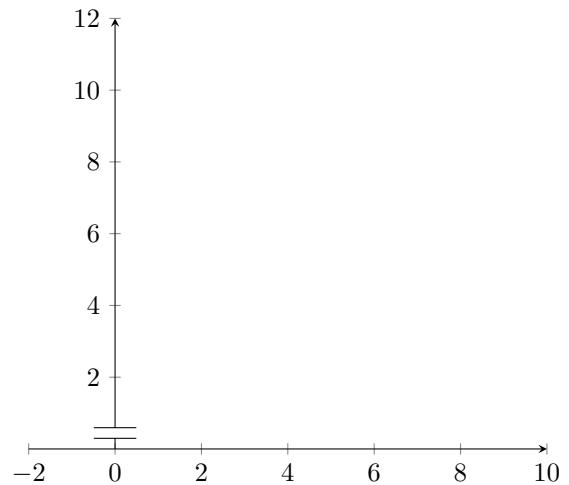


Figure 13.98: PGFplots: axis/axes range

## 13.5.7.1.6 tick tick pos ticklabel pos

```
\begin{tikzpicture}
 \begin{axis}[
 xtick pos = upper,
 yticklabel pos = upper,
]
 % empty
 \end{axis}
\end{tikzpicture}
```

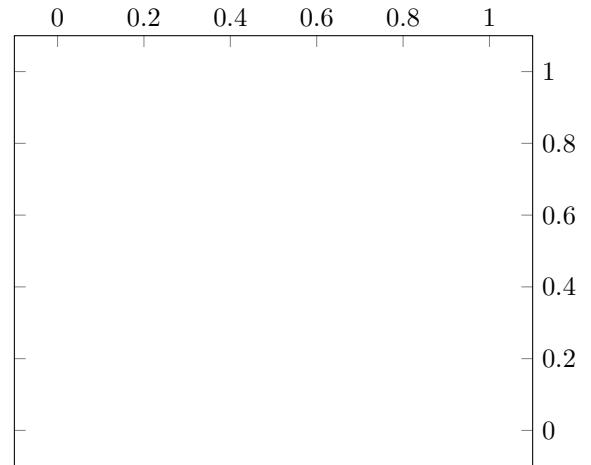


Figure 13.99: PGFplots: axis/axes range

tick distance  
tick align: inside, center, outside

```
\begin{tikzpicture}
 \begin{axis}[
 axis lines=center,
 xmin=-1,xmax=3,
 ymin=-3,ymax=3,
 xtick distance=.8,
 ytick distance=1.1,
 tick align=inside,
]
 % empty
 \end{axis}
\end{tikzpicture}
```

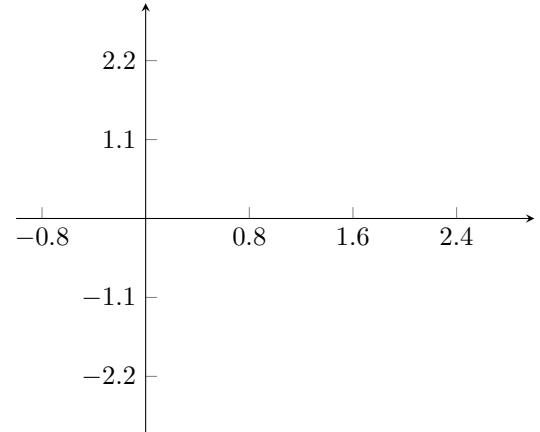


Figure 13.100: PGFplots: axis/axes range

minor tick num

```
\begin{tikzpicture}
 \begin{axis}[
 axis y line=none, axis x line=center,
 ymin=0,ymax=0,xmin=-3,xmax=3,
 xtick distance=2,tick align=inside,
 minor tick num=2,
]
 % empty
 \end{axis}
\end{tikzpicture}
```

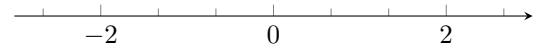


Figure 13.101: PGFplots: axis/axes range

xtick=\emptyset|data|{\<coordinates>}

```
\begin{tikzpicture}
\begin{axis}[
 axis y line=none, axis x line=center,
 ymin=0, ymax=0, xmin=-3, xmax=3,
 xtick={-2.5,0,1}, minor
→ xtick={1/3,2/3},
 tick align=inside,
]
% empty
\end{axis}
\end{tikzpicture}
```

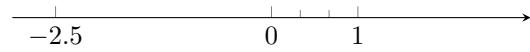


Figure 13.102: PGFplots: axis/axes range

```
extra x ticks={<coordinates>}
```

```
\begin{tikzpicture}
\begin{axis}[
 axis y line=none, axis x line=center,
 ymin=0, ymax=0, xmin=-2.3, xmax=4.9,
 xtick distance=2, minor tick num=1,
 extra x ticks={e,pi},
 extra x tick labels={e,π},
 tick align=inside,
]
% empty
\end{axis}
\end{tikzpicture}
```

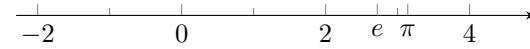


Figure 13.103: PGFplots: axis/axes range

```
ticklabels=<labels> extra x tick labels=<labels>
hide obscured x ticks = false for origin x tick label
```

```
\begin{tikzpicture}
\begin{axis}[
 axis y line=none, axis x line=center,
 ymin=0, ymax=0, xmin=-2.3*pi, xmax=2.3*pi,
 xtick distance=pi,
 xticklabels={-2π,-π,0,π,2π},
]
% empty
\end{axis}
\end{tikzpicture}
```

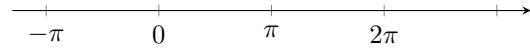


Figure 13.104: PGFplots: axis/axes range

### 13.5.7.2 addplot+

What does addplot+ do exactly?

<https://tex.stackexchange.com/questions/275959/what-does-addplot-do-exactly>

<https://tikz.dev/pgfplots/reference-addplot#/addplot+>

Every `\addplot` directive receives a pre-defined style (line color, marker style etc) through a pre-defined cycle list that is automatically chosen depending on the index of the current `\addplot` instruction. If you want to add some of your styles manually (like I want red colour instead of blue, say), you can add them through options to `\addplot` like `\addplot[<your options>]`. Now the question is whether you want your own style (your options) to be appended to or replace one of these cycle lists assigned. This is decided by the `+` sign. If you use `\addplot+ [<your options>]`, your style is appended to and by `\addplot[<your options>]`, your options will replace the assigned cycle list.

### 13.5.7.3 point

`only marks` only points without lines

zero y axis range `ymin=0, ymax=0` and `axis y line=none`, making 1D x axis

```
\begin{tikzpicture}
\begin{axis}[
 xlabel=x,
 axis y line=none,
 axis x line=center,
 tick align=inside,
 xmin=-1.5, xmax=4.9, ymin=0, ymax=0,
 xtick distance=1,
 x=1cm
]
\addplot+ coordinates {(e,0)}
 node [pin=90:{\$e\$}] {};
\addplot+ coordinates {(\pi,0)}
 node [pin=90:{\$\\pi\$}] {};
\addplot+ coordinates {(-1,0)};
\end{axis}
\end{tikzpicture}
```



Figure 13.105: PGFplots: 1D points with pins

---

```
\begin{tikzpicture}
\begin{axis}[
 xlabel=x, ylabel=y,
 axis lines=center,
 tick align=inside,
 xmin=-1.5, xmax=4.9, ymin=-3.3,
 ymax=3.9,
 xtick distance=1, ytick distance=1,
 axis equal image
]
\addplot+ [only marks] coordinates {
 (-1,-2) (\pi,\pi/4) (3,2) (0,0)};
\addplot+ coordinates {(2,1)};
\addplot+ coordinates {(3,-2)};
\end{axis}
\end{tikzpicture}
```

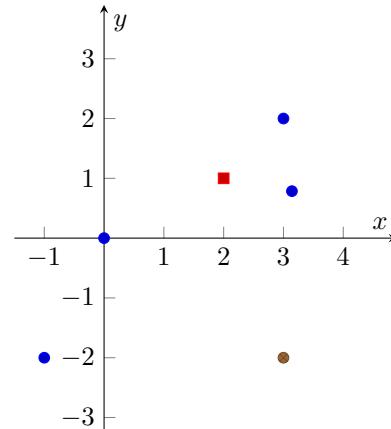


Figure 13.106: PGFplots: 2D points

---

```
\begin{tikzpicture}
\begin{axis}[axis equal image]
\addplot+ [only marks]
 table [x=xdata,y=ydata]
\end{axis}
\end{tikzpicture}
```

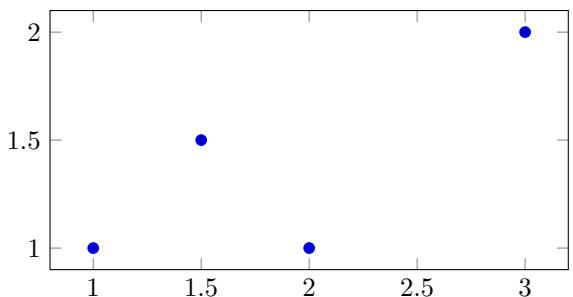


Figure 13.107: PGFplots: data points

```
axis equal image to fix aspect ratio 1:1
```

```
\begin{tikzpicture}
\begin{axis}[title=5 sampling points,
 xlabel=x,ylabel=y,
 axis equal image]
\addplot+ [only
 → marks,domain=-2:2,samples=5]
 {x^2};
\end{axis}
\end{tikzpicture}
```

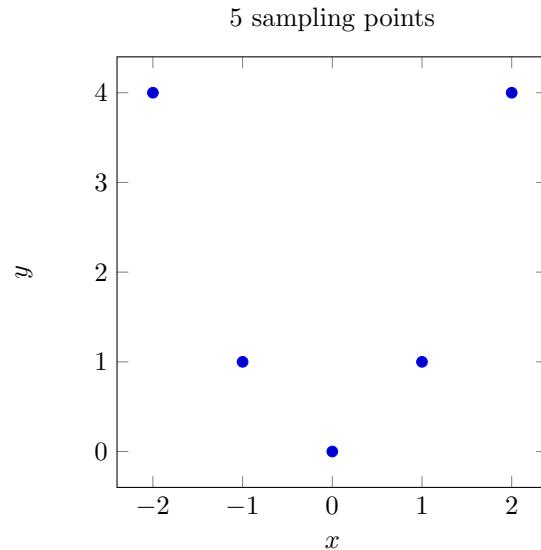


Figure 13.108: PGFplots: function sampling 5 points

```
\begin{tikzpicture}
\begin{axis}[title=55 sampling points,
 xlabel=x,ylabel=y,
 axis equal image]
\addplot+ [only
 → marks,domain=-2:2,samples=55]
 {x^2};
\end{axis}
\end{tikzpicture}
```

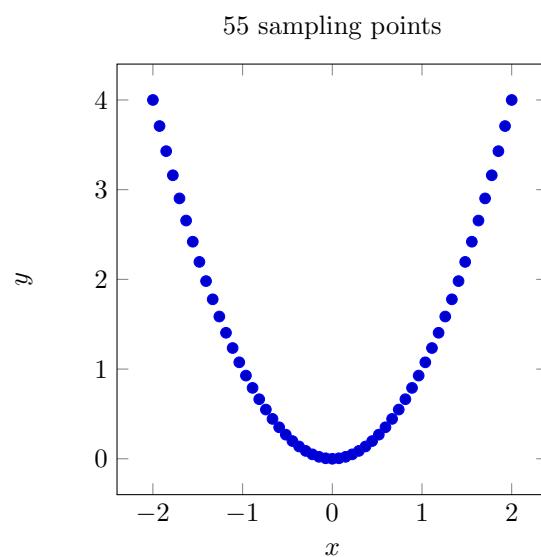


Figure 13.109: PGFplots: function sampling 55 points

```
\begin{tikzpicture}
\begin{axis}[
 trig format plots=rad, % angle in
 → radian
 axis equal image]
\addplot+ [only
 → marks,variable=t,domain=0:pi*3/2,
 samples=20] ({cos(t)},{sin(t)});
\end{axis}
\end{tikzpicture}
```

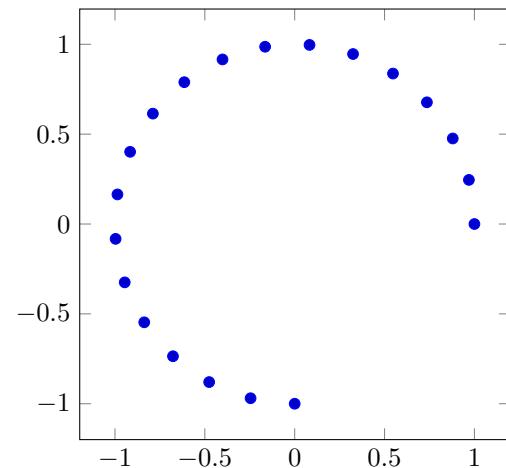


Figure 13.110: PGFplots: parametric function sampling

```
\begin{tikzpicture}
\begin{axis}[
 xlabel=x,ylabel=y,zlabel=z,
 axis lines=center,
 tick align=inside,
 xmin=-1.5,xmax=3.9,
 ymin=-1.5,ymax=3.9,
 zmin=-0.5,zmax=3.9,
 xtick distance=1,
 ytick distance=1,
 ztick distance=1,
 % width=10cm,
 % scale only axis,
 view={120}{30}, % perspective angle
 axis equal image,]
\addplot3+ coordinates{(1,0,0)};
\addplot3+ coordinates{(0,1,0)};
\addplot3+ coordinates{(0,0,1)};
\end{axis}
\end{tikzpicture}
```



Figure 13.111: PGFplots: 3D points

```
\begin{tikzpicture}
\begin{axis}[
 xlabel=x,ylabel=y,zlabel=z,
 grid=major
]
\addplot3+ [only marks] {x^2+y^2};
\end{axis}
\end{tikzpicture}
```

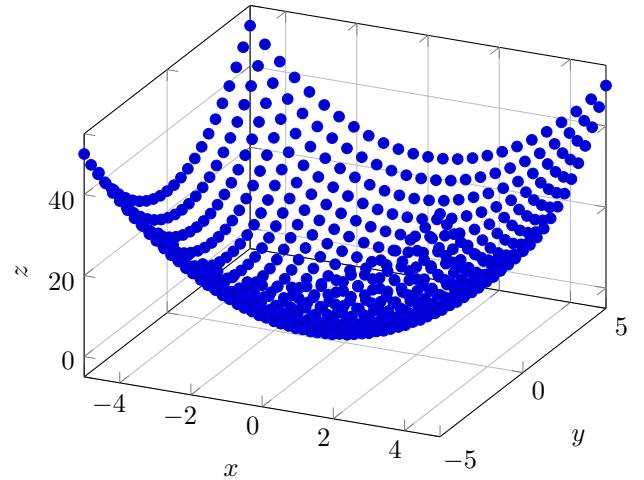


Figure 13.112: PGFplots: 3D function sampling points

#### 13.5.7.4 line

```
\begin{tikzpicture}
\begin{axis}[ytick=data]
\addplot coordinates {
(1,0.15) (2,0.21) (3,0.33) (4,0.4)
(2.5,.1) (3.5,.1)
};
\end{axis}
\end{tikzpicture}
```

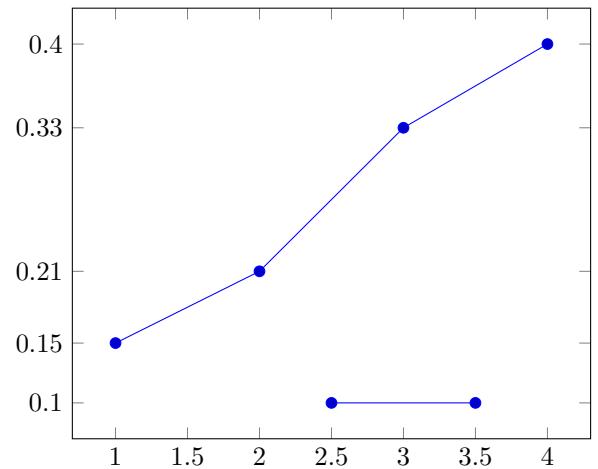


Figure 13.113: lines connecting adjacent points

`smooth` to make smooth curves or lines

```
\begin{tikzpicture}
 \begin{axis}[ytick=data]
 \addplot+ [smooth] coordinates {
 (1,0.15) (2,0.21) (3,0.33)
 (4,0.4)};
 \end{axis}
\end{tikzpicture}
```

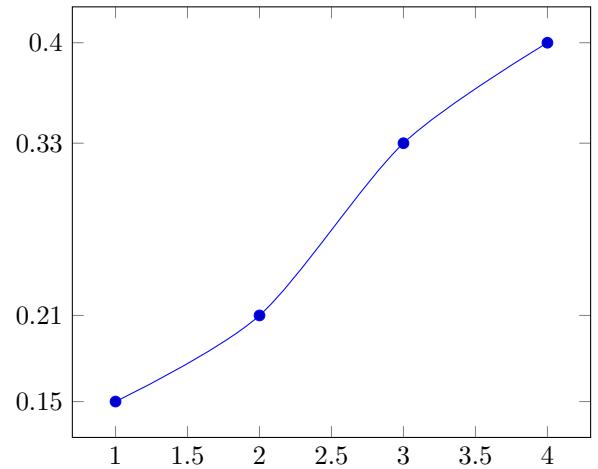
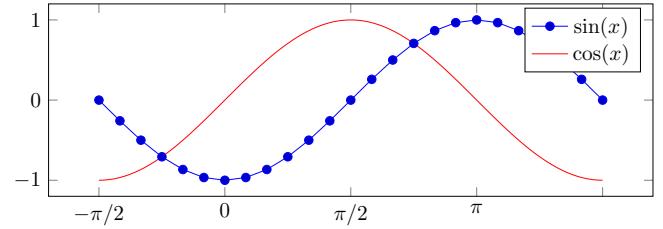


Figure 13.114: smooth lines connecting adjacent points

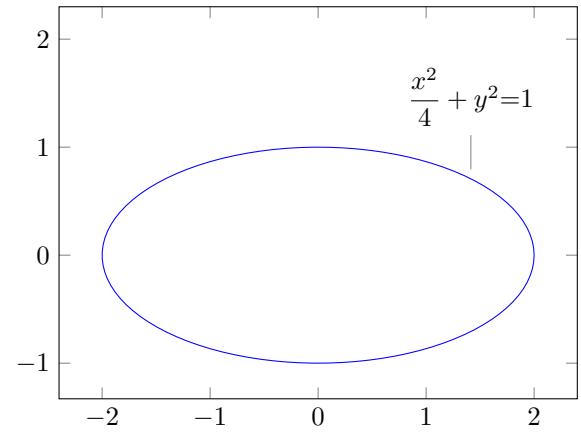
`no markers` to make no markers or points on the curves or lines

`\addlegendentry` to add legends

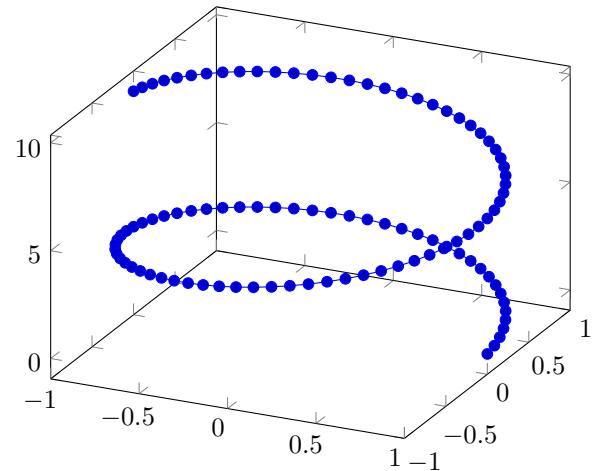
```
\begin{tikzpicture}
 \begin{axis}[
 trig format plots=rad,
 xtick distance=pi/2, ytick distance=1,
 xticklabels={-$-\pi$,$-\pi/2$,0,$\pi/2$,π},
 width=10cm, scale only axis,
 axis equal image]
 \addplot+ [domain=-pi:pi] {sin(x)};
 \addlegendentry{$\sin(x)$}
 \addplot+ [no
 markers,domain=-pi:pi,samples=100]
 {cos(x)};
 \addlegendentry{$\cos(x)$}
 \end{axis}
\end{tikzpicture}
```

Figure 13.115:  $\sin(x)$  and  $\cos(x)$ 

```
\begin{tikzpicture}
 \begin{axis}[
 trig format plots=rad,
 ymax=2.3,
 axis equal image]
 \addplot+ [no markers,
 variable=t,
 domain=0:2*pi,
 samples=100]
 ({2*cos(t)},{sin(t)});
 \node
 [pin=90:$\frac{x^2}{4}+y^2=1$]
 at ({2*cos(45)},{sin(45)}) {};
 \end{axis}
\end{tikzpicture}
```

Figure 13.116:  $\frac{x^2}{4} + y^2 = 1$

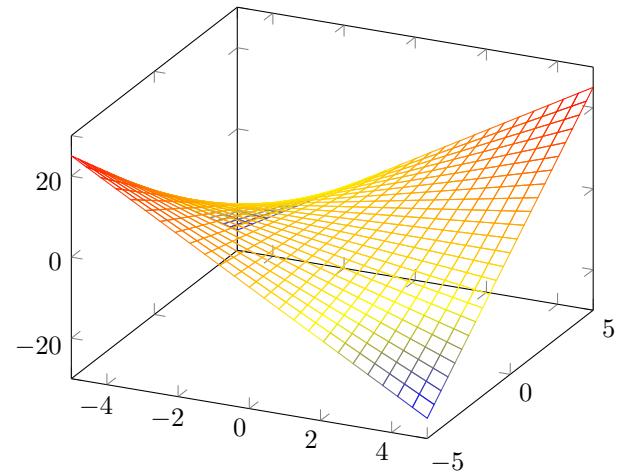
```
\begin{tikzpicture}
 \begin{axis}[trig format plots=rad]
 \addplot3+ [variable=t,
 domain=0:3*pi,
 samples=100,
 samples y=0]
 ({cos(t)},{sin(t)},{t});
 \end{axis}
\end{tikzpicture}
```

Figure 13.117:  $(\cos(t), \sin(t), t)$ 

### 13.5.7.5 plane

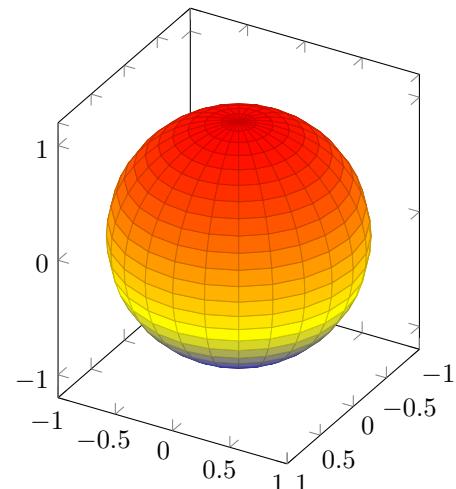
[mesh]

```
\begin{tikzpicture}
 \begin{axis}
 \addplot3+ [no markers, mesh] {x*y};
 \end{axis}
\end{tikzpicture}
```

Figure 13.118:  $f(x, y) = xy$ 

[surf] surface

```
\begin{tikzpicture}
 \begin{axis}[
 view={120}{30},
 trig format plots=rad,
 width=10cm,
 scale only axis,
 axis equal image
]
 \addplot3+ [no markers,
 surf,
 domain=0:2*pi,
 domain y=0:pi]
 ({sin(y)*cos(x)},{sin(y)*sin(x)},{cos(y)});
 \end{axis}
\end{tikzpicture}
```

Figure 13.119:  $x^2 + y^2 + z^2 = 1$

### 13.5.7.6 polar coordinate

```
data cs=polar|polarrad
```

```
\begin{tikzpicture}
\begin{axis}[
 title={$\rho=\cos 2\theta$},
 axis equal image
]
\addplot+ [no markers,
 data cs=polar,
 domain=0:360,
 samples=360
] (\x,{cos(2*\x)});
\end{axis}
\end{tikzpicture}
```

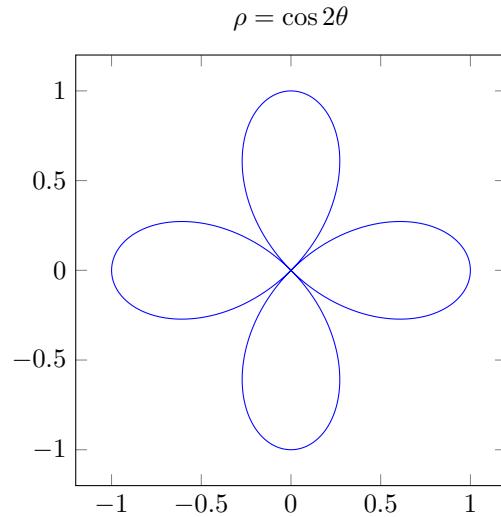


Figure 13.120: polar coordinate  $\rho = \cos 2\theta$

```
\usepgfplotslibrary{polar} to use \begin{polaraxis}
```

```
\usepgfplotslibrary{polar}
\begin{tikzpicture}
\begin{polaraxis}
\addplot+ coordinates
{(0,0) (60,1) (90,{sqrt(3)/2})} --
cycle;
\end{polaraxis}
\end{tikzpicture}
```

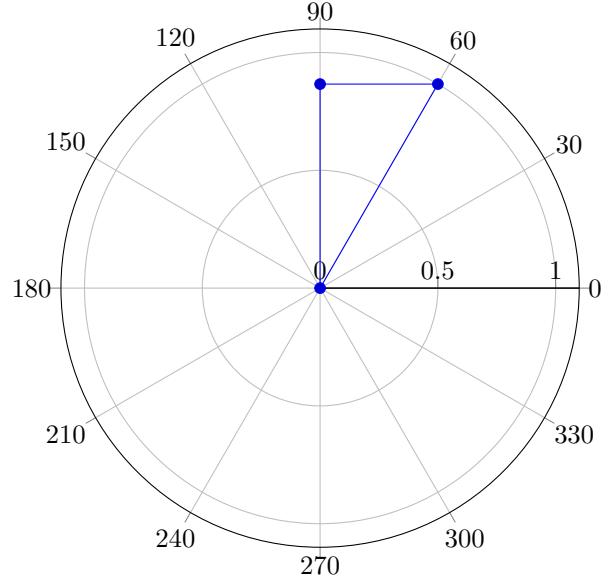


Figure 13.121: polar coordinate axes

<https://zhuanlan.zhihu.com/p/128341873>

### 13.5.8 Arnav Bandekar: Using pgfplots to make economic graphs in LaTeX

<https://towardsdatascience.com/using-pgfplots-to-make-economic-graphs-in-latex-bcdc8e27c0eb>

### 13.5.9 PGFplots gallery

<https://pgfplots.sourceforge.net/gallery.html>

```
\begin{tikzpicture}
\begin{axis}[
 xmin=-3, xmax=3,
 ymin=-3, ymax=3,
 extra x ticks={-1,1},
 extra y ticks={-2,2},
 extra tick style={grid=major},
]
\draw[red] \pgfextra{
 \pgfpathellipse{\pgfplotspointaxisxy{0}{0}}
 {\pgfplotspointaxisdirectionxy{1}{0}}
 {\pgfplotspointaxisdirectionxy{0}{2}}
 % see also the documentation of
 % 'axis direction cs' which
 % allows a simpler way to draw this
 \pgfpathellipse{\pgfplotspointaxisxy{0}{0}}
 {\pgfplotspointaxisdirectionxy{1}{1}}
 {\pgfplotspointaxisdirectionxy{0}{2}}
};
\addplot [only marks,mark=*] coordinates {
 (0,0) };
\end{axis}
\end{tikzpicture}
```

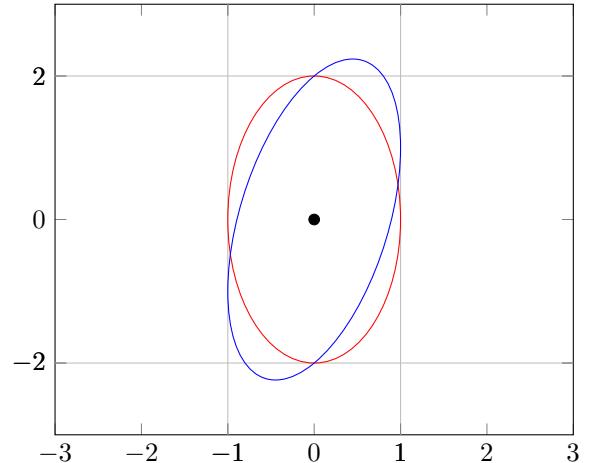


Figure 13.122: declare function

## 13.6 TikZplotLib / tikzplotlib

Python<sup>[12]</sup>

```
library(reticulate)

Warning: package 'reticulate' was built under R version 4.2.3
virtualenv_list()
virtualenv_python()
use_virtualenv("r-reticulate")

conda_list()
use_condaenv(condaenv = 'sandbox-3.9')

install TikZplotLib
virtualenv_install("r-reticulate", "tikzplotlib")

import TikZplotLib (it will be automatically discovered in "r-reticulate")
tikzplotlib <- import("tikzplotlib")
```

Error: ImportError: cannot import name 'common\_texification' from 'matplotlib.backends.backend\_pgf' (D:\Users)

<https://github.com/NixOS/nixpkgs/issues/289305>

The “solution” is to use **matplotlib 3.6**, but I guess in nixpkgs a single version is used at a time. The last working upgrade is from 0911608 I guess (I tried using virtualenv + pip + nix-ld + export LD\_LIBRARY\_PATH="LD\_LIBRARY\_PATH:NIX\_LD\_LIBRARY\_PATH")

<https://stackoverflow.com/questions/60882638/install-a-particular-version-of-python-package-in-a-virtualenv-created-with-reti>

```
#reticulate::virtualenv_install(packages = c("matplotlib==3.6.0"))
```

```

reticulate::conda_list()

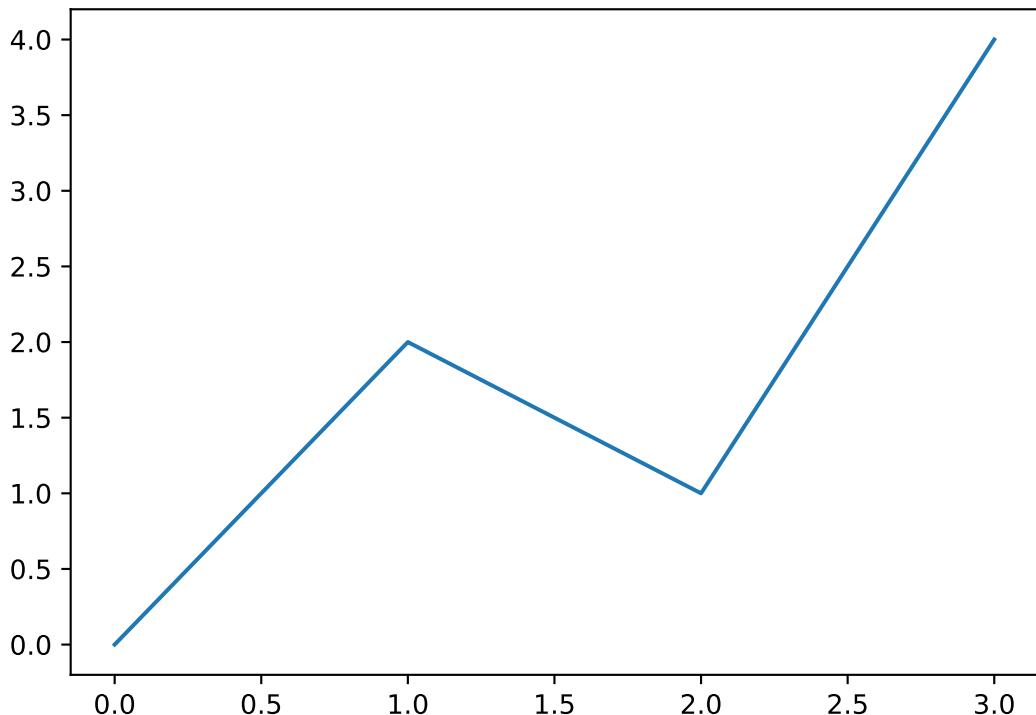
name python
1 base D:\\Anaconda3\\python.exe
2 fiftyone D:\\Anaconda3\\envs\\fiftyone\\python.exe
3 keras D:\\Anaconda3\\envs\\keras\\python.exe
4 labelme D:\\Anaconda3\\envs\\labelme\\python.exe
5 manim D:\\Anaconda3\\envs\\manim\\python.exe
6 mmyolo D:\\Anaconda3\\envs\\mmyolo\\python.exe
7 r-reticulate D:\\Anaconda3\\envs\\r-reticulate\\python.exe
8 rsconnect-jupyter D:\\Anaconda3\\envs\\rsconnect-jupyter\\python.exe
9 sandbox D:\\Anaconda3\\envs\\sandbox\\python.exe
10 sandbox-3.9 D:\\Anaconda3\\envs\\sandbox-3.9\\python.exe

reticulate::use_condaenv(condaenv = 'sandbox-3.9')

import matplotlib.pyplot as plt

plt.plot([0, 2, 1, 4])
plt.show()

```



```

import tikzplotlib

tikzplotlib.save("test.tex")
tikzplotlib.get_tikz_code()

'% This file was created with tikzplotlib v0.10.1.\n\\begin{tikzpicture}\n\\end{tikzpicture}\n'
import matplotlib.pyplot as plt

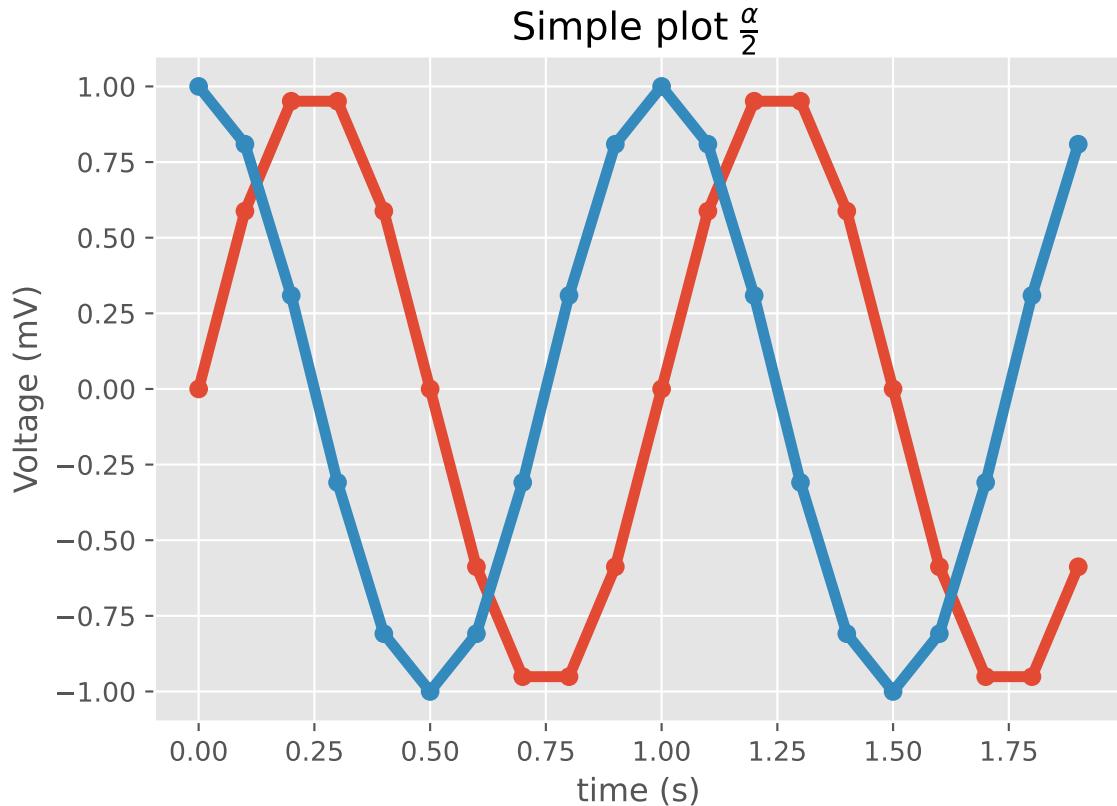
plt.close()

import matplotlib.pyplot as plt
import numpy as np

plt.style.use("ggplot")

```

```
t = np.arange(0.0, 2.0, 0.1)
s = np.sin(2 * np.pi * t)
s2 = np.cos(2 * np.pi * t)
plt.plot(t, s, "o-", lw=4.1)
plt.plot(t, s2, "o-", lw=4.1)
plt.xlabel("time (s)")
plt.ylabel("Voltage (mV)")
plt.title("Simple plot $\frac{\alpha}{2}$")
plt.grid(True)
plt.show()
```



```
import tikzplotlib

tikzplotlib.save("test.tex")
tikzplotlib.get_tikz_code()

'% This file was created with tikzplotlib v0.10.1.\n\\begin{tikzpicture}\n\n\\end{tikzpicture}\n'
```

## 13.7 animation

<https://zhuanlan.zhihu.com/p/338402487>

# Chapter 14

## xy-pic

<https://bookdown.org/yihui/rmarkdown-cookbook/install-latex-pkgs.html>

```
tinytex::install_tinytex()
```

### 14.1 pure LaTeX equation environment not rendering xy-pic in HTML

the following xymatrix from LaTeX package xy for xy-pic is not shown or rendered in HTML:

\$\backslash\$LaTeX\$ can only be used in HTML, not PDF

$$\begin{array}{ccccc} U & & & & \\ \searrow & \swarrow & & & \\ & (x,y) & \downarrow \varphi & & \\ & \downarrow \psi & & & \\ X & \times_Z & Y & \xrightarrow{p} & X \\ \downarrow q & & & & \downarrow f \\ Y & \xrightarrow{g} & Z & & \end{array}$$

### 14.2 R figure knit with TikZ engine

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

<https://bookdown.org/yihui/rmarkdown-cookbook/html-scroll.html>

<https://tex.stackexchange.com/questions/669083/how-can-i-draw-the-following-commutative-diagram>

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K \longrightarrow 1 \\ & & \text{id} \downarrow & & \theta \downarrow & & \text{id} \downarrow \\ 1 & \longrightarrow & H & \longrightarrow & H \times K & \longrightarrow & K \longrightarrow 1 \end{array}$$

Figure 14.1: xy-pic or xymatrix

$$\begin{array}{ccccccc}
 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K \longrightarrow 1 \\
 & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\
 1 & \longrightarrow & H & \longrightarrow & H \times K & \longrightarrow & K \longrightarrow 1
 \end{array}$$

Figure 14.2: tikz-cd

$$\begin{array}{ccccccc}
 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K \longrightarrow 1 \\
 & & \downarrow \text{id} & & \downarrow \theta & & \downarrow \text{id} \\
 1 & \longrightarrow & H & \longrightarrow & H \times K & \longrightarrow & K \longrightarrow 1
 \end{array}$$

```
\xymatrix{
 1 \ar[r] &
 H \ar[r]^{\alpha} \ar[d]_{\text{id}} & G \ar[r]^{\beta} \ar[d]_{\theta} & K \ar[r] & 1 \\
 1 \longrightarrow H \longrightarrow H \times K \longrightarrow K \longrightarrow 1
}
```

Figure 14.3: xy-pic or xymatrix



# Chapter 15

## statistics

### 15.1 Hung Hung

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUOqfNyA6mOD6lo2cc6clZZP>

<https://www.youtube.com/watch?v=3S2r4XBzKts>

population

普查 vs. 統計

random variable

$X$

sample has randomness

probability function

$$P_x(E) \in [0, 1]$$

event = subset of sample space

$E$

input event with output probability in 0 to 1

$$P_x : \{E_i\}_{i \in I} \rightarrow [0, 1]$$

target of interest

- probability function

but events are hard to be listed or enumerated

$$X : \{\omega_i\}_{i \in I} \rightarrow \mathbb{R}$$

CDF = cumulative distribution function

$$F_x(x) = P_x((-\infty, x]) = P_x(X \leq x)$$

real function is much easier to be operable, there is differentiation or difference operation

target of interest

- CDF = cumulative distribution function
- probability function

target of interest

- $P_x(\cdot)$  PF = probability function
  - $f_x(x) = P_x(X=x)$  PMF = probability mass function
  - $f_x(x) = \frac{d}{dx}P_x(X \leq x)$  PDF = probability density function
- $F_x(x)$  CDF = cumulative distribution function<sup>[15.1.2.1]</sup>
- $M_x(\xi)$  MGF = moment generating function<sup>[15.1.2.7.1]</sup>
- $\varphi_x(\xi)$  CF = characteristic function<sup>[15.1.2.7.2]</sup>

$$X \sim F_x(x) \stackrel{\text{FToC}}{\iff} \begin{array}{c} \uparrow \\ \varphi_x(\xi) \end{array} \quad \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge \begin{array}{c} \uparrow\downarrow \\ f_x(x) \end{array} \stackrel{\text{FToC}}{\iff} \begin{array}{c} \uparrow\downarrow \\ M_x(\xi) \end{array} \stackrel{\text{FToC}}{\iff} \begin{array}{c} \uparrow\downarrow \\ P_x \end{array} \quad \begin{array}{c} \nearrow \\ \rightarrow \end{array} \quad \begin{array}{c} \leftarrow \\ \rightarrow \end{array}$$

inversion formula :  $\uparrow \quad \uparrow \quad \uparrow$

$\Leftrightarrow \wedge \text{ supp}(f_x) \text{ is bounded } \{ \mu_n | n \in \mathbb{N} \}$

In population,

$$X \sim F_x(x)$$

by sampling,

$$X_1, \dots, X_i, \dots, X_n = X_i \sim F_x(x)$$

or

$$X_1, \dots, X_i, \dots, X_n = X_i \stackrel{\text{i.i.d.}}{\sim} F_x(x)$$

i.i.d. = independently identically distributed

and inference back

parametrically

$$\hat{X} \sim \hat{F}_x(x) = \hat{F}_x(x|\theta)$$

or nonparametrically

$$\hat{X} \sim \hat{F}_x(x)$$

inference is function of samples, or called random function, to estimate unknown parameters

$$\hat{\Theta} \leftarrow (X_1, \dots, X_i, \dots, X_n) = T(X_1, \dots, X_n) = T(\dots, X_i, \dots) = T(X_i)$$

correspondng CDF for inference or estimation function of sampling random variables

$$T(X_1, \dots, X_n) = T \sim F_T(t)$$

wish to be unbiased and consistent

$$\begin{cases} E(\hat{\Theta}) = \theta \Leftrightarrow E(\hat{\Theta}) - \theta = 0 & \text{unbiasedness} \\ V(\hat{\Theta}) = 0 & \text{consistency} \end{cases}$$

unbiasedness usually harder than consistency, thus usually first considered consistency.

modeling or parameterizing with unknown parameter  $\theta$

$$F_x(x) \stackrel{M}{=} F_x(x|\theta) = F_x(x;\theta)$$

parameterization is to reduce unknown parameters from infinite ones to finite ones

e.g. for normally distributed data

$$f_x(x) \stackrel{M}{=} f_x(x|\theta) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sqrt{2\pi\sigma^2}} = f_x(x|\mu, \sigma^2) = f_x(x|\mu, \sigma)$$

the price of parameterization is guess wrong model.

For some non-negative data, instead of normal distribution, consider distributions skewed to the right

- gamma
- exponential
- Weibull
- log-normal

topics

1.  $P_x$  probability theory
2.  $f_x(x) \stackrel{M}{=} f_x(x|\theta) = f_x(x; \theta)$  various univariable distribution
3.  $f_{\mathbf{x}}(\mathbf{x}) \stackrel{M}{=} f_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}) = f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\theta})$  multivariable distribution
4.  $T(X_1, \dots, X_n)$  inference
  - point estimation

$$\hat{\mu} = \begin{cases} \bar{X} & \rightarrow \mu \\ \text{median}(X_1, \dots, X_n) & \rightarrow \mu \\ \vdots \end{cases}$$

- interval estimation = hypothesis testing

$$\begin{cases} H_0 : \theta = \theta_0 & \leftarrow T \in \{0, 1\} \\ H_1 : \theta \neq \theta_0 \end{cases}$$

5. how to find  $T$
6. behavior of random function  $T(X_1, \dots, X_n) = T \sim F_T(t)$ 
  - statistical properties of  $T$
  - asymptotic properties

$$n \rightarrow \infty \begin{cases} \text{CLT} = \text{central limit theorem} \\ \text{LLN} = \text{law of large number} \end{cases}$$

### 15.1.1 probability theory

<https://www.youtube.com/watch?v=HBmTDtMBr3c>

**Definition 15.1.** sample space: The set  $S$  of all possible outcomes of an experiment is called the sample space

$$S = \{\omega_i\}_{i \in I}$$

**Definition 15.2.** event: An event  $E$  is any collection of possible outcomes of an experiment, i.e. any subset of  $S$

$$E \subseteq S$$

set operation

commutativity, associativity, distributivity

De Morgan law

pairwise disjoint = mutually exclusive

partition

#### 15.1.1.1 probability function

probability function axioms = probability function definition

Kolmogorov axioms of probability<sup>6</sup> p.72

**Definition 15.3.** probability function: Given a sample space  $S$  and its event  $E$ , a probability function is a function  $P$  satisfying

$$\begin{cases} P(S) = 1 \\ \forall E \subseteq S (P(E) \geq 0) \\ E_1, \dots, E_i, \dots \text{ are pairwise disjoint} \Rightarrow P\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} E_i \end{cases}$$

tossing a dice

theorems

$$P(\emptyset) = 0$$

$$P(E) \leq 1$$

$$P(E^C) = P(\bar{E}) = 1 - P(E)$$

$$P(E_2 \cap \bar{E}_1) = P(E_2) - P(E_2 \cap E_1)$$

$$E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$$

addition rule<sup>6</sup> p.75 and extended addition rule<sup>6</sup> p.76

inclusion-exclusion principle = sieve principle

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n \left( (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} P\left(\bigcap_{i \in \{i_1, \dots, i_k\}} E_i\right) \right)$$

symmetric difference<sup>6</sup> p.75

union probability upper-bounded by sum of individual probability

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \leq P(E_1) + P(E_2)$$

$$E_1 \cap E_2 = \emptyset \Leftrightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Boole inequality

$$P\left(\bigcup_{i \in I} E_i\right) \leq \sum_{i \in I} E_i$$

$$P\left(\widehat{H_0} \mid H_0\right) = P(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha = \text{type 1 error}$$

multiple hypothesis testing

How to control the family-wise error rate?

Ideally,

FWER = family-wise error rate

$$\begin{aligned}
\alpha &= P \left( \overbrace{H_0^1}^{\cup} \cup \cdots \cup \overbrace{H_0^m}^{\cap} \middle| H_0^1 \cap \cdots \cap H_0^m \right) = P(\text{reject any } H_0^i \mid \text{any } H_0^j \text{ is true}) \\
&= P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - P \left( \overbrace{\bigcup_{i=1}^m H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) \\
&= 1 - P \left( \bigcap_{i=1}^m \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - P \left( \text{not to reject any } H_0^i \middle| \bigcap_{j=1}^m H_0^j \right) \\
&\stackrel{H_0^j \text{ pairwise independent}}{=} 1 - \prod_{i=1}^m P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = 1 - \prod_{i=1}^m \left( 1 - P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) \right) \\
&\stackrel{\forall i, j \left[ P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = \alpha_0 \right]}{=} 1 - \prod_{i=1}^m (1 - \alpha_0) = 1 - (1 - \alpha_0)^m \\
\alpha &= 1 - (1 - \alpha_0)^m \\
\alpha_0 &= 1 - (1 - \alpha)^{\frac{1}{m}} = 1 - \sqrt[m]{1 - \alpha} \\
&\Downarrow \\
\text{set } P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) &= \alpha_0 = 1 - \sqrt[m]{1 - \alpha}
\end{aligned}$$

But condition  $H_0^j$  pairwise independent is too strong.

Practically,

$$\begin{aligned}
\alpha &= P \left( \overbrace{H_0^1}^{\cup} \cup \cdots \cup \overbrace{H_0^m}^{\cap} \middle| H_0^1 \cap \cdots \cap H_0^m \right) = P(\text{reject any } H_0^i \mid \text{any } H_0^j \text{ is true}) \\
&= P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) \stackrel{P \left( \bigcup_{i \in I} E_i \right) \leq \sum_{i \in I} E_i}{\leq} \sum_{i=1}^m P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) \stackrel{\text{Boole inequality}}{\leq} \sum_{i=1}^m \alpha_0 = m\alpha_0 \stackrel{\Downarrow}{=} \alpha \\
\text{let } \forall i, j \left[ P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = \alpha_0 \right] &\Rightarrow \sum_{i=1}^m P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = \sum_{i=1}^m \alpha_0 = m\alpha_0 \Rightarrow \alpha_0 = \frac{\alpha}{m} \\
&\Downarrow \\
\text{set } P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) &= \alpha_0 = \frac{\alpha}{m}
\end{aligned}$$

Bonferroni correction

$$P \left( \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) = \frac{\alpha}{m} \Rightarrow P \left( \bigcup_{i=1}^m \overbrace{H_0^i}^{\cap} \middle| \bigcap_{j=1}^m H_0^j \right) \leq \alpha$$

Bonferroni inequality<sup>6</sup> p.77

Bonferroni inequality and Boole inequality are equivalent inequalities

birthday problem<sup>6</sup> p.78

### 15.1.1.2 conditional probability

### 15.1.2 univariable distribution

$$\begin{cases} P_X(E) & \forall E \subseteq S \\ P_X(X \leq x) & \forall x \in \mathbb{R} \end{cases}$$

$$\begin{cases} P_x(X \in E) & \forall E \subseteq S \\ P_x(X \leq x) = P_x((-\infty, x]) = F_x(x) & \forall x \in \mathbb{R} \end{cases}$$

$$\begin{aligned} P_x(X \leq x) &= P_x((-\infty, x]) \\ &= P_x\left(\bigcup_{\epsilon>0}(-\infty, x-\epsilon]\right) = \lim_{\epsilon \rightarrow 0} P_x((-\infty, x-\epsilon]) \\ &\leftrightarrow P_x((-\infty, x)) = P_x(E), E = (-\infty, x) \end{aligned}$$

### 15.1.2.1 cumulative distribution function

CDF = cumulative distribution function

$$F_x(x) = P_x((-\infty, x]) = P_x(X \leq x)$$

$$X \sim P_x \leftrightarrow F_x(x)$$

$$X \sim F_x(x) \leftrightarrow P_x$$

**Definition 15.4.** CDF = cumulative distribution function: A cumulative distribution function is a function  $F : \mathbb{R} \rightarrow [0, 1]$  satisfying

$$F_x(x) = P_x((-\infty, x]) = P_x(X \leq x)$$

**Theorem 15.1.** CDF = cumulative distribution function:  $F(x)$  is a cumulative distribution function iff

$$\begin{cases} \lim_{x \rightarrow -\infty} F(x) = 0 & \lim_{x \rightarrow +\infty} F(x) = 1 \quad (01) [0, 1] \\ \forall x_1 < x_2 [F(x_1) \leq F(x_2)] & (nd) \text{ non-decreasing} \\ \lim_{x \rightarrow x_0^+} F(x) = F(x_0) & (rc) \text{ right-continuous} \end{cases}$$

**Definition 15.5.** RV = r.v. = random variable

$$\begin{cases} X \text{ is a continuous RV} & \lim_{x \rightarrow x_0} F_x(x) = F_x(x_0) \\ X \text{ is a discrete RV} & F_x \text{ is a step function of } x \end{cases}$$

<sup>6</sup> p.103

**Definition 15.6.** RV = r.v. = random variable

<sup>6</sup> p.104

**Definition 15.7.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned} \mathcal{R}_x &= \left\{ x \middle| \left\{ \begin{array}{l} \omega \in S \\ x \in X(\omega) \end{array} \right\} \right\} \\ &= \{x | \forall \omega \in S [x \in X(\omega)]\} \\ &= \{x | x \in X(\Omega)\} = X(\Omega) \end{aligned}$$

### 15.1.2.2 probability density function

$$\begin{cases} P_x(X \leq x) = P_x((-\infty, x]) = F_x(x) \\ P_x(X = x) = P_x(x) = ? \end{cases}$$

**Definition 15.8.** PDF = probability density function

PMF = probability mass function

$$\begin{cases} f_x(x) = \frac{d}{dx} F_x(x) & X \text{ continuous RV} \\ f_x(x) = F_x(x) - F_x(x^-) & X \text{ discrete RV} \end{cases}$$

$$\begin{cases} f_x(x) = \text{derivative of } F_x(x) & X \text{ continuous} \\ f_x(x) = \text{difference of } F_x(x) & X \text{ discrete} \end{cases}$$

$$\begin{cases} f_x(x) = \frac{d}{dx} F_x(x) & \Leftrightarrow F_x(x) = \int_{-\infty}^x f_x(t) dt \\ f_x(x) = F_x(x) - F_x(x^-) & \Leftrightarrow F_x(x) = \sum_{t \leq x} f_x(t) \end{cases}$$

$$\begin{cases} X \sim P_x \Leftrightarrow F_x(x) \leftrightarrow f_x(x) & \text{e.g. probability theory} \\ X \sim F_x(x) \leftrightarrow P_x & \Rightarrow F_x(x) \stackrel{M}{=} F_x(x|\theta) \text{ e.g. survival analysis} \\ X \sim f_x(x) \leftrightarrow F_x(x) \leftrightarrow P_x & \Rightarrow f_x(x) \stackrel{M}{=} f_x(x|\theta) \text{ e.g. general statistics} \end{cases}$$


---

**Theorem 15.2.** *PDF = probability density function or PMF = probability mass function:  $f(x)$  is a probability density function or probability mass function iff*

$$\begin{cases} \forall x \in \mathbb{R} [f(x) \geq 0] \\ \begin{cases} \int_{-\infty}^{+\infty} f(x) dx = 1 & t \text{ continuous} \\ \sum_{x \in X(\Omega)} f(x) = 1 & t \text{ discrete} \end{cases} \end{cases}$$


---

$$\forall E \subseteq S \left[ P_x(X \in E) = \begin{cases} \int_{x \in E} f_x(x) dx & X \text{ continuous} \\ \sum_{x \in E} f_x(x) & X \text{ discrete} \end{cases} \right]$$


---

<https://www.youtube.com/watch?v=KIXB1j-3M2k>

$$\begin{aligned} P_x(X = x) &= \lim_{\epsilon \rightarrow 0} P_x([x - \epsilon, x + \epsilon]) \\ &= \lim_{\epsilon \rightarrow 0} P_x(x - \epsilon \leq X \leq x + \epsilon) \\ &= \lim_{\epsilon \rightarrow 0} [F_x(x + \epsilon) - F_x(x - \epsilon)] \\ &= \begin{cases} F_x(x) - F_x(x) = 0 & X \text{ continuous} \\ F_x(x) - F_x(x^-) = f_x(x) & X \text{ discrete} \end{cases} \end{aligned}$$

$$X \sim F_x(x) \leftrightarrow P_x$$

$$Y = g(X)$$

$$\begin{cases} Y \sim F_Y(y) \leftrightarrow f_Y(y) & \Rightarrow F_Y(y) \stackrel{M}{=} F_Y(y|\theta) \\ Y \sim f_Y(y) \leftrightarrow F_Y(y) \leftrightarrow P_Y & \Rightarrow f_Y(y) \stackrel{M}{=} f_Y(y|\theta) \end{cases}$$

### 15.1.2.3 range vs. support

**Definition 15.9.** range of r.v. = range of RV = the range of a random variable

$$\begin{aligned} \mathcal{R}_x &= \left\{ x \middle| \left\{ \begin{array}{l} \omega \in S \\ x \in X(\omega) \end{array} \right\} \right\} \\ &= \{x | \forall \omega \in S [x \in X(\omega)]\} \\ &= \{x | x \in X(\Omega)\} = X(\Omega) \end{aligned}$$

**Definition 15.10.** support

$$\text{supp}(f) = \left\{ x \left| \begin{array}{l} f : D \rightarrow \mathcal{R} \\ x \in D \\ f_x(x) \neq 0 \end{array} \right. \right\}$$

**Definition 15.11.** support of r.v. = support of RV = the support of a random variable

$$\text{supp}(f_x) = \left\{ x \left| \begin{array}{l} x \in X(\Omega) \\ f_x(x) \neq 0 \end{array} \right. \right\} \stackrel{f_x(x) \geq 0}{=} \left\{ x \left| \begin{array}{l} x \in X(\Omega) \\ f_x(x) > 0 \end{array} \right. \right\}$$

#### 15.1.2.4 continuous monotone transformation

**Theorem 15.3.** Random variable  $Y$  is monotone transformation of random variable  $X$ , i.e.  $\begin{cases} X \sim F_x(x) \leftrightarrow f_x(x) \\ Y = g(X) \begin{cases} \forall x_1 < x_2 [g(x_1) < g(x_2)] \\ \forall x_1 < x_2 [g(x_1) > g(x_2)] \end{cases} \end{cases}$  then

$$f_Y(y) = f_x(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Proof:

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) \\ &= P(g(X) \leq y) \begin{cases} \forall x_1 < x_2 [g(x_1) < g(x_2)] \Leftrightarrow \forall g(x_1) < g(x_2) [x_1 < x_2] \\ \forall x_1 < x_2 [g(x_1) > g(x_2)] \Leftrightarrow \forall g(x_1) > g(x_2) [x_1 < x_2] \end{cases} \\ &= \begin{cases} P_X(X \leq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ P_X(X \geq g^{-1}(y) = x) & \forall y_1 > y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} P_X(X \leq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ P_X(X \geq g^{-1}(y) = x) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ &= \begin{cases} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ 1 - F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\ F_Y(y) &= \begin{cases} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ 1 - F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) = \begin{cases} \frac{d}{dy} F_X(g^{-1}(y)) & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{d}{dy} [1 - F_X(g^{-1}(y))] & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\
&= \begin{cases} \frac{dF_x(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{-dF_x(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\
&= \begin{cases} \frac{dF_x(g^{-1}(y))}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) < g^{-1}(y_2)] \\ \frac{dF_x(g^{-1}(y))}{dg^{-1}(y)} \frac{-dg^{-1}(y)}{dy} & \forall y_1 < y_2 [g^{-1}(y_1) > g^{-1}(y_2)] \end{cases} \\
&= \begin{cases} f_x(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} & \begin{cases} \frac{dg^{-1}(y)}{dy} \geq 0 \\ f_x(g^{-1}(y)) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \\ f_x(g^{-1}(y)) \frac{-dg^{-1}(y)}{dy} & \begin{cases} \frac{-dg^{-1}(y)}{dy} \geq 0 \\ f_x(g^{-1}(y)) \geq 0 \end{cases} \Rightarrow f_Y(y) \geq 0 \end{cases} \\
f_Y(y) &= \begin{cases} f_x(g^{-1}(y)) \frac{dg^{-1}(y)}{dy} & \frac{dg^{-1}(y)}{dy} \geq 0 \\ f_x(g^{-1}(y)) \frac{-dg^{-1}(y)}{dy} & \frac{-dg^{-1}(y)}{dy} \geq 0 \end{cases} \\
f_Y(y) &= f_x(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|
\end{aligned}$$

□

segment  $g(X)$  into monotone functions

For example,  $\begin{cases} g(x) = x^2 \\ Y = g(X) \end{cases} \Rightarrow Y = g(X) = X^2$ ,

$$\begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases}$$

$$\begin{aligned}
Y &= g(X) = X^2 \\
\Rightarrow Y &= \begin{cases} X^2 = g(X) & X \geq 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2 [X_1^2 < X_2^2] \\ X^2 = g(X) & X < 0 \Leftrightarrow X \in (-\infty, 0) \Rightarrow \forall X_1 < X_2 [X_1^2 > X_2^2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & X \geq 0 \Rightarrow \forall X_1^2 < X_2^2 [X_1 < X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 < X_2] \\ -\sqrt{Y} = g^{-1}(Y) & X < 0 \Rightarrow \forall X_1^2 < X_2^2 [X_1 > X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 > X_2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \geq 0 \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X < 0 \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases}
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= P_Y(Y \leq y) = P(X^2 \leq y) \\
&= P(\{X^2 \leq y\} \cap (\{X < 0\} \cup \{X \geq 0\})) \\
&= P((\{X^2 \leq y\} \cap \{X < 0\}) \cup (\{X^2 \leq y\} \cap \{X \geq 0\})) \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\
&\quad - P((\{X^2 \leq y\} \cap \{X < 0\}) \cap (\{X^2 \leq y\} \cap \{X \geq 0\})) \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) - P(\emptyset) \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) - 0 \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\
&= P(\{-X \leq \sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\
&= P(\{X \geq -\sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\
&= P_X(-\sqrt{y} \leq X < 0) + P_X(0 \leq X \leq \sqrt{y}) \\
&= [F_X(0) - F_X(-\sqrt{y})] + [F_X(\sqrt{y}) - F_X(0)] \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}$$

□

Another example,  $\begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases}$ ,

$$\begin{aligned}
Y &= g(X) = X^2 \\
\Rightarrow Y &= \begin{cases} X^2 = g(X) & X \geq 0 \Leftrightarrow X \in [0, +\infty) \Rightarrow \forall X_1 < X_2 [X_1^2 < X_2^2] \\ X^2 = g(X) & -1 \leq X < 0 \Leftrightarrow X \in [-1, 0) \Rightarrow \forall X_1 < X_2 [X_1^2 > X_2^2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow \forall X_1^2 < X_2^2 [X_1 < X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 < X_2] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in [-1, 0) \Rightarrow \forall X_1^2 < X_2^2 [X_1 > X_2] \Rightarrow \forall Y_1 < Y_2 [X_1 > X_2] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in [0, \infty) \Rightarrow X \in [0, \infty) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases} \\
\Rightarrow X &= \begin{cases} \sqrt{Y} = g^{-1}(Y) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ \sqrt{Y} = g^{-1}(Y) & Y \in [0, 1] \Rightarrow X \in [0, 1] \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) < g^{-1}(Y_2)] \\ -\sqrt{Y} = g^{-1}(Y) & Y \in (0, 1] \Rightarrow X \in [-1, 0) \Rightarrow \forall Y_1 < Y_2 [g^{-1}(Y_1) > g^{-1}(Y_2)] \end{cases}
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= P_Y(Y \leq y) = P(X^2 \leq y) \begin{cases} Y = g(X) = X^2 \\ X \in [-1, \infty) \end{cases} \\
&= P(\{X^2 \leq y\} \cap (\{X < 0\} \cup \{X \geq 0\})) = \dots \text{ as } \begin{cases} Y = g(X) = X^2 \\ X \in (-\infty, +\infty) \end{cases} \\
&= P(\{X^2 \leq y\} \cap \{X < 0\}) + P(\{X^2 \leq y\} \cap \{X \geq 0\}) \\
&= P(\{-X \leq \sqrt{y}\} \cap \{X < 0\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\}) \\
&= \begin{cases} P(\{-X \leq \sqrt{y}\} \cap \{X < 0\} \cap \{X > 1\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\} \cap \{X > 1\}) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P(\{-X \leq \sqrt{y}\} \cap \{X < 0\} \cap \{X \geq -1\}) + P(\{X \leq \sqrt{y}\} \cap \{X \geq 0\} \cap \{X \leq 1\}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} P(\emptyset) + P_x(1 < X \leq \sqrt{y}) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P(\{X \geq -\sqrt{y}\} \cap \{X < 0\} \cap \{X \geq -1\}) + P_x(0 \leq X \leq \min\{\sqrt{y}, 1\}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} 0 + [F_x(\sqrt{y}) - F_x(1)] & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P_x(\max\{-1, -\sqrt{y}\} \leq X < 0) + P_x(0 \leq X \leq \sqrt{y}) & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} F_x(\sqrt{y}) - F_x(1) & Y \in (1, \infty) \Rightarrow X \in (1, \infty) \\ P_x(-\sqrt{y} \leq X < 0) + [F_x(\sqrt{y}) - F_x(0)] & Y \in [0, 1] \Rightarrow \begin{cases} X \in [0, 1] \\ X \in [-1, 0] \end{cases} \end{cases} \\
&= \begin{cases} F_x(\sqrt{y}) - F_x(1) & y > 1 \\ F_x(\sqrt{y}) - F_x(-\sqrt{y}) & -1 \leq y \leq 1 \end{cases}
\end{aligned}$$

□

### 15.1.2.5 discrete monotone transformation

$$\begin{cases} Y = g(X) = X^2 \\ X \text{ discrete} \Rightarrow Y \text{ discrete} \end{cases}$$

$$\begin{aligned}
f_Y(y) &= P_Y(Y = y) \\
&= P(X^2 = y) \\
&= P(\{X = \sqrt{y}\} \cup \{X = -\sqrt{y}\}) \\
&= P(\{X = \sqrt{y}\}) + P(\{X = -\sqrt{y}\}) - P(\{X = \sqrt{y}\} \cap \{X = -\sqrt{y}\}) \\
&= P_x(X = \sqrt{y}) + P_x(X = -\sqrt{y}) - P(\emptyset) \\
&= P_x(X = \sqrt{y}) + P_x(X = -\sqrt{y}) - 0 \\
&= P_x(X = \sqrt{y}) + P_x(X = -\sqrt{y}) \\
&= f_x(\sqrt{y}) + f_x(-\sqrt{y})
\end{aligned}$$

□

**Theorem 15.4.** *discrete monotone transformation*

$$\begin{aligned}
&\begin{cases} Y = g(X) \\ X \text{ discrete} \Rightarrow Y \text{ discrete} \end{cases} \\
&\Downarrow \\
f_Y(y) &= \sum_{\{x|g(x)=y\}} f_x(x) = \sum_{\{x|x=g^{-1}(y)\}} f_x(x)
\end{aligned}$$

Proof:

$$\begin{aligned}
f_Y(y) &= \text{P}_Y(Y = y) \\
&= \text{P}(g(X) = y) = \sum_{t \in \{x|g(x)=y\}} f_X(t) = \sum_{x \in \{x|g(x)=y\}} f_X(x) = \sum_{\{x|g(x)=y\}} f_X(x) \\
&= \text{P}_X(X = g^{-1}(y)) = \sum_{t \in \{x|x=g^{-1}(y)\}} f_X(t) = \sum_{x \in \{x|x=g^{-1}(y)\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x) \\
f_Y(y) &= \text{P}_Y(Y = y) \\
&= \text{P}(g(X) = y) = \sum_{t \in \{x|g(x)=y\}} f_X(t) = \sum_{x \in \{x|g(x)=y\}} f_X(x) = \sum_{\{x|g(x)=y\}} f_X(x) \\
&= \text{P}_X(X = g^{-1}(y)) = \sum_{t \in \{x|x=g^{-1}(y)\}} f_X(t) = \sum_{x \in \{x|x=g^{-1}(y)\}} f_X(x) = \sum_{\{x|x=g^{-1}(y)\}} f_X(x)
\end{aligned}$$

□

**Theorem 15.5.** probability integral transformation

$$\begin{array}{c}
\left\{ \begin{array}{l} X \text{ continuous} \quad (c) \\ X \sim F_X(x) \quad (d) \\ Y = F_X(X) \quad (t) \end{array} \right. \\
\Downarrow \\
F_Y(y) = y, \forall y \in [0, 1] \\
\Downarrow \text{def.} \\
Y \sim U = U(y) \Leftrightarrow Y \sim U(y) \Leftrightarrow Y \text{ is uniformly distributed on } [0, 1]
\end{array}$$

Proof:

$$\begin{aligned}
F_Y(y) &= \text{P}_Y(Y \leq y) \stackrel{(t)}{=} \text{P}(F_X(X) \leq y), \forall x_1 < x_2 [F_X(x_1) < F_X(x_2)] \Rightarrow \left\{ \begin{array}{l} \exists F_X^{-1} : Y \rightarrow X \\ \forall y_1 < y_2 [F_X^{-1}(y_1) < F_X^{-1}(y_2)] \end{array} \right. \\
&= \text{P}_X(X \leq F_X^{-1}(y) = x) = \text{P}_X(X \leq x), x = F_X^{-1}(y) \\
&= \text{P}_X(X \leq x) = F_X(x) \stackrel{x=F_X^{-1}(y)}{\equiv} F_X(F_X^{-1}(y)) = y \\
F_Y(y) &= y
\end{aligned}$$

□

Note:

According to Theorem 15.5,

$$\begin{aligned}
U &= F_X(X) \stackrel{15.5}{\sim} U(u) \text{ on } [0, 1] \\
\Rightarrow X &= F_X^{-1}(U) \wedge X \sim F_X(x) \Rightarrow F_X^{-1}(U) = X \sim F_X(x) \Rightarrow F_X^{-1}(U) \sim F_X(x) \\
\Rightarrow X &= F_X^{-1}(U) \sim F_X(x) \\
&\text{i.e. uniform random variables substituted into the inverse of } F_X, \\
&\text{we can get random variables follow } F_X(x)
\end{aligned}$$

#### 15.1.2.6 expected value

$$\text{E}(g(X)) = \text{E}[g(X)] = \text{E}g(X) = \mathbb{E}[g(X)] = \mathbb{E}g(X)$$

**Definition 15.12.** expected value: The expected value of a random variable  $g(X)$  is

$$\text{E}(g(X)) = \text{E}[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}$$

**Definition 15.13.** expected value or expectation function: The expected value of a random variable  $X$  is

$$E(X) = E[X] = \begin{cases} \int_{-\infty}^{+\infty} xf_X(x) dx = 1 & X \text{ continuous} \\ \sum_{x \in X(\Omega)} xf_X(x) = 1 & X \text{ discrete} \end{cases}$$

**Theorem 15.6.** *the rule of the lazy statistician*

*the law of the unconscious statistician = the LOTUS*

$$E(g(X)) = E[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} g(x) f_X(x) & X \text{ discrete} \end{cases}$$

Proof:<sup>7</sup> p.162 for p.119

Discrete case:

to be proved

Continuous case:

to be proved

□

By linearity of  $\int$  and  $\sum$ , expected values have the following properties or theorems,

- $E[a_1g_1(X_1) + a_2g_2(X_2) + c] = a_1E[g_1(X_1)] + a_2E[g_2(X_2)] + c$
- $\forall x \in \mathbb{R} [g(x) \geq 0] \Rightarrow E[g(X)] \geq 0$
- $\forall x \in \mathbb{R} [g_1(x) \geq g_2(x)] \Rightarrow E[g_1(X)] \geq E[g_2(X)]$
- $\forall x \in \mathbb{R} [a \leq g(x) \leq b] \Rightarrow a \leq E[g(X)] \leq b$

**Theorem 15.7.**  $E[X]$  minimizes Euclidean distance  $E[(X - b)^2]$  over  $b$ , i.e.

$$E[X] = \arg \min_b E[(X - b)^2]$$

Proof:

$$\begin{aligned}
E[(X - b)^2] &= E[(X - E[X] + E[X] - b)^2] \\
&= E[\{(X - E[X]) + (E[X] - b)\}^2] \\
&= E[(X - E[X])^2 + 2(X - E[X])(E[X] - b) + (E[X] - b)^2] \\
&= E[(X - E[X])^2] + 2(E[X] - b)E[(X - E[X])] + E[(E[X] - b)^2] \\
&= E[(X - E[X])^2] + 2(E[X] - b)E[X - E[X]] + (E[X] - b)^2 \\
&= E[(X - E[X])^2] + 2(E[X] - b)(E[X] - E[X]) + (E[X] - b)^2 \\
&= E[(X - E[X])^2] + 2(E[X] - b)0 + (E[X] - b)^2 \\
&= E[(X - E[X])^2] + 0 + (E[X] - b)^2 \\
&= E[(X - E[X])^2] + (E[X] - b)^2 \stackrel{(E[X]-b)^2 \geq 0}{\geq} E[(X - E[X])^2]
\end{aligned}$$

$\Downarrow$

$$E[(X - b)^2] \geq E[(X - E[X])^2]$$

$\Downarrow$

$$E[(X - b)^2] = E[(X - E[X])^2] \text{ holds if } (E[X] - b)^2 = 0 \Rightarrow b = E[X] \Rightarrow E[X] = \arg \min_b E[(X - b)^2]$$

□

Note:

When  $b = E[X]$ ,  $E[(X - b)^2]$  has minimum loss  $E[(X - E[X])^2] = V[X] = V(X)$ , i.e. defintion of variance appears.

---

**Theorem 15.8.** median  $[X]$  minimizes  $E[|X - b|]$  over  $b$ , i.e.

$$\text{median}[X] = \arg \min_b E[|X - b|]$$

Proof:

to be proved

□

Note:

When  $b = \text{median}[X]$ ,  $E[(X - b)^2]$  has minimum loss  $E[|X - \text{median}[X]|]$ , i.e. defintion of MAD(mean absolute deviation) in robust statistics appears.

---

**Definition 15.14.** indicator function

$$\begin{aligned}
1(E) &= 1(x \in E) = 1(\{x \in E\}) = 1(\{x | x \in E\}) = \begin{cases} 1 & E \\ 0 & \bar{E} \end{cases} = \begin{cases} 1 & \text{if } E \\ 0 & \text{if } \bar{E} = E^C \end{cases} \\
&= \begin{cases} 1 & \text{if event } E \text{ occurs} \\ 0 & \text{if event } E \text{ does not occur} \end{cases}
\end{aligned}$$

Note:

**Theorem 15.9.** probability as expected value

$$\mathrm{P}_x(E) = \mathrm{P}(x \in E) = \mathrm{E}[1(X \in E)]$$

Proof:

$$\begin{aligned}\mathrm{P}_x(E) &= \mathrm{P}(x \in E) = \int_{x \in E} f_x(x) dx = \int_E f_x(x) dx \\ &= \int 1(x \in E) f_x(x) dx \\ &= \int g(x) f_x(x) dx, g(x) = 1(x \in E) \\ &= \mathrm{E}[g(X)], g(X) = 1(X \in E) \\ &= \mathrm{E}[1(X \in E)] \\ \mathrm{P}_x(E) &= \mathrm{P}(x \in E) = \mathrm{E}[1(X \in E)]\end{aligned}$$

□

Iverson bracket [https://en.wikipedia.org/wiki/Iverson\\_bracket](https://en.wikipedia.org/wiki/Iverson_bracket)

$$\begin{cases} v(p(x)) = \text{T} \Leftrightarrow [p(x)] = 1 \\ v(p(x)) = \text{F} \Leftrightarrow [p(x)] = 0 \end{cases}$$

$$[p(x)] = \begin{cases} 1 & v(p(x)) = \text{T} \\ 0 & v(\neg p(x)) = \text{T} \end{cases} = \begin{cases} 1 & p(x) \\ 0 & \neg p(x) \end{cases}$$

negation = NOT

$$[\neg p] = 1 - [p]$$

in set theory or domain of events,

$$1(\overline{E}) = 1 - 1(E)$$

conjunction = AND

$$[p \wedge q] = [p][q]$$

in set theory or domain of events,

$$1(E_1 \cap E_2) = 1(E_1) 1(E_2)$$

disjunction = OR

$$[p \vee q] = [p] + [q] - [p][q] = [p] + [q] - [p \wedge q]$$

Proof:

in set theory or domain of events,

$$\begin{aligned}1(E_1 \cup E_2) &\stackrel{\text{de Moivre}}{=} 1(\overline{\overline{E}_1 \cap \overline{E}_2}) \\ &= 1 - 1(\overline{E}_1 \cap \overline{E}_2) = 1 - 1(\overline{E}_1) 1(\overline{E}_2) \\ &= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\ &= 1 - [1 - 1(E_1)][1 - 1(E_2)] \\ &= 1 - [1 - 1(E_1) - 1(E_2) + 1(E_1) 1(E_2)] \\ &= 1(E_1) + 1(E_2) - 1(E_1) 1(E_2) \\ &= 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)\end{aligned}$$

$$\mathbf{1}(E_1 \cup E_2) = \mathbf{1}(E_1) + \mathbf{1}(E_2) - \mathbf{1}(E_1)\mathbf{1}(E_2) = \mathbf{1}(E_1) + \mathbf{1}(E_2) - \mathbf{1}(E_1 \cap E_2)$$

□

implication = conditional

$$\begin{aligned} [p \rightarrow q] &= [\neg p \vee q] \\ &= [\neg p] + [q] - [\neg p][q] \\ &= 1 - [p] + [q] - (1 - [p])[q] \\ &= 1 - [p] + [p][q] \end{aligned}$$

exclusive disjunction = XOR

$$\begin{aligned} [p \vee q] &= [p \oplus q] = |[p] - [q]| = ([p] - [q])^2 \\ &= [p](1 - [q]) + (1 - [p])[q] \end{aligned}$$

biconditional = XNOR

$$[p \leftrightarrow q] = [p \odot q] = [\neg(p \oplus q)] = [\neg(p \vee q)] = ([p] + (1 - [q]))((1 - [p]) + [q])$$

Kronecker delta

$$\delta_{ij} = [i = j]$$

single-argument notation

$$\delta_i = \delta_{i0} = \begin{cases} 1 & i = j = 0 \\ 0 & i \neq j = 0 \end{cases}$$

sign function

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 = [x > 0] - [x < 0] \\ -1 & x < 0 \end{cases}$$

absolute function

$$\begin{aligned} |x| &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases} = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases} \\ &= \begin{cases} x \cdot 1 & x > 0 \\ x \cdot 0 & x = 0 \\ x \cdot (-1) & x < 0 \end{cases} = \begin{cases} x \cdot \operatorname{sgn}(x) & x > 0 \\ x \cdot \operatorname{sgn}(x) & x = 0 \\ x \cdot \operatorname{sgn}(x) & x < 0 \end{cases} \\ &= x \cdot \operatorname{sgn}(x) = x([x > 0] - [x < 0]) = x[x > 0] - x[x < 0] \end{aligned}$$

binary min and max function

$$\max(x, y) = x[x > y] + y[x \leq y]$$

$$\min(x, y) = x[x \leq y] + y[x > y]$$

binary max function

$$\max(x, y) = \frac{x + y + |x - y|}{2}$$

floor and ceiling functions

floor function

$$\begin{aligned} \lfloor x \rfloor &= n, n \leq x < n + 1 \\ &= \sum_{n \in \mathbb{N}} n [n \leq x < n + 1] \end{aligned}$$

ceiling function

$$\begin{aligned} \lceil x \rceil &= n, n - 1 < x \leq n \\ &= \sum_{n \in \mathbb{N}} n [n - 1 < x \leq n] \end{aligned}$$

Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} = [x > 0] = 1_{(0, \infty)}(x)$$

or conveniently define “unit step function”

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} = [x \geq 0] = 1_{[0, \infty)}(x)$$

ramp function = rectified linear unit activation function = ReLU

$$\text{ReLU}(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} = x [x \geq 0]$$

indicator function

$$A \subseteq X \Rightarrow \begin{cases} 1_A : X \rightarrow \{0, 1\} \\ 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \end{cases} \Leftrightarrow x \in X \xrightarrow{1_A} \{0, 1\} = [x \in A] = \begin{cases} 1 & v(x \in A) = T \\ 0 & v(\neg(x \in A)) = T \end{cases}$$

$A, B \subseteq \Omega$ ,

$$A = B \Leftrightarrow 1_A = 1_B$$

$$A = \Omega \Leftrightarrow 1_A(x) = 1$$

$$A = \emptyset \Leftrightarrow 1_A(x) = 0$$

**Theorem 15.10.** *subset indicator order*

$$A \subset B \Rightarrow 1_A(x) \leq 1_B(x)$$

Proof:

$$\begin{aligned}
& \forall x (1_A(x) = 1 \Rightarrow 1_B(x) = 1) \\
\Leftrightarrow & \forall x (\neg 1_A(x) = 1 \vee 1_B(x) = 1) \\
\Leftrightarrow & \forall x (\neg 1_A(x) = 1 \wedge \neg 1_B(x) = 1) \\
\Rightarrow & \neg \exists x (1_A(x) = 1 \wedge 1_B(x) = 0) \\
\Rightarrow & \neg \exists x (1_B(x) = 0 < 1 = 1_A(x)) \\
\Rightarrow & \neg \exists x (1_B(x) < 1_A(x)) \\
\Rightarrow & \forall x (1_B(x) \geq 1_A(x))
\end{aligned}$$

□

in set theory or domain of events,

$$1(E_1 \cap E_2) = 1(E_1) 1(E_2)$$

$$1(\overline{E}) = 1 - 1(E)$$

$$1(E_1 \cup E_2) = 1(E_1) + 1(E_2) - 1(E_1) 1(E_2) = 1(E_1) + 1(E_2) - 1(E_1 \cap E_2)$$


---

expectation in many perspectives

$$Y = g(X)$$

$$\int_{-\infty}^{+\infty} y f_Y(y) dy = E[Y] = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

$$E_Y[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy = E[Y] = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx = E_X[g(X)]$$

$$E_Y[Y] = E_X[g(X)]$$


---

$$\begin{aligned}
& E[a_1 g_1(X_1) + a_2 g_2(X_2) + c], \begin{cases} Y_1 = g_1(X_1) \\ Y_2 = g_2(X_2) \end{cases} \\
= & E[a_1 Y_1 + a_2 Y_2 + c]
\end{aligned}$$

$$\begin{aligned}
E[a_1 g_1(X_1) + a_2 g_2(X_2) + c] &= a_1 E[g_1(X_1)] + a_2 E[g_2(X_2)] + c \\
&= E[a_1 Y_1 + a_2 Y_2 + c] = a_1 E[Y_1] + a_2 E[Y_2] + c
\end{aligned}$$

$$a_1 E[g_1(X_1)] + a_2 E[g_2(X_2)] + c = a_1 E_{X_1}[g_1(X_1)] + a_2 E_{X_2}[g_2(X_2)] + c$$

$$a_1 E[Y_1] + a_2 E[Y_2] + c = a_1 E_{Y_1}[Y_1] + a_2 E_{Y_2}[Y_2] + c$$

$$a_1 E_{X_1}[g_1(X_1)] + a_2 E_{X_2}[g_2(X_2)] + c = a_1 E_{Y_1}[Y_1] + a_2 E_{Y_1}[Y_2] + c$$

### 15.1.2.7 moment

**Definition 15.15.**  $n^{\text{th}}$  moment: For each integer  $n$ , the  $n^{\text{th}}$  moment of  $X$  is  $E[X^n]$ .

The  $n^{\text{th}}$  central moment of  $X$  is  $\mu_n = E[(X - E[X])^n]$ .

$$\begin{aligned} E[X^n] &= \begin{cases} \int_{-\infty}^{+\infty} x^n f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^n f_X(x) & X \text{ discrete} \end{cases} \\ \mu = E[X^1] = E[X] &= \begin{cases} \int_{-\infty}^{+\infty} x^1 f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) & X \text{ discrete} \end{cases} \\ \mu_n = E[(X - E[X])^n] &= \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^n f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^n f_X(x) & X \text{ discrete} \end{cases} \end{aligned}$$


---

1<sup>st</sup> moment of  $X$  = mean

$$\mu = E[X^1] = E[X] = \begin{cases} \int_{-\infty}^{+\infty} x^1 f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) & X \text{ discrete} \end{cases}$$

1<sup>st</sup> central moment of  $X$  = 0

$$\begin{aligned} \mu_1 = E[(X - E[X])^1] &= \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^1 f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^1 f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu) f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu) f_X(x) & X \text{ discrete} \end{cases} \\ &= \begin{cases} \int_{-\infty}^{+\infty} x f_X(x) dx - \int_{-\infty}^{+\infty} \mu f_X(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} x f_X(x) - \sum_{x \in X(\Omega)} \mu f_X(x) & X \text{ discrete} \end{cases} \\ &= \begin{cases} E[X] - \mu \int_{-\infty}^{+\infty} f_X(x) dx & X \text{ continuous} \\ E[X] - \mu \sum_{x \in X(\Omega)} f_X(x) & X \text{ discrete} \end{cases} = \begin{cases} E[X] - \mu \cdot 1 & X \text{ continuous} \\ E[X] - \mu \cdot 1 & X \text{ discrete} \end{cases} \\ &= \begin{cases} E[X] - \mu & X \text{ continuous} \\ E[X] - \mu & X \text{ discrete} \end{cases} = \begin{cases} E[X] - E[X] & X \text{ continuous} \\ E[X] - E[X] & X \text{ discrete} \end{cases} \\ &= \begin{cases} 0 & X \text{ continuous} \\ 0 & X \text{ discrete} \end{cases} = 0 \end{aligned}$$

$$E[(X - E[X])] = 0$$

$$E[X - E[X]] = 0$$

$$\forall X (E[X - E[X]] = 0)$$

For normal distribution, actually for any distribution,

$$\begin{aligned} X &\sim n(0, 1) = \mathcal{N}(0, 1^2) \\ &\Downarrow \\ E[X - E[X]] &= 0 \end{aligned}$$


---

2<sup>nd</sup> central moment of  $X$  = variance

$$\begin{aligned}\mu_2 &= E[(X - E[X])^2] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x) & X \text{ discrete} \end{cases} \\ &= E[(X - E[X])^2] = V[X] = V(X)\end{aligned}$$

For normal distribution,

$$\begin{aligned}X &\sim n(0, 1) = \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ &\Downarrow \\ V[X] &= V(X) = 1\end{aligned}$$

variance properties

$$V[aX + b] = a^2 V[X]$$

Proof:

to be proved

□

3<sup>rd</sup> central moment of  $X$

$$\mu_3 = E[(X - E[X])^3] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^3 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^3 f_x(x) & X \text{ discrete} \end{cases}$$

skewness

偏度

$$\begin{aligned}\text{skewness}[X] &= \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{\left(E[(X - E[X])^2]\right)^{\frac{3}{2}}} \\ &= \begin{cases} \frac{\int_{-\infty}^{+\infty} (x - \mu)^3 f_x(x) dx}{\left(\int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx\right)^{\frac{3}{2}}} & X \text{ continuous} \\ \frac{\sum_{x \in X(\Omega)} (x - \mu)^3 f_x(x)}{\left(\sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x)\right)^{\frac{3}{2}}} & X \text{ discrete} \end{cases}\end{aligned}$$

For normal distribution,

$$\begin{aligned}X &\sim n(0, 1) = \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, V^2[X] = 1^2) \\ &\Downarrow \\ \text{skewness}[X] &= \frac{E[(X - E[X])^3]}{(V[X])^{\frac{3}{2}}} = \frac{E[(X - E[X])^3]}{1^{\frac{3}{2}}} = 0\end{aligned}$$

Proof:

to be proved

□

$4^{\text{th}}$  central moment of  $X$

$$\mu_4 = \mathbb{E}[(X - \mathbb{E}[X])^4] = \begin{cases} \int_{-\infty}^{+\infty} (x - \mu)^4 f_x(x) dx & X \text{ continuous} \\ \sum_{x \in X(\Omega)} (x - \mu)^4 f_x(x) & X \text{ discrete} \end{cases}$$

kurtosis

峰度

$$\begin{aligned} \text{kurtosis}[X] &= \frac{\mu_4}{\mu_2^2} = \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{(\mathbb{V}[X])^2} = \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{\left(\mathbb{E}[(X - \mathbb{E}[X])^2]\right)^2} \\ &= \begin{cases} \frac{\int_{-\infty}^{+\infty} (x - \mu)^4 f_x(x) dx}{\left(\int_{-\infty}^{+\infty} (x - \mu)^2 f_x(x) dx\right)^2} & X \text{ continuous} \\ \frac{\sum_{x \in X(\Omega)} (x - \mu)^4 f_x(x)}{\left(\sum_{x \in X(\Omega)} (x - \mu)^2 f_x(x)\right)^2} & X \text{ discrete} \end{cases} \end{aligned}$$

For normal distribution,

$$\begin{aligned} X \sim n(0, 1) &= \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, \mathbb{V}^2[X] = 1^2) \\ &\Downarrow \\ \text{kurtosis}[X] &= \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{(\mathbb{V}[X])^2} = \frac{[(X - \mathbb{E}[X])^4]}{1^2} = 3 \end{aligned}$$

Proof:

to be proved

□

For normal distribution,

$$X \sim n(0, 1) = \mathcal{N}(0, 1^2) = \mathcal{N}(\mu = 0, \mathbb{V}^2[X] = 1^2)$$

$$\begin{cases} \mu = \mathbb{E}[X] & = 0 \\ \mu_1 = \mathbb{E}[X - \mathbb{E}[X]] & = 0 \\ \text{variance}[X] = \mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] & = 1 \\ \text{skewness}[X] = \frac{\mathbb{E}[(X - \mathbb{E}[X])^3]}{(\mathbb{V}[X])^{\frac{3}{2}}} & = 0 \\ \text{kurtosis}[X] = \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{(\mathbb{V}[X])^2} & = 3 \end{cases}$$

$$\begin{aligned}
\mu &= \mathbb{E}[X] = 0 \\
\mu_1 &= \mathbb{E}[X - \mathbb{E}[X]] = 0 \\
\text{variance } [X] &= \mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = 1 \\
\text{skewness } [X] &= \frac{\mathbb{E}[(X - \mathbb{E}[X])^3]}{(\mathbb{V}[X])^{\frac{3}{2}}} = 0 \\
\text{kurtosis } [X] &= \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{(\mathbb{V}[X])^2} = 3
\end{aligned}$$


---

$$X \sim F_x(x) \leftrightarrow f_x(x) \rightarrow \{\mu_n | n \in \mathbb{N}\} = \left\{ \mu_n \middle| \begin{cases} n \in \mathbb{N} \\ \mu_n = \mathbb{E}[(X - \mathbb{E}[X])^n] \end{cases} \right\}$$


---

#### 15.1.2.7.1 moment generating function

**Definition 15.16.** MGF = moment generating function: The moment generating function of  $X$  is  $M(\xi) = M_x(\xi) = \mathbb{E}[e^{\xi X}]$ , provided that the expression exists for  $t \approx 0$ .

$$M(t) = M_x(t) = \mathbb{E}[e^{tX}]$$

$$M(\xi) = M_x(\xi) = \mathbb{E}[e^{\xi X}]$$

**Theorem 15.11.** moment generating function(MGF) generating moment

$$M_x^{(n)}(\xi) = \mathbb{E}[X^n]$$

where

$$M_x^{(n)}(\xi) = \frac{d^n}{d\xi^n} M_x(\xi)$$

Proof:

to be proved

□

$$X \sim F_x(x) \leftrightarrow f_x(x) \rightarrow \{\mu_n | n \in \mathbb{N}\} = \left\{ \mu_n \middle| \begin{cases} n \in \mathbb{N} \\ \mu_n = \mathbb{E}[(X - \mathbb{E}[X])^n] \end{cases} \right\}$$

$$\begin{array}{ccccccc}
X & \sim & F_x(x) & \leftrightarrow & f_x(x) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \\
& & & & \downarrow & \nearrow & \\
& & & & M_x(\xi) & &
\end{array}$$

$$\begin{array}{ccccccc}
X & \sim & F_x(x) & \leftrightarrow & f_x(x) & & \\
& & & & \downarrow & \searrow & \\
& & & & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\}
\end{array}$$

**Theorem 15.12.** If  $X$  and  $Y$  have bounded support, then  $\forall u [F_x(u) = F_y(u)]$  iff  $\forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n])$ .

$$\forall u [F_X(u) = F_Y(u)] \Rightarrow \forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n])$$

$$\begin{cases} \forall n \in \mathbb{N} (\mathbb{E}[X^n] = \mathbb{E}[Y^n]) \\ \begin{cases} \text{supp}(f_X) \text{ is bounded} & \Rightarrow \forall u [F_X(u) = F_Y(u)] \\ \text{supp}(f_Y) \text{ is bounded} \end{cases} \end{cases}$$

Proof:

to be proved

□

**Theorem 15.13.** If  $M_X(t)$  and  $M_Y(t)$  exist, then  $\forall u [F_X(u) = F_Y(u)]$  iff  $\forall t \approx 0 [M_X(t) = M_Y(t)]$ .

$$\forall u [F_X(u) = F_Y(u)] \Rightarrow \forall t \approx 0 [M_X(t) = M_Y(t)]$$

$$\begin{cases} \forall t \approx 0 [M_X(t) = M_Y(t)] \\ \begin{cases} \exists M_X(t) \in \mathbb{R} & \Rightarrow \forall u [F_X(u) = F_Y(u)] \\ \exists M_Y(t) \in \mathbb{R} \end{cases} \end{cases}$$

Proof:

to be proved

□

$$\begin{array}{ccc} X & \sim & F_X(x) \\ & & \leftrightarrow f_X(x) \\ & \uparrow & \downarrow \\ \forall \xi \approx 0 [M_X(\xi) \in \mathbb{R}] & \wedge & M_X(\xi) \end{array} \rightarrow \begin{array}{c} \nwarrow \nearrow \\ \leftarrow \wedge \text{ supp}(f_X) \text{ is bounded} \\ \{\mu_n | n \in \mathbb{N}\} \end{array}$$

#### 15.1.2.7.2 characteristic function

**Definition 15.17.** CF = characteristic function: The characteristic function of  $X$  is  $\varphi(\xi) = \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}]$ , provided that the expression always exists.

$$\varphi(t) = \varphi_X(t) = \mathbb{E}[e^{itX}]$$

$$\varphi(\xi) = \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}]$$

Note:

1.  $\varphi(\xi) = \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}]$  always exists.

$$\forall X (\varphi(\xi) = \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}] \in \mathbb{R})$$

2. moment generating function to characteristic function

$$M(\xi) = M_X(\xi) = \mathbb{E}[e^{\xi X}] \in \mathbb{R} \Rightarrow M_X(i\xi) = \varphi_X(\xi)$$

3. inversion theorem or inversion formula

For  $a < b$ ,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^{+T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_x(t) dt \\ &= P(a < X < b) + \frac{1}{2} [P(X = a) + P(X = b)] \end{aligned}$$

i.e.  $\varphi_x(\xi)$  determines  $F_x(x)$ .

$$\begin{array}{ccccccc} X & \sim & F_x(x) & \leftrightarrow & f_x(x) & \leftrightarrow & P_x \\ \uparrow \downarrow & & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge & \uparrow \downarrow & \searrow \nearrow & \leftarrow \rho \wedge \text{supp}(f_x) \text{ is bounded} \\ \varphi_x(\xi) & & \Leftrightarrow & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \\ \\ X & \sim & F_x(x) & \xrightleftharpoons[\text{inversion formula}]{\text{FToC}} & f_x(x) & \leftrightarrow & P_x \\ \uparrow \downarrow & & \forall \xi \approx 0 [M_x(\xi) \in \mathbb{R}] \wedge & \uparrow \downarrow & \searrow \nearrow & \leftarrow \rho \wedge \text{supp}(f_x) \text{ is bounded} \\ \varphi_x(\xi) & & \Leftrightarrow & M_x(\xi) & \rightarrow & \{\mu_n | n \in \mathbb{N}\} \end{array}$$


---

<https://www.youtube.com/watch?v=fSbs6im6wqY>

MGF theorems

<https://www.youtube.com/watch?v=fSbs6im6wqY&t=524s>

**Theorem 15.14.** If

CLT

### 15.1.2.8 common families of distributions

mean and variance of discrete probability distributions<sup>[43.5]</sup>

#### 15.1.2.8.1 discrete distribution

##### 15.1.2.8.1.1 discrete uniform distribution

$$\begin{aligned} X &\sim \mathcal{DU}(1, N) \\ &\Updownarrow \\ \begin{cases} f_x(x|N) = \frac{1}{N} \\ x \in X(\Omega) = \{1, 2, \dots, N\} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \frac{N+1}{2} \\ V[X] = \frac{(N+1)(N-1)}{12} \end{cases} \end{aligned}$$

$$\begin{aligned} E[X] &= \sum_{x \in X(\Omega)} x f_x(x|N) = \sum_{x=1}^N x \frac{1}{N} \\ &= \frac{1}{N} \sum_{x=1}^N x = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2} \\ E[X] &= \frac{N+1}{2} \end{aligned}$$

$$\begin{aligned}
V[X] &= \sum_{x \in X(\Omega)} (x - E[X])^2 f_x(x|N) = \sum_{x=1}^N \left( x - \frac{N+1}{2} \right)^2 \frac{1}{N} \\
&= \frac{1}{N} \sum_{x=1}^N \left[ x^2 - (N+1)x + \left( \frac{N+1}{2} \right)^2 \right] \\
&= \frac{1}{N} \left[ \sum_{x=1}^N x^2 - (N+1) \sum_{x=1}^N x + N \left( \frac{N+1}{2} \right)^2 \right] \\
&= \frac{1}{N} \left[ \frac{N(N+1)(2N+1)}{6} - (N+1) \frac{N(N+1)}{2} + N \left( \frac{N+1}{2} \right)^2 \right] \\
&= (N+1) \left[ \frac{2N+1}{6} - \frac{N+1}{2} + \frac{N+1}{4} \right] = (N+1) \frac{4N+2-(3N+3)}{12} \\
&= \frac{(N+1)(N-1)}{12} \\
V[X] &= \frac{(N+1)(N-1)}{12}
\end{aligned}$$


---

$$X \sim \mathcal{DU}(a, b) \Leftrightarrow \begin{cases} f_x(x|N) = \frac{1}{N} = \frac{1}{b-a+1} & N = b-a+1 \\ x \in X(\Omega) = \{a, a+1, \dots, b\} & \end{cases} \Leftrightarrow F_x(x|a, b) = \frac{\lfloor x \rfloor - a + 1}{b - a + 1}$$

#### 15.1.2.8.1.2 hypergeometric distribution

$$\begin{aligned}
X &\sim \mathcal{HG}(N, M, K) \\
&\Updownarrow \\
\begin{cases} f_x(x|N, M, K) = \frac{\binom{N-M}{K-x} \binom{M}{x}}{\binom{N}{K}} \\ x \in X(\Omega) = \{\max\{0, K-(N-M)\}, \dots, \min\{K, M\}\} \end{cases} \\
&\Downarrow K \ll N, M \\
&x \in X(\Omega) = \{0, 1, \dots, K\} \\
&\Downarrow \\
\begin{cases} E[X] = \frac{KM}{N} \\ V[X] = \frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)} \end{cases}
\end{aligned}$$

#### 15.1.2.8.1.3 Bernoulli distribution

$$\begin{aligned}
X &\sim \mathcal{B}(p), p = P(X=1) \\
&\Updownarrow \\
\begin{cases} f_x(x|p) = (1-p)^{1-x} p^x \\ x \in X(\Omega) = \{0, 1\} \end{cases} \\
&\Downarrow \\
\begin{cases} E[X] = p \\ V[X] = p(1-p) \\ M_x(\xi) = (1-p) + pe^\xi \end{cases}
\end{aligned}$$

#### 15.1.2.8.1.4 binomial distribution independent and identical Bernoulli trials

$$\begin{aligned}
 X &\sim \text{b}(n, p), p = \text{P}(X = 1) \\
 &\Updownarrow \\
 \begin{cases} f_x(x|n, p) = \binom{n}{x} (1-p)^{n-x} p^x \\ x \in X(\Omega) = \{0, 1, \dots, n\} \end{cases} \\
 &\Downarrow \\
 \begin{cases} \text{E}[X] = np \\ \text{V}[X] = np(1-p) \\ M_x(\xi) = [(1-p) + pe^\xi]^n \end{cases}
 \end{aligned}$$

reparameterization technique

$$\begin{aligned}
 M_x(\xi) &= \text{E}[e^{\xi X}] = \sum_{x \in X(\Omega)} e^{\xi x} f_x(x|n, p) \\
 &= \sum_{x=1}^n e^{\xi x} \binom{n}{x} (1-p)^{n-x} p^x = \sum_{x=1}^n \binom{n}{x} (1-p)^{n-x} (pe^\xi)^x \\
 &= \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p) + pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p) + pe^\xi} \right]^x [(1-p) + pe^\xi]^{n-x} [(1-p) + pe^\xi]^x \\
 &= \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p) + pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p) + pe^\xi} \right]^x [(1-p) + pe^\xi]^n \\
 &= [(1-p) + pe^\xi]^n \sum_{x=1}^n \binom{n}{x} \left[ \frac{1-p}{(1-p) + pe^\xi} \right]^{n-x} \left[ \frac{pe^\xi}{(1-p) + pe^\xi} \right]^x \\
 &= [(1-p) + pe^\xi]^n \sum_{x=1}^n \binom{n}{x} [p^*]^{n-x} [1-p^*]^x, p^* = \frac{1-p}{(1-p) + pe^\xi} \\
 &= [(1-p) + pe^\xi]^n \sum_{x=1}^n f_x(x|n, p^*), X \sim \text{b}(n, p^*) \\
 &= [(1-p) + pe^\xi]^n \cdot 1 = [(1-p) + pe^\xi]^n \\
 M_x(\xi) &= [(1-p) + pe^\xi]^n
 \end{aligned}$$

#### 15.1.2.8.1.5 Poisson distribution count = number of events

an unbounded discrete distribution we first see or met

$$\begin{aligned}
 X &\sim \mathcal{P}(\lambda) \\
 &\Updownarrow \\
 \begin{cases} f_x(x|\lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \\ x \in X(\Omega) = \{0, 1, \dots\} \end{cases} \\
 &\Downarrow \\
 \begin{cases} \text{E}[X] = \lambda \\ \text{V}[X] = \lambda = \text{E}[X] \\ M_x(\xi) = \exp[\lambda(e^\xi - 1)] = e^{\lambda(e^\xi - 1)} \end{cases}
 \end{aligned}$$

the Poisson postulates

$$\begin{cases}
 N_t & \text{a r.v. denoting the of events in } [0, t] \\
 N_0 = N_{t=0} = 0 & \text{reset the count at the initial point} \\
 \forall s < t [N_s \perp N_t - N_s] & \text{disjoint intervals independent} \\
 N_s = N_{t+s} - N_t & \text{depends on length instead of initial point} \\
 \lim_{t \rightarrow 0} \frac{P(N_t = 1)}{t} = \lambda & \Rightarrow \forall t \approx 0 [P(N_t = 1) \approx \lambda t] \\
 \lim_{t \rightarrow 0} \frac{P(N_t > 1)}{t} = 0 & \text{no coincidence for small } t \\
 \text{solve the differential equations } \Downarrow \text{ with probability axioms} \\
 f_x(x|\lambda t) = P(N_t = x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}
 \end{cases}$$

#### 15.1.2.8.1.6 negative binomial distribution also an unbounded discrete distribution

Count the number of independent and identical Bernoulli trials until  $r$  a fixed number of success.

$$\begin{aligned}
 X &\sim \mathcal{NB}(r, p) \\
 &\Downarrow \\
 \begin{cases} f_x(x|r, p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \\ x \in X(\Omega) = \{r, r+1, \dots\} \end{cases} \\
 &\Downarrow \\
 \begin{cases} E[X] = \\ V[X] = \\ M_x(\xi) = \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 Y = X - r, X \sim \mathcal{NB}(r, p) \Rightarrow X = Y + r \\
 &\Downarrow \\
 \begin{cases} f_Y(y|r, p) = \binom{r+y-1}{r-1} (1-p)^y p^r \\ y \in Y(\Omega) = \{0, 1, \dots\} \end{cases} \\
 &\Downarrow \\
 \begin{cases} E[Y] = r \frac{1-p}{p} = r \left( \frac{1}{p} - 1 \right) \\ V[Y] = r \frac{1-p}{p^2} = r \left( \frac{1}{p^2} - \frac{1}{p} \right) \\ M_Y(\xi) = \left[ \frac{p}{1 - (1-p)e^\xi} \right]^r \end{cases}
 \end{aligned}$$

Note:

$$\begin{aligned}
 Y = X - r \Rightarrow f_Y(y|r, p) &= P(Y = y) \\
 &= P(X - r = y) \\
 &= P(X = r + y)
 \end{aligned}$$

reparameterization technique

$$\begin{aligned}
M_Y(\xi) &= \mathbb{E}[e^{\xi Y}] = \sum_{y \in Y(\Omega)} e^{\xi y} f_Y(y|r, p) \\
&= \sum_{y=0}^{\infty} e^{\xi y} \binom{r+y-1}{r-1} (1-p)^y p^r \\
&= p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} [(1-p)e^{\xi}]^y \\
&= p^r \sum_{y=0}^{\infty} \binom{r+y-1}{r-1} [1-p^*]^y [p^*]^r \frac{1}{[p^*]^r}, 1-p^* = (1-p)e^{\xi} \\
&= \left[ \frac{p}{p^*} \right]^r \sum_{y=0}^{\infty} f_Y(y|r, p^*), Y \sim \mathcal{NB}(r, p^*), p^* = 1 - (1-p)e^{\xi} \\
&= \left[ \frac{p}{p^*} \right]^r \cdot 1 = \left[ \frac{p}{p^*} \right]^r = \left[ \frac{p}{1 - (1-p)e^{\xi}} \right]^r
\end{aligned}$$

Note:

For sufficiently small  $t$  such that

$$0 \leq p^* = (1-p)e^{\xi} \leq 1$$

or else  $t \gg 1$

$$p^* = (1-p)e^{\xi} > 1$$

#### 15.1.2.8.1.7 geometric distribution

$$\begin{array}{c}
X \sim \mathcal{NB}(r, p) \\
\Updownarrow \\
\begin{cases} f_X(x|r, p) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \\ x \in X(\Omega) = \{r, r+1, \dots\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathbb{E}[X] = \\ \text{V}[X] = \\ M_X(\xi) = \end{cases}
\end{array}$$

$r = 1$ ,

$$\begin{array}{c}
X \sim \mathcal{NB}(r=1, p) = \mathcal{G}(p) \\
\Updownarrow \\
\begin{cases} f_X(x|r=1, p) = \left[ \binom{x-1}{r-1} (1-p)^{x-r} p^r \right]_{r=1} = \binom{x-1}{1-1} (1-p)^{x-1} p^1 = (1-p)^{x-1} p \\ x \in X(\Omega) = \{r, r+1, \dots\}_{r=1} = \{1, 2, \dots\} \end{cases} \\
\Downarrow \\
\begin{cases} \mathbb{E}[X] = \\ \text{V}[X] = \\ M_X(\xi) = \end{cases}
\end{array}$$

the only “memoryless” discrete distribution, and there is also a “memoryless” continuous distribution.  
memoryless property

$$\forall s > t [\mathbb{P}(X > s | X > t) = \mathbb{P}(X > s - t)]$$

Survival depends on length instead of initial point; it might be proper assumption for stuff survival, but might not be proper for human survival.

### 15.1.2.8.2 continuous distribution

#### 15.1.2.8.2.1 uniform distribution = continuous uniform distribution

$$\begin{aligned} X &\sim \mathcal{U}(a, b) \\ \Updownarrow \\ \begin{cases} f_X(x|a, b) = \frac{1(a \leq x \leq b)}{b-a} = \frac{1(x \in [a, b])}{b-a} \\ x \in X(\Omega) = [a, b] \end{cases} \\ \Downarrow \\ \begin{cases} E[X] = \frac{a+b}{2} \\ V[X] = \frac{(b-a)^2}{12} \\ M_X(\xi) = \frac{e^{b\xi} - e^{a\xi}}{(b-a)\xi} \end{cases} \end{aligned}$$

$$a = 0, b = 1$$

$$\begin{aligned} X &\sim \mathcal{U}(a=0, b=1) = \mathcal{U}(0, 1) \\ \Updownarrow \\ \begin{cases} f_X(x|a=0, b=1) = \left[ \frac{1(a \leq x \leq b)}{b-a} \right]_{a=0, b=1} = 1(0 \leq x \leq 1) = 1(x \in [0, 1]) \\ x \in X(\Omega) = [a, b] |_{a=0, b=1} = [0, 1] \end{cases} \\ \Downarrow \\ \begin{cases} E[X] = \frac{1}{2} \\ V[X] = \frac{1}{12} \\ M_X(\xi) = \frac{e^\xi - 1}{\xi} \end{cases} \end{aligned}$$

#### 15.1.2.8.2.2 gamma distribution

#### 15.1.2.8.2.3 exponential distribution

#### 15.1.2.8.2.4 Chi-square distribution

#### 15.1.2.8.2.5 Weibull distribution <https://www.youtube.com/watch?v=ojZb6nZWdvI>

#### 15.1.2.8.2.6 normal distribution

$$\begin{aligned} X &\sim \mathcal{N}(\mu, \sigma^2) \\ \Updownarrow \\ \begin{cases} f_X(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} = \frac{\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\ \Downarrow \\ \begin{cases} E[X] = \mu \\ V[X] = \sigma^2 \\ M_X(\xi) = e^{(\mu\xi + \frac{\sigma^2}{2}\xi^2)} = e^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases} \end{aligned}$$

#### 15.1.2.8.2.7 beta distribution random success probability

#### 15.1.2.8.2.8 Cauchy distribution <https://www.youtube.com/watch?v=ojZb6nZWdvI&t=33m34s>

[https://en.wikipedia.org/wiki/Cauchy\\_distribution](https://en.wikipedia.org/wiki/Cauchy_distribution)

$$\begin{aligned}
 X &\sim \mathcal{C}(\theta, \sigma) \\
 &\Updownarrow \\
 \left\{ \begin{array}{l} f_X(x|\theta, \sigma) = \frac{1}{\pi\sigma} \left[ 1 + \left( \frac{x-\theta}{\sigma} \right)^2 \right]^{-1} = \frac{1}{\pi\sigma \left[ 1 + \left( \frac{x-\theta}{\sigma} \right)^2 \right]} = \frac{1}{\pi} \left[ \frac{\sigma}{(x-\theta)^2 + \sigma^2} \right] \\ x \in X(\Omega) = \mathbb{R} \end{array} \right. \\
 &\Downarrow \\
 \left\{ \begin{array}{ll} E[X] & \text{diverges} \\ V[X] & \text{diverges} \\ M_X(\xi) & \text{diverges} \end{array} \right.
 \end{aligned}$$

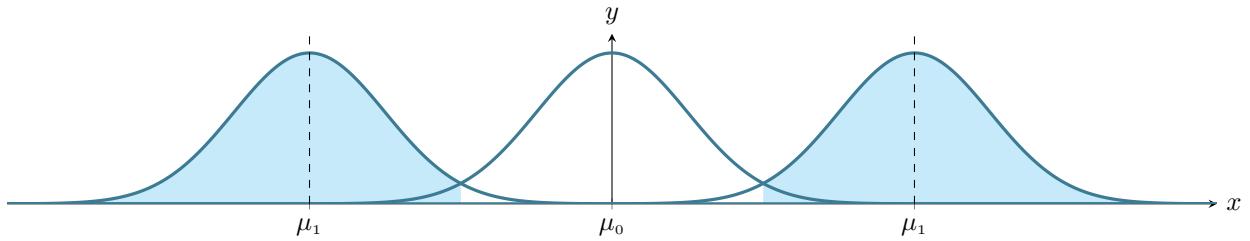


Figure 15.1: not completed: Cauchy distribution vs. normal distribution

heavy tail

*t* distribution is also heavy tail

box plot

#### 15.1.2.8.2.9 log-normal distribution

#### 15.1.2.9 exponential family

**Definition 15.18.** exponential family: A family of PDF/PMF is called exponential family if

$$f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right)$$

with  $\boldsymbol{\theta} = \boldsymbol{\theta}(\theta_1, \dots, \theta_k) = (\theta_1, \dots, \theta_k)$  for some  $h(x), c(\boldsymbol{\theta}), w_j(\boldsymbol{\theta}), t_j(x)$ , where

$$h(x) c(\boldsymbol{\theta}) \geq 0 \Rightarrow f(x|\boldsymbol{\theta}) \geq 0$$

and parameters  $\boldsymbol{\theta}$  and statistic or real number  $x$  can be separated.

$$\mathcal{E}^f = \left\{ f \left| f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right. \right\}$$

- exponential family
  - **normal distribution**<sup>[15.1.2.8.2.6]</sup>
  - **gamma distribution**<sup>[15.1.2.8.2.2]</sup>
  - **log-normal distribution**<sup>[15.1.2.8.2.9]</sup>
- non-exponential family
  - **Cauchy distribution**<sup>[15.1.2.8.2.8]</sup> <https://www.youtube.com/watch?v=oJZb6nZWdvI&t=1h7m43s>

### 15.1.2.9.1 binomial distribution with known $n$

$$f(x|p) = \binom{n}{x} (1-p)^{n-x} p^x$$

or

$$f(x|p) = \binom{n}{x} (1-p)^{n-x} p^x = f(x|n=n, p)$$


---

not

$$f(x|n, p) = \binom{n}{x} (1-p)^{n-x} p^x$$


---

$$\begin{aligned} f(x|p) &= \binom{n}{x} (1-p)^{n-x} p^x \\ &= \binom{n}{x} (1-p)^n \left( \frac{p}{1-p} \right)^x \\ &= \binom{n}{x} (1-p)^n e^{(\ln \frac{p}{1-p})x} \\ &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \begin{cases} h(x) = \binom{n}{x} \\ c(\boldsymbol{\theta}) = c(\theta_1) = c(p) = (1-p)^n \\ w_1(\boldsymbol{\theta}) = w(\theta_1) = w(p) = \ln \frac{p}{1-p} \\ t_1(x) = x \\ k = 1 \end{cases} \\ &\Downarrow \\ f(x|p) &= \binom{n}{x} (1-p)^{n-x} p^x \in \mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \end{aligned}$$

why known  $n$ ?

known  $n$

$$\binom{n}{x} = h(x)$$

unknown  $n$

$$\binom{n}{x} = h(x, n) \neq h_1(n) h_2(x)$$

### 15.1.2.9.2 continuous uniform distribution not in exponential family

$$X \sim \mathcal{U}(a, b)$$

$$\begin{aligned} f_x(x|a, b) &= \frac{1(x \in [a, b])}{b-a} \\ &= \frac{1}{b-a}, x \in [a, b] \end{aligned}$$

$$\frac{1}{b-a} = h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \begin{cases} h(x) = 1 \\ c(\boldsymbol{\theta}) = c(\theta_1, \theta_2) = c(a, b) = \frac{1}{b-a} \\ w_1(\theta_1) = w_2(\theta_2) = 0 \\ t_1(x) = t_2(x) = x \\ k = 2 \end{cases}$$

however,

$$1(x \in [a, b]) \neq h(x) c(\boldsymbol{\theta}) = h(x) c(a, b)$$

thus

$$f_x(x|a, b) = \frac{1(x \in [a, b])}{b - a} \notin \mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

#### 15.1.2.9.3 normal distribution is in exponential family

$$\begin{aligned} X &\sim \mathcal{N}(\mu, \sigma^2) \\ &\Downarrow \\ \begin{cases} f_x(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} = \frac{e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} = \frac{\exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi}} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \mu \\ V[X] = \sigma^2 \\ M_X(\xi) = e^{(\mu\xi + \frac{\sigma^2}{2}\xi^2)} = e^{\mu\xi + \frac{\sigma^2}{2}\xi^2} \end{cases} \end{aligned}$$


---

$$\begin{aligned} f_x(x|\mu, \sigma^2) &= f_x(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)} = \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} = \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} e^{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2} \\ &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)}, \begin{cases} h(x) = 1 \\ c(\boldsymbol{\theta}) = c(\theta_1, \theta_2) = c(\mu, \sigma) = \frac{e^{\frac{-1}{2}(\frac{\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}} \\ w_1(\theta_1, \theta_2) = w_1(\mu, \sigma) = \frac{\mu}{\sigma^2} \\ w_2(\theta_2) = w_2(\sigma) = \frac{-1}{2\sigma^2} \\ t_1(x) = x \\ t_2(x) = x^2 \\ k = 2 \end{cases} \\ &\Downarrow \end{aligned}$$

$$f_x(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \in \mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

#### 15.1.2.9.4 normal distribution with unknown equal mean and standard deviation in curved exponential family

$$\begin{aligned} X &\sim \mathcal{N}(\mu, \sigma^2 = \mu^2) = \mathcal{N}(\mu, \mu^2) \\ &\Downarrow \\ \begin{cases} f_x(x|\mu, \mu^2) = f_x(x|\mu) = \left[ \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \right]_{\sigma=\mu} = \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\mu})^2} \\ x \in X(\Omega) = \mathbb{R} \end{cases} \\ &\Downarrow \\ \begin{cases} E[X] = \mu \\ V[X] = \mu^2 \\ M_X(\xi) = e^{(\mu\xi + \frac{\mu^2}{2}\xi^2)} = e^{\mu\xi + \frac{\mu^2}{2}\xi^2} \end{cases} \end{aligned}$$


---

$$\begin{aligned}
f_x(x|\mu, \mu^2) &= f_x(x|\mu) = \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\mu})^2} \\
&= \frac{1}{\mu\sqrt{2\pi}} e^{\frac{-1}{2\mu^2}(x^2 - 2x\mu + \mu^2)} = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} e^{\frac{-1}{2\mu^2}x^2 + \frac{\mu}{\mu^2}x} = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} e^{\frac{1}{\mu}x - \frac{1}{2\mu^2}x^2} \\
&= h(x) c(\boldsymbol{\theta}) \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x), \quad \begin{cases} h(x) = 1 \\ c(\boldsymbol{\theta}) = c(\theta_1) = c(\mu) = \frac{e^{\frac{-1}{2}}}{\mu\sqrt{2\pi}} \\ w_1(\theta_1) = w_1(\mu) = \frac{1}{\mu} \\ w_2(\theta_1) = w_2(\mu) = \frac{-1}{2\mu^2} \\ t_1(x) = x \\ t_2(x) = x^2 \\ k = 2 > 1 = p \end{cases} \\
&\Downarrow \begin{cases} p = \dim \boldsymbol{\theta} \\ k = \dim \mathbf{w} \end{cases} \\
f_x(x|\mu, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} \in \mathcal{C}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^{k>p} w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \\
&\subset \mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}
\end{aligned}$$


---

curved exponential family

$$\mathcal{C}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^{k>p} w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$


---

exponential family

$$\mathcal{E}^f = \left\{ f \mid f = f(x|\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$


---

<https://tex.stackexchange.com/questions/145969/filling-specified-area-by-random-dots-in-tikz>

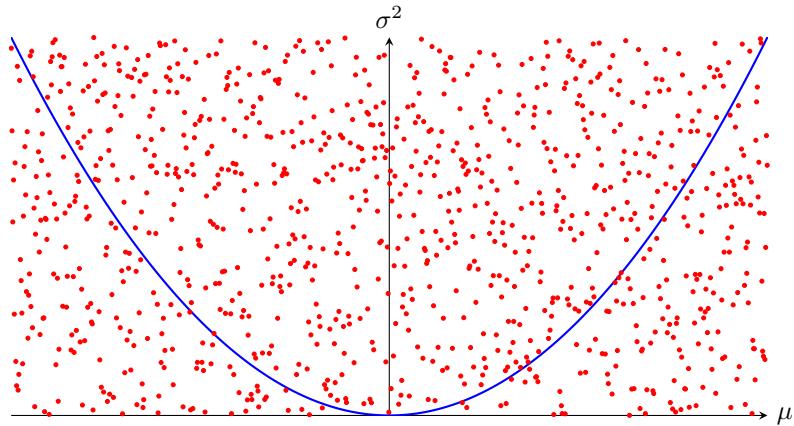


Figure 15.2: curved exponential family vs. exponential family

#### 15.1.2.9.5 properties of exponential family

### 15.1.2.9.5.1 fundamentals of statistical inference

**Lemma 15.1.** *Leibnitz rule*

$$\begin{aligned}
 \frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx &= \frac{d \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx}{d\theta} \\
 &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx \\
 &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx \\
 \frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} g(x, \theta) dx &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx \\
 \\ 
 \frac{d}{d\theta} \int_{a(\theta)=a}^{b(\theta)=b} g(x, \theta) dx &= g(x, b(\theta)) \frac{db(\theta)}{d\theta} - g(x, a(\theta)) \frac{da(\theta)}{d\theta} + \int_{a(\theta)}^{b(\theta)} \frac{dg(x, \theta)}{d\theta} dx \\
 &= g(x, b) \frac{db}{d\theta} - g(x, a) \frac{da}{d\theta} + \int_a^b \frac{dg(x, \theta)}{d\theta} dx \\
 &= g(x, b) 0 - g(x, a) 0 + \int_a^b \frac{dg(x, \theta)}{d\theta} dx = 0 + 0 + \int_a^b \frac{dg(x, \theta)}{d\theta} dx \\
 &= \int_a^b \frac{dg(x, \theta)}{d\theta} dx \\
 \frac{d}{d\theta} \int_{X(\Omega) \perp \theta} g(x, \theta) dx &= \frac{d}{d\theta} \int_{a(\theta)=a}^{b(\theta)=b} g(x, \theta) dx = \int_a^b \frac{dg(x, \theta)}{d\theta} dx = \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx \\
 \frac{d}{d\theta} \int_{X(\Omega) \perp \theta} g(x, \theta) dx &= \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx \\
 \\ 
 \frac{d}{d\theta} \int_{X(\Omega) \perp \theta} g(x, \theta) dx &= \int_{X(\Omega) \perp \theta} \frac{dg(x, \theta)}{d\theta} dx
 \end{aligned}$$

point estimation

**Lemma 15.2.** *parameter-independent expectation:*

Assume the domain of  $X$  independent of  $\theta$ , then

$$E \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = 0, \forall i = 1, \dots, p$$

Proof:

$$\begin{aligned}
 E \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\
 &= \int \left[ \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} \right] f(x|\boldsymbol{\theta}) dx \\
 &= \int \left[ \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right] f(x|\boldsymbol{\theta}) dx = \int \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\
 &= \int \frac{f(x|\boldsymbol{\theta})}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx = \int 1 \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx = \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx \\
 &\stackrel{\text{Leibnitz rule}}{=} \frac{\partial}{\partial \theta_i} \int f(x|\boldsymbol{\theta}) dx
 \end{aligned}$$

□

Note:

$$\begin{aligned}
 \mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right) f(x|\boldsymbol{\theta}) dx \\
 X \sim f_x(x|\boldsymbol{\theta}) \Rightarrow \mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= \int \frac{\partial \ln f_x(x|\boldsymbol{\theta})}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f_x(x|\boldsymbol{\theta})}{\partial \theta_i}}{f_x(x|\boldsymbol{\theta})} \right) f_x(x|\boldsymbol{\theta}) dx \\
 &= \int \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} dx \\
 X \sim f_x(x|\boldsymbol{\theta}) \Rightarrow \mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] &= \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx = \int \left( \frac{\frac{\partial f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i}}{f_x(x|\boldsymbol{\theta}^*)} \right) f(x|\boldsymbol{\theta}) dx \\
 &= \int \frac{\partial f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} \left( \frac{f_x(x|\boldsymbol{\theta})}{f_x(x|\boldsymbol{\theta}^*)} \right) dx
 \end{aligned}$$


---

$$\mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \mathbb{E}_{\boldsymbol{\theta}^*} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}^*) dx$$

$$\mathbb{E} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] = \int \frac{\partial \ln f_x(x|\boldsymbol{\theta}^*)}{\partial \theta_i} f_x(x|\boldsymbol{\theta}) dx$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] \not\equiv 0$$

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \ln f_x(X|\boldsymbol{\theta}^*)}{\partial \theta_i} \right] \underset{\boldsymbol{\theta}^*=\boldsymbol{\theta}}{\equiv} 0$$

as the fundamental to estimate parameters.

interval estimation

**Lemma 15.3.** *parameter-independent variance:*

Assume the domain of  $X$  independent of  $\theta$ , then

$$\mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right], \forall i = 1, \dots, p$$

Proof:

$$\begin{aligned}
 \mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] &= \int \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta}) dx = \int \frac{\partial}{\partial \theta_i} \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} f(x|\boldsymbol{\theta}) dx \\
 &= \int \left\{ \frac{\partial}{\partial \theta_i} \left( \frac{1}{f(x|\boldsymbol{\theta})} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right) \right\} f(x|\boldsymbol{\theta}) dx = \int \left\{ \frac{\partial}{\partial \theta_i} \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right) \right\} f(x|\boldsymbol{\theta}) dx \\
 &= \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} f(x|\boldsymbol{\theta}) - \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{[f(x|\boldsymbol{\theta})]^2} f(x|\boldsymbol{\theta}) dx = \int \frac{\frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} [f(x|\boldsymbol{\theta})]^2 - \left[ \frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i} \right]^2 [f(x|\boldsymbol{\theta})]}{[f(x|\boldsymbol{\theta})]^2} dx \\
 &= \int \frac{\partial^2 f(x|\boldsymbol{\theta})}{\partial \theta_i^2} dx - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \stackrel{\text{Leibnitz rule}}{=} \frac{\partial^2}{\partial \theta_i^2} \int f(x|\boldsymbol{\theta}) dx - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \\
 &= \frac{\partial^2}{\partial \theta_i^2} 1 - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx = 0 - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx = - \int \left( \frac{\frac{\partial f(x|\boldsymbol{\theta})}{\partial \theta_i}}{f(x|\boldsymbol{\theta})} \right)^2 f(x|\boldsymbol{\theta}) dx \\
 &= - \int \left( \frac{\partial \ln f(x|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 f(x|\boldsymbol{\theta}) dx = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right]
 \end{aligned}$$

□

$$\begin{cases} \mathbb{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = 0 \\ \mathbb{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\mathbb{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right], \forall i = 1, \dots, p \end{cases}$$

exponential family expectation and variance

**Theorem 15.15.** *exponential family expectation*

$$X \in \mathcal{E}^f = \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\}$$

$\Downarrow \forall i = 1, \dots, p$

$$\mathbb{E} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}$$

Proof:

$$\begin{aligned} f(x|\boldsymbol{\theta}) &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \\ \ln f(x|\boldsymbol{\theta}) &= \ln h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\ &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \ln e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\ &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\ \ln f(x|\boldsymbol{\theta}) &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\ \ln f(X|\boldsymbol{\theta}) &= \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\ \frac{\partial}{\partial \theta_i} \ln f(X|\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_i} \left[ \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \right] \\ &= \frac{\partial}{\partial \theta_i} \ln h(X) + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\ &= 0 + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k \frac{\partial}{\partial \theta_i} \{w_j(\boldsymbol{\theta}) t_j(X)\} \\ &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) \left( \frac{\partial}{\partial \theta_i} t_j(X) \right) \right\} \\ &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) 0 \right\} \\ &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) \right\} \\ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \\
0 &\stackrel{\text{lemma: } E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = 0}{=} E\left[\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i}\right] = E\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}\right] \\
&= E\left[\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}\right] + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&\stackrel{E[c]=c}{=} \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k E\left[\frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} E[t_j(X)] \\
0 &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] \\
E\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] &= -\frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i}
\end{aligned}$$

□

**Theorem 15.16.** exponential family variance

$$\begin{aligned}
X \in \mathcal{E}^f &= \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \\
&\Downarrow \forall i = 1, \dots, p \\
V\left[\sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X)\right] &= \frac{-\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - E\left[\sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X)\right]
\end{aligned}$$

Proof:

same as the above

$$\begin{aligned}
f(x|\boldsymbol{\theta}) &= h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \\
\ln f(x|\boldsymbol{\theta}) &= \ln h(x) c(\boldsymbol{\theta}) e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
&= \ln h(x) + \ln c(\boldsymbol{\theta}) + \ln e^{\sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x)} \\
&= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
\ln f(x|\boldsymbol{\theta}) &= \ln h(x) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \\
\ln f(X|\boldsymbol{\theta}) &= \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
\frac{\partial}{\partial \theta_i} \ln f(X|\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_i} \left[ \ln h(X) + \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \right] \\
&= \frac{\partial}{\partial \theta_i} \ln h(X) + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(X) \\
&= 0 + \frac{\partial}{\partial \theta_i} \ln c(\boldsymbol{\theta}) + \sum_{j=1}^k \frac{\partial}{\partial \theta_i} \{w_j(\boldsymbol{\theta}) t_j(X)\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) \left( \frac{\partial}{\partial \theta_i} t_j(X) \right) \right\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) + w_j(\boldsymbol{\theta}) 0 \right\} \\
&= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} w_j(\boldsymbol{\theta}) \right) t_j(X) \right\} \\
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} &= \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \\
V \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] &= V \left[ \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] \\
V[aX+b] &\stackrel{a^2 V[X]}{=} V \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] \\
V \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] &= V \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right]
\end{aligned}$$

$$\begin{aligned}
& \text{V} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = \text{V} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = \text{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} - \text{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] \right)^2 \right] \\
& \stackrel{\text{lemma: } \text{E} \left[ \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] = 0}{=} \text{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} - 0 \right)^2 \right] = \text{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right] \\
& \stackrel{\text{lemma: } \text{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\text{E} \left[ \left( \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right)^2 \right]}{=} -\text{E} \left[ \frac{\partial^2 \ln f(X|\boldsymbol{\theta})}{\partial \theta_i^2} \right] = -\text{E} \left[ \frac{\partial}{\partial \theta_i} \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} \right] \\
& \frac{\partial \ln f(X|\boldsymbol{\theta})}{\partial \theta_i} = \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \\
& = -\text{E} \left[ \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} + \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \right] \\
& = -\text{E} \left[ \frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} + \frac{\partial}{\partial \theta_i} \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] \\
& = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right\} \right] \\
& = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \left\{ \left( \frac{\partial}{\partial \theta_i} \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} \right) t_j(X) + \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial}{\partial \theta_i} t_j(X) \right\} \right] \\
& = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \left\{ \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) + \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} 0 \right\} \right] \\
& = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) \right] \\
& \text{V} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) \right]
\end{aligned}$$

□

exponential family expectation and variance

sense of downgrading by differentiation instead of integration

$$\begin{aligned}
X \in \mathcal{E}^f &= \left\{ f \middle| f = h(x) c(\boldsymbol{\theta}) \exp \left( \sum_{j=1}^k w_j(\boldsymbol{\theta}) t_j(x) \right) \right\} \\
\Rightarrow \begin{cases} \text{E} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = \frac{-\partial \ln c(\boldsymbol{\theta})}{\partial \theta_i} \\ \text{V} \left[ \sum_{j=1}^k \frac{\partial w_j(\boldsymbol{\theta})}{\partial \theta_i} t_j(X) \right] = -\frac{\partial^2 \ln c(\boldsymbol{\theta})}{\partial \theta_i^2} - \text{E} \left[ \sum_{j=1}^k \frac{\partial^2 w_j(\boldsymbol{\theta})}{\partial \theta_i^2} t_j(X) \right], \forall i = 1, \dots, p \end{cases}
\end{aligned}$$

### 15.1.3 multivariable distribution

<https://www.youtube.com/watch?v=y-Oi5voWQKo>

univariable random vector

discrete: PMF equals probability function

$$f_X(x) = \text{P}(X = x)$$

continuous: PDF equals probability intensity

$$f_x(x) dx = dP(X \leq x)$$

Given  $(X, Y) = (X_1, X_2) = \langle X, Y \rangle = \langle X_1, X_2 \rangle \sim f_{XY} = f_{XY}(x, y) = f_{X_1 X_2} = f_{X_1 X_2}(x_1, x_2)$

discrete:

**Definition 15.19.** JPMF = joint probability mass function: the JPMF of  $(X_1, X_2) = \langle X_1, X_2 \rangle$  is

$$f_{X_1 X_2} = f_{X_1 X_2}(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$$

joint vs. marginal

**Theorem 15.17.** joint probability mass function can inference marginal probability mass function:

The marginal PMF of  $X_1$ ,  $f_{X_1}(x_1) = P(X_1 = x_1)$  is given by

$$f_{X_1}(x_1) = \sum_{x_2 \in X_2(\Omega)} f_{X_1 X_2}(x_1, x_2) = \sum_{x_2 \in X_2(\Omega)} P(X_1 = x_1, X_2 = x_2)$$

Proof:

$$\begin{aligned} f_{X_1}(x_1) &= P(X_1 = x_1) \\ &= P(\{X_1 = x_1\} \cap \{X_2 \in (-\infty, \infty)\}) \\ &= P\left(\bigcup_{x_2 \in (-\infty, \infty)} \{X_1 = x_1 \wedge X_2 = x_2\}\right) \\ &= \bigcup_{x_2 \in (-\infty, \infty)} P(X_1 = x_1 \wedge X_2 = x_2) = \bigcup_{x_2 \in X_2(\Omega)} P(X_1 = x_1, X_2 = x_2) \\ &= \bigcup_{x_2 \in X_2(\Omega)} P(X_1 = x_1, X_2 = x_2) = \sum_{x_2 \in X_2(\Omega)} f_{X_1 X_2}(x_1, x_2) \end{aligned}$$

□

continuous:

**Definition 15.20.** JCDF = joint cumulative distribution function: the JCDF of  $(X_1, X_2) = \langle X_1, X_2 \rangle$  is

$$F_{X_1 X_2} = F_{X_1 X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

**Definition 15.21.** JPDF = joint probability density function: the JPDF of  $(X_1, X_2) = \langle X_1, X_2 \rangle$  is

$$f_{X_1 X_2} = f_{X_1 X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1 X_2}(x_1, x_2)$$

joint vs. marginal

**Theorem 15.18.** joint cumulative distribution function can inference marginal cumulative distribution function:

The marginal CDF of  $X_1$  is

$$F_{X_1}(x_1) = F_{X_1 X_2}(x_1, \infty)$$

Proof:

$$\begin{aligned} F_{X_1}(x_1) &= P(X_1 \leq x_1) \\ &= P(X_1 \leq x_1 \wedge X_2 \leq \infty) \\ &= P(X_1 \leq x_1, X_2 \leq \infty) \\ &= F_{X_1 X_2}(x_1, \infty) \end{aligned}$$

□

Note:

$$F_{X_1}(x_1)$$

---


$$F_{X_1}(x_1) = F_{X_1 X_2}(x_1, \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f_{X_1 X_2}(u_1, u_2) du_1 du_2$$


---

$$F_{X_1}(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f_{X_1 X_2}(u_1, u_2) du_1 du_2$$


---

$$f_{X_1}(x_1)$$

According to the fundamental theorem of calculus,

---


$$f_{X_1}(x_1) = \frac{d}{dx_1} F_{X_1}(x_1) = \frac{d}{dx_1} F_{X_1 X_2}(x_1, \infty) = \frac{d}{dx_1} \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f_{X_1 X_2}(u_1, u_2) du_1 du_2 = \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, u_2) du_2$$


---

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, u_2) du_2$$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, x_2) dx_2$$


---

$$f_{X_1}(x_1) = \begin{cases} \int_{-\infty}^{\infty} f_{X_1 X_2}(x_1, x_2) dx_2 & \text{continuous} \\ \sum_{x_2 \in X_2(\Omega)} f_{X_1 X_2}(x_1, x_2) & \text{discrete} \end{cases}$$

**Theorem 15.19.** A function  $f(x, y)$  is a joint PDF/PMF or JPDF/JPMF iff

$$\begin{cases} f(x, y) \geq 0 & \text{(ne) non-negative} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1 & \text{continuous} \\ \sum_{x \in X(\Omega_X)} \sum_{y \in Y(\Omega_Y)} f(x, y) = 1 & \text{discrete} \end{cases} \quad (1) \text{ total event}$$

**Definition 15.22.** expected value: The expected value of a random vector  $g(X, Y)$  is

$$E_{X,Y}[g(X, Y)] = E[g(X, Y)] = \begin{cases} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{XY}(x, y) dx dy & \text{continuous} \\ \sum_{x \in X(\Omega_X)} \sum_{y \in Y(\Omega_Y)} g(x, y) f_{XY}(x, y) & \text{discrete} \end{cases}$$

Def: 15.14

---


$$\begin{aligned} P((X, Y) \in E) &= E[1((X, Y) \in E)] = \begin{cases} \int_{y \in Y(\Omega_Y)} \int_{x \in X(\Omega_X)} 1((X, Y) \in E) f_{XY}(x, y) dx dy & \text{continuous} \\ \sum_{x \in X(\Omega_X)} \sum_{y \in Y(\Omega_Y)} 1((X, Y) \in E) g(x, y) f_{XY}(x, y) & \text{discrete} \end{cases} \\ &= \begin{cases} \iint_{(X, Y) \in E} f_{XY}(x, y) dx dy & \text{continuous} \\ \sum_{(X, Y) \in E} f_{XY}(x, y) & \text{discrete} \end{cases} \end{aligned}$$


---

example

$$f_{XY}(x, y) = e^{-y} \mathbf{1}(0 < x < y < \infty) \Rightarrow P(X + Y \leq 1)$$

Check if

$$f_{XY}(x, y) = e^{-y} \mathbf{1}(0 < x < y < \infty)$$

is a JPDF:

$$\begin{cases} f_{XY}(x, y) = e^{-y} \mathbf{1}(0 < x < y < \infty) & \begin{cases} e^{-y} > 0 \\ 1(0 < x < y < \infty) = \begin{cases} 0 \\ 1 \end{cases} \end{cases} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = ? & \begin{cases} 0 & (ne) \text{ non-negative} \\ 1 & (1) \text{ total event} \end{cases} \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y} \mathbf{1}(0 < x < y < \infty) dx dy, x \in (0, \infty) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} e^{-y} \mathbf{1}(0 < x < y < \infty) dx dy, y > x \in (0, \infty) \\ &= \int_0^{\infty} \int_0^{\infty} e^{-y} \mathbf{1}(0 < x < y < \infty) dx dy \stackrel{\text{Fubini}}{=} \int_0^{\infty} \int_0^{\infty} e^{-y} \mathbf{1}(0 < x < y < \infty) dy dx, y > x \\ &= \int_0^{\infty} \int_x^{\infty} e^{-y} dy dx = \int_0^{\infty} \int_x^{\infty} e^{-y} dy dx = \int_0^{\infty} [-e^{-y}]_{y=x}^{\infty} dx = \int_0^{\infty} [-e^{-\infty} - (-e^{-x})] dx \\ &= \int_0^{\infty} [-0 - (-e^{-x})] dx = \int_0^{\infty} e^{-x} dx = [-e^{-x}]_{x=0}^{\infty} = [-e^{-\infty} - (-e^0)] = [-0 - (-1)] = 1 \end{aligned}$$

$$\begin{cases} f_{XY}(x, y) = e^{-y} \mathbf{1}(0 < x < y < \infty) \geq 0 & (ne) \text{ non-negative} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1 & (1) \text{ total event} \end{cases}$$

$$P(X + Y \leq 1)$$

<https://tex.stackexchange.com/questions/75933/how-to-draw-the-region-of-inequality>

<https://tex.stackexchange.com/questions/352511/how-to-fill-in-inequality-where-all-inequalities-overlap>

interpolation dashed lines<sup>[13.5.6]</sup>

$$\begin{aligned} P(X + Y \leq 1) &= \iint_{(X,Y) \in E} f_{XY}(x, y) dx dy, E = \{X + Y \leq 1\} \\ &= \iint_E e^{-y} \mathbf{1}(0 < x < y < \infty) dx dy, E = \{X + Y \leq 1\} \\ &= \int_0^{0.5} \int_x^{1-x} e^{-y} dy dx = \int_0^{0.5} [-e^{-y}]_{y=x}^{1-x} dx = \int_0^{0.5} [-e^{-(1-x)} - (-e^{-x})] dx \\ &= \int_0^{0.5} [e^{-x} - e^{x-1}] dx = [-e^{-x} - e^{x-1}]_{x=0}^{0.5} = [-e^{-0.5} - e^{0.5-1}] - [-e^{-0} - e^{0-1}] \\ &= 1 + e^{-1} - 2e^{-0.5} \doteq 0.154818\dots \end{aligned}$$

□

<https://www.youtube.com/watch?v=r-bsPw2-sRg>

<https://www.youtube.com/watch?v=EwMFsvERFVw>

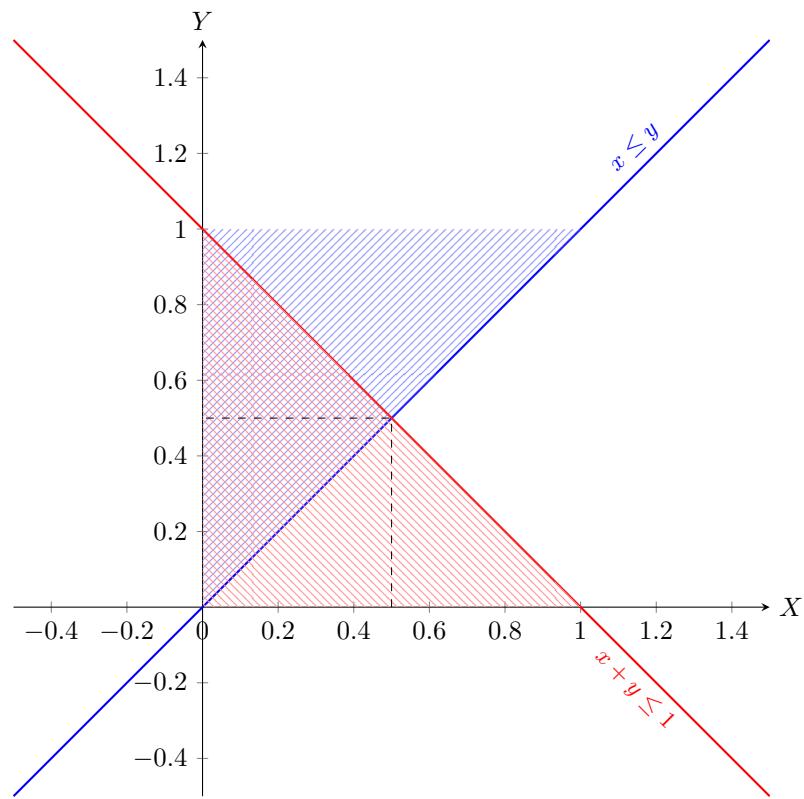
<https://www.youtube.com/watch?v=Os6HJLDtYgM>

<https://www.youtube.com/watch?v=8GefzYp6iBE>

<https://www.youtube.com/watch?v=nOSXlzxXrPY>

<https://www.youtube.com/watch?v=hwU7JLCRCgo>

<https://www.youtube.com/watch?v=duKMBB1bdCU>

Figure 15.3:  $X + Y \leq 1$ 

<https://www.youtube.com/watch?v=H7wBXhvjZfg>

<https://www.youtube.com/playlist?list=PLTp0eSi9MdkNZB4kyLSzIXIUy9JQOJ5AM>

<https://www.youtube.com/watch?v=qgef6G9rzts>

<https://www.youtube.com/watch?v=telisdm9Aus>

<https://www.youtube.com/watch?v=qeRsAId5f5U>

[https://www.youtube.com/watch?v=cuR-HsEq\\_fs](https://www.youtube.com/watch?v=cuR-HsEq_fs)

<https://www.youtube.com/watch?v=Ue1mgEVDwq0>

[https://www.youtube.com/watch?v=1meJoxJ5\\_UA](https://www.youtube.com/watch?v=1meJoxJ5_UA)

<https://www.youtube.com/watch?v=UB0kwppDucI>

<https://www.youtube.com/watch?v=YrqcdCPM1nw>

[https://www.youtube.com/watch?v=GCw1T\\_lUunw](https://www.youtube.com/watch?v=GCw1T_lUunw)

<https://www.youtube.com/watch?v=MF0mZ5MpcSw>

<https://www.youtube.com/watch?v=zX8L8yGIYaU>

<https://www.youtube.com/watch?v=XJgvphHKXwY>

<https://www.youtube.com/watch?v=8Qzqf51O6ZE>

<https://www.youtube.com/watch?v=7x8QN1pYT7c>

<https://www.youtube.com/watch?v=oTf3n-OD7EI>

<https://www.youtube.com/watch?v=Igo2EJPz3sU>

<https://www.youtube.com/watch?v=NVmOifoMqw>

<https://www.youtube.com/watch?v=HrrBA3v-xTQ>

<https://www.youtube.com/watch?v=9HbtJ3ZIPxQ>

<https://www.youtube.com/watch?v=elPnSU4AF1o>

[https://www.youtube.com/watch?v=\\_OVHGnQ7Rug](https://www.youtube.com/watch?v=_OVHGnQ7Rug)

[https://www.youtube.com/watch?v=pqHEXA AW\\_vk](https://www.youtube.com/watch?v=pqHEXA AW_vk)  
[https://www.youtube.com/watch?v=BTK\\_1Lz5ox8](https://www.youtube.com/watch?v=BTK_1Lz5ox8)  
<https://www.youtube.com/watch?v=iATKGYnlomU>  
<https://www.youtube.com/watch?v=5LlTeUeAqDc>  
<https://www.youtube.com/watch?v=usCaJRQ2i6E>  
<https://www.youtube.com/watch?v=p8NXibyDKDo>

## 15.2 Chen, Lin-An

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUPXtNAX9lro-leGgEK0OSEW>  
[https://en.wikipedia.org/wiki/Inverse\\_distance\\_weighting](https://en.wikipedia.org/wiki/Inverse_distance_weighting) Shepard interpolation

# Chapter 16

## covariance matrix

### 16.1 vector direct product

- scalar = rank-0 tensor
- vector = rank-1 tensor
- matrix = rank-2 tensor
- vector direct product = rank-1 tensor times rank-1 tensor equals rank-2 tensor: increasing rank
- vector inner product = rank-1 tensor times rank-1 tensor equals rank-0 tensor: decreasing rank

scalar = rank-0 tensor

vector = rank-1 tensor

matrix = rank-2 tensor

#### 16.1.1 vector direct product: increasing rank

vector direct product = rank-1 tensor times rank-1 tensor equals rank-2 tensor: increasing rank

$$\begin{aligned} U \otimes V &= UV^\top = U_i V_j \\ &= \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \otimes \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} (V_1 \quad V_2 \quad V_3) = \begin{pmatrix} U_1 V_1 & U_1 V_2 & U_1 V_3 \\ U_2 V_1 & U_2 V_2 & U_2 V_3 \\ U_3 V_1 & U_3 V_2 & U_3 V_3 \end{pmatrix} \\ &= \begin{pmatrix} U_1 V^\top \\ U_2 V^\top \\ U_3 V^\top \end{pmatrix} = (UV_1 \quad UV_2 \quad UV_3) \end{aligned}$$

#### 16.1.2 vector inner product: decreasing rank

vector inner product = rank-1 tensor times rank-1 tensor equals rank-0 tensor: decreasing rank

$$\begin{aligned} U \cdot V &= V^\top U = V_i U_i \\ &= \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = (V_1 \quad V_2 \quad V_3) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = V_1 U_1 + V_2 U_2 + V_3 U_3 \end{aligned}$$

### 16.1.3 tensor direct product: increasing rank

$$S \otimes T = S_{ik}T_{jl} \quad (16.1)$$

$$(i, j), (k, l) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\} \quad (16.2)$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \otimes \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} S_{11}T_{11} & S_{11}T_{12} & S_{12}T_{11} & S_{12}T_{12} \\ S_{11}T_{21} & S_{11}T_{22} & S_{12}T_{21} & S_{12}T_{22} \\ S_{21}T_{11} & S_{21}T_{12} & S_{22}T_{11} & S_{22}T_{12} \\ S_{21}T_{21} & S_{21}T_{22} & S_{22}T_{21} & S_{22}T_{22} \end{pmatrix} \quad (16.3)$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} S_{11}T & S_{12}T \\ S_{21}T & S_{22}T \end{pmatrix} \quad (16.4)$$

$$= \begin{pmatrix} S_{11} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} & S_{12} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \\ S_{21} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} & S_{22} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \end{pmatrix} \quad (16.5)$$

## 16.2 covariance matrix and its properties

8

$$\begin{aligned} C[\mathbf{X}] = \text{Cov}[\mathbf{X}] = V[\mathbf{X}] &= E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^\top] \\ &= E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X}^\top - E(\mathbf{X})^\top)] \\ &= E[\mathbf{X}\mathbf{X}^\top - E(\mathbf{X})\mathbf{X}^\top - E\mathbf{X}(E(\mathbf{X})^\top) + E(\mathbf{X})E(\mathbf{X})^\top] \\ &= E[\mathbf{X}\mathbf{X}^\top] - E[E(\mathbf{X})\mathbf{X}^\top] - E[\mathbf{X}E(\mathbf{X})^\top] + E[E(\mathbf{X})E(\mathbf{X})^\top] \\ &= E[\mathbf{X}\mathbf{X}^\top] - E(\mathbf{X})E[\mathbf{X}^\top] - E[\mathbf{X}]E(\mathbf{X})^\top + E(\mathbf{X})E(\mathbf{X})^\top \\ &= E[\mathbf{X}\mathbf{X}^\top] - E(\mathbf{X})E(\mathbf{X})^\top - E(\mathbf{X})E(\mathbf{X})^\top + E(\mathbf{X})E(\mathbf{X})^\top \\ &= E[\mathbf{X}\mathbf{X}^\top] - E(\mathbf{X})E(\mathbf{X})^\top \end{aligned}$$

$$\begin{aligned} \mathbf{X} = [X]_{1 \times 1} = X \Rightarrow C(X) = C[\mathbf{X}] &= E[\mathbf{X}\mathbf{X}^\top] - E(\mathbf{X})E(\mathbf{X})^\top \\ &= E[XX] - E(X)E(X) \\ &= E(X^2) - [E(X)]^2 = V(X) \end{aligned}$$

### 16.2.1 $V[\mathbf{X} + \mathbf{b}] = V[\mathbf{X}]$

$$\begin{aligned} V[\mathbf{X} + \mathbf{b}] &= E[((\mathbf{X} + \mathbf{b}) - E(\mathbf{X} + \mathbf{b}))((\mathbf{X} + \mathbf{b}) - E(\mathbf{X} + \mathbf{b}))^\top] \\ &\stackrel{E(\mathbf{X}+\mathbf{b})=E(\mathbf{X})+\mathbf{b}}{=} E[(\mathbf{X} + \mathbf{b} - E(\mathbf{X}) - \mathbf{b})(\mathbf{X} + \mathbf{b} - E(\mathbf{X}) - \mathbf{b})^\top] \\ &= E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^\top] = V[\mathbf{X}] \end{aligned}$$

### 16.2.2 $V[A\mathbf{X}] = AV[\mathbf{X}]A^\top$

$$\begin{aligned} V[A\mathbf{X}] &= E[((A\mathbf{X}) - E(A\mathbf{X}))((A\mathbf{X}) - E(A\mathbf{X}))^\top] \\ &\stackrel{E(A\mathbf{X})=AE(\mathbf{X})}{=} E[(A\mathbf{X} - AE(\mathbf{X}))(A\mathbf{X} - AE(\mathbf{X}))^\top] \\ &= E[A(\mathbf{X} - E(\mathbf{X}))(A(\mathbf{X} - E(\mathbf{X})))^\top] \\ &= E[A(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^\top A^\top] \\ &= AE[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^\top] A^\top = AV[\mathbf{X}]A^\top \end{aligned}$$

### 16.2.3 $V[A\mathbf{X} + \mathbf{b}] = AV[\mathbf{X}]A^\top$

$$V[A\mathbf{X} + \mathbf{b}] = V[A\mathbf{X}] = AV[\mathbf{X}]A^\top$$

# Chapter 17

## Gosper algorithm

### 17.1 Zhuli

<https://www.youtube.com/watch?v=0LFg5dvPOoc>

### 17.2 TaylorCatAlice

<https://www.bilibili.com/video/BV1ZX4y1o7EB>

<https://www.bilibili.com/video/BV1wv4y1H7m8>

<https://www.bilibili.com/video/BV1Ua4y1q7Q9>



# Chapter 18

## Lorentz transformation

### 18.1 Einstein

<https://www.youtube.com/watch?v=FvqutkaPmas>

<https://wap.hillpublisher.com/UpFile/202204/20220414165340.pdf>

### 18.2 Bondi *k*-calculus

<https://www.youtube.com/watch?v=Ghql2UNIWYA>

[https://en.wikipedia.org/wiki/Bondi\\_k-calculus](https://en.wikipedia.org/wiki/Bondi_k-calculus)

<https://www.youtube.com/watch?v=bZnmF-UKvq0>

### 18.3 wordline in Minkowski space

<https://www.bilibili.com/video/BV1sV4y1Y7fX>

#### 18.3.1 Wick rotation

<https://ncatlab.org/nlab/show/Wick+rotation>

##### 18.3.1.1 Osterwalder-Schrader reconstruction theorem

<https://ncatlab.org/nlab/show/Osterwalder-Schrader+theorem>



# Chapter 19

## R

### 19.1 TonyKuoYJ

郭耀仁 認識 R 的美好

<https://bookdown.org/tonykuoyj/eloquentr/getting-started.html>

<https://bookdown.org/tonykuoyj/eloquentr/easy-installation.html#about-packages>

```
install.packages()
```

```
library()
```

<https://bookdown.org/tonykuoyj/eloquentr/getting-started.html>

#### 19.1.1 quick intro

Ctrl + Alt + I to insert a new code chunk in RStudio

Ctrl + Enter to run the current line

Ctrl + Shift + Enter to run the current chunk

```
R.version
```

```

platform x86_64-w64-mingw32
arch x86_64
os mingw32
crt ucrt
system x86_64, mingw32
status
major 4
minor 2.1
year 2022
month 06
day 23
svn rev 82513
language R
version.string R version 4.2.1 (2022-06-23 ucrt)
nickname Funny-Looking Kid
```

```
a <- 23 # prime
a
```

```
[1] 23
combine <- c(11, 13) # twin prime
combine
```

```
[1] 11 13
```

```
?c
help(c)
```

Ctrl + L to clean R console  
 path with slash / in R, differing backslash \ in M\$ Windows

### 19.1.1.1 function

```
add <- function(x, y) {
 return(x + y)
}

add(11, 13)
```

```
[1] 24
```

$$BMI = \frac{BW \text{ [Kg]}}{BH \text{ [m]}^2}$$

```
get_bmi <- function (bw, bh) {
 return (bw/(bh/100)^2)
}

get_bmi(70, 170)
```

```
[1] 24.22145
```

### 19.1.2 R style

<https://bookdown.org/tonykuoyj/eloquentr/styleguide.html>

snake\_case rather than camelCase

### 19.1.3 data workflow or forward pipe

from *chaining method* in *object-oriented programming* to **functional programming**

#### 19.1.3.1 %>% operator

```
abs(-5:5)
```

```
[1] 5 4 3 2 1 0 1 2 3 4 5
```

```
install.packages("magrittr")
```

```
library(magrittr)
```

```
##
Attaching package: 'magrittr'
The following object is masked _by_ '.GlobalEnv':
##
add
-5:5 %>% abs()
```

```
[1] 5 4 3 2 1 0 1 2 3 4 5
```

```
with readability but too many lines
```

```
sys_date <- Sys.Date()
sys_date_yr <- format(sys_date, format = "%Y")
sys_date_num <- as.numeric(sys_date_yr)
sys_date_num
```

```
[1] 2024
less line but also less readability
sys_date_num <- as.numeric(format(Sys.Date(), format = "%Y"))
sys_date_num

[1] 2024
use %>% operator to demonstrate data workflow or forward pipe
sys_date_num <- Sys.Date() %>%
 format(format = "%Y") %>%
 as.numeric()
sys_date_num

[1] 2024
```

### 19.1.4 data processing with dplyr

<https://bookdown.org/tonykuoyj/eloquentr/dplyr.html>

some functions functioning like those in **SQL**

```
library(dplyr)

Warning: package 'dplyr' was built under R version 4.2.3
##
Attaching package: 'dplyr'
The following objects are masked from 'package:stats':

filter, lag
The following objects are masked from 'package:base':

intersect, setdiff, setequal, union
install.packages("gapminder")

library(gapminder)

Warning: package 'gapminder' was built under R version 4.2.3
head(gapminder)

A tibble: 6 x 6
country continent year lifeExp pop gdpPercap
<fct> <fct> <int> <dbl> <int> <dbl>
1 Afghanistan Asia 1952 28.8 8425333 779.
2 Afghanistan Asia 1957 30.3 9240934 821.
3 Afghanistan Asia 1962 32.0 10267083 853.
4 Afghanistan Asia 1967 34.0 11537966 836.
5 Afghanistan Asia 1972 36.1 13079460 740.
6 Afghanistan Asia 1977 38.4 14880372 786.

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 filter(year == 2007)

A tibble: 142 x 6
country continent year lifeExp pop gdpPercap
<fct> <fct> <int> <dbl> <int> <dbl>
1 Afghanistan Asia 2007 43.8 31889923 975.
2 Albania Europe 2007 76.4 3600523 5937.
3 Algeria Africa 2007 72.3 33333216 6223.
```

```

4 Angola Africa 2007 42.7 12420476 4797.
5 Argentina Americas 2007 75.3 40301927 12779.
6 Australia Oceania 2007 81.2 20434176 34435.
7 Austria Europe 2007 79.8 8199783 36126.
8 Bahrain Asia 2007 75.6 708573 29796.
9 Bangladesh Asia 2007 64.1 150448339 1391.
10 Belgium Europe 2007 79.4 10392226 33693.
i 132 more rows

```

```

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 filter(year == 2007) %>%
 select(country)

```

```

A tibble: 142 x 1
country
<fct>
1 Afghanistan
2 Albania
3 Algeria
4 Angola
5 Argentina
6 Australia
7 Austria
8 Bahrain
9 Bangladesh
10 Belgium
i 132 more rows

```

```

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 mutate(pop_in_thousands = pop / 1000)

```

```

A tibble: 1,704 x 7
country continent year lifeExp pop gdpPercap pop_in_thousands
<fct> <fct> <int> <dbl> <int> <dbl> <dbl>
1 Afghanistan Asia 1952 28.8 8425333 779. 8425.
2 Afghanistan Asia 1957 30.3 9240934 821. 9241.
3 Afghanistan Asia 1962 32.0 10267083 853. 10267.
4 Afghanistan Asia 1967 34.0 11537966 836. 11538.
5 Afghanistan Asia 1972 36.1 13079460 740. 13079.
6 Afghanistan Asia 1977 38.4 14880372 786. 14880.
7 Afghanistan Asia 1982 39.9 12881816 978. 12882.
8 Afghanistan Asia 1987 40.8 13867957 852. 13868.
9 Afghanistan Asia 1992 41.7 16317921 649. 16318.
10 Afghanistan Asia 1997 41.8 22227415 635. 22227.
i 1,694 more rows

```

```

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 arrange(year)

```

```

A tibble: 1,704 x 6
country continent year lifeExp pop gdpPercap
<fct> <fct> <int> <dbl> <int> <dbl>

```

```

1 Afghanistan Asia 1952 28.8 8425333 779.
2 Albania Europe 1952 55.2 1282697 1601.
3 Algeria Africa 1952 43.1 9279525 2449.
4 Angola Africa 1952 30.0 4232095 3521.
5 Argentina Americas 1952 62.5 17876956 5911.
6 Australia Oceania 1952 69.1 8691212 10040.
7 Austria Europe 1952 66.8 6927772 6137.
8 Bahrain Asia 1952 50.9 120447 9867.
9 Bangladesh Asia 1952 37.5 46886859 684.
10 Belgium Europe 1952 68 8730405 8343.
i 1,694 more rows

```

total population in the world in 2007

```

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 filter(year == 2007) %>%
 summarise(ttl_pop = sum(as.numeric(pop)))

```

```

A tibble: 1 x 1
ttl_pop
<dbl>
1 6251013179

```

total population group by the continents in 2007

```

library(gapminder)
library(dplyr)
library(magrittr)

gapminder %>%
 filter(year == 2007) %>%
 group_by(continent) %>%
 summarise(ttl_pop = sum(as.numeric(pop)))

```

```

A tibble: 5 x 2
continent ttl_pop
<fct> <dbl>
1 Africa 929539692
2 Americas 898871184
3 Asia 3811953827
4 Europe 586098529
5 Oceania 24549947

```

### 19.1.5 visualization statically with ggplot2

```

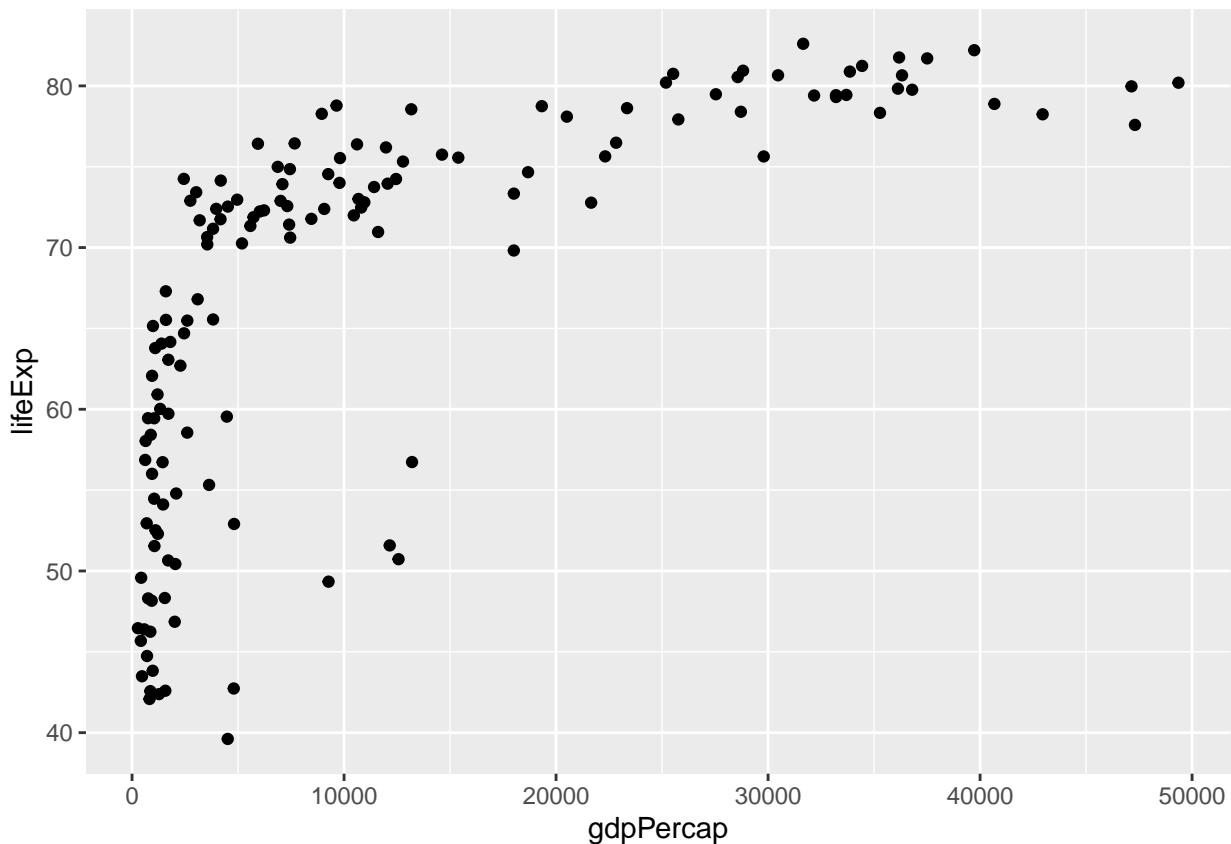
library(ggplot2)

Warning: package 'ggplot2' was built under R version 4.2.3

library(gapminder)

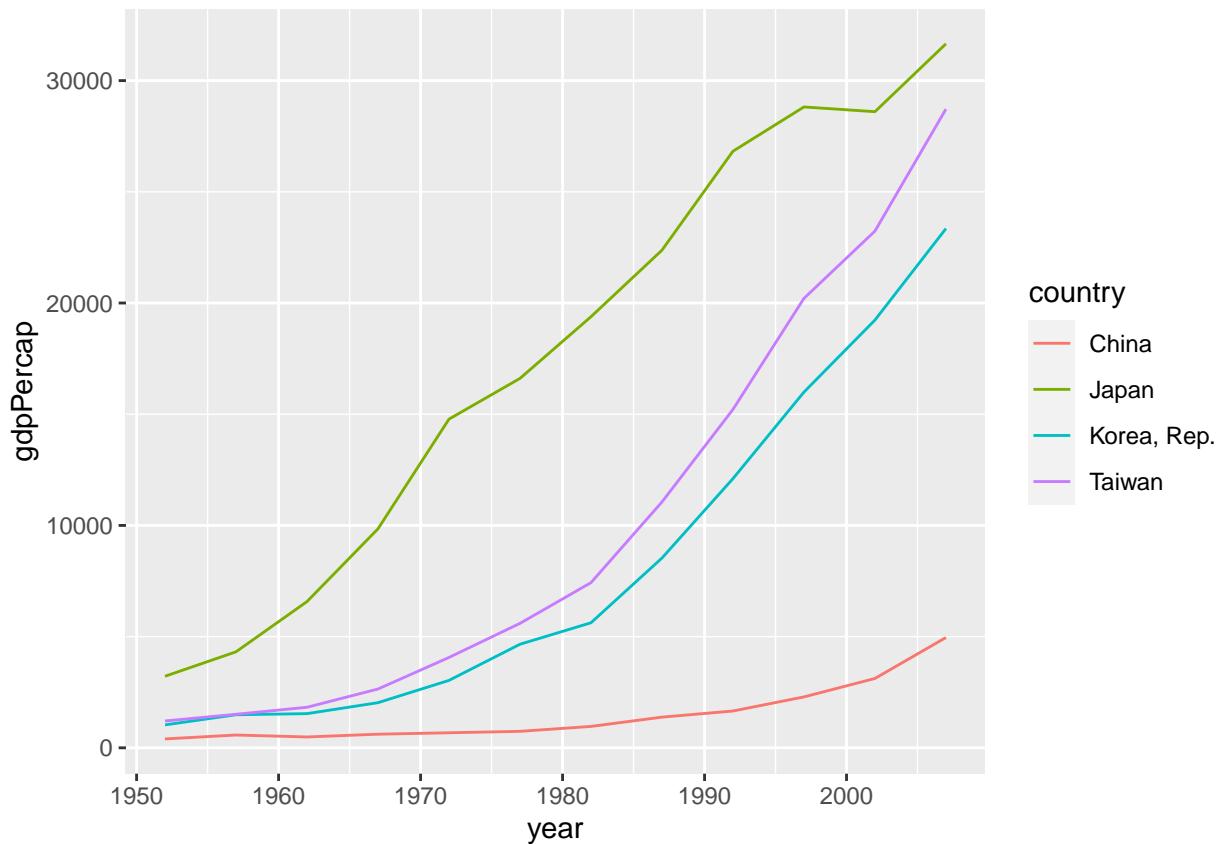
gapminder_2007 <- gapminder %>%
 filter(year == 2007)
scatter_plot <- ggplot(gapminder_2007, aes(x = gdpPercap, y = lifeExp)) +
 geom_point()
scatter_plot

```



```
library(ggplot2)
library(gapminder)

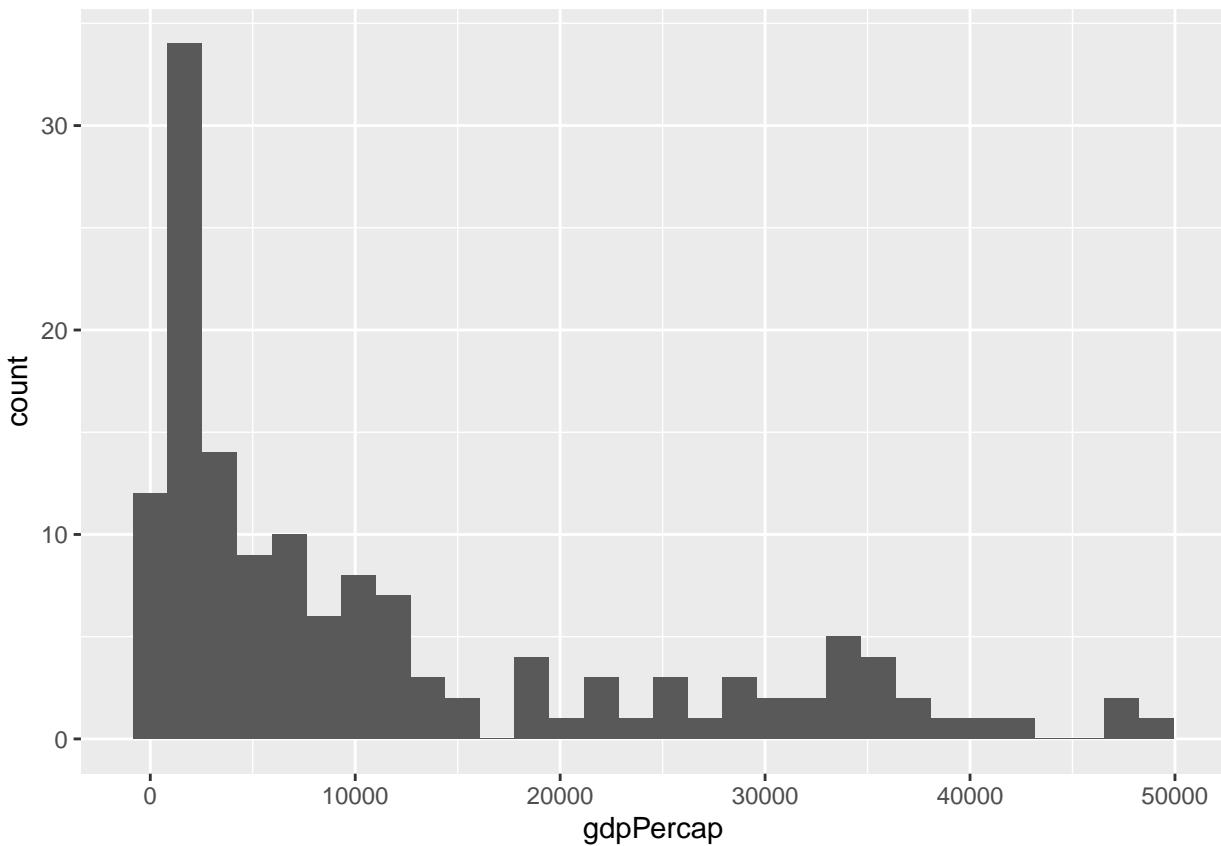
north_asia <- gapminder %>%
 filter(country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
line_plot <- ggplot(north_asia, aes(x = year, y = gdpPercap, colour = country)) +
 geom_line()
line_plot
```



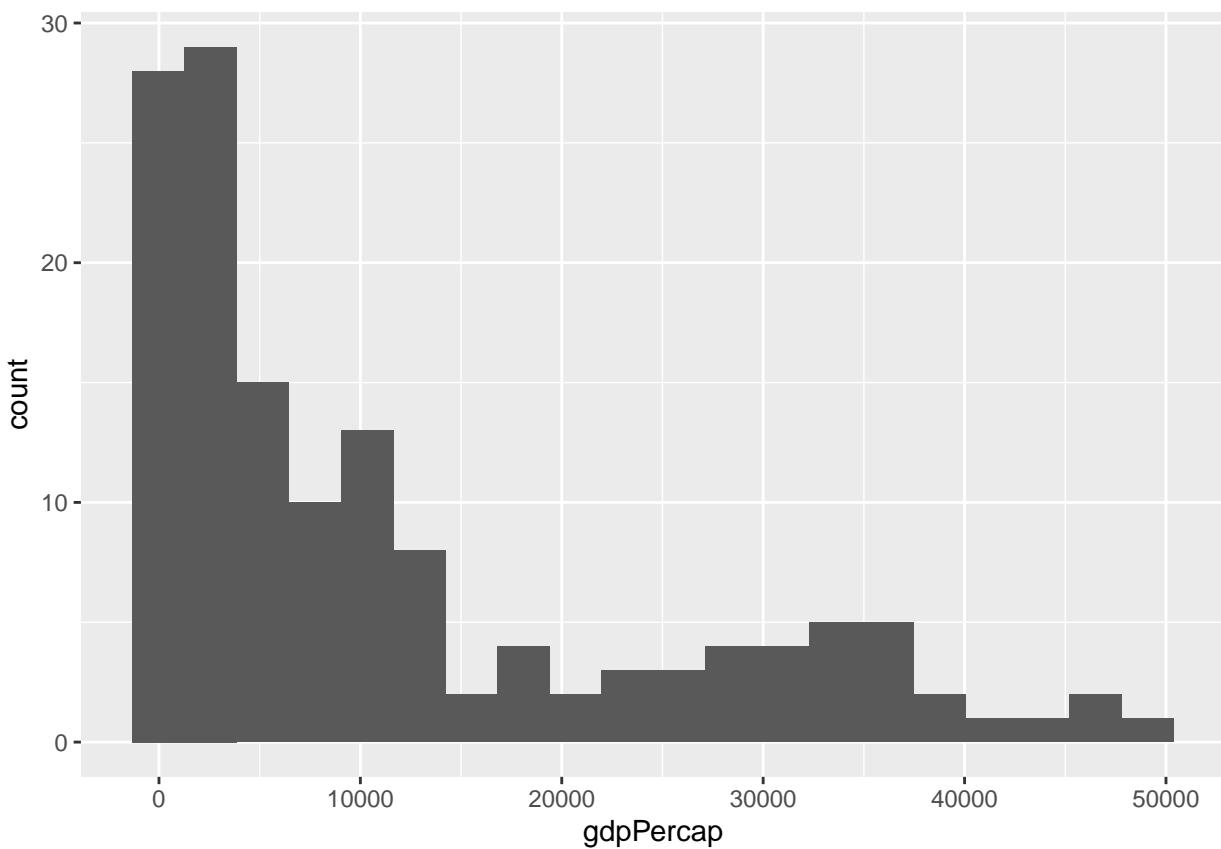
```
library(ggplot2)
library(gapminder)

hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +
 geom_histogram()
hist_plot
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

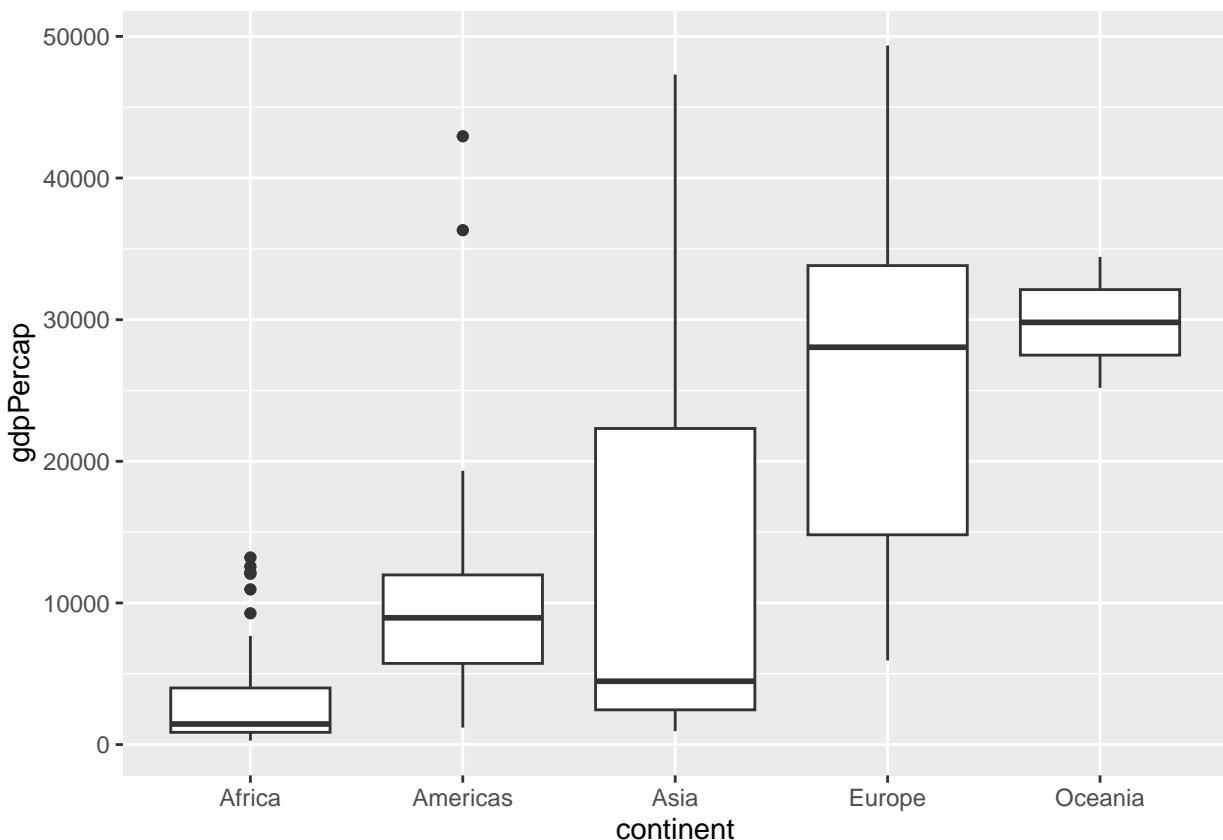


```
hist_plot <- ggplot(gapminder_2007, aes(x = gdpPercap)) +
 geom_histogram(bins = 20)
hist_plot
```



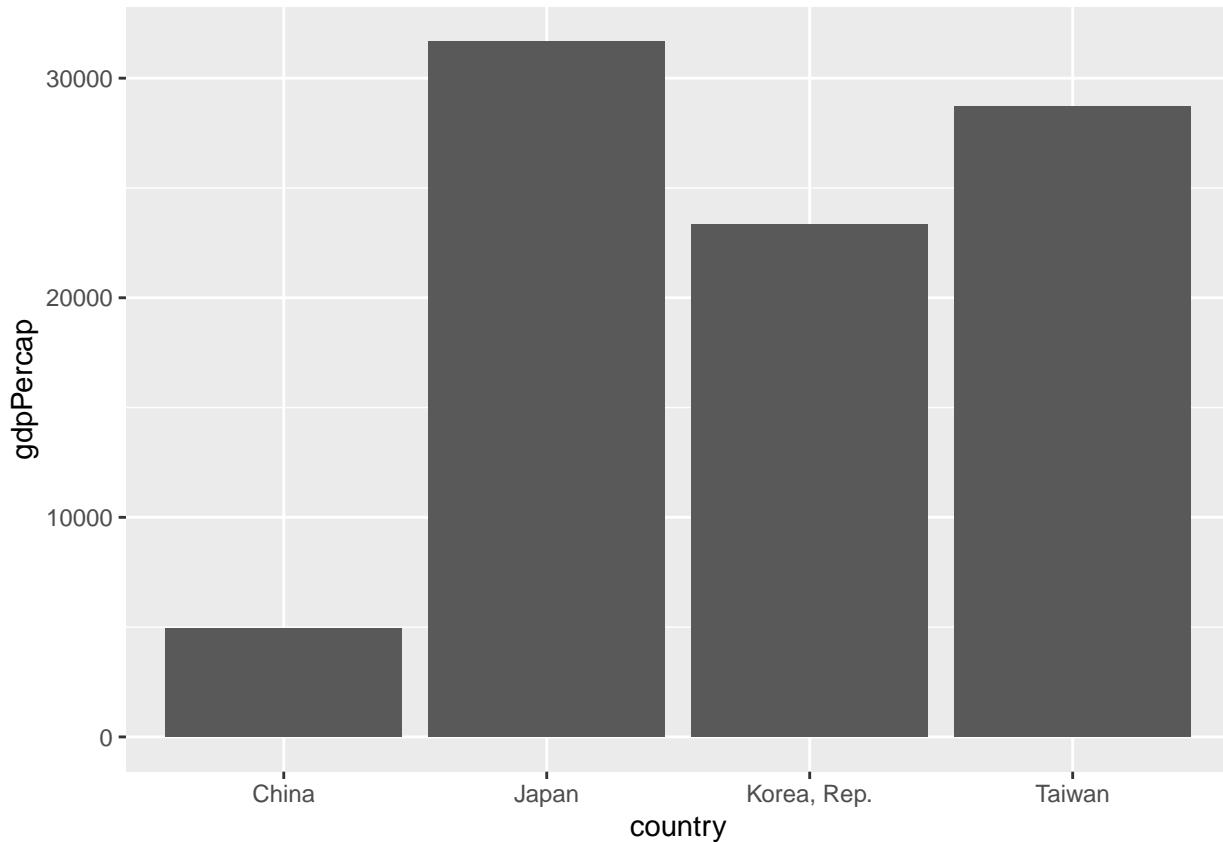
```
library(ggplot2)
library(gapminder)
```

```
box_plot <- ggplot(gapminder_2007, aes(x = continent, y = gdpPercap)) +
 geom_boxplot()
box_plot
```



```
library(ggplot2)
library(gapminder)

gdpPercap_2007_na <- gapminder %>%
 filter(year == 2007 & country %in% c("China", "Japan", "Taiwan", "Korea, Rep."))
bar_plot <- ggplot(gdpPercap_2007_na, aes(x = country, y = gdpPercap)) +
 geom_bar(stat = "identity")
bar_plot
```



### 19.1.6 loop

<https://bookdown.org/tonykuojy/eloquentr/for.html>

```
month.name
```

```
[1] "January" "February" "March" "April" "May" "June"
[7] "July" "August" "September" "October" "November" "December"
```

```
month.name[1]
```

```
[1] "January"
```

```
for (month in month.name) {
 print(month)
}
```

```
[1] "January"
[1] "February"
[1] "March"
[1] "April"
[1] "May"
[1] "June"
[1] "July"
[1] "August"
[1] "September"
[1] "October"
[1] "November"
[1] "December"
```

### 19.1.7 variable type

<https://bookdown.org/tonykuojy/eloquentr/variable-types.html>

[https://www.w3schools.com/r/r\\_data\\_types.asp](https://www.w3schools.com/r/r_data_types.asp)

- numeric
- integer

- complex = complex number
- character
- logical = boolean

```
class(2L)
```

```
[1] "integer"
```

```
class(2.0L)
```

```
[1] "integer"
```

```
class(2.3L)
```

```
[1] "numeric"
```

time: POSIXct POSIXt

```
class(Sys.time())
```

```
[1] "POSIXct" "POSIXt"
```

```
0 %in% -5:5
```

```
[1] TRUE
```

### 19.1.7.1 date

1970-01-01 = 0L

```
date_of_origin <- as.Date("1970-01-01")
as.integer(date_of_origin)
```

```
[1] 0
```

check if type of x is Date

```
inherits(x, what = "Date")
```

convert character to Date

```
as.Date("01-01-1970", format = "%m-%d-%Y")
```

### 19.1.7.2 time

1970-01-01 00:00:00 GMT = 0L

tz = time zone

```
time_of_origin <- as.POSIXct("1970-01-01 00:00:00", tz = "GMT")
as.integer(time_of_origin)
```

```
[1] 0
```

check if type of x is time

```
inherits(x, what = "POSIXct")
```

convert character to time

```
as.POSIXct("1970-01-01 00:00:00", tz = "GMT")
```

### 19.1.7.3 quotient %/% operator

[https://www.w3schools.com/r/r\\_operators.asp](https://www.w3schools.com/r/r_operators.asp)

```
7 %/% 3
```

```
[1] 2
```

### 19.1.8 data type

<https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html>

- 1D
  - `vector`<sup>[19.1.8.1]</sup>
  - `factor`<sup>[19.1.8.2]</sup>
- 2D
  - `matrix`<sup>[19.1.8.3]</sup>
  - `data frame`<sup>[19.1.8.4]</sup>
- $n$ D
  - `array`<sup>[19.2.6.1]</sup>
  - `list`<sup>[19.2.6.2]</sup>

#### 19.1.8.1 vector

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons
```

```
[1] "spring" "summer" "autumn" "winter"
favorite_season <- four_seasons[3]
favorite_season
```

```
[1] "autumn"
favorite_seasons <- four_seasons[c(-2, -4)]
favorite_seasons
```

```
[1] "spring" "autumn"
```

only one variable type for a vector

```
lucky_numbers <- c(7L, 24)
class(lucky_numbers[1])
```

```
[1] "numeric"
```

```
lucky_numbers <- c(7L, FALSE)
lucky_numbers
```

```
[1] 7 0
```

```
class(lucky_numbers[2])
```

```
[1] "integer"
```

```
mixed_vars <- c(TRUE, 7L, 24, "spring")
mixed_vars
```

```
[1] "TRUE" "7" "24" "spring"
```

```
class(mixed_vars[1])
```

```
[1] "character"
```

```
class(mixed_vars[2])
```

```
[1] "character"
```

```
class(mixed_vars[3])
```

```
[1] "character"
```

```
four_seasons <- c("spring", "summer", "autumn", "winter")
my_favorite_seasons <- four_seasons == "spring" | four_seasons == "autumn"
four_seasons[my_favorite_seasons]
```

### 19.1.8.1.1 logic

```
[1] "spring" "autumn"

rep(7L, times = 8)
```

### 19.1.8.1.2 rep repeat

```
[1] 7 7 7 7 7 7 7 7
rep("R", times = 10)

[1] "R" "R" "R" "R" "R" "R" "R" "R" "R" "R"
```

seq(from = 7, to = 77, by = 7)

### 19.1.8.1.3 seq sequence

```
[1] 7 14 21 28 35 42 49 56 63 70 77
11:20
```

```
[1] 11 12 13 14 15 16 17 18 19 20
```

### 19.1.8.2 factor

<https://bookdown.org/tonykuoyj/eloquentr/vector-factor.html#factor>

```
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons
```

```
[1] "spring" "summer" "autumn" "winter"
four_seasons_factor <- factor(four_seasons)
four_seasons_factor
```

```
[1] spring summer autumn winter
Levels: autumn spring summer winter
four_seasons <- c("spring", "summer", "autumn", "winter")
four_seasons_factor <- factor(four_seasons, ordered = TRUE, levels = c("summer", "winter", "spring",
 ↪ "autumn"))
four_seasons_factor
```

```
[1] spring summer autumn winter
Levels: summer < winter < spring < autumn
temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE, levels = c("cold", "warm", "hot"))
temp_factors
```

```
[1] warm hot cold
Levels: cold < warm < hot
```

if no levels specified, the levels will be specified alphabetically, sometimes not really true

```
temperatures <- c("warm", "hot", "cold")
temp_factors <- factor(temperatures, ordered = TRUE)
temp_factors
```

```
[1] warm hot cold
Levels: cold < hot < warm
```

### 19.1.8.3 matrix

<https://bookdown.org/tonykuoyj/eloquentr/matrix-dataframe-more.html>

```

my_mat <- matrix(1:6, nrow = 2)
my_mat

[,1] [,2] [,3]
[1,] 1 3 5
[2,] 2 4 6
class(my_mat)

[1] "matrix" "array"
my_mat2 <- matrix(1:6, nrow = 2, byrow = TRUE)
my_mat2

[,1] [,2] [,3]
[1,] 1 2 3
[2,] 4 5 6
my_mat2[2, 3]

[1] 6
my_mat2[2,]

[1] 4 5 6
my_mat2[, 3]

[1] 3 6
filter <- my_mat2 < 6 & my_mat2 > 1
my_mat2[filter]

[1] 4 2 5 3

```

boolean will become value in a matrix, like vector

```

my_mat3 <- matrix(c(1, 2, TRUE, FALSE, 3, 4), nrow = 2)
my_mat3

[,1] [,2] [,3]
[1,] 1 1 3
[2,] 2 0 4
class(my_mat3[, 2])

[1] "numeric"

```

#### 19.1.8.4 data frame

- variable: column
- observation: row
- value: cell

```

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season)
great_nba_teams

team_name wins losses is_champion season
1 Chicago Bulls 72 10 TRUE 1995-96
2 Golden State Warriors 73 9 FALSE 2015-16

```

```

great_nba_teams[1, 1]

[1] "Chicago Bulls"
great_nba_teams[1,]

team_name wins losses is_champion season
1 Chicago Bulls 72 10 TRUE 1995-96
great_nba_teams[, 1]

[1] "Chicago Bulls" "Golden State Warriors"
stringsAsFactors = TRUE

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season, stringsAsFactors = TRUE)
great_nba_teams[, 1]

[1] Chicago Bulls Golden State Warriors
Levels: Chicago Bulls Golden State Warriors
stringsAsFactors = FALSE

team_name <- c("Chicago Bulls", "Golden State Warriors")
wins <- c(72, 73)
losses <- c(10, 9)
is_champion <- c(TRUE, FALSE)
season <- c("1995-96", "2015-16")

great_nba_teams <- data.frame(team_name, wins, losses, is_champion, season, stringsAsFactors = FALSE)
great_nba_teams[, 1]

[1] "Chicago Bulls" "Golden State Warriors"

great_nba_teams$team_name

```

#### 19.1.8.4.1 selecting variable or column

```

[1] "Chicago Bulls" "Golden State Warriors"
great_nba_teams[, "team_name"]

[1] "Chicago Bulls" "Golden State Warriors"

filter <- great_nba_teams$is_champion == TRUE
great_nba_teams[filter,]

```

#### 19.1.8.4.2 filtering observation or row

```

team_name wins losses is_champion season
1 Chicago Bulls 72 10 TRUE 1995-96

str(great_nba_teams)

'data.frame': 2 obs. of 5 variables:
$ team_name : chr "Chicago Bulls" "Golden State Warriors"
$ wins : num 72 73

```

#### 19.1.8.4.3 check mixed data type

```
$ losses : num 10 9
$ is_champion: logi TRUE FALSE
$ season : chr "1995-96" "2015-16"
```

## 19.2 W3School

<https://www.w3schools.com/r/default.asp>

### 19.2.1 same multiple variable

[https://www.w3schools.com/r/r\\_variables\\_multiple.asp](https://www.w3schools.com/r/r_variables_multiple.asp)

```
Assign the same value to multiple variables in one line
var1 <- var2 <- var3 <- "Orange"

Print variable values
var1

[1] "Orange"

var2

[1] "Orange"

var3

[1] "Orange"
```

### 19.2.2 legal variable name

[https://www.w3schools.com/r/r\\_variables\\_name.asp](https://www.w3schools.com/r/r_variables_name.asp)

# Legal variable names:

```
myvar <- "John"
my_var <- "John"
myVar <- "John"
MYVAR <- "John"
myvar2 <- "John"
.myvar <- "John"
```

## Illegal variable names:

```
2myvar <- "John"
my-var <- "John"
my var <- "John"
_my_var <- "John"
my_v@ar <- "John"
TRUE <- "John"
```

### 19.2.3 complex number

[https://www.w3schools.com/r/r\\_data\\_types.asp](https://www.w3schools.com/r/r_data_types.asp)

[https://www.w3schools.com/r/r\\_numbers.asp](https://www.w3schools.com/r/r_numbers.asp)

### 19.2.4 escape character

[https://www.w3schools.com/r/r\\_strings\\_esc.asp](https://www.w3schools.com/r/r_strings_esc.asp)

### 19.2.5 global assignment <<-

```
my_function <- function() {
 txt <<- "fantastic"
 paste("R is", txt)
}
```

```
my_function()

[1] "R is fantastic"
print(txt)

[1] "fantastic"

txt <- "awesome"
my_function <- function() {
 txt <<- "fantastic"
 paste("R is", txt)
}

my_function()

[1] "R is fantastic"
paste("R is", txt)

[1] "R is fantastic"
```

## 19.2.6 data type

### 19.2.6.1 array

[https://www.w3schools.com/r/r\\_arrays.asp](https://www.w3schools.com/r/r_arrays.asp)

```
An array with one dimension with values ranging from 1 to 24
thisarray <- c(1:24)
thisarray

[1] 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

An array with more than one dimension
multiarray <- array(thisarray, dim = c(4, 3, 2))
multiarray

, , 1
##
[,1] [,2] [,3]
[1,] 1 5 9
[2,] 2 6 10
[3,] 3 7 11
[4,] 4 8 12
##
, , 2
##
[,1] [,2] [,3]
[1,] 13 17 21
[2,] 14 18 22
[3,] 15 19 23
[4,] 16 20 24

multiarray[2, 3, 2]

[1] 22
```

### 19.2.6.2 list

[https://www.w3schools.com/r/r\\_lists.asp](https://www.w3schools.com/r/r_lists.asp)

## 19.3 Apan Liao

<https://www.youtube.com/playlist?list=PL5AC0ADBF65924EAD>

### 19.3.1 data input

[https://www.youtube.com/watch?v=STcIxf\\_vUWY&list=PL5AC0ADBF65924EAD&index=1](https://www.youtube.com/watch?v=STcIxf_vUWY&list=PL5AC0ADBF65924EAD&index=1)

- `scan()`
- `read`
  - `read.table()`
  - `read.csv()`

### 19.3.2 descriptive statistics

[https://www.youtube.com/watch?v=GL3Wv\\_45LaU&list=PL5AC0ADBF65924EAD&index=2](https://www.youtube.com/watch?v=GL3Wv_45LaU&list=PL5AC0ADBF65924EAD&index=2)

## Chapter 20

# Laplace transform

<https://www.youtube.com/watch?v=lg90shB1TrU>

<https://www.youtube.com/watch?v=WEkuV55B4q4>



# Chapter 21

## conic section

conic section 圓錐曲線 / 圓錐截痕

[https://en.wikipedia.org/wiki/Conic\\_section](https://en.wikipedia.org/wiki/Conic_section)

<https://tex.stackexchange.com/questions/222882/drawing-minimal-xy-axis>

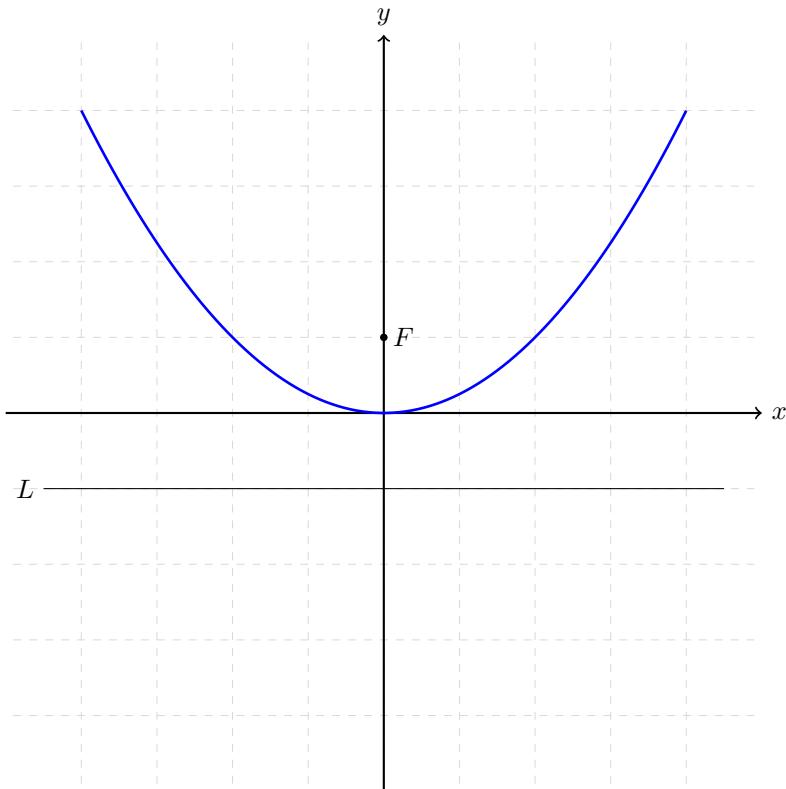


Figure 21.1: parabola defined by focus, directrix, eccentricity

### 21.1 Cartesian coordinate: focus, directrix, eccentricity

focus, directrix, eccentricity 焦點, 準線, 離心率

$$\begin{cases} F = (0, y_F) \\ L = y - y_L = 0 \\ \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\|(x, y) - (0, y_F)\|}{\|y - y_L\|} \end{cases} \quad \begin{array}{l} F : \text{focus} \\ L : \text{directrix} \\ \epsilon : \text{eccentricity} \end{array}$$

$$0 \leq \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{\sqrt{(y - y_L)^2}} \quad (21.1)$$

$$\epsilon^2 = \frac{x^2 + (y - y_F)^2}{(y - y_L)^2} = \frac{x^2 + y^2 - 2y_F y + y_F^2}{y^2 - 2y_L y + y_L^2} \quad (21.2)$$

$$0 = x^2 + (1 - \epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \quad (21.3)$$

$$\stackrel{\epsilon \neq 1}{=} x^2 + (1 - \epsilon^2) \left[ y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2} \right] \quad (21.4)$$

$$= x^2 + (1 - \epsilon^2) \quad (21.5)$$

$$\left[ y^2 - \frac{2(y_F - \epsilon^2 y_L)}{1 - \epsilon^2} y + \left( \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 - \left( \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{y_F^2 - \epsilon^2 y_L^2}{1 - \epsilon^2} \right] \quad (21.6)$$

$$= x^2 + (1 - \epsilon^2) \left[ \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2}{(1 - \epsilon^2)^2} \right] \quad (21.7)$$

$$= x^2 + (1 - \epsilon^2) \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 + \frac{(y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2}{1 - \epsilon^2} \quad (21.8)$$

$$\begin{aligned} & (y_F^2 - \epsilon^2 y_L^2)(1 - \epsilon^2) - (y_F - \epsilon^2 y_L)^2 \\ &= (1 - \epsilon^2) y_F^2 - (\epsilon^2 - \epsilon^4) y_L^2 - y_F^2 + 2\epsilon^2 y_F y_L - \epsilon^4 y_L^2 \\ &= -\epsilon^2 y_F^2 - \epsilon^2 y_L^2 + 2\epsilon^2 y_F y_L = -\epsilon^2 (y_F - y_L)^2 \end{aligned}$$

$$\begin{aligned} \frac{\epsilon^2 (y_F - y_L)^2}{1 - \epsilon^2} &\stackrel{\epsilon \neq 1}{=} x^2 + (1 - \epsilon^2) \left( y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2} \right)^2 \\ &\stackrel{\epsilon \neq 0, 1}{=} 1 \begin{cases} \left( \frac{x - 0}{\epsilon (y_F - y_L)} \right)^2 + \left( \frac{y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\epsilon (y_F - y_L)} \right)^2 & 1 - \epsilon^2 > 0 \stackrel{\epsilon > 0}{\Rightarrow} 0 < \epsilon < 1 \\ -\left( \frac{x - 0}{\epsilon (y_F - y_L)} \right)^2 + \left( \frac{y - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\epsilon (y_F - y_L)} \right)^2 & 1 - \epsilon^2 < 0 \stackrel{\epsilon > 0}{\Rightarrow} \epsilon > 1 \end{cases} \end{aligned}$$

$$\epsilon = 0 \text{ or } \lim_{|y_L| \rightarrow \infty} \epsilon = 0$$

$$r = \overline{PF} = \|(x, y) - (0, y_F)\| = \|(x, y - y_F)\| = \sqrt{x^2 + (y - y_F)^2}$$

$$\epsilon = \frac{r}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{|y - y_L|}$$

$$\lim_{|y_L| \rightarrow \infty} \epsilon = \lim_{|y_L| \rightarrow \infty} \frac{r}{d(P, L)} = \lim_{|y_L| \rightarrow \infty} \frac{\sqrt{x^2 + (y - y_F)^2}}{|y - y_L|} = 0$$

$$\epsilon = 1$$

$$\begin{aligned} 0 &= x^2 + (1 - \epsilon^2) y^2 - 2(y_F - \epsilon^2 y_L) y + (y_F^2 - \epsilon^2 y_L^2) \\ &\stackrel{\epsilon = 1}{=} x^2 + (1 - 1^2) y^2 - 2(y_F - 1^2 y_L) y + (y_F^2 - 1^2 y_L^2) \\ &= x^2 - 2(y_F - y_L) y + (y_F^2 - y_L^2) \\ &= x^2 - 2(y_F - y_L) y + (y_F + y_L)(y_F - y_L) \\ x^2 &= 2(y_F - y_L) \left( y - \frac{y_F + y_L}{2} \right) \end{aligned}$$

Let one curve vertex  $P = V = (0, 0)$  on the curve, and fix the directrix  $L$  or  $y_L$ ,

$$\epsilon \neq 1$$

$$\begin{aligned} 1 &\stackrel{P(x,y)=V(0,0)}{=} 0 + \left( \frac{0 - \frac{y_F - \epsilon^2 y_L}{1 - \epsilon^2}}{\frac{\epsilon(y_F - y_L)}{1 - \epsilon^2}} \right)^2 \\ &\Rightarrow y_F - \epsilon^2 y_L = \pm \epsilon (y_F - y_L) \\ &\Rightarrow \begin{cases} (1 - \epsilon) y_F = \epsilon (\epsilon - 1) y_L & + \\ (1 + \epsilon) y_F = \epsilon (\epsilon + 1) y_L & - \end{cases} \\ &\Rightarrow y_F = \begin{cases} -\epsilon y_L & + \\ \epsilon y_L & - \end{cases} \end{aligned}$$

$$\epsilon = 1$$

$$\begin{aligned} x^2 &= 2(y_F - y_L) \left( y - \frac{y_F + y_L}{2} \right) \\ &\stackrel{P(x,y)=V(0,0)}{=} 0^2 = 2(y_F - y_L) \left( 0 - \frac{y_F + y_L}{2} \right) \\ &\Rightarrow 0 = (y_F - y_L)(y_F + y_L) \\ &\Rightarrow y_F = \mp y_L \end{aligned}$$

or by definition of eccentricity (21.1)

$$\begin{aligned} 0 \leq \epsilon &= \frac{\overline{PF}}{d(P, L)} = \frac{\overline{PF}}{\overline{PP'}} = \frac{\|(x, y) - (0, y_F)\|}{\|(x, y) - (x, y_L)\|} = \frac{\|(x, y - y_F)\|}{\|(0, y - y_L)\|} = \frac{\sqrt{x^2 + (y - y_F)^2}}{\sqrt{(y - y_L)^2}} \\ &\stackrel{P(x,y)=V(0,0)}{=} \frac{\sqrt{0^2 + (0 - y_F)^2}}{\sqrt{(0 - y_L)^2}} = \sqrt{\left(\frac{y_F}{y_L}\right)^2} \\ \epsilon^2 &= \left(\frac{y_F}{y_L}\right)^2 \Rightarrow y_F = \mp \epsilon y_L \end{aligned}$$

actually,

$$y_F = -\epsilon y_L$$

## 21.2 two-definition equivalence for ellipse and hyperbola

<https://math.stackexchange.com/questions/1833973/prove-that-the-directrix-focus-and-focus-focus-definitions-are-equivalent>

<https://www.geogebra.org/calculator/zkppuxwp>

$$\begin{cases} P = (x, y) \\ F = (x_F, y_F) = (\alpha, \varphi) & F' = (x_{F'}, y_{F'}) = (\chi, \psi) \\ L = A'x + B'y + C' = 0 \end{cases}$$

### 21.2.1 first definition for conic sections including ellipses and hyperbolas

distance from a point to a line<sup>[^22^]</sup>

$$0 \leq \epsilon = \frac{\overline{PF}}{d(P, L)} = \frac{\sqrt{(x - x_F)^2 + (y - y_F)^2}}{\frac{|A'x + B'y + C'|}{\sqrt{A'^2 + B'^2}}} = \frac{\sqrt{(x - \alpha)^2 + (y - \varphi)^2}}{|Ax + By + C|}, \begin{cases} A = \frac{A'}{\sqrt{A'^2 + B'^2}} \\ B = \frac{B'}{\sqrt{A'^2 + B'^2}} \\ C = \frac{C'}{\sqrt{A'^2 + B'^2}} \end{cases}$$

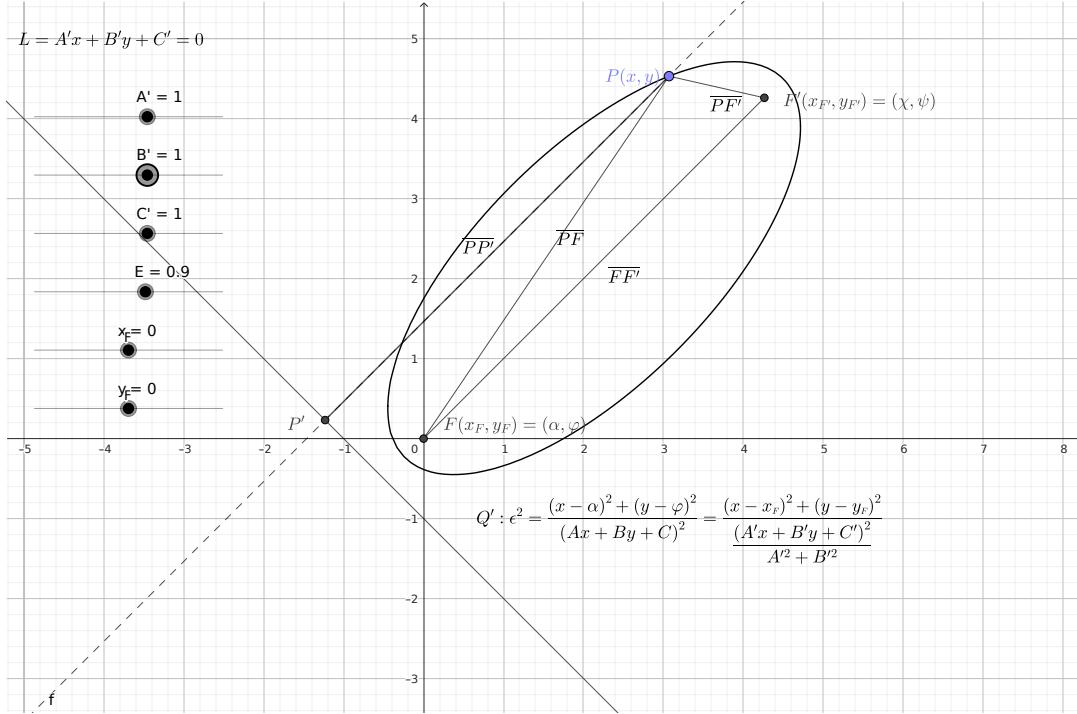


Figure 21.2: conic sections

$$A^2 + B^2 = \left( \frac{A'}{\sqrt{A'^2 + B'^2}} \right)^2 + \left( \frac{B'}{\sqrt{A'^2 + B'^2}} \right)^2 = 1$$

or allowing  $\epsilon < 0$  by squaring the definition

$$\epsilon^2 = \frac{(x - \alpha)^2 + (y - \varphi)^2}{(Ax + By + C)^2} = \frac{(x - x_F)^2 + (y - y_F)^2}{\frac{(A'x + B'y + C')^2}{A'^2 + B'^2}}$$

$$(x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2$$

### 21.2.2 second definition for ellipses and hyperbolas

$$2c = \overline{FF'} = \|(x_F, y_F) - (x_{F'}, y_{F'})\| = \|(\alpha, \varphi) - (\chi, \psi)\| \\ = \sqrt{(\alpha - \chi)^2 + (\chi - \psi)^2}$$

$$D = \begin{cases} \sqrt{(x - x_F)^2 + (y - y_F)^2} + \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} & \text{ellipse} \\ \sqrt{(x - x_F)^2 + (y - y_F)^2} - \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} & \text{hyperbola} \end{cases} \\ = \sqrt{(x - x_F)^2 + (y - y_F)^2} \pm \sqrt{(x - x_{F'})^2 + (y - y_{F'})^2} \\ = \sqrt{(x - \alpha)^2 + (y - \varphi)^2} \pm \sqrt{(x - \chi)^2 + (y - \psi)^2}$$

$$(x - \alpha)^2 + (y - \varphi)^2 = \left( D \mp \sqrt{(x - \chi)^2 + (y - \psi)^2} \right)^2 \\ = D^2 \mp 2D \sqrt{(x - \chi)^2 + (y - \psi)^2} \\ + (x - \chi)^2 + (y - \psi)^2$$

$$\begin{aligned}
D^2 &= (x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2 \\
&\quad \pm 2\sqrt{\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right]} \\
&\quad (x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2 - D^2 \\
&= \mp 2\sqrt{\left[(x - \alpha)^2 + (y - \varphi)^2\right]\left[(x - \chi)^2 + (y - \psi)^2\right]} \\
&\quad \left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right]^2 + D^4 \\
&\quad - 2D^2 \left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right] \\
&= 4 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \left[(x - \chi)^2 + (y - \psi)^2\right] \\
&\quad \left[(x - \alpha)^2 + (y - \varphi)^2\right]^2 + \left[(x - \chi)^2 + (y - \psi)^2\right]^2 \\
&\quad + 2 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \left[(x - \chi)^2 + (y - \psi)^2\right] + D^4 \\
&\quad - 2D^2 \left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right] \\
&= 4 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \left[(x - \chi)^2 + (y - \psi)^2\right] \\
0 &= \left[(x - \alpha)^2 + (y - \varphi)^2\right]^2 + \left[(x - \chi)^2 + (y - \psi)^2\right]^2 \\
&\quad - 2 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \left[(x - \chi)^2 + (y - \psi)^2\right] + D^4 \\
&\quad - 2D^2 \left[(x - \alpha)^2 + (y - \varphi)^2 + (x - \chi)^2 + (y - \psi)^2\right] \\
0 &= \left\{ \left[(x - \alpha)^2 + (y - \varphi)^2\right] - \left[(x - \chi)^2 + (y - \psi)^2\right] \right\}^2 + D^4 \\
&\quad - 2D^2 \left\{ \left[(x - \alpha)^2 + (y - \varphi)^2\right] + \left[(x - \chi)^2 + (y - \psi)^2\right] \right\} \\
0 &= \left\{ \left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right] \right\}^2 + D^4 \\
&\quad - 2D^2 \left\{ \left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right] \right\} \\
&\quad - 4D^2 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \\
&\quad (2D)^2 \left[(x - \alpha)^2 + (y - \varphi)^2\right] \\
&= \left\{ \left[(x - \chi)^2 + (y - \psi)^2\right] - \left[(x - \alpha)^2 + (y - \varphi)^2\right] - D^2 \right\}^2 \\
&= \left\{ \left[(x - \chi)^2 - (x - \alpha)^2\right] + \left[(y - \psi)^2 - (y - \varphi)^2\right] - D^2 \right\}^2 \\
&= \left\{ (2x - \chi - \alpha)(\alpha - \chi) + (2y - \psi - \varphi)(\varphi - \psi) - D^2 \right\}^2 \\
&= \left\{ 2(\alpha - \chi)x - (\alpha^2 - \chi^2) + 2(\varphi - \psi)y - (\varphi^2 - \psi^2) - D^2 \right\}^2 \\
&= \left\{ 2(\alpha - \chi)x + 2(\varphi - \psi)y - [(\alpha^2 - \chi^2) + (\varphi^2 - \psi^2) + D^2] \right\}^2
\end{aligned}$$

$D \neq 0$

$$\begin{aligned}
&(x - \alpha)^2 + (y - \varphi)^2 \\
&= \left[ \frac{\alpha - \chi}{D}x + \frac{\varphi - \psi}{D}y - \left( \frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2} \right) \right]^2
\end{aligned}$$

$$\begin{cases} (x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2 \\ (x - \alpha)^2 + (y - \varphi)^2 = \left[ \frac{\alpha - \chi}{D}x + \frac{\varphi - \psi}{D}y - \left( \frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2} \right) \right]^2 \end{cases}$$

$$(A, B, C) \rightleftharpoons (\chi, \psi, D)$$

$$\begin{cases}
\epsilon A = \pm \frac{\alpha - \chi}{D} & \chi \pm \epsilon AD = \alpha \\
\epsilon B = \pm \frac{\varphi - \psi}{D} & \psi \pm \epsilon BD = \varphi \\
\epsilon C = \mp \left( \frac{\alpha^2 - \chi^2}{2D} + \frac{\varphi^2 - \psi^2}{2D} + \frac{D}{2} \right) & \\
\end{cases}$$

$$\begin{aligned}
2\epsilon C &= \mp \left( \frac{\alpha - \chi}{D} (\alpha + \chi) + \frac{\varphi - \psi}{D} (\varphi + \psi) + D \right) \\
&= \mp (\pm \epsilon A(\alpha + \chi) \pm \epsilon B(\varphi + \psi) + D) \\
\mp \epsilon (A\alpha + B\varphi + 2C) &= \pm \epsilon A\chi \pm \epsilon B\psi + D
\end{aligned}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A \\ 0 & 1 & \pm \epsilon B \\ \pm \epsilon A & \pm \epsilon B & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ D \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \\ \mp \epsilon (A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & \pm \epsilon B & 1 \mp \epsilon^2 A^2 & \mp \epsilon (2A\alpha + B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & 0 & 1 \mp \epsilon^2 A^2 \mp \epsilon^2 B^2 & \mp \epsilon (2A\alpha + 2B\varphi + 2C) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \pm \epsilon A & \alpha \\ 0 & 1 & \pm \epsilon B & \varphi \\ 0 & 0 & 1 & \frac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} \end{pmatrix}$$

$$A^2 + B^2 = \left( \frac{A'}{\sqrt{A'^2 + B'^2}} \right)^2 + \left( \frac{B'}{\sqrt{A'^2 + B'^2}} \right)^2 = 1$$

$$\begin{cases}
\chi = \alpha \mp \epsilon AD = \alpha \mp \epsilon \frac{A'}{\sqrt{A'^2 + B'^2}} D \\
\psi = \varphi \mp \epsilon BD = \varphi \mp \epsilon \frac{B'}{\sqrt{A'^2 + B'^2}} D \\
D = \frac{\mp 2\epsilon (A\alpha + B\varphi + C)}{1 \mp \epsilon^2 (A^2 + B^2)} = \frac{\mp 2\epsilon}{1 \mp \epsilon^2} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} \quad A^2 + B^2 = 1
\end{cases}$$

actually, only one of two solutions is true

$$\begin{cases}
\chi = \alpha - \epsilon AD = \alpha - \epsilon \frac{A'}{\sqrt{A'^2 + B'^2}} D = \alpha - \frac{2\epsilon^2}{\epsilon^2 - 1} \frac{A'^2 \alpha + A'B'\varphi + A'C'}{A'^2 + B'^2} \\
\psi = \varphi - \epsilon BD = \varphi - \epsilon \frac{B'}{\sqrt{A'^2 + B'^2}} D = \varphi - \frac{2\epsilon^2}{\epsilon^2 - 1} \frac{A'B'\alpha + B'^2 \varphi + B'C'}{A'^2 + B'^2} \\
D = \frac{-2\epsilon (A\alpha + B\varphi + C)}{1 - \epsilon^2 (A^2 + B^2)} = \frac{-2\epsilon}{1 - \epsilon^2} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}} = \frac{2\epsilon}{\epsilon^2 - 1} \frac{A'\alpha + B'\varphi + C'}{\sqrt{A'^2 + B'^2}}
\end{cases}$$

$$\begin{cases}
\chi = \frac{(\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2(A'^2\alpha + A'B'\varphi + A'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\
\psi = \frac{(\epsilon^2 - 1)(A'^2 + B'^2)\varphi - 2\epsilon^2(A'B'\alpha + B'^2\varphi + B'C')}{(\epsilon^2 - 1)(A'^2 + B'^2)} \\
\left| \frac{D}{d(F, L)} \right| = \left| \frac{2\epsilon}{1 - \epsilon^2} \right| \Rightarrow \left( \frac{D}{d(F, L)} \right)^2 = \left( \frac{2\epsilon}{1 - \epsilon^2} \right)^2
\end{cases}$$

$$\begin{aligned}
&(\epsilon^2 - 1)(A'^2 + B'^2)\alpha - 2\epsilon^2(A'^2\alpha + A'B'\varphi + A'C') \\
&= (-(\epsilon^2 + 1)A'^2 + (\epsilon^2 - 1)B'^2)\alpha - 2\epsilon^2(A'B'\varphi + A'C') \\
&= (-(\epsilon^2 + 1)A'^2 + (\epsilon^2 - 1)B'^2)\alpha - 2\epsilon^2(A'B'\varphi + A'C')
\end{aligned}$$

Can the above be more simplified?

$$\begin{aligned}
 \overline{FF'}^2 &= (\alpha - \chi)^2 + (\varphi - \psi)^2 \\
 &= (\alpha - (\alpha - \epsilon AD))^2 + (\varphi - (\varphi - \epsilon BD))^2 \\
 &= (\epsilon D)^2 (A^2 + B^2) \\
 &= (\epsilon D)^2
 \end{aligned}$$

### 21.2.3 eccentricity and its equivalent representation

$$\left(\frac{c}{a}\right)^2 = \left(\frac{\overline{PF}}{d(P, L)}\right)^2 = \epsilon^2 = \left(\frac{\overline{FF'}}{D}\right)^2 = \left(\frac{2c}{D}\right)^2 \Rightarrow D = 2a$$

$$\left(\frac{D}{d(F, L)}\right)^2 = \left(\frac{2\epsilon}{1 - \epsilon^2}\right)^2$$

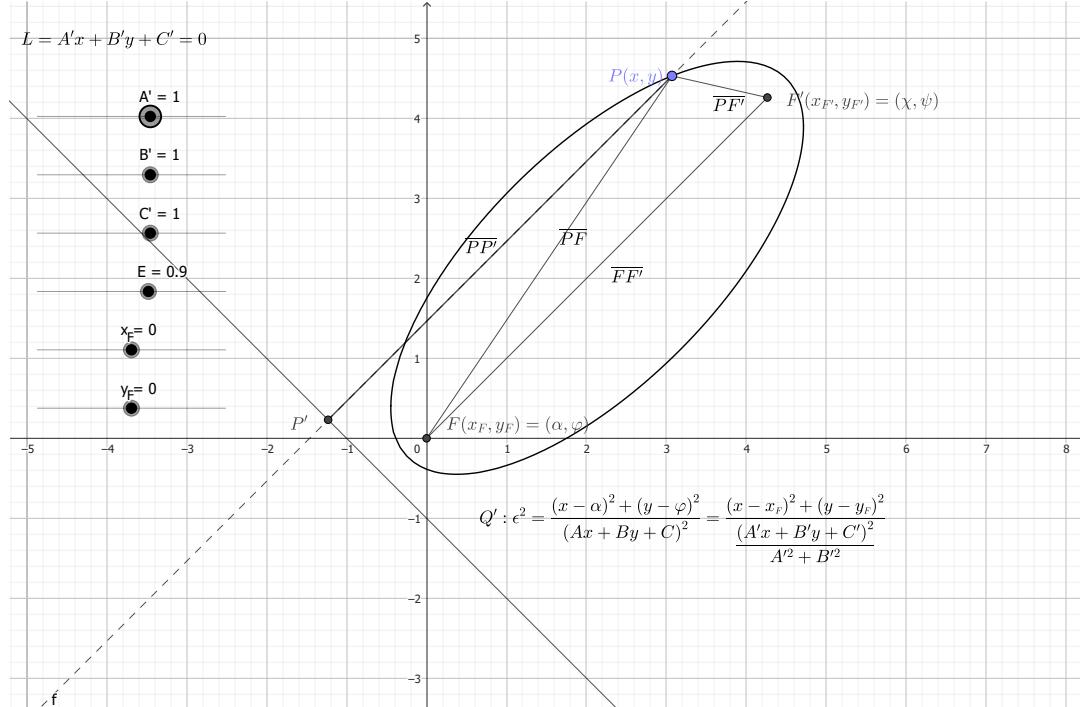


Figure 21.3: conic sections: ellipse

## 21.3 Cartesian coordinate: standard form / standard equation

|           |                                                                        |                                                                      |
|-----------|------------------------------------------------------------------------|----------------------------------------------------------------------|
| circle    | $\left(\frac{y - k}{a}\right)^2 + \left(\frac{x - h}{a}\right)^2 = 1$  | $b = a$                                                              |
| ellipse   | $\left(\frac{y - k}{b}\right)^2 + \left(\frac{x - h}{a}\right)^2 = 1$  | vertical $b > a$                                                     |
|           | $\left(\frac{y - k}{b}\right)^2 + \left(\frac{x - h}{a}\right)^2 = 1$  | horizontal $a > b$                                                   |
| parabola  | $(y - k) - 4c(x - h)^2 = 0$                                            | vertical                                                             |
|           | $-4c(y - k)^2 + (x - h) = 0$                                           | horizontal                                                           |
| hyperbola | $\left(\frac{y - k}{b}\right)^2 - \left(\frac{x - h}{a}\right)^2 = 1$  | vertical $\frac{x - h}{a} = 0 \Rightarrow \frac{y - k}{b} = \pm 1$   |
|           | $-\left(\frac{y - k}{b}\right)^2 + \left(\frac{x - h}{a}\right)^2 = 1$ | horizontal $\frac{y - k}{b} = 0 \Rightarrow \frac{x - h}{a} = \pm 1$ |

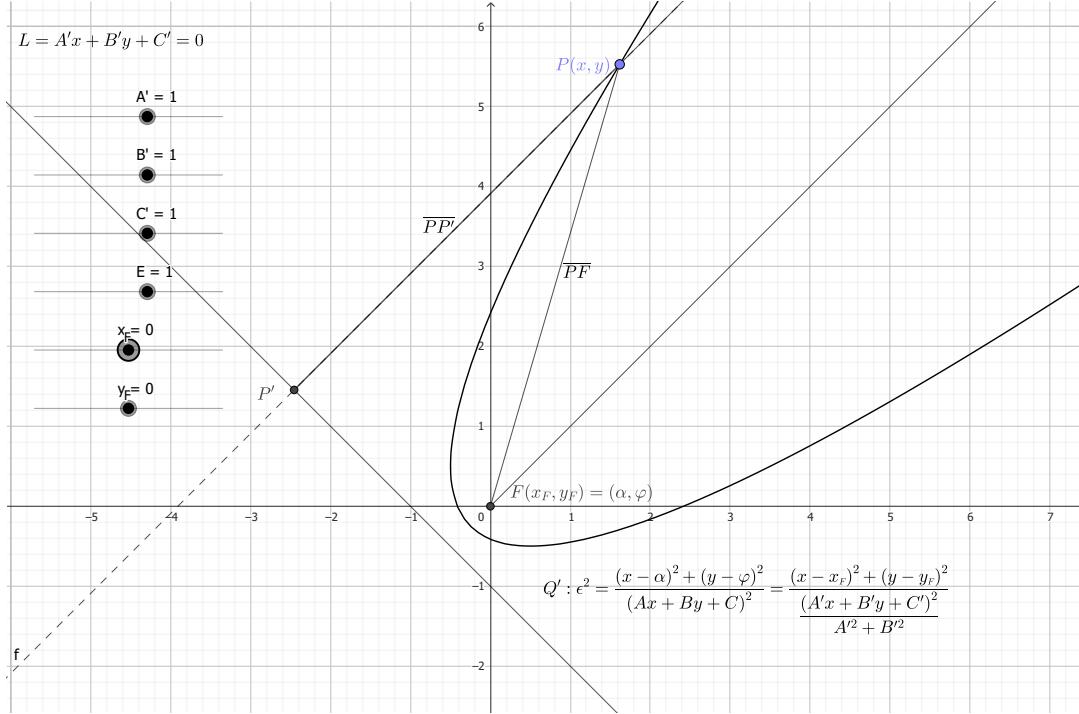


Figure 21.4: conic sections: parabola

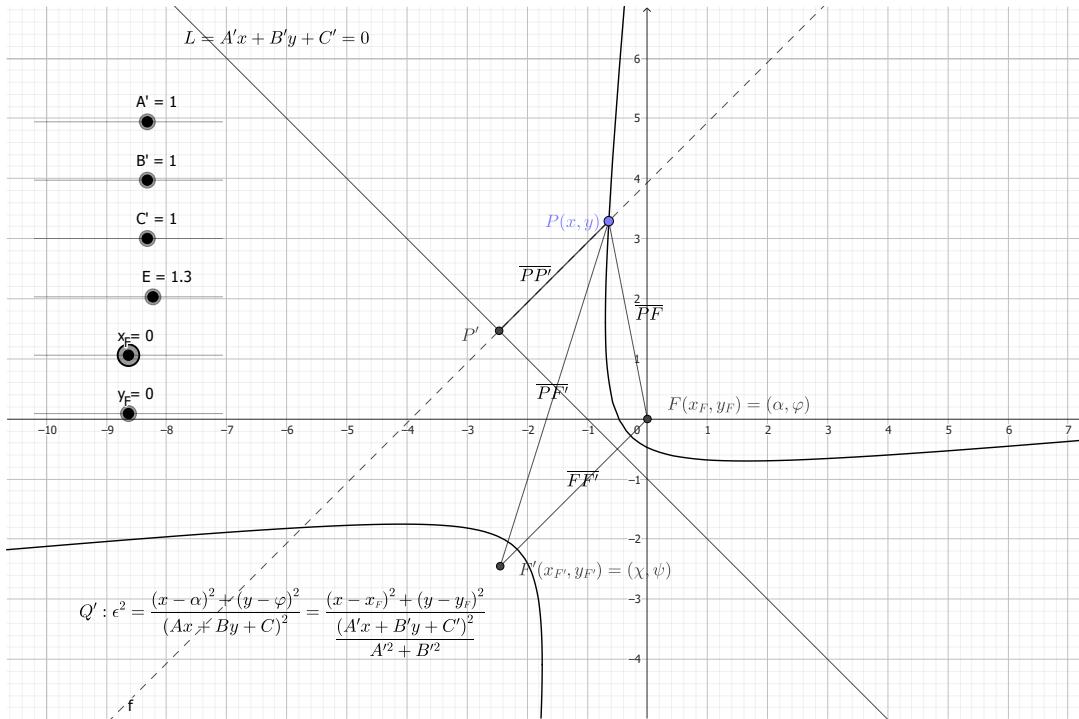


Figure 21.5: conic sections: hyperbola

## 21.4 parametric equation

|           |                                                                    |                                                                                                                                                                                                                                                                                                   |
|-----------|--------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| circle    | $\left(\frac{y-k}{a}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$  | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & a & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ a \\ 1 \end{pmatrix}$       |
| ellipse   | $\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$  | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & 0 & h \\ 0 & \sin t & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$       |
| parabola  | $(y-k) - 4c(x-h)^2 = 0$                                            | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 4c & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} t & 0 & h \\ 0 & t^2 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4c \\ 1 \end{pmatrix}$                     |
|           | $-4c(y-k)^2 + (x-h) = 0$                                           | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 4c & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} t^2 & 0 & h \\ 0 & t & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4c \\ 1 \end{pmatrix}$                     |
| hyperbola | $\left(\frac{y-k}{b}\right)^2 - \left(\frac{x-h}{a}\right)^2 = 1$  | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pm \cosh t \\ \sinh t \\ 1 \end{pmatrix} = \begin{pmatrix} \tan t & 0 & h \\ 0 & \sec t & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ |
|           | $-\left(\frac{y-k}{b}\right)^2 + \left(\frac{x-h}{a}\right)^2 = 1$ | $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & h \\ 0 & b & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pm \cosh t \\ \sinh t \\ 1 \end{pmatrix} = \begin{pmatrix} \sec t & 0 & h \\ 0 & \tan t & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ |

tangent half-angle formula<sup>[24]</sup>

## 21.5 polar coordinate

$$(x - \alpha)^2 + (y - \varphi)^2 = [\epsilon(Ax + By + C)]^2$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$(r \cos \theta - \alpha)^2 + (r \sin \theta - \varphi)^2 = [\epsilon(Ar \cos \theta + Br \sin \theta + C)]^2$$

$$\text{If } \begin{cases} F = (x_F, y_F) = (\alpha, \varphi) = (0, 0) \\ L = Ax + By + C = x + p = 0 \end{cases}$$

$$\begin{aligned} (r \cos \theta)^2 + (r \sin \theta)^2 &= [\epsilon(r \cos \theta + p)]^2 \\ r^2 &= \\ r &= \pm \epsilon(r \cos \theta + p) \\ &= \pm(r\epsilon \cos \theta + \epsilon p) \\ r(1 \mp \epsilon \cos \theta) &= \epsilon p \\ r &= \frac{\epsilon p}{1 \mp \epsilon \cos \theta} \end{aligned}$$

<https://www.geogebra.org/calculator/azksjxbq>

$r = \frac{\epsilon p}{1 - \epsilon \cos \theta}$  will not cross  $L = x + p = 0$  on graphs, so maybe it is a more correct solution

$$r = \frac{\epsilon p}{1 - \epsilon \cos \theta}$$

## 21.6 Cartesian coordinate: general form / quadratic equation

<https://ccjou.wordpress.com/2013/05/24/%E9%99%90%E9%9D%A2%E9%9D%A2/>

[https://en.wikipedia.org/wiki/Matrix\\_representation\\_of\\_conic\\_sections](https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections)

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$(x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y) \begin{pmatrix} ax + (b/2)y \\ (b/2)x + cy \end{pmatrix} = ax^2 + bxy + cy^2$$

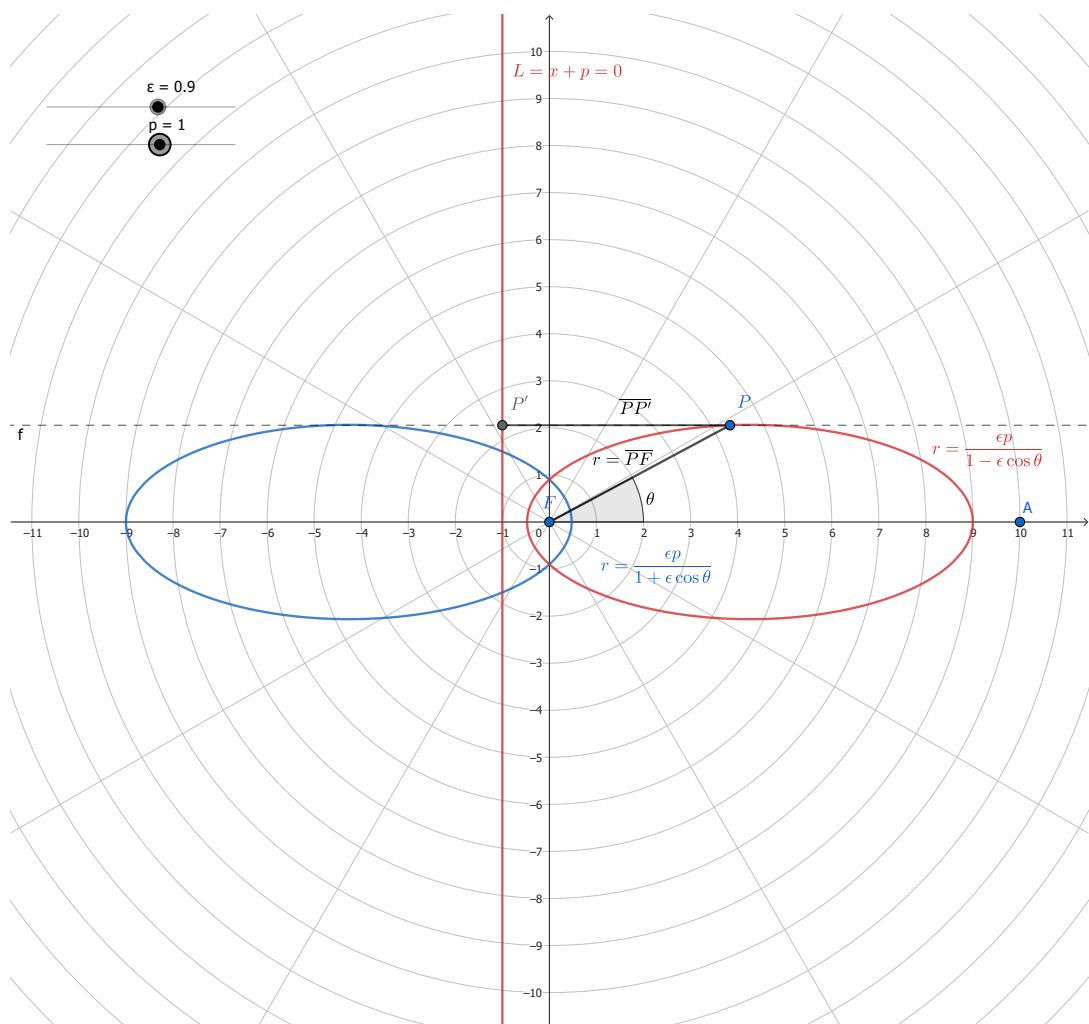


Figure 21.6: polar conic sections: ellipse

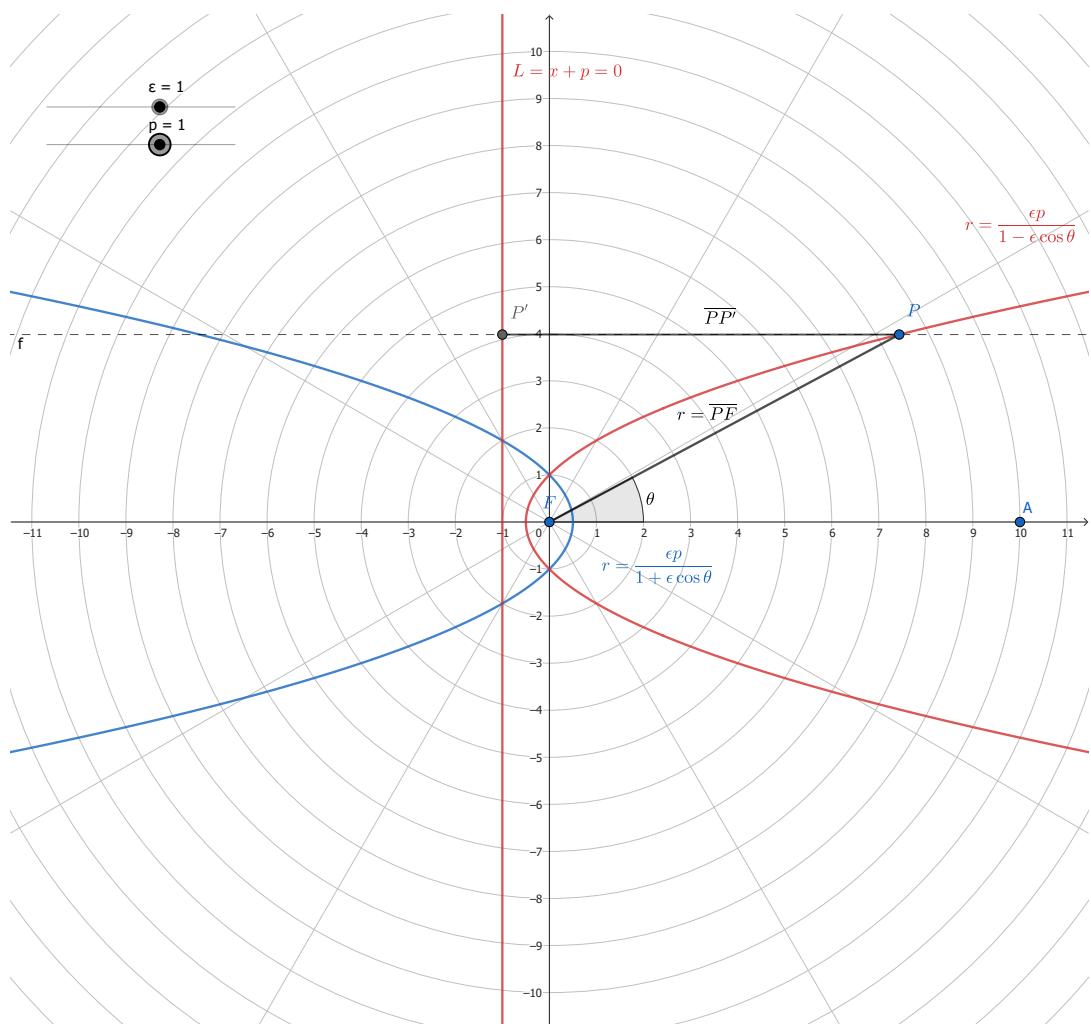


Figure 21.7: polar conic sections: parabola

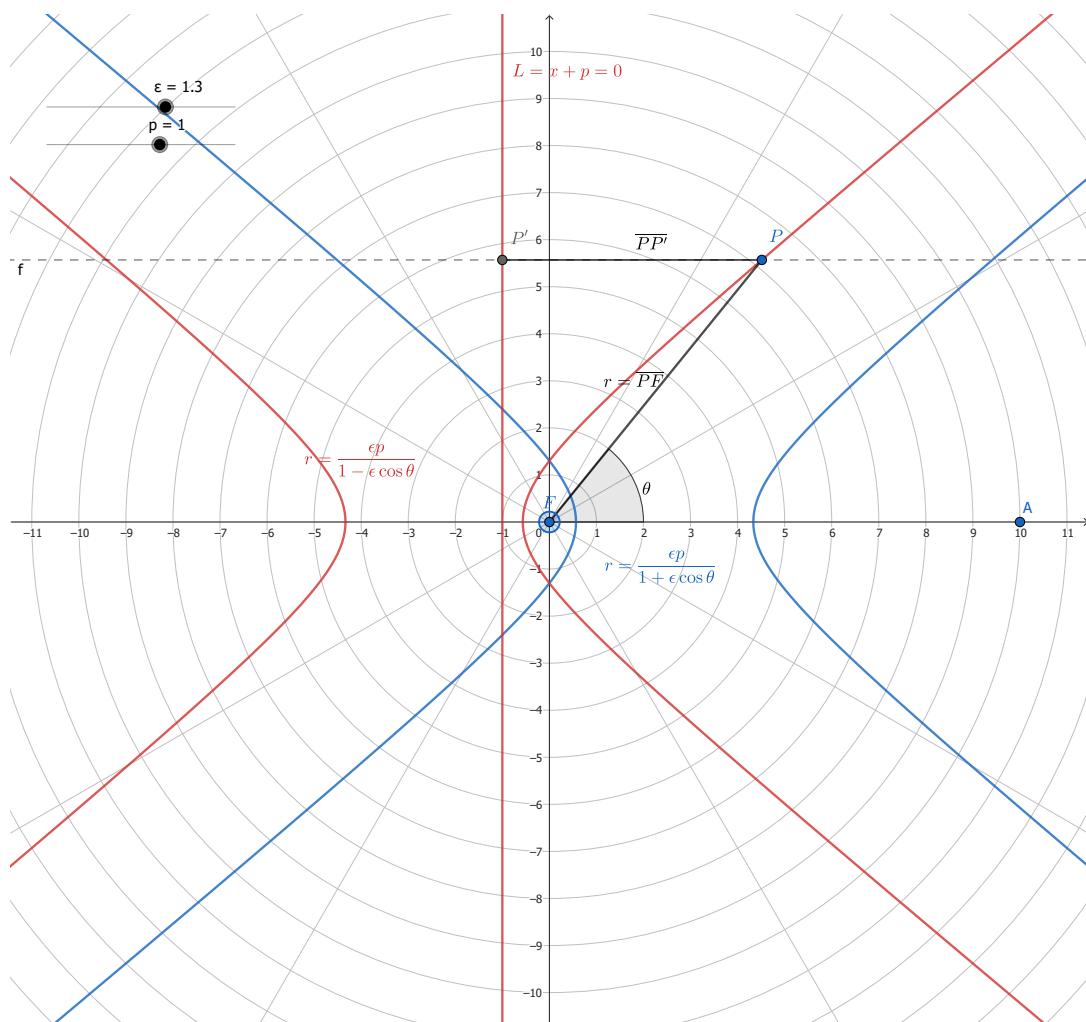


Figure 21.8: polar conic sections: hyperbola

$$0 = (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f$$

$$= \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f, \quad \begin{cases} A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} & A \text{ real symmetric} \\ \mathbf{b} = \begin{pmatrix} d \\ e \end{pmatrix} \\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases}$$

real symmetric matrix diagonalizable<sup>[23]</sup>

## 21.7 homogeneous coordinate

X homogeneous coordinate

[homogeneous coordinate](#) O: HTML, X: PDF becoming web link

O homogeneous coordinate<sup>[25]</sup>

X homogeneous coordinate

X homogeneous coordinate<sup>[21.7]</sup>

<https://ccjou.wordpress.com/2013/05/24/%E5%9C%A8%E9%9D%A2%E9%83%A8%E7%BD%AE/>

$$(x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} a & b/2 & 0 \\ b/2 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$(d \ e) \begin{pmatrix} x \\ y \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \kappa \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} \alpha x + \beta y + \gamma \\ \delta x + \epsilon y + \zeta \\ \eta x + \theta y + \kappa \end{pmatrix}, \quad \begin{cases} \gamma + \eta = d \\ \zeta + \theta = e \end{cases}$$

$$= (x \ y \ 1) \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & \zeta \\ \eta & \theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (x \ y \ 1) \begin{pmatrix} 0 & 0 & d/2 \\ 0 & 0 & e/2 \\ d/2 & e/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$0 = ax^2 + bxy + cy^2 + dx + ey + f$$

$$= (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f = \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f$$

$$= (x \ y \ 1) \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (\mathbf{x}^\top \ 1) M \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

$$0 = ax^2 + bxy + cy^2 + dx + ey + f$$

$$= (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (d \ e) \begin{pmatrix} x \\ y \end{pmatrix} + f = \mathbf{x}^\top A \mathbf{x} + \mathbf{b}^\top \mathbf{x} + f$$

$$= (x \ y \ 1) \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (\mathbf{x}^\top \ 1) M \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, M = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

[https://en.wikipedia.org/wiki/Matrix\\_representation\\_of\\_conic\\_sections](https://en.wikipedia.org/wiki/Matrix_representation_of_conic_sections)

$$0 = Q = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

$$= [x \ y \ 1] \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{x}_h^\top A_Q \mathbf{x}_h$$

$$= [x \ y] \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [D \ E] \begin{bmatrix} x \\ y \end{bmatrix} + F = \mathbf{x}^\top A_{Q,33} \mathbf{x} + \mathbf{b}^\top \mathbf{x} + F$$

## 21.8 TalyorCatAlice: projective geometry

<https://www.bilibili.com/video/BV1pK42117UZ>

<https://www.bilibili.com/video/BV1cx4y1f7w6>

<https://www.bilibili.com/video/BV1sy421h7aF>

<https://www.bilibili.com/video/BV1zv421y7iH>

<https://www.bilibili.com/video/BV1ZH4y1h7vC>

## 21.9 double cone

<https://space.bilibili.com/87052889/channel/collectiondetail?sid=1691820>

## 21.10 theorem proof without analytic method

<https://www.zhihu.com/question/470672139/answer/2873265380>

## Chapter 22

# distance from a point to a line

點到直線距離

**Theorem 22.1.**

$$\begin{cases} P = P(x_0, y_0) \\ L = L(x, y) = Ax + By + C = 0, A^2 + B^2 \neq 0 \end{cases} \Downarrow d(P, L) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

[https://en.wikipedia.org/wiki/Distance\\_from\\_a\\_point\\_to\\_a\\_line](https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line)

<https://highscope.ch.ntu.edu.tw/wordpress/?p=47407>

<https://web.math.sinica.edu.tw/mathmedia/HTMLArticle18.jsp?mID=40312>

Proofs:

### 22.1 by shortest $\overline{PP'}$

$$\begin{aligned} P' &= P'(x, y) \in L = Ax + By + C = 0 \\ \Rightarrow y &= \frac{-1}{B}(Ax + C) \end{aligned}$$

$$\begin{aligned} \overline{PP'}^2(x, y) &= (x_0 - x)^2 + (y_0 - y)^2 \\ &= (x_0 - x)^2 + \left( y_0 - \frac{-1}{B}(Ax + C) \right)^2 \\ &= (x - x_0)^2 + \left( \frac{A}{B}x + \frac{C}{B} + y_0 \right)^2 = \overline{PP'}^2(x) \end{aligned}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \overline{PP'}^2(x) = 2(x - x_0) + 2 \left( \frac{A}{B}x + \frac{C}{B} + y_0 \right) \frac{A}{B} \\ &= \frac{2}{B^2} (B^2(x - x_0) + A^2x + AC + ABy_0) \\ &= \frac{2}{B^2} [(A^2 + B^2)x - (B^2x_0 - ABy_0 - AC)] \\ x &= \frac{B^2x_0 - ABy_0 - AC}{A^2 + B^2} \end{aligned}$$

or by completing the square to find  $x$ .

$$\begin{aligned}
& \overline{PP'}^2 \left( x = \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2} \right) \\
&= \left( \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2} - x_0 \right)^2 + \left( \frac{A B^2 x_0 - AB y_0 - AC}{A^2 + B^2} + \frac{C}{B} + y_0 \right)^2 \\
&= \left( \frac{-A^2 x_0 - AB y_0 - AC}{A^2 + B^2} \right)^2 + \left( \frac{A (B^2 x_0 - AB y_0 - AC) + C (A^2 + B^2) + B (A^2 + B^2) y_0}{B (A^2 + B^2)} \right)^2 \\
&= \left( \frac{-A (Ax_0 + By_0 + C)}{A^2 + B^2} \right)^2 + \left( \frac{AB^2 x_0 + B^3 y_0 + B^2 C}{B (A^2 + B^2)} \right)^2 \\
&= \frac{A^2 (Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2} + \frac{B^2 (Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2} \\
&= \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2} \\
\overline{PP'} &= \overline{PP'} \left( x = \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2} \right) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

## 22.2 by perpendicular foot

$$y = \frac{-A}{B}x - \frac{C}{B} = \frac{-1}{B}(Ax + C), \text{ if } B \neq 0$$

$$L_{\perp} : \left( y = \frac{B}{A}x + K \right) \perp \left( y = \frac{-A}{B}x - \frac{C}{B} \right) : L$$

$$L_{\perp} = L_{\perp}(x, y) = Bx - Ay + K = 0$$

$$P = P(x_0, y_0) \in L_{\perp} = B(x - x_0) - A(y - y_0) = 0$$

$$L_{\perp} = Bx - Ay - (Bx_0 - Ay_0) = 0$$

perpendicular foot = foot of the perpendicular  $P'$

$$\begin{aligned}
P' \in (L_{\perp} \cap L) &= \begin{cases} L = Ax + By + C = 0 \\ L_{\perp} = Bx - Ay - (Bx_0 - Ay_0) = 0 \end{cases} \\
&= \begin{cases} Ax + By = -C \\ Bx - Ay = Bx_0 - Ay_0 \end{cases} \\
P' = P'(x, y) &= \left( \frac{\begin{vmatrix} -C & B \\ Bx_0 - Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}}, \frac{\begin{vmatrix} A & -C \\ B & Bx_0 - Ay_0 \end{vmatrix}}{\begin{vmatrix} A & B \\ B & -A \end{vmatrix}} \right) \\
&= \left( \frac{\begin{vmatrix} C & B \\ -Bx_0 + Ay_0 & -A \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}}, \frac{\begin{vmatrix} A & C \\ B & -Bx_0 + Ay_0 \end{vmatrix}}{\begin{vmatrix} A & -B \\ B & A \end{vmatrix}} \right) \\
&= \left( \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2}, \frac{-AB x_0 + A^2 y_0 - BC}{A^2 + B^2} \right)
\end{aligned}$$

$$\begin{aligned}
d(P, L) &= \overrightarrow{PP'} \\
&= \left\| (x_0, y_0) - \left( \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2}, \frac{-AB x_0 + A^2 y_0 - BC}{A^2 + B^2} \right) \right\| \\
&= \sqrt{\left( x_0 - \frac{B^2 x_0 - AB y_0 - AC}{A^2 + B^2} \right)^2 + \left( y_0 - \frac{-AB x_0 + A^2 y_0 - BC}{A^2 + B^2} \right)^2} \\
&= \sqrt{\left( \frac{A^2 x_0 + AB y_0 + AC}{A^2 + B^2} \right)^2 + \left( \frac{AB x_0 + B^2 y_0 + BC}{A^2 + B^2} \right)^2} \\
&= \sqrt{\frac{A^2 (Ax_0 + By_0 + C)^2 + B^2 (Ax_0 + By_0 + C)^2}{(A^2 + B^2)^2}} = \sqrt{\frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}} \\
&= \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

## 22.3 by normal vector

$$\begin{cases} \vec{n} = (A, B) \perp L = Ax + By + C = 0 \\ \vec{PP'} = P' - P = (x - x_0, y - y_0) \end{cases}$$

$P$ 到 $L$ 的距離 $d(P, L)$ 即為 $L$ 線上一點 $P'$ 對應之 $\vec{PP'}$ 在 $L$ 法向量 $\vec{n}$ 方向上的投影長

$$\begin{aligned}
\vec{PP'} \cdot \vec{n} &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| \cos \theta \\
\left| \vec{PP'} \cdot \vec{n} \right| &= \left\| \vec{PP'} \right\| \left\| \vec{n} \right\| |\cos \theta| \\
\left\| \vec{PP'} \right\| |\cos \theta| &= \left| \vec{PP'} \cdot \hat{n} \right| = \frac{\left| \vec{PP'} \cdot \vec{n} \right|}{\left\| \vec{n} \right\|} = \frac{|(x - x_0, y - y_0) \cdot (A, B)|}{\|(A, B)\|} \\
&= \frac{|A(x - x_0) + B(y - y_0)|}{\sqrt{A^2 + B^2}} = \frac{|-Ax_0 - By_0 + Ax + By|}{\sqrt{A^2 + B^2}} \\
&\stackrel{Ax+By+C=0}{=} \frac{|-Ax_0 - By_0 - C|}{\sqrt{A^2 + B^2}} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\end{aligned}$$

PDF LaTeX \usepackage{fdsymbol} to have \overrightharpoon vector; however, there are too many side effects, including ugly mathptmx  $\sum, \dots$

```

\usepackage{fdsymbol} % vector over accent, but will use mathptmx
% replace the rather ugly mathptmx \sum operator with the equivalent Computer Modern one
\let\sum\relax
\DeclareSymbolFont{CMLargesymbols}{OMX}{cmex}{m}{n}
\DeclareMathSymbol{\sum}{\mathop}{CMLargesymbols}{50}

```

## 22.4 by Cauchy inequality

$$\begin{aligned}
Ax + By + C &= 0 \\
Ax + By &= -C \\
(Ax + By) - (Ax_0 + By_0) &= -C - (Ax_0 + By_0) \\
A(x - x_0) + B(y - y_0) &= -(Ax_0 + By_0 + C)
\end{aligned}$$

$$\begin{aligned}\overline{PP'}^2 &= (x_0 - x)^2 + (y_0 - y)^2 \\ [A^2 + B^2] \overline{PP'}^2 &= [A^2 + B^2] \left[ (x_0 - x)^2 + (y_0 - y)^2 \right] \\ &\geq [A(x - x_0) + B(y - y_0)]^2 \\ &= [-(Ax_0 + By_0 + C)]^2 = (Ax_0 + By_0 + C)^2 \\ \overline{PP'}^2 &\geq \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2} \\ \overline{PP'} &\geq \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}\end{aligned}$$

# Chapter 23

## real symmetric matrix diagonalizable

<https://ccjou.wordpress.com/2011/02/09/%E6%97%A5%E5%AF%BC%E7%9F%A5%E6%9D%A1%E5%8A%A8%E5%88%86%E6%9E%90/>

<https://tex.stackexchange.com/questions/30619/what-is-the-best-symbol-for-vector-matrix-transpose>

### Theorem 23.1.

實對稱矩陣的特徵值皆是實數，且對應特徵向量是實向量。

$$\begin{array}{ll}
 \left\{ \begin{array}{l} A \in \mathcal{M}_{n \times n}(\mathbb{R}) \\ A^\top = A \end{array} \right. & \begin{array}{l} \text{real matrix} \\ \text{symmetric matrix} \end{array} \\
 \left\{ \begin{array}{l} Ax = \lambda x \end{array} \right. & \begin{array}{l} \text{real symmetric matrix} \\ \left\{ \begin{array}{ll} \lambda \in \mathbb{C} & \text{complex eigenvalue} \\ \mathbf{0} \neq x \in \mathbb{C}^n & \text{complex eigenvector} \end{array} \right. \end{array} \\
 \Downarrow & \\
 \left\{ \begin{array}{ll} \lambda \in \mathbb{R} & \text{real eigenvalue (1)} \\ x \in \mathbb{R}^n & \text{real eigenvector (2)} \end{array} \right. & 
 \end{array}$$

*Proof.* (1)

$$\begin{aligned}
 Ax &= \lambda x \\
 \bar{A}\bar{x} &= \overline{Ax} = \overline{\lambda x} = \bar{\lambda}\bar{x} \\
 \bar{x}^\top \bar{A}^\top &= (\bar{A}\bar{x})^\top = (\bar{\lambda}\bar{x})^\top = \bar{\lambda}\bar{x}^\top \\
 \bar{x}^\top A &\stackrel{\text{symmetric}}{=} \bar{x}^\top A^\top \stackrel{\text{real}}{=} \\
 \bar{x}^\top A &= \bar{\lambda}\bar{x}^\top \\
 \lambda \bar{x}^\top x &= \bar{x}^\top (\lambda x) \stackrel{Ax=\lambda x}{=} \bar{x}^\top Ax = \bar{\lambda}\bar{x}^\top x \\
 \lambda \bar{x}^\top x &= \bar{\lambda}\bar{x}^\top x \\
 (\lambda - \bar{\lambda}) \bar{x}^\top x &= 0 \wedge \begin{cases} \bar{x}^\top x = \sum_{i=1}^n |x_i|^2 \\ x \neq \mathbf{0} \end{cases} \Rightarrow \bar{x}^\top x \neq 0 \\
 \lambda - \bar{\lambda} &= 0 \\
 \lambda = \bar{\lambda} &\Leftrightarrow \lambda \in \mathbb{R}
 \end{aligned}$$

□

*Proof.* (1) fast concept

$$\begin{aligned}
(\bar{A}\bar{x})^\top \bar{x} &= (\bar{x}^\top \bar{A}^\top) \bar{x} \stackrel{\text{symmetric}}{=} (\bar{x}^\top \bar{A}) \bar{x} = \bar{x}^\top (\bar{A}x) \\
(L) &= (\bar{A}\bar{x})^\top \bar{x} = \bar{x}^\top (\bar{A}x) = (R) \\
(L) &= (\bar{A}\bar{x})^\top \bar{x} \stackrel{Ax=\lambda x}{=} (\bar{\lambda}\bar{x})^\top \bar{x} = \bar{\lambda}\bar{x}^\top \bar{x} \\
(R) &= \bar{x}^\top (\bar{A}x) \stackrel{\text{real}}{=} \bar{x}^\top (Ax) \stackrel{Ax=\lambda x}{=} \bar{x}^\top (\lambda x) = \lambda \bar{x}^\top \bar{x} \\
&\quad \bar{\lambda}\bar{x}^\top \bar{x} = (\bar{A}\bar{x})^\top \bar{x} = \bar{x}^\top (\bar{A}x) = \lambda \bar{x}^\top \bar{x} \\
&\quad \bar{\lambda}\bar{x}^\top \bar{x} = \lambda \bar{x}^\top \bar{x}
\end{aligned}$$

□

*Proof.* (2)

???

推論特徵空間  $N(A - \lambda I)$  ( $A - \lambda I$  的零空間) 為  $\mathbb{R}^n$  的子空間，故  $x \in N(A - \lambda I)$  是一個非零實向量。

□

**Theorem 23.2.**

實對稱矩陣對應相異特徵值的特徵向量互為正交。

$$\left\{
\begin{array}{ll}
\begin{cases} A \in \mathcal{M}_{n \times n}(\mathbb{R}) & \text{real matrix} \\ A^\top = A & \text{symmetric matrix} \end{cases} & \text{real symmetric matrix} \\
Ax = \lambda x & \text{23.1 } \begin{cases} \lambda \in \mathbb{R} & \text{real eigenvalue} \\ x \in \mathbb{R}^n & \text{real eigenvector} \end{cases} \\
\begin{cases} Ax_1 = \lambda_1 x_1 & (e_1) \\ Ax_2 = \lambda_2 x_2 & (e_2) \end{cases} & \lambda_1 \neq \lambda_2 \\
& \Downarrow \\
& x_1^\top x_2 = 0 \Leftrightarrow x_1 \perp x_2
\end{array}
\right.$$

*Proof.* (1)

$$\begin{aligned}
Ax_2 &= \lambda_2 x_2 \\
x_1^\top Ax_2 &\stackrel{x_1^\top}{=} x_1^\top \lambda_2 x_2 = \lambda_2 x_1^\top x_2 = (1) \\
Ax_1 &= \lambda_1 x_1 \\
x_1^\top A^\top &= (Ax_1)^\top = (\lambda_1 x_1)^\top = \lambda_1 x_1^\top \\
x_1^\top A^\top &= \lambda_1 x_1^\top \\
x_1^\top Ax_2 &\stackrel{\text{symmetric}}{=} x_1^\top A^\top x_2 \stackrel{x_2}{=} \lambda_1 x_1^\top x_2 = (2) \\
\lambda_2 x_1^\top x_2 &\stackrel{(1)}{=} x_1^\top Ax_2 \stackrel{(2)}{=} \lambda_1 x_1^\top x_2 \\
\lambda_2 x_1^\top x_2 &= \lambda_1 x_1^\top x_2 \\
(\lambda_2 - \lambda_1) x_1^\top x_2 &= 0 \wedge \lambda_1 \neq \lambda_2 \\
x_1^\top x_2 &= 0
\end{aligned}$$

□

*Proof.* (1) fast concept

$$\begin{aligned}
(Ax_1)^\top x_2 &= (x_1^\top A^\top) x_2 \stackrel{\text{symmetric}}{=} (x_1^\top A) x_2 = x_1^\top (Ax_2) \\
(L) &= (Ax_1)^\top x_2 = x_1^\top (Ax_2) = (R) \\
(L) &= (Ax_1)^\top x_2 \stackrel{(e_1)}{=} (\lambda_1 x_1)^\top x_2 = \lambda_1 x_1^\top x_2 \\
(R) &= x_1^\top (Ax_2) \stackrel{(e_2)}{=} x_1^\top (\lambda_2 x_2) = \lambda_2 x_1^\top x_2 \\
\lambda_1 x_1^\top x_2 &= (Ax_1)^\top x_2 = x_1^\top (Ax_2) = \lambda_2 x_1^\top x_2 \\
\lambda_1 x_1^\top x_2 &= \lambda_2 x_1^\top x_2
\end{aligned}$$

□

**Theorem 23.3.**

$$\begin{cases}
 \begin{cases}
 A \in \mathcal{M}_{n \times n}(\mathbb{R}) & \text{real matrix} \\
 A^\top = A & \text{symmetric matrix}
 \end{cases} & \text{real symmetric matrix} \\
 Ax_1 = \lambda x_1 & (e) \\
 x_2^\top x_1 = 0 \Leftrightarrow x_2 \perp x_1 & (o)
 \end{cases}$$

$\Downarrow$   
 $Ax_2 \perp x_1 \Leftrightarrow (Ax_2)^\top x_1 = 0$

*Proof.*

$$\begin{aligned}
 (Ax_2)^\top x_1 &= (x_2^\top A^\top) x_1 \stackrel{\text{symmetric}}{=} (x_2^\top A) x_1 \\
 &= x_2^\top (Ax_1) \stackrel{(e)}{=} x_2^\top (\lambda x_1) \\
 &= \lambda x_2^\top x_1 \stackrel{(o)}{=} \lambda \cdot 0 = 0
 \end{aligned}$$

□



## Chapter 24

# tangent half-angle formula

[https://en.wikipedia.org/wiki/Tangent\\_half-angle\\_formula](https://en.wikipedia.org/wiki/Tangent_half-angle_formula)

<https://zh.wikipedia.org/zh-tw/正切半角公式>

正切半形公式又稱萬能公式

以切表弦公式，簡稱以切表弦



# **Chapter 25**

## **homogeneous coordinate**

### **25.1 Cem Yuksel**

<https://youtu.be/EKN7dTJ4ep8?si=8woajZxbqPfEXhdK&t=2263>

<https://youtu.be/1z1S2kQKXD8?si=71o339yBtIQYhWtj&t=3082>



# **Chapter 26**

## **Archimedean property**

### **26.1 integer Archimedean property**

### **26.2 rational Archimedean property**

<https://math.stackexchange.com/questions/3699023/proof-the-the-field-of-rational-numbers-has-the-archimedean-property>

<https://math.stackexchange.com/questions/1919829/proving-the-archimedean-properties-of-rational-numbers>

### **26.3 real Archimedean property**

<https://www.youtube.com/watch?v=6wBVk1fBKXQ&t=2326>



# Chapter 27

## Matplotlib / matplotlib

```
library(reticulate)

Warning: package 'reticulate' was built under R version 4.2.3
conda_list()
use_condaenv(condaenv = 'sandbox-3.9')

install Seaborn
conda_install("r-reticulate", "seaborn")

import Seaborn (it will be automatically discovered in "r-reticulate")
seaborn <- import("seaborn")
```

- tikzplotlib<sup>[13.6]</sup>: Python<sup>[12]</sup> matplotlib<sup>[27]</sup> export to TikZ<sup>[13]</sup> .tex

### 27.1 Timothy H. Wu

巫孟叡

- API = application programming interface
  - functional<sup>[27.1.1]</sup>
  - object-oriented<sup>[27.1.2]</sup>
    - \* figure
    - \* axes
    - \* subplot

[https://matplotlib.org/stable/tutorials/introductory/quick\\_start.html](https://matplotlib.org/stable/tutorials/introductory/quick_start.html)

<https://pbpython.com/effective-matplotlib.html>

<https://tex.stackexchange.com/questions/84847/can-i-use-webp-images-in-latex>

You probably need to convert the image to png.

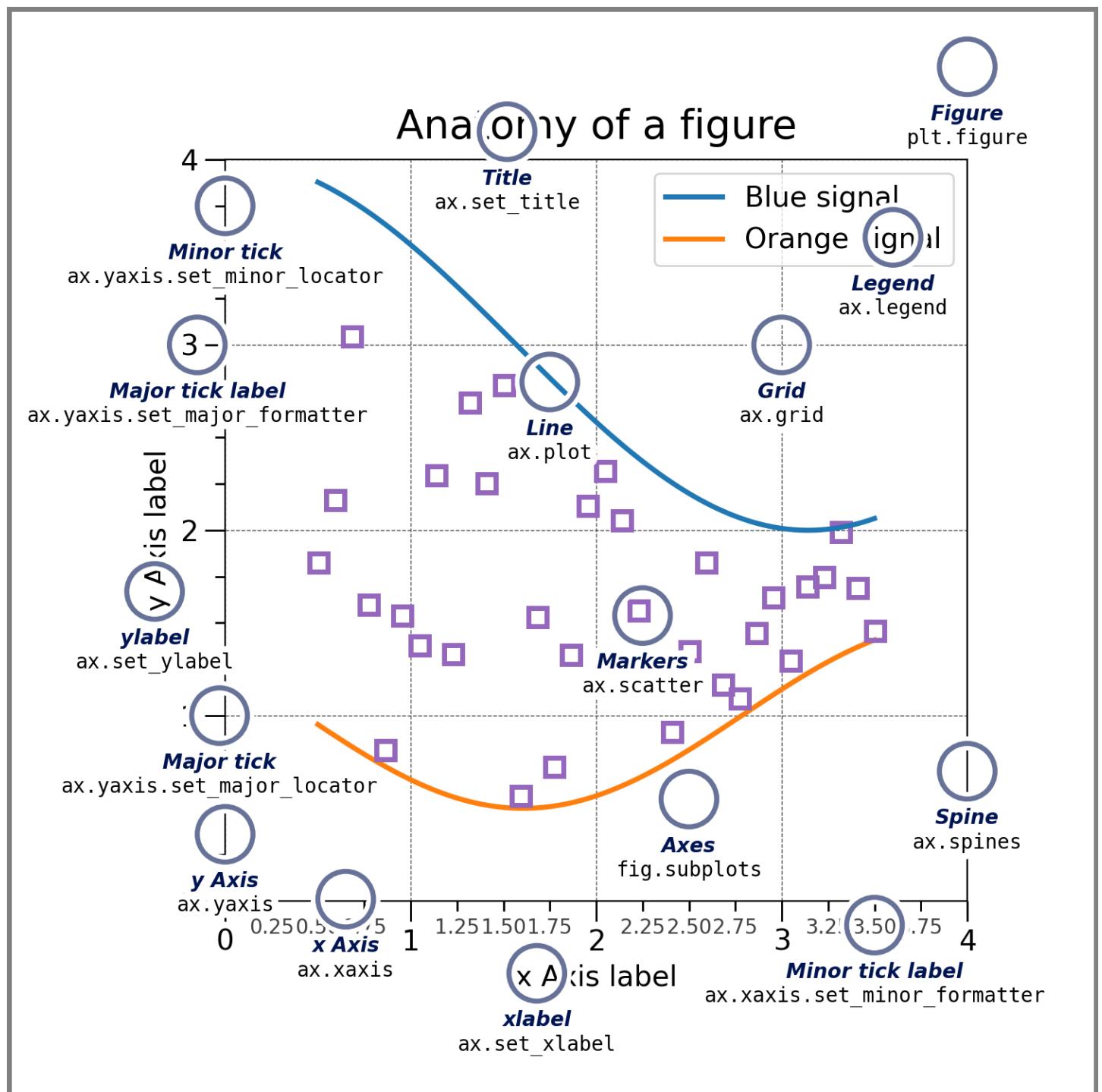


Figure 27.1: matplotlib figure anatomy

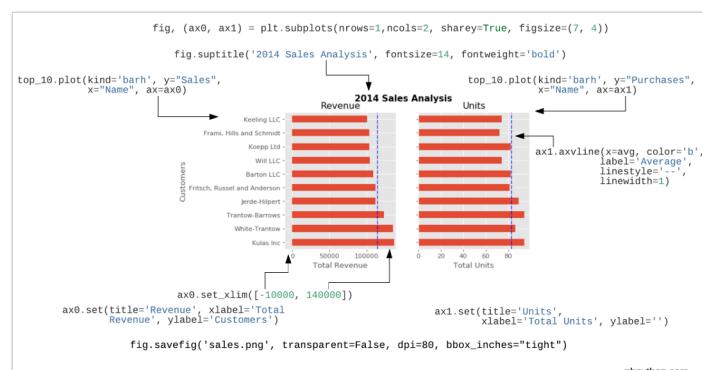
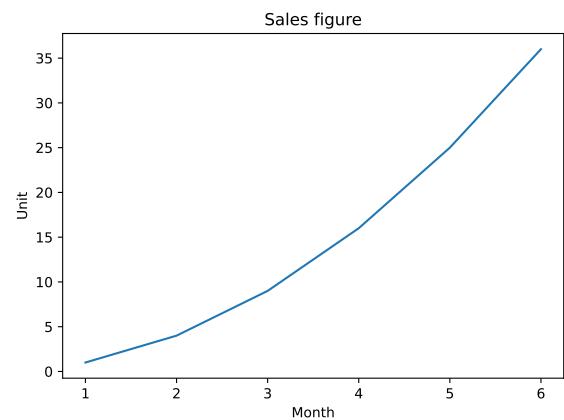


Figure 27.2: matplotlib subplot anatomy

### 27.1.1 funcitonal API

```
import matplotlib.pyplot as plt

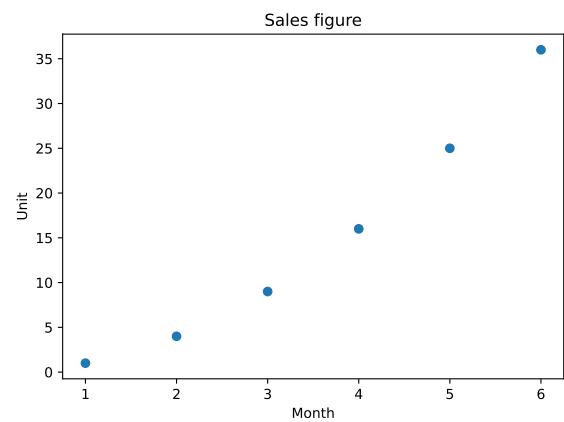
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2) # this doesn't actually show
 ← the plot.
plt.show() This is automatically called for
 ← Jupyter notebook.
```



To plot a scatterplot, call `scatter()` instead of `plot()`.

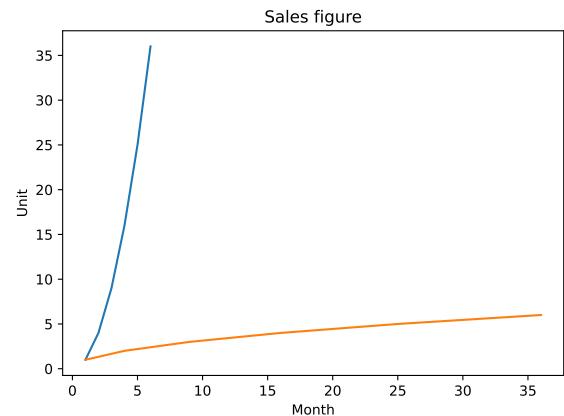
```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.scatter(a1, a2) # instead of plt.plot(),
 ← use scatter() to show scatter plot
```



```
import matplotlib.pyplot as plt

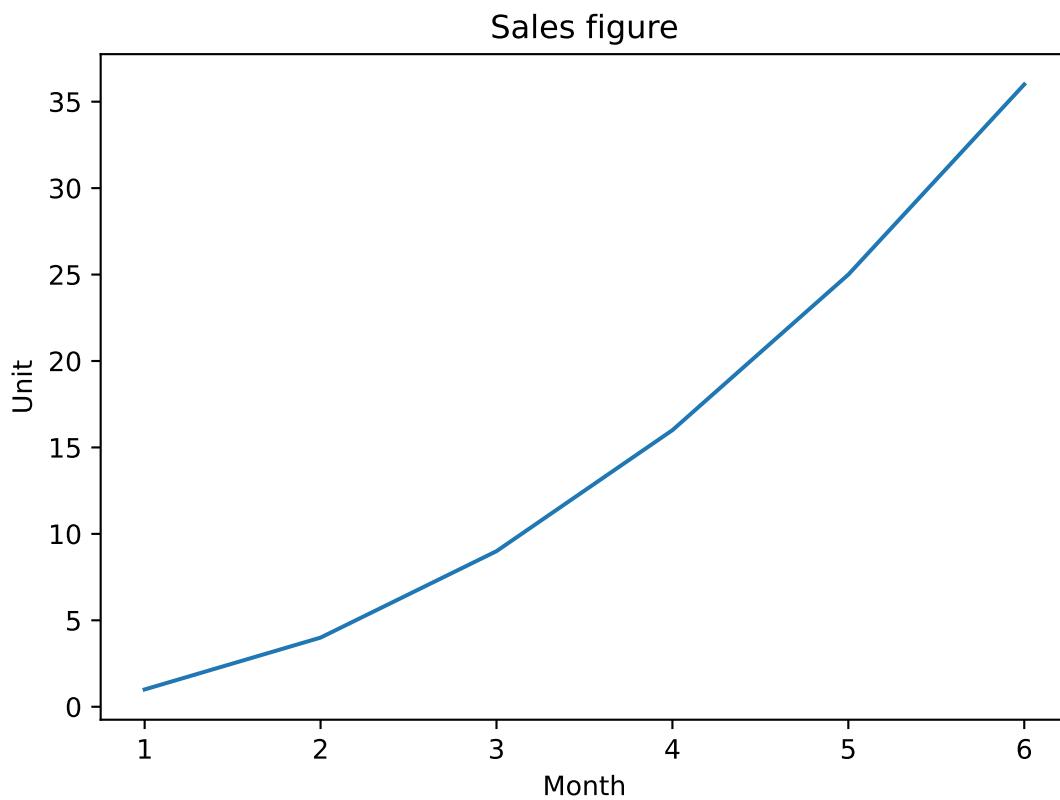
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2)
plt.plot(a2, a1)
```



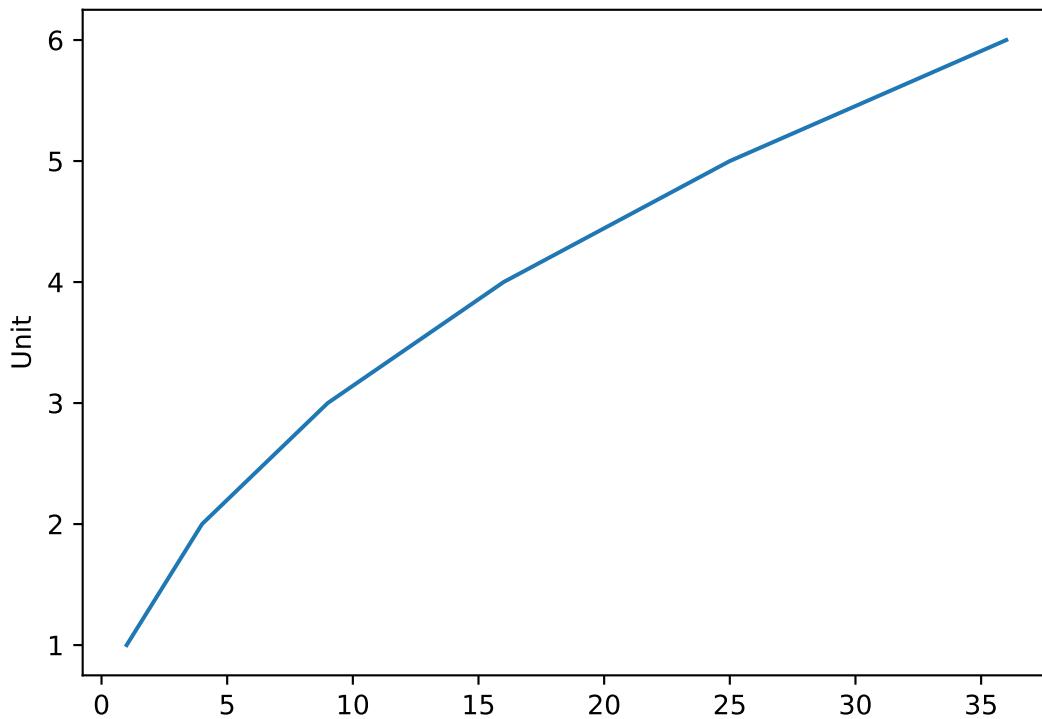
The behavior of the functional API is stateful. What's stateful? An example is when you read a text file. When you `open()` a text file to read, the library read the next line every time you call `readline()`. It remembers where you left off, despite the fact that you do not give it the position to read from. This behavior of the library is called **stateful**. The way we've used Matplotlib is also **stateful**. And everytime, `plt.show()` is called (and it automatically gets called on cell ends), some state about plots is reset. We can see that here:

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
plt.title('Sales figure')
plt.xlabel('Month')
plt.ylabel('Unit')
plt.plot(a1, a2)
```



```
plt.plot(a2, a1)
plt.ylabel('Unit')
```



It makes two graphs instead of one. Also note that `ylabel()` was called after `plot()`, and it is still shown before `plot.show()` but Sales Figure plot title and other labels don't show up on this graph. Because every time `plot.show()` is called, things are reset. This is a **stateful API** we're using. The functional APIs are used when you plot Matplotlib by calling on `pyplot` module level API (module level functions).

### 27.1.2 object-oriented API

In object-oriented API, we're getting two type of objects. One is `Figure`, the other one is `Axes`. `Figure` is the *canvas* of the plot. In English, axes is the plural form of axis. We're talking about the axis in x axis and y axis. Since one `plot` consists of both axis, in Matplotlib the object that represents one plot is called `Axes`. Since it's an object. We'll call it "a" axes.

```
import matplotlib.pyplot as plt
```

```
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
print("fig is of type:", type(fig))
```

```
fig is of type: <class 'matplotlib.figure.Figure'>
```

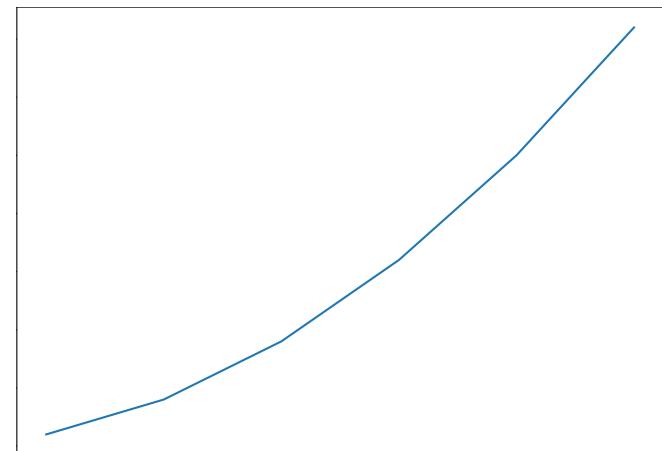
```
ax1 = fig.add_axes([0, 0, 1, 1]) # [left, bottom, width, height]
print("ax1 is of type:", type(ax1))
ax1.plot(a1, a2)
```

```
ax1 is of type: <class 'matplotlib.axes._axes.Axes'>
```

1. Call `figure()` to get a `Figure` type
  2. Call `add_axes()` to get a `ax1` type `ax1 = fig.add_axes([0, 0, 1, 1]) # [left, bottom, width, height]`
- The list given to `add_axes()` is the rectangular region of where to show the plot:
    - Bottom left corner at  $x=0, y=0$ , width and height of both 1, 1

```
import matplotlib.pyplot as plt

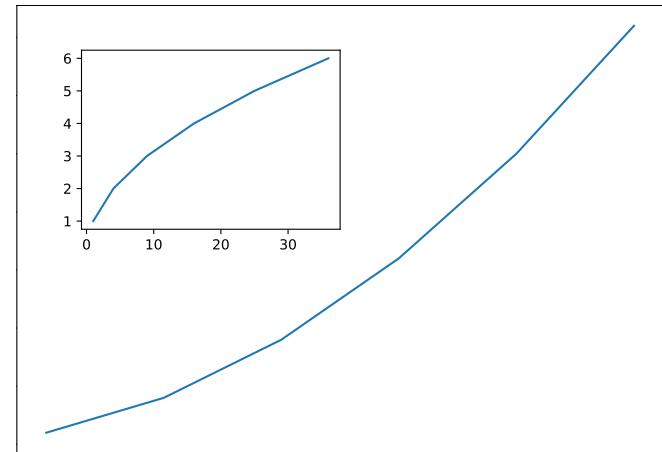
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
print("fig is of type:", type(fig))
ax1 = fig.add_axes([0, 0, 1, 1]) # [left,
bottom, width, height]
print("ax1 is of type:", type(ax1))
ax1.plot(a1, a2)
```




---

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax1 = fig.add_axes([0, 0, 1, 1])
ax1.plot(a1, a2)
ax2 = fig.add_axes([0.1, 0.5, .4, .4])
ax2.plot(a2, a1)
```

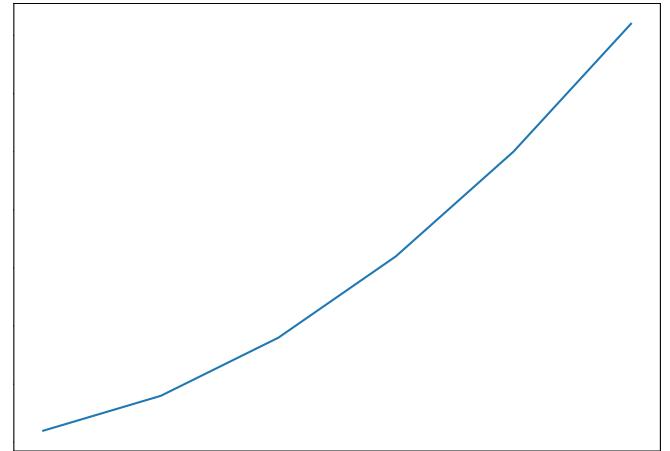


```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
ax.set_xlabel('Time')
ax.set_ylabel('Unit')
ax.set_title('Sales figure')
alternatively:
ax.set(xlabel='Time', ylabel='Unit',
title='Sales figure')

```



### 27.1.2.1 Configure the figure size and DPI

Get image size for the figure object. 6 by 4 is the default.

```

fig.get_size_inches()
import matplotlib.pyplot as plt

fig = plt.figure()
fig.get_size_inches()

array([6.5, 4.5])

```

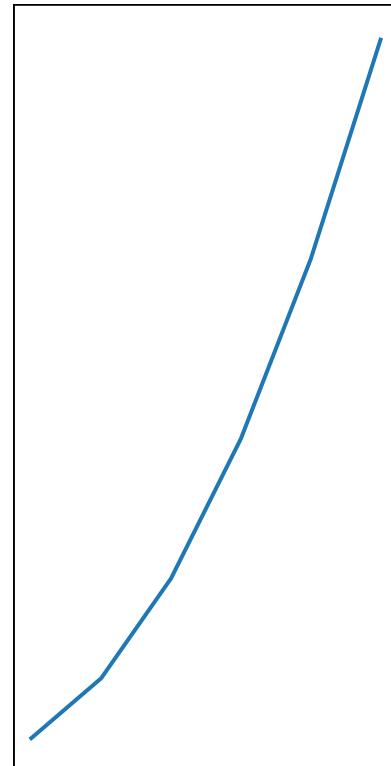
```
fig = plt.figure(figsize=(2, 4))
```

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure(figsize=(2, 4))
you can also set after getting the figure
fig.set_size_inches((12, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)

```



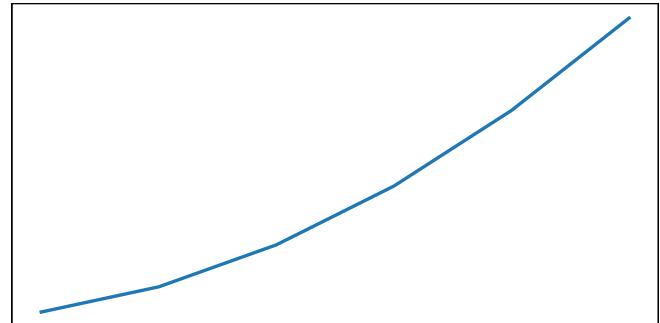
```
fig.set_size_inches((4, 2))
```

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
you can also set after getting the figure
fig.set_size_inches((4, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)

```



DPI = dots per inch

Get image DPI for the figure object. 100 is the default here.

```

fig.get_dpi()
import matplotlib.pyplot as plt

fig = plt.figure()
fig.get_dpi()

100.0

```

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
you can also set after getting the figure
fig.set_size_inches((12, 2))
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
plt.title('Sales figure')
fig.get_dpi()

100.0

```

## 27.1.2.2 subplot

```

import matplotlib.pyplot as plt

Subplots handles add_axes for you according to the number of rows and columns
fig, axes = plt.subplots(nrows=2, ncols=2, figsize=(6, 3))
axes is a numpy array, you can use it like using a list.
print(axes)

[[<Axes: > <Axes: >]
[<Axes: > <Axes: >]]

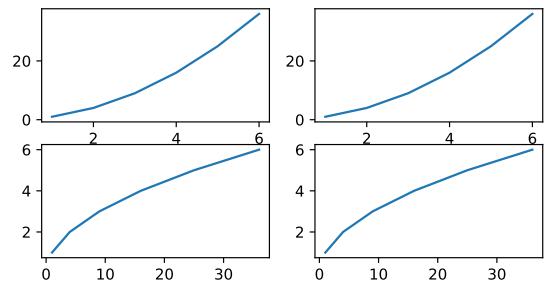
```

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
Subplots handles add_axes for you according
to the number of rows and columns
fig, axes = plt.subplots(nrows=2, ncols=2,
axes is a numpy array, you can use it like
using a list.
print(axes)
axes[0][0].plot(a1, a2)
axes[0][1].plot(a1, a2)
axes[1][0].plot(a2, a1)
axes[1][1].plot(a2, a1)

```



### 27.1.2.3 color and linestyle

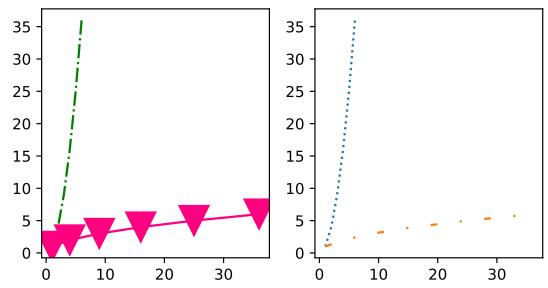
- Color
  - <https://matplotlib.org/stable/tutorials/colors/colors.html>
- line-style (ls)
  - [https://matplotlib.org/stable/gallery/lines\\_bars\\_and\\_markers/linestyles.html](https://matplotlib.org/stable/gallery/lines_bars_and_markers/linestyles.html)
- marker
  - [https://matplotlib.org/stable/api/markers\\_api.html](https://matplotlib.org/stable/api/markers_api.html)
- linewidth (lw)

```

import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
Subplots handles add_axes for you according
to the number of rows and columns
fig, axes = plt.subplots(nrows=1, ncols=2,
axes is a numpy array
print(axes)
axes[0].plot(a1, a2, color='green',
linestyle='-.')
axes[0].plot(a2, a1, color=(1, 0, 0.5),
marker='v', markersize=20)
axes[1].plot(a1, a2, linestyle='dotted')
axes[1].plot(a2, a1, linestyle=(0, (3, 10, 1,
10)))

```



### 27.1.2.4 other inputs

- NumPy array
- Pandas series

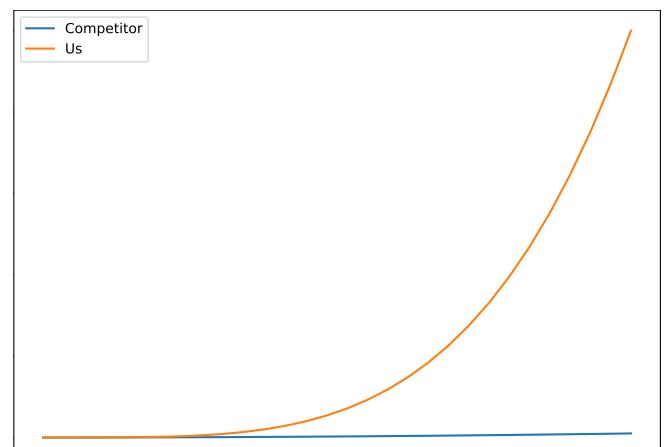
### 27.1.2.5 legend

```

import matplotlib.pyplot as plt
import numpy as np

x = np.linspace(0, 10, 30)
y = x * x
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
line1 = ax.plot(x, y, label="Competitor")
line2 = ax.plot(x, y**2, label="Us")
ax.legend()

```



### 27.1.2.6 customize style

predefined styles

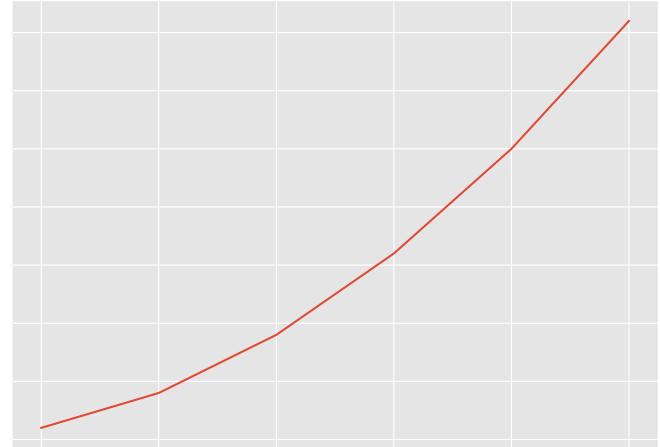
```
import matplotlib.pyplot as plt

print(plt.style.available)
```

```
['Solarize_Light2', '_classic_test_patch', '_mpl-gallery', '_mpl-gallery-nogrid', 'bmh', 'classic', 'dark...', 'fivethirtyeight', 'ggplot', 'seaborn', 'seaborn-bright', 'seaborn-colorblind', 'seaborn-dark', 'seaborn-dark...', 'seaborn-deep', 'seaborn-fiveth...', 'seaborn-notebook', 'seaborn-paper', 'seaborn-pastel', 'seaborn-poster', 'seaborn-ticks', 'seaborn-white', 'seaborn-white...', 'seaborn-zipkin']
```

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
plt.style.use('ggplot')
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
```



restore default style

```
import matplotlib.pyplot as plt

plt.style.use('default') # But strangely enough figure size gets changed still

import matplotlib.pyplot as plt

plt.rcParams["figure.figsize"] = (6, 4)
plt.rcParams["figure.dpi"] = 100
```

### 27.1.2.7 save to file

`savefig` saves image to file. We also set the `bbox_inches` parameter to `tight` to make sure the image doesn't get out of the image bound.

save to .png

```
import matplotlib.pyplot as plt

a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
fig.savefig('out.png', bbox_inches = 'tight')
```

save to .pdf

```
import matplotlib.pyplot as plt

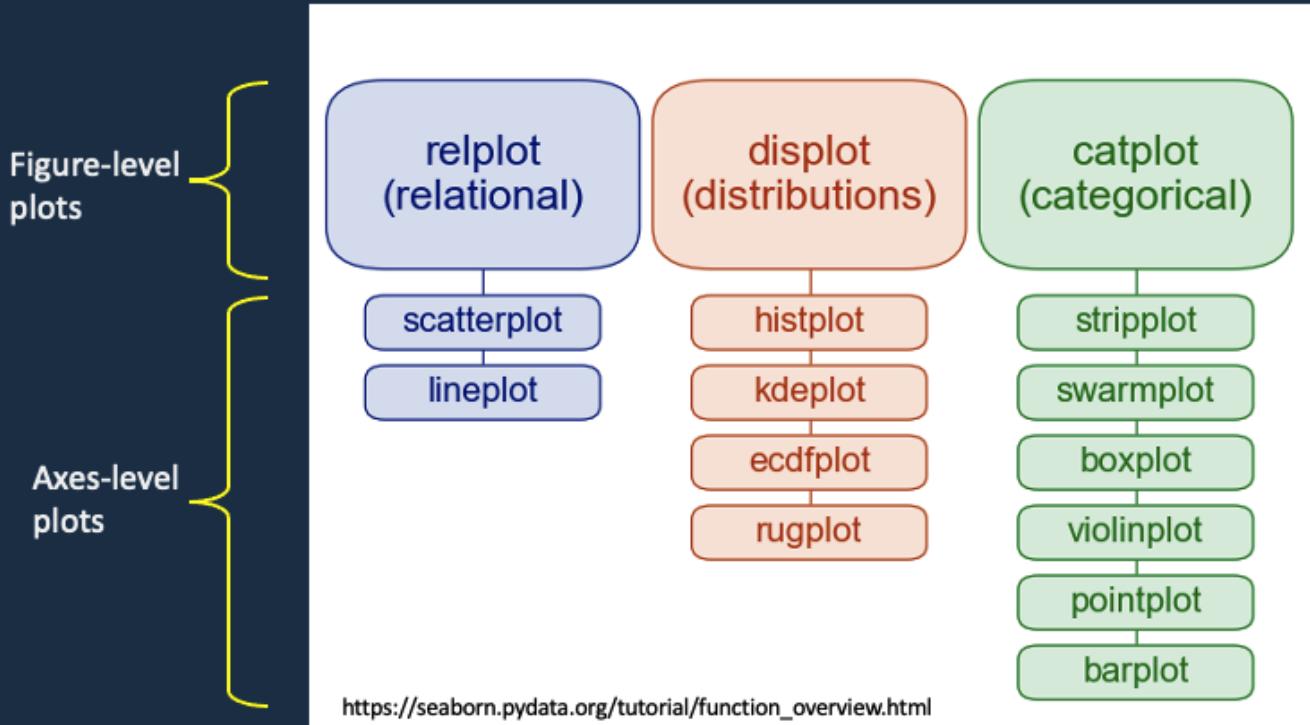
a1 = list(range(1, 7)) # [1, 2, 3, 4, 5, 6]
a2 = [1, 4, 9, 16, 25, 36]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(a1, a2)
fig.savefig('out.pdf', bbox_inches = 'tight')
```

## 27.1.3 Seaborn

- Kimberly Fessel

- visually explained
- Seaborn
- Matplotlib
- Pandas
- iPyWidgets

## Three main type of Seaborn plots



Yes, that correspond to Matplotlib's concept of figure and axes

Figure 27.3: seaborn plot type

- Seaborn
  - figure-level plot
  - axes-level plot

R or RStudio run Python with installing packages or modules by using `reticulate`, R package

and directly using Anaconda `conda` environment for convenience, instead of `virtualenv`

[https://rstudio.github.io/reticulate/articles/python\\_packages.html](https://rstudio.github.io/reticulate/articles/python_packages.html)

```
library(reticulate)
conda_list()
use_condaenv(condaenv = 'sandbox-3.9')

install Seaborn
conda_install("r-reticulate", "seaborn")

import Seaborn (it will be automatically discovered in "r-reticulate")
seaborn <- import("seaborn")
```

### 27.1.3.1 basic

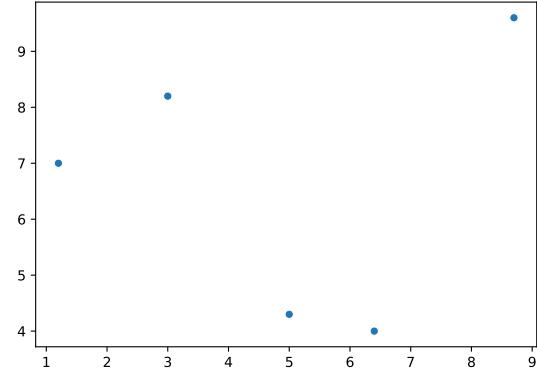
```
import matplotlib.pyplot as plt # need it sometimes
import seaborn as sns

sns.set_theme() # set the default theme
```

---

```
import seaborn as sns

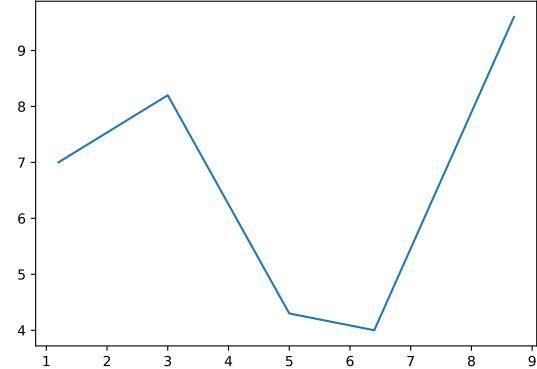
x = [3, 5, 1.2, 8.7, 6.4]
y = [8.2, 4.3, 7, 9.6, 4]
sns.scatterplot(x=x, y=y)
```




---

```
import seaborn as sns

x = [3, 5, 1.2, 8.7, 6.4]
y = [8.2, 4.3, 7, 9.6, 4]
sns.lineplot(x=x, y=y)
```



### 27.1.3.2 data frame

```
sns.load_dataset
```

---

```
import seaborn as sns

tips = sns.load_dataset("tips")
tips
```

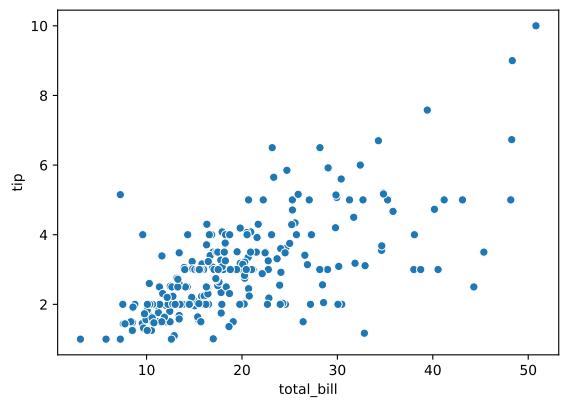
|        | total_bill             | tip  | sex    | smoker | day  | time   |
|--------|------------------------|------|--------|--------|------|--------|
| ## 0   | 16.99                  | 1.01 | Female | No     | Sun  | Dinner |
| ## 1   | 10.34                  | 1.66 | Male   | No     | Sun  | Dinner |
| ## 2   | 21.01                  | 3.50 | Male   | No     | Sun  | Dinner |
| ## 3   | 23.68                  | 3.31 | Male   | No     | Sun  | Dinner |
| ## 4   | 24.59                  | 3.61 | Female | No     | Sun  | Dinner |
| ## ..  | ...                    | ...  | ...    | ...    | ...  | ...    |
| ## 239 | 29.03                  | 5.92 | Male   | No     | Sat  | Dinner |
| ## 240 | 27.18                  | 2.00 | Female | Yes    | Sat  | Dinner |
| ## 241 | 22.67                  | 2.00 | Male   | Yes    | Sat  | Dinner |
| ## 242 | 17.82                  | 1.75 | Male   | No     | Sat  | Dinner |
| ## 243 | 18.78                  | 3.00 | Female | No     | Thur | Dinner |
| ##     |                        |      |        |        |      |        |
| ##     | [244 rows x 7 columns] |      |        |        |      |        |

---

```
data
```

```
import seaborn as sns

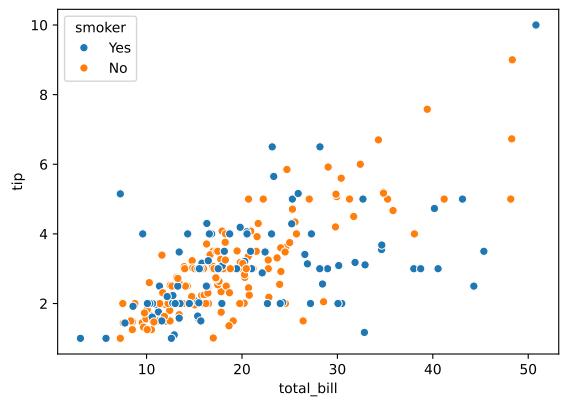
tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
 x='total_bill',
 y='tip')
```



hue

```
import seaborn as sns

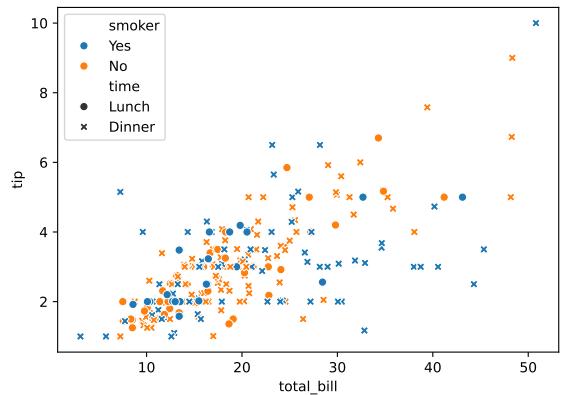
tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
 x='total_bill',
 y='tip',
 hue='smoker')
```



style

```
import seaborn as sns

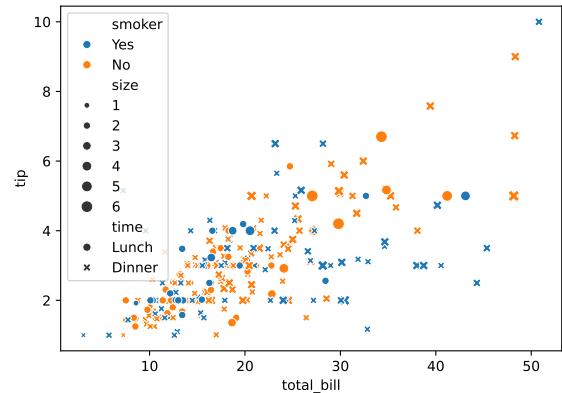
tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
 x='total_bill',
 y='tip',
 hue='smoker',
 style='time')
```



size

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.scatterplot(data=tips,
 x='total_bill',
 y='tip',
 hue='smoker',
 style='time',
 size='size')
```



- Legend is covering up the graph, it's getting out of hand. Let's tune the range of x
- `sns.scatterplot()` actually returns something that resembles Matplotlib axis. So we use a Matplotlib axis function:

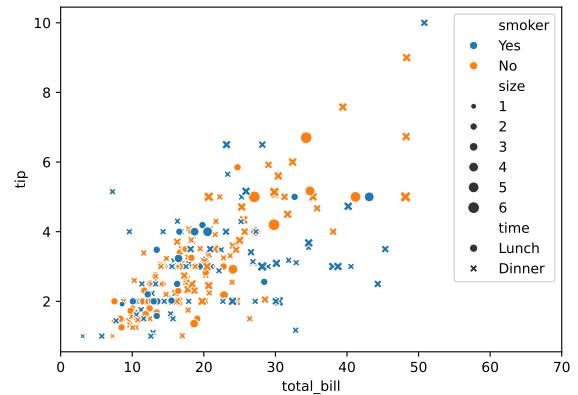
```
ax = sns.scatterplot(...)
```

<https://stackoverflow.com/questions/26597116/seaborn-plots-not-showing-up>

```
import matplotlib.pyplot as plt
import seaborn as sns

tips = sns.load_dataset("tips")
ax = sns.scatterplot(data=tips,
 x='total_bill',
 y='tip',
 hue='smoker',
 style='time',
 size='size')
ax.set_xlim(0, 70)
alternatively:
ax.set(xlim=(0, 70))
plt.show()
```

## (0.0, 70.0)

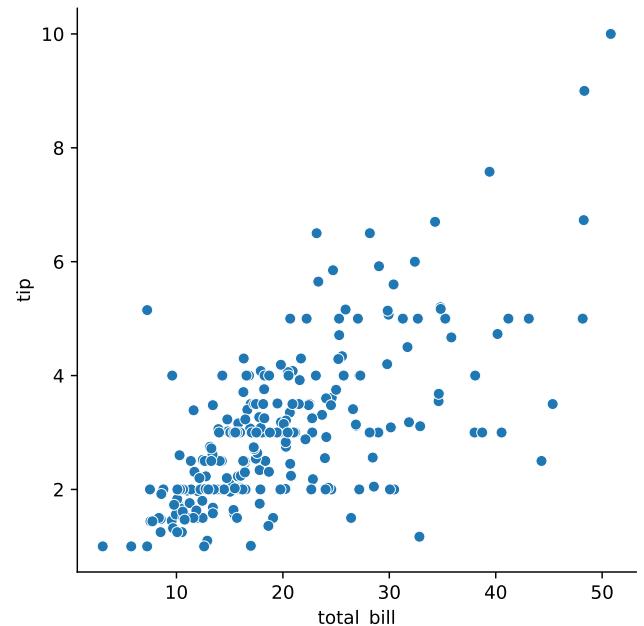


### 27.1.3.3 axis-level plot and figure-level plot

```
sns.relplot
```

```
import seaborn as sns

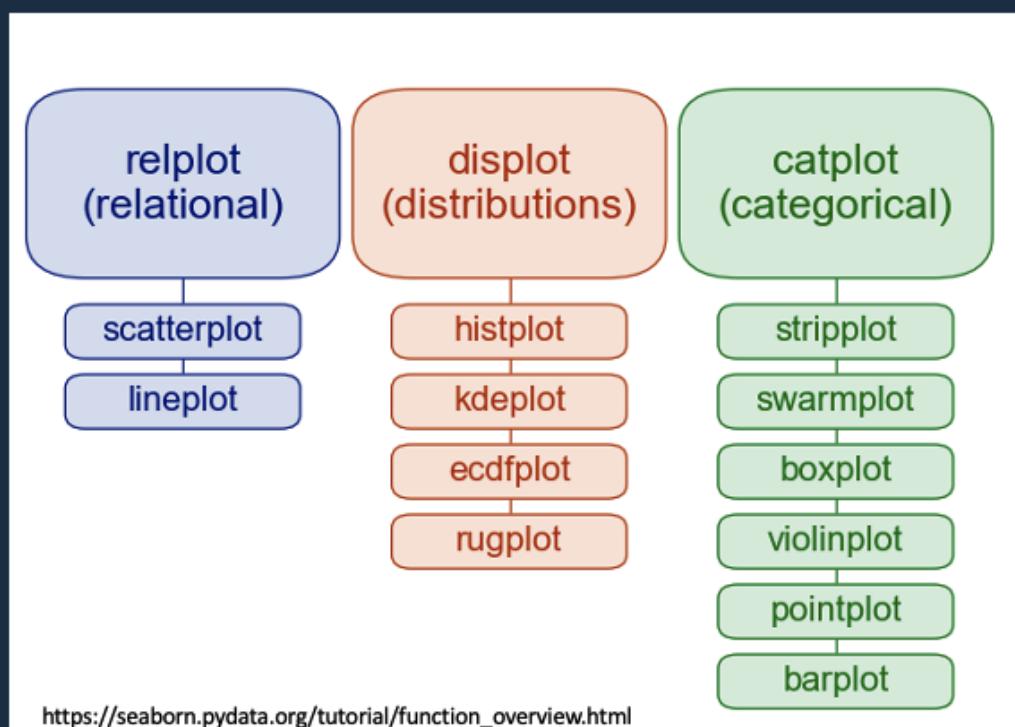
tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
 data=tips,
 x='total_bill',
 y='tip')
```



## Three main type of Seaborn plots

Figure-level  
plots

Axes-level  
plots

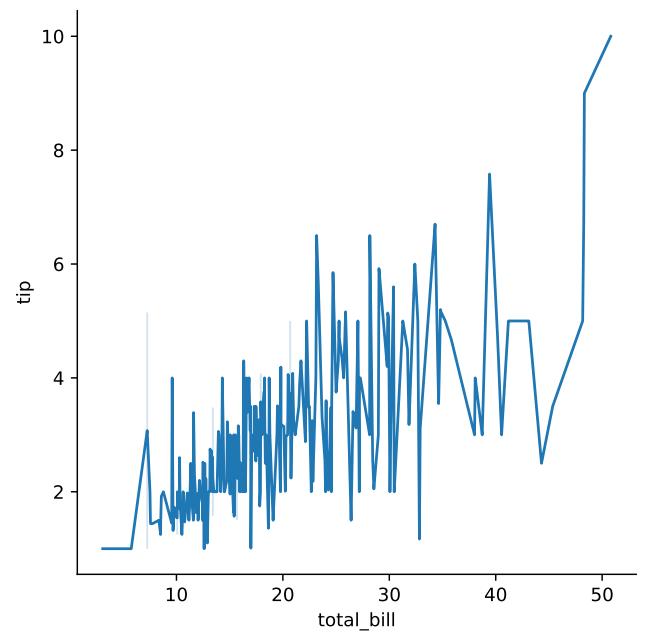


Yes, that correspond to Matplotlib's concept of figure and axes

Figure 27.4: seaborn plot type

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='line',
 data=tips,
 x='total_bill',
 y='tip')
```



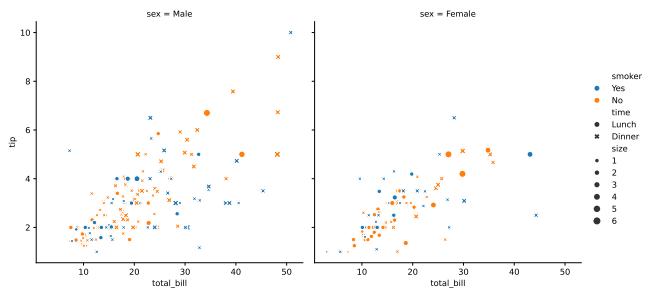
With figure-level plot, we can draw more than one plot (one axes).

Here we specify that different `sex` be on different column by specifying `col=sex`.

`col=sex`

```
import seaborn as sns

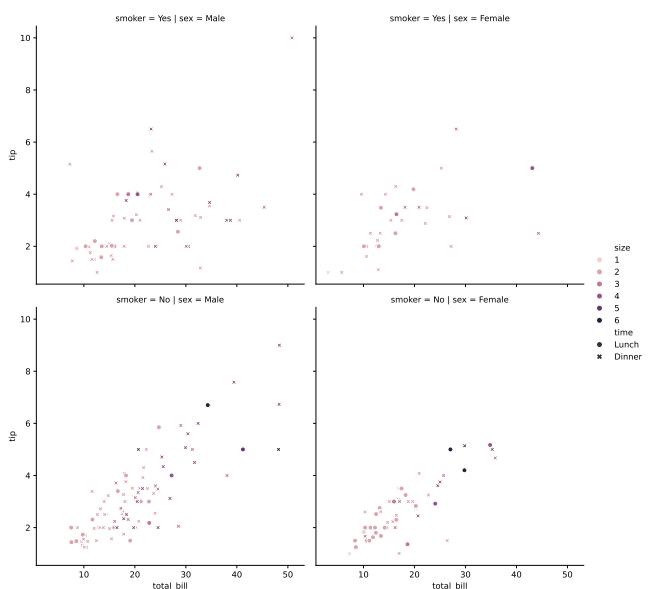
tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
 data=tips,
 x='total_bill',
 y='tip',
 hue='smoker',
 style='time',
 size='size',
 col='sex')
```



`col=sex row=smoker`

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.relplot(kind='scatter',
 data=tips,
 x='total_bill',
 y='tip',
 style='time',
 hue='size',
 row='smoker',
 col='sex')
```



#### 27.1.3.4 accessing figure and axes objects

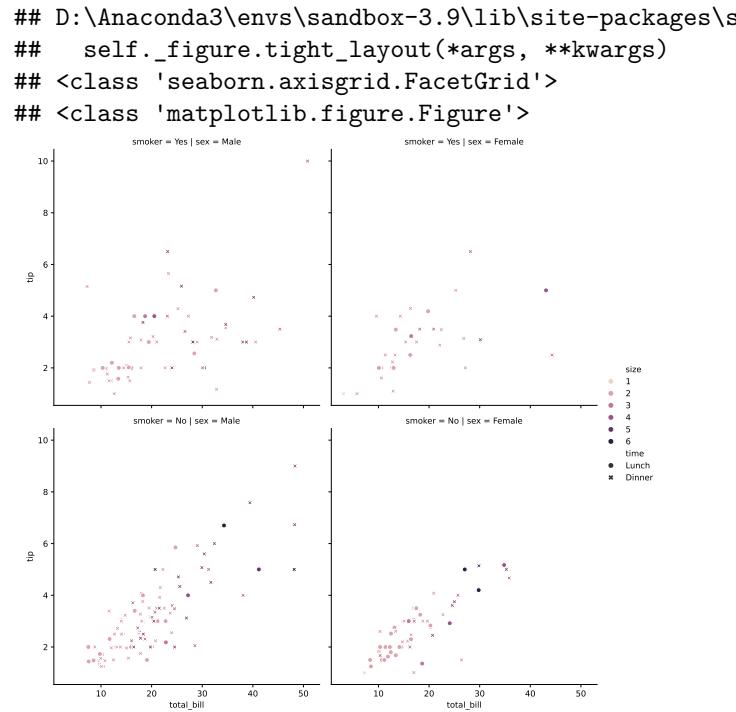
Recall that Seaborn uses Matplotlib to draw the graphics. So underneath the Seaborn library, you can still access Matplotlib's figure object and axes objects if necessary. The call to figure-level plot returns an object.

```

import seaborn as sns

tips = sns.load_dataset("tips")
g = sns.relplot(kind='scatter',
 data=tips,
 x='total_bill',
 y='tip',
 style='time',
 hue='size',
 row='smoker',
 col='sex')
print(type(g))
print(type(g.fig)) # g.fig gets you the Figure
g.fig

```



### 27.1.3.5 distribution plot

```
sns.displot
```

#### 27.1.3.5.1 histogram

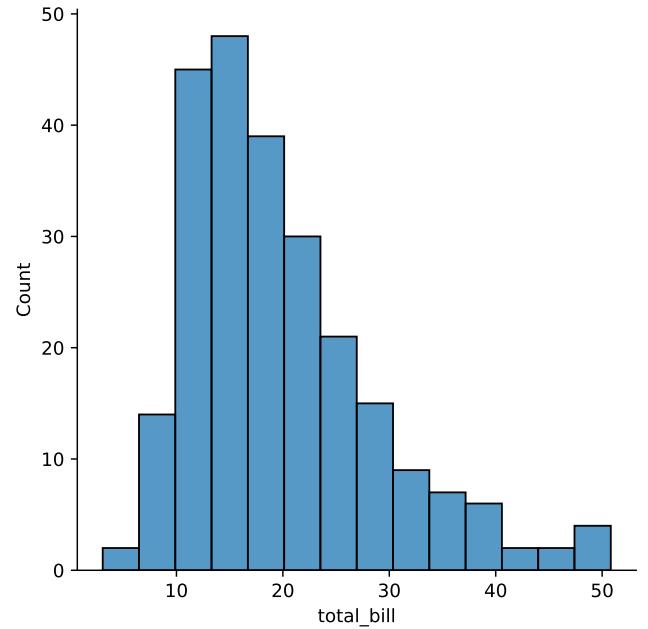
---

```

import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist',
 data=tips,
 x='total_bill')

```

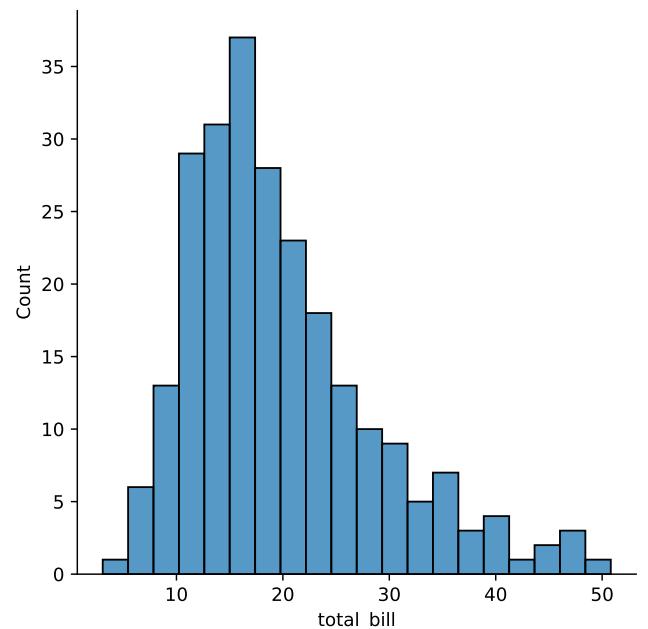



---

```
bins
```

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20,
 data=tips,
 x='total_bill')
```

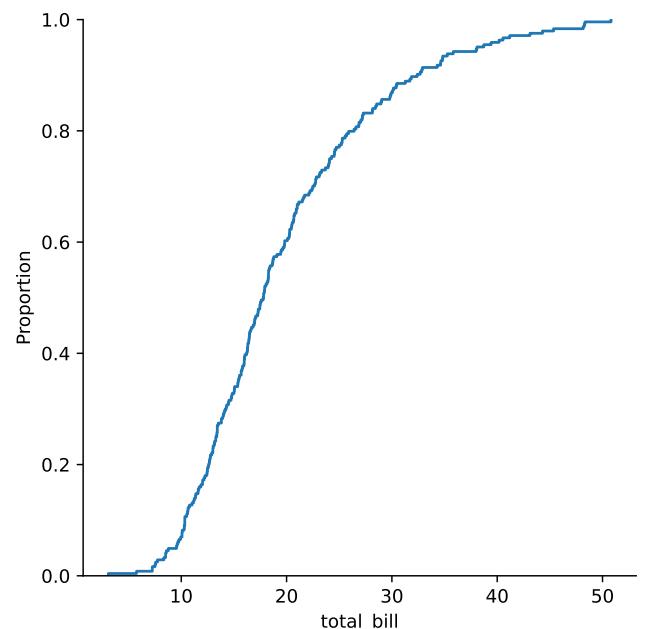


#### 27.1.3.5.2 ECDF = empirical cumulative distrutive function

---

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='ecdf',
 data=tips,
 x='total_bill')
```

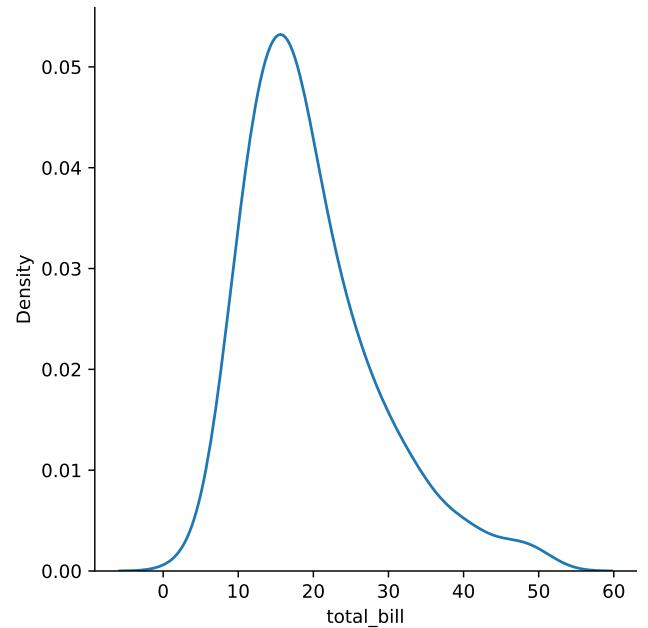


#### 27.1.3.5.3 KDE = kernel density estimation

---

```
import seaborn as sns

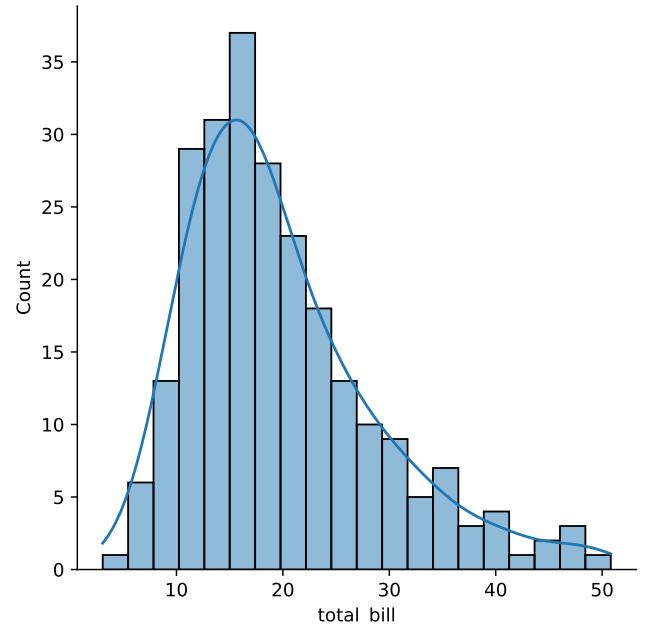
tips = sns.load_dataset("tips")
sns.displot(kind='kde',
 data=tips,
 x='total_bill')
```



histogram with KDE

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20, kde='true',
 data=tips,
 x='total_bill')
```



## '% This file was created with tikzplotlib v0.10.1

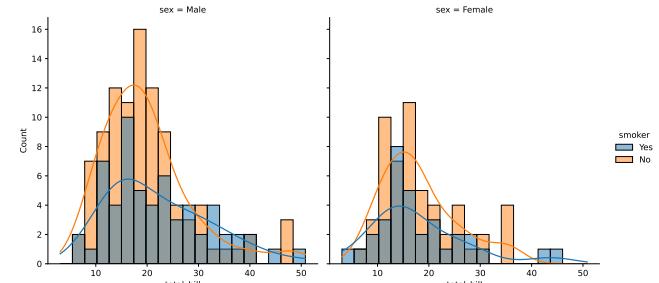
TikZ / tikzpicture with PGFplots axis by transforming Matplotlib-based Seaborn plot to .tex via Python package tikzplotlib

xlabel={total\_bill}, changed to xlabel={total bill}, without \_ in text

X xlabel={Count}, changed to ylabel={\$\text{Count}\$},: not necessary to be changed, the problem is \_ in xlabel name

```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.displot(kind='hist', bins=20, kde='true',
 data=tips,
 x='total_bill',
 hue='smoker',
 col='sex'
)
```



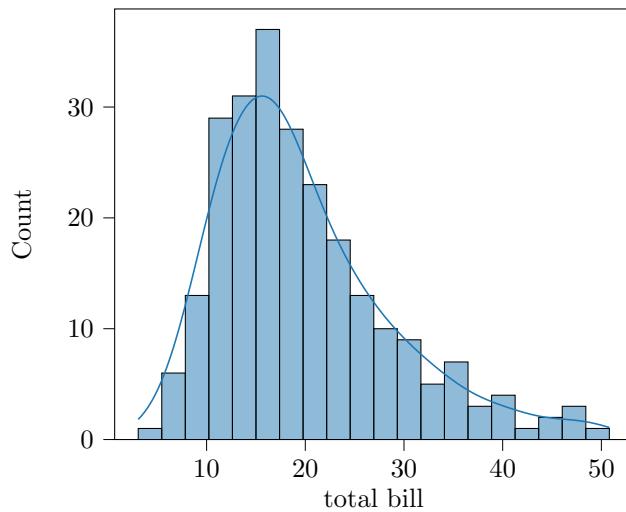
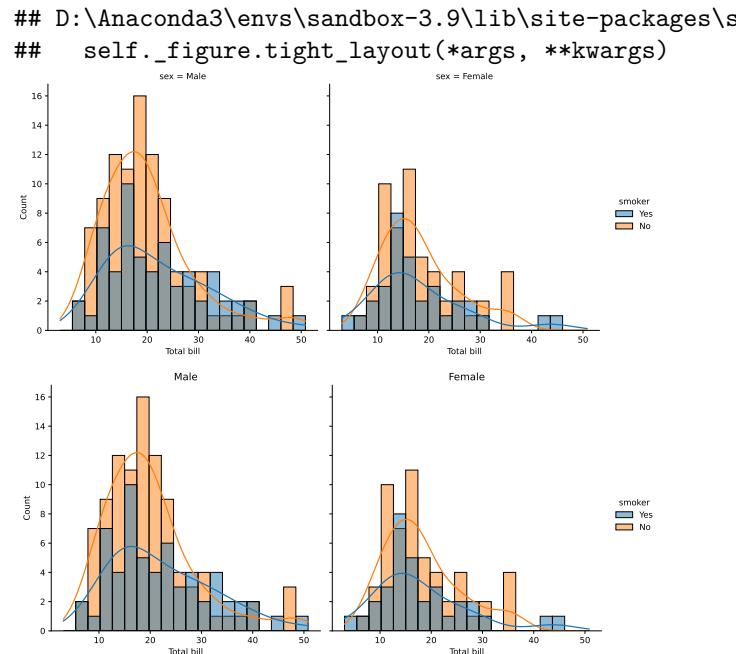


Figure 27.5: tikzplotlib

Let's customize the label and title.

```
import seaborn as sns

tips = sns.load_dataset("tips")
g = sns.displot(kind='hist', bins=20,
 kde=True,
 data=tips,
 x='total_bill',
 hue='smoker',
 col='sex')
g.set(xlabel='Total bill')
g.axes[0][0].set(title='Male')
g.axes[0][1].set(title='Female')
g.fig
```

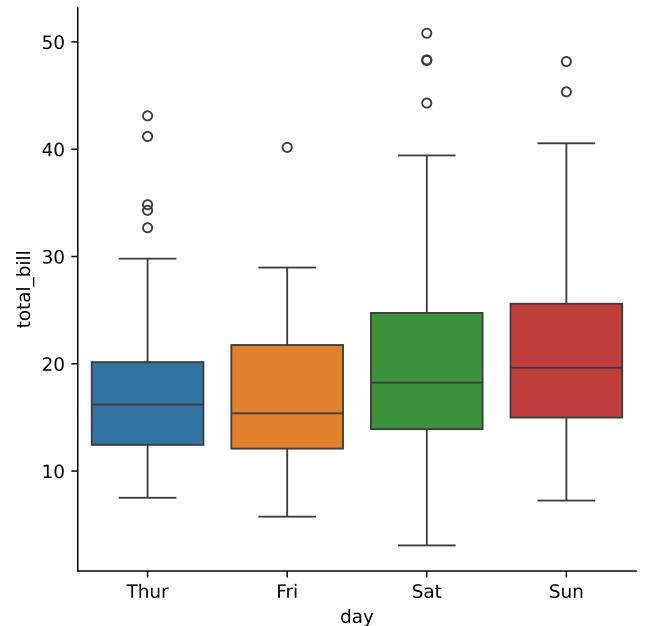


### 27.1.3.6 categorical plot

```
sns.catplot
```

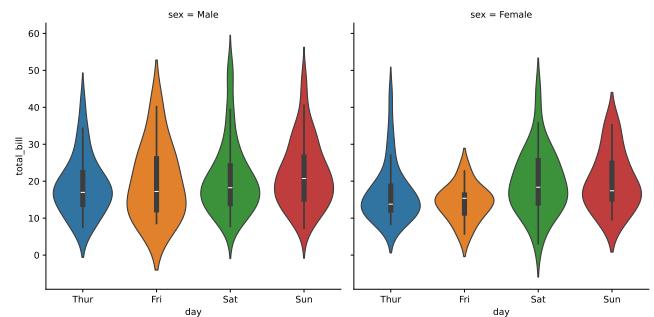
```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.catplot(kind='box',
 data=tips,
 x="day", hue="day",
 y="total_bill"
)
```



```
import seaborn as sns

tips = sns.load_dataset("tips")
sns.catplot(kind='violin',
 data=tips,
 x="day", hue="day",
 y="total_bill",
 col='sex'
)
```



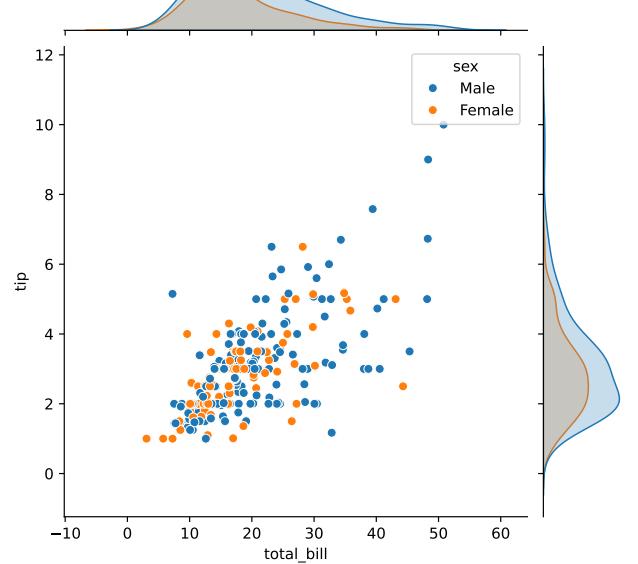
### 27.1.3.7 joint plot

```
sns.jointplot
```

```
<seaborn.axisgrid.JointGrid object at 0x0000012D8
```

```
import matplotlib.pyplot as plt
import seaborn as sns

tips = sns.load_dataset("tips")
sns.jointplot(data=tips,
 x="total_bill",
 y="tip",
 hue='sex'
)
plt.show()
```

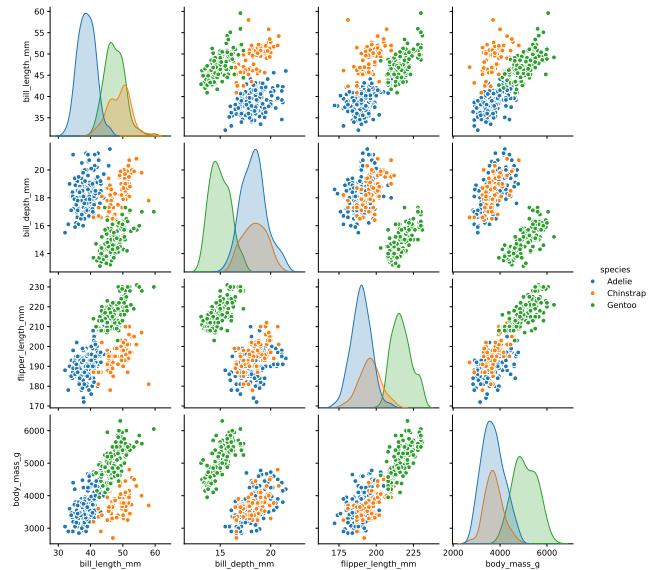


### 27.1.3.8 pair plot

```
sns.pairplot
```

```
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
 hue='species')
```



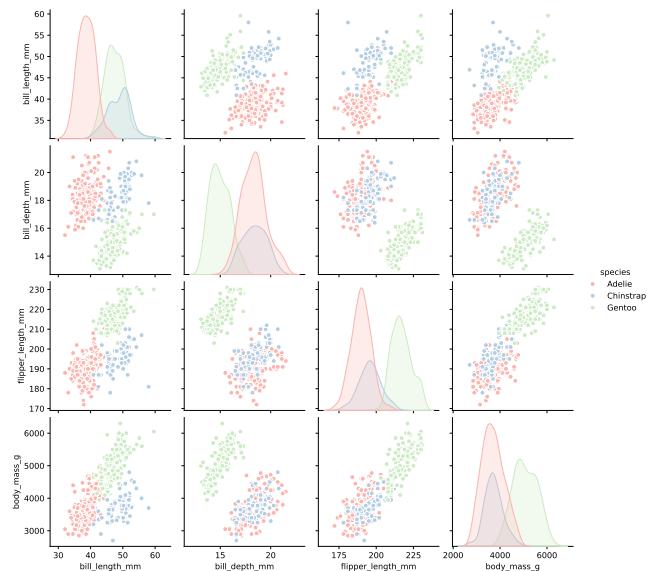
### 27.1.3.9 palette

Changing the color with Matplotlib's color

A list of color: <https://matplotlib.org/stable/tutorials/colors/colormaps.html>

```
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
 hue='species',
 palette='Pastel1')
```



### 27.1.3.10 volcano plot

### 27.1.3.11 heatmap

## 27.2 export

### 27.2.1 .svg

<https://stackoverflow.com/questions/24525111/how-can-i-get-the-output-of-a-matplotlib-plot-as-an-svg>

```
import matplotlib.pyplot as plt
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
 hue='species')

plt.savefig("test.svg")
plt.savefig("test.svg", dpi=1200)
```

## 27.2.2 .eps

<https://stackoverflow.com/questions/16183462/saving-images-in-python-at-a-very-high-quality>

The PostScript backend does not support transparency; partially transparent artists will be rendered opaque.

```
import matplotlib.pyplot as plt
import seaborn as sns

penguins = sns.load_dataset("penguins")
sns.pairplot(data=penguins,
 hue='species')

plt.savefig("test.eps")
plt.savefig("test.eps", dpi=1200)
```

# **Chapter 28**

## **survival analysis**

**28.1 Python package `tableone`**

**28.2 Python package `lifelines`**



# Chapter 29

## Manim

### 29.1 VSCode extension: Manim Sideview

<https://marketplace.visualstudio.com/items?itemName=Rickaym.manim-sideview>

ffmpeg.exe placed in the same folder with .py

VSCode Ctrl + Shift + P: open Mobject gallery

### 29.2 installation

<https://docs.manim.community/en/stable/installation.html>

#### 29.2.1 Conda

conda install -c conda-forge manim

### 29.3 quickstart

<https://docs.manim.community/en/stable/tutorials/quickstart.html>

[https://www.w3schools.com/tags/att\\_video\\_autoplay.asp](https://www.w3schools.com/tags/att_video_autoplay.asp)

[https://www.w3schools.com/tags/att\\_video\\_loop.asp](https://www.w3schools.com/tags/att_video_loop.asp)

```
from manim import *

class CreateCircle(Scene):
 def construct(self):
 circle = Circle() # create a circle
 circle.set_fill(PINK, opacity=0.5) #
 # set the color and transparency
 self.play(Create(circle)) # show the
 # circle on screen
```

manim -pql scene.py CreateCircle



# Chapter 30

## ggplot2

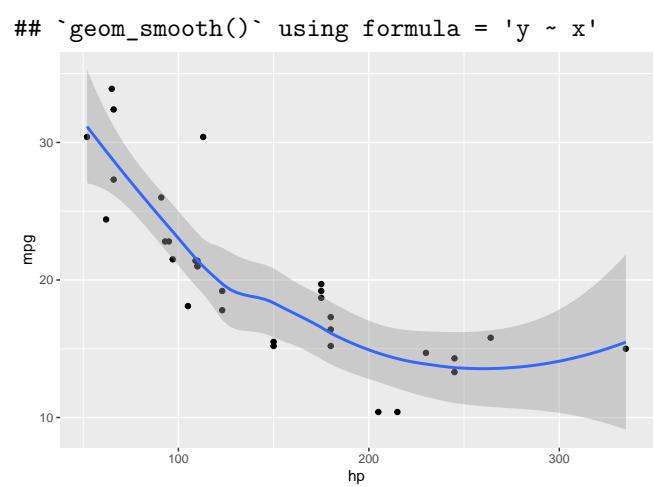
<https://bookdown.org/xiangyun/msg/system.html#chap:system>

Modern Statistical Graphics [section 5.1](#)

- <https://www.rdocumentation.org/> to search function
- ggplot2
  - <https://ggplot2.tidyverse.org/index.html> to search ggplot2 function
  - panel = layer
    - \* geom = geometric objects / geometry = element
      - .element
    - \* statistic
    - \* scale
    - \* coordinate system
    - \* facet

```
library(ggplot2)

p <- ggplot(aes(x = hp, y = mpg), data =
 mtcars) +
 geom_point() # layer of scatterplot
p + geom_smooth(method = "loess") # add layer
 of smooth
```

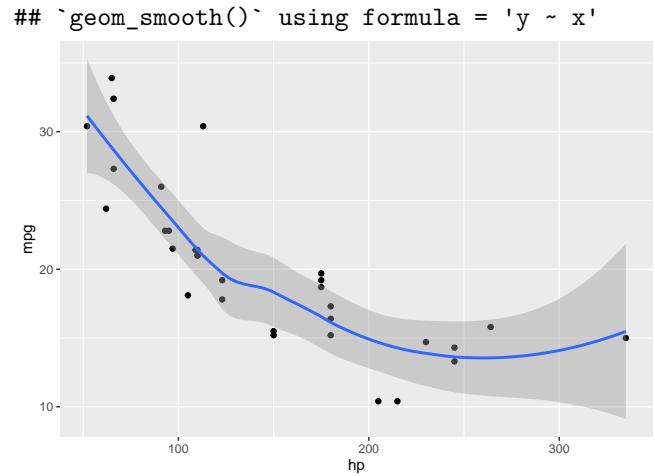


### 30.1 geom

<https://bookdown.org/xiangyun/msg/system.html#section-13>

```
library(ggplot2)

ggplot(aes(x = hp, y = mpg), data = mtcars) +
 geom_point() +
 geom_smooth(method = "loess")
```



points [https://ggplot2.tidyverse.org/reference/geom\\_point.html?q=geom\\_point#null](https://ggplot2.tidyverse.org/reference/geom_point.html?q=geom_point#null)

`geom_point`

smoothed conditional means [https://ggplot2.tidyverse.org/reference/geom\\_smooth.html?q=geom\\_sm#null](https://ggplot2.tidyverse.org/reference/geom_smooth.html?q=geom_sm#null)

Aids the eye in seeing patterns in the presence of overplotting. `geom_smooth()` and `stat_smooth()` are effectively aliases: they both use the same arguments. Use `stat_smooth()` if you want to display the results with a non-standard geom.

`geom_smooth`

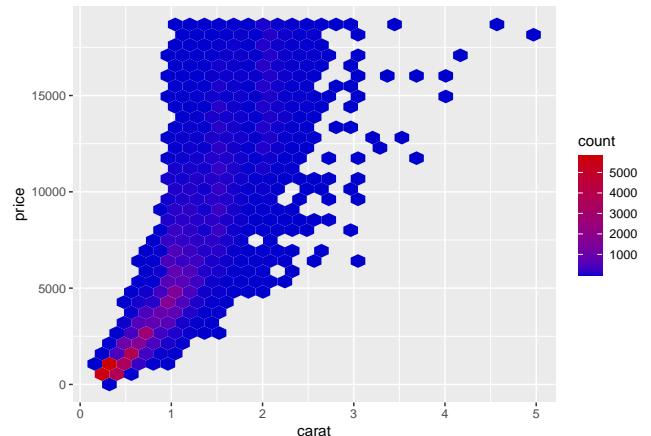
`stat_smooth`

`method` Smoothing method (function) to use, accepts either NULL or a character vector, e.g. "lm", "glm", "gam", "loess" or a function, e.g. MASS::rlm or mgcv::gam, stats::lm, or stats::loess. "auto" is also accepted for backwards compatibility. It is equivalent to NULL.

```
install.packages("hexbin")
```

```
library(ggplot2)

ggplot(aes(x = carat, y = price), data =
 diamonds) +
 geom_hex() +
 scale_fill_gradient(low = "blue3", high =
 "red3")
```

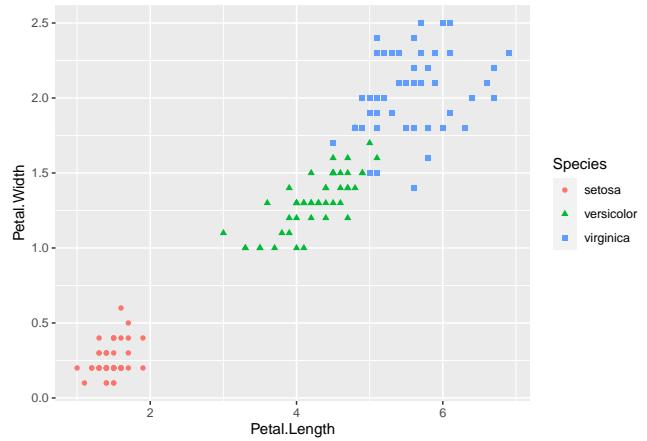


gradient color scales [https://ggplot2.tidyverse.org/reference/scale\\_gradient.html?q=scale\\_fill\\_gradient#ref-usage](https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradient#ref-usage)

[https://ggplot2.tidyverse.org/reference/scale\\_gradient.html?q=scale\\_fill\\_gradient#ref-examples](https://ggplot2.tidyverse.org/reference/scale_gradient.html?q=scale_fill_gradient#ref-examples)

```
library(ggplot2)

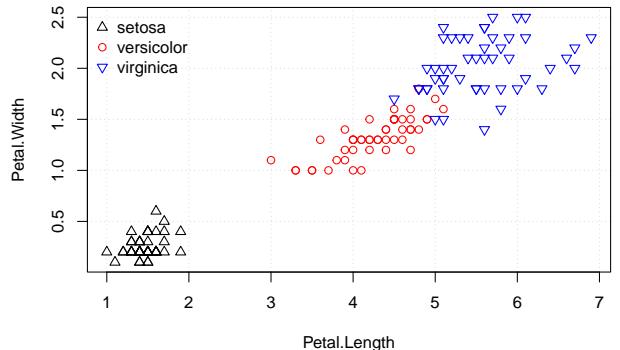
ggplot(aes(x = Petal.Length, y = Petal.Width),
 data = iris) +
 geom_point(aes(color = Species, shape =
 Species))
```



basic plot system

<https://bookdown.org/xiangyun/msg/elements.html#sec:points>

```
iris species converted to type integer 1, 2,
3 for further using vectors
idx <- as.integer(iris[["Species"]])
plot(iris[, 3:4],
 pch = c(24, 21, 25)[idx],
 col = c("black", "red", "blue")[idx],
 panel.first = grid()
)
legend("topleft",
 legend = levels(iris[["Species"]]),
 col = c("black", "red", "blue"), pch = c(24,
 21, 25), bty = "n"
)
```



plot <https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/plot.default>

**pch** a vector of **plotting characters** or symbols: see [points](#).

**col** The **colors** for lines and points. Multiple colors can be specified so that each point can be given its own color. If there are fewer colors than points they are recycled in the standard fashion. Lines will all be plotted in the first colour specified.

**panel.first** an ‘expression’ to be evaluated after the plot axes are set up but before any plotting takes place. This can be useful for drawing **background grids** or **scatterplot smooths**. Note that this works by lazy evaluation: passing this argument from other plot methods may well not work since it may be evaluated too early.

legend <https://www.rdocumentation.org/packages/graphics/versions/3.6.2/topics/legend>

**bty** the **type of box** to be drawn around the legend. The allowed values are “o” (the default) and “n”.

<https://stackoverflow.com/questions/10108073/plot-legends-without-border-and-with-white-background>

Use option **bty = "n"** in legend to remove the box around the legend.

**legend** a character or expression vector of length  $\geq 1$  to appear in the legend. Other objects will be coerced by [as.graphicsAnnot](#)



### 30.1.1 basic plot system decomposition

|       | Sepal.Length | Sepal.Width | Petal.Length | Petal.Width |
|-------|--------------|-------------|--------------|-------------|
| ## 1  | 5.1          | 3.5         | 1.4          |             |
| ## 2  | 4.9          | 3.0         | 1.4          |             |
| ## 3  | 4.7          | 3.2         | 1.3          |             |
| ## 4  | 4.6          | 3.1         | 1.5          |             |
| ## 5  | 5.0          | 3.6         | 1.4          |             |
| ## 6  | 5.4          | 3.9         | 1.7          |             |
| ## 7  | 4.6          | 3.4         | 1.4          |             |
| ## 8  | 5.0          | 3.4         | 1.5          |             |
| ## 9  | 4.4          | 2.9         | 1.4          |             |
| ## 10 | 4.9          | 3.1         | 1.5          |             |
| ## 11 | 5.4          | 3.7         | 1.5          |             |
| ## 12 | 4.8          | 3.4         | 1.6          |             |
| ## 13 | 4.8          | 3.0         | 1.4          |             |
| ## 14 | 4.3          | 3.0         | 1.1          |             |
| ## 15 | 5.8          | 4.0         | 1.2          |             |
| ## 16 | 5.7          | 4.4         | 1.5          |             |
| ## 17 | 5.4          | 3.9         | 1.3          |             |
| ## 18 | 5.1          | 3.5         | 1.4          |             |
| ## 19 | 5.7          | 3.8         | 1.7          |             |
| ## 20 | 5.1          | 3.8         | 1.5          |             |
| ## 21 | 5.4          | 3.4         | 1.7          |             |
| ## 22 | 5.1          | 3.7         | 1.5          |             |
| ## 23 | 4.6          | 3.6         | 1.0          |             |
| ## 24 | 5.1          | 3.3         | 1.7          |             |
| ## 25 | 4.8          | 3.4         | 1.9          |             |
| ## 26 | 5.0          | 3.0         | 1.6          |             |
| ## 27 | 5.0          | 3.4         | 1.6          |             |
| ## 28 | 5.2          | 3.5         | 1.5          |             |
| ## 29 | 5.2          | 3.4         | 1.4          |             |
| ## 30 | 4.7          | 3.2         | 1.6          |             |
| ## 31 | 4.8          | 3.1         | 1.6          |             |
| ## 32 | 5.4          | 3.4         | 1.5          |             |
| ## 33 | 5.2          | 4.1         | 1.5          |             |
| ## 34 | 5.5          | 4.2         | 1.4          |             |
| ## 35 | 4.9          | 3.1         | 1.5          |             |
| ## 36 | 5.0          | 3.2         | 1.2          |             |
| ## 37 | 5.5          | 3.5         | 1.3          |             |
| ## 38 | 4.9          | 3.6         | 1.4          |             |
| ## 39 | 4.4          | 3.0         | 1.3          |             |
| ## 40 | 5.1          | 3.4         | 1.5          |             |
| ## 41 | 5.0          | 3.5         | 1.3          |             |
| ## 42 | 4.5          | 2.3         | 1.3          |             |
| ## 43 | 4.4          | 3.2         | 1.3          |             |
| ## 44 | 5.0          | 3.5         | 1.6          |             |
| ## 45 | 5.1          | 3.8         | 1.9          |             |
| ## 46 | 4.8          | 3.0         | 1.4          |             |
| ## 47 | 5.1          | 3.8         | 1.6          |             |
| ## 48 | 4.6          | 3.2         | 1.4          |             |
| ## 49 | 5.3          | 3.7         | 1.5          |             |
| ## 50 | 5.0          | 3.3         | 1.4          |             |
| ## 51 | 7.0          | 3.2         | 4.7          |             |
| ## 52 | 6.4          | 3.2         | 4.5          |             |
| ## 53 | 6.9          | 3.1         | 4.9          |             |
| ## 54 | 5.5          | 2.3         | 4.0          |             |
| ## 55 | 6.5          | 2.8         | 4.6          |             |
| ## 56 | 5.7          | 2.8         | 4.5          |             |
| ## 57 | 6.3          | 3.3         | 4.7          |             |
| ## 58 | 4.9          | 2.4         | 3.3          |             |
| ## 59 | 6.6          | 2.9         | 4.6          |             |
| ## 60 | 5.2          | 2.7         | 3.9          |             |
| ## 61 | 5.0          | 2.0         | 3.5          |             |
| ## 62 | 5.9          | 3.0         | 4.2          |             |
| ## 63 | 6.0          | 2.2         | 4.0          |             |
| ## 64 | 6.1          | 2.9         | 4.7          |             |
| ## 65 | 5.6          | 2.0         | 2.6          |             |

iris\$Species

```
[1] setosa setosa setosa setosa setosa
[7] setosa setosa setosa setosa setosa
[13] setosa setosa setosa setosa setosa
[19] setosa setosa setosa setosa setosa
[25] setosa setosa setosa setosa setosa
[31] setosa setosa setosa setosa setosa
[37] setosa setosa setosa setosa setosa
[43] setosa setosa setosa setosa setosa
[49] setosa setosa versicolor versicolor
[55] versicolor versicolor versicolor versicolor
[61] versicolor versicolor versicolor versicolor
[67] versicolor versicolor versicolor versicolor
[73] versicolor versicolor versicolor versicolor
[79] versicolor versicolor versicolor versicolor
[85] versicolor versicolor versicolor versicolor
[91] versicolor versicolor versicolor versicolor
[97] versicolor versicolor versicolor versicolor
[103] virginica virginica virginica virginica
[109] virginica virginica virginica virginica
[115] virginica virginica virginica virginica
[121] virginica virginica virginica virginica
[127] virginica virginica virginica virginica
[133] virginica virginica virginica virginica
[139] virginica virginica virginica virginica
[145] virginica virginica virginica virginica
Levels: setosa versicolor virginica
```

```
iris[["Species"]]
```

```
[1] setosa setosa setosa setosa setosa
[7] setosa setosa setosa setosa setosa
[13] setosa setosa setosa setosa setosa
[19] setosa setosa setosa setosa setosa
[25] setosa setosa setosa setosa setosa
[31] setosa setosa setosa setosa setosa
[37] setosa setosa setosa setosa setosa
[43] setosa setosa setosa setosa setosa
[49] setosa setosa versicolor versicolor
[55] versicolor versicolor versicolor versicolor
[61] versicolor versicolor versicolor versicolor
[67] versicolor versicolor versicolor versicolor
[73] versicolor versicolor versicolor versicolor
[79] versicolor versicolor versicolor versicolor
[85] versicolor versicolor versicolor versicolor
[91] versicolor versicolor versicolor versicolor
[97] versicolor versicolor versicolor versicolor
[103] virginica virginica virginica virginica
[109] virginica virginica virginica virginica
[115] virginica virginica virginica virginica
[121] virginica virginica virginica virginica
[127] virginica virginica virginica virginica
[133] virginica virginica virginica virginica
[139] virginica virginica virginica virginica
[145] virginica virginica virginica virginica
Levels: setosa versicolor virginica
```

```
as.integer(iris[["Species"]])
```

```
levels(iris[["Species"]])
```

```
[1] "setosa" "versicolor" "virginica"
```

```
idx <- as.integer(iris[["Species"]])
idx
```

```
idx <- as.integer(iris[["Species"]])
c(24, 21, 25)[idx]
```

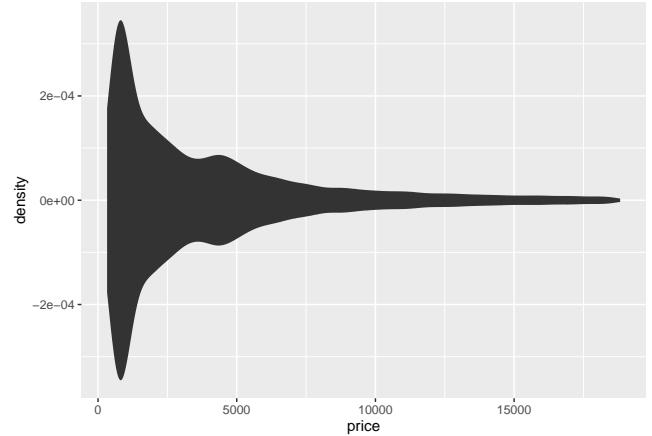
## 30.2 statistic

<https://bookdown.org/xiangyun/msg/system.html#section-14>

```
library(ggplot2)

ggplot(diamonds, aes(x = price)) +
 stat_density(aes(ymax = ..density.., ymin =
 ↵ -..density..),
 geom = "ribbon", position = "identity"
)
```

```
Warning: The dot-dot notation (`..density..`) was
i Please use `after_stat(density)` instead.
This warning is displayed once every 8 hours.
Call `lifecycle::last_lifecycle_warnings()` to see
generated.
```



smoothed density estimates [https://ggplot2.tidyverse.org/reference/geom\\_density.html](https://ggplot2.tidyverse.org/reference/geom_density.html)

Computes and draws kernel density estimate, which is a smoothed version of the histogram. This is a useful alternative to the histogram for continuous data that comes from an underlying smooth distribution.

`geom_density`

stat\_density

[https://ggplot2.tidyverse.org/reference/geom\\_density.html#ref-examples](https://ggplot2.tidyverse.org/reference/geom_density.html#ref-examples)

### 30.3 scale

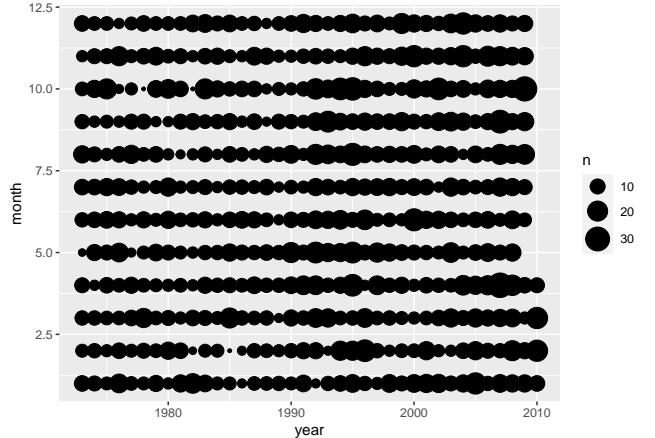
ggplot2

<https://bookdown.org/xiangyun/msg/system.html#section-15>

```
Warning: package 'ggplot2' was built under R version 3.6.3
Warning: The dot-dot notation (`..n..`) was deprecated in lifecycle v0.2.0.
Please use `after_stat(n)` instead.
This warning is displayed once every 8 hours.
Call `lifecycle::last_lifecycle_warnings()` to see all.
generated.
```

```
library(ggplot2)

data(quake6, package = "MSG")
ggplot(quake6, aes(x = year, y = month)) +
 stat_sum(aes(size = ..n..)) +
 scale_size(range = c(1, 8))
```



count overlapping points [https://ggplot2.tidyverse.org/reference/geom\\_count.html?q=stat\\_sum#ref-usage](https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-usage)

This is a variant `geom_point()` that counts the number of observations at each location, then maps the count to point area. It is useful when you have discrete data and overplotting.

`geom_count`

`stat_sum`

[https://ggplot2.tidyverse.org/reference/geom\\_count.html?q=stat\\_sum#ref-examples](https://ggplot2.tidyverse.org/reference/geom_count.html?q=stat_sum#ref-examples)

scales for area or radius [https://ggplot2.tidyverse.org/reference/scale\\_size.html?q=scale\\_size#null](https://ggplot2.tidyverse.org/reference/scale_size.html?q=scale_size#null)

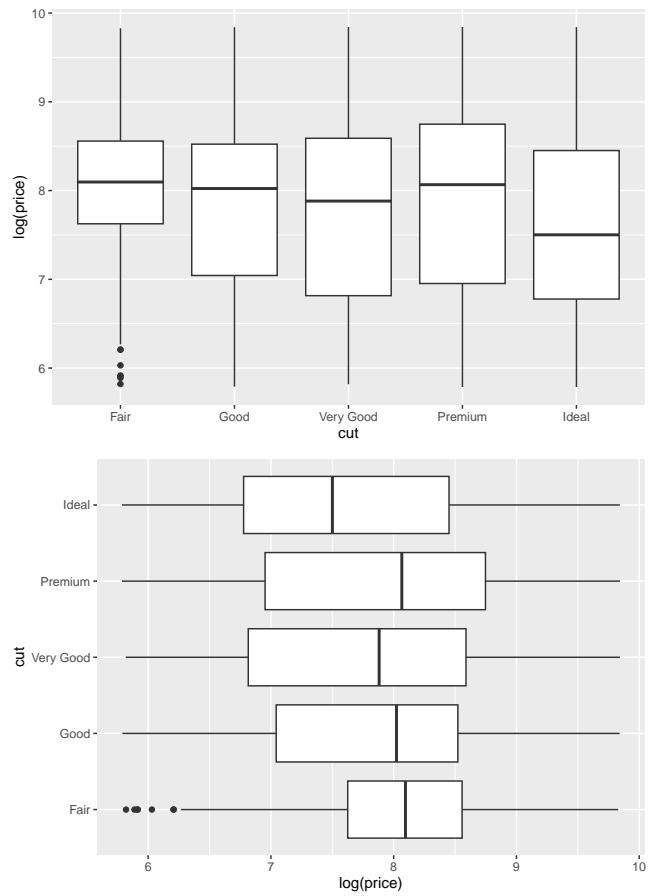
`scale_size`

## 30.4 coordinate system

<https://bookdown.org/xiangyun/msg/system.html#section-16>

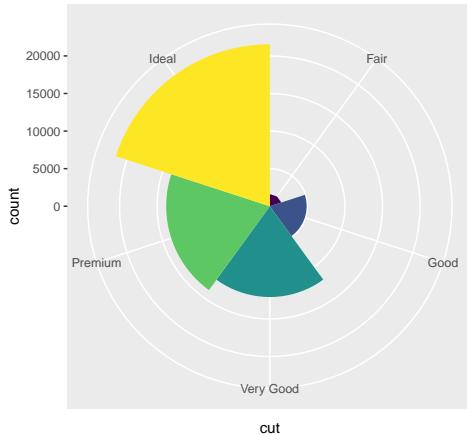
```
library(ggplot2)

p <- ggplot(aes(x = cut, y = log(price)), data
 ~ diamonds) +
 geom_boxplot()
p
p + coord_flip()
```



```
library(ggplot2)

ggplot(aes(x = cut, fill = cut), data =
 ~ diamonds) +
 coord_polar() +
 geom_bar(width = 1, show.legend = FALSE)
```

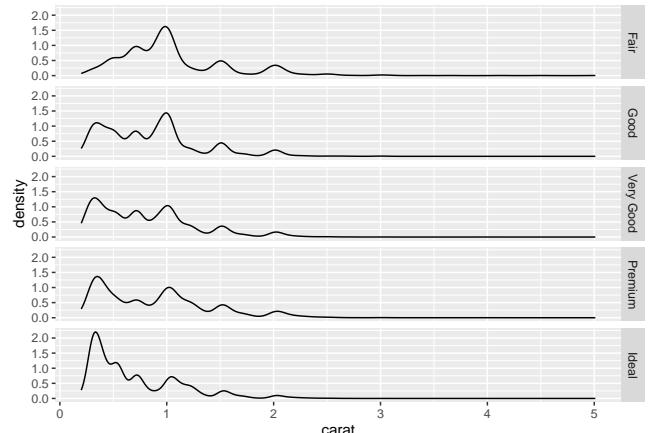


## 30.5 facet

<https://bookdown.org/xiangyun/msg/system.html#subsec:facet>

```
library(ggplot2)

ggplot(aes(x = carat), data = diamonds) +
 geom_density() +
 facet_grid(cut ~ .)
```

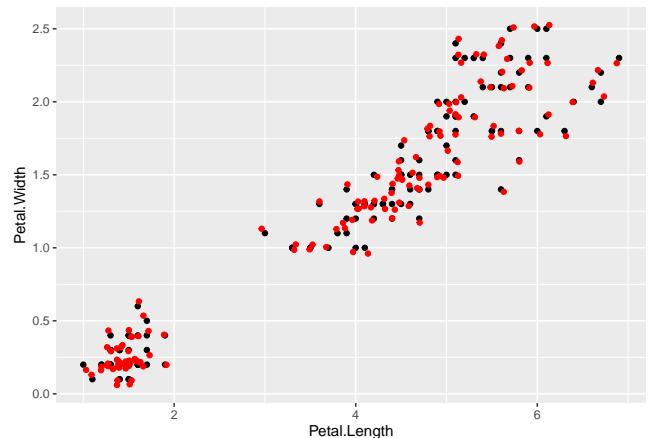


## 30.6 jitter

<https://bookdown.org/xiangyun/msg/system.html#section-17>

```
library(ggplot2)

ggplot(aes(x = Petal.Length, y = Petal.Width),
 data = iris) +
 geom_point() +
 geom_jitter(color = "red")
```



## 30.7 font

<https://bookdown.org/xiangyun/msg/system.html#subsec:font>

# Chapter 31

## notational system for design

<sup>9</sup> p.51

NSD = notational system for design

5

### 31.1 graphic notation

<sup>9</sup> p.51

- $X$ : treatment or exposure to an agent or an event of interest
- $P$ : **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$ : **observation** or process of measurement
- $R$ : **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $g$ : **groups**
  - $k$ : **kinds** of treatments, exposures, or placebos
  - $t$ : **time** or sequential order

<https://tex.stackexchange.com/questions/591882/citation-within-a-latex-figure-caption-in-rmarkdown>

```
(ref:rudolph) *nice* cite: [@Lam94].
(ref:campbell1963) *nice* cite: [@campbell1963].
(ref:campbell1963) ([@campbell1963]
(ref:campbell1963) \ [@campbell1963]
```

5 5

### 31.2 pre-experimental design

<sup>5</sup> p.6

#### 31.2.1 one-shot case study

$X \ O$

#### 31.2.2 one-group pretest-posttest design

$O \ X \ O$

paired  $t$  test

<sup>9</sup> p.62

|       |     |       |
|-------|-----|-------|
| $O$   | $X$ | $O$   |
| $O_t$ | $X$ | $O_t$ |
| $O_0$ | $X$ | $O_1$ |

|                                                                                              | Sources of Invalidity |          |         |                 |            |           |           |                                             |                              |                                |                       |                         |
|----------------------------------------------------------------------------------------------|-----------------------|----------|---------|-----------------|------------|-----------|-----------|---------------------------------------------|------------------------------|--------------------------------|-----------------------|-------------------------|
|                                                                                              | Internal              |          |         | External        |            |           |           |                                             |                              |                                |                       |                         |
|                                                                                              | History               | Maturat. | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturat., etc. | Interaction of Testing and X | Interaction of Selection and X | Reactive Arrangements | Multiple-X Interference |
| <i>Pre-Experimental Designs:</i>                                                             |                       |          |         |                 |            |           |           |                                             |                              |                                |                       |                         |
| 1. One-Shot Case Study<br><i>X O</i>                                                         | —                     | —        |         |                 | —          | —         |           |                                             | —                            |                                |                       |                         |
| 2. One-Group Pretest-Posttest Design<br><i>O X O</i>                                         | —                     | —        | —       | —               | ?          | +         | +         | —                                           | —                            | —                              | ?                     |                         |
| 3. Static-Group Comparison<br><i>X O</i><br><i>— O</i>                                       | +                     | ?        | +       | +               | +          | +         | —         | —                                           | —                            |                                |                       |                         |
| <i>True Experimental Designs:</i>                                                            |                       |          |         |                 |            |           |           |                                             |                              |                                |                       |                         |
| 4. Pretest-Posttest Control Group Design<br><i>R O X O</i><br><i>R O O</i>                   | +                     | +        | +       | +               | +          | +         | +         | +                                           | —                            | ?                              | ?                     |                         |
| 5. Solomon Four-Group Design<br><i>R O X O</i><br><i>R O O</i><br><i>R X O</i><br><i>R O</i> | +                     | +        | +       | +               | +          | +         | +         | +                                           | +                            | ?                              | ?                     |                         |
| 6. Posttest-Only Control Group Design<br><i>R X O</i><br><i>R O</i>                          | +                     | +        | +       | +               | +          | +         | +         | +                                           | +                            | ?                              | ?                     |                         |

Figure 31.1: pre- and true experimental designs ( <sup>5</sup> p.8)

$$\begin{array}{ccc} O & X & O \\ O_{gt} & X_g & O_{gt} \\ O_{10} & X & O_{11} \end{array}$$

### 31.2.3 static-group comparison

$$\begin{array}{cc} X & O \\ X_g & O_{gt} \\ X & O_{11} \\ & O_{21} \end{array}$$


---

$$\begin{array}{cc} X & O \\ X_g & O_{gt} \\ X & O_{11} \\ & O_{01} \end{array}$$

## 31.3 true experimental design

<sup>5</sup> p.13

### 31.3.1 posttest-only control group design

basic experimental design

two-sample *t* test

<sup>9</sup> p.53

$$\begin{array}{ccc} R & X & O \\ R & & O \end{array}$$

|                                                                                                                                                                                                 | Sources of Invalidity |            |         |                 |            |           |           |                                               |          |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|------------|---------|-----------------|------------|-----------|-----------|-----------------------------------------------|----------|
|                                                                                                                                                                                                 | History               | Maturation | Testing | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. |          |
|                                                                                                                                                                                                 |                       |            |         |                 |            |           |           |                                               | External |
| <i>Quasi-Experimental Designs:</i>                                                                                                                                                              |                       |            |         |                 |            |           |           |                                               |          |
| 7. Time Series<br>$O \ O \ O \ O X O \ O \ O \ O$                                                                                                                                               | -                     | +          | +       | ?               | +          | +         | +         | +                                             | - ? ?    |
| 8. Equivalent Time Samples Design<br>$X_1O \ X_2O \ X_1O \ X_2O, \text{ etc.}$                                                                                                                  | +                     | +          | +       | +               | +          | +         | +         | +                                             | - ? -    |
| 9. Equivalent Materials Samples Design<br>$M_aX_1O \ M_bX_2O \ M_cX_1O \ M_dX_2O, \text{ etc.}$                                                                                                 | +                     | +          | +       | +               | +          | +         | +         | +                                             | - ? ? -  |
| 10. Nonequivalent Control Group Design<br>$\begin{array}{c} O \quad X \quad O \\ \hline O \quad O \end{array}$                                                                                  | +                     | +          | +       | +               | ?          | +         | +         | -                                             | - ? ?    |
| 11. Counterbalanced Designs<br>$\begin{array}{cccc} X_1O & X_2O & X_1O & X_2O \\ \hline X_2O & X_1O & X_2O & X_1O \\ X_3O & X_4O & X_3O & X_4O \\ \hline X_4O & X_3O & X_4O & X_3O \end{array}$ | +                     | +          | +       | +               | +          | +         | +         | -                                             |          |
| 12. Separate-Sample Pretest-Posttest Design<br>$\begin{array}{c} R \ O \ (X) \\ R \quad X \ O \end{array}$                                                                                      | -                     | -          | +       | ?               | +          | +         | -         | -                                             | + + +    |
| 12a. $\begin{array}{c} R \ O \ (X) \\ R \quad X \ O \end{array}$                                                                                                                                | +                     | -          | +       | ?               | +          | +         | -         | +                                             | + + +    |
| 12b. $\begin{array}{c} R \ O_1 \quad O_2 \ (X) \\ R \quad O_2 \ (X) \quad O_1 \end{array}$                                                                                                      | -                     | +          | +       | ?               | +          | +         | -         | ?                                             | + + +    |
| 12c. $\begin{array}{c} R \ O_1 \quad X \quad O_2 \quad O_3 \\ R \quad X \quad O_3 \quad O_2 \end{array}$                                                                                        | -                     | -          | +       | ?               | +          | +         | +         | -                                             | + + +    |

Figure 31.2: quasi-experimental designs ( <sup>5</sup> p.40)

|                                                                                                                                                                                                                                  | Sources of Invalidity |          |          |                 |            |           |           |                                               |  |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|----------|----------|-----------------|------------|-----------|-----------|-----------------------------------------------|--|
|                                                                                                                                                                                                                                  | History               | Maturity | Testing  | Instrumentation | Regression | Selection | Mortality | Interaction of Selection and Maturation, etc. |  |
|                                                                                                                                                                                                                                  | Internal              |          | External |                 |            |           |           |                                               |  |
| <i>Quasi-Experimental Designs</i>                                                                                                                                                                                                |                       |          |          |                 |            |           |           |                                               |  |
| <i>Continued:</i>                                                                                                                                                                                                                |                       |          |          |                 |            |           |           |                                               |  |
| 13. Separate-Sample<br>Pretest-Posttest<br>Control Group<br>Design                                                                                                                                                               | + + + + + + + -       |          | + + +    |                 |            |           |           |                                               |  |
| $R \quad O \quad (X)$<br>$R \quad X \quad O$<br>$\hline R \quad O$<br>$R \quad O$                                                                                                                                                |                       |          |          |                 |            |           |           |                                               |  |
| 13a. $\begin{cases} R \quad O \quad (X) \\ R \quad X \quad O \\ \hline R \quad O \quad (X) \\ R \quad X \quad O \\ \hline R \quad O \\ R \quad O \\ \hline R' \quad O \\ R \quad O \\ \hline R \quad O \\ R \quad O \end{cases}$ | + + + + + + + +       |          | + + +    |                 |            |           |           |                                               |  |
| 14. Multiple Time-Series                                                                                                                                                                                                         | + + + + + + + +       |          | - - ?    |                 |            |           |           |                                               |  |
| $\begin{matrix} O & O & O & X & O & O & O \\ \hline O & O & O & O & O & O & O \end{matrix}$                                                                                                                                      |                       |          |          |                 |            |           |           |                                               |  |
| 15. Institutional Cycle<br>Design                                                                                                                                                                                                |                       |          |          |                 |            |           |           |                                               |  |
| Class A X $O_1$                                                                                                                                                                                                                  |                       |          |          |                 |            |           |           |                                               |  |
| Class B <sub>1</sub> R $O_2$ X $O_3$                                                                                                                                                                                             |                       |          |          |                 |            |           |           |                                               |  |
| Class B <sub>2</sub> R X $O_4$                                                                                                                                                                                                   |                       |          |          |                 |            |           |           |                                               |  |
| Class C $O_5$ X                                                                                                                                                                                                                  |                       |          |          |                 |            |           |           |                                               |  |
| $O_2 < O_1$                                                                                                                                                                                                                      | + - + + ? - ?         |          | + ? +    |                 |            |           |           |                                               |  |
| $O_5 < O_4$                                                                                                                                                                                                                      |                       |          |          |                 |            |           |           |                                               |  |
| $O_2 < O_3$                                                                                                                                                                                                                      | - - - ? ? + +         |          | - ? +    |                 |            |           |           |                                               |  |
| $O_2 < O_4$                                                                                                                                                                                                                      |                       |          |          |                 |            |           |           |                                               |  |
| $O_6 = O_7$                                                                                                                                                                                                                      | -                     |          |          |                 |            |           |           |                                               |  |
| $O_{2y} = O_{2o}$                                                                                                                                                                                                                | + -                   |          |          |                 |            |           |           |                                               |  |
| 16. Regression<br>Discontinuity                                                                                                                                                                                                  | + + + ? + + ? +       |          | + - + +  |                 |            |           |           |                                               |  |

• General Population Controls for Class B, etc.

Figure 31.3: quasi-experimental designs continued ( <sup>5</sup> p.56)

or, with a placebo or an active control,

$$\begin{array}{ccc} R & X & O \\ R & P & O \end{array}$$


---

$$R \quad X_g \quad O_{gt}$$


---

$$\begin{array}{lll} R & X_g = X_1 = X & O_{gt} = O_{11} \\ R & X_g = X_2 = \emptyset & O_{gt} = O_{21} \end{array}$$

or, with a placebo or an active control

$$\begin{array}{lll} R & X_g = X_1 = X & O_{gt} = O_{11} \\ R & X_g = X_2 = P & O_{gt} = O_{21} \end{array}$$


---

$$\begin{array}{ccc} R & X & O_{11} \\ R & & O_{21} \end{array}$$

or, with a placebo or an active control

$$\begin{array}{ccc} R & X & O_{11} \\ R & P & O_{21} \end{array}$$


---

$$\begin{array}{ccc} R & X_g & O_{gt} \\ R & X & O_{11} \\ R & P & O_{21} \end{array}$$


---

$$\begin{array}{cccc} R & X & O \\ R & X_g & O_{gt} \\ R & X & O_{11} \\ R & P & O_{21} \end{array}$$

### 31.3.2 pretest-posttest control group design

$$\begin{array}{cccc} R & O & X & O \\ R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \end{array}$$

### 31.3.3 Solomon four-group design

<sup>9</sup> p.52

Solomon 4-group design = pretest-posttest + posttest-only control group design

$$\begin{array}{ccccc} R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \\ R & & X & O_{31} \\ R & & & O_{41} \end{array}$$


---

$$\begin{array}{cccc} R & O & X & O \\ R & O_{gt} & X_g & O_{gt} \\ R & O_{10} & X & O_{11} \\ R & O_{20} & & O_{21} \\ R & & X & O_{31} \\ R & & & O_{41} \end{array}$$

## 31.4 quasi-experimental design

<sup>5</sup> p.34

## 31.5 correlational and ex post facto designs

<sup>5</sup> p.64

## 31.6 graphic notation, advanced

<sup>9</sup> p.74

- $X$ : treatment or exposure to an agent or an event of interest
- $P$ : **placebo**, i.e. blank treatment or exposure, or standard treatment, or exposure as an active control
- $O$ : **observation** or process of measurement
- $R$ : **randomization**, i.e. random assignment of research subjects to separate treatment or exposure groups
- subscript
  - $g$ : **groups**
  - $k$ : **kinds** of treatments, exposures, or placebos
  - $t$ : **time** or sequential order
- $V$ : **variable(s)**
  - $B(V)$ : **blocking** by the variable(s)
  - $M(V)$ : **matching** by the variable(s)
  - $S(V)$ : **stratifying** by the variable(s)
  - $L(V/L)$ : **limiting** to the level(s) of the variable(s)
- $M^*$ : research **material(s)** selected
- -: cohort

# Chapter 32

## design of experiment

experimental design = experiment design = design of experiments = DoE

<sup>9</sup> p.72

question-design-analysis loop

### 32.1 notational system for design<sup>[31]</sup>

graphic notation, advanced<sup>[31.6]</sup>

### 32.2 terminology

- population
  - sample
    - \* subsample
- unit
  - experimental unit
    - \* response
    - \* block: group of similar experimental unit (<sup>10</sup> p.74)
  - observational unit / measurement unit <sup>1</sup>
- replication (<sup>9</sup> p.76): an independent observation of the treatment (<sup>10</sup> p.74)
  - treatment replication: experimental-unit-to-experimental-unit variation
  - measurement replication = subsample: measurement-to-measurement variation
- replicate
  - experimental replicate
  - biological replicate
  - technical replicate

---

<sup>10</sup> p.73

$Y_{ij}$ : the response observed from the  $j^{\text{th}}$  experimental unit assigned to the  $i^{\text{th}}$  treatment

$\mu_i$ : the mean response to the  $i^{\text{th}}$  treatment

$\mathcal{E}_{ij}$ : the noise from other possible natural variation or nonrandom and random error

$$Y_{ij} = \mu_i + \mathcal{E}_{ij}, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ treatments} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ experimental units per treatment} \end{cases}$$

Each treatment has  $n_j$  experimental units, so there are totally  $n_i n_j$  experimental units.

If experimental units cannot be homogeneous, we can try to

- stratify them
- group them, and measure group to group variation
- block them

---

<sup>1</sup><https://passel2.unl.edu/view/lesson/2e09f0055f13/6>

here  $n_j$  blocks each with  $n_i$  experimental units where **each treatment occurs once in each block**

$$\begin{aligned} Y_{ij} &= \mu_i + \varepsilon_{ij} \\ &= \mu_i + b_j + \varepsilon_{ij}^*, \begin{cases} i \in \mathbb{N} \cap [1, n_i] & \mathbb{N} \ni n_i \text{ experimental units per block} \\ j \in \mathbb{N} \cap [1, n_j] & \mathbb{N} \ni n_j \text{ blocks} \end{cases} \end{aligned}$$

where

$$\varepsilon_{ij} = b_j + \varepsilon_{ij}^*$$

i.e. the variation between groups or blocks of experimental units has been identified and isolated from  $\varepsilon_{ij}^*$ , which represents the variability of experimental units within a block. By isolating the block effect from the experimental units, the within-block variation can be used to compare treatment effects, which involves computing the estimated standard errors of contrasts of the treatments.

$$\begin{aligned} Y_{ij} - Y_{i'j} &= (\mu_i + b_j + \varepsilon_{ij}^*) \\ &\quad - (\mu_{i'} + b_j + \varepsilon_{i'j}^*) \\ &= (\mu_i - \mu_{i'}) + (\varepsilon_{ij}^* - \varepsilon_{i'j}^*) \end{aligned}$$

which does not depend on the block effect  $b_j$  or free of block effects. The result of this difference is that the variance of the difference of two treatment responses within a block depends on the within-block variation among the experimental units and not the between-block variation.

### 32.2.1 replication vs. subsample

It is very important to distinguish between a **subsample** and a **replication** since the error variance estimated from between subsamples is in general considerably smaller than the error variance estimated from replications or between experimental units. (10 p.77)

<https://www.researchgate.net/post/What-is-Experimental-Unit-Replicate-Total-sample-size-treatment-size>

### 32.2.2 replication vs. repeated measurements

#### 32.2.3 replication, replicate

##### 32.2.3.1 technical replicate, biological replicate

[https://www.youtube.com/watch?v=c\\_cpl5YsBV8](https://www.youtube.com/watch?v=c_cpl5YsBV8)

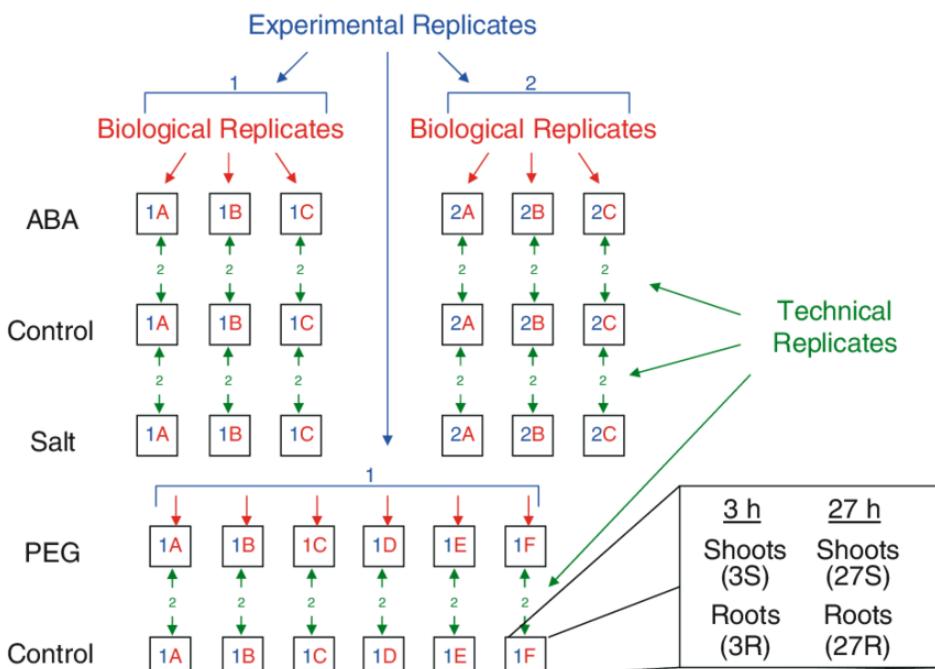


Figure 32.1: experimental, biological, technical replicates (11)

### 32.2.4 Latin square design

LSD = Latin square design

<sup>6</sup> p.505~507

<https://tex.stackexchange.com/questions/501671/how-to-get-math-mode-curly-braces-in-tikz>

```
\usepackage{pgfplots} in engine.opts=list(extra.preamble=c("\usepackage{pgfplots}"))
\usetikzlibrary{decorations}
```

|                         |   |   |   |
|-------------------------|---|---|---|
| $p = 4 \text{ columns}$ |   |   |   |
| $p = 4 \text{ rows}$    | A | B | C |
|                         | B | C | D |
|                         | C | D | A |
|                         | D | A | B |

$p = |\{A, B, C, D\}| = 4 \text{ treatments}$

Figure 32.2: Latin square example

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases}$$

$$\varepsilon_{ijk} \stackrel{\text{i.i.d.}}{\sim} \text{n}(0, \sigma^2)$$

$\rho_i$ :  $i^{\text{th}}$  row

$\kappa_j$ :  $j^{\text{th}}$  column

$\tau_k$ :  $k^{\text{th}}$  treatment

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \end{cases}$$

$$= \mu + \rho_i + \kappa_j + \tau_k + \varepsilon_{ijk}, \begin{cases} i \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ rows} \\ j \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ columns} \\ k \in \mathbb{N} \cap [1, p] & \mathbb{N} \ni p \text{ treatments} \end{cases}$$

### 32.2.5 model assumption and experimental unit, measurement/observational unit

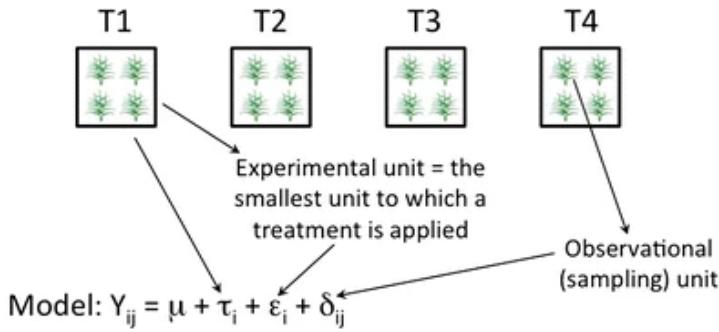
$$Y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ij} + \Delta_{ijk}$$

## 32.3 experiment structure

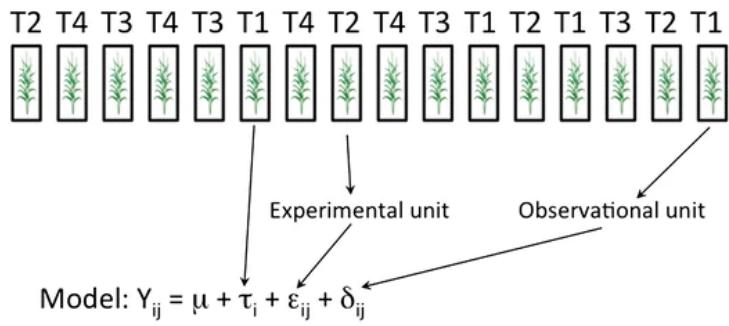
### 32.3.1 treatment structure

<sup>10</sup> p.77

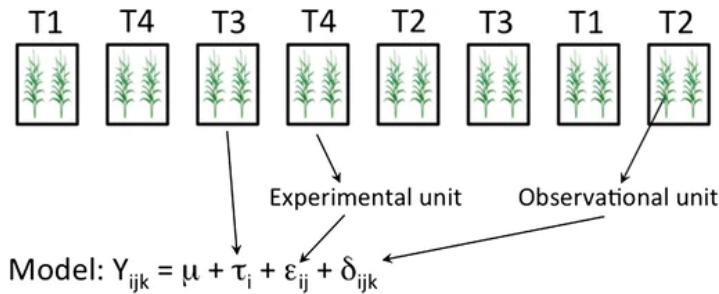
- 1-way treatment structure
- 2-way treatment structure
- factorial arrangement treatment structure
- *fractional* factorial arrangement treatment structure



| ANOVA Source of variation       | df        |
|---------------------------------|-----------|
| Treatments + Experimental error | 3 (fixed) |
| Observational error             | 12        |

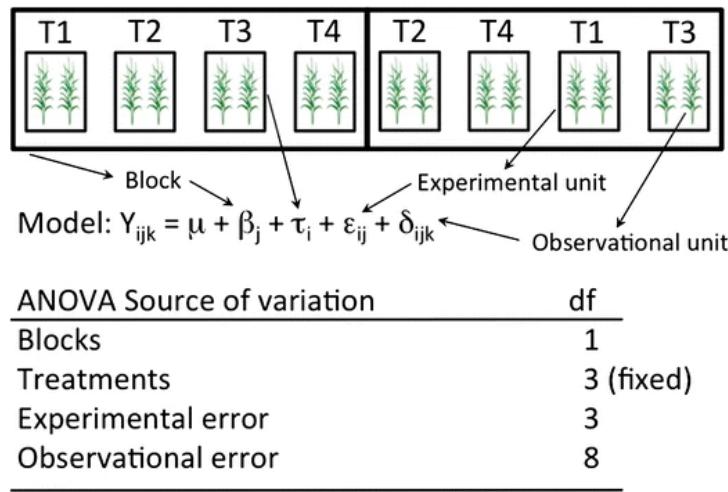
Figure 32.3: model assumption and experimental unit 1 ( [12](#) fig.1)

| ANOVA Source of variation            | df        |
|--------------------------------------|-----------|
| Treatments                           | 3 (fixed) |
| Error (experimental + observational) | 12        |

Figure 32.4: model assumption and experimental unit 2 ( [12](#) fig.2)

| ANOVA Source of variation | df        |
|---------------------------|-----------|
| Treatments                | 3 (fixed) |
| Experimental error        | 4         |
| Observational error       | 8         |

Figure 32.5: model assumption and experimental unit 3 ( [12](#) fig.3)

Figure 32.6: model assumption and experimental unit 4 ( <sup>12</sup> fig.4)

- factorial arrangement with one or more controls

### 32.3.2 design structure

<sup>10</sup> p.77

- CRD = completely randomized design
- RCBD = randomized complete block design
  - ? why not called CRBD = completely randomized block design
- LSD = Latin square design<sup>[32.2.4]</sup>
- IBD = incomplete block design
  - BIBD = balanced IBD
- various combinations and generalizations

### 32.3.3 size of experimental unit

- split-plot design
  - split-split-plot design
  - split-split-split-plot design
- repeated measures design
  - cross-over design
  - change-over design
- nested design = hierarchical design
- variations and combinations
  - SSEU = several sizes of experimental units

## 32.4 approach to experimentation

<sup>9</sup> p.75

- approach to experimentation
  - best-guess approach
  - one-factor-at-a-time approach = OFAT
  - factorial approach

## 32.5 sample size estimation

## 32.6 statistical analysis plan

## 32.7 protocol

<sup>9</sup> p.95

- study objective

- study endpoint
  - primary endpoint
  - secondary endpoint(s)
- experimental unit(s)
- treatment structure<sup>[32.3.1]</sup>
- design structure<sup>[32.3.2]</sup>
- potential confounder(s)
- randomization
- blinding
- chance reduction
- sample size estimation<sup>[32.5]</sup>
- data collection
- data management system
- statistical analysis plan<sup>[32.6]</sup>
- DSMB / DSMC = data and safety monitoring board / committee

## 32.8 DoE course with six sigma and Minitab

<https://zhuanlan.zhihu.com/p/265914617>

<https://www.zhihu.com/question/416312693/answer/1426399810>

### 32.8.1 evolution

- Fisher
- Rao

# Chapter 33

## quine

<https://www.youtube.com/watch?v=vdZr-O3nTPY>

<https://www.bilibili.com/video/BV1Wu4m1u7SF>

```
s = 's = %r\nprint(s%%s)'
print(s%s)

s = 's = %r\nprint(s%%s)'
print(s%s)
```

This snippet is a clever example of a quine. A quine is a computer program that takes no input and produces a copy of its own source code as its output. The given code in Python is written to print its own source when executed. Let's break it down:

`s = 's = %r\nprint(s%%s)':` This line defines a string `s` that contains a format string. `%r` is a placeholder that gets replaced with the `repr()` of the argument provided to the `%` operator, which in this case will be the string `s` itself. This means it will insert the string representation of `s` into the format string at `%r`.

`print(s%s):` This line prints the result of `s%s`. Here, the `%` operator is used to format the string `s` with itself. The `%s` inside the `print` statement is replaced by the string `s`, leading to the entire string being printed out, including the `print` statement itself.

This is because the format operation replaces `%r` with the representation of the string `s`, and `%%` is a way to escape the `%` sign in format strings, resulting in a single `%` in the output. This output is exactly the same as the source code, making it a quine.

### 33.1 %r

The `%r` in Python string formatting represents the “representation” of a value, which is typically the way you would see it if you were to type it into a Python interpreter. It uses the `repr()` function to convert the value to a string. This is useful for debugging, among other things, because it shows strings with quotes around them and escapes special characters. Essentially, `%r` gives you the “developer’s view” of what a variable looks like.

Here's a simple example to illustrate `%r` versus `%s` in string formatting:

```
my_str = "Hello, World!\nNew line character is represented with \\n"
print("Using %s: %s" % my_str)
```

```
Using %s: Hello, World!
New line character is represented with \n
print("Using %r: %r" % my_str)
```

```
Using %r: 'Hello, World!\nNew line character is represented with \\n'
```

In this example:

The `%s` specifier tells Python to convert the object using `str()`, which is designed to be readable and outputs the string `"Hello, World!\nNew line character is represented with \n"`, interpreting the escape character `\n` as a newline.

The `%r` specifier tells Python to convert the object using `repr()`, which aims to generate output that could be used to recreate the object, outputting the string `'Hello, World!\nNew line character is represented with \\n'`, preserving the actual escape characters in the output.

Notice how `%r` preserves the string exactly as it is, including the quotes and escaped characters, making it clear it's a string and showing the escape sequence explicitly.

# Chapter 34

## quaternion

### 34.1 TaylorCatAlice

<https://www.bilibili.com/video/BV1PV411P7w4>

[https://en.wikipedia.org/wiki/Blackboard\\_bold](https://en.wikipedia.org/wiki/Blackboard_bold)

#### 34.1.1 complex / dionion / bionion

$$\begin{aligned} c &= a + bi = a + ib, \begin{cases} c \in \mathbb{C} \\ i^2 = -1 \\ a, b \in \mathbb{R} \Leftrightarrow \langle a, b \rangle \in \mathbb{R}^2 \end{cases} \\ z &= x + yi = x + iy, \begin{cases} z \in \mathbb{C} \\ i^2 = -1 \\ x, y \in \mathbb{R} \Leftrightarrow \langle x, y \rangle \in \mathbb{R}^2 \end{cases} \\ &= \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}} i \right) = r (\cos \theta + i \sin \theta) = re^{i\theta} \end{aligned}$$

Also see complex group representation<sup>[37.7]</sup>.

#### 34.1.2 trionion / triernion / triplex / ternion

<https://zh.wikipedia.org/zh-tw/%E4%B8%89%E5%85%83%E6%95%B8>

<https://math.stackexchange.com/questions/1784166/why-are-there-no-triernions-3-dimensional-analogue-of-complex-numbers-quate>

<https://math.stackexchange.com/questions/32100/is-there-a-third-dimension-of-numbers/4453131>

$$\begin{aligned} t &= a + bi + cj = a + ib + jc, \begin{cases} t \in \mathbb{T} \\ i^2 = -1 \\ j^2 = -1 \end{cases} \\ w &= x + yi + zj = x + iy + jz, \begin{cases} w \in \mathbb{T} \\ i^2 = -1 \\ j^2 = -1 \end{cases} \\ &= \sqrt{x^2 + y^2 + z^2} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} i + \frac{z}{\sqrt{x^2 + y^2 + z^2}} j \right) = ? \end{aligned}$$

---

$$\begin{cases} A(BC) = (AB)C & (a) \text{ associativity} \\ A(B+C) = AB + AC & (d) \text{ distributivity} \end{cases}$$

$$\begin{aligned} \mathbb{T} &\ni ij = X + Yi + Zj \in \mathbb{T} \\ -j &= (i^2) j \stackrel{(a)}{=} i(ij) = i(X + Yi + Zj) \stackrel{(d)}{=} -Y + Xi + Zij \\ ij &= \frac{Y}{Z} - \frac{X}{Z}i - \frac{1}{Z}j \Rightarrow \begin{cases} X = \frac{Y}{Z} \\ Y = -\frac{X}{Z} \\ Z = -\frac{1}{Z} \end{cases} \Rightarrow Z^2 = -1 \Rightarrow Z \notin \mathbb{R} \\ -i &= i(j^2) \stackrel{(a)}{=} (ij)j = (X + Yi + Zj)j \stackrel{(d)}{=} -Z + Xj + Yij \\ ij &= \frac{Z}{Y} - \frac{1}{Y}i - \frac{X}{Y}j \Rightarrow \begin{cases} X = \frac{Z}{Y} \\ Y = -\frac{1}{Y} \\ Z = -\frac{X}{Y} \end{cases} \Rightarrow Y^2 = -1 \Rightarrow Y \notin \mathbb{R} \end{aligned}$$

### 34.1.3 quaternion

<https://en.wikipedia.org/wiki/Quaternion>

$$\begin{aligned} q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ &= w = t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\ &=? \\ q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ &= w = t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\ &= t + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (e_1 \ e_2 \ e_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + x, \begin{cases} e_1 = i = i \\ e_2 = j = j \\ e_3 = k = k \end{cases} \\ &= t + \frac{ix + jy + kz}{r} r, \begin{cases} r^2 = x^2 + y^2 + z^2 \\ \|n\|^2 = \left( \frac{ix + jy + kz}{r} \right)^2 = -1 \end{cases} \Rightarrow |q|^2 = t^2 + r^2 \\ &= \sqrt{t^2 + r^2} \left( \frac{t}{\sqrt{t^2 + r^2}} + n \frac{r}{\sqrt{t^2 + r^2}} \right) = |q| \left( \cos \frac{\theta}{2} + n \sin \frac{\theta}{2} \right) = |q| e^{n \frac{\theta}{2}} \end{aligned}$$


---

The quaternion set is denoted  $\mathbb{H}$  for Sir R.W. **Hamilton**, because he suddenly and strikingly realized

$$\begin{cases} ij = k \\ k \in \mathbb{H} \end{cases} \Rightarrow ij \in \mathbb{H} \text{ for closure property}$$


---

for the sake of rigorosity, see **group theory**<sup>[37]</sup>

$$k^2 = -1$$

$$\begin{aligned} ij &= k \\ ijk &= i(jk) \stackrel{(a)}{=} (ij)k = kk = k^2 = -1 \\ kij &= (ki)j \stackrel{(a)}{=} k(ij) = kk = k^2 = -1 \end{aligned}$$

$$\begin{aligned}
& ij = k \\
& -j = (i^2) j \stackrel{(a)}{=} i(ij) = ik \\
& -i = i(j^2) \stackrel{(a)}{=} (ij)j = kj \\
\\
& -j = (i^2) j \stackrel{(a)}{=} i(ij) = ik \\
& 1 = -j^2 = j(-j) = j(ik) \stackrel{(a)}{=} (ji)k \\
& k = [1]k = [(ji)k]k \stackrel{(a)}{=} (ji)(k^2) = (ji)(-1) \\
& -k = ji \\
& -i = i(j^2) \stackrel{(a)}{=} (ij)j = kj \\
& 1 = (-i)i = (kj)i \stackrel{(a)}{=} k(ji) = kji \\
& 1 = kji
\end{aligned}$$

There is no more **commutativity**<sup>[34.1.3.2.1]</sup>, i.e.

$$AB \not\equiv BA$$

but

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

satisfying **anticommutativity**<sup>[34.1.3.2.2]</sup>.

$$\begin{cases} ij = k \\ ji = -k \end{cases} \Leftrightarrow ji = -k = -ij \\
\Rightarrow ji = -ij \\
\Leftrightarrow ij + ji = 0$$

$$\begin{cases} kij = -1 \\ kji = 1 \end{cases} \Leftrightarrow kij = -1 = -kji \\
\Rightarrow kij = -kji \\
\Leftrightarrow kij + kji = 0$$


---

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \quad \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\
&= w = t + xi + yj + zk = t + ix + jy + kz, \quad \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\
&= t + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (\mathbf{i} \ \mathbf{j} \ \mathbf{k}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + \mathbf{x}, \quad \begin{cases} e_1 = \mathbf{i} = i \\ e_2 = \mathbf{j} = j \\ e_3 = \mathbf{k} = k \end{cases}
\end{aligned}$$

| . | 1 | i  | j  | k  |
|---|---|----|----|----|
| 1 | 1 | i  | j  | k  |
| i | i | -1 | k  | -j |
| j | j | -k | -1 | i  |
| k | k | j  | -i | -1 |

| .  | 1 | i  | j  | k  | -1 | -i | -j | -k |
|----|---|----|----|----|----|----|----|----|
| 1  | 1 | i  | j  | k  |    |    |    |    |
| i  | i | -1 | k  | -j |    |    |    |    |
| j  | j | -k | -1 | i  |    |    |    |    |
| k  | k | j  | -i | -1 |    |    |    |    |
| -1 |   |    |    |    |    |    |    |    |
| -i |   |    |    |    |    |    |    |    |
| -j |   |    |    |    |    |    |    |    |
| -k |   |    |    |    |    |    |    |    |

Figure 34.1: quaternion basis group table

| .  | 1 | -1 | i | -i | j | -j | k | -k |
|----|---|----|---|----|---|----|---|----|
| 1  |   |    |   |    |   |    |   |    |
| -1 |   |    |   |    |   |    |   |    |
| i  |   |    |   |    |   |    |   |    |
| -i |   |    |   |    |   |    |   |    |
| j  |   |    |   |    |   |    |   |    |
| -j |   |    |   |    |   |    |   |    |
| k  |   |    |   |    |   |    |   |    |
| -k |   |    |   |    |   |    |   |    |

Figure 34.2: quaternion basis group table 2

### 34.1.3.1 true origin of ( dot product & cross product ) / ( inner product & outer product )

product of two pure imaginary quaternions

$$\begin{aligned}
 \mathbf{x}_1 \mathbf{x}_2 &= (x_{11}i + x_{12}j + x_{13}k)(x_{21}i + x_{22}j + x_{23}k) \\
 &= x_{11}x_{21}i^2 + x_{11}x_{22}ij + x_{11}x_{23}ik \\
 &\quad + x_{12}x_{21}ji + x_{12}x_{22}j^2 + x_{12}x_{23}jk \\
 &\quad + x_{13}x_{21}ki + x_{13}x_{22}kj + x_{13}x_{23}k^2 \\
 &= -(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}) \\
 &\quad + (x_{12}x_{23} - x_{13}x_{22})jk + (x_{13}x_{21} - x_{11}x_{23})ki + (x_{11}x_{22} - x_{12}x_{21})ij \\
 &= -(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}) \\
 &\quad + (x_{12}x_{23} - x_{13}x_{22})i + (x_{13}x_{21} - x_{11}x_{23})j + (x_{11}x_{22} - x_{12}x_{21})k \\
 &= -(\mathbf{x}_1 \cdot \mathbf{x}_2) + (\mathbf{x}_1 \times \mathbf{x}_2), \begin{cases} \mathbf{x}_1 \cdot \mathbf{x}_2 = x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} \\ \mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} i & j & k \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} \end{cases} \\
 &= -\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{x}_2
 \end{aligned}$$

product of two general quaternions / ordinary quaternions = Grassmann product

$$\begin{aligned}
q_1 q_2 &= (q_{10} + q_{11}\mathbf{i} + q_{12}\mathbf{j} + q_{13}\mathbf{k})(q_{20} + q_{21}\mathbf{i} + q_{22}\mathbf{j} + q_{23}\mathbf{k}) \\
&= (x_{10} + \mathbf{x}_1)(x_{20} + \mathbf{x}_2), \quad \begin{cases} x_{i\mu} = q_{i\mu} & \mu \in \{0\} \cup (\mathbb{N} \cap [1, 3]) \\ \mathbf{x}_i = x_{ij}\mathbf{e}_j & i, j \in \mathbb{N} \cap [1, 3], \end{cases} \quad \begin{cases} \mathbf{e}_1 = \mathbf{i} \\ \mathbf{e}_2 = \mathbf{j} \\ \mathbf{e}_3 = \mathbf{k} \end{cases} \\
&= x_{10}x_{20} + x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_2 \\
&= x_{10}x_{20} + x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 - \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{x}_2 \\
&= (x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2) \\
x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2 &= (q_{10} \quad q_{11} \quad q_{12} \quad q_{13}) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = q_1^\mu \eta_{\mu\nu} q_2^\nu \\
&= (q_{10} \quad q_{11} \quad q_{12} \quad q_{13}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = \mathbf{q}_1^\top H \mathbf{q}_2, H = [\eta_{\mu\nu}]_{4 \times 4} = \eta_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
q_1 q_2 &= (x_{10}x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10}\mathbf{x}_2 + x_{20}\mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2) \\
QP &= (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})
\end{aligned}$$

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

Minkowski metric tensor

$$\eta = H = [\eta_{\mu\nu}]_{4 \times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta_{\mu\nu}$$

and quaternions as 4-vectors or four-vectors

$$\mathbf{q}_1^\top = (q_{10} \quad q_{11} \quad q_{12} \quad q_{13})$$

$$\mathbf{q}_2 = \begin{pmatrix} q_{20} \\ q_{21} \\ q_{22} \\ q_{23} \end{pmatrix}$$

### 34.1.3.2 commutativity vs. anticommutativity

#### 34.1.3.2.1 commutativity 交換律 = 交換性 = 對易性

$$AB = BA \Leftrightarrow AB - BA = 0$$

$$AB = BA \Rightarrow AB \equiv BA$$

#### 34.1.3.2.2 anticommutativity 反交換律 = 反交換性 = 反對易性

$$AB + BA = 0 \Leftrightarrow AB = -BA$$

$$AB = -BA \Rightarrow AB \neq BA$$

### 34.1.3.3 bracket

#### 34.1.3.3.1 self-invented bracket 自創括號 = 自創括

### 34.1.3.3.1.1 commutative bracket 交換括號 = 對易式

$$[X, Y] = \frac{XY - YX}{2}$$

### 34.1.3.3.1.2 anticommutative bracket 反交換括號 = 反對易式

$$\{X, Y\} = \frac{XY + YX}{2}$$

### 34.1.3.3.2 Poisson bracket [https://en.wikipedia.org/wiki/Poisson\\_bracket](https://en.wikipedia.org/wiki/Poisson_bracket)

### 34.1.3.3.3 Lagrange bracket

### 34.1.3.3.4 Lie bracket

### 34.1.3.4 triple product

product = double product = Grassmann product

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b})$$

pure imaginary

$$\begin{aligned} ab &= (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \\ &\stackrel{\begin{cases} a_0 = 0 \\ b_0 = 0 \end{cases}}{=} (00 - \mathbf{a} \cdot \mathbf{b}) + (0\mathbf{b} + 0\mathbf{a} + \mathbf{a} \times \mathbf{b}) \\ &= -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b} \end{aligned}$$

pure imaginary product can get both ( real & imaginary ) / ( scalar & vector ) parts

$$ab = \mathbf{a}\mathbf{b} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}, \begin{cases} a = 0 + \mathbf{a} = \mathbf{a} \\ b = 0 + \mathbf{b} = \mathbf{b} \end{cases}$$


---

triple product

[https://en.wikipedia.org/wiki/Triple\\_product](https://en.wikipedia.org/wiki/Triple_product)

pure imaginary

$$\begin{cases} a = 0 + \mathbf{a} = \mathbf{a} \\ b = 0 + \mathbf{b} = \mathbf{b} \\ c = 0 + \mathbf{c} = \mathbf{c} \end{cases}$$

$$\begin{aligned} abc &= \mathbf{a}\mathbf{b}\mathbf{c} = (\mathbf{a}\mathbf{b})\mathbf{c} = (-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= \mathbf{a}(\mathbf{b}\mathbf{c}) = \mathbf{a}(-\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \times \mathbf{c}) \end{aligned}$$

$$\begin{aligned} abc &= (\mathbf{a}\mathbf{b})\mathbf{c} = (-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \times \mathbf{b})\mathbf{c} \\ &= -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (-(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \\ &= [-(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}] + [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}] \\ &= \mathbf{a}(\mathbf{b}\mathbf{c}) = \mathbf{a}(-\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \times \mathbf{c}) \\ &= -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{b} \times \mathbf{c}) \\ &= -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + (-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times (\mathbf{b} \times \mathbf{c})) \\ &= [-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] + [\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})] \end{aligned}$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

$$\begin{aligned} & \begin{cases} -(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) \end{cases} \\ \Rightarrow & \begin{cases} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) & (s) \\ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) & (v) \end{cases} \end{aligned}$$


---

permutation

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots \\ \sigma(x_1) & \sigma(x_2) & \cdots \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$


---

$$(s) \Rightarrow \begin{cases} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) & \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, s_1 \\ (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) & \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, s_2 \\ (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) & \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, s_3 \\ (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) & \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, s_4 \\ (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) & \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, s_5 \\ (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) & \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, s_6 \end{cases}$$

$$\stackrel{\text{commutative}}{\Rightarrow} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \stackrel{s_1}{=} (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} \stackrel{s_2}{=} (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \stackrel{s_3}{=} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$\stackrel{\text{anticommutative}}{=} -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} \stackrel{s_6}{=} -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} \stackrel{s_5}{=} -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \stackrel{s_4}{=} -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$$

$$\Leftrightarrow \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$= -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$


---

$$\begin{aligned}
 (v) \Rightarrow & \left\{ \begin{array}{l} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) \\ (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) - \mathbf{b} (\mathbf{c} \cdot \mathbf{a}) \\ (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} - (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \\ (\mathbf{b} \times \mathbf{a}) \times \mathbf{c} - (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} = \mathbf{b} \times (\mathbf{a} \times \mathbf{c}) - \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) \\ (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = \mathbf{a} \times (\mathbf{c} \times \mathbf{b}) - \mathbf{a} (\mathbf{c} \cdot \mathbf{b}) \\ (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} - (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} = \mathbf{c} \times (\mathbf{b} \times \mathbf{a}) - \mathbf{c} (\mathbf{b} \cdot \mathbf{a}) \end{array} \right. \begin{array}{l} \left( \begin{matrix} a & b & c \\ a & b & c \\ a & b & c \\ b & c & a \\ a & b & c \\ c & a & b \end{matrix} \right), v_1 \\ \left( \begin{matrix} a & b & c \\ b & c & a \\ a & b & c \\ a & b & c \\ b & a & c \\ a & b & c \end{matrix} \right), v_2 \\ \left( \begin{matrix} a & b & c \\ c & a & b \\ a & b & c \\ a & b & c \\ a & c & b \\ a & b & c \end{matrix} \right), v_3 \\ \left( \begin{matrix} a & b & c \\ b & a & c \\ a & b & c \\ a & c & b \\ a & b & c \\ c & b & a \end{matrix} \right), v_4 \\ \left( \begin{matrix} a & b & c \\ a & c & b \\ a & b & c \\ a & c & b \\ a & b & c \\ c & b & a \end{matrix} \right), v_5 \\ \left( \begin{matrix} a & b & c \\ c & b & a \\ a & b & c \\ a & b & c \\ a & c & b \\ a & b & c \end{matrix} \right), v_6 \end{array} \\
 \Rightarrow & \left\{ \begin{array}{l} -Z - C = X - A \quad v_1 \\ -X - A = Y - B \quad v_2 \\ -Y - B = Z - C \quad v_3 \\ Z - C = -Y - B \quad v_4 \\ Y - B = -X - A \quad v_5 \\ X - A = -Z - C \quad v_6 \end{array} \right. , \left. \begin{array}{l} \text{and scalar-vector product commutative} \\ \times \quad \text{anticommutative} \end{array} \right. , \\
 & \left\{ \begin{array}{l} X = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} \\ Y = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = -(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = -\mathbf{b} \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} \\ Z = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{b} \times \mathbf{a}) = (\mathbf{b} \times \mathbf{a}) \times \mathbf{c} \\ A = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) = \mathbf{a} (\mathbf{c} \cdot \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} = (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} \\ B = \mathbf{b} (\mathbf{c} \cdot \mathbf{a}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} \\ C = \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) = \mathbf{c} (\mathbf{b} \cdot \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \end{array} \right. \\
 \Rightarrow & \left\{ \begin{array}{l} -Z - C = X - A \quad v_1 = v_6 \\ -X - A = Y - B \quad v_2 = v_5 \\ -Y - B = Z - C \quad v_3 = v_4 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} Z + X = A - C \\ X + Y = B - A \\ Y + Z = C - B \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} X + Y = B - A \\ Y + Z = C - B \\ Z + X = A - C \end{array} \right. \\
 \Leftrightarrow & \left\{ \begin{array}{l} 2(X + Y + Z) = 0 \Rightarrow X + Y + Z = 0 \\ Y + Z = C - B \Rightarrow X = B - C \Leftrightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{c} \cdot \mathbf{a}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \quad \text{"back cab"} \\ Z + X = A - C \Rightarrow Y = C - A \Leftrightarrow \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) - \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) \\ X + Y = B - A \Rightarrow Z = A - B \Leftrightarrow \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) - \mathbf{b} (\mathbf{c} \cdot \mathbf{a}) \end{array} \right.
 \end{aligned}$$

### 34.1.3.5 differential operator

[https://en.wikipedia.org/wiki/Differential\\_operator](https://en.wikipedia.org/wiki/Differential_operator)

#### 34.1.3.5.1 4-differential operator 4-differential operator / four-differential operator = d'Alembert operator

$$\begin{aligned}
 D &= \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} = \partial_t + i \partial_x + j \partial_y + k \partial_z \\
 &= \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} = \partial_t + i \partial_x + j \partial_y + k \partial_z \\
 &= \frac{\partial}{\partial x_0} + \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \partial_0 + \mathbf{e}_i \partial_i = \partial_0 + \nabla
 \end{aligned}$$

$$D = \partial_0 + i \partial_1 + j \partial_2 + k \partial_3 = \partial_0 + \nabla = D_0 + \mathbf{D}$$

#### 34.1.3.5.2 nabla nabla = spatial differential operator = 3-differential operator / three-differential operator

$$\nabla = \mathbf{e}_i \partial_i = \sum_{i=1}^3 \mathbf{e}_i \partial_i = \sum_{i=1}^3 \mathbf{e}_i \frac{\partial}{\partial x_i} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^\top = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

#### 34.1.3.5.3 Laplace operator Laplace operator = Laplacian

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

#### 34.1.3.5.4 d'Alembert operator

$$\square = \square_c = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\square_1 = \square_{c=1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial t^2} - \nabla^2$$

#### 34.1.3.6 electromagnetism

Maxwell

##### 34.1.3.6.1 4-potential electromagnetic 4-potential / four-potential

$$A = A_0 + iA_1 + jA_2 + kA_3 = A_0 + \mathbf{A}$$

4-differential operator<sup>[34.1.3.5.1]</sup>

$$\mathbf{D} = \partial_0 + i\partial_1 + j\partial_2 + k\partial_3 = \partial_0 + \nabla = \mathbf{D}_0 + \mathbf{D}$$

$$QP = (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})$$

commutative bracket<sup>[34.1.3.3.1.1]</sup>

$$\begin{aligned} [\mathbf{D}, A] &= \frac{\mathbf{D}A - AD}{2} \\ 2[\mathbf{D}, A] &= DA - AD \\ &= (\partial_0 + i\partial_1 + j\partial_2 + k\partial_3)(A_0 + iA_1 + jA_2 + kA_3) \\ &\quad - (A_0 + iA_1 + jA_2 + kA_3)(\partial_0 + i\partial_1 + j\partial_2 + k\partial_3) \\ DA &= (\mathbf{D}_0 A_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0 \mathbf{A} + A_0 \mathbf{D} + \mathbf{D} \times \mathbf{A}) \\ &= (\mathbf{D}_0 A_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0 \mathbf{A} + \mathbf{D} A_0 + \mathbf{D} \times \mathbf{A}) \\ &= \mathbf{D}_0 (A_0 + \mathbf{A}) - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 + \mathbf{D} \times \mathbf{A} \\ &= \mathbf{D}_0 A - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 + \mathbf{D} \times \mathbf{A} \\ &= \partial_0 A - \nabla \cdot \mathbf{A} + \nabla A_0 + \nabla \times \mathbf{A} \\ AD &= (A_0 \mathbf{D}_0 - \mathbf{A} \cdot \mathbf{D}) + (A_0 \mathbf{D} + \mathbf{D}_0 \mathbf{A} + \mathbf{A} \times \mathbf{D}) \\ &= (\mathbf{D}_0 A_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D} A_0 + \mathbf{D}_0 \mathbf{A} - \mathbf{D} \times \mathbf{A}) \\ &= (\mathbf{D}_0 A_0 - \mathbf{D} \cdot \mathbf{A}) + (\mathbf{D}_0 \mathbf{A} + \mathbf{D} A_0 - \mathbf{D} \times \mathbf{A}) \\ &= \mathbf{D}_0 (A_0 + \mathbf{A}) - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 - \mathbf{D} \times \mathbf{A} \\ &= \mathbf{D}_0 A - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 - \mathbf{D} \times \mathbf{A} \\ &= \partial_0 A - \nabla \cdot \mathbf{A} + \nabla A_0 - \nabla \times \mathbf{A} \end{aligned}$$

$$\begin{aligned} \mathbf{D}A &= \mathbf{D}_0 A - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 + \mathbf{D} \times \mathbf{A} = \partial_0 A - \nabla \cdot \mathbf{A} + \nabla A_0 + \nabla \times \mathbf{A} \\ AD &= \mathbf{D}_0 A - \mathbf{D} \cdot \mathbf{A} + \mathbf{D} A_0 - \mathbf{D} \times \mathbf{A} = \partial_0 A - \nabla \cdot \mathbf{A} + \nabla A_0 - \nabla \times \mathbf{A} \end{aligned}$$

$$\begin{aligned} \mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D} &= 2\mathbf{D} \times \mathbf{A} = 2\boldsymbol{\nabla} \times \mathbf{A} \\ [\mathbf{D}, \mathbf{A}] &= \frac{\mathbf{D}\mathbf{A} - \mathbf{A}\mathbf{D}}{2} = \mathbf{D} \times \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} \end{aligned}$$

anticommutative bracket<sup>[34.1.3.3.1.2]</sup>

$$\begin{aligned} \mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D} &= 2(\mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0) = 2(\partial_0\mathbf{A} - \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla}\mathbf{A}_0) \\ \{\mathbf{D}, \mathbf{A}\} &= \frac{\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}}{2} = \mathbf{D}_0\mathbf{A} - \mathbf{D} \cdot \mathbf{A} + \mathbf{D}\mathbf{A}_0 = \partial_0\mathbf{A} - \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla}\mathbf{A}_0 \end{aligned}$$

commutation and anticommutation on differential operator and any quaternion

$$\begin{aligned} [\mathbf{D}, \mathbf{Q}] &= \boldsymbol{\nabla} \times \mathbf{Q} \\ \{\mathbf{D}, \mathbf{Q}\} &= \partial_0\mathbf{Q} - \boldsymbol{\nabla} \cdot \mathbf{Q} + \boldsymbol{\nabla}\mathbf{Q}_0 \end{aligned}$$

or more evident

$$\begin{aligned} [\mathbf{D}, \mathbf{Q}] &= \boldsymbol{\nabla} \times \mathbf{Q} \\ \{\mathbf{D}, \mathbf{Q}\} &= \partial_0\mathbf{Q} - \boldsymbol{\nabla} \cdot \mathbf{Q} + \boldsymbol{\nabla}\mathbf{Q}_0 \\ &= \partial_0(\mathbf{Q}_0 + \mathbf{Q}) - \boldsymbol{\nabla} \cdot \mathbf{Q} + \boldsymbol{\nabla}\mathbf{Q}_0 \\ &= (\partial_0\mathbf{Q}_0 - \boldsymbol{\nabla} \cdot \mathbf{Q}) + (\partial_0\mathbf{Q} + \boldsymbol{\nabla}\mathbf{Q}_0) \\ &= \left( \frac{\partial\mathbf{Q}_0}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{Q} \right) + \left( \frac{\partial\mathbf{Q}}{\partial t} + \boldsymbol{\nabla}\mathbf{Q}_0 \right) \end{aligned}$$

### 34.1.3.6.2 Maxwell compromise for both quaternion and 3-vector electric potential and vector potential

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{i}\mathbf{A}_1 + \mathbf{j}\mathbf{A}_2 + \mathbf{k}\mathbf{A}_3 = \mathbf{A}_0 + \mathbf{A} = \mathbf{U} + \mathbf{A}$$

electric quaternion and electric field

$$\begin{aligned} \mathbf{E} &= -\{\mathbf{D}, \mathbf{A}\} \\ &= -(\partial_0\mathbf{A} - \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla}\mathbf{A}_0) \\ &= -\partial_0\mathbf{A} + \boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\nabla}\mathbf{A}_0 \\ &= -\partial_t(\mathbf{U} + \mathbf{A}) + \boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\nabla}\mathbf{U} \\ &= -\frac{\partial\mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\nabla}\mathbf{U} - \frac{\partial\mathbf{A}}{\partial t} \\ &= \mathbf{E}_0 + \mathbf{E}, \quad \begin{cases} \mathbf{E}_0 = -\frac{\partial\mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} \\ \mathbf{E} = -\boldsymbol{\nabla}\mathbf{U} - \frac{\partial\mathbf{A}}{\partial t} \end{cases} \quad \text{electric field 3-vector} \end{aligned}$$

magnetic field

$$\mathbf{B} = [\mathbf{D}, \mathbf{A}] = \boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{vmatrix} = \mathbf{B}$$

Work on time? Yes.

$$q\mathbf{E} = q\mathbf{E}_0 + q\mathbf{E} = q\mathbf{E}_0 + \mathbf{F}_E$$

force equivalent on time

---


$$q\mathbf{E}_0$$

$$\begin{cases} E = -\{\mathbf{D}, A\} = E_0 + \mathbf{E} \\ B = +[\mathbf{D}, A] = B_0 + \mathbf{B} = 0 + \mathbf{B} = \mathbf{B} \quad B_0 = 0 \end{cases}$$


---

for any quaternion commuting and anticommutating with differential operator

$$\begin{aligned} [\mathbf{D}, Q] &= \nabla \times \mathbf{Q} \\ \{\mathbf{D}, Q\} &= \partial_0 Q - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\ &= \partial_0 (Q_0 + \mathbf{Q}) - \nabla \cdot \mathbf{Q} + \nabla Q_0 \\ &= (\partial_0 Q_0 - \nabla \cdot \mathbf{Q}) + (\partial_0 \mathbf{Q} + \nabla Q_0) \\ &= \left( \frac{\partial Q_0}{\partial t} - \nabla \cdot \mathbf{Q} \right) + \left( \frac{\partial \mathbf{Q}}{\partial t} + \nabla Q_0 \right) \end{aligned}$$


---

$$\begin{cases} [\mathbf{D}, E] = \nabla \times \mathbf{E} = (0) + (\nabla \times \mathbf{E}) \\ \{\mathbf{D}, E\} = (\partial_0 E_0 - \nabla \cdot \mathbf{E}) + (\partial_0 \mathbf{E} + \nabla E_0) = \left( \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ [\mathbf{D}, B] = \nabla \times \mathbf{B} = (0) + (\nabla \times \mathbf{B}) \\ \{\mathbf{D}, B\} = (\partial_0 B_0 - \nabla \cdot \mathbf{B}) + (\partial_0 \mathbf{B} + \nabla B_0) \stackrel{B_0=0}{=} -\nabla \cdot \mathbf{B} + \partial_0 \mathbf{B} = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases}$$

by comparing ( real & imaginary ) / ( scalar & vector ) parts,

Maxwell equations without source terms

$$\begin{aligned} \begin{cases} [\mathbf{D}, B] = +\{\mathbf{D}, E\} \Leftrightarrow (0) + (\nabla \times \mathbf{B}) = \left( \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ [\mathbf{D}, E] = -\{\mathbf{D}, B\} \Leftrightarrow (0) + (\nabla \times \mathbf{E}) = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases} \\ \Leftrightarrow \begin{cases} \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} = 0 & \Leftrightarrow \nabla \cdot \mathbf{E} = \frac{\partial E_0}{\partial t} \\ \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 = \nabla \times \mathbf{B} & \Leftrightarrow \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \text{ 動電生磁} \\ -\nabla \cdot \mathbf{B} = 0 & \Leftrightarrow \nabla \cdot \mathbf{B} = 0 \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} & \Leftrightarrow \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \text{ 動磁生電} \end{cases} \end{aligned}$$

### 34.1.3.7 Joule heat vs. Thomson heat (Kelvin heat?)

The Lord Kelvin = William Thomson

**34.1.3.7.1 thermoelectric effect** thermoelectric effect = Seebeck effect = Peltier effect = Thomson effect

### 34.1.3.8 source term

$$J = J_0 + iJ_1 + jJ_2 + kJ_3 = J_0 + \mathbf{J} = \rho + \mathbf{J}$$

Maxwell equations with source terms

$$\begin{aligned} \begin{cases} [\mathbf{D}, B] = J + \{\mathbf{D}, E\} \Leftrightarrow (0) + (\nabla \times \mathbf{B}) = \left( \rho + \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} \right) + \left( \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \right) \\ [\mathbf{D}, E] = 0 - \{\mathbf{D}, B\} \Leftrightarrow (0) + (\nabla \times \mathbf{E}) = (-\nabla \cdot \mathbf{B}) + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \end{cases} \\ \Leftrightarrow \begin{cases} \rho + \frac{\partial E_0}{\partial t} - \nabla \cdot \mathbf{E} = 0 & \Leftrightarrow \nabla \cdot \mathbf{E} = \rho + \frac{\partial E_0}{\partial t} \\ \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 = \nabla \times \mathbf{B} & \Leftrightarrow \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} + \nabla E_0 \text{ 動電生磁} \\ -\nabla \cdot \mathbf{B} = 0 & \Leftrightarrow \nabla \cdot \mathbf{B} = 0 \\ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} & \Leftrightarrow \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \text{ 動磁生電} \end{cases} \end{aligned}$$

### 34.1.4 quaternion group

[https://en.wikipedia.org/wiki/Quaternion\\_group](https://en.wikipedia.org/wiki/Quaternion_group)

group theory<sup>[37]</sup>

or please first see quaternion group representation<sup>[37.8]</sup>.

<https://www.bilibili.com/video/BV1rj41117VW>

#### 34.1.4.1 2D rotation

##### 34.1.4.1.1 matrix

$$\mathbf{r} = (x, y) = \langle x, y \rangle = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$$

$$\mathbf{r}' = (x', y') = \langle x', y' \rangle = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix}$$

$$\begin{aligned} \mathbf{r}' &= \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} = r \begin{pmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{pmatrix} \\ &= r \begin{pmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta \\ \sin \alpha \cos \theta + \cos \alpha \sin \theta \end{pmatrix} = r \begin{pmatrix} \cos \theta \cos \alpha - \sin \theta \sin \alpha \\ \sin \theta \cos \alpha + \cos \theta \sin \alpha \end{pmatrix} \\ &= r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\ &= R \mathbf{r}, \quad \begin{cases} R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = R(\theta) = R_\theta \\ \mathbf{r} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}, \mathbf{r}' = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} \end{cases} \end{aligned}$$

orthonormal matrix

$$\mathbf{r}' = O \mathbf{r}$$

$$\begin{aligned} |\mathbf{r}'|^2 &= |\mathbf{r}|^2 \\ \mathbf{r}' \cdot \mathbf{r}' &= \mathbf{r} \cdot \mathbf{r} \\ \mathbf{r}'^\top \mathbf{r}' &= \mathbf{r}^\top \mathbf{r} \\ (\mathbf{O}\mathbf{r})^\top (\mathbf{O}\mathbf{r}) &= \\ \mathbf{r}^\top \mathbf{O}^\top \mathbf{O} \mathbf{r} &= \\ \mathbf{r}^\top \mathbf{O}^\top \mathbf{O} \mathbf{r} &= \mathbf{r}^\top \mathbf{r} \\ \mathbf{O}^\top \mathbf{O} &= 1 = I = I_2 \end{aligned}$$

$$\begin{aligned} \mathbf{R}^\top \mathbf{R} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^\top \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = I \\ \mathbf{R}^\top \mathbf{R} = 1 &\Rightarrow \mathbf{R} \in \{O | \mathbf{O}^\top \mathbf{O} = 1\} \end{aligned}$$

[https://en.wikipedia.org/wiki/Transformation\\_matrix#Affine\\_transformations](https://en.wikipedia.org/wiki/Transformation_matrix#Affine_transformations)

reflection matrix

$$\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & P_x^\top P_x = P_x^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_x \in \{O | O^\top O = 1\} \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & P_y^\top P_y = P_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow P_y \in \{O | O^\top O = 1\} \end{cases}$$

translation matrix?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$O(2)$  group

$$\begin{aligned} O(2) &= \{1, R, P_x, P_y\} \\ &= \{I_2, R_\theta, P_x, P_y\} \subseteq \{O | O^\top O = 1\} \end{aligned}$$

$$\begin{aligned} 1 &= O^\top O \\ 1 &= \det 1 = \det I = \det(I_2) \\ &= \det(O^\top O) = (\det O^\top)(\det O) = (\det O)(\det O) = (\det O)^2 \\ 1 &= (\det O)^2 \\ \det O &= \pm 1 \end{aligned}$$

$$\det R = \det R_\theta = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta - (-\sin^2 \theta) = \cos^2 \theta + \sin^2 \theta = 1$$

$$\begin{cases} P_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \det P_x = -1 \\ P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \det P_y = -1 \end{cases}$$

special orthonormal group of degree 2

$$\begin{aligned} SO(2) &= \{1, R\} = \{I_2, R_\theta\} \subseteq \left\{ O \middle| \begin{cases} O^\top O = 1 & O \\ \det O = 1 & S \end{cases} \right\} \\ &\subset \{1, R, P_x, P_y\} = O(2) \subseteq \{O | O^\top O = 1\} \end{aligned}$$

See also Lie group<sup>[34.2.8]</sup>

Mathemaniac: Lie group [https://www.youtube.com/playlist?list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP](https://www.youtube.com/playlist?list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP)  
[https://www.youtube.com/watch?v=erA0jb9dSm0&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=2](https://www.youtube.com/watch?v=erA0jb9dSm0&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=2)

#### 34.1.4.1.2 complex

$$z = r(\cos \alpha + i \sin \alpha) = r e^{i\alpha}$$

$$z' = r(\cos(\alpha + \theta) + i \sin(\alpha + \theta)) = r e^{i(\alpha+\theta)}$$

$$\begin{aligned} z' &= z_\theta z \\ z_\theta &= \frac{z'}{z} = \frac{r' e^{i(\alpha+\theta)}}{r e^{i\alpha}} = \frac{r'}{r} e^{i\theta} = \frac{r'}{r} (\cos \theta + i \sin \theta) \end{aligned}$$

$$\begin{aligned}
z_\theta z &= \left[ \frac{r'}{r} (\cos \theta + i \sin \theta) \right] [r (\cos \alpha + i \sin \alpha)] \\
&= r' [(\cos \theta \cos \alpha - \sin \theta \sin \alpha) + i (\sin \theta \cos \alpha + \cos \theta \sin \alpha)] \\
&= r' [\cos(\alpha + \theta) + i \sin(\alpha + \theta)] = z'
\end{aligned}$$

$$\hat{z}_\theta = z_\theta \left( \frac{r'}{r} = 1 \right) = e^{i\theta} = \cos \theta + i \sin \theta$$

$$\hat{z}_\theta^* = \overline{\hat{z}_\theta} = e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\hat{z}_\theta^* \hat{z}_\theta = e^{i\theta} e^{-i\theta} = e^{i\theta + (-i\theta)} = e^{i0} = e^0 = 1$$

unitary group of degree 1

$$U(1) = \{1, \hat{z}_\theta\} = \{e^{i0}, e^{i\theta}\}$$

**34.1.4.1.3**  $SO(2) \cong U(1)$   $\mathbb{C} \leftrightarrow \mathcal{M}_{2 \times 2}(\mathbb{R}) = \mathcal{M}_2(\mathbb{R})$  complex group representation<sup>[37.7]</sup>

$$x + yi \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

$$\begin{aligned}
U(1) &= \{1, \hat{z}_\theta\} = \{e^{i0}, e^{i\theta}\} \\
&= \{\cos 0 + i \sin 0, \cos \theta + i \sin \theta\} \\
&\leftrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos 0 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin \theta \right\} \\
&= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 1 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin \theta \right\} \\
&= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\} = \{I_2, R_\theta\} = \{1, R\} = SO(2)
\end{aligned}$$

$$U(1) \cong SO(2) \Leftrightarrow SO(2) \cong U(1)$$

unitary group of degree 1 and special orthonormal group of degree 2 are isomorphism

#### 34.1.4.2 3D rotation

##### 34.1.4.2.1 matrix

###### 34.1.4.2.1.1 construction with 2D rotation matrix

$$\begin{aligned}
r' &= \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \\ z' \end{pmatrix} \stackrel{z' = z}{=} \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \\ z \end{pmatrix} = \begin{pmatrix} R(\theta) & \\ & 1 \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} \\
&= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} = R_z(\theta) \mathbf{r}, \quad \begin{cases} R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \\ z \end{pmatrix} \end{cases}
\end{aligned}$$

$$\begin{aligned} \mathbf{r}' &= \begin{pmatrix} x' \\ r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} \stackrel{x' \equiv x}{=} \begin{pmatrix} x \\ r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} = \begin{pmatrix} 1 & & \\ & R(\theta) & \end{pmatrix} \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} = R_x(\theta) \mathbf{r}, \quad \begin{cases} R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} x \\ r \cos \alpha \\ r \sin \alpha \end{pmatrix} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathbf{r}' &= \begin{pmatrix} r \sin(\alpha + \theta) \\ y' \\ r \cos(\alpha + \theta) \end{pmatrix} \stackrel{y' \equiv y}{=} \begin{pmatrix} r \sin(\alpha + \theta) \\ y \\ r \cos(\alpha + \theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} r \sin \alpha \\ y \\ r \cos \alpha \end{pmatrix} = R_y(\theta) \mathbf{r}, \quad \begin{cases} R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} r \sin \alpha \\ y \\ r \cos \alpha \end{pmatrix} \end{cases} \end{aligned}$$

#### 34.1.4.2.1.2 Euler angle $z \rightarrow x \rightarrow z : \alpha \rightarrow \beta \rightarrow \gamma$

$$\begin{aligned} &R_z(\gamma) R_x(\beta) R_z(\alpha) \\ &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \cos \beta \sin \alpha & \cos \beta \cos \alpha & -\sin \beta \\ \sin \beta \sin \alpha & \sin \beta \cos \alpha & \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos \gamma \cos \alpha - \sin \gamma \cos \beta \sin \alpha & -\cos \gamma \sin \alpha - \sin \gamma \cos \beta \cos \alpha & \sin \gamma \sin \beta \\ \sin \gamma \cos \alpha + \cos \gamma \cos \beta \sin \alpha & -\sin \gamma \sin \alpha - \cos \gamma \cos \beta \cos \alpha & -\cos \gamma \sin \beta \\ \sin \beta \sin \alpha & \sin \beta \cos \alpha & \cos \beta \end{pmatrix} \end{aligned}$$

$x \rightarrow y \rightarrow z : \alpha \rightarrow \beta \rightarrow \gamma$

$$\begin{aligned} &R_z(\gamma) R_y(\beta) R_x(\alpha) \\ &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta \sin \alpha & \sin \beta \cos \alpha \\ 0 & \cos \alpha & -\sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \gamma \cos \beta & \cos \gamma \sin \beta \sin \alpha - \sin \gamma \cos \alpha & \cos \gamma \sin \beta \cos \alpha + \sin \gamma \sin \alpha \\ \sin \gamma \cos \beta & \sin \gamma \sin \beta \sin \alpha + \cos \gamma \cos \alpha & \sin \gamma \sin \beta \cos \alpha - \cos \gamma \sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{pmatrix} \end{aligned}$$

#### 34.1.4.2.1.3 3D rotation about an arbitrary axis <https://math.stackexchange.com/questions/4550704/rotation-around-an-arbitrary-axis>

spherical coordinate unit vector

$$\begin{cases} \hat{x} = r \sin \theta \cos \phi & \stackrel{r \equiv 1}{=} \sin \theta \cos \phi \\ \hat{y} = r \sin \theta \sin \phi & \stackrel{r \equiv 1}{=} \sin \theta \sin \phi \\ \hat{z} = r \cos \theta & \stackrel{r \equiv 1}{=} \cos \theta \end{cases}$$

although I prefer  $\theta$  and  $\phi$  switched back to be compatible with 2D coordinate

$$\begin{cases} \hat{x} = r \sin \phi \cos \theta & \stackrel{r=1}{=} \sin \phi \cos \theta \\ \hat{y} = r \sin \phi \sin \theta & \stackrel{r=1}{=} \sin \phi \sin \theta \\ \hat{z} = r \cos \phi & \stackrel{r=1}{=} \cos \phi \end{cases}$$

or cos first in  $x, y$ -plane

$$\begin{cases} \hat{x} = r \cos \phi \cos \theta & \stackrel{r=1}{=} \cos \phi \cos \theta \\ \hat{y} = r \cos \phi \sin \theta & \stackrel{r=1}{=} \cos \phi \sin \theta \\ \hat{z} = r \sin \phi & \stackrel{r=1}{=} \sin \phi \end{cases}$$


---

still use the most convention

$$\hat{\mathbf{n}} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$


---

$$\begin{cases} \hat{\mathbf{n}} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \\ \hat{\mathbf{u}} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} \Leftarrow \cos \theta \sin \theta - \cos \theta \sin \theta = 0 \Rightarrow \hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = 0 \Leftrightarrow \hat{\mathbf{u}} \perp \hat{\mathbf{n}} \\ \hat{\mathbf{v}} = \hat{\mathbf{n}} \times \hat{\mathbf{u}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \end{vmatrix} = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \hat{\mathbf{v}} \perp \hat{\mathbf{n}} \\ \hat{\mathbf{v}} \perp \hat{\mathbf{u}} \end{cases} \end{cases}$$

$S = \{\hat{\mathbf{n}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}\} = \{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}}\}$  is a basis of the spherical coordinate

$$[\mathbf{V}]_S = \begin{pmatrix} u \\ v \\ n \end{pmatrix}$$

$$\begin{aligned} \mathbf{V} &= (\hat{\mathbf{u}} \quad \hat{\mathbf{v}} \quad \hat{\mathbf{n}}) \begin{pmatrix} u \\ v \\ n \end{pmatrix} = u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + n\hat{\mathbf{n}} \\ &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} u \\ v \\ n \end{pmatrix} = S [\mathbf{V}]_S \\ \begin{pmatrix} u \\ v \\ n \end{pmatrix} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^{-1} \mathbf{V} \\ [\mathbf{V}]_S &= S^{-1} \mathbf{V} \end{aligned}$$

$$\begin{aligned}
S^{-1} &= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^{-1}, S \in \{O | O^T O = 1\} \\
&= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}^T \\
&= \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}^T \\ \hat{\mathbf{v}}^T \\ \hat{\mathbf{n}}^T \end{pmatrix} \\
S^{-1}S &= \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1
\end{aligned}$$

$\hat{\mathbf{n}}$  as  $z$  direction

$$\begin{aligned}
[\mathbf{V}]_S &= S^{-1} \mathbf{V} \\
[\mathbf{V}']_S &= R_z(\gamma) [\mathbf{V}]_S = R_z(\gamma) S^{-1} \mathbf{V} \\
\mathbf{V}' &= S [\mathbf{V}']_S = SR_z(\gamma) [\mathbf{V}]_S = SR_z(\gamma) S^{-1} \mathbf{V} \\
\mathbf{V}' &= SR_z(\gamma) S^{-1} \mathbf{V}
\end{aligned}$$

<https://www.symbolab.com/>

$$\begin{aligned}
&SR_z(\gamma) S^{-1} \\
&= \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \\
&= \begin{pmatrix} c_1 c_2 & -s_2 & s_1 c_2 \\ c_1 s_2 & c_2 & s_1 s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 c_2 & c_1 s_2 & -s_1 \\ -s_2 & c_2 & 0 \\ s_1 c_2 & s_1 s_2 & c_1 \end{pmatrix}, \begin{cases} c_1 = \cos \theta & s_1 = \sin \theta \\ c_2 = \cos \phi & s_2 = \sin \phi \\ c_3 = \cos \gamma & s_3 = \sin \gamma \end{cases} \\
&= \begin{pmatrix} c_1 c_2 & -s_2 & s_1 c_2 \\ c_1 s_2 & c_2 & s_1 s_2 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_3 c_1 c_2 + s_3 s_2 & c_3 c_1 s_2 - c_2 s_3 & -c_3 s_1 \\ c_1 c_2 s_3 - c_3 s_2 & c_3 c_2 + c_1 s_3 s_2 & -s_3 s_1 \\ c_2 s_1 & s_2 s_1 & c_1 \end{pmatrix} \\
&= \begin{pmatrix} c_1 c_2 (c_1 c_3 s_2 + c_2 s_3) - s_2 (c_2 c_3 - c_1 s_2 s_3) + c_2 s_2 s_1^2 & c_1 s_2 (c_1 c_2 c_3 - s_2 s_3) + c_2 (-c_1 c_2 s_3 - c_3 s_2) + c_2 s_2 s_1^2 & c_1 c_2 s_1 - s_1 (c_1 c_2 c_3 - s_2 s_3) + c_2 s_2 s_1^2 \\ -c_1 c_2 c_3 s_1 - s_2 s_1 s_3 + c_1 c_2 s_1 & c_1^2 c_3 s_2^2 + s_2^2 s_1^2 + c_2^2 c_3 & c_1 s_2 s_1 - s_1 (c_1 c_3 s_2 + c_2 s_1 s_3) + c_1 c_2 s_1 \\ & -c_1 c_3 s_2 s_1 + c_2 s_1 s_3 + c_1 s_2 s_1 & c_3 s_1^2 + c_1^2 \end{pmatrix}
\end{aligned}$$

#### 34.1.4.2.2 quaternion <https://math.stackexchange.com/questions/328117/how-does-one-derive-this-rotation-quaternion-formula>

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\
w &= t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ i^2 = j^2 = k^2 = -1 = ijk \Rightarrow ij = k \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\
&= t + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (i \ j \ k) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_0 + (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0 + \mathbf{x}, \begin{cases} \mathbf{e}_1 = \mathbf{i} = i \\ \mathbf{e}_2 = \mathbf{j} = j \\ \mathbf{e}_3 = \mathbf{k} = k \end{cases}
\end{aligned}$$

$$q_1 q_2 = (x_{10} x_{20} - \mathbf{x}_1 \cdot \mathbf{x}_2) + (x_{10} \mathbf{x}_2 + x_{20} \mathbf{x}_1 + \mathbf{x}_1 \times \mathbf{x}_2)$$

$$QP = (Q_0 P_0 - \mathbf{Q} \cdot \mathbf{P}) + (Q_0 \mathbf{P} + P_0 \mathbf{Q} + \mathbf{Q} \times \mathbf{P})$$

$$q = q_0 + \mathbf{q} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$$

$$\begin{aligned} q^* &= \bar{q} = \overline{q_0 + \mathbf{q}} = q_0 - \mathbf{q} \\ &= \overline{q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}} = q_0 - q_1 \mathbf{i} - q_2 \mathbf{j} - q_3 \mathbf{k} \\ &= q_0 - q_1 \mathbf{i} - q_2 \mathbf{j} - q_3 \mathbf{k} \end{aligned}$$

$$v = v_0 + \mathbf{v} = v_0 + v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} = v_0 + v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

$$\begin{aligned} q\mathbf{v} &= (q_0 v_0 - \mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + v_0 \mathbf{q} + \mathbf{q} \times \mathbf{v}) \\ &\stackrel{v_0=0}{=} (q_0 0 - \mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + 0 \mathbf{q} + \mathbf{q} \times \mathbf{v}) \\ &= (-\mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v}) \\ q\mathbf{v} &= (-\mathbf{q} \cdot \mathbf{v}) + (q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v}) \end{aligned}$$

$$\begin{aligned} \mathbf{v}\bar{q} &= (v_0 q_0 - \mathbf{v} \cdot \bar{\mathbf{q}}) + (v_0 \bar{\mathbf{q}} + q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\ &\stackrel{v_0=0}{=} (0 q_0 - \mathbf{v} \cdot \bar{\mathbf{q}}) + (0 \bar{\mathbf{q}} + q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\ &= (-\mathbf{v} \cdot \bar{\mathbf{q}}) + (q_0 \mathbf{v} + \mathbf{v} \times \bar{\mathbf{q}}) \\ &= (-\mathbf{v} \cdot (-\mathbf{q})) + (q_0 \mathbf{v} + \mathbf{v} \times (-\mathbf{q})) \\ &= (\mathbf{v} \cdot \mathbf{q}) + (q_0 \mathbf{v} - \mathbf{v} \times \mathbf{q}) \\ \mathbf{v}\bar{q} &= (\mathbf{v} \cdot \mathbf{q}) + (q_0 \mathbf{v} - \mathbf{v} \times \mathbf{q}) \end{aligned}$$

<https://math.stackexchange.com/questions/41574/can-eulers-identity-be-extended-to-quaternions>

$$\begin{aligned} q &= a + bi + cj + dk = a + ib + jc + kd \\ &= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\ &= a + \theta \frac{ib + jc + kd}{\theta}, \theta^2 = b^2 + c^2 + d^2 \end{aligned}$$

$$\begin{aligned} \left( \frac{ib + jc + kd}{r} \right)^2 &= \frac{-b^2 - c^2 - d^2 + bc(ij + ji) + cd(jk + kj) + db(ki + ik)}{r^2} \\ &= \frac{-b^2 - c^2 - d^2 + bc(k - k) + cd(i - i) + db(j - j)}{r^2} \\ &= \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\ &= \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\ \frac{ib + jc + kd}{r} &= \pm \sqrt{-1} \end{aligned}$$

$$\begin{aligned} e^q &= e^{a+ib+jc+kd} \\ &= e^{a+r\sqrt{-1}} = e^a e^{r\sqrt{-1}} = e^{a+\theta\sqrt{-1}} = e^a e^{\theta\sqrt{-1}} \\ &= e^a (\cos r + \sqrt{-1} \sin r) = e^a (\cos \theta + \sqrt{-1} \sin \theta) \\ &= e^a \left( \cos r + \frac{ib + jc + kd}{r} \sin r \right) = e^a \left[ \cos r + (ib + jc + kd) \frac{\sin r}{r} \right] \\ &= e^a \left( \cos \theta + \frac{ib + jc + kd}{\theta} \sin \theta \right) = e^a \left[ \cos \theta + (ib + jc + kd) \frac{\sin \theta}{\theta} \right] \end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd \\
&= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\
&= \sqrt{a^2 + r^2} \left( \frac{a}{\sqrt{a^2 + r^2}} + \frac{ib + jc + kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right) \\
&= \rho (\cos \phi + \sqrt{-1} \sin \phi), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} \end{cases} \\
&= \rho e^{\phi \sqrt{-1}} \\
q &= \rho e^{\phi \sqrt{-1}}, \begin{cases} q = a + ib + jc + kd \\ \rho = \sqrt{a^2 + r^2} = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \arctan \frac{\pm \sqrt{b^2 + c^2 + d^2}}{a} \end{cases} \\
\rho e^{-\phi \sqrt{-1}} &= \rho [\cos(-\phi) + \sqrt{-1} \sin(-\phi)] \\
&= \rho [\cos \phi - \sqrt{-1} \sin \phi] \\
&= \sqrt{a^2 + r^2} \left[ \frac{a}{\sqrt{a^2 + r^2}} - \frac{ib + jc + kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right] \\
&= a - (ib + jc + kd) = a - ib - jc - kd = \bar{q} = q^*
\end{aligned}$$

### 34.1.5 octonion

## 34.2 Krasjet

<https://github.com/Krasjet/quaternion>

<https://krasjet.github.io/quaternion/>

[https://krasjet.github.io/quaternion/bonus\\_gimbal\\_lock.pdf](https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf)

### 34.2.1 Rodrigues rotation

$$\mathbf{v} \rightarrow \mathbf{v}'$$

$$\mathbf{v} \xrightarrow{\text{rotate about } \mathbf{u}} \mathbf{v}'$$

$$\mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}'$$

$$\begin{cases} \mathbf{v} = \mathbf{v}_{\parallel n} + \mathbf{v}_{\perp n} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \Rightarrow \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} \\ \mathbf{v}' = \mathbf{v}'_{\parallel n} + \mathbf{v}'_{\perp n} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \Rightarrow \mathbf{v}'_{\perp} = \mathbf{v}' - \mathbf{v}'_{\parallel} \end{cases}$$

$$\begin{aligned}
\mathbf{v}_{\parallel n} &= \mathbf{v}_{\parallel} = \text{proj}_n \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \hat{\mathbf{n}} \\
&= \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{\parallel \hat{\mathbf{n}}} &= \mathbf{v}_{\parallel} = \text{proj}_{\hat{\mathbf{n}}} \mathbf{v} = \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} \hat{\mathbf{n}} \stackrel{\|\hat{\mathbf{n}}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \\
&= \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} \frac{\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|} = \frac{\mathbf{v} \cdot \hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|^2} \hat{\mathbf{n}} \stackrel{\|\hat{\mathbf{n}}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}}
\end{aligned}$$

$$\mathbf{n} = \hat{\mathbf{n}}$$

$$\begin{aligned}
\mathbf{v}_{\parallel n} &= \mathbf{v}_{\parallel} = \text{proj}_n \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \hat{\mathbf{n}} \stackrel{\|\mathbf{n}\|=1}{=} (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} = \mathbf{v} \cdot \hat{\mathbf{n}} \hat{\mathbf{n}} \\
&= \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|} \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \stackrel{\|\mathbf{n}\|=1}{=} (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} = \mathbf{v} \cdot \mathbf{n} \mathbf{n}
\end{aligned}$$

$$\mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$$

$$\mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}' = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp}$$

$$\mathbf{v}_{\parallel} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}'_{\parallel}$$

$$\mathbf{v}_{\parallel} = \mathbf{v}_{\parallel \mathbf{n}} = \mathbf{v}'_{\parallel \mathbf{n}} = \mathbf{v}'_{\parallel}$$

$$\mathbf{v}_{\parallel} = \mathbf{v}'_{\parallel}$$

$$\mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel}$$

$$\begin{cases} \mathbf{u} = \mathbf{n} \times \mathbf{v}_{\perp} \\ \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \end{cases} \quad \begin{cases} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \\ \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{cases}$$

$$\begin{aligned} \mathbf{v}'_{\perp} &= \mathbf{v}'_{\parallel \mathbf{v}_{\perp}} + \mathbf{v}'_{\parallel \mathbf{u}} \\ &= (\cos \theta) \mathbf{v}_{\perp} + (\sin \theta) \mathbf{u}, \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \\ &= \cos \theta \mathbf{v}_{\perp} + \sin \theta \mathbf{u} \\ &= \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}) \\ \mathbf{v}'_{\perp} &= \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}) \end{aligned}$$

$$\mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp}$$


---

$$\mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp})$$


---

$$\begin{aligned} \mathbf{v}' &= \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp}, \begin{cases} \mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \end{cases} \\ &= \mathbf{v}_{\parallel} + \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \\ &= \mathbf{v}_{\parallel} + \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \begin{cases} \mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \\ \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \end{cases} \\ &= (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \cos \theta [\mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}] + \sin \theta [\mathbf{n} \times (\mathbf{v} - \mathbf{v}_{\parallel})] \\ &= (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \cos \theta \mathbf{v} - \cos \theta (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta [\mathbf{n} \times \mathbf{v} - \mathbf{n} \times \mathbf{v}_{\parallel}] \\ &= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta [\mathbf{n} \times \mathbf{v} - \mathbf{0}] \\ &= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v} \end{aligned}$$


---

$$\mathbf{v}' = \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v}$$


---

$$\begin{aligned}
\mathbf{v}' &= \cos \theta \mathbf{v} + (1 - \cos \theta) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + \left[ 1 - \left( 1 - 2 \sin^2 \frac{\theta}{2} \right) \right] (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left[ 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + \left[ 2 \sin^2 \frac{\theta}{2} \right] (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left[ 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right] \mathbf{n} \times \mathbf{v} \\
&= \cos \theta \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} \right] \\
&= \cos \theta \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} + \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \right] \\
&= \left[ 2 \cos^2 \frac{\theta}{2} - 1 \right] \mathbf{v} + 2 \sin \frac{\theta}{2} \left[ \left( \cos \frac{\theta}{2} \right) \mathbf{n} \times \mathbf{v} + \left( \sin \frac{\theta}{2} \right) (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \right]
\end{aligned}$$

### 34.2.2 quaternion operation

$$\begin{aligned}
q &= a + bi + cj + dk \\
\varrho &= \alpha + \beta i + \gamma j + \delta k
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} &\leftarrow q\varrho = (a + bi + cj + dk) (\alpha + \beta i + \gamma j + \delta k) \\
&= a(\alpha + \beta i + \gamma j + \delta k) \\
&\quad + bi(\alpha + \beta i + \gamma j + \delta k) \\
&\quad + cj(\alpha + \beta i + \gamma j + \delta k) \\
&\quad + dk(\alpha + \beta i + \gamma j + \delta k) \\
&= a\alpha + a\beta i + a\gamma j + a\delta k \\
&\quad + b\alpha i - b\beta + b\gamma k - b\delta j \\
&\quad + c\alpha j - c\beta k - c\gamma + c\delta i \\
&\quad + d\alpha k + d\beta j - d\gamma i - d\delta \\
&= (a\alpha - b\beta - c\gamma - d\delta) \\
&\quad + (b\alpha + a\beta - d\gamma + c\delta) i \\
&\quad + (c\alpha + d\beta + a\gamma - b\delta) j \\
&\quad + (d\alpha - c\beta + b\gamma + a\delta) k \\
\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} &\leftarrow \\
&= (\alpha a - \beta b - \gamma c - \delta d) \\
&\quad + (\beta a + \alpha b + \delta c - \gamma d) i \\
&\quad + (\gamma a - \delta b + \alpha c + \beta d) j \\
&\quad + (\delta a + \gamma b - \beta c + \alpha d) k \\
\begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &\leftarrow \\
qv &\leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_i v
\end{aligned}$$

concept like quaternion group representation<sup>[37.8]</sup>

$$L(q) = Q_i = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} a + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} b \\ + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} c + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} d \\ \leftrightarrow 1a + ib + jc + kd$$

$$ij \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow k$$

$$ji \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -k$$

$$ki \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow j$$

$$Q_i = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} -b & c & d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix}, \begin{cases} a = q_0 \\ b \\ c \\ d \\ Q_i = \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{cases}$$

Grassmann product

$$qv \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_i v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q}^\top \mathbf{v} \\ \mathbf{q} v_0 + Q_i \mathbf{v} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v} \end{pmatrix}$$

$$vq \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_r v$$

$$R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ + c \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \leftrightarrow a1 + bi + cj + dk$$

$$ij \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow -k$$

$$ji \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow k$$

$$ki \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow -j$$

$$\begin{aligned} R(q) = Q_r &= \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \leftrightarrow \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} \\ &= \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \begin{cases} \alpha = a+bi & \bar{\alpha} = a-bi \\ \beta = c+di & \bar{\beta} = c-di \end{cases} \end{aligned}$$

$$Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & \begin{pmatrix} - (b & c & d) \\ a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \end{pmatrix} = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix}, \begin{cases} a = q_0 \\ \mathbf{q} = \begin{pmatrix} b \\ c \\ d \end{pmatrix} \\ Q_i = \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \end{cases}$$

$$vq \leftrightarrow \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{q}^\top v \\ v_0 \mathbf{q} + Q_i^\top v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{v} \cdot \mathbf{q} \\ v_0 \mathbf{q} + q_0 \mathbf{v} + \mathbf{v} \times \mathbf{q} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} - \mathbf{q} \times \mathbf{v} \end{pmatrix}$$

$$\begin{cases} qv \leftrightarrow Q_i v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q}^\top v \\ \mathbf{q} v_0 + Q_i v \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} + \mathbf{q} \times \mathbf{v} \end{pmatrix} \\ vq \leftrightarrow Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{q}^\top v \\ v_0 \mathbf{q} + Q_i^\top v \end{pmatrix} = \begin{pmatrix} v_0 q_0 - \mathbf{v} \cdot \mathbf{q} \\ v_0 \mathbf{q} + q_0 \mathbf{v} + \mathbf{v} \times \mathbf{q} \end{pmatrix} = \begin{pmatrix} q_0 v_0 - \mathbf{q} \cdot \mathbf{v} \\ \mathbf{q} v_0 + q_0 \mathbf{v} - \mathbf{q} \times \mathbf{v} \end{pmatrix} \end{cases}$$

$$\begin{cases} qv = \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} = L(q)v = Q_i v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} \\ vq = \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = R(q)v = Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^\top \\ \mathbf{q} & Q_i^\top \end{pmatrix} \begin{pmatrix} v_0 \\ v \end{pmatrix} \end{cases}$$

$$\begin{aligned}
& \left\{ \begin{array}{l} \mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}' \\ \left\{ \begin{array}{l} \mathbf{v} = \mathbf{v}_{\parallel n} + \mathbf{v}_{\perp n} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \Rightarrow \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} \\ \mathbf{v}' = \mathbf{v}'_{\parallel n} + \mathbf{v}'_{\perp n} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \Rightarrow \mathbf{v}'_{\perp} = \mathbf{v}' - \mathbf{v}'_{\parallel} \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{u} = \mathbf{n} \times \mathbf{v}_{\perp} \\ \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{v}'_{\parallel} = \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} = \cos \theta \mathbf{v}_{\perp} + \sin \theta (\mathbf{n} \times \mathbf{v}_{\perp}), \theta = \angle \mathbf{v}'_{\perp} \mathbf{v}_{\perp} \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{array} \right. \end{array} \right. \\
& \Leftrightarrow \left\{ \begin{array}{l} \mathbf{v} \xrightarrow{\text{rotate about } \mathbf{n}} \mathbf{v}' \\ \left\{ \begin{array}{l} \mathbf{v} = \begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}_{\perp n} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \Rightarrow \mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} \\ \mathbf{v}' = \begin{pmatrix} 0 \\ \mathbf{v}' \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp n} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp} \end{pmatrix} = \mathbf{v}'_{\parallel} + \mathbf{v}'_{\perp} \Rightarrow \mathbf{v}'_{\perp} = \mathbf{v}' - \mathbf{v}'_{\parallel} \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{u} = \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ 00 - 0 \\ \mathbf{n} \mathbf{0} + 0 \mathbf{v}_{\perp} + \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ n_0 v_{\perp 0} - \mathbf{n} \cdot \mathbf{v}_{\perp} \\ n \mathbf{v}_{\perp 0} + n_0 \mathbf{v}_{\perp} + \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = n \mathbf{v}_{\perp} \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{u} \perp \mathbf{n} \\ \mathbf{u} \perp \mathbf{v}_{\perp} \end{array} \right. \\ \left\{ \begin{array}{l} \|\mathbf{u}\| = \|\mathbf{n} \times \mathbf{v}_{\perp}\| = \|\mathbf{n}\| \|\mathbf{v}_{\perp}\| \sin \frac{\pi}{2} = \|\mathbf{v}_{\perp}\| \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{n} \perp \mathbf{v}_{\perp} \\ \|\mathbf{n}\| = 1 \end{array} \right. \\ \left\{ \begin{array}{l} \mathbf{v}'_{\parallel} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\parallel} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} = \mathbf{v}_{\parallel} \\ \mathbf{v}'_{\perp} = \begin{pmatrix} 0 \\ \mathbf{v}'_{\perp} \end{pmatrix} = \cos \theta \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} + \sin \theta \begin{pmatrix} 0 \\ \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \cos \theta \mathbf{v}_{\perp} + \sin \theta n \mathbf{v}_{\perp} = (\cos \theta + \sin \theta n) \mathbf{v}_{\perp} \end{array} \right. \end{array} \right. 
\end{aligned}$$

$$\mathbf{v}'_{\perp} = (\cos \theta + \sin \theta n) \mathbf{v}_{\perp}$$

$$\mathbf{v}'_{\perp} = p \mathbf{v}_{\perp}, p = \cos \theta + \sin \theta n = \cos \theta \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + \sin \theta \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}$$

$$p = \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} = \cos \theta + \sin \theta n, \|\mathbf{n}\| = 1$$

$$|p| = \left\| \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}^T \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}} = \sqrt{\cos^2 \theta + \sin^2 \theta \|\mathbf{n}\|^2} = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$|p| = \left\| \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} \right\| = 1$$

$$p^* = \bar{p} = \begin{pmatrix} p_0 \\ -\mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -\sin \theta \mathbf{n} \end{pmatrix} = \cos \theta - \sin \theta n$$

$$\begin{aligned}
p^* p &= \bar{p} p = \begin{pmatrix} p_0 \\ -\mathbf{p} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} p_0 p_0 - (-\mathbf{p}) \cdot \mathbf{p} \\ -\mathbf{p} p_0 + p_0 \mathbf{p} + (-\mathbf{p}) \times \mathbf{p} \end{pmatrix} \\
&= \begin{pmatrix} p_0 p_0 + \mathbf{p} \cdot \mathbf{p} \\ \mathbf{0} + \mathbf{0} \end{pmatrix} = p_0 p_0 + \mathbf{p} \cdot \mathbf{p} = p_0^2 + \|\mathbf{p}\|^2 = |p|^2 = 1^2 = 1 \\
&= \begin{pmatrix} p_0 p_0 - \mathbf{p} \cdot (-\mathbf{p}) \\ \mathbf{p} p_0 + p_0 (-\mathbf{p}) + \mathbf{p} \times (-\mathbf{p}) \end{pmatrix} = p \bar{p} = pp^*
\end{aligned}$$

$$p^* p = \bar{p} p = 1 = p \bar{p} = pp^* = |p|^2 = p_0^2 + \|\mathbf{p}\|^2 \Rightarrow p^* = \frac{1}{p} = p^{-1}$$

$$\begin{aligned}
p^2 = pp &= \begin{pmatrix} p_0 p_0 - \mathbf{p} \cdot \mathbf{p} \\ \mathbf{p} p_0 + p_0 \mathbf{p} + \mathbf{p} \times \mathbf{p} \end{pmatrix} = \begin{pmatrix} \cos^2 \theta - \sin^2 \theta \|\mathbf{n}\|^2 \\ 2(\sin \theta \mathbf{n}) \cos \theta + \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2 \theta - \sin^2 \theta \\ 2 \sin \theta \cos \theta \mathbf{n} \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \mathbf{n} \end{pmatrix} \\
\begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix}^2 &= \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \mathbf{n} \end{pmatrix} \\
q^2 = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{n} \end{pmatrix}^2 &= \begin{pmatrix} \cos \theta \\ \sin \theta \mathbf{n} \end{pmatrix} = p
\end{aligned}$$

$$qv_{\parallel} = v_{\parallel} q$$

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel \mathbf{n}} &= \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel} = \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} \alpha 0 - \beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \beta \mathbf{n} 0 + \alpha \mathbf{v}_{\parallel} + \beta \mathbf{n} \times \mathbf{v}_{\parallel} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} + \mathbf{0} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} 0\alpha - \mathbf{v}_{\parallel} \cdot \beta \mathbf{n} \\ \mathbf{v}_{\parallel} \alpha + 0\beta \mathbf{n} + \mathbf{v}_{\parallel} \times \beta \mathbf{n} \end{pmatrix} = \begin{pmatrix} -\mathbf{v}_{\parallel} \cdot \beta \mathbf{n} \\ \alpha \mathbf{v}_{\parallel} + \mathbf{0} \end{pmatrix} = \begin{pmatrix} -\beta \mathbf{n} \cdot \mathbf{v}_{\parallel} \\ \alpha \mathbf{v}_{\parallel} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mathbf{v}_{\parallel} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} = v_{\parallel} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} = v_{\parallel \mathbf{n}} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \\
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\parallel} &= v_{\parallel} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}
\end{aligned}$$

$$qv_{\perp} = v_{\perp} q^*$$

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp \mathbf{n}} &= \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp} = \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} \\
&= \begin{pmatrix} \alpha 0 - \beta \mathbf{n} \cdot \mathbf{v}_{\perp} \\ \beta \mathbf{n} 0 + \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \mathbf{v}_{\perp} + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} \\
&= \begin{pmatrix} 0\alpha - \mathbf{v}_{\perp} \cdot (-\beta \mathbf{n}) \\ \mathbf{v}_{\perp} \alpha + 0(-\beta \mathbf{n}) + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ \alpha \mathbf{v}_{\perp} + \mathbf{v}_{\perp} \times (-\beta \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \mathbf{v}_{\perp} + \beta \mathbf{n} \times \mathbf{v}_{\perp} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mathbf{v}_{\perp} \end{pmatrix} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp \mathbf{n}} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} \\
\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} v_{\perp} &= v_{\perp} \begin{pmatrix} \alpha \\ -\beta \mathbf{n} \end{pmatrix} = v_{\perp} \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}^* = v_{\perp} \overline{\begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix}}
\end{aligned}$$

$$\begin{aligned}
q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n &\in \left\{ p \left| \begin{array}{l} p = \cos \theta + \sin \theta n \\ n = \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix}, \|\mathbf{n}\| = 1 \end{array} \right. \right\} \subset \left\{ \begin{pmatrix} \alpha \\ \beta \mathbf{n} \end{pmatrix} \middle| \begin{array}{l} \alpha, \beta \in \mathbb{R} \\ \|\mathbf{n}\| = 1 \end{array} \right. \right\} \\
&\Rightarrow \begin{cases} q^* q = \bar{q} q = 1 = q \bar{q} = q q^* = |q|^2 = q_0^2 + \|\mathbf{q}\|^2 \Rightarrow q^* = \frac{1}{q} = q^{-1} & (u) \text{ unit quaternion} \\ q v_{\parallel} = v_{\parallel} q & (c) \text{ commutativity} \\ q v_{\perp} = v_{\perp} q^* = v_{\perp} \bar{q} \Leftrightarrow \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{n} \end{pmatrix} v_{\perp} = v_{\perp} \begin{pmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \mathbf{n} \end{pmatrix} & (a) \text{ anticommutativity} \end{cases}
\end{aligned}$$

$$\begin{aligned}
v' &= v'_{\parallel} + v'_{\perp}, \begin{cases} v'_{\parallel} = v_{\parallel} \\ v'_{\perp} = p v_{\perp} = q^2 v_{\perp} \end{cases} \quad q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, |n| = 1 \\
&= v_{\parallel} + q^2 v_{\perp} \stackrel{(u)}{=} 1 v_{\parallel} + q q v_{\perp} \\
&\stackrel{(a)}{=} q q^* v_{\parallel} + q v_{\perp} q^* \stackrel{(c)}{=} q v_{\parallel} q^* + q v_{\perp} q^* \\
&= q (v_{\parallel} + v_{\perp}) q^* = q (v) q^* = q v q^*
\end{aligned}$$


---

$$\begin{aligned}
v' &= qvq^{-1}, q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, |n| = 1 \\
&= qvq^* = q(v_{\parallel} + v_{\perp})q^* \\
&= qv_{\parallel}q^* + qv_{\perp}q^* = qq^*v_{\parallel} + qqv_{\perp} = 1v_{\parallel} + q^2v_{\perp} \\
&= v_{\parallel} + pv_{\perp}, p = q^2 = \cos \theta + \sin \theta n
\end{aligned}$$


---

$$\begin{aligned}
v' &= qvq^* = q(vq^*) \leftrightarrow L(q) R(q^*) v = Q_i Q_r v = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \\
&= (qv)q^* \leftrightarrow R(q^*)L(q)v = Q_r Q_i v = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \\
\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} &= \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i^T \\ \mathbf{q} q_0 + Q_i \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i Q_i^T \end{pmatrix} = L(q) R(q^*) \\
\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} &= \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i \\ \mathbf{q} q_0 + Q_i^T \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i^T Q_i \end{pmatrix} = R(q^*) L(q)
\end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n = \cos \frac{\theta}{2} + bi + cj + dk \\
\Rightarrow Q_i Q_i^T &= \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} = \begin{pmatrix} a^2 + d^2 + c^2 & -cb & -db \\ -cb & d^2 + a^2 + b^2 & -dc \\ -db & -dc & c^2 + b^2 + a^2 \end{pmatrix} \\
&= Q_i^T Q_i = \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} = \begin{pmatrix} a^2 + d^2 + c^2 & -cb & -db \\ -cb & d^2 + a^2 + b^2 & -dc \\ -db & -dc & c^2 + b^2 + a^2 \end{pmatrix} \\
\mathbf{q}^T Q_i^T &= (b \quad c \quad d) \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} = (ba \quad ca \quad da) \\
&= \mathbf{q}^T Q_i = (b \quad c \quad d) \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} = (ba \quad ca \quad da) \\
Q_i \mathbf{q} &= \begin{pmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ab \\ ac \\ ad \end{pmatrix} \\
&= Q_i^T \mathbf{q} = \begin{pmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ab \\ ac \\ ad \end{pmatrix} \\
\begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} &= \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i^T \\ \mathbf{q} q_0 + Q_i \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i Q_i^T \end{pmatrix} = L(q) R(q^*) \\
&= \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i^T \end{pmatrix} \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & Q_i \end{pmatrix} = \begin{pmatrix} q_0^2 - \mathbf{q}^T \mathbf{q} & -q_0 \mathbf{q}^T - \mathbf{q}^T Q_i \\ \mathbf{q} q_0 + Q_i^T \mathbf{q} & -\mathbf{q} \mathbf{q}^T + Q_i^T Q_i \end{pmatrix} = R(q^*) L(q) \\
&\Downarrow \\
L(q) R(q^*) &= R(q^*) L(q) \\
&\Updownarrow \\
v' &= qvq^* = q(vq^*) = (qv)q^*
\end{aligned}$$


---

$$a = \cos \frac{\theta}{2} \Rightarrow \theta = 2 \arccos a = 2 \cos^{-1} a$$

$$\sin \frac{\theta}{2} n = \sin \frac{2 \arccos a}{2} n = \sin (\cos^{-1} a) n$$

$$n = \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\mathbf{n}} \end{pmatrix}, \|n\| = 1, \mathbf{n} = \begin{pmatrix} \sin \gamma \cos \varphi \\ \sin \gamma \sin \varphi \\ \cos \gamma \end{pmatrix}, \begin{cases} \varphi = \angle \hat{\mathbf{n}}_{xy} \hat{\mathbf{x}} \\ \gamma = \angle \hat{\mathbf{n}} \hat{\mathbf{z}} \end{cases}$$

$$\begin{aligned}
q &= a + bi + cj + dk \\
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n, n = \begin{pmatrix} 0 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sin \gamma \cos \varphi \\ \sin \gamma \sin \varphi \\ \cos \gamma \end{pmatrix}, \|n\| = 1, \begin{cases} \varphi = \angle \hat{\mathbf{n}}_{xy} \hat{\mathbf{x}} \\ \gamma = \angle \hat{\mathbf{n}} \hat{\mathbf{z}} \end{cases} \\
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{n}, \mathbf{n} = (\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k} \\
&= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\
&= \cos \frac{\theta}{2} + \mathbf{n} \sin \frac{\theta}{2} \\
q^{-1} &= q^* = \cos \frac{\theta}{2} - \mathbf{n} \sin \frac{\theta}{2} \\
&= \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\
\mathbf{v}' &= v' = qvq^{-1} = q\mathbf{v}q^{-1}, \begin{cases} v = \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \\ v' = \mathbf{v}' = v'_1 \mathbf{i} + v'_2 \mathbf{j} + v'_3 \mathbf{k} \end{cases} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \\
&= q(\theta, \gamma, \varphi) \mathbf{v} q^{-1}(\theta, \gamma, \varphi) \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= q(\theta, n_z, n_x) \mathbf{v} q^{-1}(\theta, n_z, n_x)
\end{aligned}$$

$$\mathbf{v}' = v' = qvq^{-1} = q\mathbf{v}q^{-1}, \begin{cases} v = \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \\ v' = \mathbf{v}' = v'_1 \mathbf{i} + v'_2 \mathbf{j} + v'_3 \mathbf{k} \end{cases}$$

$$\begin{cases} q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \\ q^{-1} = \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \end{cases}$$

### 34.2.3 matrix form

$$q \stackrel{|q|=1}{=} a + bi + cj + dk, \begin{cases} a = \cos \frac{\theta}{2} \\ b = \sin \frac{\theta}{2} \sin \gamma \cos \varphi \\ c = \sin \frac{\theta}{2} \sin \gamma \sin \varphi \\ d = \sin \frac{\theta}{2} \cos \gamma \end{cases}$$

$$L(q) = Q_l = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

$$R(q) = Q_r = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

$$\begin{aligned}
R(q^*) &= \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \\
v' &= qvq^* = (qv)q^* \leftrightarrow R(q^*)L(q)v \\
&= q(vq^*) \leftrightarrow L(q)R(q^*)v \\
&= \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} v \\
&= \begin{pmatrix} a^2 + b^2 + c^2 + d^2 & 0 & 0 & 0 \\ 0 & b^2 + a^2 - d^2 - c^2 & -2ad + 2bc & 2ac + 2bd \\ 0 & 2ad + 2bc & c^2 - d^2 + a^2 - b^2 & 2cd - 2ab \\ 0 & -2ac + 2bd & 2cd + 2ab & d^2 - c^2 - b^2 + a^2 \end{pmatrix} v \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 0 & 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 0 & 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{pmatrix} v \\
\begin{pmatrix} 0 \\ v' \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 0 & 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 0 & 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\
\hline
\mathbf{v}' &= \begin{pmatrix} 1 - 2(d^2 + c^2) & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & 1 - 2(d^2 + b^2) & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & 1 - 2(c^2 + b^2) \end{pmatrix} \mathbf{v}
\end{aligned}$$

$$q \stackrel{|q|=1}{=} a + bi + cj + dk, \quad \begin{cases} a = \cos \frac{\theta}{2} \\ b = \sin \frac{\theta}{2} \sin \gamma \cos \varphi \\ c = \sin \frac{\theta}{2} \sin \gamma \sin \varphi \\ d = \sin \frac{\theta}{2} \cos \gamma \end{cases}$$

#### 34.2.4 exponential form

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd \\
&= a + r \frac{ib + jc + kd}{r}, \quad r^2 = b^2 + c^2 + d^2 \\
&\stackrel{|q|=1}{=} \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{n}, \quad \begin{cases} a = \cos \frac{\theta}{2} & \cos \frac{\theta}{2} = a \\ r = \sin \frac{\theta}{2} & \sin \frac{\theta}{2} = r \\ \frac{ib + jc + kd}{r} = \mathbf{n} & \mathbf{n} = \frac{ib + jc + kd}{r} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{n}\|^2 &= \left( \frac{ib + jc + kd}{r} \right)^2 = \frac{-b^2 - c^2 - d^2 + bc(ij + ji) + cd(jk + kj) + db(ki + ik)}{r^2} \\
&= \frac{-b^2 - c^2 - d^2 + bc(k - k) + cd(i - i) + db(j - j)}{r^2} \\
&= \frac{-b^2 - c^2 - d^2 + bc0 + cd0 + db0}{r^2} = \frac{-b^2 - c^2 - d^2}{r^2} \\
&= \frac{-b^2 - c^2 - d^2}{b^2 + c^2 + d^2} = -1 \\
\mathbf{n} &= \frac{ib + jc + kd}{r} = \pm \sqrt{-1}
\end{aligned}$$

$$\begin{aligned}
q &= a + bi + cj + dk = a + ib + jc + kd \\
&= a + r \frac{ib + jc + kd}{r}, r^2 = b^2 + c^2 + d^2 \\
&= \sqrt{a^2 + r^2} \left( \frac{a}{\sqrt{a^2 + r^2}} + \frac{ib + jc + kd}{r} \frac{r}{\sqrt{a^2 + r^2}} \right) \\
&= \rho (\cos \phi + \sqrt{-1} \sin \phi), \begin{cases} \rho = \sqrt{a^2 + r^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \frac{\theta}{2} \end{cases} \\
&= \rho e^{\phi \sqrt{-1}} |q|^{1/2} e^{\mathbf{n} \frac{\theta}{2}} = e^{\mathbf{n} \frac{\theta}{2}} \\
q &= \rho e^{\phi \sqrt{-1}} |q|^{1/2} e^{\mathbf{n} \frac{\theta}{2}}, \begin{cases} q = a + ib + jc + kd \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} = 1 \\ \tan \frac{\theta}{2} = \frac{r}{a} \Leftrightarrow \frac{\theta}{2} = \arctan \frac{r}{a} = \arctan \frac{\pm \sqrt{b^2 + c^2 + d^2}}{a} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}' &= v' = qvq^{-1} = q\mathbf{v}q^{-1}, \begin{cases} v = \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \\ v' = \mathbf{v}' = v'_1\mathbf{i} + v'_2\mathbf{j} + v'_3\mathbf{k} \end{cases}, \mathbf{n} = \begin{pmatrix} \sin \gamma \cos \varphi \\ \sin \gamma \sin \varphi \\ \cos \gamma \end{pmatrix}, \begin{cases} \varphi = \angle \hat{\mathbf{n}}_{xy} \hat{\mathbf{x}} \\ \gamma = \angle \hat{\mathbf{n}} \hat{\mathbf{z}} \end{cases} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [(\sin \gamma \cos \varphi) \mathbf{i} + (\sin \gamma \sin \varphi) \mathbf{j} + (\cos \gamma) \mathbf{k}] \right\} \\
&= q(\theta, \gamma, \varphi) \mathbf{v} q^{-1}(\theta, \gamma, \varphi) = q \left( \frac{\theta}{2}, \gamma, \varphi \right) \mathbf{v} q^{-1} \left( \frac{\theta}{2}, \gamma, \varphi \right) \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} \pm \sqrt{1 - n_x^2 - n_z^2} \mathbf{j} + n_z \mathbf{k}] \right\} \\
&= \left\{ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \right\} \mathbf{v} \\
&\quad \left\{ \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \right\} \\
&= q(n_x, n_y, \theta) \mathbf{v} q^{-1}(n_x, n_y, \theta) = q \left( n_x, n_y, \frac{\theta}{2} \right) \mathbf{v} q^{-1} \left( n_x, n_y, \frac{\theta}{2} \right) \\
&= e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}' &= q\mathbf{v}q^{-1}, \\
\begin{cases} q = q(n_x, n_y, \theta) &= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \\ q^{-1} = q^* = \bar{q} &= \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \end{cases}, \\
\begin{cases} n_x &= \sin \gamma \cos \varphi \\ n_y &= \sin \gamma \sin \varphi \end{cases} &\Rightarrow \pm \sqrt{1 - n_x^2 - n_y^2} = n_z = \cos \gamma
\end{aligned}$$

$$\mathbf{v}' = e^{\mathbf{n} \frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n} \frac{\theta}{2}}$$

$$q = \rho e^{\phi\sqrt{-1}} = |q| e^{\mathbf{n}\phi}, \begin{cases} q = a + ib + jc + kd = a + r \frac{ib + jc + kd}{r} = a + r\mathbf{n} \\ \rho = \sqrt{a^2 + r^2} = |q| = \sqrt{a^2 + b^2 + c^2 + d^2} \\ \tan \phi = \frac{r}{a} \Leftrightarrow \phi = \arctan \frac{r}{a} = \arctan \frac{\pm\sqrt{b^2 + c^2 + d^2}}{a} \end{cases}$$

$$= e^{\ln|q|} e^{\mathbf{n}\phi} = e^{\ln|q| + \mathbf{n}\phi}$$

### 34.2.5 double cover

<https://www.bilibili.com/video/BV1rj41117VW/?t=15m15s>

$$\begin{array}{ccccccccccccccccc} \theta & & 0 & \rightarrow & \frac{\pi}{2} & \rightarrow & \pi & \rightarrow & \frac{3\pi}{2} & \rightarrow & 2\pi & \rightarrow & \frac{5\pi}{2} & \rightarrow & 3\pi & \rightarrow & \frac{7\pi}{2} & \rightarrow & 4\pi \\ \theta & & 0 & \rightarrow & \frac{\pi}{4} & \rightarrow & \frac{\pi}{2} & \rightarrow & \frac{3\pi}{4} & \rightarrow & \pi & \rightarrow & \frac{5\pi}{4} & \rightarrow & \frac{3\pi}{2} & \rightarrow & \frac{7\pi}{4} & \rightarrow & 2\pi \\ \frac{\theta}{2} & & 0 & \rightarrow & \frac{1}{4} & \rightarrow & \frac{1}{2} & \rightarrow & \frac{3}{4} & \rightarrow & 1 & \rightarrow & \frac{5}{4} & \rightarrow & \frac{3}{2} & \rightarrow & \frac{7}{4} & \rightarrow & 1 \\ e^{\mathbf{n}\frac{\theta}{2}} & & 1 & \rightarrow & e^{\mathbf{n}\frac{\pi}{4}} & \rightarrow & \mathbf{n} & \rightarrow & e^{\mathbf{n}\frac{3\pi}{4}} & \rightarrow & -1 & \rightarrow & e^{\mathbf{n}\frac{5\pi}{4}} & \rightarrow & -\mathbf{n} & \rightarrow & e^{\mathbf{n}\frac{7\pi}{4}} & \rightarrow & 1 \\ e^{-\mathbf{n}\frac{\theta}{2}} & & 1 & \rightarrow & e^{\mathbf{n}\frac{-\pi}{4}} & \rightarrow & -\mathbf{n} & \rightarrow & e^{\mathbf{n}\frac{-3\pi}{4}} & \rightarrow & -1 & \rightarrow & e^{\mathbf{n}\frac{-5\pi}{4}} & \rightarrow & \mathbf{n} & \rightarrow & e^{\mathbf{n}\frac{-7\pi}{4}} & \rightarrow & 1 \\ \mathbf{v}' = e^{\mathbf{n}\frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n}\frac{\theta}{2}} & \mathbf{v} & \rightarrow & & \rightarrow & -\mathbf{n} \mathbf{v} \mathbf{n} & \rightarrow & & & \rightarrow & \mathbf{v} & \rightarrow & & & \rightarrow & -\mathbf{n} \mathbf{v} \mathbf{n} & \rightarrow & & \mathbf{v} \end{array}$$

$$SU(2) = \left\{ 1(\cdot) 1, e^{\mathbf{n}\frac{\theta}{2}}(\cdot) e^{-\mathbf{n}\frac{\theta}{2}} \right\}$$

3D rotation about an arbitrary axis<sup>[34.1.4.2.1.3]</sup>

$$\mathbf{V}' = SR_z(\gamma) S^{-1} \mathbf{V}$$

$$\mathbf{v}' = SR_z(\theta) S^{-1} \mathbf{v}$$

$$SO(3) = \left\{ 1, SR_z(\theta) S^{-1} \right\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, SR_z(\theta) S^{-1} \right\}$$

$$SU(2) \cong 2 \cdot SO(3)$$

Special unitary group of degree 2  $SU(2)$  is the boss of special orthonormal group of degree 3  $SO(3)$ ,  $SU(2)$  double covers  $SO(3)$ .

[https://en.wikipedia.org/wiki/Special\\_unitary\\_group#The\\_group\\_SU\(2\)](https://en.wikipedia.org/wiki/Special_unitary_group#The_group_SU(2))

[https://en.wikipedia.org/wiki/Unitary\\_group](https://en.wikipedia.org/wiki/Unitary_group)

“Particle is group representation, spin is square root of 4-vector.”?

「粒子是群表示，自旋是四維向量開根號。」？

See also **spinor**<sup>[72]</sup> and **Lie group**<sup>[34.2.8]</sup>

<https://zhuanlan.zhihu.com/p/143972449>

### 34.2.6 right-hand vs. left-hand coordinates

Krasjet 四元數與三維旋轉 p.63 73

#### 34.2.6.1 clock

$$\mathbf{v}' = e^{\mathbf{n}\frac{\theta}{2}} \mathbf{v} e^{-\mathbf{n}\frac{\theta}{2}}$$

$$\mathbf{v}' = e^{-\mathbf{n}\frac{\theta}{2}} \mathbf{v} e^{\mathbf{n}\frac{\theta}{2}}$$

right-hand = anticlockwise

$$[\mathbf{v}']_R = \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_R [\mathbf{v}]_R \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_R$$

$$[\mathbf{v}']_L = \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_R [\mathbf{v}]_L \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_R$$

$$[\mathbf{v}']_L = \left[ e^{\mathbf{n} \frac{\theta}{2}} \right]_L [\mathbf{v}]_L \left[ e^{-\mathbf{n} \frac{\theta}{2}} \right]_L$$

### 34.2.6.2 Rodrigues rotation in right-hand vs. left-hand coordinates

$$\mathbf{u} = \mathbf{n} \times \mathbf{v}_\perp$$

$$[\mathbf{u}]_R = [\mathbf{n}]_R \times_R [\mathbf{v}_\perp]_R = [\mathbf{n}]_L \times_L [\mathbf{v}_\perp]_L = [\mathbf{u}]_L$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_R \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times_L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[[\mathbf{u}]_L]_R = -[\mathbf{u}]_R$$

$$\left[ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_L \right]_R = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_R$$

$$\det O = \pm 1$$

$\det O = -1$  will change handedness of the coordinate.

### 34.2.7 gimbal lock

[https://krasjet.github.io/quaternion/bonus\\_gimbal\\_lock.pdf](https://krasjet.github.io/quaternion/bonus_gimbal_lock.pdf)

<https://www.zhihu.com/question/47736315/answer/3146533323>

### 34.2.8 Lie group

<https://zhuanlan.zhihu.com/p/684801875>

<https://www.zhihu.com/question/541029773/answer/3327615524>

#### 34.2.8.1 Mathemaniac

[https://www.youtube.com/playlist?list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP](https://www.youtube.com/playlist?list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP)

[https://www.youtube.com/watch?v=ZRca3Ggpy\\_g&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=3](https://www.youtube.com/watch?v=ZRca3Ggpy_g&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=3)

[https://www.youtube.com/watch?v=ZRca3Ggpy\\_g&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=3&t=7m23s](https://www.youtube.com/watch?v=ZRca3Ggpy_g&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=3&t=7m23s)

$$\begin{aligned} \mathbf{v}' &= q \mathbf{v} q^{-1}, \\ \begin{cases} q = q(n_x, n_y, \theta) &= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \\ q^{-1} = q^* = \bar{q} &= \cos \frac{\theta}{2} - \sin \frac{\theta}{2} [n_x \mathbf{i} + n_y \mathbf{j} \pm \sqrt{1 - n_x^2 - n_y^2} \mathbf{k}] \end{cases}, \\ \begin{cases} n_x &= \sin \gamma \cos \varphi \\ n_y &= \sin \gamma \sin \varphi \end{cases} &\Rightarrow \pm \sqrt{1 - n_x^2 - n_y^2} = n_z = \cos \gamma \end{aligned}$$

$q$  or  $q(\cdot) q^{-1} = e^{\mathbf{n} \frac{\theta}{2}}(\cdot) e^{-\mathbf{n} \frac{\theta}{2}}$  corresponding 3D manifold is a solid ball by coordinate  $(n_x, n_y, \theta)$

$$q = q(n_x, n_y, \theta) = q\left(\begin{pmatrix} n_x \\ n_y \\ \theta \end{pmatrix}\right) = q\left(\begin{pmatrix} n_x \\ n_y \\ \theta \end{pmatrix}\right) = q\left(\begin{pmatrix} n_x \\ n_y \\ \theta \end{pmatrix}\right) = q\left(\begin{pmatrix} \sin \gamma \cos \varphi \\ \sin \gamma \sin \varphi \\ \theta \end{pmatrix}\right), \begin{cases} \gamma \in [0, \pi] \\ \varphi \in [-\pi, \pi] \\ \theta \in [-\pi, \pi] \end{cases}$$

[https://www.youtube.com/watch?v=9CBS5CAynBE&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=4](https://www.youtube.com/watch?v=9CBS5CAynBE&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=4)

See also (by Elliot Schneider) Taylor expansion or Taylor series<sup>[45.1]</sup> Elliot Schneider section or

<https://www.youtube.com/watch?v=HQsZG8Yxb7w>

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=12m25s>

[https://www.youtube.com/watch?v=B2PJh2K-jdU&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=5](https://www.youtube.com/watch?v=B2PJh2K-jdU&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=5)

See also (by Elliot Schneider) <https://www.youtube.com/watch?v=0kY3Wpvutfs> Hamiltonian flow and its relation to matrix exponential

<https://www.youtube.com/watch?v=0kY3Wpvutfs&t=22m45s>

[https://www.youtube.com/watch?v=gj4kvpy1eCE&list=PLDcSwjT2BF\\_WDki-WvmJ\\_\\_Q0nLIHuNPbP&index=6](https://www.youtube.com/watch?v=gj4kvpy1eCE&list=PLDcSwjT2BF_WDki-WvmJ__Q0nLIHuNPbP&index=6)

See also Shinonome Masaki<sup>[72.2]</sup> especially

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 與 SO(3) 的夢幻聯動 <https://zhuanlan.zhihu.com/p/341625428>

### 34.2.8.2 Liang, Can-Bin

<https://www.bilibili.com/video/BV1pV4y1a7y5>

<https://www.youtube.com/playlist?list=PLstdOGDXMaWLhtcJwsGiw6V9YLGmCmmhO>

周彬

<https://www.youtube.com/playlist?list=PLstdOGDXMaWK4JpSoKjH8mnfxXwh9dtb6>

### 34.2.8.3 special unitary group

### 34.2.9 geometric algebra and Clifford algebra

[https://en.wikipedia.org/wiki/Geometric\\_algebra](https://en.wikipedia.org/wiki/Geometric_algebra)

[https://en.wikipedia.org/wiki/Clifford\\_algebra](https://en.wikipedia.org/wiki/Clifford_algebra)

John Vince\_2009\_Geometric Algebra\_ An Algebraic System for Computer Games and Animation

<https://www.amazon.com/Geometric-Algebra-Algebraic-Computer-Animation/dp/1848823789>

### 34.2.10 dual quaternion

### 34.2.11 interpolation

34.2.11.1 Lerp

34.2.11.2 Nlerp

34.2.11.3 Slerp

## 34.3 visualization

<https://zhuanlan.zhihu.com/p/684379741>

<https://space.bilibili.com/3546564685466124/video>

## 34.4 3Blue1Brown

### 34.4.1 Ben Eater

<https://eater.net/quaternions/video/intro>

<https://eater.net/quaternions>

### 34.4.2 3B1B

<https://www.youtube.com/watch?v=zjMuIxRvygQ>

<https://www.youtube.com/watch?v=d4EgbgTm0Bg>

### 34.4.3 Sutrabla

[https://www.youtube.com/watch?v=syQnn\\_xuB8U](https://www.youtube.com/watch?v=syQnn_xuB8U)

<https://www.youtube.com/watch?v=4mXL751ko0w>

<https://www.newscientist.com/article/mg20427391-600-alices-adventures-in-algebra-wonderland-solved/>

<https://threejs.org/>

## 34.5 CCJou: LA Revelation

<https://ccjou.wordpress.com/2014/04/21/四元數/>



# **Chapter 35**

## **DICOM**

### **35.1 Innolitics: DICOM Standard Browser**

<https://dicom.innolitics.com/ciods>

<https://dicom.innolitics.com/ciods/parametric-map/parametric-map-multi-frame-functional-groups/52009229/0048021a/040072a>

### **35.2 David Clunie: Medical Image Format Site**

<https://www.dclunie.com/>

### **35.3 Hsiao, Chia-Hung**

<https://www.youtube.com/@user-zp9yy1ln6u/videos>



# **Chapter 36**

## **tendon pathophysiology**

Mark E. Schweitzer, MD

<https://www.youtube.com/watch?v=uIo58pQgxhY>

### **36.1 histology**

MSk MRI

Lee, Jee Eun

<https://www.youtube.com/@MSKMRI/playlists>



# Chapter 37

## group theory

<https://www.bilibili.com/video/BV1hm4y1v75R>

[https://en.wikipedia.org/wiki/Group\\_\(mathematics\)](https://en.wikipedia.org/wiki/Group_(mathematics))

### 37.1 matrix group

subset of two-by-two matrices at least excluding zero matrix

$$\mathcal{M} = (\mathcal{M}, \cdot) \subset (\mathcal{M}_{2 \times 2}(\mathbb{C}) - \{0\}, \cdot) = \mathcal{M}_{2 \times 2}(\mathbb{C}) - \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

matrix multiplication

$$\begin{aligned} & \forall \langle M_1, M_2 \rangle \in \mathcal{M}^2, \exists M_1 M_2 \in \mathcal{M} [M_1 M_2 = M_1 \cdot M_2] \\ \Leftrightarrow & \cdot : \mathcal{M} \times \mathcal{M} = \mathcal{M}^2 \rightarrow \mathcal{M} \\ \Leftrightarrow & \cdot : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \\ \Leftrightarrow & \mathcal{M} \times \mathcal{M} \stackrel{\cdot}{\rightarrow} \mathcal{M} \\ \Leftrightarrow & \mathcal{M}^2 \stackrel{\cdot}{\rightarrow} \mathcal{M} \end{aligned}$$

matrix group

$$\begin{cases} \forall \langle M_1, M_2, M_3 \rangle \in \mathcal{M}^3 [M_1 (M_2 M_3) = (M_1 M_2) M_3] & \text{associativity} \\ \exists I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}, \forall M \in \mathcal{M} [IM = M] & \text{left unit element} \\ \forall M \in \mathcal{M}, \exists M^{-1} \in \mathcal{M} [M^{-1} M = I] & \text{left inverse (element)} \end{cases}$$

$\Rightarrow \mathcal{M} = (\mathcal{M}, \cdot)$  is a matrix group

### 37.2 group definition and basic theorem

[https://en.wikipedia.org/wiki/Group\\_\(mathematics\)#Elementary\\_consequences\\_of\\_the\\_group\\_axioms](https://en.wikipedia.org/wiki/Group_(mathematics)#Elementary_consequences_of_the_group_axioms)

**Definition 37.1** (group). group definition by a set and a binary operation on the set

$$\begin{cases} \circ : G \times G = G^2 \rightarrow G & \text{binary operation} \\ \forall \langle g_1, g_2, g_3 \rangle \in G^3 [g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3] & \text{associativity} \\ \exists e = \in G, \forall g \in G [e \circ g = g] & \text{left unit element} \\ \forall g \in G, \exists g^{-1} \in G [g^{-1} \circ g = e] & \text{left inverse (element)} \end{cases}$$

$\Leftrightarrow G = (G, \circ)$  is a group

**Theorem 37.1.** group left inverses equal right inverses

Proof:

to be proved

□

### 37.3 EpicOrganism = AIRoswell = Pan, Yi-Wen<sup>1</sup>

<https://space.bilibili.com/14316464/video>

<https://space.bilibili.com/14316464/channel/collectiondetail?sid=1768137>

<https://www.bilibili.com/video/BV1mC4y1Z78k>

<https://www.bilibili.com/video/BV1Sp4y1w76M>

<https://www.bilibili.com/video/BV1cc411o7Hc>

<https://www.bilibili.com/video/BV1d8411v7t9>

<https://www.bilibili.com/video/BV1Rw411e7km>

<https://www.bilibili.com/video/BV1Q84y127ZQ>

<https://www.bilibili.com/video/BV1hm4y1g7A9>

<https://www.bilibili.com/video/BV1s94y1L7MU>

<https://www.bilibili.com/video/BV1j84y1d7eY>

<https://www.bilibili.com/video/BV18z4y1P7oB>

<https://www.bilibili.com/video/BV1cN4y1m77z>

<https://www.bilibili.com/video/BV18N4y1a7a3>

<https://www.bilibili.com/video/BV1Ma4y1r7nT>

<https://www.bilibili.com/video/BV11N4y187QW>

### 37.4 PKU-CFCS

<https://www.bilibili.com/video/BV11r421E7kD>

### 37.5 Mathemaniac

[https://www.youtube.com/playlist?list=PLDcSwjT2BF\\_VuNbn8HiHZKKy59SgnIAeO](https://www.youtube.com/playlist?list=PLDcSwjT2BF_VuNbn8HiHZKKy59SgnIAeO)

### 37.6 Polya enumeration theorem

<https://www.bilibili.com/video/BV17s4y1R7fW>

### 37.7 complex group representation

#### 37.7.1 complex basis group

$$\begin{aligned} G &= \{1, i, -1, -i\} \\ &= \{i^0, i^1, i^2, i^3\} \end{aligned}$$

$$\begin{aligned} \forall \langle g_1, g_2 \rangle \in G^2, \exists g_1 g_2 \in G [g_1 g_2 = g_1 \cdot g_2] \\ \Leftrightarrow \cdot : G \times G = G^2 \rightarrow G \end{aligned}$$

---

<sup>1</sup><https://www.linkedin.com/in/yiwen-pan-16a90076>

<https://tex.stackexchange.com/questions/627708/tikz-how-to-put-tables-within-arbitrarily-placed-nodes>

| .  | 1  | i  | -1 | -i |
|----|----|----|----|----|
| 1  | 1  | i  | -1 | -i |
| i  | i  | -1 | -i | 1  |
| -1 | -1 | -i | 1  | i  |
| -i | -i | 1  | i  | -1 |

| .  | 1  | i  | -1 | -i |
|----|----|----|----|----|
| 1  | 1  | i  | -1 | -i |
| i  | i  | -1 | -i | 1  |
| -1 | -1 | -i | 1  | i  |
| -i | -i | 1  | i  | -1 |

Figure 37.1: complex basis group table

### 37.7.2 $\mathbb{C} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = I$$

$$\begin{aligned} c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ &= xI + yJ, J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R}), \langle x, y \rangle \in \mathbb{R}^2 \end{aligned}$$


---

$$J^2 = -I$$


---

$$J^2 = -I$$

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ca + cd & cb + d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} a^2 + bc = -1 & b = 0 \Rightarrow a^2 = -1 \Rightarrow a \in \mathbb{R} \Rightarrow b \neq 0 \\ ab + bd = 0 & (b = 0) \vee (a = -d) \xrightarrow{b \neq 0} a = -d \quad \text{if } a = d \\ ca + cd = 0 & \\ cb + d^2 = -1 & a^2 = d^2 \Rightarrow (a = d) \vee (a = -d) \end{cases} \Rightarrow a = d = 0 \Rightarrow bc = -1$$

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} = J(a, b), b \neq 0$$

$$J(a, b) = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix}, b \neq 0$$


---

$$J(a = 1, b) = \begin{pmatrix} 1 & b \\ -2 & -1 \end{pmatrix} \Rightarrow J^2(a = 1, b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$xI + yJ(a = 1, b) = \begin{pmatrix} x + y & yb \\ y \cdot \frac{-2}{b} & x - y \end{pmatrix}$$


---

$$J(a = 0, b) = \begin{pmatrix} 0 & b \\ \frac{-1}{b} & 0 \end{pmatrix}$$

$$J(a=0, b=1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{aligned} J(a=0, b=-1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J(a=0, b=1) \\ \Rightarrow J^2(a=0, b=-1) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \end{aligned}$$

$$\begin{aligned} J &= J(a=0, b=-1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} 1 \leftrightarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \leftrightarrow J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases} \\ \Rightarrow x + yi &\leftrightarrow xI + yJ \\ &= x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{aligned}$$

$$x + yi \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

realizing

$$\mathbb{C} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R}) = \mathcal{M}_2(\mathbb{R})$$

### 37.7.3 ( determinant of complex group representation ) equivalent to ( squared modulus of complex number )

$$\det(xI + yJ) = \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 = |x + yi|^2$$

#### 37.7.3.1 Lagrange identity

Lagrange identity

generalization of Brahmagupta–Fibonacci identity

specialization of Binet–Cauchy identity

cf. Euler identity<sup>[37.8.1.1]</sup>

$$\begin{aligned} &\det[(aI + bJ)(cI + dJ)] \\ &= \det \left[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \right] \\ &= \det \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix} = |(ac - bd) + (ad + bc)i|^2 = (ac - bd)^2 + (ad + bc)^2 \\ &= \left[ \det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right] \left[ \det \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \right] = |a + bi|^2 |c + di|^2 = (a^2 + b^2)(c^2 + d^2) \end{aligned}$$

$$|a + bi|^2 |c + di|^2 = (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

$$\begin{aligned}
& \det [(x_1 I + y_1 J)(x_2 I + y_2 J)] \\
&= \det \left[ \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \right] \\
&= \det \begin{pmatrix} x_1 x_2 - y_1 y_2 & -x_1 y_2 - y_1 x_2 \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix} = |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i|^2 = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2 \\
&= \left[ \det \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \right] \left[ \det \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \right] = |x_1 + y_1 i|^2 |x_2 + y_2 i|^2 = (x_1^2 + y_1^2) (x_2^2 + y_2^2)
\end{aligned}$$

$$|x_1 + y_1 i|^2 |x_2 + y_2 i|^2 = (x_1^2 + y_1^2) (x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2$$

### 37.7.4 Euler formula proved by complex group representation

<https://www.bilibili.com/video/BV1mM4y1J79a>

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \cos n \frac{\theta}{n} & -\sin n \frac{\theta}{n} \\ \sin n \frac{\theta}{n} & \cos n \frac{\theta}{n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{n} & -\sin \frac{\theta}{n} \\ \sin \frac{\theta}{n} & \cos \frac{\theta}{n} \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} &= \lim_{n \rightarrow \infty} \begin{pmatrix} \cos \frac{\theta}{n} & -\sin \frac{\theta}{n} \\ \sin \frac{\theta}{n} & \cos \frac{\theta}{n} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{n} \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} \\
&= I + \frac{\theta}{n} R_{\frac{\pi}{2}}
\end{aligned}$$


---

$$\lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} = \lim_{n \rightarrow \infty} \begin{pmatrix} \cos \frac{\theta}{n} & -\sin \frac{\theta}{n} \\ \sin \frac{\theta}{n} & \cos \frac{\theta}{n} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta}{n} \\ \frac{\theta}{n} & 1 \end{pmatrix} = I + \frac{\theta}{n} R_{\frac{\pi}{2}}$$


---

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \lim_{n \rightarrow \infty} \begin{pmatrix} x' \\ y' \end{pmatrix} = \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \begin{pmatrix} x \\ y \end{pmatrix} = \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \right] \left[ \lim_{n \rightarrow \infty} \begin{pmatrix} x \\ y \end{pmatrix} \right] = \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}}^n \right] \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n \rightarrow \infty} \left[ \lim_{n \rightarrow \infty} R_{\frac{\theta}{n}} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \rightarrow \infty} \left[ I + \frac{\theta}{n} R_{\frac{\pi}{2}} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \lim_{n \rightarrow \infty} \left[ I + \frac{\theta}{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]^n \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n \rightarrow \infty} \left[ I + \frac{\theta J}{n} \right]^n \begin{pmatrix} x \\ y \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= e^{J\theta} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

$$\begin{cases} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} x' \\ y' \end{pmatrix} = e^{J\theta} \begin{pmatrix} x \\ y \end{pmatrix} \end{cases}$$

$$\Rightarrow e^{J\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \stackrel{x+y \leftrightarrow}{=} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = xI + yJ$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

□

## 37.8 quaternion group representation

[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Two-dimensional\\_irreducible\\_representation\\_over\\_a\\_splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Two-dimensional_irreducible_representation_over_a_splitting_field)

$$\begin{aligned} q &= a + bi + cj + dk = a + ib + jc + kd, \begin{cases} q \in \mathbb{H} \\ a, b, c, d \in \mathbb{R} \end{cases} \Leftrightarrow \langle a, b, c, d \rangle \in \mathbb{R}^4 \\ w &= t + xi + yj + zk = t + ix + jy + kz, \begin{cases} w \in \mathbb{H} \\ t, x, y, z \in \mathbb{R} \end{cases} \Leftrightarrow \langle t, x, y, z \rangle \in \mathbb{R}^4 \\ &= a1 + bi + cj + dk = t1 + xi + yj + zk = x_0 1 + e_i x_i \end{aligned}$$

37.8.1  $\mathbb{H} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C})$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_2 = 1$$

$$\begin{aligned} \mathbf{e} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} = \begin{pmatrix} 0 & b \\ \frac{-1}{b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \Rightarrow \mathbf{e}_2 = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = j & a = 0 \\ \begin{pmatrix} a & b \\ \frac{-a^2 - 1}{b} & -a \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} & a \neq 0 \end{cases} \\ \begin{pmatrix} \alpha^2 + \beta^2 & 0 \\ 0 & \beta^2 + \alpha^2 \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = \mathbf{e}^2 = -1 = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\Downarrow \\ \alpha^2 + \beta^2 = -1 &\Leftrightarrow \beta^2 + \alpha^2 = -1 \end{aligned}$$

$$\alpha^2 + \beta^2 = -1 \Rightarrow \langle \alpha, \beta \rangle \notin \mathbb{R}^2 \Rightarrow \langle \alpha, \beta \rangle \in \mathbb{R}^2 \Rightarrow \alpha^2 + \beta^2 \geq 0$$

quaternion group has no irreducible two-dimensional representation over the reals <sup>2</sup>

$$\langle \alpha, \beta \rangle \in \mathbb{C}^2 - \mathbb{R}^2$$

$$\alpha^2 + \beta^2 = -1 = \beta^2 + \alpha^2$$

$$\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} \begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} \beta^2 + \alpha^2 & 0 \\ 0 & \alpha^2 + \beta^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$ij = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} = k$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e}_1 = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

<sup>2</sup>[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Four-dimensional\\_irreducible\\_representation\\_over\\_a\\_non-splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field)

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, k = \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$$

$$jk = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = i$$


---

|                           |                                                                     |                                                                  |                                                                   |
|---------------------------|---------------------------------------------------------------------|------------------------------------------------------------------|-------------------------------------------------------------------|
| $\alpha^2 + \beta^2 = -1$ | $\begin{cases} \alpha = \sqrt{-1} \\ \beta = 0 \end{cases}$         | $\begin{cases} \alpha = \sqrt{-2} \\ \beta = 1 \end{cases}$      | $\beta = \alpha^2, n \in \{1, 2, 4, 5\}$                          |
| 1                         | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                      | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                   | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                    |
| -1                        | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                    | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                 | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                  |
| i                         | $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$   | $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$  | $\begin{pmatrix} \sqrt{-2} & 1 \\ 1 & -\sqrt{-2} \end{pmatrix}$   |
| -i                        | $\begin{pmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$ | $\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$  | $\begin{pmatrix} -\sqrt{-2} & -1 \\ -1 & \sqrt{-2} \end{pmatrix}$ |
| j                         | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                     | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                  | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                   |
| -j                        | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                     | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                  | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                   |
| k                         | $\begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$ | $\begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & -\sqrt{-2} \\ -\sqrt{-2} & -1 \end{pmatrix}$ |
| -k                        | $\begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix}$   | $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$   | $\begin{pmatrix} -1 & \sqrt{-2} \\ \sqrt{-2} & 1 \end{pmatrix}$   |

---

$$\begin{aligned} -1 &= \alpha^2 + \beta^2 \\ &\stackrel{\beta=\alpha^2}{=} \alpha^2 + \alpha^4 \\ \alpha^4 + \alpha^2 + 1 &= 0, \alpha^4 + \alpha^2 + 1 = (\alpha^2 + \alpha + 1)(\alpha^2 - \alpha + 1) \\ (\alpha^2 - 1)(\alpha^4 + \alpha^2 + 1) &= 0 \\ \alpha^6 - 1 &= 0 \\ \alpha^6 &= 1 = e^{2\pi k \sqrt{-1}}, k \in \mathbb{Z} \\ \alpha &= e^{2\pi \frac{n}{6} \sqrt{-1}}, n \in \{0, 1, 2, 3, 4, 5\} - \{0, 3\} \\ &= e^{\pi \frac{n}{3} \sqrt{-1}}, n \in \{1, 2, 4, 5\} \end{aligned}$$


---

|                           |                                                                     |                                                             |                                                                   |
|---------------------------|---------------------------------------------------------------------|-------------------------------------------------------------|-------------------------------------------------------------------|
| $\alpha^2 + \beta^2 = -1$ | $\begin{cases} \alpha = i \\ \beta = 0 \end{cases}$                 | $\begin{cases} \alpha = \sqrt{2}i \\ \beta = 1 \end{cases}$ | $\omega = e^{i\pi \frac{n}{3}}, n \in \{1, 2, 4, 5\}$             |
| 1                         | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                      | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$              | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                    |
| -1                        | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                    | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$            | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                  |
| i                         | $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$   | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$             | $\begin{pmatrix} \sqrt{2}i & 1 \\ 1 & -\sqrt{2}i \end{pmatrix}$   |
| -i                        | $\begin{pmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$ | $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$             | $\begin{pmatrix} -\sqrt{2}i & -1 \\ -1 & \sqrt{2}i \end{pmatrix}$ |
| j                         | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                     | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$             | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                   |
| -j                        | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                     | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$             | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                   |
| k                         | $\begin{pmatrix} \beta & -\alpha \\ -\alpha & -\beta \end{pmatrix}$ | $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$            | $\begin{pmatrix} 1 & -\sqrt{2}i \\ -\sqrt{2}i & -1 \end{pmatrix}$ |
| -k                        | $\begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix}$   | $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$              | $\begin{pmatrix} -1 & \sqrt{2}i \\ \sqrt{2}i & 1 \end{pmatrix}$   |

realizing

$$\mathbb{H} \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{C}) = \mathcal{M}_2(\mathbb{C})$$

### 37.8.1.1 Euler identity

cf. Lagrange identity<sup>[37.7.3.1]</sup>

$$\begin{aligned} & \det(a + bi + cj + dk) \\ &= \det \left[ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ &= \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} = \begin{vmatrix} a + bi & -c - di \\ c - di & a - bi \end{vmatrix} \\ &= (a^2 + b^2) + (c^2 + d^2) = a^2 + b^2 + c^2 + d^2 \end{aligned}$$

$$\det(a + bi + cj + dk) = \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} = a^2 + b^2 + c^2 + d^2$$

$$\begin{aligned} & \det[(q_{10} + q_{11}i + q_{12}j + q_{13}k)(q_{20} + q_{21}i + q_{22}j + q_{23}k)] = \det[(a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k)] \\ &= \det \left\{ \left[ q_{10} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_{11} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + q_{12} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + q_{13} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right. \\ &\quad \left. \left[ q_{20} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_{21} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + q_{22} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + q_{23} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right\} \\ &= \det \left\{ \left[ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right. \\ &\quad \left. \left[ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \gamma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \right\} \\ &= \det \left\{ \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} \begin{pmatrix} \alpha + \beta i & -\gamma - \delta i \\ \gamma - \delta i & \alpha - \beta i \end{pmatrix} \right\} \\ &= \det \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix} \det \begin{pmatrix} \alpha + \beta i & -\gamma - \delta i \\ \gamma - \delta i & \alpha - \beta i \end{pmatrix} = (a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\ &= \det \left\{ \begin{pmatrix} [a + bi][\alpha + \beta i] - [c + di][\gamma - \delta i] & [a + bi][- \gamma - \delta i] - [c + di][\alpha - \beta i] \\ [c - di][\alpha + \beta i] + [a - bi][\gamma - \delta i] & [c - di][- \gamma - \delta i] + [a - bi][\alpha - \beta i] \end{pmatrix} \right\} \\ &= \det \left\{ \begin{pmatrix} ((a\alpha - b\beta - c\gamma - d\delta) + i(a\beta + b\alpha + c\delta - d\gamma)) & -(a\gamma - b\delta + c\alpha + d\beta) - i(a\delta + b\gamma - c\beta + d\alpha) \\ ((a\gamma - b\delta + c\alpha + d\beta) - i(a\delta + b\gamma - c\beta + d\alpha)) & (a\alpha - b\beta - c\gamma - d\delta) - i(a\beta + b\alpha + c\delta - d\gamma) \end{pmatrix} \right\} \\ &= \det \{(a\alpha - b\beta - c\gamma - d\delta) + (a\beta + b\alpha + c\delta - d\gamma)i + (a\gamma - b\delta + c\alpha + d\beta)j + (a\delta + b\gamma - c\beta + d\alpha)k\} \\ &= (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2 \\ &= (q_{10}q_{20} - q_{11}q_{21} - q_{12}q_{22} - q_{13}q_{23})^2 + (q_{10}q_{21} + q_{11}q_{20} + q_{12}q_{23} - q_{13}q_{22})^2 \\ &\quad + (q_{10}q_{22} - q_{11}q_{23} + q_{12}q_{20} + q_{13}q_{21})^2 + (q_{10}q_{23} + q_{11}q_{22} - q_{12}q_{21} + q_{13}q_{20})^2 \\ &\quad (a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\ &= (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 \\ &\quad + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2 \end{aligned}$$

**Theorem 37.2.** For any two integers greater than zero, their multiplication can be the summation of squared four integers greater than zero.

$$\forall \langle m_1, m_2 \rangle \in (\mathbb{N} \cup \{0\})^2, \exists \langle k_1, k_2, k_3, k_4 \rangle \in (\mathbb{N} \cup \{0\})^4 [m_1 m_2 = k_1^2 + k_2^2 + k_3^2 + k_4^2]$$

Proof:

$$\text{Let } \begin{cases} m_1 = a^2 + b^2 + c^2 + d^2 & \langle a, b, c, d \rangle \in (\mathbb{N} \cup \{0\})^4 \\ m_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & \langle \alpha, \beta, \gamma, \delta \rangle \in (\mathbb{N} \cup \{0\})^4 \end{cases} \xrightarrow{\text{closure property}} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 + d^2 \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{pmatrix} \in (\mathbb{N} \cup \{0\})^2,$$

$$\begin{aligned}
m_1 m_2 &= (a^2 + b^2 + c^2 + d^2) (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\
&\stackrel{\text{Euler identity}}{=} (a\alpha - b\beta - c\gamma - d\delta)^2 + (a\beta + b\alpha + c\delta - d\gamma)^2 \\
&\quad + (a\gamma - b\delta + c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta + d\alpha)^2 \\
&= |a\alpha - b\beta - c\gamma - d\delta|^2 \\
&\quad + |a\beta + b\alpha + c\delta - d\gamma|^2 \\
&\quad + |a\gamma - b\delta + c\alpha + d\beta|^2 \\
&\quad + |a\delta + b\gamma - c\beta + d\alpha|^2 \\
&= k_1^2 + k_2^2 + k_3^2 + k_4^2, \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} |a\alpha - b\beta - c\gamma - d\delta| \\ |a\beta + b\alpha + c\delta - d\gamma| \\ |a\gamma - b\delta + c\alpha + d\beta| \\ |a\delta + b\gamma - c\beta + d\alpha| \end{pmatrix} \in (\mathbb{N} \cup \{0\})^4
\end{aligned}$$

$\therefore \begin{cases} \langle a, b, c, d \rangle \in (\mathbb{N} \cup \{0\})^4 \\ \langle \alpha, \beta, \gamma, \delta \rangle \in (\mathbb{N} \cup \{0\})^4 \end{cases}$   
 $\xrightarrow{\text{closure property}} \begin{pmatrix} |a\alpha - b\beta - c\gamma - d\delta| \\ |a\beta + b\alpha + c\delta - d\gamma| \\ |a\gamma - b\delta + c\alpha + d\beta| \\ |a\delta + b\gamma - c\beta + d\alpha| \end{pmatrix} \in (\mathbb{N} \cup \{0\})^4$

□

### 37.8.2 $\mathbb{H} \rightarrow \mathcal{M}_{4 \times 4}(\mathbb{R})$

[https://groupprops.subwiki.org/wiki/Linear\\_representation\\_theory\\_of\\_quaternion\\_group#Four-dimensional\\_irreducible\\_representation\\_over\\_a\\_non-splitting\\_field](https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_quaternion_group#Four-dimensional_irreducible_representation_over_a_non-splitting_field)

$$\mathbb{H} \rightarrow \mathcal{M}_2(\mathbb{C}) \xrightarrow{\mathbb{C} \rightarrow \mathcal{M}_2(\mathbb{R})} \mathcal{M}_{4 \times 4}(\mathbb{R}) = \mathcal{M}_4(\mathbb{R})$$

$$\mathbb{C} \rightarrow \mathcal{M}_2(\mathbb{R}) \Leftarrow \begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$$


---

$$1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$-1 \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$i \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$-i \rightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \rightarrow \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$j \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$-j \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$k \rightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \rightarrow \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$-k \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rightarrow \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$


---

some examinations

$$ij \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \leftarrow k$$

$$ji \rightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftarrow -k$$

$$i^2 = ii \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \leftarrow -1$$


---

|    |                                                     |                                                                                                                                                       |                                                                   |                                                                                                                                                       |
|----|-----------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|
|    | $\begin{cases} \alpha = i \\ \beta = 0 \end{cases}$ | $\begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$ | $\begin{cases} \alpha = \sqrt{2}i \\ \beta = 1 \end{cases}$       | $\begin{cases} 1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}$ |
| 1  | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$      | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$                                                      | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$                    | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$                                                      |
| -1 | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$    | $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$                                                  | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$                  | $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$                                                  |
| i  | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$     | $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} \sqrt{2}i & 1 \\ 1 & -\sqrt{2}i \end{pmatrix}$   | $\begin{pmatrix} 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \\ 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix}$                        |
| -i | $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$     | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} -\sqrt{2}i & -1 \\ -1 & \sqrt{2}i \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$                                                |
| j  | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$     | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$                   | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$                                                    |
| -j | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$     | $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$                   | $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$                                                    |
| k  | $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$    | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} 1 & -\sqrt{2}i \\ -\sqrt{2}i & -1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ -\sqrt{2} & 0 & 0 & -1 \end{pmatrix}$                      |
| -k | $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$      | $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$                                                    | $\begin{pmatrix} -1 & \sqrt{2}i \\ \sqrt{2}i & 1 \end{pmatrix}$   | $\begin{pmatrix} -1 & 0 & 0 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \end{pmatrix}$                      |

some examinations

$$ij \rightarrow \begin{pmatrix} 0 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & 1 \\ 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ -\sqrt{2} & 0 & 0 & -1 \end{pmatrix} \leftarrow k$$



# Chapter 38

## tensor

### 38.1 Einstein summation convention

#### 38.1.1 dummy index

#### 38.1.2 free index

### 38.2 EpicOrganism = AIRoswell = Pan, Yi-Wen<sup>1</sup>

<https://space.bilibili.com/14316464/video>

<https://www.bilibili.com/video/BV1T5411D7mS>

<https://www.bilibili.com/video/BV1QA4y1X7Xk>

<https://www.bilibili.com/video/BV1K34y1i75w>

<https://www.bilibili.com/video/BV1ZU411o7xL>

### 38.3 EigenChris

#### 38.3.1 tensor for beginner

<https://www.youtube.com/playlist?list=PLJHszsWbB6hrkmmq57lX8BV-o-YIOFsiG>

#### 38.3.2 tensor calculus

<https://www.youtube.com/playlist?list=PLJHszsWbB6hpk5h8lSfBkVrpjsqvUGTCx>

### 38.4 Elliot Schneider

- Elliot Schneider: [Physics with Elliot](#)

#### 38.4.1 Fundamentals of Cartesian Tensors

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2176662131>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076341>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2154478208/posts/2174076344>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745848>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745896>

---

<sup>1</sup><https://www.linkedin.com/in/yiwen-pan-16a90076>

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745882>

vector notation

$$\vec{V} = \overrightarrow{V} = \overset{\text{harpoon}}{V} = \mathbf{V} = \mathbf{V}$$

\boldsymbol{} is relatively more elegant.

\del or \nabla

vector del = vector nabla

$$\vec{\nabla} = \overrightarrow{\nabla} = \overset{\text{harpoon}}{\nabla} = \nabla = \nabla$$

scalar del = scalar nabla

$$\nabla$$

vector field

e.g. [magnetic] vector potential

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \vec{A}(\vec{r}) = \mathbf{A}((x_1, x_2, x_3)) = \mathbf{A}(x_1, x_2, x_3) = \begin{pmatrix} A_1(x_1, x_2, x_3) \\ A_2(x_1, x_2, x_3) \\ A_3(x_1, x_2, x_3) \end{pmatrix} \\ &= \mathbf{A}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_1\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \\ A_2\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \\ A_3\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) \end{pmatrix} = \begin{pmatrix} A_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ A_2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ A_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{pmatrix} \end{aligned}$$

scalar field

e.g. temperature

$$\begin{aligned} T(\mathbf{r}) &= T(\vec{r}) = T((x_1, x_2, x_3)) = T(x_1, x_2, x_3) \\ &= T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

or generally

$$f(\mathbf{r}) = f(\vec{r}) = f(x_1, x_2, x_3) = f(\dots, x_i, \dots)$$

**Definition 38.1.** directional derivative

$$\begin{aligned} \nabla_{\mathbf{V}} f(\mathbf{r}) &= \lim_{\epsilon \rightarrow 0} \frac{f(\mathbf{r} + \epsilon \mathbf{V}) - f(\mathbf{r})}{\epsilon} \\ &= \nabla_{\vec{V}} f(\vec{r}) = \lim_{\epsilon \rightarrow 0} \frac{f(\vec{r} + \epsilon \vec{V}) - f(\vec{r})}{\epsilon} \end{aligned}$$

1D directional derivative

$$\begin{aligned}
f_x(\mathbf{r}) = f'(x) &= \begin{cases} \frac{\partial f}{\partial x} & f = f(\dots, x, \dots) \\ \frac{df}{dx} & f = f(x) \end{cases} \\
&= \nabla_{\frac{\vec{x}}{\|\vec{x}\|}} f(\vec{r}) = \nabla_{\frac{\vec{x}}{\|\vec{x}\|}} f(\mathbf{r}) = \nabla_{\frac{\vec{x}}{\|\vec{x}\|}} f(\vec{r}) = \nabla_{\frac{\vec{x}}{\|\vec{x}\|}} f(\mathbf{r}) \\
&= \nabla_{\hat{x}} f(\vec{r}) = \nabla_{\hat{x}} f(\mathbf{r}) = \begin{cases} \lim_{\epsilon \rightarrow 0} \frac{f(\mathbf{r} + \epsilon \hat{x}) - f(\mathbf{r})}{\epsilon} & f = f(\dots, x, \dots) \\ \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon} & f = f(x) \end{cases}
\end{aligned}$$


---

According to  $\text{d}f^{[41]}$ ,

$$\text{d}f = f'(x) dx = \left( \frac{df}{dx} \right) dx = \frac{df}{dx} dx$$

is the differential of  $f$ , read as “ $\text{d } f$ ”.

$$\begin{aligned}
\text{d}f &= \left( \frac{df}{dx} \right) dx = \frac{df}{dx} dx = \left( \frac{df}{dx} \hat{x} \right) \cdot (dx \hat{x}) \\
&= \nabla_{\hat{x}} f(\mathbf{r}) \cdot d\hat{x} \\
&= \nabla f \cdot d\mathbf{r} = \vec{\nabla} f \cdot d\vec{r} \\
&\stackrel{\text{commutative}}{=} d\mathbf{r} \cdot \nabla f = d\vec{r} \cdot \vec{\nabla} f
\end{aligned}$$


---

$$\text{d}f = d\mathbf{r} \cdot \nabla f = d\vec{r} \cdot \vec{\nabla} f$$

$\nabla f$  is the gradient of  $f$ , read as “ $\text{del } f$ ”.

According to  $\text{d}f^{[41]}$ ,

$$\text{d}f = f'(x) dx = \left( \frac{df}{dx} \right) dx = \frac{df}{dx} dx$$

is the differential of  $f$ , read as “ $\text{d } f$ ”.

2D directional derivative

According to  $\text{d}f^{[41]}$ ,

$$\text{d}f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\begin{aligned}
\text{d}f &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
&= \left( \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} \right) \cdot (dx \hat{x} + dy \hat{y}) \\
&= \nabla f \cdot d\mathbf{r}, \begin{cases} \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} \\ d\mathbf{r} = dx \hat{x} + dy \hat{y} \end{cases}
\end{aligned}$$


---

example:

$$f(\mathbf{r}) = f(x_1, x_2) = f(x, y) = \sqrt{x^2 + y^2}$$

using Feynman method of differentiation / derivative technique<sup>[44.1]</sup>,

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} = \frac{\partial \sqrt{x^2 + y^2}}{\partial x} \hat{x} + \frac{\partial \sqrt{x^2 + y^2}}{\partial y} \hat{y} \\
&\stackrel{\text{Feynman}}{=} \frac{\frac{1}{2} \cdot 2x}{\sqrt{x^2 + y^2}} \hat{x} + \frac{\frac{1}{2} \cdot 2y}{\sqrt{x^2 + y^2}} \hat{y} \\
&= \frac{x}{\sqrt{x^2 + y^2}} \hat{x} + \frac{y}{\sqrt{x^2 + y^2}} \hat{y} \\
&= \frac{x \hat{x} + y \hat{y}}{\sqrt{x^2 + y^2}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\vec{r}}{\|\vec{r}\|}
\end{aligned}$$


---

geometric meaning of a gradient

$\nabla f$  points in the direction of “steepest ascent” of  $f$ .

directional derivative: from definition to calculation

$$\begin{aligned}
df &= d\mathbf{r} \cdot \nabla f = d\vec{r} \cdot \vec{\nabla} f \\
&= \epsilon \mathbf{V} \cdot \nabla f = \epsilon \vec{V} \cdot \vec{\nabla} f \\
df &= f(\mathbf{r} + \epsilon \mathbf{V}) - f(\mathbf{r}) = f(\vec{r} + \epsilon \vec{V}) - f(\vec{r})
\end{aligned}$$

$$\begin{aligned}
\nabla_{\mathbf{V}} f(\mathbf{r}) &= \nabla_{\vec{V}} f(\vec{r}) \\
&= \lim_{\epsilon \rightarrow 0} \frac{f(\mathbf{r} + \epsilon \mathbf{V}) - f(\mathbf{r})}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{f(\vec{r} + \epsilon \vec{V}) - f(\vec{r})}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{df}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{df}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{d\mathbf{r} \cdot \nabla f}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{d\vec{r} \cdot \vec{\nabla} f}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{\epsilon \mathbf{V} \cdot \nabla f}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\epsilon \vec{V} \cdot \vec{\nabla} f}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{\mathbf{V} \cdot \nabla f}{1} = \lim_{\epsilon \rightarrow 0} \frac{\vec{V} \cdot \vec{\nabla} f}{1} \\
&= \mathbf{V} \cdot \nabla f = \vec{V} \cdot \vec{\nabla} f
\end{aligned}$$


---

$$\nabla_{\mathbf{V}} f = \nabla_{\mathbf{V}} f(\mathbf{r}) = \mathbf{V} \cdot \nabla f = \nabla_{\vec{V}} f = \nabla_{\vec{V}} f(\vec{r}) = \vec{V} \cdot \vec{\nabla} f$$


---

$$\begin{aligned}
\nabla_{\mathbf{V}} f &= \mathbf{V} \cdot \nabla f \\
&= \nabla_{\vec{V}} f = \vec{V} \cdot \vec{\nabla} f
\end{aligned}$$


---

in the view of operator acting on scalar field(s)

$$\mathbf{V} = V_x \hat{x} + V_y \hat{y}$$

grad or gradient = vector del or vector nabla as a vector operator acting on scalar field(s)

$$\begin{aligned}
\nabla f &= \vec{\nabla} f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} = \nabla_{\hat{x}} f \hat{x} + \nabla_{\hat{y}} f \hat{y} \\
&= \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} = \hat{x} \nabla_{\hat{x}} f + \hat{y} \nabla_{\hat{y}} f \\
&= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) f = (\hat{x} \nabla_{\hat{x}} + \hat{y} \nabla_{\hat{y}}) f \\
&\Downarrow \\
\nabla &= \vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} = \hat{x} \nabla_{\hat{x}} + \hat{y} \nabla_{\hat{y}}
\end{aligned}$$

directional derivative = scalar del or scalar nabla as a scalar operator acting on scalar field(s)

$$\begin{aligned}
 \nabla_{\mathbf{V}} f &= \nabla_{\vec{v}} f = \mathbf{V} \cdot \nabla f = \vec{V} \cdot \vec{\nabla} f \\
 &= (V_x \hat{\mathbf{x}} + V_y \hat{\mathbf{y}}) \cdot (\nabla_{\hat{\mathbf{x}}} f \hat{\mathbf{x}} + \nabla_{\hat{\mathbf{y}}} f \hat{\mathbf{y}}) \\
 &= V_x \nabla_{\hat{\mathbf{x}}} f + V_y \nabla_{\hat{\mathbf{y}}} f = (V_x \nabla_{\hat{\mathbf{x}}} + V_y \nabla_{\hat{\mathbf{y}}}) f \\
 &= V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} = \left( V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} \right) f \\
 &\Downarrow \\
 \nabla_{\mathbf{V}} &= \nabla_{\vec{v}} = V_x \nabla_{\hat{\mathbf{x}}} + V_y \nabla_{\hat{\mathbf{y}}} = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y}
 \end{aligned}$$


---

product rule

$$\begin{aligned}
 \nabla_{\mathbf{V}} (fg) &= \nabla_{\vec{v}} (fg) = \mathbf{V} \cdot \nabla (fg) = \vec{V} \cdot \vec{\nabla} (fg) \\
 &= \mathbf{V} \cdot [(\nabla f) g + f \nabla g] = \mathbf{V} \cdot [g \nabla f + f \nabla g] = \mathbf{V} \cdot [f \nabla g + g \nabla f] \\
 &= f (\mathbf{V} \cdot \nabla g) + g (\mathbf{V} \cdot \nabla f) = g (\mathbf{V} \cdot \nabla f) + f (\mathbf{V} \cdot \nabla g) \\
 &= f (\nabla_{\mathbf{V}} g) + g (\nabla_{\mathbf{V}} f) = g (\nabla_{\mathbf{V}} f) + f (\nabla_{\mathbf{V}} g) \\
 &= f \mathbf{V} \cdot \nabla g + g \mathbf{V} \cdot \nabla f = g \mathbf{V} \cdot \nabla f + f \mathbf{V} \cdot \nabla g \\
 &= f \nabla_{\mathbf{V}} g + g \nabla_{\mathbf{V}} f = g \nabla_{\mathbf{V}} f + f \nabla_{\mathbf{V}} g \\
 \nabla_{\mathbf{V}} (fg) &= f \nabla_{\mathbf{V}} g + g \nabla_{\mathbf{V}} f
 \end{aligned}$$


---

two kinds of linearities or bilinearity of directional derivative operator

$$\begin{aligned}
 \nabla_{\mathbf{V}} (f + \lambda g) &= \nabla_{\vec{v}} (f + \lambda g) = \mathbf{V} \cdot \nabla (f + \lambda g) = \vec{V} \cdot \vec{\nabla} (f + \lambda g) \\
 &= \mathbf{V} \cdot [\nabla f + \nabla (\lambda g)] = \mathbf{V} \cdot [\nabla f + \lambda \nabla g] \\
 &= \mathbf{V} \cdot \nabla f + \lambda (\mathbf{V} \cdot \nabla g) \\
 &= \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{V}} g \\
 \nabla_{\mathbf{V}} (f + \lambda g) &= \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{V}} g
 \end{aligned}$$

$$\begin{aligned}
 \nabla_{\mathbf{V} + \lambda \mathbf{W}} f &= \nabla_{\vec{v} + \lambda \vec{w}} f = (\mathbf{V} + \lambda \mathbf{W}) \cdot \nabla f = (\vec{V} + \lambda \vec{W}) \cdot \vec{\nabla} f \\
 &= \mathbf{V} \cdot \nabla f + \lambda \mathbf{W} \cdot \nabla f = \vec{V} \cdot \vec{\nabla} f + \lambda \vec{W} \cdot \vec{\nabla} f \\
 &= \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{W}} f = \nabla_{\vec{v}} f + \lambda \nabla_{\vec{w}} f \\
 \nabla_{\mathbf{V} + \lambda \mathbf{W}} f &= \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{W}} f
 \end{aligned}$$


---

$$\begin{cases} \nabla_{\mathbf{V}} (f + \lambda g) = \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{V}} g \\ \nabla_{\mathbf{V} + \lambda \mathbf{W}} f = \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{W}} f \end{cases}$$

$$\begin{cases} \nabla_{\mathbf{V}} (f + \lambda g) = \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{V}} g \Leftrightarrow \nabla_{\mathbf{V}} (\lambda f + \mu g) = \lambda \nabla_{\mathbf{V}} f + \mu \nabla_{\mathbf{V}} g \\ \nabla_{\mathbf{V} + \lambda \mathbf{W}} f = \nabla_{\mathbf{V}} f + \lambda \nabla_{\mathbf{W}} f \Leftrightarrow \nabla_{\lambda \mathbf{V} + \mu \mathbf{W}} f = \lambda \nabla_{\mathbf{V}} f + \mu \nabla_{\mathbf{W}} f \end{cases}$$

$$\begin{aligned}
 \nabla_{\lambda \mathbf{V} + \mu \mathbf{W}} f &= \nabla_{\lambda \mathbf{V}} f + \nabla_{\mu \mathbf{W}} f \\
 &= \lambda \nabla_{\mathbf{V}} f + \mu \nabla_{\mathbf{W}} f \\
 &= \lambda \frac{\partial f}{\partial V} + \mu \frac{\partial f}{\partial W} \\
 &= \lambda \frac{\partial f}{\partial V} + \mu \frac{\partial f}{\partial W} \\
 \nabla_{\mathbf{V}} f &= \nabla_{V_x \hat{\mathbf{x}} + V_y \hat{\mathbf{y}}} f = \nabla_{V_x \hat{\mathbf{x}}} f + \nabla_{V_y \hat{\mathbf{y}}} f \\
 &= V_x \nabla_{\hat{\mathbf{x}}} f + V_y \nabla_{\hat{\mathbf{y}}} f \\
 &= V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} \\
 &= V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y}
 \end{aligned}$$

directional derivative of a 3D vector field

$$\begin{aligned}
\nabla_{\vec{v}} \mathbf{A} &= \nabla_{\vec{v}} \vec{A} = \mathbf{V} \cdot \nabla \mathbf{A} = \vec{V} \cdot \vec{\nabla} \vec{A} \\
&= (\hat{x}V_x + \hat{y}V_y + \hat{z}V_z) \cdot \left( \hat{x} \frac{\partial \mathbf{A}}{\partial x} + \hat{y} \frac{\partial \mathbf{A}}{\partial y} + \hat{z} \frac{\partial \mathbf{A}}{\partial z} \right) \\
&= (\hat{x}V_x + \hat{y}V_y + \hat{z}V_z) \cdot \left( \hat{x} \frac{\partial \vec{A}}{\partial x} + \hat{y} \frac{\partial \vec{A}}{\partial y} + \hat{z} \frac{\partial \vec{A}}{\partial z} \right) \\
\nabla \mathbf{A} &= \vec{\nabla} \vec{A} = \hat{x} \frac{\partial \mathbf{A}}{\partial x} + \hat{y} \frac{\partial \mathbf{A}}{\partial y} + \hat{z} \frac{\partial \mathbf{A}}{\partial z} = \hat{x} \frac{\partial \vec{A}}{\partial x} + \hat{y} \frac{\partial \vec{A}}{\partial y} + \hat{z} \frac{\partial \vec{A}}{\partial z} \\
&= \hat{x} \frac{\partial (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)}{\partial x} \\
&\quad + \hat{y} \frac{\partial (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)}{\partial y} \\
&\quad + \hat{z} \frac{\partial (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)}{\partial z} \\
&= \hat{x}\hat{x} \frac{\partial A_x}{\partial x} + \hat{x}\hat{y} \frac{\partial A_y}{\partial x} + \hat{x}\hat{z} \frac{\partial A_z}{\partial x} \\
&\quad + \hat{y}\hat{x} \frac{\partial A_x}{\partial y} + \hat{y}\hat{y} \frac{\partial A_y}{\partial y} + \hat{y}\hat{z} \frac{\partial A_z}{\partial y} \\
&\quad + \hat{z}\hat{x} \frac{\partial A_x}{\partial z} + \hat{z}\hat{y} \frac{\partial A_y}{\partial z} + \hat{z}\hat{z} \frac{\partial A_z}{\partial z} \\
&= \hat{x} \otimes \hat{x} \frac{\partial A_x}{\partial x} + \hat{x} \otimes \hat{y} \frac{\partial A_y}{\partial x} + \hat{x} \otimes \hat{z} \frac{\partial A_z}{\partial x} \\
&\quad + \hat{y} \otimes \hat{x} \frac{\partial A_x}{\partial y} + \hat{y} \otimes \hat{y} \frac{\partial A_y}{\partial y} + \hat{y} \otimes \hat{z} \frac{\partial A_z}{\partial y} \\
&\quad + \hat{z} \otimes \hat{x} \frac{\partial A_x}{\partial z} + \hat{z} \otimes \hat{y} \frac{\partial A_y}{\partial z} + \hat{z} \otimes \hat{z} \frac{\partial A_z}{\partial z}
\end{aligned}$$

$\nabla A = \vec{\nabla} \vec{A}$  is a rank-2 tensor.

$\otimes$  is tensor product.

$\nabla_v A = \nabla_{\vec{v}} \vec{A} = V \cdot \nabla A = \vec{V} \cdot \vec{\nabla} A$  yields a vector, instead of a scalar.

Dan Fleisch: What's a Tensor?

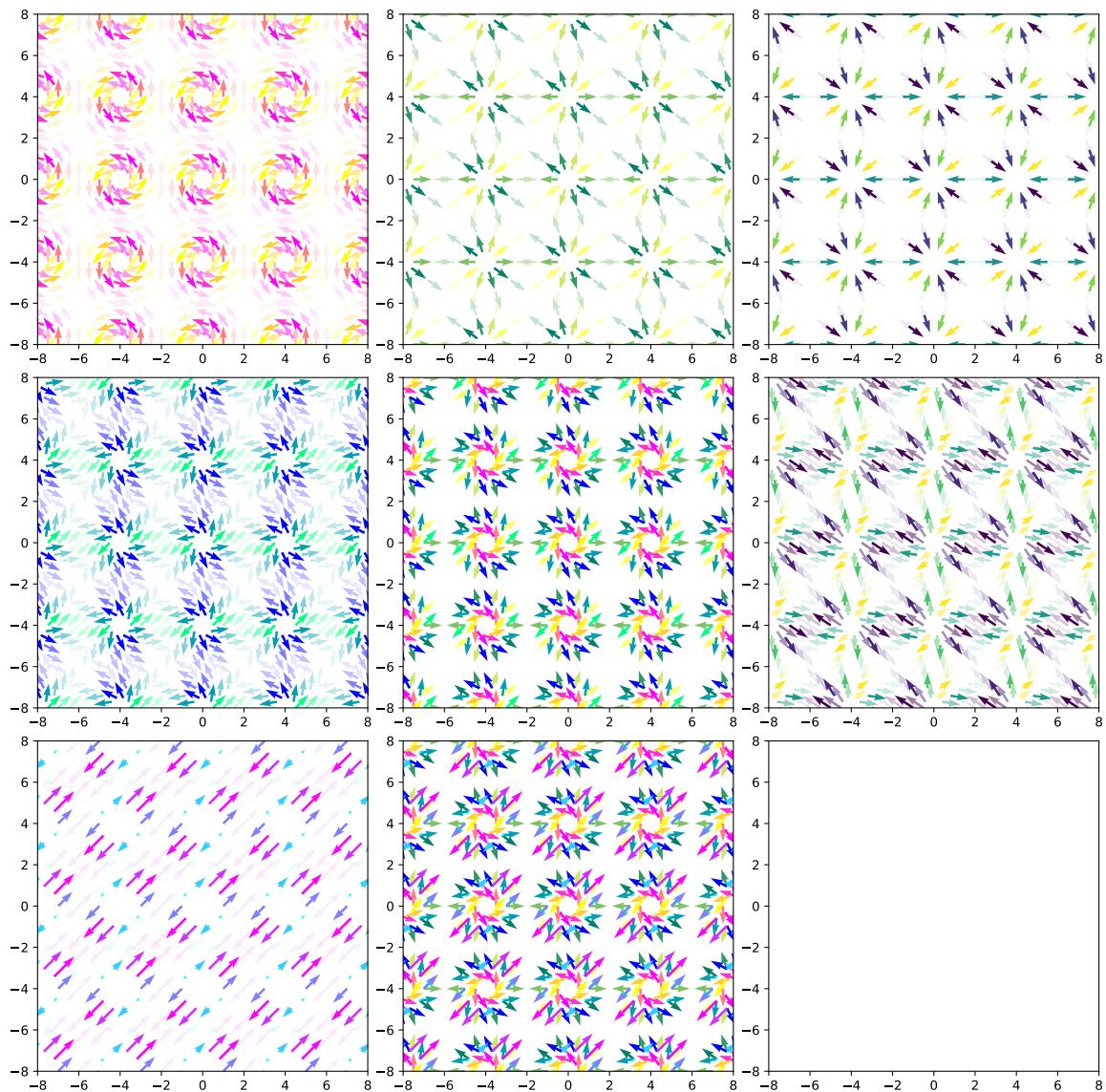
<https://www.youtube.com/watch?v=f5liqUk0ZTw>

UdiProd: tensor

<https://www.youtube.com/watch?v=YxXyN2ifK8A&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=1>

<https://www.youtube.com/watch?v=A95jdIUUW0&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=2>

<https://www.youtube.com/watch?v=51ARho2bvQY&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=3>



```

import numpy as np
import matplotlib.pyplot as plt

A = np.array([[0,-1],
 [1, 0]])

B = np.array([[1, 0],
 [0, 1]])

C = np.array([[1,-1],
 [1, 1]])

D = np.array([[1, 1],
 [1, 1]])

E = np.array([[-1, 0],
 [0,-1]])

F = np.array([[3, 1],
 [-2, 2]])

t = np.arange(0, 2 * np.pi, np.pi/5)

Creating arrow
X = np.cos(t)
Y = np.sin(t)

X, Y = np.meshgrid(x, y)

V = np.array([X, Y])

u = x
v = y

u, v = np.matmul(A, V)
u, v = np.matmul(A, np.array([X, Y]))
dx, dy = np.matmul(B, np.array([X, Y]))
dx2, dy2 = np.matmul(C, np.array([X, Y]))
dx3, dy3 = np.matmul(D, np.array([X, Y]))
dx4, dy4 = np.matmul(E, np.array([X, Y]))
dx5, dy5 = np.matmul(F, np.array([X, Y]))

n = -2

Defining color
color = np.sqrt(((v-n)/2)*2 + ((u-n)/2)*2)
color = (u**2+v**2)**(1/2)
color = u*v

Creating plot
fig, ax = plt.subplots(3,3,figsize =(15,15),
 gridspec_kw = {'wspace':0.1,
 'hspace':0.1})

ax.quiver(X, Y, u, v, color, alpha = 1)
for i in range(-2,3):
 for j in range(-2,3):
 # ax.quiver(1.0*X+4*i, 1.0*Y+4*j, u,
 # v), color, alpha = 1)
 # ax.quiver(1.5*X+4*i, 1.5*Y+4*j, u,
 # v), color, alpha = 1)
 # ax.quiver(2.0*X+4*i, 2.0*Y+4*j, u,
 # v), color, alpha = 1)

 ax[0,0].quiver(0.5*X+4*i, 0.5*Y+4*j,
 u, v, u*v, alpha = 0.50, cmap='spring') ## (-8.0, 8.0, -8.0, 8.0)
 ax[0,0].quiver(1.0*X+4*i, 1.0*Y+4*j,
 u, v, u*v, alpha = 0.50, cmap='spring') ## (-8.0, 8.0, -8.0, 8.0)

```

<https://courses.physicswithelliot.com/products/part-i-fundamentals-of-cartesian-tensors/categories/2155140686/posts/2176745916>

### 38.4.2 Fundamentals of Curvilinear Tensors

### 38.4.3 Fundamentals of Spacetime Tensors

## 38.5 Lin, Hsiu-Hau / Porcupine Lin / Hedgehog Note



# Chapter 39

## dual space

dual space and linear functional

<https://ccjou.wordpress.com/2011/06/13/%E7%BA%BF%E6%8D%80%E6%95%99%E5%8A%A8%E4%BD%9C/>

### 39.1 linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (b_i) = b_i$$

$$(a_{i1} \quad \cdots \quad a_{ij} \quad \cdots \quad a_{in}) \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = (b_i) = b_i$$

$$(\cdots \quad a_{ij} \quad \cdots) \begin{pmatrix} \vdots \\ x_j \\ \vdots \end{pmatrix} = b_i$$

$$\mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$

$$a_{ij}x_j = \mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots$$

if finite,

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$

## 39.2 matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

$$AX = B$$

$$(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}) \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix} = (b_{ik}) = b_{ik}$$

$$(a_{i1} \quad \cdots \quad a_{ij} \quad \cdots \quad a_{in}) \begin{pmatrix} x_{1k} \\ \vdots \\ x_{jk} \\ \vdots \\ x_{nk} \end{pmatrix} = (b_{ik}) = b_{ik}$$

$$(\cdots \quad a_{ij} \quad \cdots) \begin{pmatrix} \vdots \\ x_{jk} \\ \vdots \end{pmatrix} = b_{ik}$$

$$a_{ij}x_j = \mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i \cdot \mathbf{x} = b_i$$

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j$$

\iddots in MathJax

<https://math.meta.stackexchange.com/questions/23273/mathjax-and-iddots-udots-or-reflectbox>

$$b_{ik} = a_{ij}x_{jk} = \mathbf{a}_i \cdot \mathbf{x}_k = \mathbf{a}_i^\top \mathbf{x}_k$$

$$\text{row}(A) \text{col}(X) = b_{\text{row}, \text{col}}$$

$$\mathbf{a}_i \cdot \mathbf{x}_k = \mathbf{a}_i^\top \mathbf{x}_k = a_{ij}x_{jk}$$

$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj}$$


---

$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = \cdots + a_{ik}x_{kj} + \cdots$$

if finite,

$$\mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = a_{i1}x_{1j} + \cdots + a_{ik}x_{kj} + \cdots + a_{in}x_{nj}$$

### 39.3 functional

( inner product or dot product ) or linear equations

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots$$

if finite,

$$\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$


---

actually, several rows

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j & = & \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j & = & a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$


---

in functional aspect,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j = \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} & = & a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$


---

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\dots, x_j, \dots) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \cdots + a_{ij}x_j + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

if finite,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \dots, x_j, \dots, x_n) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$


---

or simply

$$f_i(\dots, x_j, \dots) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ik}x_{kj} = \cdots + a_{ij}x_j + \cdots$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^\infty \rightarrow \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^\infty \rightarrow \mathbb{F}$$

or more abstract notation,

$$f_i : F^\infty \rightarrow F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^\infty \rightarrow \mathbb{R}$$


---

if finite,

$$f_i(x_1, \dots, x_j, \dots, x_n) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ij}x_j = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n$$

$$\begin{aligned} f_i(x_1, x_2, \dots, x_n) &= f_i(x_1, \dots, x_j, \dots, x_n) = f_i(x_j) = f_i(\mathbf{x}) \\ &= \mathbf{a}_i \cdot \mathbf{x}_j = \mathbf{a}_i^\top \mathbf{x}_j = a_{ij}x_j = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n \end{aligned}$$

if scalar with complex as the field,

$$f_i : \mathbb{C}^n \rightarrow \mathbb{C}$$

if scalar with a field,

$$f_i : \mathbb{F}^n \rightarrow \mathbb{F}$$

or more abstract notation,

$$f_i : F^n \rightarrow F$$

if scalar with real as the field,

$$f_i : \mathbb{R}^n \rightarrow \mathbb{R}$$


---

functionals are a set of fuctions mapping  $n$ -dimensional vectors to scalars

$$f_i : F^n \rightarrow F$$


---

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(\dots, x_j, \dots) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = \dots + a_{ij}x_j + \dots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^\infty \rightarrow F\} = \left\{ \begin{array}{c} f_i(\dots, x_j, \dots) = f_i(x_j) = f_i(\mathbf{x}) = \mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j, \\ \vdots \end{array} \right\}$$

if finite,

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f_i(x_1, \dots, x_j, \dots, x_n) & = & f_i(x_j) & = & f_i(\mathbf{x}) & = & \mathbf{a}_i \cdot \mathbf{x} & = & \mathbf{a}_i^\top \mathbf{x} = a_{ij}x_j = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

$$f = \{f_i | f_i : F^n \rightarrow F\} = \left\{ \begin{array}{l} \vdots \\ f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n, \\ \vdots \end{array} \right\}$$

linear functionals are a set of functions mapping vectors to scalars linearly

$$f = \{f_i | f_i : F^n \rightarrow F\} = \left\{ \begin{array}{l} \vdots \\ f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n, \\ \vdots \end{array} \right\}$$

## 39.4 definition of linear functional

functionals generalized to general vector space

$$f = \{f_i | f_i : V \rightarrow F\}$$

linear functionals generalized to general vector space is a linear transformation

$$f = \left\{ f_i \left| \begin{array}{l} f_i : V \rightarrow F \\ \forall \langle \mathbf{x}, \mathbf{y} \rangle \in V^2 [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : V \rightarrow F \\ \forall \mathbf{x}, \mathbf{y} \in V [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$\text{if } \begin{cases} F = \mathbb{C} \\ V = \mathbb{C}^n \end{cases},$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : \mathbb{C}^n \rightarrow \mathbb{C} \\ \forall \langle \mathbf{x}, \mathbf{y} \rangle \in (\mathbb{C}^n)^2 [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

$$f = \left\{ f_i \left| \begin{array}{l} f_i : \mathbb{C}^n \rightarrow \mathbb{C} \\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in \mathbb{C}^n, c \in \mathbb{C} [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \right\}$$

then

$$f_i(\mathbf{x}) = a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n$$

satisfying

$$\begin{aligned} f_i(\mathbf{x} + \mathbf{y}) &= a_{i1}(x + y)_1 + \dots + a_{ij}(x + y)_j + \dots + a_{in}(x + y)_n \\ &= a_{i1}(x_1 + y_1) + \dots + a_{ij}(x_j + y_j) + \dots + a_{in}(x_n + y_n) \\ &= (a_{i1}x_1 + a_{i1}y_1) + \dots + (a_{ij}x_j + a_{ij}y_j) + \dots + (a_{in}x_n + a_{in}y_n) \\ &= (a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n) + (a_{i1}y_1 + \dots + a_{ij}y_j + \dots + a_{in}y_n) \\ &= f_i(\mathbf{x}) + f_i(\mathbf{y}) \end{aligned}$$

$$\begin{aligned} f_i(c\mathbf{x}) &= a_{i1}(cx)_1 + \dots + a_{ij}(cx)_j + \dots + a_{in}(cx)_n \\ &= a_{i1}(cx_1) + \dots + a_{ij}(cx_j) + \dots + a_{in}(cx_n) \\ &= c(a_{i1}x_1) + \dots + c(a_{ij}x_j) + \dots + c(a_{in}x_n) \\ &= c(a_{i1}x_1 + \dots + a_{ij}x_j + \dots + a_{in}x_n) \\ &= cf_i(\mathbf{x}) \end{aligned}$$

different functional has different  $a_{ij}$

let

$$a_{ij} = f_i(\mathbf{e}_j), \mathbf{e}_j = \left\langle \underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0 \right\rangle = (0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0)^\top = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

if  $f_i$  is a linear functional, then

$$\begin{aligned} f_i(\mathbf{x}) &= f_i(x_1 \mathbf{e}_1 + \dots + x_j \mathbf{e}_j + \dots + x_n \mathbf{e}_n) \\ &= f_i(x_1 \mathbf{e}_1) + \dots + f_i(x_j \mathbf{e}_j) + \dots + f_i(x_n \mathbf{e}_n) \\ &= x_1 f_i(\mathbf{e}_1) + \dots + x_j f_i(\mathbf{e}_j) + \dots + x_n f_i(\mathbf{e}_n) \\ &= x_1 a_{i1} + \dots + x_j a_{ij} + \dots + x_n a_{in} \\ &= a_{i1} x_1 + \dots + a_{ij} x_j + \dots + a_{in} x_n \\ &= f_i(\mathbf{x}) \end{aligned}$$

### 39.5 set of all linear transformations is a vector space

<https://ccjou.wordpress.com/2011/04/08/%E7%BA%BF%E6%8D%A2%E5%85%83%E5%95%9C/>

vector space<sup>[40]</sup>

<https://math.stackexchange.com/questions/2381942/the-set-of-all-linear-maps-tv-w-is-a-vector-space>

$$T : V \rightarrow W \Leftrightarrow \forall \mathbf{v} \in V, \exists! \mathbf{w} \in W [ \mathbf{w} = T(\mathbf{v}) ]$$

$$\begin{cases} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})] \\ \forall \mathbf{v} \in V, c \in F [T(c\mathbf{v}) = cT(\mathbf{v})] \end{cases} \quad \text{linearity} \\ \Leftrightarrow T \text{ is a linear transformation} \end{cases}$$

$$\begin{cases} V, W \text{ are vector spaces, both over } F \\ T, U : V \rightarrow W \\ \begin{cases} T : V \rightarrow W \\ U : V \rightarrow W \end{cases} \\ T, U \text{ are both linear transformations} \\ \mathbf{v} \in V \\ c \in F \end{cases}$$

There is still linearity over linear transformations

(va)

$$\begin{aligned} (T + U)(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u} + \mathbf{v}) + U(\mathbf{u} + \mathbf{v}) \\ &= [T(\mathbf{u}) + T(\mathbf{v})] + [U(\mathbf{u}) + U(\mathbf{v})] \\ &= [T(\mathbf{u}) + U(\mathbf{u})] + [T(\mathbf{v}) + U(\mathbf{v})] \\ &= (T + U)(\mathbf{u}) + (T + U)(\mathbf{v}) \end{aligned}$$

(sm)

$$\begin{aligned}
(T + U)(cv) &= T(cv) + U(cv) \\
&= cT(v) + cU(v) \\
&= c[T(v) + U(v)] \\
&= c(T + U)(v)
\end{aligned}$$

so we can define

$$\begin{cases} (T + U)(v) = T(v) + U(v) & \text{linear transformation addition} \\ (cT)(v) = cT(v) & \text{scalar linear transformation multiplication} \end{cases}$$


---

the set of all linear transformations is a vector space

$\mathcal{T}$  is the set of all linear transformations

$$\left\{
\begin{array}{ll}
F & (f) F \text{ is a field} \\
\mathcal{T} \neq \emptyset & (ne) \text{ nonempty set} \\
+ : \mathcal{T} \times \mathcal{T} \xrightarrow{\perp} \mathcal{T} \Leftrightarrow \forall T, U \in \mathcal{T}, \exists S \in \mathcal{T} [S = T + U] & (va) \text{ vector addition} \\
\cdot : F \times \mathcal{T} \xrightarrow{\perp} \mathcal{T} \Leftrightarrow \forall c \in F, \forall T \in \mathcal{T}, \exists U \in \mathcal{T} [U = cT = c \cdot T] & (sm) \text{ scalar multiplication} \\
\begin{cases} \forall S, T, U \in \mathcal{T} [S + (T + U) = (S + T) + U] & (a) \\ \forall T, U \in \mathcal{T} [T + U = U + T] & (c) \\ \exists O \in \mathcal{T}, \forall T \in \mathcal{T} [O + T = T] & (e) \\ \forall T \in \mathcal{T}, \exists! -T \in \mathcal{T} [(-T) + T = O] & (i) \end{cases} & (va) \text{ vector addition axioms} \\
\begin{cases} \forall b, c \in F, T \in \mathcal{T} [b(cT) = (bc)T] & (a) \\ \exists! 1 \in F, \forall T \in \mathcal{T} [1T = T] & (e) \\ \forall c \in F, T, U \in \mathcal{T} [c(T + U) = cT + cU] & (dv) \\ \forall b, c \in F, T \in \mathcal{T} [(b + c)T = bT + cT] & (ds) \end{cases} & (sm) \text{ scalar multiplication axioms} \\
\Leftrightarrow \mathcal{T} = \mathcal{T}(F, +, \cdot) = (\mathcal{T}, F, +, \cdot) \text{ is a vector space over the field } F & \\
\Leftrightarrow \mathcal{T} \text{ is a vector space} &
\end{array}
\right.$$

Selected proofs of 8 vector space axioms due to some trivial field and vector space properties:

(va) (a)

$$\begin{aligned}
(S + (T + U))(v) &= S(v) + (T + U)(v) \\
&= S(v) + T(v) + U(v) \\
&= (S + T)(v) + U(v) \\
&= ((S + T) + U)(v)
\end{aligned}$$

(va) (c)

$$\begin{aligned}
(T + U)(v) &= T(v) + U(v) \\
&= U(v) + T(v) \\
&= (U + T)(v)
\end{aligned}$$

(va) (e)

$$O(v) = 0w \in W$$

$$\begin{aligned}
(O + T)(v) &= O(v) + T(v) \\
&= 0w + T(v) \\
&= T(v)
\end{aligned}$$

$$O_1(v) - O_2(v) = 0w - 0w = 0w \Rightarrow O_1(v) = O_2(v)$$

$(sm)(dv)$

$$\begin{aligned}(c(T+U))(\mathbf{v}) &= c(T+U)(\mathbf{v}) \\ &= c[T(\mathbf{v}) + U(\mathbf{v})] \\ &= cT(\mathbf{v}) + cU(\mathbf{v}) \\ &= (cT + cU)(\mathbf{v})\end{aligned}$$

The set of all linear tranformations  $\mathcal{T}$  is a vector space.

□

## 39.6 definition of dual space

$$\begin{aligned}V^* &= L(V, F) \\ &= \left\{ f_i \middle| \begin{array}{l} f_i : V \rightarrow F \\ \left\{ \begin{array}{l} \forall \mathbf{x}, \mathbf{y} \in V [f_i(\mathbf{x} + \mathbf{y}) = f_i(\mathbf{x}) + f_i(\mathbf{y})] \\ \forall \mathbf{x} \in V, c \in F [f_i(c\mathbf{x}) = cf_i(\mathbf{x})] \end{array} \right. \\ \text{(L) linearity} \end{array} \right\} \quad \text{functional mapping vector to field scalar}\end{aligned}$$

$\Leftrightarrow V^*$  is a dual space, a set of linear functionals  $f_i$  mapping vectors in the vector space  $V$  to scalars in the field  $F$

vector space<sup>[40]</sup>

<https://web.math.sinica.edu.tw/mathmedia/HTMLArticle18.jsp?mID=31304>

[https://web.math.sinica.edu.tw/mathmedia/author18.jsp?query\\_filter=%E9%BE%94%E6%98%87](https://web.math.sinica.edu.tw/mathmedia/author18.jsp?query_filter=%E9%BE%94%E6%98%87)

## 39.7 double dual

double dual = second dual

<https://ccjou.wordpress.com/2014/04/10/%E9%BE%94%E6%98%87/>

<https://www.zhihu.com/question/444079322/answer/1749490720>

Hough Transform

<https://www.cnblogs.com/php-rearch/p/6760683.html>

# Chapter 40

## vector space

<https://www.bilibili.com/video/BV1ez4y1n7h8>

<https://ccjou.wordpress.com/2010/04/15/%E5%90%8C%E6%9E%9A%E7%9F%A5%E9%80%9A/>

### 40.1 What is a vector?

What is a vector? or What is an element in a vector space?

Binary operations defined on a vector space satisfying some properties is more important than what is a vector.

ultimate answer: double dual concept<sup>[40.4.1.2]</sup>

### 40.2 vector space definition

<https://tex.stackexchange.com/a/141489> multiline node

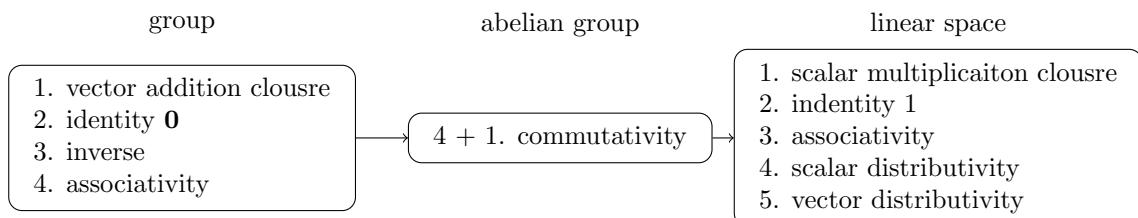


Figure 40.1: vector space construction

$$\left\{ \begin{array}{ll} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ + : V \times V = V^2 \xrightarrow{*} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (va) \text{ vector addition} \\ \cdot : F \times V \xrightarrow{*} V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \left\{ \begin{array}{ll} \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{array} \right. & (va) \text{ axioms} \\ \left\{ \begin{array}{ll} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{array} \right. & (sm) \text{ axioms} \end{array} \right.$$

$\Leftrightarrow V = V(F, +, \cdot) = (V, F, +, \cdot)$  is a vector space over the field  $F$

$\Leftrightarrow V$  is a vector space

#### 40.2.1 commutative group structure of vector space

(va) axioms = vector addition axioms

$$\begin{aligned}
& V = (V, +) \text{ is a commutative group} \Leftrightarrow V = (V, +) \text{ is an abelian group} \\
\Leftrightarrow & \left\{ \begin{array}{ll} V = (V, +) = (V, +_V) \text{ is a group} & (g) \text{ group} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{array} \right. \\
\Leftrightarrow & \left\{ \begin{array}{lll} + : V \times V = V^2 \xrightarrow{\dagger} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (cl) \text{ closure} \\ \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \end{array} \right. \quad (g)
\end{aligned}$$

$V$  is a vector space

$\Leftrightarrow V = V(F, +, \cdot) = (V, F, +, \cdot)$  is a vector space over the field  $F$

$$\begin{aligned}
& \Leftrightarrow \left\{ \begin{array}{ll} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ V = (V, +) \text{ is a commutative group} \Leftrightarrow V = (V, +) \text{ is an abelian group} & (va) \text{ vector addition} \\ \cdot : F \times V \xrightarrow{\dagger} V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \left\{ \begin{array}{ll} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{array} \right. & (sm) \text{ axioms} \end{array} \right. \\
& \Leftrightarrow \left\{ \begin{array}{ll} F \text{ is a field} & (f) \text{ field} \\ V \neq \emptyset & (ne) \text{ nonempty set} \\ \left\{ \begin{array}{ll} V = (V, +) = (V, +_V) \text{ is a group} & (g) \text{ group} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \text{ commutativity} \end{array} \right. & (va) \text{ vector addition} \\ \cdot = \cdot_{F \times V} : F \times V \xrightarrow{\dagger} V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (sm) \text{ scalar multiplication} \\ \left\{ \begin{array}{ll} \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = (st)\mathbf{v}] & (a) \text{ associativity} \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{array} \right. & (sm) \text{ axioms} \end{array} \right. \\
& \Leftrightarrow \left\{ \begin{array}{ll} F = F(+_F, \cdot_F) = (F, +_F, \cdot_F) = (F, +, \cdot) \text{ is a field} & (f) \\ V \neq \emptyset & (ne) \\ \left\{ \begin{array}{ll} + : V \times V = V^2 \xrightarrow{\dagger} V \Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in V, \exists! \mathbf{w} \in V [\mathbf{w} = \mathbf{u} + \mathbf{v}] & (cl) \text{ closure} \\ \exists! \mathbf{0} \in V, \forall \mathbf{v} \in V [\mathbf{0} + \mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V [(-\mathbf{v}) + \mathbf{v} = \mathbf{0}] & (i) \text{ inverse} \\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V [\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}] & (a) \text{ associativity} \\ \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}] & (c) \end{array} \right. & (va) \\ \cdot : F \times V \xrightarrow{\dagger} V \Leftrightarrow \forall s \in F, \forall \mathbf{v} \in V, \exists! \mathbf{u} \in V [\mathbf{u} = s\mathbf{v} = s \cdot \mathbf{v}] & (cl) \text{ closure} \\ \exists! 1 \in F, \forall \mathbf{v} \in V [1\mathbf{v} = \mathbf{v}] & (e) \text{ identity} \\ \forall s, t \in F, \mathbf{v} \in V [s(t\mathbf{v}) = s \cdot_{F \times V} (t \cdot_{F \times V} \mathbf{v}) = (s \cdot_F t) \cdot_{F \times V} \mathbf{v} = (st)\mathbf{v}] & (a) \text{ associativity} \quad (sm) \\ \forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = (s+_F t)\mathbf{v} = s\mathbf{v} +_V t\mathbf{v}] & (ds) \text{ scalar distributivity} \\ \forall s \in F, \mathbf{u}, \mathbf{v} \in V [s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}] & (dv) \text{ vector distributivity} \end{array} \right. \end{aligned}$$

#### 40.2.2 scalar distributivity

(sm) (ds)

$$\forall s, t \in F, \mathbf{v} \in V [(s+t)\mathbf{v} = s\mathbf{v} + t\mathbf{v}]$$

$$\forall s, t \in F, \mathbf{v} \in V [(s+_F t)\mathbf{v} = s\mathbf{v} +_V t\mathbf{v}]$$

$$\forall s, t \in F, \mathbf{v} \in V [(s +_F t)\mathbf{v} = s\mathbf{v} +_V t\mathbf{v}]$$

## 40.3 linearity

$$\begin{aligned} & \begin{cases} f(x+y) = f(x) + f(y) & \text{additivity} \\ f(\lambda x) = \lambda f(x) & \text{homogeneity} \end{cases} \\ \Leftrightarrow & f(\lambda x + y) = \lambda f(x) + f(y) \\ \Leftrightarrow & f \text{ is linear} \end{aligned}$$

### 40.3.1 linear structure of vector space

$$\forall s \in F, \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\forall s \in F, \forall \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\forall s \in F, \langle \mathbf{u}, \mathbf{v} \rangle \in V^2 [ \mathbf{u} + s\mathbf{v} \in V ]$$

$$\begin{aligned} & \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} \in V] & \text{vector addition closure} \\ \forall s \in F, \mathbf{v} \in V [s\mathbf{v} \in V] & \text{scalar multiplication closure} \end{cases} \\ \Leftrightarrow & \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [\mathbf{u} + \mathbf{v} \in V] & (a) \text{ additivity} \\ \forall s \in F, \mathbf{v} \in V [s\mathbf{v} \in V] & (h) \text{ homogeneity} \end{cases} \\ \Leftrightarrow & \forall s \in F, \mathbf{u}, \mathbf{v} \in V [ \mathbf{u} + s\mathbf{v} \in V ] \quad (l) \text{ linearity} \end{aligned}$$

### 40.3.2 linear transformation or linear map

$$\begin{aligned} & \begin{cases} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \begin{cases} \forall \mathbf{u}, \mathbf{v} \in V [T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})] & (a) \text{ additivity} \\ \forall \mathbf{v} \in V, c \in F [T(c\mathbf{v}) = cT(\mathbf{v})] & (h) \text{ homogeneity} \end{cases} \end{cases} \quad (L) \\ \Leftrightarrow & \begin{cases} V, W \text{ are vector spaces} \\ T : V \rightarrow W \\ \forall \mathbf{u}, \mathbf{v} \in V, c \in F [T(\mathbf{u} + c\mathbf{v}) = T(\mathbf{u}) + cT(\mathbf{v})] \quad (l) \text{ linearity} \end{cases} \\ \Leftrightarrow & T \text{ is a linear map from } V \text{ to } W \\ \Leftrightarrow & T \text{ is a linear transformation} \end{aligned}$$

## 40.4 vector space example

- arrow vector
- number
  - integer
  - real
  - complex
  - quaternion
- function
  - polynomial function
  - continuous function
- matrix
  - real matrix
  - complex matrix
- reciprocal space

<https://www.bilibili.com/video/BV1NC4y1J7UL>

applications in different disciplines

- math
  - recursive number series
  - Fourier series
- physics

- electrical circuit: linear response / superposition theorem in linear circuit / linear network
- chemistry
  - balancing chemical equation

#### 40.4.1 reciprocal space

reciprocal space = 倒易空間

$$\begin{cases} \mathbf{e}_1 = \mathbf{a} & \mathbf{a} \times \mathbf{b} \neq \mathbf{0} \\ \mathbf{e}_2 = \mathbf{b} & \mathbf{b} \times \mathbf{c} \neq \mathbf{0} \\ \mathbf{e}_3 = \mathbf{c} & \mathbf{c} \times \mathbf{a} \neq \mathbf{0} \end{cases} \Rightarrow \begin{cases} \mathbf{e}'_1 = \frac{\mathbf{b} \times \mathbf{c}}{\Omega} \\ \mathbf{e}'_2 = \frac{\mathbf{c} \times \mathbf{a}}{\Omega} \\ \mathbf{e}'_3 = \frac{\mathbf{a} \times \mathbf{b}}{\Omega} \end{cases},$$

$$\Omega = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

reciprocal space as dual space and contravariant vector

$$\begin{aligned} \text{span} \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} &= \text{span} \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} = V \\ &= \mathbb{R}^3 = \text{span} \{ \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 \} = \text{span} \left\{ \frac{\mathbf{b} \times \mathbf{c}}{\Omega}, \frac{\mathbf{c} \times \mathbf{a}}{\Omega}, \frac{\mathbf{a} \times \mathbf{b}}{\Omega} \right\} \\ &= \text{span} \{ \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \} = \text{span} \{ \mathbf{e}^* \}_{* \in \{1, 2, 3\}} = V^* \end{aligned}$$

##### 40.4.1.1 Kronecker delta

$$\begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{pmatrix} = [\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{e}^1 \cdot \mathbf{e}_1 & \mathbf{e}^1 \cdot \mathbf{e}_2 & \mathbf{e}^1 \cdot \mathbf{e}_3 \\ \mathbf{e}^2 \cdot \mathbf{e}_1 & \mathbf{e}^2 \cdot \mathbf{e}_2 & \mathbf{e}^2 \cdot \mathbf{e}_3 \\ \mathbf{e}^3 \cdot \mathbf{e}_1 & \mathbf{e}^3 \cdot \mathbf{e}_2 & \mathbf{e}^3 \cdot \mathbf{e}_3 \end{pmatrix}$$

Kronecker delta

$$\mathbf{e}_i \cdot \mathbf{e}'_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker delta tensor = Kronecker tensor

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\mathbf{v} = v_a \mathbf{a} + v_b \mathbf{b} + v_c \mathbf{c} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$

$$\mathbf{e}^1 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_1 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_1 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_1 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_1 = v_1$$

$$\mathbf{e}^2 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_2 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_2 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_2 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_2 = v_2$$

$$\mathbf{e}^3 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_3 = v_1 \mathbf{e}_1 \cdot \mathbf{e}'_3 + v_2 \mathbf{e}_2 \cdot \mathbf{e}'_3 + v_3 \mathbf{e}_3 \cdot \mathbf{e}'_3 = v_3$$

$$\begin{aligned} \mathbf{v} &= v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \\ &= (\mathbf{v} \cdot \mathbf{e}'_1) \mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}'_2) \mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}'_3) \mathbf{e}_3 \\ &= (\mathbf{e}^1 \cdot \mathbf{v}) \mathbf{e}_1 + (\mathbf{e}^2 \cdot \mathbf{v}) \mathbf{e}_2 + (\mathbf{e}^3 \cdot \mathbf{v}) \mathbf{e}_3 \\ &= \mathbf{e}^1(\mathbf{v}) \mathbf{e}_1 + \mathbf{e}^2(\mathbf{v}) \mathbf{e}_2 + \mathbf{e}^3(\mathbf{v}) \mathbf{e}_3 \end{aligned}$$

$$\begin{cases} \mathbf{e}^1(\mathbf{v}) = \mathbf{e}^1 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_1 = v_1 \\ \mathbf{e}^2(\mathbf{v}) = \mathbf{e}^2 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_2 = v_2 \\ \mathbf{e}^3(\mathbf{v}) = \mathbf{e}^3 \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{e}'_3 = v_3 \end{cases}$$

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i = \delta_{j,i} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

reciprocal space is a dual space of its original vector space

$$\begin{aligned} V &= \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3\} \\ &= \left\{ \sum_{j=1}^3 v_j \mathbf{e}_j \right\} = \left\{ v_j \mathbf{e}_j \middle| \begin{array}{l} v_j \in F \\ \mathbf{e}_j \in F^3 \end{array} \right\} = \{\mathbf{v} | \mathbf{v} \in V\} \\ V^* &= \text{span}\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\} = \{v^{*1} \mathbf{e}^1 + v^{*2} \mathbf{e}^2 + v^{*3} \mathbf{e}^3\} \\ &= \left\{ \sum_{i=1}^3 v^{*i} \mathbf{e}^i \right\} = \left\{ v^{*i} \mathbf{e}^i \middle| \begin{array}{l} v^{*i} \in F \\ \mathbf{e}^i \in F^3 \end{array} \right\} = \{\mathbf{v}^* | \mathbf{v}^* \in V^*\} \\ \mathbf{v}^*(\mathbf{v}) &= (v^{*i} \mathbf{e}^i)(\mathbf{v}), \mathbf{v} \in V \\ &= (v^{*1} \mathbf{e}^1 + v^{*2} \mathbf{e}^2 + v^{*3} \mathbf{e}^3)(\mathbf{v}) \\ &= v^{*1} \mathbf{e}^1(\mathbf{v}) + v^{*2} \mathbf{e}^2(\mathbf{v}) + v^{*3} \mathbf{e}^3(\mathbf{v}) \\ &= v^{*1} v_1 + v^{*2} v_2 + v^{*3} v_3 \in F \end{aligned}$$

element in dual space is a functional or mapping from its original vector space to the field

$$\mathbf{v}^* : V \rightarrow F$$

$$V \xrightarrow{\mathbf{v}^*} F$$

$$V^* = \{\mathbf{v}^* | \mathbf{v}^* : V \rightarrow F\}$$

$$\begin{array}{ccccccc} & & V & = & \{ & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{v} & \cdots \} \\ \mathbf{e}^1 & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{\mathbf{e}^2\} & F & \supseteq & \{ & 1 & 0 & 0 & v_1 & \cdots \} \\ \mathbf{e}^3 & V & = & \{ & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{v} & \cdots \} \\ \mathbf{v}^* & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & F & \supseteq & \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i} v_i & \cdots \} \end{array}$$

$$\begin{aligned} V^* &= \{\mathbf{v}^* | \mathbf{v}^* \in V^*\} = \{\mathbf{v}^* | \mathbf{v}^* : V \rightarrow F\} \\ &= \left\{ \mathbf{v}^* \middle| V \xrightarrow{\mathbf{v}^*} F \right\} \\ &= \{\omega | \omega : V \rightarrow F\} \\ &= \left\{ \omega^i \mathbf{e}^i \middle| \begin{array}{l} \omega^i \in F \\ \mathbf{e}^i \in F^3 \end{array} \right\} \end{aligned}$$

By defining vector addition and scalar multiplication on the dual space

$$\begin{cases} + : V^* \times V^* \rightarrow V^* \Leftrightarrow \forall \omega_1, \omega_2 \in V^*, \exists! (\omega_1 + \omega_2) \in V^* [(\omega_1 + \omega_2)(\mathbf{v}) = \omega_1(\mathbf{v}) + \omega_2(\mathbf{v})] \\ \cdot : F \times V^* \rightarrow V^* \Leftrightarrow \forall k \in F, \forall \omega \in V^*, \exists! (k\omega) \in V^* [(k\omega)(\mathbf{v}) = k \cdot \omega(\mathbf{v})] \\ \forall \omega \in V^*, \exists! \mathbf{0} \in V^* [(\omega + \mathbf{0})(\mathbf{v}) = \omega(\mathbf{v}) + \mathbf{0}(\mathbf{v}) = \omega(\mathbf{v})] \end{cases}$$

the dual space also becomes a vector space.

#### 40.4.1.2 double dual concept

double dual space = second dual space

$$\begin{aligned} V^{**} &= (V^*)^* \\ &= \{\omega^* | \omega^* : V^* \rightarrow F\} \\ &= \{\omega^* | \omega^* \in V^{**}\} \end{aligned}$$

$$\begin{aligned} V^{**} &= (V^*)^* = \text{span} \{e^\mu\}_{\mu \in \{1, 2, 3\}}^* \\ &= \text{span} \{e^1, e^2, e^3\}^* \\ &= \text{span} \{e^{1*}, e^{2*}, e^{3*}\} \\ &= \text{span} \{e^{\nu*}\}_{\nu \in \{1, 2, 3\}} \end{aligned}$$

$$\begin{aligned} \omega^*(\omega) &= (\omega^{*\nu} e^{\nu*})(\omega), \omega \in V^* \\ &= (\omega^{*1} e^{1*} + \omega^{*2} e^{2*} + \omega^{*3} e^{3*})(\omega) \\ &= \omega^{*1} e^{1*}(\omega) + \omega^{*2} e^{2*}(\omega) + \omega^{*3} e^{3*}(\omega) \\ &= \omega^{*1} \omega_1 + \omega^{*2} \omega_2 + \omega^{*3} \omega_3 \in F \end{aligned}$$


---

$$V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\}$$

$$\begin{array}{ccccccc} & V^* & = & \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\ \begin{matrix} e^{1*} \\ e^{2*} \\ e^{3*} \\ \omega^* \\ \vdots \end{matrix} & : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ V^{**} = \{ & F & \supseteq & \{ & 1 & 0 & 0 & \omega^1 & \dots \} \\ & V^* & = & \{ & e_1 & e_2 & e_3 & v & \dots \} \\ & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ & F & \supseteq & \{ & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*} \omega^\mu & \dots \} \end{array}$$


---

$$\begin{cases} e^{1*}(\omega) = e^{1*} \cdot \omega = \omega \cdot e^{1*} = \omega(e^{1*}) \\ e^{2*}(\omega) = e^{2*} \cdot \omega = \omega \cdot e^{2*} = \omega(e^{2*}) \\ e^{3*}(\omega) = e^{3*} \cdot \omega = \omega \cdot e^{3*} = \omega(e^{3*}) \end{cases}$$

$$\omega^*(\omega) = \omega^* \cdot \omega = \omega \cdot \omega^* = \omega(\omega^*)$$

i.e.  $f$  acts on  $x$  equivalent to  $x$  acts on  $f$

$$x(f) = x \cdot f = f \cdot x = f(x)$$

$$e^\mu(e_\nu) = e^\mu \cdot e_\nu = \delta_\nu^\mu = \delta_\nu^\mu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$$\begin{aligned} e^{\nu*}(e^\mu) &\stackrel{\text{def.}}{=} e_\nu \cdot e^\mu = e^\mu \cdot e_\nu = e^\mu(e_\nu) \\ &\Downarrow \\ V^{**} &= \text{span} \{e^{\nu*}\}_{\nu \in \{1, 2, 3\}} \cong \text{span} \{e_\nu\}_{\nu \in \{1, 2, 3\}} = V \\ V^{**} &\cong V \\ &\Downarrow \begin{cases} V^{**} \cong V & V, V^{**} \text{ are isomorphism} \\ & \text{independent of choice of bases} \end{cases} \\ &V, V^{**} \text{ are naturally isomorphism} \end{aligned}$$


---

$$V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\} \cong V = \{v | v : V^* \rightarrow F\}$$

$$\begin{array}{ccccccc}
 & V^* & = \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\
 \begin{matrix} e^{1*} \\ e^{2*} \\ e^{3*} \\ \omega^* \end{matrix} : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\
 V^{**} = \{ & F & \supseteq \{ & 1 & 0 & 0 & \omega^1 & \dots \} \\
 & V^* & = \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & F & \supseteq \{ & \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*}\omega^{\mu} & \dots \} \\
 & & V^* & = \{ & e^1 & e^2 & e^3 & v^* & \dots \} \\
 & e_1 & : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \cong V = \{ & e_2 & \} & F & \supseteq \{ & 1 & 0 & 0 & v^{*1} & \dots \} \\
 & e_3 & \} & V^* & = \{ & e^1 & e^2 & e^3 & v^* & \dots \} \\
 & v & : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & \vdots & F & \supseteq \{ & v_1 & v_2 & v_3 & v_{\mu}v^{*\mu} & \dots \}
 \end{array}$$


---

$$V \cong V^{**}$$


---

$$V \cong V^{**} = \{\omega^* | \omega^* : V^* \rightarrow F\}$$

$$V = \{v | v : V^* \rightarrow F\}$$

i.e. **vector space** is a set of functionals or mappings from its dual space to the field, answering **What is a vector?**<sup>[40.1]</sup>, and satifying Fig: 40.1.

## 40.5 EpicOrganism = AIRoswell = Pan, Yi-Wen<sup>1</sup>

<https://space.bilibili.com/14316464/video>

<https://www.bilibili.com/video/BV1dC4y1171H>

<https://www.bilibili.com/video/BV14H4y1C7A9>

<https://www.bilibili.com/video/BV1MW421N7HS>

## 40.6 field

[https://web.math.sinica.edu.tw/math\\_media/d312/31202.pdf](https://web.math.sinica.edu.tw/math_media/d312/31202.pdf)

## 40.7 module

<https://web.math.sinica.edu.tw/mathmedia/HTMLarticle18.jsp?mID=31304>

## 40.8 subspace

---

<sup>1</sup><https://www.linkedin.com/in/yiwen-pan-16a90076>



# Chapter 41

$\mathrm{d}f$

## 41.1 $\mathrm{d}f$ decomposed with partials as a set of basis in vector space

$$f = \{f_i\} = \{f_1, f_2, \dots\} = \{f, g, \dots\}$$

$$\mathbf{v} : f \rightarrow F$$

$$\mathbf{v}(af + bg) = a\mathbf{v}(f) + b\mathbf{v}(g)$$

$$\mathbf{v}(fg) = f|_P \mathbf{v}(g) + \mathbf{v}(f)g|_P$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [f(x)g(x)]|_{x=x_0} = f(x_0) \frac{\mathrm{d}}{\mathrm{d}x} g(x)|_{x=x_0} + \frac{\mathrm{d}}{\mathrm{d}x} f(x)|_{x=x_0} g(x_0)$$

$$V = \{\mathbf{v} | \mathbf{v} : f \rightarrow F\}$$

$$\begin{aligned} f &= f(\mathbf{x}) \\ &= f(x_1, \dots, x_j, \dots, x_n) \\ &= f(x^1, \dots, x^j, \dots, x^n) \end{aligned}$$

$$\mathbf{x} = \langle x^1, \dots, x^j, \dots, x^n \rangle$$

$$\mathbf{x}(t) = \langle x^1(t), \dots, x^j(t), \dots, x^n(t) \rangle$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\mathrm{d}x^1}{\mathrm{d}t} \frac{\partial f}{\partial x^1} + \dots + \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} + \dots + \frac{\mathrm{d}x^n}{\mathrm{d}t} \frac{\partial f}{\partial x^n} \\ &= \dots + \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} + \dots = \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial f}{\partial x^j} = \frac{\mathrm{d}x^j}{\mathrm{d}t} \partial_j f \end{aligned}$$

$$\begin{aligned} V &= \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_j, \dots, \mathbf{e}_n\} \\ &= \text{span}\left\{\frac{\partial}{\partial x^1}|_P, \dots, \frac{\partial}{\partial x^j}|_P, \dots, \frac{\partial}{\partial x^n}|_P\right\} \\ &= \text{span}\{\boldsymbol{\partial}_1, \dots, \boldsymbol{\partial}_j, \dots, \boldsymbol{\partial}_n\} \\ &= \left\{\boldsymbol{\partial}_t \middle| \boldsymbol{\partial}_t = a_j \mathbf{e}_j = a_j \boldsymbol{\partial}_j = a_j \frac{\partial}{\partial x^j}|_P\right\} \\ &= \left\{\frac{\partial}{\partial t}|_P \middle| \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \dots + a_j \frac{\partial}{\partial x^j}|_P + \dots + a_n \frac{\partial}{\partial x^n}|_P\right\} \end{aligned}$$

## 41.2 dual space of span of partials

$$V^* = \{\omega_f | \omega_f : V \rightarrow F\}$$

$$\omega_f(e_j) = \omega_f(\partial_j) = \omega_f\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial f}{\partial x^j}|_P \in F$$

$$\begin{aligned}\omega_{fg}(\partial_j) &= \frac{\partial fg}{\partial x^j}|_P = f|_P \frac{\partial g}{\partial x^j}|_P + \frac{\partial f}{\partial x^j}|_P g|_P \\ &= f|_P \omega_g(\partial_j) + \omega_f(\partial_j) g|_P\end{aligned}$$

$$\omega_{x^i}(\partial_j) = \omega_{x^i}\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\begin{aligned}V^* = \{\omega_f | \omega_f : V \rightarrow F\} &= \left\{ \omega_f \left| \begin{array}{l} \omega_f(e_j) = \omega_f(\partial_j) = \omega_f\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial f}{\partial x^j}|_P \in F \\ \omega_{fg}(\partial_j) = f|_P \omega_g(\partial_j) + \omega_f(\partial_j) g|_P \\ \omega_{x^i}(\partial_j) = \omega_{x^i}\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\} \\ &= \{df|df : V \rightarrow F\} = \left\{ df \left| \begin{array}{l} df(e_j) = df(\partial_j) = df\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial f}{\partial x^j}|_P \in F \\ dfg(\partial_j) = f|_P (dg) + (df) g|_P \\ dx^i(\partial_j) = dx^i\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\}\end{aligned}$$

$$dx^i\left(\frac{\partial}{\partial x^j}|_P\right) = \delta_{ij} = e^i \cdot e_j \Rightarrow \begin{cases} e^i = dx^i \\ e_j = \frac{\partial}{\partial x^j}|_P \end{cases}$$

$$\begin{aligned}V^* = \{df|df : V \rightarrow F\} &= \left\{ df \left| \begin{array}{l} df(e_j) = df(\partial_j) = df\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial f}{\partial x^j}|_P \in F \\ dfg(\partial_j) = f|_P (dg) + (df) g|_P \\ dx^i(\partial_j) = dx^i\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \end{array} \right. \right\} \\ &= \text{span}\{dx^1, \dots, dx^i, \dots, dx^n\} = \text{span}\{e^1, \dots, e^j, \dots, e^n\}\end{aligned}$$

## 41.3 directional derivative

$$\begin{aligned}df(v) &= df(v_j e_j) = v_j df(e_j) \\ &= v_j df(\partial_j) = v_j \frac{\partial f}{\partial x^j}|_P \\ &= v_1 \frac{\partial f}{\partial x^1}|_P + \dots + v_j \frac{\partial f}{\partial x^j}|_P + \dots + v_n \frac{\partial f}{\partial x^n}|_P \\ &= (v_1 \quad \dots \quad v_j \quad \dots \quad v_n) \nabla f\end{aligned}$$


---

$$\widehat{PQ} = C(t) - C(0) = Q - P$$

$$v = \frac{\partial}{\partial t}|_P$$

$$\begin{aligned}df(sv) &= df\left(s \frac{\partial}{\partial t}|_P\right) = s \frac{\partial f}{\partial t}|_P \\ &= sv(f) = s \cdot \lim_{t \rightarrow 0} \frac{f(C(t)) - f(C(0))}{t} \\ &\approx s \cdot \frac{f(Q) - f(P)}{s} = f(Q) - f(P) = \Delta f\end{aligned}$$

## 41.4 coefficient of linear combination for vector space and dual space

$$\begin{aligned}
 V &= \{v|v : f \rightarrow F\} \\
 &= \text{span}\{e_1, \dots, e_j, \dots, e_n\} \\
 &= \text{span}\left\{\frac{\partial}{\partial x^1}|_P, \dots, \frac{\partial}{\partial x^j}|_P, \dots, \frac{\partial}{\partial x^n}|_P\right\} = \text{span}\{\partial_1, \dots, \partial_j, \dots, \partial_n\} \\
 &= \left\{\partial_t \middle| \partial_t = a_j e_j = a_j \partial_j = a_j \frac{\partial}{\partial x^j}|_P\right\} \\
 &= \left\{\frac{\partial}{\partial t}|_P \middle| \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \dots + a_j \frac{\partial}{\partial x^j}|_P + \dots + a_n \frac{\partial}{\partial x^n}|_P\right\} \\
 V^* &= \{df|df : V \rightarrow F\} \\
 &= \text{span}\{e^1, \dots, e^i, \dots, e^n\} \\
 &= \text{span}\{dx^1, \dots, dx^i, \dots, dx^n\} \\
 &= \{df|df = b^i e^i = b^i dx^i\} \\
 &= \{df|df = b^1 dx^1 + \dots + b^i dx^i + \dots + b^n dx^n\}
 \end{aligned}$$


---

or more simply to be comparison

$$\begin{array}{ll}
 V &= \text{span}\{e_j = \partial_j\} = \{v = \partial_t|_P = a_j e_j = a_j \partial_j|_P : f \rightarrow F\} \\
 V^* &= \text{span}\{e^i = dx^i\} = \{\omega = df = b^i e^i = b^i dx^i : V \rightarrow F\}
 \end{array}$$


---

$$\begin{aligned}
 &\left\{ dx^i(\partial_j) = dx^i\left(\frac{\partial}{\partial x^j}|_P\right) = \frac{\partial x^i}{\partial x^j}|_P = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \right. \\
 &\left. \partial_t = a_j e_j = a_j \partial_j \Leftrightarrow \frac{\partial}{\partial t}|_P = a_1 \frac{\partial}{\partial x^1}|_P + \dots + a_j \frac{\partial}{\partial x^j}|_P + \dots + a_n \frac{\partial}{\partial x^n}|_P \right. \\
 &\Rightarrow \left\{ \begin{array}{l} dx^i(\partial_t) = dx^i\left(\frac{\partial}{\partial t}|_P\right) = \frac{\partial x^i}{\partial t}|_P \\ dx^i(\partial_t) = dx^i(a_j \partial_j) = a_j dx^i(\partial_j) = a_j \delta_{ij} = a_i \end{array} \right. \Rightarrow a_i = dx^i(\partial_t) = \frac{\partial x^i}{\partial t}|_P \\
 &\Rightarrow a_i = \frac{\partial x^i}{\partial t}|_P \Rightarrow a_j = \frac{\partial x^j}{\partial t}|_P = \partial_t x^j|_P \\
 &\Rightarrow \frac{\partial}{\partial t}|_P = a_i \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t}|_P \frac{\partial}{\partial x^i}|_P = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}|_P \Rightarrow \frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \\
 &\Rightarrow \partial_t|_P = \frac{\partial x^j}{\partial t} \partial_j|_P \Leftrightarrow \partial_t|_P = \partial_t x^j \partial_j|_P
 \end{aligned}$$

$$\begin{aligned}
 df &= b^i e^i = b^i dx^i \\
 \frac{\partial f}{\partial x^j} &= df(\partial_j) = df(e_j) = b^i e^i \cdot e_j = b^i \delta_{ij} = b^j \\
 b^j &= \frac{\partial f}{\partial x^j} \\
 b^i &= \frac{\partial f}{\partial x^i} = \partial_i f \\
 df &= b^i e^i = b^i dx^i = \frac{\partial f}{\partial x^i} dx^i \\
 df &= \frac{\partial f}{\partial x^i} dx^i \\
 df &= \partial_i f dx^i
 \end{aligned}$$


---

$$\begin{array}{ll}
 V &= \text{span}\{e_j = \partial_j\} = \{v = \partial_t|_P = a_j e_j = a_j \partial_j|_P : f \rightarrow F\} \\
 V^* &= \text{span}\{e^i = dx^i\} = \{\omega = df = b^i e^i = b^i \partial_i f dx^i : V \rightarrow F\}
 \end{array}$$

## 41.5 change of basis / change of coordinate

$$\frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} \stackrel{t=x'^j}{\Rightarrow} \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} = \frac{\partial x^1}{\partial x'^j} \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^j} \frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^j} \frac{\partial}{\partial x^3}$$

$$\begin{aligned} df &= \frac{\partial f}{\partial x^i} dx^i \\ f &= x'^j \Downarrow \\ dx'^j &= \frac{\partial x'^j}{\partial x^i} dx^i \end{aligned}$$


---

$$\begin{cases} \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} = \sum_i \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i} \\ dx'^j = \frac{\partial x'^j}{\partial x^i} dx^i = \sum_i \frac{\partial x'^j}{\partial x^i} dx^i \end{cases}$$

## 41.6 ambiguity with partial notation

<https://www.youtube.com/watch?v=mICbKwwHzlI>

## 41.7 1-form

1-form = one-form

## 41.8 What is Math: differential geometry

<https://www.youtube.com/playlist?list=PLXo8Tdaw0czOWyRD-esa6mNajlPZnjHQs>

## 41.9 Liang, Can-bin: differential geometry and general relativity

<https://www.bilibili.com/video/BV1o4411L72E>

<https://www.youtube.com/playlist?list=PLstdOGDXMaWIKCWheiNIRumejII0gItYM>

<https://www.youtube.com/playlist?list=PLstdOGDXMaWICAkLFdCX24pwcWww5YzyQ>

# Chapter 42

## determinant

<https://www.bilibili.com/video/BV13e411m7Js>

<https://www.youtube.com/watch?v=Sv7VseMsOQc>

### 42.1 induction of determinant axioms

### 42.2 determinant axioms

determinant axioms

$$\begin{aligned} & \left\{ \begin{array}{l} \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u} + s\mathbf{v}, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u}, \mathbf{v} + s\mathbf{u}) \end{array} \right. \quad \text{translation invariance} \\ & \left\{ \begin{array}{l} \det(s\mathbf{u}, \mathbf{v}) = s \det(\mathbf{u}, \mathbf{v}) \\ \det(\mathbf{u}, s\mathbf{v}) = s \det(\mathbf{u}, \mathbf{v}) \end{array} \right. \quad \text{scaling} \\ & \left\{ \begin{array}{l} \det(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) = \det(\mathbf{u}_1, \mathbf{v}) + \det(\mathbf{u}_2, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2) = \det(\mathbf{u}, \mathbf{v}_1) + \det(\mathbf{u}, \mathbf{v}_2) \end{array} \right. \quad \text{decomposition} \\ \Leftrightarrow & \left\{ \begin{array}{l} \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u} + s\mathbf{v}, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{u}, \mathbf{v} + s\mathbf{u}) \\ \det(\mathbf{u}_1 + s\mathbf{u}_2, \mathbf{v}) = \det(\mathbf{u}_1, \mathbf{v}) + s \det(\mathbf{u}_2, \mathbf{v}) \\ \det(\mathbf{u}, \mathbf{v}_1 + s\mathbf{v}_2) = \det(\mathbf{u}, \mathbf{v}_1) + s \det(\mathbf{u}, \mathbf{v}_2) \end{array} \right. \quad \begin{array}{l} \text{translation invariance} \\ \text{linearity} \end{array} \end{aligned}$$

### 42.3 determinant theorems or properties

### 42.4 ellipse area

### 42.5 Cramer rule geometry perspective



# Chapter 43

## hypergeometric function

### 43.1 linear space of function

<https://www.bilibili.com/video/BV1PX4y167RS>

quantum state<sup>[45.2.3]</sup>

Taylor vs. Fourier<sup>[@ref(taylor-vs.-fourier)]</sup>

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \dots = \sum_{k=0}^{\infty} a_k x^k$$

$$f(x) = x_0x^0 + x_1x^1 + x_2x^2 + \dots = \sum_{k=0}^{\infty} x_k x^k$$

Def: 45.3

$$\langle f|g \rangle = \int_a^b \overline{f(x)} g(x) dx \stackrel{f,g:\mathbb{R} \rightarrow \mathbb{R}}{=} \int_a^b f(x) g(x) dx$$

Dirac bracket<sup>[45.5]</sup>

$$\langle x^2|x \rangle \stackrel{x^2,x:\mathbb{R} \rightarrow \mathbb{R}}{=} \int_a^b x^2 x dx = \int_a^b x^3 dx = \left[ \frac{x^4}{4} \right]_a^b \not\equiv 0$$

$$x^0 \not\perp x^1, x^1 \not\perp x^2, \dots$$

$$\langle x_m|x^n \rangle = \int_a^b x_m x^n dx = \delta_{mn}$$

$$\langle 1|x^n \rangle = \int_a^b x_0 x^n dx = \delta_{0n} \Rightarrow x_0 = \delta(x) = \delta(x-0)$$

$$\langle x_m|x^n \rangle = \int_a^b x_m x^n dx = \delta_{mn} \Rightarrow x_m = \frac{(-1)^m}{m!} \delta^{(m)}(x)$$

$$|f\rangle = 1|f\rangle = \left( \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \right) |f\rangle = \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| f \rangle$$

$$\begin{aligned}
|f\rangle &= 1|f\rangle = \left( \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \right) |f\rangle = \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| f \rangle \\
&= 1|f\rangle = \left( \sum_n |x^n\rangle \langle x^n| \right) |f\rangle = \sum_n |x^n\rangle \langle x^n| f \rangle = \sum_n \langle x^n| f \rangle |x^n\rangle \\
\langle x^n| f \rangle &= \langle x^n| f \rangle = \int_a^b x_n f(x) dx = \int_a^b \frac{(-1)^n}{n!} \delta^{(n)}(x) f(x) dx = \frac{f^{(n)}(0)}{n!} \\
|f\rangle &= \sum_n \langle x^n| f \rangle |x^n\rangle = \sum_n \frac{f^{(n)}(0)}{n!} |x^n\rangle \\
|f\rangle &= \sum_n \frac{f^{(n)}(0)}{n!} |x^n\rangle \\
&\Downarrow \\
f(x) &= \sum_n \frac{f^{(n)}(0)}{n!} x^n
\end{aligned}$$

## 43.2 beta function

<https://www.bilibili.com/video/BV1pa4y1P7Da>

$$\begin{aligned}
\binom{n}{k} &= C_k^n = \frac{n!}{(n-k)!k!} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \begin{cases} n \in \mathbb{N} \\ k \in (\{0\} \cup \mathbb{N}) \end{cases} \\
\binom{r}{k} &= \begin{cases} \frac{r(r-1)\cdots(r-k+1)}{k!} & k \geq 0, k \in \mathbb{Z} \\ 0 & k < 0, k \in \mathbb{Z} \end{cases}
\end{aligned}$$

$$\sum_{k=0}^n \binom{r}{k} (\cdot)$$

$$\sum_{k=-\infty}^n \binom{r}{k} (\cdot) = (0 + 0 + \cdots) + \sum_{k=0}^n \binom{r}{k} (\cdot)$$

$$\sum_{k=-\infty}^{\infty} \binom{r}{k} (\cdot)$$


---

$$n! = \Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x} dx$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)} \\ [\Gamma(z)\Gamma(1-z)]_{z=-n} &= \left[ \frac{\pi}{\sin(\pi z)} \right]_{z=-n}, n \in \mathbb{N} \\ \Gamma(-n)n! &= \Gamma(n+1) = \Gamma(-n)\Gamma(1-(-n)) = \frac{\pi}{\sin(\pi(-n))} = \frac{\pi}{-\sin(n\pi)} \\ \Gamma(-n) &= \frac{-\pi}{n! \sin(n\pi)} = \frac{-\pi}{n!0} \rightarrow -\infty, n \in \mathbb{N}\end{aligned}$$

$$\begin{aligned}\binom{n}{k} &= C_k^n = \frac{n!}{(n-k)!k!} \\ &= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}\end{aligned}$$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}$$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \stackrel{k \leq 0}{=} \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(1)} = \frac{\Gamma(n+1)}{\Gamma(n+1)1} = 1 & k = 0 \\ \frac{\Gamma(n+1)}{\Gamma(n-k+1)(-\infty)} = 0 & k \leq -1, k \in \mathbb{Z} \end{cases}$$

beta function =  $\beta$  function

**Definition 43.1.** beta function =  $\beta$  function

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\begin{aligned}\binom{n}{k} &= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \\ \left[ \binom{n}{k} \right]_{\begin{cases} n=a+b \\ k=a \end{cases}} &= \left[ \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \right]_{\begin{cases} n=a+b \\ k=a \end{cases}} \\ \binom{a+b}{a} &= \frac{\Gamma(a+b+1)}{\Gamma(a+b-a+1)\Gamma(a+1)} \\ &= \frac{\Gamma(a+b+1)}{\Gamma(b+1)\Gamma(a+1)}\end{aligned}$$

$$\begin{aligned}B(p, q) &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\ [B(p, q)]_{\begin{cases} p=a+1 \\ q=b+1 \end{cases}} &= \left[ \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \right]_{\begin{cases} p=a+1 \\ q=b+1 \end{cases}} \\ B(a+1, b+1) &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+1+b+1)} \\ &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma([a+b+1]+1)} = \frac{\Gamma(a+1)\Gamma(b+1)}{[a+b+1]\Gamma(a+b+1)}\end{aligned}$$

$$\begin{aligned}\binom{a+b}{a} &= \frac{\Gamma(a+b+1)}{\Gamma(b+1)\Gamma(a+1)} = \frac{1}{\frac{\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+1)}} \\ &= \frac{1}{[a+b+1]\frac{\Gamma(b+1)\Gamma(a+1)}{[a+b+1]\Gamma(a+b+1)}} \\ &= \frac{1}{[a+b+1]B(a+1, b+1)}\end{aligned}$$


---

[https://en.wikipedia.org/wiki/Beta\\_function](https://en.wikipedia.org/wiki/Beta_function)  
[https://en.wikipedia.org/wiki/Beta\\_function#Other\\_identities\\_and\\_formulas](https://en.wikipedia.org/wiki/Beta_function#Other_identities_and_formulas)  
[https://en.wikipedia.org/wiki/Beta\\_function#Multivariate\\_beta\\_function](https://en.wikipedia.org/wiki/Beta_function#Multivariate_beta_function)  
<https://www.bilibili.com/video/BV1pa4y1P7Da/?t=4m>

#### 43.2.1 Wallis product formula

[https://en.wikipedia.org/wiki/Wallis\\_product](https://en.wikipedia.org/wiki/Wallis_product)  
<https://www.bilibili.com/video/BV1pa4y1P7Da/?t=4m10s>  
<https://www.math.sinica.edu.tw/mathmedia/journals/4739?keywords%5B%5D=Paul+Dirac>

### 43.3 gamma function

<https://www.bilibili.com/video/BV1cT411H7Hp>

### 43.4 recursion

<https://www.bilibili.com/video/BV1FV4y1Z7jm>  
<https://www.bilibili.com/video/BV1Sg4y1L7DF>

### 43.5 mean and variance of discrete probability distributions

<https://www.bilibili.com/video/BV1Tk4y1n7NX>

### 43.6 CORDIC = coordinate rotation digital computer

<https://www.bilibili.com/video/BV1ge411e7K7>  
<https://space.bilibili.com/11008987/channel/collectiondetail?sid=2053177>  
<https://www.bilibili.com/video/BV1AW4y1A7HN>  
<https://www.bilibili.com/video/BV115411i7VS>

# Chapter 44

## Feynman method

### 44.1 Feynman method of differentiation / derivative technique

<https://www.bilibili.com/video/BV1hG411Z7Cb>

分式微分不是難而是煩

#### 44.1.1 principle

**Theorem 44.1.** Feynman method of differentiation / derivative

$$\begin{aligned} f(t) &= k [u(t)]^a [v(t)]^b [w(t)]^c \dots \\ f'(t) &= f(t) \left[ a \frac{u'(t)}{u(t)} + b \frac{v'(t)}{v(t)} + c \frac{w'(t)}{w(t)} + \dots \right] \end{aligned}$$

Proof:

$$f(t) = k [u(t)]^a [v(t)]^b [w(t)]^c \dots$$

$$f = k u^a v^b w^c \dots = k \cdot u^a \cdot v^b \cdot w^c \dots$$

$$f = k u^a v^b w^c \dots = k \cdot u^a \cdot v^b \cdot w^c \dots$$

$$\begin{aligned} \ln f &= \ln(k u^a v^b w^c \dots) = \ln k + \ln u^a + \ln v^b + \ln w^c + \dots \\ &= \ln k + a \ln u + b \ln v + c \ln w + \dots \end{aligned}$$

$$\frac{d}{dt} \ln f = \frac{d}{dt} (\ln k + a \ln u + b \ln v + c \ln w + \dots)$$

$$\begin{aligned} \frac{d}{dt} \frac{f}{f} &= 0 + \frac{d}{dt} (a \ln u) + \frac{d}{dt} (b \ln v) + \frac{d}{dt} (c \ln w) + \dots \\ &= a \frac{d}{dt} \ln u + b \frac{d}{dt} \ln v + c \frac{d}{dt} \ln w + \dots \end{aligned}$$

$$\begin{aligned} &= a \frac{d}{dt} u + b \frac{d}{dt} v + c \frac{d}{dt} w \\ &= a \frac{u'}{u} + b \frac{v'}{v} + c \frac{w'}{w} + \dots \end{aligned}$$

$$\frac{f'}{f} = a \frac{u'}{u} + b \frac{v'}{v} + c \frac{w'}{w} + \dots$$

$$f' = f \left( a \frac{u'}{u} + b \frac{v'}{v} + c \frac{w'}{w} + \dots \right)$$

$$f'(t) = f(t) \left[ a \frac{u'(t)}{u(t)} + b \frac{v'(t)}{v(t)} + c \frac{w'(t)}{w(t)} + \dots \right]$$

□

#### 44.1.2 examples

$$f(x) = x^x$$

$$(x^x)' = x^x + x^x \ln x$$

$$\begin{aligned} f(x) &= x^x \\ \ln f(x) &= x \ln x \\ \frac{d}{dx} \ln f(x) &= \frac{d}{dx} [x \ln x] \\ \frac{f'(x)}{f(x)} &= [x \ln x]' \\ f'(x) &= f(x) [x \ln x]' = x^x [(x)' \ln x + x (\ln x)'] \\ &= x^x \left[ 1 \ln x + x \frac{1}{x} \right] = x^x [\ln x + 1] = x^x [1 + \ln x] \\ &= x^x + x^x \ln x \end{aligned}$$

□

3D delta function<sup>[45.2.2]</sup>

$$\Delta \left( \frac{1}{r} \right) = \nabla^2 \left( \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = \nabla \cdot \nabla \left( \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right)$$

$$\frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\begin{aligned} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}} &= \frac{\partial}{\partial x} \left[ (x^2 + y^2 + z^2)^{-\frac{1}{2}} \right] \\ &= (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left[ \frac{-1}{2} \frac{2x}{x^2 + y^2 + z^2} \right] \\ &= \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \end{aligned}$$

$$\nabla \left( \frac{1}{r} \right) = \nabla \left( \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)$$

$$\begin{aligned} \nabla \cdot \nabla \left( \frac{1}{r} \right) &= \nabla \cdot \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\ &= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \\ &\quad \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} &= \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \left[ 1 \cdot \frac{1}{x} + \frac{-3}{2} \frac{2x}{x^2 + y^2 + z^2} \right] \\ &= \frac{-1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= \frac{-(x^2 + y^2 + z^2) + 3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \end{aligned}$$

$$\begin{aligned}\nabla \cdot \nabla \left( \frac{1}{r} \right) &= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \\ &= \frac{+2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0\end{aligned}$$

$$\Delta \left( \frac{1}{r} \right) = \nabla \cdot \nabla \left( \frac{1}{r} \right) = 0$$

□

$$\begin{aligned}& \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} + \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \\ & \quad \frac{d}{dt} \left[ \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} + \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \right] \\ & \quad \frac{d}{dt} \left[ \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \right] \\ & \quad \frac{d}{dt} \left[ \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \right] \\ & = \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \cdot [ \\ & = \frac{6[1+2t^2](t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \cdot [1 \cdot \frac{1}{1+2t^2} \cdot 4t, [1+2t^2]] \rightarrow \begin{cases} \text{exponential :} & 1 \\ \text{linear to denominator :} & \frac{1}{1+2t^2} \\ \text{differentiation} & 4t \end{cases} \\ & = \frac{6(1+2t^2)[(t^3-t)^2]}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \cdot [\frac{4t}{1+2t^2} + 2 \cdot \frac{1}{t^3-t} \cdot (3t^2-1) \\ & , [(t^3-t)^2] \rightarrow \begin{cases} \text{exponential :} & 2 \\ \text{linear to denominator :} & \frac{1}{t^3-t} \\ \text{differentiation} & 3t^2-1 \end{cases} \\ & = \frac{6(1+2t^2)(t^3-t)^2}{[\sqrt{t+5t^2}](4t)^{\frac{3}{2}}} \cdot [\frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} + \frac{-1}{2} \cdot \frac{1}{t+5t^2} \cdot (1+10t) \\ & , [\sqrt{t+5t^2}] \rightarrow \begin{cases} \text{exponential :} & -\frac{1}{2} \\ \text{linear to denominator :} & \frac{1}{t+5t^2} \\ \text{differentiation} & 1+10t \end{cases} \\ & = \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}[(4t)^{\frac{3}{2}}]} \cdot \left[ \frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} - \frac{1+10t}{2t+10t^2} + \frac{-3}{2} \cdot \frac{1}{4t} \cdot 4 \right] \\ & , [(4t)^{\frac{3}{2}}] \rightarrow \begin{cases} \text{exponential :} & -\frac{3}{2} \\ \text{linear to denominator :} & \frac{1}{4t} \\ \text{differentiation} & 4 \end{cases} \\ & = \frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{\frac{3}{2}}} \left[ \frac{4t}{1+2t^2} + \frac{6t^2-2}{t^3-t} - \frac{1+10t}{2t+10t^2} - \frac{3}{2t} \right]\end{aligned}$$

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \right] \\ &= \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \left[ \frac{1}{2} \frac{2}{1+2t} + (-1) \frac{1+\left[\frac{1}{2} \frac{2t}{\sqrt{1+t^2}}\right]}{t+\sqrt{1+t^2}} \right] \end{aligned}$$

□

## 44.2 Feynman method of integration / integral technique

### 44.2.1 principle

<https://www.bilibili.com/video/BV1Lj411L79X/?t=2m38s>

<https://www.youtube.com/watch?v=GW86SShcYbM>

$$I = \int f(x) dx$$

$$I(t) = \int f(x, t) dx$$

$$\begin{aligned} I &= \int f(x) dx \\ I(t) &= \int f(x, t) dx \\ \frac{d}{dt} I(t) &= I'(t) = \frac{d}{dt} \int f(x, t) dx \\ &\stackrel{\text{Leibniz integral rule}}{=} f(x, b(x), t) \frac{db(x)}{dx} - f(x, a(x), t) \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial t} f(x, t) dx \\ &\stackrel{\dots}{=} \int \frac{\partial}{\partial t} f(x, t) dx = \int \frac{\partial f(x, t)}{\partial t} dx = \int f_t(x, t) dx \\ I(t) &= \int I'(t) dt = \int I'(t) dt + C = \left[ \int I'(t) dt \right](t) \\ &\Downarrow \text{if } f(x, t=0) = f(x, 0) = f(x) \\ I &\stackrel{f(x,0)=f(x)}{=} \left[ \int I'(t) dt \right](0) \end{aligned}$$

### 44.2.2 Dirichlet integral

as a example by Feynman method of integration / integral technique

[https://en.wikipedia.org/wiki/Dirichlet\\_integral](https://en.wikipedia.org/wiki/Dirichlet_integral)

<https://www.bilibili.com/video/BV1Lj411L79X/?t=4m38s>

$$\int_0^\infty \frac{\sin x}{x} dx$$

<https://www.youtube.com/watch?v=ZZccexuOpb4k>

<https://blog.csdn.net/zhuoqingjoking97298/article/details/127950915>

$$\int_{-\infty}^\infty \frac{\sin x}{x} dx$$

$$\int_{-\infty}^\infty \frac{\sin x}{x} dx$$

$f(x) = \frac{\sin x}{x}$  is an even function  $\Leftrightarrow f(-x) = \frac{\sin(-x)}{(-x)} = \frac{-\sin(x)}{-x} = \frac{\sin x}{x} = f(x)$

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{\sin x}{x} dx &= \int_{-\infty}^0 \frac{\sin x}{x} dx + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{x=-\infty}^{x=0} \frac{\sin x}{x} dx + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{x=-\infty}^{x=0} \frac{\sin x}{x} dx \quad (, x' = -x \Leftrightarrow x = -x') + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{(-x')=-\infty}^{(-x')=0} \frac{\sin(-x')}{(-x')} d(-x') + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{x'=-(\infty)}^{x'=-0} \frac{-\sin x'}{-x'} (-dx') + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{x'=\infty}^{x'=0} \frac{\sin x'}{x'} (-dx') + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= - \int_{x'=\infty}^{x'=0} \frac{\sin x'}{x'} dx' + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_{x'=0}^{x'=\infty} \frac{\sin x'}{x'} dx' + \int_0^{\infty} \frac{\sin x}{x} dx \\
 &= \int_0^{\infty} \frac{\sin x}{x} dx + \int_0^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx
 \end{aligned}$$

---


$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx$$


---

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

$$\begin{aligned}
I &= \int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty f(x) dx, f(x) = \frac{\sin x}{x} \\
\int_0^\infty e^{-tx} \frac{\sin x}{x} dx &= \int_0^\infty e^{-tx} f(x) dx, \text{ Laplacian transform of } f(x) \\
&= \int_0^\infty f(x, t) dx, \begin{cases} f(x, t) = e^{-tx} f(x) \\ f(x, 0) = e^{-0x} f(x) = 1 \cdot f(x) = f(x) \end{cases} \Downarrow \\
I(t) &= \int_0^\infty f(x, t) dx = \int_0^\infty e^{-tx} f(x) dx = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx \Rightarrow I(0) = I \\
\frac{d}{dt} I(t) &= I'(t) = \frac{d}{dt} \int_0^\infty f(x, t) dx = \int_0^\infty \frac{\partial f(x, t)}{\partial t} dx = \int_0^\infty \frac{\partial e^{-tx}}{\partial t} \frac{\sin x}{x} dx \\
&= \int_0^\infty \frac{\sin x}{x} \left[ \frac{\partial e^{-tx}}{\partial t} \right] dx = \int_0^\infty \frac{\sin x}{x} [-xe^{-tx}] dx = \int_0^\infty \sin x [-e^{-tx}] dx \\
&= - \int_0^\infty e^{-tx} \sin x dx, \text{ Laplacian transform of } -\sin x \\
&= \int_0^\infty e^{-tx} (-\sin x) dx = \int_{x=0}^{x=\infty} e^{-tx} d\cos x, I'(t) = \int_0^\infty e^{-tx} (-\sin x) dx \\
&= [e^{-tx} \cos x]_{x=0}^{x=\infty} - \int_{x=0}^{x=\infty} \cos x de^{-tx} \\
&= [e^{-t\infty} \cos \infty] - [e^{-t0} \cos 0] - \int_{x=0}^{x=\infty} \cos x (-te^{-tx} dx) \\
&= [0 \cos \infty] - [1 \cdot 1] + \int_0^\infty te^{-tx} \cos x dx \\
&= 0 - 1 + \int_0^\infty te^{-tx} \cos x dx = -1 + \int_{x=0}^{x=\infty} te^{-tx} d\sin x \\
&= -1 + [te^{-tx} \sin x]_{x=0}^{x=\infty} - \int_{x=0}^{x=\infty} \sin x d(te^{-tx}) \\
&= -1 + [0 - 0] - \int_0^\infty \sin x (te^{-tx} (-t) dx) = -1 - \int_0^\infty t^2 e^{-tx} (-\sin x) dx \\
&= -1 - t^2 \int_0^\infty e^{-tx} (-\sin x) dx = -1 - t^2 I'(t) \\
I'(t) &= -1 - t^2 I'(t) \\
I'(t) &= \frac{-1}{1+t^2} \\
I(t) &= \int \frac{-1}{1+t^2} dt \stackrel{t=\tan \theta}{=} \int \frac{-1}{1+\tan^2 \theta} d\tan \theta = - \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta = - \int d\theta = -\theta + C \\
&= -\arctan t + C = -\tan^{-1} t + C \\
\int_0^\infty e^{-tx} \frac{\sin x}{x} dx &= I(t) = -\arctan t + C \\
\Downarrow \Rightarrow I &= \int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty e^{-0x} \frac{\sin x}{x} dx = I(0) = -\arctan 0 + C = -0 + C \Rightarrow C = I \\
\Rightarrow 0 &= \int_0^\infty 0 \frac{\sin x}{x} dx = \int_0^\infty e^{-\infty x} \frac{\sin x}{x} dx = I(\infty) = -\arctan \infty + C = \frac{-\pi}{2} + C \\
0 &= \frac{-\pi}{2} + C \Rightarrow C = \frac{\pi}{2} \\
I &= I(0) = C = \frac{\pi}{2}, I = \int_0^\infty \frac{\sin x}{x} dx \\
\int_0^\infty \frac{\sin x}{x} dx &= \frac{\pi}{2}
\end{aligned}$$


---

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$


---

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^{\infty} \frac{\sin x}{x} dx = 2 \cdot \frac{\pi}{2} = \pi$$


---

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

□

#### 44.2.2.1 residue method

<https://www.bilibili.com/video/BV1Lj411L79X/?t=7m26s>

#### 44.2.2.2 sinc function

[https://en.wikipedia.org/wiki/Sinc\\_function](https://en.wikipedia.org/wiki/Sinc_function)

[https://en.wikipedia.org/wiki/Anti-aliasing\\_filter](https://en.wikipedia.org/wiki/Anti-aliasing_filter)

[https://en.wikipedia.org/wiki/Borwein\\_integral](https://en.wikipedia.org/wiki/Borwein_integral)

<https://blog.udn.com/paraquat/22455342>

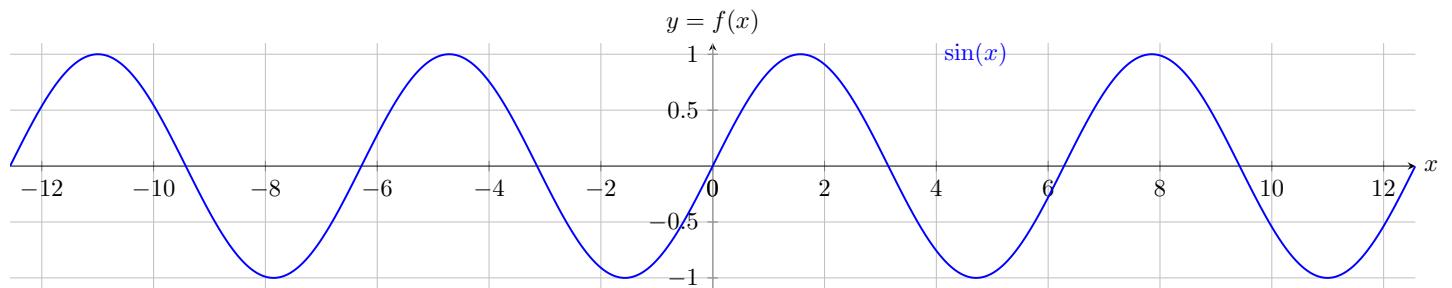


Figure 44.1:  $\sin(x)$

<https://tex.stackexchange.com/questions/235006/declaring-sinc-in-tikz>

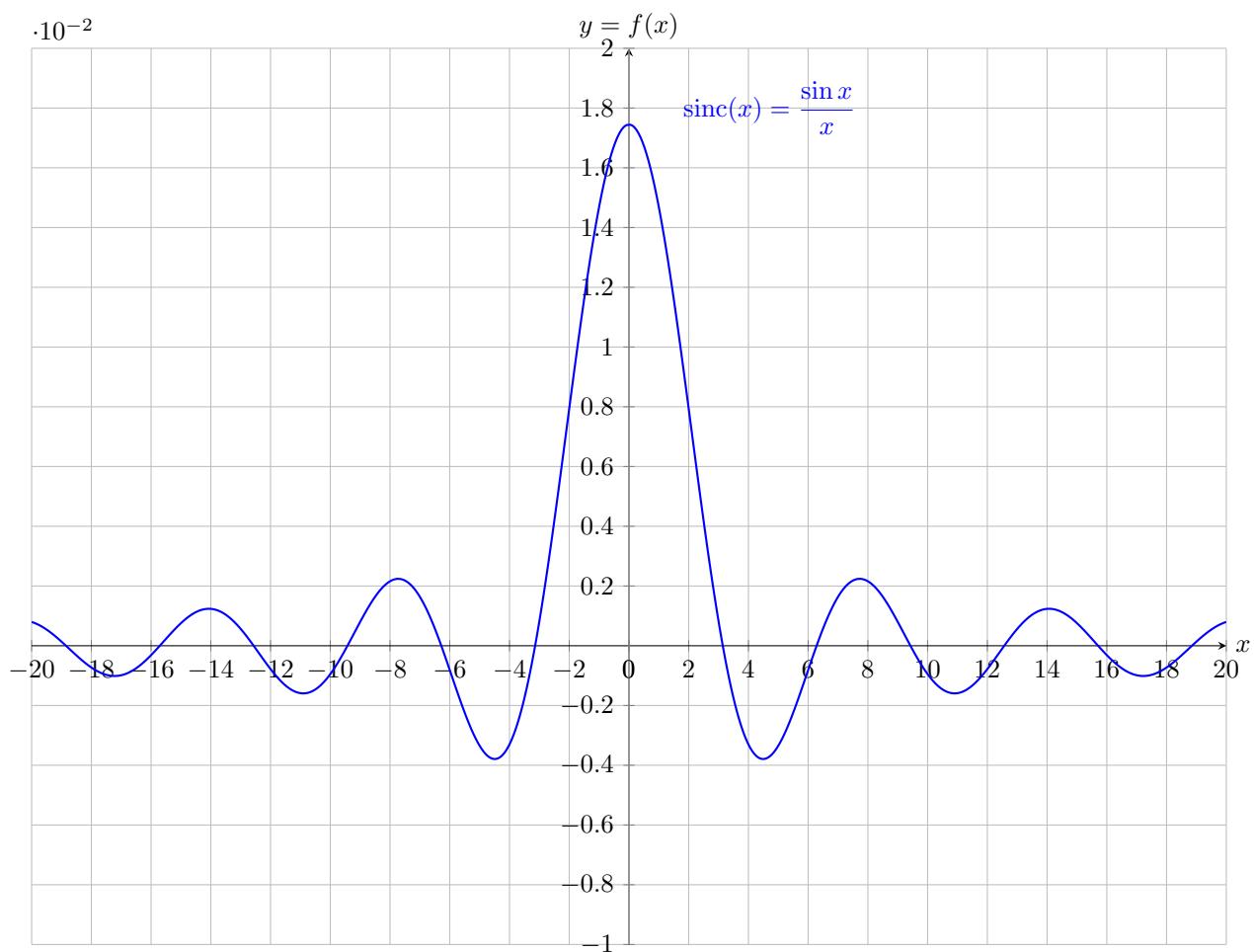
#### 44.2.3 Gaussian integral

<https://www.youtube.com/watch?v=jP-6j6mEpRg>

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

$$f(x) = e^{-x^2} \text{ is an even function} \Leftrightarrow f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$$

Figure 44.2:  $\text{sinc}(x)$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2} dx &= \int_{-\infty}^0 e^{-x^2} dx + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{x=-\infty}^{x=0} e^{-x^2} dx + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{x=-\infty}^{x=0} e^{-x^2} dx, (x' = -x \Leftrightarrow x = -x') + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{(-x')=-\infty}^{(-x')=0} e^{-(-x')^2} d(-x') + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{x'=-(-\infty)}^{x'=0} e^{-(x')^2} (-dx') + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{x'=\infty}^{x'=0} e^{-(x')^2} dx' + \int_0^{\infty} e^{-x^2} dx \\
&= - \int_{x'=\infty}^{x'=0} e^{-(x')^2} dx' + \int_0^{\infty} e^{-x^2} dx \\
&= \int_{x'=0}^{x'=\infty} e^{-(x')^2} dx + \int_0^{\infty} e^{-x^2} dx \\
&= \int_0^{\infty} e^{-x^2} dx + \int_0^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx
\end{aligned}$$


---

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx$$


---

$$\int_0^{\infty} e^{-x^2} dx$$

$$I(t) = \left[ \int_0^t e^{-x^2} dx \right]^2$$

$$\begin{aligned}
I'(t) &= \frac{dI(t)}{dt} = \frac{d \left[ \int_0^t e^{-x^2} dx \right]^2}{dt} = \frac{d \left[ \int_0^t e^{-x^2} dx \right]}{dt} \frac{d \left[ \int_0^t e^{-x^2} dx \right]}{dt} = 2 \left[ \int_0^t e^{-x^2} dx \right] \left[ \frac{d}{dt} \int_0^t e^{-x^2} dx \right] \\
&= 2 \left[ \int_0^t e^{-x^2} dx \right] \left[ \frac{d}{dt} \int_0^t e^{-x^2} dx \right] \stackrel{\text{FToC}}{=} 2 \left[ \int_0^t e^{-x^2} dx \right] [e^{-t^2}] \\
&= 2 \int_0^t e^{-x^2} e^{-t^2} dx = 2 \int_0^t e^{-(x^2+t^2)} dx = 2 \int_{x=0}^{x=t} e^{-(x^2+t^2)} dx, x' = \frac{x}{t} \Leftrightarrow x = tx' \\
&= 2 \int_{tx'=0}^{tx'=t} e^{-((tx')^2+t^2)} dt x' = 2 \int_{x'=0}^{x'=1} e^{-t^2((x')^2+1)} t dx' = 2 \int_0^1 t e^{-t^2(x^2+1)} dx \\
&= - \int_0^1 (-2t) e^{-t^2(x^2+1)} dx = - \int_0^1 \frac{\partial}{\partial t} \frac{e^{-t^2(x^2+1)}}{x^2+1} dx = - \frac{d}{dt} \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2+1} dx \\
&\quad \left[ \int_0^t e^{-x^2} dx \right]^2 = I(t) = - \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2+1} dx + C \\
0 &= \left[ \int_0^0 e^{-x^2} dx \right]^2 = I(0) = \int_0^1 \frac{1}{x^2+1} dx + C \stackrel{x=\tan\theta}{=} [-\arctan x]_0^1 + C = \frac{-\pi}{4} + C \Rightarrow C = \frac{\pi}{4} \\
\left[ \int_0^t e^{-x^2} dx \right]^2 &= I(t) = - \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2+1} dx + C = - \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2+1} dx + \frac{\pi}{4} \\
\left[ \int_0^{\infty} e^{-x^2} dx \right]^2 &= \lim_{t \rightarrow \infty} I(t) = I(\infty) = - \int_0^1 \frac{e^{-\infty^2(x^2+1)}}{x^2+1} dx + \frac{\pi}{4} = - \int_0^1 \frac{0}{x^2+1} dx + \frac{\pi}{4} = \frac{\pi}{4} \\
\left[ \int_0^{\infty} e^{-x^2} dx \right]^2 &= \frac{\pi}{4} \\
\int_0^{\infty} e^{-x^2} dx &= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^\infty e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$$

□

#### 44.2.4 other examples

<https://www.youtube.com/watch?v=GW86SShcYbM>

$$\int_0^1 \ln x dx$$

$$I = \int_0^1 \ln x dx = \int_0^1 \ln(x) dx, f(x) = \ln x$$

$$I(t) = \int_0^1 \ln t x dx = \int_0^1 f(tx) dx = \int_0^1 f(x,t) dx, \begin{cases} f(x,t) = f(tx) = \ln tx \\ f(x,1) = f(x) = \ln x \end{cases}$$

$$\begin{aligned} I(t) &= \int_0^1 f(x,t) dx = \int_0^1 f(tx) dx = \int_0^1 \ln t x dx \Rightarrow I(1) = I \\ \frac{d}{dt} I(t) &= I'(t) = \frac{d}{dt} \int_0^1 f(x,t) dx = \int_0^1 \frac{\partial f(x,t)}{\partial t} dx = \int_0^1 \frac{\partial \ln t x}{\partial t} dx \\ &= \int_0^1 \frac{1}{tx} x dx = \int_0^1 \frac{1}{t} dx = \frac{1}{t} \int_0^1 dx = \frac{1}{t} [x]_0^1 = \frac{1}{t} \\ I(t) &= \int I'(t) dt = \int \frac{1}{t} dt = \ln |t| + C \\ \Downarrow \Rightarrow I &= \int_0^1 \ln x dx = \int_0^1 \ln 1 x dx = I(1) = \ln |1| + C = 0 + C \Rightarrow C = I \end{aligned}$$

no more known boundary condition

$$\begin{aligned} \Im(t) &= \int_0^1 x^t dx = \left[ \frac{x^{t+1}}{t+1} \right]_{x=0}^1 = \frac{1}{t+1} \\ \frac{-1}{(t+1)^2} &= \frac{d}{dt} \frac{1}{t+1} = \frac{d}{dt} \Im(t) = \Im'(t) = \frac{d}{dt} \int_0^1 x^t dx = \int_0^1 \frac{\partial x^t}{\partial t} dx = \int_0^1 x^t \ln x dx \\ -1 &= \left[ \frac{-1}{(t+1)^2} \right]_{t=0} = \Im'(0) = \int_0^1 x^0 \ln x dx = \int_0^1 1 \ln x dx = \int_0^1 \ln x dx \\ &\quad \int_0^1 \ln x dx = -1 \end{aligned}$$

<https://zhuanlan.zhihu.com/p/687355703>

<https://www.zhihu.com/question/646881575/answer/3417318090>

TaylorCatAlice: Feynman method of integration and residue theorem

<https://www.bilibili.com/video/BV1Lj411L79X>

MatheManiac: complex integral

[https://www.youtube.com/watch?v=EyBDtUtyshk&list=PLDcSwjT2BF\\_UDdkQ3KQjX5SRQ2DLLwv0R&index=11](https://www.youtube.com/watch?v=EyBDtUtyshk&list=PLDcSwjT2BF_UDdkQ3KQjX5SRQ2DLLwv0R&index=11)

<https://www.youtube.com/watch?v=EyBDtUtyshk>

## 44.3 Feynman method of path integral

Elliot Schneider: Physics with Elliot

<https://www.youtube.com/watch?v=W8QZ-yxebFA>

<https://www.youtube.com/watch?v=Se-CpexiJLQ>

Feynman method of path integral in quantum dynamics

<https://www.youtube.com/watch?v=Sp5SvdDh2u8>

history of path integral

[https://en.wikipedia.org/wiki/Path\\_integral\\_formulation](https://en.wikipedia.org/wiki/Path_integral_formulation)

<https://www.youtube.com/watch?v=7yjv-gLHFqg>



# Chapter 45

## Hilbert space

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

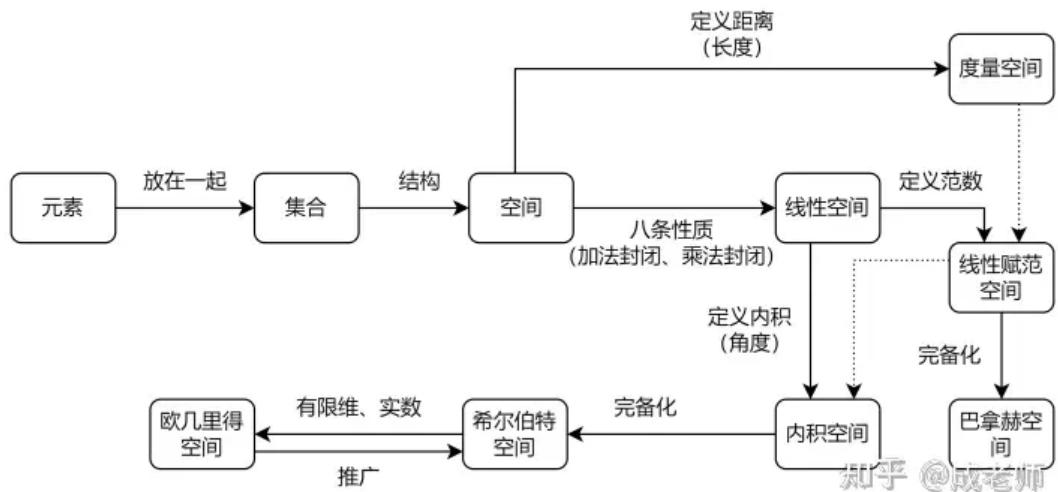


Figure 45.1: Euclid space construction

1

### 45.1 Taylor expansion or Taylor series

<https://www.bilibili.com/video/BV15s4y1g7HQ>

**Lemma 45.1.** Newton-Leibniz formula = N-LF, equivalent to first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1

牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF, equivalent to 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 ??

$$\int_a^b f'(x) dx = f(b) - f(a) \quad (45.1)$$

<sup>1</sup><https://zhuanlan.zhihu.com/p/85867887>

$$\begin{aligned}
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 & \left\{ \begin{array}{ll} f' : [a, b] \rightarrow \mathbb{R} & (4) \\ a < b & (3) \\ f' \text{ continuous on } [a, b] & (5) \end{array} \right. \Rightarrow \exists c \in (a, b) \left( f'(c) = \frac{\int_a^b f'(x) dx}{b-a} \right) \quad \text{MVTi1 ??} \\
 & \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \} \quad 45.1 \\
 \Rightarrow & \updownarrow \\
 & \int_a^b f'(x) dx = f(b) - f(a) \quad 45.1
 \end{aligned}$$

(↓):

$$\begin{aligned}
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \} \quad \text{MVTd1 ??} \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 & [a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \dots \cup [x_{n-1}, x_n] \quad 45.2 \\
 \stackrel{??}{\Rightarrow} & \int_a^b f'(x) dx = f(b) - f(a)
 \end{aligned}$$

(↑):

$$\begin{aligned}
 & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \quad \text{N-LF 45.1} \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
 & \left\{ \begin{array}{ll} f' : [a, b] \rightarrow \mathbb{R} & (4) \\ a < b & (3) \\ f' \text{ continuous on } [a, b] & (5) \end{array} \right. \Rightarrow \exists c \in (a, b) \left( f'(c) = \frac{\int_a^b f'(x) dx}{b-a} \right) \quad \text{MVTi1 ??} \\
 \Rightarrow & \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \}
 \end{aligned}$$

Proof: (↓)

重疊端點分割  $[a, b]$  成  $n$  部分聯集 ( $n \in \mathbb{N}$ )

$$[a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \dots \cup [x_{n-1}, x_n] \quad (45.2)$$

$$\begin{aligned}
& \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \} \quad \text{MVTd1 ??} \\ f \text{ differentiable on } (a, b) & (2) \end{array} \right. \\
& [a, b] = [x_0, x_n] = [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \quad 45.2 \\
\Rightarrow & \left\{ \begin{array}{ll} (f : [x_0, x_n] \rightarrow \mathbb{R}) = (f : [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] \rightarrow \mathbb{R}) & \Leftarrow (0) \\ f \text{ continuous on } [x_0, x_1] \cup [x_0, x_2] \cup \cdots \cup [x_{n-1}, x_n] & \Leftarrow (1) \\ f \text{ differentiable on } (x_0, x_1) \cup (x_0, x_2) \cup \cdots \cup (x_{n-1}, x_n) & \Leftarrow (2) \end{array} \right. \\
\Rightarrow & \left\{ \begin{array}{ll} f : [x_0, x_1] \rightarrow \mathbb{R}, f : [x_0, x_2] \rightarrow \mathbb{R}, \dots, f : [x_{n-1}, x_n] \rightarrow \mathbb{R} & \Leftarrow (0) \\ f \text{ continuous on } [x_0, x_1], [x_0, x_2], \dots, [x_{n-1}, x_n] & \Leftarrow (1) \\ f \text{ differentiable on } (x_0, x_1), (x_0, x_2), \dots, (x_{n-1}, x_n) \Rightarrow f \text{ differentiable on } (x_0, x_1), (x_0, x_2), \dots, (x_{n-1}, x_n) & \Leftarrow (2) \end{array} \right. \\
\stackrel{??}{\Rightarrow} & \left\{ \begin{array}{l} \exists c_1 \in (x_0, x_1) \{ f'(c_1)(x_1 - x_0) = f(x_1) - f(x_0) \} \\ \exists c_2 \in (x_1, x_2) \{ f'(c_2)(x_2 - x_1) = f(x_2) - f(x_1) \} \\ \vdots \\ \exists c_k \in (x_{k-1}, x_k) \{ f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1}) \} \quad \forall k \in \mathbb{N} \cap [1, n] \\ \vdots \\ \exists c_n \in (x_{n-1}, x_n) \{ f'(c_n)(x_n - x_{n-1}) = f(x_n) - f(x_{n-1}) \} \end{array} \right. \\
\Rightarrow & f'(c_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1}) \\
\Rightarrow & \sum_k f'(c_k)(x_k - x_{k-1}) = \sum_k f(x_k) - f(x_{k-1}) \\
\Rightarrow & \sum_{k=1}^n f'(c_k)(x_k - x_{k-1}) = \sum_{k=1}^n f(x_k) - f(x_{k-1}) \\
& = [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \cdots + [f(x_n) - f(x_{n-1})] \\
& = f(x_n) - f(x_0) = f(b) - f(a) \\
\Rightarrow & \sum_{k=1}^n f'(c_k)(x_k - x_{k-1}) = f(b) - f(a) \stackrel{\Delta x_k = x_k - x_{k-1}}{\Leftrightarrow} \sum_{k=1}^n f'(c_k) \Delta x_k = f(b) - f(a) \\
\Rightarrow & \lim_{n \rightarrow \infty} \lim_{\Delta x_k = \frac{x_n - x_0}{n}} \sum_{k=1}^n f'(c_k) \Delta x_k = \lim_{n \rightarrow \infty} \lim_{\Delta x_k = \frac{b-a}{n} \rightarrow 0} f(b) - f(a) = f(b) - f(a) \\
\Rightarrow & \int_a^b f'(x) dx = \int_{x_0}^{x_n} f'(x) dx = f(b) - f(a) \\
\Rightarrow & \int_a^b f'(x) dx = f(b) - f(a)
\end{aligned}$$

$$\int_a^b f'(x) dx = f(b) - f(a)$$

得到 牛頓-萊布尼茨公式 Newton-Leibniz formula = N-LF [eq:N-LF]

□

Proof: (↑)

$$\begin{aligned}
& \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1} \\
& \left\{ \begin{array}{ll} f \text{ differentiable on } (a, b) & (2) \\ f : [a, b] \rightarrow \mathbb{R} & a < b \quad (3) \Rightarrow \exists c \in (a, b) \left( f(c) = \frac{\int_a^b f(x) dx}{b-a} \right) \end{array} \right. \quad \text{MVTi1 ??} \\
\Rightarrow & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1} \\
& \left\{ \begin{array}{ll} f \text{ differentiable on } (a, b) & (2) \\ f : [a, b] \rightarrow \mathbb{R} & a < b \quad (3) \Rightarrow \exists c \in (a, b) \left\{ f(c)(b-a) = \int_a^b f(x) dx \right\} \end{array} \right. \quad \text{MVTi1 ??} \\
\Rightarrow & \left\{ \begin{array}{ll} f : [a, b] \rightarrow \mathbb{R} & (0) \\ f \text{ continuous on } [a, b] & (1) \stackrel{??}{\Rightarrow} \int_a^b f'(x) dx = f(b) - f(a) \end{array} \right. \quad \text{N-LF 45.1} \\
& \left\{ \begin{array}{ll} f \text{ differentiable on } (a, b) & (2) \\ \text{if } f' : [a, b] \rightarrow \mathbb{R} & (4) \Rightarrow \exists c \in (a, b) \left\{ f'(c)(b-a) = \int_a^b f'(x) dx \right\} \end{array} \right. \quad \text{MVTi1 ??} \\
& \left\{ \begin{array}{ll} \text{if } f' \text{ continuous on } [a, b] & (5) \end{array} \right. \\
\Rightarrow & \exists c \in (a, b) \left\{ f'(c)(b-a) = \int_a^b f'(x) dx = f(b) - f(a) \right\} \\
\Rightarrow & \exists c \in (a, b) \{ f'(c)(b-a) = f(b) - f(a) \}
\end{aligned}$$

$$\exists c \in (a, b) [f'(c)(b-a) = f(b) - f(a)]$$

得到 第一微分均值定理 / 拉格朗日均值定理 first mean value theorem for derivatives / Lagrange mean value theorem = MVTdL / MVTd1 [thm:MVTd1]

□

$$\begin{aligned}
& \int_a^b f'(x) dx = f(b) - f(a) \\
& \int_{x_0}^x f'(t) dt = f(x) - f(x_0) \\
& f(x) = f(x_0) + \int_{x_0}^x f'(t) dt
\end{aligned}$$


---

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt$$


---

**Lemma 45.2.** univariable product rule

$$d(uv) = (du)v + u dv = v du + u dv$$

$$\frac{d(uv)}{dt} = v \frac{du}{dt} + u \frac{dv}{dt}$$

**Lemma 45.3.** integration by parts

$$\begin{aligned} \int \frac{d(uv)}{dt} dt &= \int v \frac{du}{dt} + u \frac{dv}{dt} dt \\ \int d(uv) &= \int v \frac{du}{dt} dt + \int u \frac{dv}{dt} dt \\ uv &= \int v du + \int u dv \\ \int u \frac{dv}{dt} dt &= uv - \int v \frac{du}{dt} dt \text{ "switching derivatives"} \end{aligned} \tag{45.3}$$

$$\int u dv = uv - \int v du \text{ "switching differentials"} \tag{45.4}$$

**Theorem 45.1.** *univariable Taylor theorem with the remainder in integral form*

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt$$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

Proof: relatively weird  $(t-x)$

$$\stackrel{45.1}{=} \frac{f(x) - f(a)}{\int_a^x f'(t) dt} = \int_a^x f'(t) \frac{d(t-x)}{dt} dt \stackrel{45.3}{=} [f'(t)(t-x)]_{t=a}^x - \int_a^x (t-x) \frac{df'(t)}{dt} dt \quad (45.5)$$

$$= [-f'(a)(a-x)] - \int_a^x f''(t)(t-x) dt = f'(a)(x-a) - \int_a^x f''(t) \frac{d\frac{(t-x)^2}{2}}{dt} dt \quad (45.6)$$

$$= f'(a)(x-a) - \left( \left[ f''(t) \frac{(t-x)^2}{2} \right]_{t=a}^x - \int_a^x \frac{(t-x)^2}{2} \frac{df''(t)}{dt} dt \right) \quad (45.7)$$

$$= f'(a)(x-a) - \left( \left[ -f''(a) \frac{(x-a)^2}{2} \right] - \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \right) \quad (45.8)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \quad (45.9)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{d\frac{(t-x)^3}{2 \cdot 3}}{dt} dt \quad (45.10)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(t) \frac{(t-x)^3}{2 \cdot 3} \right]_{t=a}^x - \int_a^x \frac{(t-x)^3}{2 \cdot 3} \frac{df'''(t)}{dt} dt \right) \quad (45.11)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(a) \frac{(x-a)^3}{2 \cdot 3} \right] - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \right) \quad (45.12)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \quad (45.13)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{d\frac{(t-x)^4}{2 \cdot 3 \cdot 4}}{dt} dt \quad (45.14)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \left( \left[ f^{(4)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} \right]_{t=a}^x - \int_a^x \frac{(t-x)^4}{2 \cdot 3 \cdot 4} \frac{df^{(4)}(t)}{dt} dt \right) \quad (45.15)$$

$$= f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} \quad (45.16)$$

$$- \left( \left[ f^{(4)}(a) \frac{(x-a)^4}{2 \cdot 3 \cdot 4} \right] - \int_a^x f^{(5)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} dt \right) \quad (45.17)$$

$$= f'(t)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} \quad (45.18)$$

$$- \left( \left[ -f^{(4)}(a) \frac{(x-a)^4}{2 \cdot 3 \cdot 4} \right] - \int_a^x f^{(5)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} dt \right) \quad (45.19)$$

$$= f'(t)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(a) \frac{(x-a)^3}{2 \cdot 3} + f^{(4)}(a) \frac{(x-a)^4}{2 \cdot 3 \cdot 4} \quad (45.20)$$

$$\vdots + \int_a^x f^{(5)}(t) \frac{(t-x)^4}{2 \cdot 3 \cdot 4} dt \quad (45.21)$$

$$= \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt \text{ remainder in integral form} \quad (45.22)$$

□

$$\begin{aligned}
& f(x) - f(a) \\
& \stackrel{45.1}{=} \int_a^x f'(t) dt = \int_a^x f'(t) \frac{d(t-x)}{dt} dt \stackrel{45.3}{=} [f'(t)(t-x)]_{t=a}^x - \int_a^x (t-x) \frac{df'(t)}{dt} dt \\
& = [-f'(a)(a-x)] - \int_a^x f''(t)(t-x) dt = f'(a)(x-a) + \int_a^x f''(t)(x-t) dt \\
& = f'(a)(x-a) - \int_a^x f''(t)(t-x) dt = \sum_{k=1}^1 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f''(t)(x-t) dt \\
& = f'(a)(x-a) - \int_a^x f''(t) \frac{d\frac{(t-x)^2}{2}}{dt} dt \\
& = f'(a)(x-a) - \left( \left[ f''(t) \frac{(t-x)^2}{2} \right]_{t=a}^x - \int_a^x \frac{(t-x)^2}{2} \frac{df''(t)}{dt} dt \right) \\
& = f'(a)(x-a) - \left( \left[ -f''(a) \frac{(x-a)^2}{2} \right] - \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \right) \\
& = f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt \\
& = \sum_{k=1}^2 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f'''(t) \frac{(t-x)^2}{2} dt = \sum_{k=1}^2 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f'''(t) \frac{(x-t)^2}{2!} dt \\
& = f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \int_a^x f'''(t) \frac{d\frac{(t-x)^3}{2 \cdot 3}}{dt} dt \\
& = f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(t) \frac{(t-x)^3}{2 \cdot 3} \right]_{t=a}^x - \int_a^x \frac{(t-x)^3}{2 \cdot 3} \frac{df'''(t)}{dt} dt \right) \\
& = f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \left( \left[ f'''(a) \frac{(x-a)^3}{2 \cdot 3} \right] - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \right) \\
& = f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(t) \frac{(x-a)^3}{2 \cdot 3} - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt \\
& = \sum_{k=1}^3 \frac{f^{(k)}(a)}{k!} (x-a)^k - \int_a^x f^{(4)}(t) \frac{(t-x)^3}{2 \cdot 3} dt = \sum_{k=1}^3 \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(4)}(t) \frac{(x-t)^3}{3!} dt \\
& \vdots \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
f(x) - f(a) & = \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
f(x) & = f(a) + \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
& = \frac{f^{(0)}(a)}{0!} (x-a)^0 + \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
& = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt
\end{aligned}$$

□

**Theorem 45.2.** univariable Taylor theorem with the remainder in Lagrange form

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

Proof:

by 連續函數 極值定理 / 最大最小值定理 / 最小最大值定理 continuous function extreme value theorem = CFEVT / extreme value theorem = EVT ?? and 連續函數 介值定理 / 中間值定理 continuous function intermediate value theorem = CFIVT / intermediate value theorem = IVT ??

$$\text{let } f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + r_n(x) \quad (45.23)$$

$$r_n(x) = \int_a^x f^{(n+1)}(t) \frac{(t-x)^n}{n!} dt \quad (45.24)$$

$$\text{let } n \in \{2k-1 | k \in \mathbb{N}\} \quad (45.25)$$

$$\text{let } f(t) \stackrel{??}{\in} [m, M] \subseteq f((a, x)) \text{ when } a < x \quad (45.26)$$

$$\int_a^x m \frac{(t-x)^n}{n!} dt \leq r_n(x) \leq \int_a^x M \frac{(t-x)^n}{n!} dt \quad (45.27)$$

$$m \int_a^x \frac{(t-x)^n}{n!} dt = M \int_a^x \frac{(t-x)^n}{n!} dt \quad (45.28)$$

$$m \left[ \frac{(t-x)^{n+1}}{(n+1)!} \right]_{t=a}^x = M \left[ \frac{(t-x)^{n+1}}{(n+1)!} \right]_{t=a}^x \quad (45.29)$$

$$m \frac{(x-a)^{n+1}}{(n+1)!} = M \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.30)$$

$$\Downarrow \quad ?? \quad (45.31)$$

$$r_n(x) \stackrel{\exists \xi \in (a, x)}{=} f^{(n+1)}(\xi) \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.32)$$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + f^{(n+1)}(\xi) \frac{(x-a)^{n+1}}{(n+1)!} \quad (45.33)$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \quad (45.34)$$

□

**Theorem 45.3.** univariable Taylor theorem with the remainder in big O form

$$f(a+h) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + O(h^{n+1})$$

Proof:

$$\text{if } |f^{(n+1)}(t)| \leq K \forall t \in (a, x) \quad (45.35)$$

$$|r_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \right| \leq \frac{K}{(n+1)!} |x-a|^{n+1} \quad (45.36)$$

$$r_n(x) \in O((x-a)^{n+1}) \quad (45.37)$$

$$\text{let } R_n(h) = r_n(a+h) \quad (45.38)$$

$$R_n(h) \in O(h^{n+1}) \quad (45.39)$$

$$f(a+h) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \quad (45.40)$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} h^k + O(h^{n+1}) \quad (45.41)$$

□

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$f(x) = \sin(x)$$

$$\sin(x) = \sum_{k=0}^n \frac{\sin^{(k)}(a)}{k!} (x-a)^k + \int_a^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$a = 0$$

$$\sin(x) = \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$\begin{aligned} \sin(x) &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\ &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} x^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\ &= \frac{\sin^{(0)}(0)}{0!} x^0 + \frac{\sin^{(1)}(0)}{1!} x^1 + \frac{\sin^{(2)}(0)}{2!} x^2 + \frac{\sin^{(3)}(0)}{3!} x^3 \\ &\quad + \int_0^x \sin^{(3+1)}(t) \frac{(x-t)^3}{3!} dt \\ &= \frac{0}{0!} x^0 + \frac{\cos(0)}{1!} x^1 + \frac{-\sin(0)}{2!} x^2 + \frac{-\cos(0)}{3!} x^3 \\ &\quad + \int_0^x \sin^{(4)}(t) \frac{(x-t)^3}{3!} dt \\ &= 0 + \frac{1}{1} x + 0 - \frac{1}{6} x^3 + \int_0^x \sin(t) \frac{(x-t)^3}{6} dt \\ &= x - \frac{1}{6} x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt \\ \sin(x) &= x - \frac{1}{6} x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt \end{aligned}$$

$$\begin{aligned} \sin(x) &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} (x-0)^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\ &= \sum_{k=0}^n \frac{\sin^{(k)}(0)}{k!} x^k + \int_0^x \sin^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\ &= \frac{\sin^{(0)}(0)}{0!} x^0 + \frac{\sin^{(1)}(0)}{1!} x^1 + \frac{\sin^{(2)}(0)}{2!} x^2 + \frac{\sin^{(3)}(0)}{3!} x^3 + \frac{\sin^{(4)}(0)}{4!} x^4 \\ &\quad + \int_0^x \sin^{(4+1)}(t) \frac{(x-t)^4}{4!} dt \\ &= \frac{0}{0!} x^0 + \frac{\cos(0)}{1!} x^1 + \frac{-\sin(0)}{2!} x^2 + \frac{-\cos(0)}{3!} x^3 + \frac{\sin(0)}{4!} x^4 \\ &\quad + \int_0^x \sin^{(5)}(t) \frac{(x-t)^4}{4!} dt \\ &= 0 + \frac{1}{1} x + 0 - \frac{1}{6} x^3 + 0 + \int_0^x \cos(t) \frac{(x-t)^4}{24} dt \\ &= x - \frac{1}{6} x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\ \sin(x) &= x - \frac{1}{6} x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \end{aligned}$$

$$\begin{aligned}
\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt \\
&= x - \frac{1}{6}x^3 + \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\
&\quad \Downarrow \\
\frac{1}{6} \int_0^x (x-t)^3 \sin(t) dt &= \frac{1}{24} \int_0^x (x-t)^4 \cos(t) dt \\
&\quad \Downarrow \\
\int_0^x (x-t)^4 \cos(t) dt &= 4 \int_0^x (x-t)^3 \sin(t) dt
\end{aligned}$$


---

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$f(x) = e^x = \exp(x)$$

$$e^x = \sum_{k=0}^n \frac{\exp^{(k)}(a)}{k!} (x-a)^k + \int_a^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$a = 0$$

$$e^x = \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} (x-0)^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

$$\begin{aligned}
e^x &= \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} (x-0)^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{\exp^{(k)}(0)}{k!} x^k + \int_0^x \exp^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{\exp(0)}{k!} x^k + \int_0^x \exp(t) \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{e^0}{k!} x^k + \int_0^x e^t \frac{(x-t)^n}{n!} dt = \sum_{k=0}^n \frac{1}{k!} x^k + \int_0^x e^t \frac{(x-t)^n}{n!} dt \\
&= \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\
e^x &= \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt
\end{aligned}$$

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt$$


---

$$\lim_{n \rightarrow \infty} \frac{1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}}{e^n}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}}{e^n}, \wedge e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\
&= \lim_{n \rightarrow \infty} \frac{1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}}{\left[ \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \right]_{x=n}} \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \frac{n^k}{k!}}{\sum_{k=0}^n \frac{n^k}{k!} + \frac{1}{n!} \int_0^n (n-t)^n e^t dt} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{\frac{1}{n!} \int_0^n (n-t)^n e^t dt}{\sum_{k=0}^n \frac{n^k}{k!}}}
\end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}}{e^n}, \wedge e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{1}{n!} \int_0^x (x-t)^n e^t dt \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \frac{n^k}{k!}}{e^n}, \wedge e^n = \sum_{k=0}^n \frac{n^k}{k!} + \frac{1}{n!} \int_0^n (n-t)^n e^t dt \Rightarrow \sum_{k=0}^n \frac{n^k}{k!} = e^n - \frac{1}{n!} \int_0^n (n-t)^n e^t dt \\
&= \lim_{n \rightarrow \infty} \frac{e^n - \frac{1}{n!} \int_0^n (n-t)^n e^t dt}{e^n} = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt \right], \text{ if } \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (t-n)^n e^{t-n} dt \in \mathbb{R} \\
&= \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt = 1 - \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt
\end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n (n-t)^n e^{t-n} dt, x = n-t \\
&= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_{t=0}^{t=n} (n-t)^n e^{t-n} dt, t = n-x \\
&= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_{n-x=0}^{n-x=n} (n-(n-x))^n e^{(n-x)-n} d(n-x) \\
&= \lim_{n \rightarrow \infty} \frac{1}{n!} \left( - \int_{x=n}^{x=0} x^n e^{-x} dx \right) = \lim_{n \rightarrow \infty} \frac{1}{n!} \left( \int_{x=0}^{x=n} x^n e^{-x} dx \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^n x^n e^{-x} dx
\end{aligned}$$

$$\int x^n e^{-x} dx$$

difficult process

$$\begin{aligned}
\int x^n e^{-x} dx &= - \int x^n e^{-x} dx \\
&= - \left[ x^n e^{-x} - \int e^{-x} dx^n \right] \\
&= - \left[ x^n e^{-x} - \int e^{-x} n x^{n-1} dx \right] \\
&= - x^n e^{-x} + n \int x^{n-1} e^{-x} dx
\end{aligned}$$

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx$$

$$\begin{aligned}
\int_0^n x^n e^{-x} dx &= [-x^n e^{-x}]_{x=0}^n + n \int_0^n x^{n-1} e^{-x} dx \\
&= [-n^n e^{-n} - (-0^n e^{-0})] + n \int_0^n x^{n-1} e^{-x} dx \\
&= -n^n e^{-n} + n \int_0^n x^{n-1} e^{-x} dx
\end{aligned}$$

$$\int_0^n x^n e^{-x} dx = -n^n e^{-n} + n \int_0^n x^{n-1} e^{-x} dx$$

more formal process with gamma function

<https://www.wolframalpha.com/input/?i2d=true&i=Divide%5BIntegrate%5BPower%5Bx%2Cn%5D%Power%5Be%2Cx-%2C-x%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D>

<https://www.wolframalpha.com/input/?i2d=true&i=Limit%5BDivide%5BIntegrate%5BPower%5Bx%2Cn%5D%Power%5Be%2Cx%5D%2C%7Bx%2C0%2Cn%7D%5D%2Cn%21%5D%2Cn-%3E%2E%88%9E%5D>

to be proved

---

<https://www.bilibili.com/video/BV1du411a7qB>

#### 45.1.1 Elliot Schneider: Taylor series

<https://www.youtube.com/watch?v=HQsZG8Yxb7w>

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=12m25s>

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

$$f(x_0 + \epsilon) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1}), \epsilon = x - x_0$$

$$f(x + \epsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \epsilon^n$$

$$\begin{aligned}
f(x + \epsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \epsilon^n = \sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n} f(x)}{n!} \epsilon^n = \sum_{n=0}^{\infty} \frac{\left(\frac{d}{dx}\right)^n f(x)}{n!} \epsilon^n \\
&= \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n f(x)}{n!} = \left( \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n}{n!} \right) f(x) = e^{\epsilon \frac{d}{dx}} f(x)
\end{aligned}$$


---

univariable Taylor operator

$$f(x + \epsilon) = e^{\epsilon \frac{d}{dx}} f(x)$$

$$e^{\epsilon \frac{d}{dx}} = \sum_{n=0}^{\infty} \frac{\left(\epsilon \frac{d}{dx}\right)^n}{n!} = 1 + \epsilon \frac{d}{dx} + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2} + \dots$$


---

$$f(x) = mx + b,$$

$$\begin{aligned} e^{\epsilon \frac{d}{dx}} f(x) &= \left(1 + \epsilon \frac{d}{dx} + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2} + \dots\right) (mx + b) \\ &= 1(mx + b) + \epsilon \frac{d}{dx}(mx + b) + \frac{1}{2} \epsilon^2 \frac{d^2}{dx^2}(mx + b) + \dots \\ &= (mx + b) + (\epsilon m) + \left(\frac{1}{2} \epsilon^2 0\right) + (0 + 0 + \dots) \\ &= (mx + b) + (\epsilon m) + (0 + 0 + \dots) \\ &= (mx + b) + (\epsilon m) + 0 \\ &= mx + b + m\epsilon = m(x + \epsilon) + b = f(x + \epsilon) \end{aligned}$$


---

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=15m34s>

multivariable ...

$$\begin{aligned} f(x + \epsilon) &= e^{\epsilon \frac{d}{dx}} f(x) \\ f(x + \epsilon_x, y + \epsilon_y, z + \epsilon_z) &= f(x, y + \epsilon_y, z + \epsilon_z) \\ &\quad + \epsilon \frac{\partial}{\partial x} f(x, y + \epsilon_y, z + \epsilon_z) \\ &\quad + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial x^2} f(x, y + \epsilon_y, z + \epsilon_z) \\ &\quad + \dots \\ f(x + \epsilon) &= e^{\epsilon \frac{d}{dx}} f(x) \\ f(x + \epsilon_x, y + \epsilon_y, z + \epsilon_z) &= f(\mathbf{r} + \boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon} \cdot \nabla} f(\mathbf{r}), \quad \begin{cases} \mathbf{r} = (x, y, z) \\ \boldsymbol{\epsilon} = (\epsilon_x, \epsilon_y, \epsilon_z) \\ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \end{cases} \\ \boldsymbol{\epsilon} \cdot \nabla &= (\epsilon_x, \epsilon_y, \epsilon_z) \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \epsilon_x \frac{\partial}{\partial x} + \epsilon_y \frac{\partial}{\partial y} + \epsilon_z \frac{\partial}{\partial z} \end{aligned}$$


---

multivariable Taylor operator

$$f(\mathbf{r} + \boldsymbol{\epsilon}) = e^{\boldsymbol{\epsilon} \cdot \nabla} f(\mathbf{r})$$


---

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=17m36s>

physics

#### 45.1.1.1 making complicated equation simple

linearize

simple pendulum

$\sin(x)$  to  $x$  linearization

potential energy

$$f(x_0 + \epsilon) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1})$$

$$U(x_0 + \epsilon) = \sum_{k=0}^n \frac{U^{(k)}(x_0)}{k!} \epsilon^k + O(\epsilon^{n+1}) = U(x_0) + U'(x_0)x + \frac{1}{2}U''(x_0)x^2 + \dots$$

$$F = \frac{-dU}{dx} = -U'(x_0) - U''(x_0)x - \dots \stackrel{\text{if } U'(x_0)=0}{=} -U''(x_0)x = -kx, k = U''(x_0)$$

#### 45.1.1.2 non-relativistic limit

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=22m32s>

$$\begin{aligned}
 E &= \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{m^2c^4 + p^2c^2} \\
 &= mc^2\sqrt{1 + \left(\frac{pc}{mc^2}\right)^2} = mc^2\sqrt{1 + \left(\frac{p}{mc}\right)^2} = mc^2\left(1 + \left(\frac{p}{mc}\right)^2\right)^{\frac{1}{2}} \\
 &= mc^2\left[\frac{1}{0!} + \frac{1}{1!}\left(\frac{1}{2}\right)\left(\left(\frac{p}{mc}\right)^2\right)^1 + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\left(\frac{p}{mc}\right)^2\right)^2 + \dots\right] \\
 &= mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \dots\right] = mc^2\left[1 + \frac{1}{2}\frac{p^2}{m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4} + \dots\right] \\
 &= mc^2 + \frac{1}{2}\frac{p^2}{m} - \frac{1}{8}\frac{p^4}{m^3c^2} + \dots = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots \\
 &= E_0 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots, \quad \begin{cases} E_0 = mc^2 \\ \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{mv^2}{2} = \frac{1}{2}mv^2 \end{cases}
 \end{aligned}$$

slappy here  $p = \gamma mv$

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=24m50s>

leading relativistic correction

$$-\frac{p^4}{8m^3c^2}$$

binding energy of hydrogen atom

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=25m40s>

fine-structure constant

#### 45.1.1.3 quantum momentum operator

<https://www.youtube.com/watch?v=HQsZG8Yxb7w&t=29m13s>

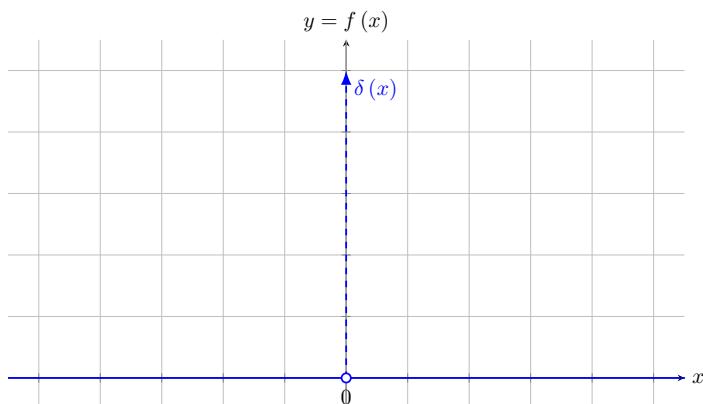
## 45.2 Dirac delta function

<https://www.bilibili.com/video/BV1qu411578c>

Dirac function = Dirac delta function

[https://tikz.net/delta\\_function/](https://tikz.net/delta_function/)

<https://tikz.dev/tikz-arrows>

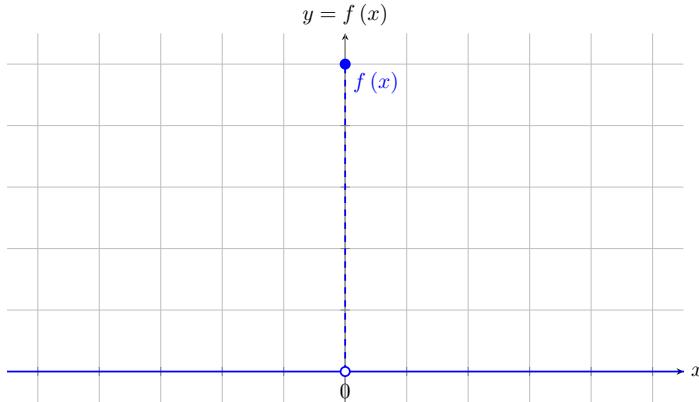


$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(x) dx = 1$$

Figure 45.2: Dirac function = Dirac delta function

$$\text{supp}(f) = \left\{ x \middle| \begin{cases} x \in \mathcal{D} \\ f(x) \neq 0 \end{cases} \right\}$$

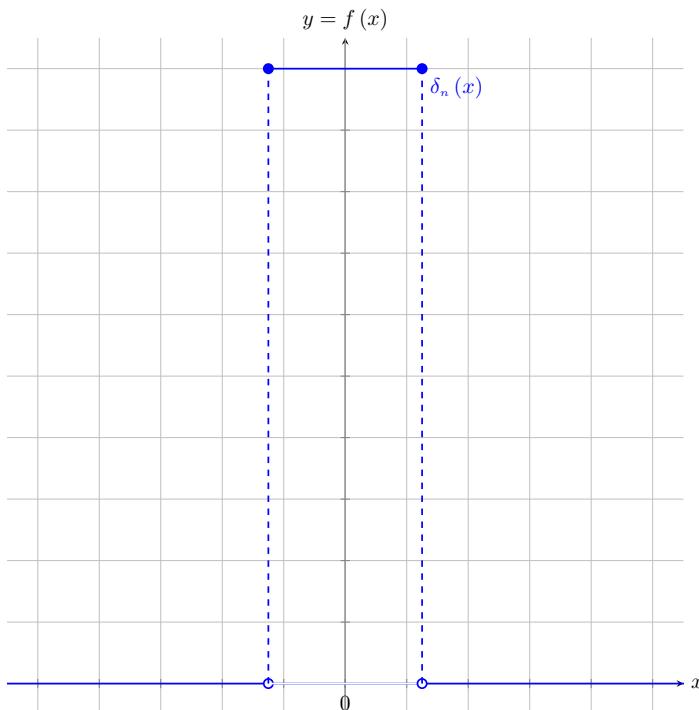
$$\text{supp}(\delta) = \left\{ x \middle| \begin{cases} x \in \mathcal{D} = \mathbb{R} \\ \delta(x) \neq 0 \end{cases} \right\} = \{0\}$$



$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Figure 45.3:  $f(x) = 1$  if  $x = 0$  else 0

<https://tex.stackexchange.com/questions/45275/tikz-get-values-for-predefined-dash-patterns>



$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \\ n & |x| \leq \frac{1}{2n} \end{cases}, \forall n \in \mathbb{N}$$

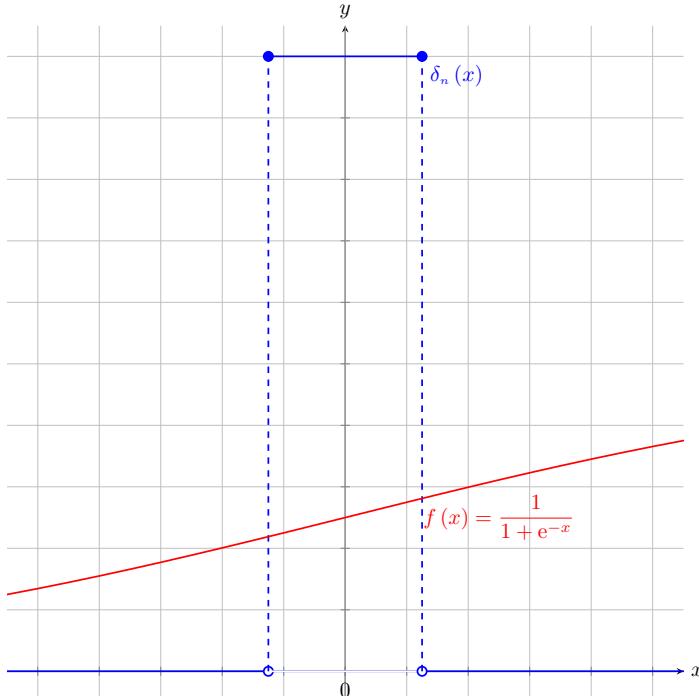
$$\delta(x) = \{\delta_n(x) | n \in \mathbb{N}\} = \{\delta_1(x), \delta_2(x), \dots\} = \lim_{n \rightarrow \infty} \delta_n(x)$$

Figure 45.4:  $\delta_n(x)$ 

$$\delta_n(x) = \begin{cases} 0 & |x| > \frac{1}{2n} \Leftrightarrow \begin{cases} x > \frac{1}{2n} \\ x < -\frac{1}{2n} \end{cases} \\ n & |x| \leq \frac{1}{2n} \Leftrightarrow -\frac{1}{2n} \leq x \leq \frac{1}{2n} \Leftrightarrow x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right] \end{cases}, \forall n \in \mathbb{N}$$

$$\int_{-\infty}^{\infty} \delta_n(x) dx = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} dx = n [x]_{-\frac{1}{2n}}^{\frac{1}{2n}} = n \left[ \frac{1}{2n} - \left( -\frac{1}{2n} \right) \right] = n \cdot \frac{1}{n} = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx \\
&= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x) dx \\
&= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx, \begin{cases} f(x) \in [m, M] \subseteq f\left(\left[\frac{-1}{2n}, \frac{1}{2n}\right]\right) \\ \Downarrow \\ m \leq f(x) \leq M \end{cases} \\
\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n m dx &\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n M dx \\
m = m \cdot 1 &= m \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx \leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq M \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = M \cdot 1 = M \\
&\Downarrow ?? \\
\int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx &\stackrel{\exists \xi_n \in (\frac{-1}{2n}, \frac{1}{2n})}{=} f(\xi_n) \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = f(\xi_n) \cdot 1 = f(\xi_n) \\
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx = \lim_{n \rightarrow \infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\
&= f(0) \\
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= f(0) \\
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= f(0)
\end{aligned}$$



$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x) dx \\
&= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \\
&= \lim_{n \rightarrow \infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x) f(x) dx &= f(0) \\
\int_{-\infty}^{\infty} \delta(x-0) f(x) dx &= f(0)
\end{aligned}$$

Figure 45.5:  $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-0) f(x) dx = f(0)$$

<https://math.stackexchange.com/questions/73010/proof-of-dirac-deltas-sifting-property>

<https://www.youtube.com/watch?v=2QaRZ7u-BgM>

SIFTing property

SIFT = scale-invariant feature transform

[https://en.wikipedia.org/wiki/Scale-invariant\\_feature\\_transform](https://en.wikipedia.org/wiki/Scale-invariant_feature_transform)

Proof:

Similarly to:

$$\begin{aligned}
 \int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx \\
 &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \delta_n(x) f(x) dx \\
 &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx, \quad \begin{cases} f(x) \in [m, M] \subseteq f\left(\left[\frac{-1}{2n}, \frac{1}{2n}\right]\right) \\ \Downarrow \\ m \leq f(x) \leq M \end{cases} \\
 \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n m dx &\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n M dx \\
 m = m \cdot 1 = m \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx &\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx \leq M \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = M \cdot 1 = M \\
 &\Downarrow ?? \\
 \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx &\stackrel{\exists \xi_n \in (\frac{-1}{2n}, \frac{1}{2n})}{=} f(\xi_n) \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n dx = f(\xi_n) \cdot 1 = f(\xi_n) \\
 \int_{-\infty}^{\infty} \delta(x) f(x) dx &= \lim_{n \rightarrow \infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx = \lim_{n \rightarrow \infty} f(\xi_n), \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\
 &= f(0) \\
 \int_{-\infty}^{\infty} \delta(x) f(x) dx &= f(0)
 \end{aligned}$$

$$f(x) \in [m, M] \subseteq f([x_0 - \frac{1}{2n}, x_0 + \frac{1}{2n}]),$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x - x_0) f(x) dx \\
&= \lim_{n \rightarrow \infty} \left[ \int_{-\infty}^{x_0 - \frac{1}{2n}} \delta_n(x - x_0) f(x) dx + \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} \delta_n(x - x_0) f(x) dx + \lim_{n \rightarrow \infty} \int_{x_0 + \frac{1}{2n}}^{\infty} \delta_n(x - x_0) f(x) dx \right] \\
&= \lim_{n \rightarrow \infty} \left[ \int_{-\infty}^{x_0 - \frac{1}{2n}} 0f(x) dx + \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx + \lim_{n \rightarrow \infty} \int_{x_0 + \frac{1}{2n}}^{\infty} 0f(x) dx \right] \\
&\quad = \lim_{n \rightarrow \infty} \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx, \begin{cases} f(x) \in [m, M] \subseteq f([x_0 - \frac{1}{2n}, x_0 + \frac{1}{2n}]) \\ \Downarrow \\ m \leq f(x) \leq M \end{cases} \\
&\quad \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nmdx \leq \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx \leq \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nMdx \\
&\quad m = m \cdot 1 = m \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} ndx \leq \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx \leq M \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} ndx = M \cdot 1 = M \\
&\quad \Downarrow ?? \\
&\quad \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx \stackrel{\exists \xi_n \in (x_0 - \frac{1}{2n}, x_0 + \frac{1}{2n})}{=} f(\xi_n) \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} ndx = f(\xi_n) \cdot 1 = f(\xi_n) \\
&\quad \int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n \rightarrow \infty} \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} nf(x) dx = \lim_{n \rightarrow \infty} f(\xi_n), \xi_n \in \left( x_0 - \frac{1}{2n}, x_0 + \frac{1}{2n} \right) \\
&\quad = f(x_0) \\
&\quad \int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)
\end{aligned}$$

□

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

$$\int_{-\infty}^{\infty} \delta(x - x') f(x) dx = f(x')$$

<https://www.youtube.com/watch?v=nDa3cqFk80o>

$$\begin{aligned}
& \left\{ \{f_n(x) | n \in \mathbb{N}\} \middle| \begin{cases} f_n : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} f_n(x) dx = 1 \end{cases} \right\} \\
&= \left\{ \left\{ \frac{n}{2} e^{-n|x|} \middle| n \in \mathbb{N} \right\}, \right. \\
&\quad \left\{ \frac{1}{\pi} \frac{n}{n^2 x^2 + 1} \middle| n \in \mathbb{N} \right\}, \\
&\quad \left\{ \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \middle| n \in \mathbb{N} \right\}, \\
&\quad \left. \left\{ \frac{1}{\pi} \frac{\sin(nx)}{x} \middle| n \in \mathbb{N} \right\}, \dots \right\} \\
& \left\{ \delta(x) \middle| \begin{cases} \delta : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} \delta(x) \cdot 1 dx = 1 \end{cases} \right\}
\end{aligned}$$

$$\left\{ \delta(x) \middle| \begin{cases} \delta : \mathbb{R} \rightarrow \mathbb{R} \\ \int_{-\infty}^{\infty} \delta(x-0) \cdot f(x) dx = f(0) \end{cases} \right\}$$

In measure theory, we can define the distance of two functions by

$$d(f, g) = \sqrt{\int_{-\infty}^{\infty} [f(x) - g(x)]^2 dx}$$

for real distance of two square delta function approximations,

$$\begin{aligned} d(\delta_m, \delta_n) &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x) - \delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2 - 2\delta_m(x)\delta_n(x) + [\delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} [\delta_m(x)]^2 dx - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \int_{-\infty}^{\infty} [\delta_n(x)]^2 dx} \\ &= \sqrt{\int_{-\infty}^{\infty} \delta_m(x)\delta_m(x) dx - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \int_{-\infty}^{\infty} \delta_n(x)\delta_n(x) dx} \\ &= \sqrt{\delta_m(\xi_m) - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + \delta_n(\xi_n)}, \begin{cases} \xi_m \in \left(\frac{-1}{2m}, \frac{1}{2m}\right) \\ \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \end{cases} \\ &= \sqrt{m - 2 \int_{-\infty}^{\infty} \delta_m(x)\delta_n(x) dx + n}, \text{ if } m > n \\ &= \sqrt{m - 2\delta_n(\xi_n) + n}, \xi_n \in \left(\frac{-1}{2n}, \frac{1}{2n}\right) \\ &= \sqrt{m - 2n + n} = \sqrt{m - n} \in \mathbb{R} \\ d(\delta_m, \delta_n) &\stackrel{m \geq n}{=} \sqrt{m - n} \in \mathbb{R} \end{aligned}$$

$\langle d(\delta_m, \delta_n) \rangle_{n \in \mathbb{N}} = \langle \sqrt{m - n} \rangle_{n \in \mathbb{N}}$  is not a Cauchy series, not even mentioned convergence

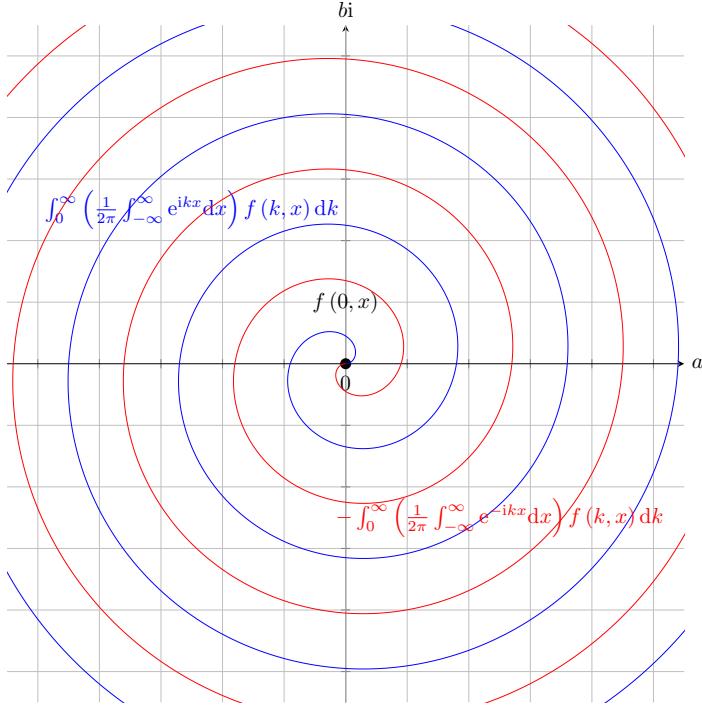
Def: 45.2

#### 45.2.1 complex delta function

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \\
&= \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^{k'-} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'+}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^0 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i0x} dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{-\infty}^0 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ - \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \right) f(k, x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 dx \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ - \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx \right) f(x) dk + \int_{k'-}^{k'+} \left( \frac{1}{2\pi} \infty \right) f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ \int_{k'-}^{k'+} \frac{1}{dk} f(0, x) dk + \int_0^{\infty} \left( \frac{1}{2\pi} e^{ikx} - e^{-ikx} \right) f(k, x) dk \right] \\
&\approx \lim_{k' \rightarrow 0} \left[ f(0, x) + \int_0^{\infty} \left( \frac{1}{2\pi} 0 \right) f(k, x) dk \right] \approx f(0, x) \\
f(0, x) &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk = \int_{-\infty}^{\infty} \delta(k) f(k, x) dk = \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk
\end{aligned}$$

<https://tex.stackexchange.com/questions/150138/how-can-i-create-a-polar-plot-on-a-cartesian-grid>



$$\begin{aligned}
f(0, x) &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk \\
&= \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk
\end{aligned}$$

Figure 45.6:  $\int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx \right) f(k, x) dk$

<sup>13</sup> p.18~20

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x) dx = 1 \end{cases}$$

According the SIFTing property above,

$$\int_{-\infty}^{\infty} \delta(x - x') f(x) dx = f(x')$$

And according to Fourier transform and inverse transform, Fourier analysis<sup>[50]</sup> with  $2\pi$  in the powers,

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x k} \left( \int_{-\infty}^{\infty} e^{-2\pi i k x'} f(x') dx' \right) dk$$

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} e^{2\pi i x k} \left( \int_{-\infty}^{\infty} e^{-2\pi i k x'} f(x') dx' \right) dk \\ &= \int_{-\infty}^{\infty} e^{2\pi i x k} f(x') dx' \left( \int_{-\infty}^{\infty} e^{-2\pi i k x'} dk \right) \\ &= \int_{-\infty}^{\infty} f(x') dx' \left( \int_{-\infty}^{\infty} e^{2\pi i x k} e^{-2\pi i k x'} dk \right) \\ &= \int_{-\infty}^{\infty} f(x') dx' \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) f(x') dx' \\ &\stackrel{\text{Fubini}}{=} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) f(x') dk dx' \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) f(x') dx' \\ f(x) &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) f(x') dx' \end{aligned}$$

comparing with SIFTing definition of Dirac delta function,

$$\int_{-\infty}^{\infty} \delta(x - x') f(x) dx = f(x')$$

thus

$$\int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk = \delta(x - x')$$

i.e.

$$\delta(x - x') = \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk$$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{2\pi i k(x-x')} dk \right) f(x') dx' \\
&= \int_{-\infty}^{\infty} \delta(x-x') f(x') dx', \begin{cases} k' = 2\pi k \Leftrightarrow k = \frac{k'}{2\pi} \\ x-x' = \varepsilon \Leftrightarrow x' = x-\varepsilon \end{cases} \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\varepsilon k'} dk' \right) f(x') dx' = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{2\pi} \int_{-K}^K e^{i\varepsilon k'} dk' \right) f(x') dx', K > 0 \\
&= \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{2\pi} \left[ \frac{e^{i\varepsilon k'}}{i\varepsilon} \right]_{k'=-K}^K \right) f(x') dx' = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{2\pi} \left[ \frac{e^{i\varepsilon K} - e^{i\varepsilon(-K)}}{i\varepsilon} \right] \right) f(x') dx' \\
&= \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi\varepsilon} \left[ \frac{e^{i\varepsilon K} - e^{-i\varepsilon K}}{2i} \right] \right) f(x') dx' = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi\varepsilon} [\sin(\varepsilon K)] \right) f(x') dx' \\
&= \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(\varepsilon K)}{\varepsilon} \right) f(x') dx' = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(K\varepsilon)}{\varepsilon} \right) f(x') dx', \begin{cases} x'-x = -\varepsilon \Leftrightarrow \varepsilon = x-x' \\ x' = -\varepsilon \Leftrightarrow \varepsilon = -x' \\ Kx' = u \Leftrightarrow x' = \frac{u}{K} \end{cases} \\
&\stackrel{x=0}{=} \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(K[-x'])}{-x'} \right) f(x') dx' = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{-\sin(Kx')}{-x'} \right) f(x') dx' \\
&= \int_{x'=-\infty}^{x'=\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(Kx')}{x'} \right) f(x') dx', \begin{cases} x' = -\varepsilon \Leftrightarrow \varepsilon = -x' \\ Kx' = u \Leftrightarrow x' = \frac{u}{K}, \wedge K > 0 \end{cases} \\
&= \int_{\frac{u}{K}=-\infty}^{\frac{u}{K}=\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(u)}{\frac{u}{K}} \right) f\left(\frac{u}{K}\right) d\frac{u}{K} = \int_{u=-\infty}^{u=\infty} \lim_{K \rightarrow \infty} \left( \frac{K}{\pi} \frac{\sin(u)}{u} \right) f\left(\frac{u}{K}\right) \frac{1}{K} du \\
&= \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{K}{\pi} \frac{\sin(u)}{u} \right) f\left(\frac{u}{K}\right) \frac{1}{K} du = \int_{-\infty}^{\infty} \lim_{K \rightarrow \infty} \left( \frac{1}{\pi} \frac{\sin(u)}{u} \right) f\left(\frac{u}{K}\right) du \\
&= \frac{1}{\pi} \lim_{K \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\sin(u)}{u} f\left(\frac{u}{K}\right) du \stackrel{\text{MVT}}{=} \frac{1}{\pi} \lim_{K \rightarrow \infty} f\left(\frac{\xi}{K}\right) \int_{-\infty}^{\infty} \frac{\sin(u)}{u} du, \exists \xi \in (-\epsilon, +\epsilon), \epsilon \in \mathbb{R}_{>0} \\
&= \frac{1}{\pi} f(0) \int_{-\infty}^{\infty} \frac{\sin(u)}{u} du \stackrel{\text{Feynman method or residue method}}{=} \frac{1}{\pi} f(0) \pi = f(0) = [f(x)]_{x=0}
\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{\sin(u)}{u} du = \pi \text{ see Feynman method of integration / integral technique}^{[44.2]}$$

According to Fourier transform and inverse transform, Fourier analysis<sup>[50]</sup> without  $2\pi$  in the powers,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\cdot\omega} \left( \int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s) ds \right) d\omega$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixk} \left( \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right) dk$$

or

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\cdot\omega} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\cdot s} f(s) ds \right) d\omega$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x') dx' \right) dk$$

complex function if analytic always with better properties than real function

$$\begin{aligned} & \left\{ \delta(k) \mid k \in \mathbb{R} \right\} \left| \begin{array}{l} \delta : \mathbb{R} \rightarrow \mathbb{C} \\ \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk = f(0, x) \end{array} \right. \\ &= \left\{ \left\{ K \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk \mid k \in \mathbb{R} \right\}, \dots \right\} \end{aligned}$$


---

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\cdot) dx$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk$$

$$\delta(x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx'} (\cdot) dk$$


---

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk$$

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x) dx$$

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ikx} e^{-ikx'} \psi(x') dx' \right) dk \\ &\stackrel{\text{Fubini}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk \right) dx' \\ &= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk \right) dx' \\ &= \int_{-\infty}^{\infty} \psi(x') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk \right) dx' \\ &= \int_{-\infty}^{\infty} \psi(x') \delta(x-x') dx' = \int_{-\infty}^{\infty} \delta(x-x') \psi(x') dx' = \psi(x) \end{aligned}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk$$


---

$$\int_{-\infty}^{\infty} |\psi(k)|^2 dk = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$

$$\begin{cases} \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi(k) dk \\ \psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \end{cases}$$

$$\begin{aligned} \psi^*(k) &= \bar{\psi}(k) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{-ikx} \psi(x)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{e^{-ikx} \psi(x)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx \end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} |\psi(k)|^2 dk = \int_{-\infty}^{\infty} \psi^*(k) \psi(k) dk \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx \right) \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx \right) \left( \int_{-\infty}^{\infty} e^{-ikx'} \psi(x') dx' \right) dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} e^{ikx} e^{-ikx'} \psi(x') dx' dx dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dx' dx dk \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dx' dx dk \\
&\stackrel{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \psi(x') dk dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \psi(x') \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*(x) \psi(x') \delta(x-x') dx' dx \\
&= \int_{-\infty}^{\infty} \psi^*(x) \int_{-\infty}^{\infty} \psi(x') \delta(x-x') dx' dx \\
&= \int_{-\infty}^{\infty} \psi^*(x) \psi(x) dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx
\end{aligned}$$


---

convolution

$$\psi(x) = \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \Rightarrow \psi(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)$$

$$\begin{aligned}
& \psi(x) = \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \\
& \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \\
& \psi_2(x-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \\
& \quad \begin{cases} \psi(x) = & \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \\ \psi_1(y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \\ \psi_2(x-y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \end{cases} \\
& \psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \right) \left( \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \right) dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik_1 y} e^{-ik_2 y} \psi_1(k_1) e^{-ikx} e^{ik_2 x} \psi_2(k_2) dk_1 dk_2 dy dx \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) e^{i(k_2-k)x} \psi_2(k_2) dk_1 dk_2 dy dx \\
&\stackrel{\text{Fubini}}{=} \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) e^{i(k_2-k)x} \psi_2(k_2) dy dx dk_1 dk_2 \\
&= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right) \left( \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right) dk_1 dk_2
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right) \left( \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right) dk_1 dk_2 \\
& = \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} \psi_1(k_1) dy \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_2-k)x} \psi_2(k_2) dx \right] dk_1 dk_2 \\
& = \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \psi_1(k_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_1-k_2)y} dy \right] \left[ \psi_2(k_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k_2-k)x} dx \right] dk_1 dk_2 \\
& = \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\psi_1(k_1) \delta(k_1 - k_2)] [\psi_2(k_2) \delta(k_2 - k)] dk_1 dk_2 \\
& \stackrel{\text{Fubini}}{=} \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\psi_1(k_1) \delta(k_1 - k_2)] [\psi_2(k_2) \delta(k_2 - k)] dk_2 dk_1 \\
& = \sqrt{2\pi} \int_{-\infty}^{\infty} [\psi_1(k_1)] \left[ \int_{-\infty}^{\infty} \delta(k_1 - k_2) \psi_2(k_2) \delta(k_2 - k) dk_2 \right] dk_1 \\
& = \sqrt{2\pi} \int_{-\infty}^{\infty} [\psi_1(k_1)] [\delta(k_1 - k) \psi_2(k)] dk_1 = \sqrt{2\pi} \psi_2(k) \int_{-\infty}^{\infty} \psi_1(k_1) \delta(k_1 - k) dk_1 \\
& = \sqrt{2\pi} \psi_2(k) \psi_1(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)
\end{aligned}$$

$$\begin{cases} \psi(x) = & \int_{-\infty}^{\infty} \psi_1(y) \psi_2(x-y) dy \\ \psi_1(y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_1 y} \psi_1(k_1) dk_1 \\ \psi_2(x-y) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik_2(x-y)} \psi_2(k_2) dk_2 \end{cases} \Downarrow \psi(k) = \sqrt{2\pi} \psi_1(k) \psi_2(k)$$

### 45.2.2 3D delta function

<https://www.youtube.com/watch?v=Y8y965ZAmQE>

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x') \delta(y - y') \delta(z - z')$$

Poisson equation

Feynman method of differentiation / derivative technique<sup>[44.1]</sup>

$$\begin{aligned}
\Delta \left( \frac{1}{r} \right) &= \nabla^2 \left( \frac{1}{\sqrt{r^2}} \right) = \nabla \cdot \nabla \left( \frac{1}{\sqrt{r \cdot r}} \right) = \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \\
&= \nabla \cdot \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) = \nabla \cdot \frac{-\mathbf{r}}{r^3} \\
&= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right) \\
&= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\
&= \frac{-2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0
\end{aligned}$$

$$\Delta \left( \frac{1}{r} \right) = -4\pi \delta(\mathbf{r})$$

<https://math.stackexchange.com/questions/3774483/derivatives-of-frac1r-and-dirac-delta-function>

$$\begin{aligned}
\Delta \frac{1}{r(\epsilon)} &= \nabla^2 \frac{1}{\sqrt{\mathbf{r}^2 + \epsilon^2}} = \nabla \cdot \nabla \frac{1}{\sqrt{x^2 + y^2 + z^2 + \epsilon^2}} \\
&= \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2 + \epsilon^2}} \right) \\
&= \nabla \cdot \left( \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}}, \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}}, \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \right) \\
&= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{-y}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} + \frac{\partial}{\partial z} \frac{-z}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{3}{2}}} \\
&= \frac{+2x^2 - y^2 - z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} + \frac{-x^2 + 2y^2 - z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} + \frac{-x^2 - y^2 + 2z^2 - \epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} \\
&= \frac{-3\epsilon^2}{(x^2 + y^2 + z^2 + \epsilon^2)^{\frac{5}{2}}} = \frac{-3\epsilon^2}{(\mathbf{r}^2 + \epsilon^2)^{\frac{5}{2}}}
\end{aligned}$$

$$\Delta \left( \frac{1}{r} \right) = \nabla^2 \left( \frac{1}{r} \right) = \nabla^2 \frac{1}{|\mathbf{r}|} = \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{0}|} = \Delta \frac{1}{|\mathbf{r} - \mathbf{0}|} = \Delta(\mathbf{r} - \mathbf{0}) = \Delta(\mathbf{r})$$

$$\Delta \frac{1}{r(\epsilon)} = \Delta_\epsilon(r) = \Delta_\epsilon(\mathbf{r}) = \Delta_\epsilon(\mathbf{r} - \mathbf{0})$$

$$\lim_{\epsilon \rightarrow 0} \Delta_\epsilon(\mathbf{r}) = \lim_{\epsilon \rightarrow 0} \Delta \frac{1}{r(\epsilon)} = \Delta \left( \frac{1}{r} \right) = \Delta(\mathbf{r})$$

$$K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} K_3 \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r} - \mathbf{0}) d\mathbf{r} = f(\mathbf{0})$$

$$K_1 \int_{\mathbb{R}^1} f(x) \delta(x) dx = \lim_{n \rightarrow \infty} K_1 \int_{\mathbb{R}^1} f(x) \delta_n(x) dx = K_1 \int_{\mathbb{R}^1} f(x) \delta_n(x - 0) dx = f(0)$$

$$\begin{aligned}
\int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} &= \int_{\mathbb{R}^3} f(\mathbf{r}) \frac{-3\epsilon^2}{(\mathbf{r}^2 + \epsilon^2)^{\frac{5}{2}}} d\mathbf{r}, \begin{cases} \mathbf{r} = r\mathbf{n}_s \\ \mathbf{n}_s = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} \end{cases} \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(r\mathbf{n}_s) \frac{-3\epsilon^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} r^2 \sigma(d\mathbf{n}_s) dr, \begin{cases} \mathbb{S}^2 = \text{unit sphere centered at the origin} \\ \sigma(d\mathbf{n}_s) \text{ is the surface measure of } \mathbb{S}^2 \\ r^2 \sigma(d\mathbf{n}_s) = r^2 \sin \phi d\phi d\theta \\ r = \epsilon s \end{cases} \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{-3\epsilon^2}{((\epsilon s)^2 + \epsilon^2)^{\frac{5}{2}}} (\epsilon s)^2 \sigma(d\mathbf{n}_s) d\epsilon s \\
&= \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{-3\epsilon^5}{\epsilon^5 (s^2 + 1)^{\frac{5}{2}}} s^2 \sigma(d\mathbf{n}_s) ds = - \int_0^\infty \int_{\mathbb{S}^2} f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} \sigma(d\mathbf{n}_s) ds \\
&= - \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} \sin \phi d\phi d\theta ds = - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds
\end{aligned}$$

$$\begin{aligned}
&\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} \\
&= \lim_{\epsilon \rightarrow 0} \left[ - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds \right] \\
&= - 4\pi f(\mathbf{0}) \int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds
\end{aligned}$$

$$\begin{aligned}
& \int \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds, s = \tan u \\
&= \int \frac{3 \tan^2 u}{(\tan^2 u + 1)^{\frac{5}{2}}} d \tan u = \int \frac{3 \tan^2 u}{(\tan^2 u + 1)^{\frac{5}{2}}} \sec^2 u du \\
&= \int \frac{3 \tan^2 u}{(\sec^2 u)^{\frac{5}{2}}} \sec^2 u du = \int \frac{3 \tan^2 u}{\sec^5 u} \sec^2 u du = \int \frac{3 \tan^2 u}{\sec^3 u} du \\
&= \int \frac{3 \left( \frac{\sin u}{\cos u} \right)^2}{\left( \frac{1}{\cos u} \right)^3} du = 3 \int \sin^2 u \cos u du = 3 \int \sin^2 u d \sin u = 3 \frac{\sin^3 u}{3} + C \\
&= \sin^3 u + C = \left( \frac{s}{\sqrt{s^2 + 1}} \right)^3 + C = \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} + C \stackrel{s \geq 0}{=} \frac{1}{(1 + s^{-2})^{\frac{3}{2}}} + C
\end{aligned}$$

$$\int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds = \left[ \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} \right]_0^\infty = \left[ \frac{1}{(1 + s^{-2})^{\frac{3}{2}}} \right]_{s=\infty} - \left[ \frac{s^3}{(s^2 + 1)^{\frac{3}{2}}} \right]_{s=0} = 1 - 0 = 1$$

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} \\
&= \lim_{\epsilon \rightarrow 0} \left[ - \int_0^\infty 4\pi f(\epsilon s \mathbf{n}_s) \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds \right] \\
&= -4\pi f(\mathbf{0}) \int_0^\infty \frac{3s^2}{(s^2 + 1)^{\frac{5}{2}}} ds = -4\pi f(\mathbf{0}) 1 = -4\pi f(\mathbf{0})
\end{aligned}$$

$$\int_{\mathbb{R}^3} f(\mathbf{r}) \Delta(\mathbf{r}) d\mathbf{r} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{r}) \Delta_\epsilon(\mathbf{r}) d\mathbf{r} = -4\pi f(\mathbf{0}) = -4\pi \int_{\mathbb{R}^3} f(\mathbf{r}) \delta(\mathbf{r}) d\mathbf{r}$$

$$\Delta \left( \frac{1}{r} \right) = \Delta(\mathbf{r}) = -4\pi \delta(\mathbf{r})$$

Big Delta reciprocal r is minus 4 pi delta r.

□

---


$$\Delta \left( \frac{1}{r} \right) = -4\pi \delta(\mathbf{r})$$


---

Poisson equation rigorous proof by general function and Green theorem

---


$$\nabla^2 \frac{1}{r} = -4\pi \delta(\mathbf{r}) \text{ 的严格证明}$$


---

**证** 在经典意义下, 函数  $1/r$  显然在原点不可导, 应当在广义函数的意义下计算  $\nabla^2(1/r)$ . 因为  $\nabla^2$  中只含有对自变量的二阶导数, 故对于任意检验函数  $\phi(\mathbf{r})$ , 根据广义函数导数的定义, 有

$$\left\langle \nabla^2 \frac{1}{r}, \phi \right\rangle = \left\langle \frac{1}{r}, \nabla^2 \phi \right\rangle = \iiint_{\mathcal{R}_3} \frac{\nabla^2 \phi}{r} d\mathbf{r} = \lim_{\varepsilon \rightarrow 0} \iiint_{r > \varepsilon} \frac{\nabla^2 \phi}{r} d\mathbf{r}.$$

上式最后一步是因为  $1/r$  在原点的奇异性比较弱, 右端的积分收敛. 注意在一定有界区域外  $\phi = 0$ , 用格林公式计算上式右端的积分, 可得到

$$\iiint_{r > \varepsilon} \frac{\nabla^2 \phi}{r} d\mathbf{r} = \iiint_{r > \varepsilon} \phi \nabla^2 \left( \frac{1}{r} \right) d\mathbf{r} + \iint_{r=\varepsilon} \left[ \frac{1}{r} \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{r} \right) \right] dS,$$

其中  $\mathbf{n}$  是区域  $r > \varepsilon$  的外法线, 即指向球心,  $\partial/\partial \mathbf{n} = -(\partial/\partial r)$ . 注意当  $r \neq 0$  时  $\nabla^2(1/r) = 0$ , 所以上式右端第一项为 0, 因此有

$$\iiint_{r > \varepsilon} \frac{\nabla^2 \phi}{r} d\mathbf{r} = - \iint_{r=\varepsilon} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\phi}{r^2} \right) dS.$$

因为检验函数的导数是有界的, 即  $|\partial \phi / \partial r| < M$ , 所以

$$\left| \iint_{r=\varepsilon} \frac{1}{r} \frac{\partial \phi}{\partial r} dS \right| \leq \frac{M}{\varepsilon} (4\pi\varepsilon^2) = 4\pi\varepsilon M \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

同时,

$$\iint_{r=\varepsilon} \frac{\phi}{r^2} dS = \iint_{r=\varepsilon} \frac{\phi(0) + [\phi(\mathbf{r}) - \phi(0)]}{r^2} dS = 4\pi\phi(0) + \iint_{r=\varepsilon} \frac{\phi(\mathbf{r}) - \phi(0)}{r^2} dS.$$

因为  $\phi(\mathbf{r})$  在原点连续, 所以上式后一个积分在  $\varepsilon \rightarrow 0$  时趋于 0. 由此就导出

$$\left\langle \nabla^2 \frac{1}{r}, \phi \right\rangle = -4\pi\phi(0),$$

所以, 在广义函数的意义下,

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}).$$

知乎 @Hsuty

Figure 45.7: Poisson equation rigorous proof by general function and Green theorem

<https://math.stackexchange.com/questions/368155/where-does-the-relation-nabla21-r-4-pi-delta3-bf-r-between-laplacian>

$$\begin{aligned} & \left\{ \delta(k) | k \in \mathbb{R} \right\} \left\{ \begin{array}{l} \delta : \mathbb{R} \rightarrow \mathbb{C} \\ \int_{-\infty}^{\infty} \delta(k-0) f(k, x) dk = f(0, x) \end{array} \right\} \\ &= \left\{ \left\{ K \int_{-\infty}^{\infty} e^{ikx} (\cdot) dk \middle| k \in \mathbb{R} \right\}, \dots \right\} \end{aligned}$$

$$\begin{aligned} & \left\{ \left\{ \delta(\mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^3 \right\} \middle| \left\{ \begin{array}{l} \delta : \mathbb{R}^3 \rightarrow \mathbb{H} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{k} - \mathbf{0}) f(\mathbf{k}, \mathbf{r}) d^3 r = f(\mathbf{0}, \mathbf{r}) \end{array} \right\} \right\} \\ &= \left\{ \left\{ K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mathbf{k} \cdot \mathbf{r}} (\cdot) d^3 \mathbf{k} \mid \mathbf{k} \in \mathbb{R}^3 \right\}, \dots \right\} \end{aligned}$$


---

Coulomb law

$$F_e = k \frac{Qq}{r^2}$$

$$\mathbf{F}_e = k \frac{Qq}{r^2} \hat{\mathbf{r}}$$

electric field from point electric charge

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = \mathbf{E}(r) = \mathbf{E} = k \frac{q}{r^2} \hat{\mathbf{r}} &= k_e \frac{q}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} = k_e q \frac{\mathbf{r}}{|\mathbf{r}|^3} = k_e q \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} \\ &= k_e q \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} = k_e q \left( -\nabla \frac{1}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} \right) = -k_e q \nabla \frac{1}{\|\mathbf{r}\|} \end{aligned}$$

$$\mathbf{E}(\mathbf{r}) = -k_e q \nabla \frac{1}{\|\mathbf{r}\|}$$

electric field from tiny point electric charge

$$\begin{aligned} d\mathbf{E}(\mathbf{r}) = d\mathbf{E}(r) = d\mathbf{E} = k \frac{dq(r)}{r^2} \hat{\mathbf{r}} &= k \frac{dq(r)}{r^2} \hat{\mathbf{r}} = k_e dq(\mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} \\ &= k_e dq(\mathbf{r}) \frac{\mathbf{r}}{(\mathbf{r}^2)^{\frac{3}{2}}} = k_e dq(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|} \end{aligned}$$

$$d\mathbf{E}(\mathbf{r}) = -k_e dq(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|}$$

$$dq = \rho dV \Leftrightarrow \rho = \frac{dq}{dV}$$

$$dq(\mathbf{r}) = \rho(\mathbf{r}) dV \Leftrightarrow \rho(\mathbf{r}) = \frac{dq(\mathbf{r})}{dV}$$

$$q = \int_q dq(\mathbf{r}) = \int_V \rho(\mathbf{r}) dV = \iiint_V \rho(\mathbf{r}) d^3 r$$

$$\begin{aligned} \mathbf{E} &= \iiint_V \mathbf{E}(\mathbf{r}) d^3 r = \iiint_V \left( -k_e \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|} \right) d^3 r \\ &= -k_e \iiint_V \left( \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r}\|} \right) d^3 r \\ &= -k_e \iiint_V \left( \rho(\mathbf{r}) \nabla \frac{1}{\|\mathbf{r} - \mathbf{0}\|} \right) d^3 r = \mathbf{E}(\mathbf{0}) \end{aligned}$$

$$\begin{aligned}
& \Downarrow \begin{cases} Q \text{ at } \mathbf{r} \\ dq \text{ or } \rho \text{ at } \mathbf{r}', \text{ totally } V \end{cases} \\
\mathbf{E}(\mathbf{r}) &= \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' \\
\nabla_r \cdot \mathbf{E}(\mathbf{r}) &= \nabla_r \cdot \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r \cdot \nabla_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' \\
&= \iiint_V \left( -k_e \rho(\mathbf{r}') \nabla_r^2 \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' = \iiint_V \left( -k_e \rho(\mathbf{r}') \Delta_r \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \right) d^3 r' \\
&= \iiint_V f(\mathbf{r}') \Delta(\mathbf{r} - \mathbf{r}') d^3 r', f(\mathbf{r}') = -k_e \rho(\mathbf{r}') \\
&= \iiint_V f(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') d^3 r', f(\mathbf{r}') = -k_e \rho(\mathbf{r}') \\
&= -4\pi f(\mathbf{r}' = \mathbf{r}) = -4\pi [-k_e \rho(\mathbf{r}' = \mathbf{r})] \\
&= -4\pi f(\mathbf{r}) = -4\pi [-k_e \rho(\mathbf{r})] = 4\pi k_e \rho(\mathbf{r}) \\
\nabla \cdot \mathbf{E}(\mathbf{r}) &= 4\pi k_e \rho(\mathbf{r})
\end{aligned}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi k_e \rho(\mathbf{r})$$


---

gravity

<https://zhuanlan.zhihu.com/p/695160620>

---

<https://www.bilibili.com/video/BV1qu411578c/?t=7m26s>

[https://en.wikipedia.org/wiki/Helmholtz\\_equation](https://en.wikipedia.org/wiki/Helmholtz_equation)

Helmholtz equation

$$(\Delta + k^2) \frac{e^{\pm i\mathbf{k} \cdot \mathbf{r}}}{r} = -4\pi \delta(\mathbf{r})$$

$$(\nabla_r^2 + \mathbf{k}^2) \frac{e^{\pm i\mathbf{k} \cdot \mathbf{r}}}{\|\mathbf{r}\|} = -4\pi \delta(\mathbf{r})$$

### 45.2.3 quantum state

<https://www.bilibili.com/video/BV1qu411578c/?t=7m45s>

<https://www.bilibili.com/video/BV1xu4y1S7W2>

### 45.2.4 rigorous definition of delta function

$$\int_{-\infty}^{\infty} \delta(x - 0) \cdot f(x) dx = f(0)$$

$$\int \delta(x) f(x) dx = f(0)$$

$$\int \delta(x)(\cdot) dx : f(x) \rightarrow f(0)$$

$$f(x) \xrightarrow{\int \delta(x)(\cdot) dx} f(0)$$

$f$  must be with good properties

$$\text{continuous linear functional} \begin{cases} \text{function-like} & \eta_\varphi(f) = \int \varphi(x) f(x) dx \\ \text{non-function-like, e.g.} & \delta_{x'}(f) = f(x') \end{cases}$$

general function with “formally” integral notation

$$\delta_{x'}(f) = \int \delta(x - x') f(x) dx = f(x')$$

<https://www.bilibili.com/video/BV1qu411578c/?t=11m36s>

Quantum mechanics is founded on Hilbert space; However, Dirac delta function is not a linear functional on Hilbert space.?

One of the reason is that we cannot find 1-1 of Dirac delta function, but many-to-one or one-to-many functions.

量子力学本不需 Dirac delta function, 其卻仍大行其道, 實廣義函數為其負重前行。

狄拉克函數不是希爾伯特空間的線性泛函. Dirac delta function is not a linear functional on Hilbert space.

狄拉克括號卻是希爾伯特空間的向量表示. Dirac bra-ket brackets are vector representation on Hilbert space.

## 45.3 Fourier expansion or Fourier series

vector space definition<sup>[40.2]</sup>

Fig: 40.1

linear space of function<sup>[43.1]</sup>

<https://www.bilibili.com/video/BV1PX4y167RS>

<https://www.bilibili.com/video/BV1m24y1A74K>

<https://www.bilibili.com/video/BV1m24y1A74K/?=3m2s>

### 45.3.1 Taylor vs. Fourier

$$f(x) = x_0 x^0 + x_1 x^1 + x_2 x^2 + \dots = \sum_{k=0}^{\infty} x_k x^k, x_k = \frac{f^{(k)}(0)}{k!}$$

$$f(r) = r_0 r^0 + r_1 r^1 + r_2 r^2 + \dots = \sum_{k=0}^{\infty} r_k r^k, r_k = \frac{f^{(k)}(0)}{k!}$$

$$f(\theta) = a_0 \cos(0\theta) + a_1 \cos(1\theta) + a_2 \cos(2\theta) + \dots = \sum_{k=0}^{\infty} a_k \cos(k\theta)$$

$$a_k = \int_{-\infty}^{\infty} \cos(k\theta) f(\theta) d\theta$$

$$\begin{aligned} f(\theta) &= a_0 \cos(0\theta) + a_1 \cos(1\theta) + a_2 \cos(2\theta) + \dots = \sum_{k=0}^{\infty} a_k \cos(k\theta) \\ &= a_0 \Re[e^{i0\theta}] + a_1 \Re[e^{i1\theta}] + a_2 \Re[e^{i2\theta}] + \dots = \sum_{k=0}^{\infty} a_k \Re[e^{ik\theta}] = \sum_{k=0}^{\infty} a_k \Re[(e^{i\theta})^k] \\ &= \Re[a_0 e^{i0\theta} + a_1 e^{i1\theta} + a_2 e^{i2\theta} + \dots] = \Re \left[ \sum_{k=0}^{\infty} a_k e^{ik\theta} \right] = \Re \left[ \sum_{k=0}^{\infty} a_k (e^{i\theta})^k \right] \end{aligned}$$

$$f(x) = x_0 x^0 + x_1 x^1 + x_2 x^2 + \dots = \sum_{k=0}^{\infty} x_k x^k$$

$$f(z) = z_0 z^0 + z_1 z^1 + z_2 z^2 + \dots = \sum_{k=0}^{\infty} z_k z^k$$

$$z = r e^{i\theta}$$

$$\begin{aligned}
 f(z) &= z_0 z^0 + z_1 z^1 + z_2 z^2 + \cdots = \sum_{k=0}^{\infty} z_k z^k \\
 f(re^{i\theta}) &= z_0 (re^{i\theta})^0 + z_1 (re^{i\theta})^1 + z_2 (re^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} z_k (re^{i\theta})^k \\
 &= f(r) = z_0 (e^{i\theta})^0 r^0 + z_1 (e^{i\theta})^1 r^1 + z_2 (e^{i\theta})^2 r^2 + \cdots = \sum_{k=0}^{\infty} z_k (e^{i\theta})^k r^k \\
 &= r_0 r^0 + r_1 r^1 + r_2 r^2 + \cdots = \sum_{k=0}^{\infty} r_k r^k, \quad r_k = z_k (e^{i\theta})^k \\
 &= f(e^{i\theta}) = z_0 r^0 (e^{i\theta})^0 + z_1 r^1 (e^{i\theta})^1 + z_2 r^2 (e^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} z_k r^k (e^{i\theta})^k \\
 &= (e^{i\theta})_0 (e^{i\theta})^0 + (e^{i\theta})_1 (e^{i\theta})^1 + (e^{i\theta})_2 (e^{i\theta})^2 + \cdots = \sum_{k=0}^{\infty} (e^{i\theta})_k (e^{i\theta})^k \\
 &, (e^{i\theta})_k = z_k r^k
 \end{aligned}$$

<https://www.bilibili.com/video/BV1nV411F7es>

## 45.4 Hilbert space construction

<https://www.zhihu.com/question/19967778/answer/2572975608>

<https://www.bilibili.com/video/BV1cV411T7hb>

**Definition 45.1.** sequence limit

$$\lim_{n \rightarrow \infty} a_n = a \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} (n > N \Rightarrow |a_n - a| < \epsilon)$$

$$x_1, x_2, \dots, x_n, x_{n+1}, \dots$$

- to find convergence
  - monotone convergence theorem
  - split into known sequence
  - fraction: Stolz theorem
  - sequence itself
    - \* ratio  $\frac{x_m}{x_n}$ , but not good for some terms being zero
    - \* difference  $x_m - x_n$

**Theorem 45.4.** Cauchy convergence theorem

$$\begin{gathered}
 \{x_n\}_{n \in \mathbb{N}} \text{ is convergent} \\
 \Updownarrow \\
 \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \in \mathbb{N} (m, n > N \Rightarrow |x_m - x_n| < \epsilon) \Leftrightarrow \{x_n\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence}
 \end{gathered}$$

**Definition 45.2.** Cauchy sequence

$$\begin{gathered}
 \{x_n\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence} \\
 \Updownarrow \\
 \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \in \mathbb{N} (m, n > N \Rightarrow |x_m - x_n| < \epsilon)
 \end{gathered}$$

$$\text{convergence} \begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

$$\text{convergence} \begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ \downarrow \text{easy proof} & \uparrow \text{hard proof, weak condition needs all real number as supporting base} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

$$\text{convergence} \begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ \downarrow & \uparrow x_n, x_m, x \in \mathbb{R} \text{ with real completeness} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases}$$

#### 45.4.1 inner product space

vector space definition<sup>[40.2]</sup>

Fig: 40.1

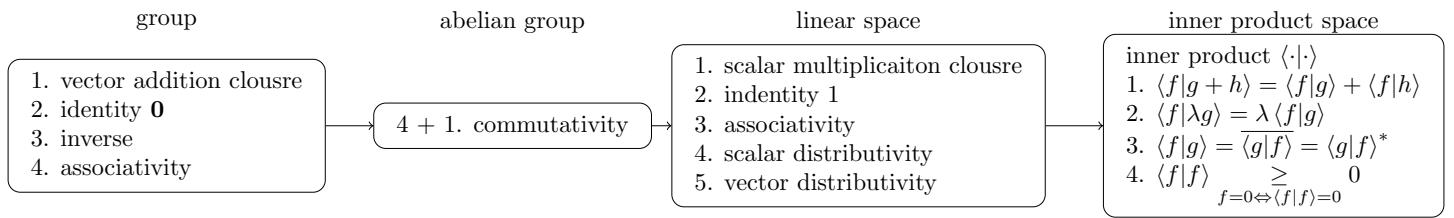


Figure 45.8: inner product space construction

$$\langle \cdot | \cdot \rangle : \begin{cases} \langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle \\ \langle f | cg \rangle = c \langle f | g \rangle \\ \langle f | g \rangle = \overline{\langle g | f \rangle} = \langle g | f \rangle^* \\ \langle f | f \rangle \geq 0 \\ f = 0 \Leftrightarrow \langle f | f \rangle = 0 \end{cases}$$

definition of distance

**Definition 45.3.** distance

$$d(f, g) = d \langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle}$$

For real number,

$$\begin{cases} d(f, g) = d \langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in \mathbb{R} \\ \langle f | g \rangle = fg = f \cdot g \end{cases} \quad \Downarrow \quad d \langle f | g \rangle = \sqrt{(f - g)^2} = |f - g|$$

For  $n$ -dimensional vector

$$\begin{cases} d(f, g) = d \langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in F^n \\ \langle f | g \rangle = \sum_{k=1}^n f_k g_k \end{cases} \quad \Downarrow \quad d \langle f | g \rangle = \sqrt{\sum_{k=1}^n (f_k - g_k)^2}$$

For continuous complex function on  $[a, b]$ ,

$$\begin{cases} d(f, g) = d\langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in C_{\mathbb{C}}([a, b]) \\ \langle f | g \rangle = \int_a^b \overline{f(x)} g(x) dx \end{cases} \quad \text{continuous complex function}$$

$\Downarrow$   
 $d\langle f | g \rangle = \sqrt{\int_a^b [f - g](x) [f - g](x) dx} = \sqrt{\int_a^b |[f - g](x)|^2 dx}$

---

For real number sequence,

$$\begin{aligned}
 & \begin{cases} d(f, g) = d\langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in \mathbb{R} \\ \langle f | g \rangle = fg = f \cdot g \end{cases} \\
 & \Downarrow \\
 & \begin{aligned} \text{convergence } & \begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases} \\ \text{convergence } & \begin{cases} |x_n - x| < \epsilon & \text{convergent sequence} \\ \Downarrow & \uparrow x_n, x_m, x \in \mathbb{R} \text{ with real completeness} \\ |x_m - x_n| < \epsilon & \text{Cauchy sequence} \end{cases} \end{aligned}
 \end{aligned}$$

For complex function sequence,

$$\begin{aligned}
 & \begin{cases} d(f, g) = d\langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in C_{\mathbb{C}}([a, b]) \\ \langle f | g \rangle = \int_a^b \overline{f(x)} g(x) dx \end{cases} \quad \text{continuous complex function}$$

$\Downarrow$   
 $d\langle f | g \rangle = \sqrt{\int_a^b [f - g](x) [f - g](x) dx} = \sqrt{\int_a^b |[f - g](x)|^2 dx}$

$$\begin{aligned} \text{function convergence } & \begin{cases} d\langle f_n | f \rangle < \epsilon & \text{convergent function sequence} \\ d\langle f_m | f_n \rangle < \epsilon & \text{Cauchy function sequence} \end{cases} \\ \text{function convergence } & \begin{cases} d\langle f_n | f \rangle < \epsilon & \text{convergent function sequence} \\ \Downarrow & \nexists \text{ not always continuous} \\ d\langle f_m | f_n \rangle < \epsilon & \text{Cauchy function sequence} \end{cases} \end{aligned}$$

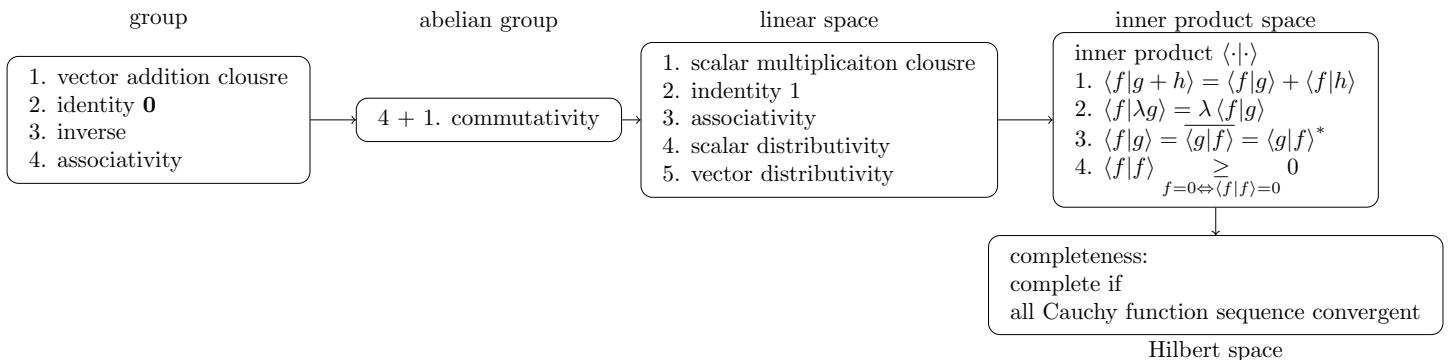


Figure 45.9: Hilbert space construction

| 度量                              |       | 范数                               |       | 内积                                                                                                                        |          |
|---------------------------------|-------|----------------------------------|-------|---------------------------------------------------------------------------------------------------------------------------|----------|
| $d(x,y) \geq 0$                 | 非负性   | $\ x\  \geq 0$                   | 非负性   | $\langle x, x \rangle \geq 0$                                                                                             | 正定性      |
| $d(x,y)=0, \text{only if } x=y$ | 非退化性  | $\ x\  = 0, \text{only if } x=0$ | 非退化性  | $\langle x, x \rangle = 0, \text{only if } x=0$                                                                           | 非退化性     |
| $d(x,y)=d(y,x)$                 | 对称性   |                                  |       | $\langle x, y \rangle = \overline{\langle y, x \rangle}$                                                                  | 共轭对称性    |
|                                 |       | $\ ax\  =  a  \cdot \ x\ $       | 齐次性   | $\begin{aligned} \langle ax + by, z \rangle \\ = a\langle x, z \rangle + b\langle y, z \rangle \end{aligned}$             | 第一变元线性   |
|                                 |       |                                  |       | $\begin{aligned} \langle z, ax + by \rangle \\ = \bar{a}\langle z, x \rangle + \bar{b}\langle z, y \rangle \end{aligned}$ | 第二变元共轭线性 |
| $d(x,y) \leq d(x,z) + d(z,y)$   | 三角不等式 | $\ x + y\  \leq \ x\  + \ y\ $   | 三角不等式 |                                                                                                                           |          |

Figure 45.10: metric vs. inner product

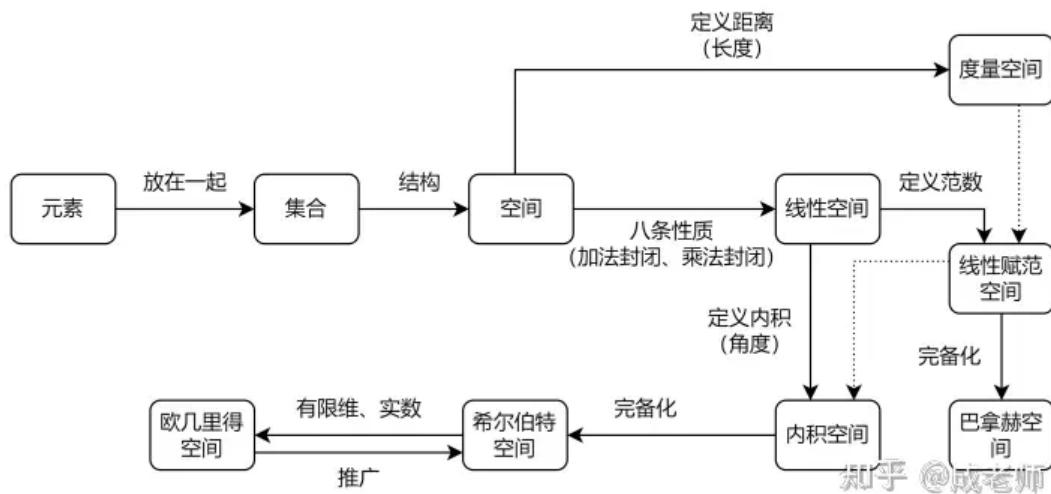


Figure 45.11: Euclid space construction

<https://zhuanlan.zhihu.com/p/85867887>

#### 45.4.2 Riesz representation theorem

<https://zhuanlan.zhihu.com/p/308879557>

### 45.5 Dirac bracket

<https://www.bilibili.com/video/BV1GN411m7A9>

reciprocal space<sup>[40.4.1]</sup>

double dual concept<sup>[40.4.1.2]</sup>

$$\begin{array}{ccccccc}
 & & V & = & \{ & e_1 & e_2 & e_3 & v & \cdots \} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 V^* = \{ & e^1 & : & F & \supseteq & \{ & 1 & 0 & 0 & v_1 & \cdots \} \\
 & e^2 & \} & F & \supseteq & \{ & e_1 & e_2 & e_3 & v & \cdots \} \\
 & e^3 & & V & = & \{ & & & & \cdots \} \\
 & v^* & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & F & \supseteq & \{ & v^{*1} & v^{*2} & v^{*3} & v^{*i}v_i & \cdots \}
 \end{array}$$

$$\begin{array}{ccccccc}
& V^* & = \{ & e^1 & e^2 & e^3 & \omega & \dots \} \\
V^{**} = \{ & \begin{matrix} e^{1*} \\ e^{2*} \\ e^{3*} \\ \omega^* \end{matrix} & : & \begin{matrix} \downarrow \\ F \\ V^* \\ \vdots \end{matrix} & \supseteq \{ & \begin{matrix} 1 & 0 & 0 & \omega^1 & \dots \end{matrix} \\
& & & & \begin{matrix} e^1 & e^2 & e^3 & \omega & \dots \end{matrix} \\
& & & & \downarrow & & \downarrow \\
& & & & \begin{matrix} \omega^{1*} & \omega^{2*} & \omega^{3*} & \omega^{\mu*}\omega^\mu & \dots \end{matrix} \\
& & & & \begin{matrix} e^1 & e^2 & e^3 & v^* & \dots \end{matrix} \\
& & & & \downarrow & & \downarrow \\
& & & & \begin{matrix} 1 & 0 & 0 & v^{*1} & \dots \end{matrix} \\
& & & & \begin{matrix} e^1 & e^2 & e^3 & v^* & \dots \end{matrix} \\
& & & & \downarrow & & \downarrow \\
& & & & \begin{matrix} v_1 & v_2 & v_3 & v_\mu v^{*\mu} & \dots \end{matrix} \\
& & & & \vdots & & \vdots \\
& & & & F & \supseteq \{ & \dots \}
\end{array}$$


---

dual space of span of partials<sup>[41.2]</sup>

coefficient of linear combination for vector space and dual space<sup>[41.4]</sup>

$$\begin{array}{ccccccc}
& V & = \{ & e_1 & e_2 & e_3 & v & \dots \} \\
V^* = \{ & \begin{matrix} e^1 \\ e^2 \\ e^3 \\ v^* \end{matrix} & : & \begin{matrix} \downarrow \\ F \\ V \\ \vdots \end{matrix} & \supseteq \{ & \begin{matrix} 1 & 0 & 0 & v_1 & \dots \end{matrix} \\
& & & & \begin{matrix} e_1 & e_2 & e_3 & v & \dots \end{matrix} \\
& & & & \downarrow & & \downarrow \\
& & & & \begin{matrix} v^{*1} & v^{*2} & v^{*3} & v^{*i}v_i & \dots \end{matrix} \\
& & & & \vdots & & \vdots \\
& & & & F & \supseteq \{ & \dots \}
\end{array}$$

inner product space<sup>[45.4.1]</sup>

$$\begin{cases} d(f, g) = d\langle f | g \rangle = \sqrt{\langle f - g | f - g \rangle} \\ f, g \in C_{\mathbb{C}}([a, b]) \end{cases} \quad \text{continuous complex function}$$

$$\begin{aligned}
(f, g) &= \langle f | g \rangle = \int_a^b \overline{f(x)} g(x) dx = \int_a^b f^*(x) g(x) dx \\
&\Downarrow \\
d\langle f | g \rangle &= \sqrt{\int_a^b \overline{[f - g](x)} [f - g](x) dx} = \sqrt{\int_a^b |[f - g](x)|^2 dx}
\end{aligned}$$

$$v^*(f) = \eta_f(\cdot) = (f, \cdot) = \langle f | \cdot \rangle = \int_a^b \overline{f(x)} (\cdot) dx = \int_a^b f^*(x) (\cdot) dx$$

$$\eta_f(g) = (f, g) = \langle f | g \rangle = \int_a^b \overline{f(x)} g(x) dx = \int_a^b f^*(x) g(x) dx$$

[https://en.wikipedia.org/wiki/Antilinear\\_map](https://en.wikipedia.org/wiki/Antilinear_map)

$$\begin{cases} v^*(f + g) = v^*(f) + v^*(g) & \text{additivity = superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \bar{\lambda} v^*(f) & \text{conjugate homogeneity = complex conjugate} \end{cases}$$

$$\begin{cases} v^*(f + g) = v^*(f) + v^*(g) & \text{additivity = superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \bar{\lambda} v^*(f) & \text{conjugate homogeneity = complex conjugate} \end{cases}$$

$$\Leftrightarrow \begin{cases} v^*(f + g) = \lambda^* v^*(f) + v^*(g) = \bar{\lambda} v^*(f) + v^*(g) & \text{antilinearity} \end{cases}$$

$$\begin{array}{ccccccc}
& V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \dots \} \\
\hat{f}^1 = \left( \begin{matrix} \hat{f}_1, \cdot \end{matrix} \right) & : & \begin{matrix} \downarrow \\ F \\ V \\ \vdots \end{matrix} & \supseteq \{ & \begin{matrix} (\hat{f}_1, \hat{f}_1) = 1 & (\hat{f}_1, \hat{f}_2) = 0 & (\hat{f}_1, \hat{f}_3) = 0 & (\hat{f}_1, f) = \dots \end{matrix} \\
\hat{f}^2 = \left( \begin{matrix} \hat{f}_2, \cdot \end{matrix} \right) & & & & \begin{matrix} \hat{f}_2 & \hat{f}_3 & f & \dots \end{matrix} \\
\hat{f}^3 = \left( \begin{matrix} \hat{f}_3, \cdot \end{matrix} \right) & & & & \begin{matrix} \hat{f}_3 & f & \dots \end{matrix} \\
f^* = (f, \cdot) & : & \begin{matrix} \downarrow \\ F \\ \vdots \end{matrix} & \supseteq \{ & \begin{matrix} (\hat{f}_1, \hat{f}_1) & (\hat{f}_1, \hat{f}_2) & (\hat{f}_1, \hat{f}_3) & (f, f) & \dots \end{matrix}
\end{array}$$

$$\begin{array}{ccccccc}
V & = \{ & \hat{f}_1 & & \hat{f}_2 & & \hat{f}_3 & & f & \cdots \} \\
\hat{f}^1 = \left\langle \hat{f}_1 \middle| \cdot \right\rangle & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{f}^2 = \left\langle \hat{f}_2 \middle| \cdot \right\rangle & } & F & \supseteq \{ & \left\langle \hat{f}_1 \middle| \hat{f}_1 \right\rangle = 1 & \left\langle \hat{f}_1 \middle| \hat{f}_2 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| \hat{f}_3 \right\rangle = 0 & \left\langle \hat{f}_1 \middle| f \right\rangle & \cdots \} \\
\hat{f}^3 = \left\langle \hat{f}_3 \middle| \cdot \right\rangle & } & V & = \{ & \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & f & \cdots \} \\
f^* = \langle f | \cdot \rangle & : & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \vdots & F & \supseteq \{ & \langle f | \hat{f}_1 \rangle & \langle f | \hat{f}_2 \rangle & \langle f | \hat{f}_3 \rangle & \langle f | f \rangle & \cdots \}
\end{array}$$

row vector is dual vector of column vector

$$f = \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow v^*(f) = \eta_f = (a^* \quad b^*) = (\bar{a} \quad \bar{b}) = \overline{\begin{pmatrix} a \\ b \end{pmatrix}}^\top = \overline{\begin{pmatrix} a \\ b \end{pmatrix}}^\top = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger$$

$$\eta_f \cdot f = (a^* \quad b^*) \begin{pmatrix} a \\ b \end{pmatrix} = a^*a + b^*b$$

<https://www.bilibili.com/video/BV1GN411m7A9/?t=2m40s>

[https://en.wikipedia.org/wiki/Riesz\\_representation\\_theorem](https://en.wikipedia.org/wiki/Riesz_representation_theorem)

**Theorem 45.5.** Riesz representation theorem

to be proved

<https://www.bilibili.com/video/BV1GN411m7A9/?t=6m20s>

$$\text{Hilbert space } V \left\{ \begin{array}{l} B = \left\{ \hat{f}_i \right\}_{i \in I} = \left\{ \dots, \hat{f}_i, \dots \right\} \\ \sum_i c_i \hat{f}_i = 0 \Rightarrow c_i = 0 \quad \text{bases are linear independent} \\ f = \sum_i c_i \hat{f}_i, c_i \in \mathbb{C} \quad \text{expansion over bases} \\ \left\{ \left( \hat{f}_i, \hat{f}_j \right) = \left\langle \hat{f}_i \middle| \hat{f}_j \right\rangle = \delta_{ij} \right. \\ \left. \text{completeness: } \forall \hat{f}_i \in B, \exists f \in V - B \left[ \left( \hat{f}_i, f \right) = \left\langle \hat{f}_i \middle| f \right\rangle = 0 \right] \right. \end{array} \right.$$

$$f = \sum_i \left( \hat{f}_i, f \right) \hat{f}_i = \sum_i \left( \hat{f}_i, \hat{f}_j \right) \hat{f}_i = \sum_i \left\langle \hat{f}_i \middle| \hat{f}_j \right\rangle \hat{f}_i$$

<https://www.bilibili.com/video/BV1GN411m7A9/?t=7m55s>

#### 45.5.1 Dirac bracket symmetry

$$f = \sum_i \left( \hat{f}_i, f \right) \hat{f}_i = \sum_i \hat{f}_i \left( \hat{f}_i, f \right)$$

$$f^* = (f, \cdot) = \left( \sum_i \left( \hat{f}_i, f \right) \hat{f}_i, \cdot \right) = \left( \sum_i \hat{f}_i \left( \hat{f}_i, f \right), \cdot \right)$$

$$\xrightarrow{\text{antilinearity}} \overline{\sum_i \left( \hat{f}_i, f \right)} \left( \hat{f}_i, \cdot \right) = \sum_i \overline{\left( \hat{f}_i, f \right)} \left( \hat{f}_i, \cdot \right)$$

$$= \sum_i \left( f, \hat{f}_i \right) \left( \hat{f}_i, \cdot \right)$$

$$\xrightarrow{\text{Dirac bracket}} \sum_i \left\langle f \middle| \hat{f}_i \right\rangle \left\langle \hat{f}_i \middle| \cdot \right\rangle = \langle f |$$

‡

$$f = \sum_i \hat{f}_i \left( \hat{f}_i, f \right)$$

$$\xrightarrow{\text{Dirac bracket}} \sum_i \left| \hat{f}_i \right\rangle \left\langle \hat{f}_i \middle| f \right\rangle = |f\rangle$$

$$\begin{cases} |f\rangle = \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| f \rangle = \sum_i \langle \hat{f}_i| f \rangle |\hat{f}_i\rangle \\ \langle f| = \sum_i \langle f| \hat{f}_i \rangle \langle \hat{f}_i| \\ \langle \hat{f}_i| \hat{f}_j \rangle = \delta_{ij} \end{cases}$$

$$V = \{ |\hat{f}_1\rangle, |\hat{f}_2\rangle, |\hat{f}_3\rangle, |f\rangle, \dots \}$$

$$V^* = \{ \langle \hat{f}_1|, \langle \hat{f}_2|, \langle \hat{f}_3|, \langle f|, \dots \}$$

$$\begin{array}{c|ccccc} V & = & |\hat{f}_1\rangle & |\hat{f}_2\rangle & |\hat{f}_3\rangle & |f\rangle & \dots \\ \hline \langle \hat{f}_1| & : \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \langle \hat{f}_2| & F & \supseteq \{ \langle \hat{f}_1| \hat{f}_1 \rangle = 1, \langle \hat{f}_1| \hat{f}_2 \rangle = 0, \langle \hat{f}_1| \hat{f}_3 \rangle = 0, \langle \hat{f}_1| f \rangle = 0, \dots \} \\ \langle \hat{f}_3| & V & = \{ |\hat{f}_1\rangle, |\hat{f}_2\rangle, |\hat{f}_3\rangle, |f\rangle, \dots \} \\ \langle f| & : \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \vdots & F & \supseteq \{ \langle f| \hat{f}_1 \rangle, \langle f| \hat{f}_2 \rangle, \langle f| \hat{f}_3 \rangle, \langle f| f \rangle, \dots \} \end{array}$$

$$\begin{cases} v^*(f+g) = v^*(f) + v^*(g) & \text{additivity = superposition} \\ v^*(\lambda f) = \lambda^* v^*(f) = \bar{\lambda} v^*(f) & \text{conjugate homogeneity = complex conjugate} \end{cases}$$

$$\Leftrightarrow \{ v^*(f+g) = \lambda^* v^*(f) + v^*(g) = \bar{\lambda} v^*(f) + v^*(g) \text{ antilinearity}$$

$$\begin{cases} (c\psi, \cdot) \rightarrow \langle c\psi| = c^* \langle \psi| & c\psi \rightarrow |c\psi\rangle = c|\psi\rangle \\ \langle \psi| \leftrightarrow |\psi\rangle \\ \langle \psi|\phi \rangle = \overline{\langle \phi|\psi \rangle} & \langle \phi|\psi \rangle = \overline{\langle \psi|\phi \rangle} \\ \langle c\psi|\phi \rangle = c^* \langle \psi|\phi \rangle & \langle \phi|c\psi \rangle = c \langle \phi|\psi \rangle \end{cases}$$

$|f\rangle \langle g| = ?$

$$|f\rangle \langle g| |h\rangle = |f\rangle \langle g|h\rangle = \langle g|h\rangle |f\rangle$$

$$\downarrow$$

$$|h\rangle \xrightarrow{|f\rangle \langle g|} |f\rangle \langle g| |h\rangle = \langle g|h\rangle |f\rangle$$

$$\langle h| |f\rangle \langle g| = \langle h|f\rangle \langle g|$$

$$\downarrow$$

$$\langle h| \xrightarrow{|f\rangle \langle g|} \langle h| |f\rangle \langle g| = \langle h|f\rangle \langle g|$$

$|f\rangle \langle g|$  is a linear transform or rank-2 tensor

$$\begin{aligned} |f\rangle &= \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| f \rangle \\ &= \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| |f\rangle \\ &= \left( \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \right) |f\rangle = 1 |f\rangle = |f\rangle \\ 1 &= \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \\ \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| &= 1 \end{aligned}$$

$$\sum_i |\hat{f}_i\rangle \langle \hat{f}_i| = 1$$

the above is completeness relationship.

---


$$\begin{cases} |f\rangle = 1|f\rangle = \left( \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \right) |f\rangle = \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| f \rangle \\ \langle f| = \langle f| 1 = \langle f| \left( \sum_i |\hat{f}_i\rangle \langle \hat{f}_i| \right) = \sum_i \langle f| \hat{f}_i \rangle \langle \hat{f}_i| \end{cases}$$


---

The sculpture is already complete within the marble block, before I start my work. It is already there, I just have to chisel away the superfluous material. Michelangelo

<https://www.goodreads.com/quotes/1191114-the-sculpture-is-already-complete-within-the-marble-block-before>

<https://www.bilibili.com/video/BV1y84y1U7Ps>

## 45.6 quantum operator

<https://www.bilibili.com/video/BV1za4y1F7xo>

## 45.7 Gram-Schmidt algorithm to find orthonormal basis

<https://www.bilibili.com/video/BV1eu4y1S77C>

## 45.8 Hermite polynomials and Legendre polynomials

<https://www.bilibili.com/video/BV1Lu411V7Rp>



# Chapter 46

## Lagrange inversion

### 46.1 Newton

<https://www.bilibili.com/video/BV1ge411e7K7>

### 46.2 Lagrange

Lagrange inversion theorem

拉格朗日反演

<https://www.bilibili.com/video/BV13V411L7XL>

<https://www.bilibili.com/video/BV1AM41167y2>



## **Chapter 47**

# **quantum color dynamics**

<https://www.youtube.com/watch?v=FL3ImtGcHqQ>



# Chapter 48

## Lagrangian

- Elliot Schneider: Physics with Elliot
  - Lagrangian fundamentals
  - Lagrangian mechanics

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458875>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458876>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458877>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492677/posts/2158458878>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458879>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458880>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458881>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492678/posts/2158458882>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458883>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458884>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458885>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492679/posts/2158458886>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458887>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458888>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458889>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492680/posts/2158458890>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2158458891>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912436>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912460>

<https://courses.physicswithelliot.com/products/fundamentals-of-lagrangian-mechanics/categories/2150492681/posts/2161912464>

<https://www.youtube.com/watch?v=0DHNGtsmmH8>

<https://www.youtube.com/watch?v=sUk9y23FPHk>

[https://www.youtube.com/watch?v=O0NYaO\\_OnH4](https://www.youtube.com/watch?v=O0NYaO_OnH4)

<https://www.youtube.com/watch?v=KVk1QNTWBx>

<https://www.youtube.com/watch?v=h2SEK6Jjv3Y>

<https://www.youtube.com/watch?v=KpIaWiWvuRs>

[https://www.youtube.com/watch?v=13hCkUiu\\_mi](https://www.youtube.com/watch?v=13hCkUiu_mi)

# Chapter 49

## Hamiltonian

- Elliot Schneider: Physics with Elliot
  - Hamiltonian mechanics

[https://www.youtube.com/watch?v=Nd4b0\\_vJZUk](https://www.youtube.com/watch?v=Nd4b0_vJZUk)

<https://www.youtube.com/watch?v=uncm8DChdhc>

[https://www.youtube.com/watch?v=\\_lz1VfI6Wxk](https://www.youtube.com/watch?v=_lz1VfI6Wxk)



# Chapter 50

## Fourier analysis

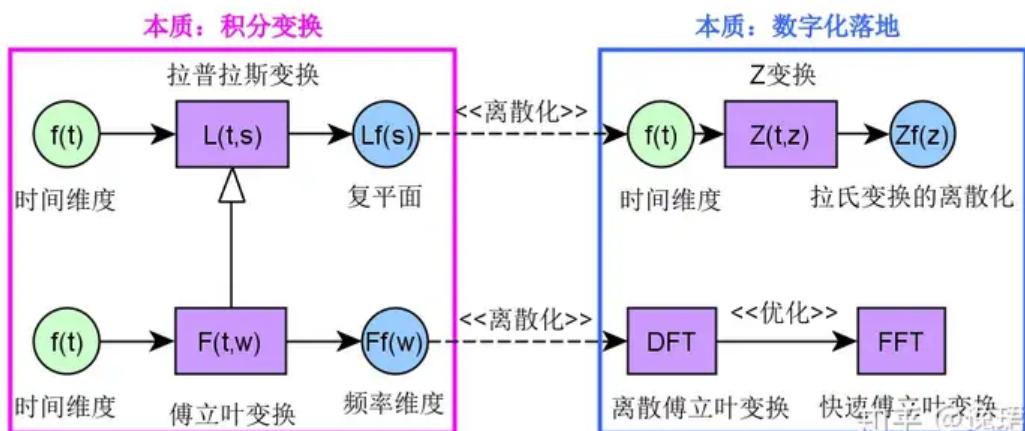


Figure 50.1: transform relationships: Fourier vs Laplace vs Z

<https://zhuanlan.zhihu.com/p/342952028>

<https://zhuanlan.zhihu.com/p/19763358>

### 50.1 basic calculation

$$\int_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(n \frac{2\pi}{\tau} t + \phi\right) dt = \left[ \frac{\sin(n \frac{2\pi}{\tau} t + \phi)}{n \frac{2\pi}{\tau}} \right]_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} = 0$$

$$\begin{aligned}\cos \alpha \cos \beta &= \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\ \sin \alpha \sin \beta &= \frac{-\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} \\ \sin \alpha \cos \beta &= \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} \\ \cos \alpha \sin \beta &= \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}\end{aligned}$$

$$\begin{aligned}
& \int \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} \int \cos \phi_0 dt & m = n = 0 \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos \phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n) + \cos(-\phi_n)}{2} dt & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos \phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n) + \cos \phi_n}{2} dt & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases} \\
&= \begin{cases} t \cos \phi_0 & m = n = 0 \\ \begin{cases} \int \frac{\cos(2n\frac{2\pi}{\tau}t + \phi_n)}{2} dt + \frac{t \cos \phi_n}{2} & m = n \\ \int \frac{\cos((m+n)\frac{2\pi}{\tau}t + \phi_n) + \cos((m-n)\frac{2\pi}{\tau}t - \phi_n)}{2} dt & m \neq n \end{cases} & mn \neq 0 \end{cases}
\end{aligned}$$

$$\cos \alpha \cos \beta = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$$

$$\sin \alpha \sin \beta = \frac{-\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$$

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\cos \alpha \sin \beta = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$$

$$\begin{aligned}
& \int_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} \tau \cos \phi_0 & m = n = 0 \\ \begin{cases} \frac{\tau \cos \phi_n}{2} & m = n \\ 0 & m \neq n \end{cases} & mn \neq 0 \end{cases} \\
&= \begin{cases} \begin{cases} \tau \cos \phi_0 & n = 0 \\ \frac{\tau \cos \phi_n}{2} & n \neq 0 \end{cases} & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{\tau \cos \phi_n} \int_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \int_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \sin\left(m\frac{2\pi}{\tau}t\right) \cos\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \int_{-\frac{\tau}{2}+\psi}^{\frac{\tau}{2}+\psi} \cos\left(m\frac{2\pi}{\tau}t\right) \sin\left(n\frac{2\pi}{\tau}t + \phi_n\right) dt \\
&= \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

$$\frac{2}{\tau \cos \phi_n} \int_{-\frac{\pi}{2} + \psi}^{\frac{\pi}{2} + \psi} \cos\left(m \frac{2\pi}{\tau} t\right) \cos\left(n \frac{2\pi}{\tau} t + \phi_n\right) dt = \begin{cases} \begin{cases} 2 & n = 0 \\ 1 & n \neq 0 \\ 0 & m \neq n \end{cases} & m = n \\ 0 & m \neq n \end{cases}$$

$$\frac{2}{\tau \cos \phi_n} \int_{-\frac{\pi}{2} + \psi}^{\frac{\pi}{2} + \psi} \sin\left(m \frac{2\pi}{\tau} t\right) \sin\left(n \frac{2\pi}{\tau} t + \phi_n\right) dt = \begin{cases} 1 & m = n \neq 0 \\ 0 & \neg(m = n \neq 0) \end{cases}$$

$$\int_{-\frac{\pi}{2} + \psi}^{\frac{\pi}{2} + \psi} \sin\left(m \frac{2\pi}{\tau} t\right) \cos\left(n \frac{2\pi}{\tau} t + \phi_n\right) dt = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_{-\frac{\pi}{2} + \psi}^{\frac{\pi}{2} + \psi} \cos\left(m \frac{2\pi}{\tau} t\right) \sin\left(n \frac{2\pi}{\tau} t + \phi_n\right) dt = \begin{cases} 0 & m = n \\ 0 & m \neq n \end{cases}$$

## 50.2 TaylorCatAlice: phase velocity vs. group velocity

<https://www.bilibili.com/video/BV1nV411F7es>

## 50.3 quantum mechanics or wave mechanics

Elliot Schneider: Physics with Elliot

<https://www.youtube.com/watch?v=W8QZ-yxebFA>

### 50.3.1 uncertainty principle

<https://www.youtube.com/watch?v=p7bzE1E5PMY>

## 50.4 Simon Xu

### 50.4.1 DFT = discrete Fourier transform

[https://www.youtube.com/watch?v=mkGsMWi\\_j4Q](https://www.youtube.com/watch?v=mkGsMWi_j4Q)

Reducible

<https://www.youtube.com/watch?v=yYEMxqreA10>

### 50.4.2 FFT = fast Fourier transform

<https://www.youtube.com/watch?v=htCj9exbGo0>

Reducible

<https://www.youtube.com/playlist?list=PLpXOY-RxVRTNBfxhIuqoZcWtg-JZKCktX>

<https://www.youtube.com/watch?v=h7apO7q16V0>

<https://www.youtube.com/watch?v=Ty0JcR6Dvis>

### 50.4.3 wavelet

<https://www.youtube.com/watch?v=ZnmvUCtUAEE>

Artem Kirsanov

<https://www.youtube.com/watch?v=jnxqHcObNK4>

### 50.4.3.1 compressed sensing MRI

Michael (Miki) Lustig

<https://www.youtube.com/watch?v=AP6JczMW8C8>

Tseng, Wen-Yih

<https://www.youtube.com/watch?v=P6tzQ9KQ9JQ&list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU&index=14>

[\(Michael Lustig\)](https://www.youtube.com/watch?v=hxdZdUQ6y2k&list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU&index=16)

## 50.5 sampling and reconstruction

<https://www.youtube.com/playlist?list=PLTp0eSi9MdkPtCLf1VxMWvUSI5JI8kAtz>

<https://www.youtube.com/watch?v=8CPPyE1rvMU>

<https://www.youtube.com/watch?v=Qd8fLSDwbQo>

<https://www.youtube.com/watch?v=0255KLvu75g>

## 50.6 signal and system

[https://www.youtube.com/playlist?list=PLX6FA3vfNTfChkbNQGxVPrIsvkC\\_DwNV6](https://www.youtube.com/playlist?list=PLX6FA3vfNTfChkbNQGxVPrIsvkC_DwNV6)

## 50.7 image compression / data compression

Reducible

<https://www.youtube.com/playlist?list=PLpXOY-RxVRTOR1PAtQUwoZN2tSs1ICSk7>

## 50.8 Lin, Chi-Kun

<https://www.youtube.com/playlist?list=PLj6E8qlqmkFuX5N1O3FKoDfoySC6Hku-2>

# Chapter 51

## axiom of choice

[https://www.youtube.com/watch?v=szfsGJ\\_PGQ0](https://www.youtube.com/watch?v=szfsGJ_PGQ0)

### 51.1 Zorn lemma

<https://zhuanlan.zhihu.com/p/670990411>



# **Chapter 52**

## **linear algebra**

### **52.1 The Art of Linear Algebra**

<https://github.com/kenjihiranabe/The-Art-of-Linear-Algebra>

<https://stackoverflow.com/questions/39173714/r-markdown-can-i-insert-a-pdf-to-the-r-markdown-file-as-an-image>

<https://stackoverflow.com/a/39177166>

<br>

# The Art of Linear Algebra

– Graphic Notes on “Linear Algebra for Everyone” –

Kenji Hiranabe \*

with the kindest help of Gilbert Strang †

translator: Kefang Liu ‡

traditionalization and rearrangement: Joey Yu Hsu, MD §

September 1, 2021/updated May 11, 2024

## Abstract

我嘗試為Gilbert Strang 在書籍“Linear Algebra for Everyone” 中介紹的矩陣的重要概念進形可視化圖釋，以促進從矩陣分解的角度對向量、矩陣計算和算法的理解。<sup>1</sup> 它們包括矩陣分解(Column-Row,  $CR$ )、高斯消去法(Gaussian Elimination,  $LU$ )、格拉姆-施密特正交化(Gram-Schmidt Orthogonalization,  $QR$ )、特徵值和對角化(Eigenvalues and Diagonalization,  $Q\Lambda Q^T$ )、和奇異值分解(Singular Value Decomposition,  $U\Sigma V^T$ )。

## 序言

我很高興能看到Kenji Hiranabe 的線性代數中的矩陣運算的圖片！這樣的圖片是展示代數的絕佳方式。我們當然可以通過列·行的點乘來想象矩陣乘法，但那絕非全部——它是“線性組合”與“秩1矩陣”組成的代數與藝術。我很感激能看到日文翻譯的書籍和Kenji 的圖片中的想法。

— Gilbert Strang  
麻省理工學院數學教授

\*twitter: @hiranabe, k-hiranabe@esm.co.jp, <https://anagileway.com>

†Massachusetts Institute of Technology, <http://www-math.mit.edu/~gs/>

‡twitter: @kfchliu, 微博用戶: 5717297833

§redwing1304@gmail.com

<sup>1</sup>“Linear Algebra for Everyone”: <http://math.mit.edu/everyone/>.

&lt;br&gt;

## 52.2 CCJou

[https://www.youtube.com/playlist?list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAc3t\\_Ju](https://www.youtube.com/playlist?list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju)

### 52.2.1 coordinate

[https://www.youtube.com/watch?v=eMUFexQsKXA&list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAc3t\\_Ju&index=20](https://www.youtube.com/watch?v=eMUFexQsKXA&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=20)

$$\begin{aligned}
 \mathbf{v} &= v^1 \mathbf{v}_1 + v^2 \mathbf{v}_2 + \cdots + v^n \mathbf{v}_n \\
 &= v^1 \mathbf{v}_1 + \cdots + v^n \mathbf{v}_n \\
 &= c^1 \beta_1 + \cdots + c^n \beta_n = c_1 \beta_1 + \cdots + c_n \beta_n = B\mathbf{c} = B[\mathbf{v}]_B \\
 &= e^1 \hat{\beta}_1 + \cdots + e^n \hat{\beta}_n = e_1 \hat{\beta}_1 + \cdots + e_n \hat{\beta}_n \\
 &= e^1 \hat{\mathbf{e}}_1 + \cdots + e^n \hat{\mathbf{e}}_n = e_1 \hat{\mathbf{e}}_1 + \cdots + e_n \hat{\mathbf{e}}_n \\
 &\stackrel{\text{e.g.}}{=} e^1 \begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} + \cdots + e^n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} = e_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + e_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}_{n \times n} \begin{pmatrix} e^1 \\ \vdots \\ e^j \\ \vdots \\ e^n \end{pmatrix}_{n \times 1} = I_{n \times n} \mathbf{e} = I\mathbf{e}
 \end{aligned}$$

$$\begin{aligned}
B\mathbf{c} &= c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^j \begin{pmatrix} | \\ \boldsymbol{\beta}_j \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^j \\ \vdots \\ c^n \end{pmatrix} \\
&= c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^j \begin{pmatrix} | \\ \boldsymbol{\beta}_j \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^j \\ \vdots \\ c^n \end{pmatrix} \\
&= c^1 \boldsymbol{\beta}_1 + c^2 \boldsymbol{\beta}_2 + \cdots + c^n \boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + c^2 \begin{pmatrix} | \\ \boldsymbol{\beta}_2 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & | & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^2 \\ \vdots \\ c^n \end{pmatrix} \\
&= c^1 \boldsymbol{\beta}_1 + c^2 \boldsymbol{\beta}_2 + \cdots + c^n \boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + c^2 \begin{pmatrix} | \\ \boldsymbol{\beta}_2 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & | & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^2 \\ \vdots \\ c^n \end{pmatrix} \\
&= c^1 \boldsymbol{\beta}_1 + \cdots + c^n \boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix} = B [\mathbf{v}]_B \\
&= c^1 \boldsymbol{\beta}_1 + \cdots + c^n \boldsymbol{\beta}_n = c^1 \begin{pmatrix} | \\ \boldsymbol{\beta}_1 \\ | \end{pmatrix} + \cdots + c^n \begin{pmatrix} | \\ \boldsymbol{\beta}_n \\ | \end{pmatrix} = \begin{pmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix} = \begin{pmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \\
B &= \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_j & \cdots & \boldsymbol{\beta}_n \\ | & & | & & | \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_n \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_n \\ | & & | \end{pmatrix} \\
[\mathbf{v}]_B &= \mathbf{c} = \begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \in F^n = \mathbb{F}^n \in \{\mathbb{R}^n, \mathbb{C}^n, \dots\}
\end{aligned}$$


---

$$\mathbf{v} = B [\mathbf{v}]_B$$


---

$$\begin{aligned}
\mathbf{v} &= B [\mathbf{v}]_B \\
B^{-1} \mathbf{v} &= B^{-1} B [\mathbf{v}]_B = I [\mathbf{v}]_B = [\mathbf{v}]_B \\
[\mathbf{v}]_B &= B^{-1} \mathbf{v}
\end{aligned}$$


---

$$[\mathbf{v}]_B = B^{-1} \mathbf{v}$$


---

$$[\mathbf{v}]_B = B^{-1} \mathbf{v} \in F^n \in \{\mathbb{R}^n, \mathbb{C}^n, \dots\}$$


---

$$\mathbf{v} = B [\mathbf{v}]_B = B' [\mathbf{v}]_{B'}$$


---

$$\begin{aligned}
B[\mathbf{v}]_B &= B'[\mathbf{v}]_{B'} \\
[\mathbf{v}]_B = I[\mathbf{v}]_B &= B^{-1}B[\mathbf{v}]_B = B^{-1}B'[\mathbf{v}]_{B'} \\
&= B^{-1}B'[\mathbf{v}]_{B'} \\
&= B'[\mathbf{v}]_{B'} = B[\mathbf{v}]_B \\
[\mathbf{v}]_{B'} = I[\mathbf{v}]_{B'} &= B'^{-1}B'[\mathbf{v}]_{B'} = B'^{-1}B[\mathbf{v}]_B \\
&= B'^{-1}B[\mathbf{v}]_B
\end{aligned}$$

### 52.2.2 linear transformation

$$\begin{aligned}
\mathbf{v} &= v^1 \mathbf{v}_1 + \cdots + v^n \mathbf{v}_n = v^j \mathbf{v}_j = \sum_{j=1}^n v^j \mathbf{v}_j \\
&= v^1 \begin{bmatrix} | \\ \mathbf{v}_1 \\ | \end{bmatrix} + \cdots + v^n \begin{bmatrix} | \\ \mathbf{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = V[\mathbf{v}]_V \\
&, \begin{cases} V = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}, \mathfrak{V} = \{\mathbf{v}_j\}_{j=1}^n = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \\ [v]_V = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in F^n \in \{\mathbb{R}^n, \mathbb{C}^n, \dots\} \end{cases} \quad \text{if } V \text{ invertible} \implies [\mathbf{v}]_V = V^{-1}\mathbf{v}
\end{aligned}$$

\mathfrak{V} = \mathfrak{V} \text{ vs. } \mathfrak{B} = \mathfrak{B}

$$\begin{aligned}
\mathbf{v} \in \mathcal{V} &\xrightarrow{T} \mathcal{W} \ni \mathbf{w} = T(\mathbf{v}) \\
T : \mathcal{V} \rightarrow \mathcal{W} &\begin{cases} T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) & (a) \text{ additivity} \\ T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) & (h) \text{ homogeneity} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathbf{w} = T(\mathbf{v}) &= T(v^1 \mathbf{v}_1 + \cdots + v^n \mathbf{v}_n) \stackrel{(a)}{=} T(v^1 \mathbf{v}_1) + \cdots + T(v^n \mathbf{v}_n) \stackrel{(h)}{=} v^1 T(\mathbf{v}_1) + \cdots + v^n T(\mathbf{v}_n) \\
&= v^1 \begin{bmatrix} | \\ T(\mathbf{v}_1) \\ | \end{bmatrix} + \cdots + v^n \begin{bmatrix} | \\ T(\mathbf{v}_n) \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix} [\mathbf{v}]_V \\
\mathbf{w} = T(\mathbf{v}) &= \underbrace{\begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix}}_{[T]_V} [\mathbf{v}]_V = T(V)[\mathbf{v}]_V, T(V) = \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix}
\end{aligned}$$

$$\mathbf{w} = T(\mathbf{v}) \xrightarrow{\mathbf{v} = V[\mathbf{v}]_V} T(V[\mathbf{v}]_V) \xrightarrow{\text{linear}} T(V)[\mathbf{v}]_V$$


---

$$\begin{aligned}
T(\mathbf{v}) = \mathbf{w} &= w^1 \mathbf{w}_1 + \cdots + w^m \mathbf{w}_m = w^j \mathbf{w}_j = \sum_{j=1}^n w^j \mathbf{w}_j \\
&= w^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + w^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = W[\mathbf{w}]_W \xrightarrow{\mathbf{w} = T(\mathbf{v})} W[T(\mathbf{v})]_W \\
&, \begin{cases} W = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix}, \mathfrak{W} = \{\mathbf{w}_j\}_{j=1}^m = \{\mathbf{w}_1, \dots, \mathbf{w}_m\} \\ [w]_W = \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} \in F^m \in \{\mathbb{R}^m, \mathbb{C}^m, \dots\} \end{cases} \quad \text{if } W \text{ invertible} \implies [\mathbf{w}]_W = W^{-1}\mathbf{w}
\end{aligned}$$

\mathfrak{W} vs. \mathfrak{V}

$$\begin{aligned}
T(\mathbf{v}_1) &= t_1^1 \mathbf{w}_1 + \cdots + t_1^m \mathbf{w}_m = t_1^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + t_1^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\mathbf{v}_1)]_W \\ | \end{bmatrix} = W[T(\mathbf{v}_1)]_W \\
&\quad \vdots
\end{aligned}$$

$$\begin{aligned}
T(\mathbf{v}_n) &= t_n^1 \mathbf{w}_1 + \cdots + t_n^m \mathbf{w}_m = t_n^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + t_n^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\mathbf{v}_n)]_W \\ | \end{bmatrix} = W[T(\mathbf{v}_n)]_W
\end{aligned}$$

$$\begin{aligned}
T(V) &= \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} & \cdots & \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{bmatrix} = WT \\
&, \begin{cases} W = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \\ T = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{bmatrix} \end{cases}
\end{aligned}$$

$$T(V) = \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix} = WT, \begin{cases} W = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \\ T = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{bmatrix} \end{cases}$$

$$T = \begin{bmatrix} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{bmatrix} \stackrel{V=\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}}{=} [T(V)]_W$$


---

$$\begin{aligned}
T = [T(V)]_W &= W^{-1}T(V) = \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{array} \right]^{-1} T(V), \text{ if } W \text{ invertible} \\
&= \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{array} \right]^{-1} T \left( \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{array} \right] \right) \\
&= \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{array} \right]^{-1} \left[ \begin{array}{ccc|c} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{array} \right] \\
&= W^{-1} \left[ \begin{array}{ccc|c} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} | & & | \\ W^{-1}T(\mathbf{v}_1) & \cdots & W^{-1}T(\mathbf{v}_n) \\ | & & | \end{array} \right] \\
&= \left[ \begin{array}{ccc|c} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{array} \right]
\end{aligned}$$


---

$$T = [T(V)]_W = \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{array} \right]^{-1} \left[ \begin{array}{ccc|c} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{array} \right] = W^{-1}T(V)$$


---

$$\begin{aligned}
T &= \left[ \begin{array}{ccc|c} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{array} \right] = \left[ \begin{array}{ccc|c} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{array} \right] = [T(V)]_W \\
&= \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{t}_1 & \cdots & \mathbf{t}_n \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} | & & | \\ [T(\mathbf{v}_1)]_W & \cdots & [T(\mathbf{v}_n)]_W \\ | & & | \end{array} \right], \mathbf{t}_j = [T(\mathbf{v}_j)]_W
\end{aligned}$$

$$\begin{aligned}
W[\mathbf{w}]_W &= \mathbf{w} = T(\mathbf{v}) = T(V)[\mathbf{v}]_V \stackrel{T(V)=WT}{=} WT[\mathbf{v}]_V = W(T[\mathbf{v}]_V) \\
W[\mathbf{w}]_W &= W(T[\mathbf{v}]_V) \\
[\mathbf{w}]_W &= I[\mathbf{w}]_W = W^{-1}W[\mathbf{w}]_W = W^{-1}W(T[\mathbf{v}]_V) = IT[\mathbf{v}]_V = T[\mathbf{v}]_V, \text{ if } W \text{ invertible} \\
&= T[\mathbf{v}]_V \\
[T(\mathbf{v})]_W &\stackrel{\mathbf{w}=T(\mathbf{v})}{=} [\mathbf{w}]_W = T[\mathbf{v}]_V \\
&= T[\mathbf{v}]_V \stackrel{T=[T(V)]_W}{=} [T(V)]_W[\mathbf{v}]_V
\end{aligned}$$


---

$$\begin{aligned}
[\mathbf{w}]_W &= T[\mathbf{v}]_V \\
&= [T(\mathbf{v})]_W = [T(V)]_W[\mathbf{v}]_V
\end{aligned}$$


---

$$\begin{aligned}
T &= \left[ \begin{array}{ccc|c} | & & | \\ \mathbf{t}_1 & \cdots & \mathbf{t}_n \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} \left[ \begin{array}{c} t_1^1 \\ \vdots \\ t_1^m \end{array} \right] & \cdots & \left[ \begin{array}{c} t_n^1 \\ \vdots \\ t_n^m \end{array} \right] \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{array} \right], \quad [\mathbf{w}]_W = T[\mathbf{v}]_V \stackrel{\Updownarrow}{=} \\
&= \left[ \begin{array}{ccc|c} \left[ \begin{array}{c} t^1_1 \\ \vdots \\ t^m_1 \end{array} \right] & \cdots & \left[ \begin{array}{c} t^1_n \\ \vdots \\ t^m_n \end{array} \right] \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} t^1_1 & \cdots & t^1_n \\ \vdots & \ddots & \vdots \\ t^m_1 & \cdots & t^m_n \end{array} \right] = [t^i_j]_{m \times n} = [t^i_j] = t^i_j \\
&= \left[ \begin{array}{ccc|c} \left[ \begin{array}{c} t_{11} \\ \vdots \\ t_{m1} \end{array} \right] & \cdots & \left[ \begin{array}{c} t_{1n} \\ \vdots \\ t_{mn} \end{array} \right] \\ | & & | \end{array} \right] = \left[ \begin{array}{ccc|c} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mn} \end{array} \right] = [t_{ij}]_{m \times n} = [t_{ij}] = t_{ij}
\end{aligned}$$


---

$$T = \begin{bmatrix} | & & | \\ \mathbf{t}_1 & \cdots & \mathbf{t}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} t^1_1 & \cdots & t^1_n \\ \vdots & \ddots & \vdots \\ t^m_1 & \cdots & t^m_n \end{bmatrix} = [t^i_j]_{m \times n} = t^i_j$$


---

$$\begin{aligned} T(\mathbf{v}_1) &= t_1^1 \mathbf{w}_1 + \cdots + t_1^m \mathbf{w}_m = t_1^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + t_1^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 \\ \vdots \\ t_1^m \end{bmatrix} \\ &= W[T(\mathbf{v}_1)]_W \\ &\quad \vdots \\ T(\mathbf{v}_n) &= t_n^1 \mathbf{w}_1 + \cdots + t_n^m \mathbf{w}_m = t_n^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + t_n^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_n^1 \\ \vdots \\ t_n^m \end{bmatrix} \\ &= W[T(\mathbf{v}_n)]_W \\ &\quad \Downarrow \\ \begin{bmatrix} | & & | \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ | & & | \end{bmatrix} &= \begin{array}{c} t_1^1 \\ \vdots \\ t_n^1 \end{array} \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots \curvearrowright \begin{array}{c} t_1^m \\ \vdots \\ t_n^m \end{array} \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | & \cdots & | \\ \mathbf{t}_1 & \cdots & \mathbf{t}_n \\ | & & | \end{bmatrix} = WT \\ &= \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ [T(\mathbf{v}_1)]_W \\ | \end{bmatrix} \cdots \begin{bmatrix} | \\ [T(\mathbf{v}_n)]_W \\ | \end{bmatrix} = W[T(V)]_W \\ T(V) &= WT = W[T(V)]_W \end{aligned}$$


---

$$\begin{aligned} WT &= T(V) \\ &\quad \Downarrow \\ T &= W^{-1}T(V) \end{aligned}$$


---

$$T = \begin{bmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & \ddots & \vdots \\ t_1^m & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} t_1^1 & \cdots & t_1^m \\ \vdots & \ddots & \vdots \\ t_n^1 & \cdots & t_n^m \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{t}_1 & \cdots & \mathbf{t}_n \\ | & & | \end{bmatrix} = [t_j^i]_{m \times n} = [t_j^i]_{m \times n} = t_j^i$$


---

$$[t_j^i]_{m \times n} = [t_j^i]_{m \times n} = [t_j^i]_{m \times n}, \begin{cases} w^i = t_j^i v^j = t_j^i v^j \\ T(\mathbf{v}_j) = t_j^i \mathbf{w}_i = t_j^i \mathbf{w}_i \end{cases}$$


---

$T = [t^i_j]_{m \times n} = t^i_j$  is the matrix representation of the linear transformation  $T(\cdot) : \mathcal{V} \rightarrow \mathcal{W}$

---

### 52.2.3 change of basis

[https://www.youtube.com/watch?v=WAtLPk55ljM&list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAc3t\\_Ju&index=22](https://www.youtube.com/watch?v=WAtLPk55ljM&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAc3t_Ju&index=22)

$$\mathbf{v} = B[\mathbf{v}]_B = B'[\mathbf{v}]_{B'}$$


---

$$\begin{array}{ccc}
\{\mathbf{v}_j\}_{j=1}^n = \mathfrak{V} \subseteq \mathcal{V} & \xrightarrow{T(\cdot)} & \mathcal{W} \quad \supseteq \mathfrak{W} = \{\mathbf{w}_j\}_{j=1}^m \\
& \Downarrow & \Downarrow \\
V[\mathbf{v}]_V = v^j \mathbf{v}_j = \mathbf{v} & \xleftarrow[T(\cdot)]{T^{-1}(\cdot)} & \mathbf{w} = w^j \mathbf{w}_j = W[\mathbf{w}]_W = T(\mathbf{v}) = T(V)[\mathbf{v}]_V \\
VT^{-1}W^{-1}\mathbf{w} = V^{-1} & \downarrow & \downarrow W \\
V^{-1}\mathbf{v} = [\mathbf{v}]_V & \xleftarrow[T]{T^{-1}} & [\mathbf{w}]_W = W^{-1}\mathbf{w} = [T(\mathbf{v})]_W = T[\mathbf{v}]_V \\
& \Downarrow & \Downarrow \\
t_j = [T(\mathbf{v}_j)]_W & \mathbb{F}^n & \xrightarrow{T} \mathbb{F}^m \\
& \Downarrow & \Downarrow \\
& & T = [T(V)]_W
\end{array}$$

$\{\mathbb{R}^n, \mathbb{C}^m, \dots\}$     $\{\mathbb{R}^m, \mathbb{C}^m, \dots\}$

Figure 52.1: coordinate under linear transformation

$$\begin{aligned}
B[\mathbf{v}]_B &= B'[\mathbf{v}]_{B'} \\
[\mathbf{v}]_B = I[\mathbf{v}]_B &= B^{-1}B[\mathbf{v}]_B = B^{-1}B'[\mathbf{v}]_{B'} \\
&= B^{-1}B'[\mathbf{v}]_{B'} \\
B'[\mathbf{v}]_{B'} &= B[\mathbf{v}]_B \\
[\mathbf{v}]_{B'} = I[\mathbf{v}]_{B'} &= B'^{-1}B'[\mathbf{v}]_{B'} = B'^{-1}B[\mathbf{v}]_B \\
&= B'^{-1}B[\mathbf{v}]_B
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} &= v^1 \mathbf{v}_1 + \cdots + v^n \mathbf{v}_n = v^j \mathbf{v}_j = \sum_{j=1}^n v^j \mathbf{v}_j = v^1 \begin{bmatrix} | \\ \mathbf{v}_1 \\ | \end{bmatrix} + \cdots + v^n \begin{bmatrix} | \\ \mathbf{v}_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = V[\mathbf{v}]_V \\
&= v'^1 \mathbf{v}'_1 + \cdots + v'^n \mathbf{v}'_n = v'^j \mathbf{v}'_j = \sum_{j=1}^n v'^j \mathbf{v}'_j = v'^1 \begin{bmatrix} | \\ \mathbf{v}'_1 \\ | \end{bmatrix} + \cdots + v'^n \begin{bmatrix} | \\ \mathbf{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \\ | & & | \end{bmatrix} \begin{bmatrix} v'^1 \\ \vdots \\ v'^n \end{bmatrix} = V'[\mathbf{v}]_{V'}
\end{aligned}$$

$$\begin{aligned}
\mathbf{w} &= w^1 \mathbf{w}_1 + \cdots + w^m \mathbf{w}_m = w^j \mathbf{w}_j = \sum_{j=1}^m w^j \mathbf{w}_j = w^1 \begin{bmatrix} | \\ \mathbf{w}_1 \\ | \end{bmatrix} + \cdots + w^m \begin{bmatrix} | \\ \mathbf{w}_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_m \\ | & & | \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = W[\mathbf{w}]_W \\
&= w'^1 \mathbf{w}'_1 + \cdots + w'^m \mathbf{w}'_m = w'^j \mathbf{w}'_j = \sum_{j=1}^m w'^j \mathbf{w}'_j = w'^1 \begin{bmatrix} | \\ \mathbf{w}'_1 \\ | \end{bmatrix} + \cdots + w'^m \begin{bmatrix} | \\ \mathbf{w}'_m \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{w}'_1 & \cdots & \mathbf{w}'_m \\ | & & | \end{bmatrix} \begin{bmatrix} w'^1 \\ \vdots \\ w'^m \end{bmatrix} = W'[\mathbf{w}]_{W'}
\end{aligned}$$

$$\mathbf{w} = W[\mathbf{w}]_W = W'[\mathbf{w}]_{W'}$$

$$\begin{aligned}
\mathbf{v}_1 &= v_1'^1 \mathbf{v}'_1 + \cdots + v_1'^n \mathbf{v}'_n = v_1'^1 \begin{bmatrix} | \\ \mathbf{v}'_1 \\ | \end{bmatrix} + \cdots + v_1'^n \begin{bmatrix} | \\ \mathbf{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} v_1'^1 \\ \vdots \\ v_1'^n \end{bmatrix} = V' [\mathbf{v}_1]_{V'} \\
&\vdots \\
\mathbf{v}_n &= v_n'^1 \mathbf{v}'_1 + \cdots + v_n'^n \mathbf{v}'_n = v_n'^1 \begin{bmatrix} | \\ \mathbf{v}'_1 \\ | \end{bmatrix} + \cdots + v_n'^n \begin{bmatrix} | \\ \mathbf{v}'_n \\ | \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} v_n'^1 \\ \vdots \\ v_n'^n \end{bmatrix} = V' [\mathbf{v}_n]_{V'} \\
&\Downarrow \\
\begin{bmatrix} | & \cdots & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & \cdots & | \end{bmatrix} &= \begin{bmatrix} v_1'^1 & & & v_1'^n & \\ \vdots & \mathbf{v}'_1 & \cdots & \vdots & \mathbf{v}'_n \\ v_n'^1 & & & v_n'^n & \end{bmatrix} \\
&= \begin{bmatrix} | & \cdots & | \\ \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} v_1'^1 & \cdots & v_n'^1 \\ \vdots & \ddots & \vdots \\ v_1'^n & \cdots & v_n'^n \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} [\mathbf{v}_1]_{V'} & \cdots & [\mathbf{v}_n]_{V'} \\ | & \cdots & | \end{bmatrix} = V' [V]_{V'} \\
V &= V' [V]_{V'} \xrightarrow{\text{if } V' \text{ invertible}} [V]_{V'} = V'^{-1} V
\end{aligned}$$


---

$$[V]_{V'} = V'^{-1} V : [V' | V] \xrightarrow{\text{Gauss-Jordan delimitation}} [I | V'^{-1} V] = [I | [V]_{V'}]$$

$$[V']_V = V^{-1} V : [V | V'] \xrightarrow{\text{Gauss-Jordan delimitation}} [I | V^{-1} V'] = [I | [V']_V]$$


---

$$\begin{cases} [V]_{V'} = V'^{-1} V : [V' | V] \xrightarrow{\text{Gauss-Jordan delimitation}} [I | V'^{-1} V] = [I | [V]_{V'}], \\ [V']_V = V^{-1} V : [V | V'] \xrightarrow{\text{Gauss-Jordan delimitation}} [I | V^{-1} V'] = [I | [V']_V] \end{cases}$$


---

$$\begin{aligned}
\mathbf{v} &= V [\mathbf{v}]_V = V' [\mathbf{v}]_{V'} \\
&= V [\mathbf{v}]_V, \wedge V = V' [V]_{V'} \\
&= V' [V]_{V'} [\mathbf{v}]_V \\
V' [\mathbf{v}]_{V'} &= V' [V]_{V'} [\mathbf{v}]_V \\
[\mathbf{v}]_{V'} &= [V]_{V'} [\mathbf{v}]_V
\end{aligned}$$


---

$$[\mathbf{v}]_{V'} = [V]_{V'} [\mathbf{v}]_V$$


---

symmetrically,

$$[\mathbf{v}]_V = [V']_V [\mathbf{v}]_{V'}$$


---

$$[\mathbf{v}]_V \rightleftharpoons [\mathbf{v}]_{V'}$$

$$\begin{aligned}
[\mathbf{v}]_V &\xrightarrow{[V]_{V'}} [\mathbf{v}]_{V'} = [V]_{V'} [\mathbf{v}]_V \\
[\mathbf{v}]_V &\xleftarrow{[V']_V} [\mathbf{v}]_{V'} = [V']_V^{-1} [\mathbf{v}]_V \\
&\Downarrow \\
[V]_{V'} &= [V']_V^{-1} \Leftrightarrow [V']_V = [V]_{V'}^{-1}
\end{aligned}$$

$$[\mathbf{v}]_V \rightarrow [\mathbf{v}]_{V'}$$

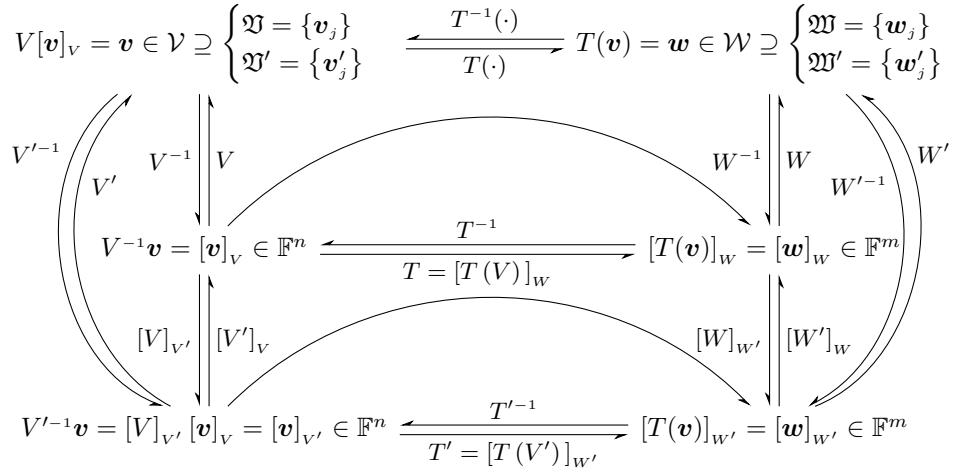


Figure 52.2: change of coordinate basis under linear transformation

$$\begin{aligned}
 & [\mathbf{v}]_V \xrightarrow{[V]_{V'}} [\mathbf{v}]_{V'} = [V]_{V'} [\mathbf{v}]_V \\
 & [\mathbf{v}]_V \xrightarrow{V} \mathbf{v} \xrightarrow{V'^{-1}} [\mathbf{v}]_{V'} = V'^{-1}V [\mathbf{v}]_V \\
 & \Downarrow \\
 & [V]_{V'} = V'^{-1}V \xrightleftharpoons{[V']_{V'} = [V]_{V'}^{-1}} [V']_V = (V'^{-1}V)^{-1} = V^{-1}V'
 \end{aligned}$$

$$[\mathbf{v}]_{V'} \rightarrow [\mathbf{w}]_{W'}$$

$$\begin{aligned}
 & [\mathbf{v}]_{V'} \xrightarrow{T'=[T(V')]_{W'}} [\mathbf{w}]_{W'} = [T(V')]_{W'} [\mathbf{v}]_{V'} \\
 & [\mathbf{v}]_{V'} \xrightarrow{[V']_V} [\mathbf{v}]_V \xrightarrow{T=[T(V)]_W} [\mathbf{w}]_W \xrightarrow{[W]_{W'}} [\mathbf{w}]_{W'} = [W]_{W'} [T(V)]_W [V']_V [\mathbf{v}]_{V'} \\
 & \Downarrow \\
 & [T(V')]_{W'} = [W]_{W'} [T(V)]_W [V']_V \\
 & = W'^{-1}W [T(V)]_W V^{-1}V' \\
 & = W'^{-1}WW^{-1}T(V)V^{-1}V' \\
 & = W'^{-1}IT(V)V^{-1}V' \\
 & = W'^{-1}T(V)V^{-1}V'
 \end{aligned}$$

$$[T(V')]_{W'} \rightleftharpoons [T(V)]_W$$

$$[T(V')]_{W'} = [W]_{W'} [T(V)]_W [V']_V$$

$$[T(V')]_{W'} = W'^{-1}T(V)V^{-1}V'$$

$$\begin{aligned}
 & W' [T(V)]_{W'} = T(V) \\
 & \Downarrow \\
 & [T(V)]_{W'} = W'^{-1}T(V)
 \end{aligned}$$

$$\begin{aligned}
 [T(V')]_{w'} &= W'^{-1}T(V)V^{-1}V' \\
 &= \{W'^{-1}T(V)\}\{V^{-1}V'\} \\
 &= \{[T(V)]_{w'}\}\{V^{-1}V'\} \\
 &= [T(V)]_{w'} V^{-1}V' \\
 &, \begin{cases} [T(V)]_{w'} = W'^{-1}T(V) & : W'[T(V)]_{w'} = T(V) \\ V^{-1}V' & : [V|V'] \xrightarrow{\text{Gauss-Jordan delimitation}} [I|V^{-1}V'] \end{cases}
 \end{aligned}$$


---

$T' \rightleftharpoons T$

$$\begin{aligned}
 [T(V')]_{w'} &= [W]_{w'} [T(V)]_w [V']_v \\
 T' &= [T(V')]_{w'} \Updownarrow T = [T(V)]_w \\
 T' &= [W]_{w'} T [V']_v \\
 &= [W'^{-1}W] T [V^{-1}V'] \\
 T' &= W'^{-1}WT V^{-1}V'
 \end{aligned}$$


---

$$T' = W'^{-1}WT V^{-1}V'$$

### 52.2.3.1 differential polynomial

#### 52.2.3.1.1 Chebyshev polynomial

#### 52.2.3.1.2 Hermite polynomial

### 52.2.4 singular value decomposition

[https://www.youtube.com/watch?v=oPHM-ZWWvlg&list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAC3t\\_Ju&index=42](https://www.youtube.com/watch?v=oPHM-ZWWvlg&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAC3t_Ju&index=42)

[https://www.youtube.com/watch?v=zagHmMPZyoo&list=PLP-JUp2VR1LsFtHT-i\\_vZ3oNFIAC3t\\_Ju&index=43](https://www.youtube.com/watch?v=zagHmMPZyoo&list=PLP-JUp2VR1LsFtHT-i_vZ3oNFIAC3t_Ju&index=43)

## 52.3 Chi, Chen-Yu

<https://www.youtube.com/playlist?list=PLJWAeYEa8SXBej3kuQMz8vV41VabZUILb>

# Chapter 53

## analysis

### 53.1 real analysis

<https://zhuanlan.zhihu.com/p/665222184>

### 53.2 Chi, Chen-Yu

<https://www.youtube.com/playlist?list=PLVJXJebpO4PhAc21JW-cYbzT3sq4s7Qg8>

<https://www.youtube.com/playlist?list=PLil-R4o6jmGihq7XzdNzb0d5hHqEJbr6L>

[https://www.youtube.com/playlist?list=PLil-R4o6jmGhUqtKbZf0LIFKd-xN\\_g\\_M](https://www.youtube.com/playlist?list=PLil-R4o6jmGhUqtKbZf0LIFKd-xN_g_M)

[https://www.youtube.com/playlist?list=PLil-R4o6jmGhkuZPmKL\\_A5Y7N4HOsaInX](https://www.youtube.com/playlist?list=PLil-R4o6jmGhkuZPmKL_A5Y7N4HOsaInX)

### 53.3 Chen, Jin-Tzu

<https://www.youtube.com/playlist?list=PLil-R4o6jmGjoxAWZurHXAY0q9yxwXv5F>

### 53.4 $\sigma$ -algebra

<https://www.zhihu.com/question/68135067/answer/259982491>

### 53.5 complex analysis

#### 53.5.1 Mathemaniac

[https://www.youtube.com/playlist?list=PLDcSwjT2BF\\_UDdkQ3KQjX5SRQ2DLLwv0R](https://www.youtube.com/playlist?list=PLDcSwjT2BF_UDdkQ3KQjX5SRQ2DLLwv0R)



# Chapter 54

## matrix calculus

14

[https://www.youtube.com/playlist?list=PLhcN-s3\\_Z7-YS6ltpJhjwqvHO1TYDbiZv](https://www.youtube.com/playlist?list=PLhcN-s3_Z7-YS6ltpJhjwqvHO1TYDbiZv)

$$\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle = [x_1 \ x_2 \ \dots \ x_n]^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(x_1, x_2, \dots, x_n) = f(\langle x_1, x_2, \dots, x_n \rangle) = f(\mathbf{x})$$

$$\mathbf{y} = \langle y_1, y_2, \dots, y_m \rangle = [y_1 \ y_2 \ \dots \ y_m]^\top = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

### 54.1 vector-by-scalar

$$\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$

### 54.2 scalar-by-vector

$$\nabla f = \frac{\partial}{\partial \mathbf{x}} f = \frac{\partial f}{\partial \mathbf{x}} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}^\top = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$f_i = f_i(x_1, x_2, \dots, x_n) = f_i(\mathbf{x})$$

$$\mathbf{f} = \langle f_1, f_2, \dots, f_m \rangle = [f_1 \ f_2 \ \dots \ f_m]^\top = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

## 54.3 vector-by-vector

### 54.3.1 numerator-layout notation

分子布局

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} \\ \frac{\partial y_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial y_p}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} & \frac{\partial \mathbf{y}}{\partial x_2} & \cdots & \frac{\partial \mathbf{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_i}{\partial x_j} \end{bmatrix}_{m \times n}$$

### 54.3.2 denominator-layout notation

分母布局

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} & \frac{\partial y_2}{\partial \mathbf{x}} & \cdots & \frac{\partial y_m}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_j}{\partial x_i} \end{bmatrix}_{n \times m}$$

<https://zhuanlan.zhihu.com/p/692195114>

# Chapter 55

## genetics

楊欣洲: genetics <https://www.youtube.com/playlist?list=PLTp0eSi9MdkPp0swo8-VVplaG8bateq7q>

### 55.1 term

- chromosome
- locus (sl.) loci (pl.)
- marker
- allele
- haplotype
- genotype
- phenotype / trait
  - endophenotype

### 55.2 marker

- large marker: STRP = short tandem repeat polymorphism, STRs = short tandem repeats
  - linkage analysis
  - paternity testing
  - taxonomy
- small marker: SNP = single-nucleotide polymorphism
  - association analysis
  - disease diagnosis
  - pharmacodynamics? drug design / pharmacogenomics = custom drug
- RFLP = restriction fragment length polymorphism
- platform
  - customized: MALDI-TOF MS(= mass spectrometry)
  - whole-genome gene chip
    - \* Affymetrix
      - Taiwan BioBank: TWB2.0
    - \* Illumina
  - NGS = next-generation sequencing

### 55.3 genome project

- HGP = Human Genome Project
  - 20~25K genes
  - 3 billion bps(base pairs)
  - ELSI = ethical, legal, and social issues
  - 99.9% bps are exactly the same in all people
  - germline mutation
    - \* male : female = 2 : 1
- HapMap = International HapMap Project
  - SNP, haplotype, tag SNP
    - \* haplotypes are combination of SNPs
    - \* tag SNPs can identify unique haplotypes

- HapMap 1 & 2
  - \* between-ancestry SNP: 1 common SNP per 5 Kb to 1 common SNP per 1 Kb
    - African
    - European
    - East Asian
      - Han Chinese
      - Japanese
  - HapMap 3: more ancestries
- 1000 Genomes Project
  - NGS
  - identify 95% genetic variants with frequencies at least 1%
  - final phase 77M SNPs
  - browser
    - \* Ensembl GRCh37
    - \* Ensembl GRCh38 [http://asia.ensembl.org/Homo\\_sapiens/Info/Index](http://asia.ensembl.org/Homo_sapiens/Info/Index)
- TWB = Taiwan BioBank
  - browser: TaiwanView <https://taiwanview.twbiobank.org.tw/index>
  - pricing: [https://www.biobank.org.tw/about\\_price.php](https://www.biobank.org.tw/about_price.php)
- TPMI = Taiwan Precision Medicine Initiative <https://tpmi.ibms.sinica.edu.tw/www/precision-medicine/>

## 55.4 linkage analysis

- Mendel 1<sup>st</sup> & 2<sup>nd</sup> laws
  - law of segregation ~ 3 : 1
  - law of assortment ~ 9 : 3 : 3 : 1
- phenotypic model by R.A. Fisher
  -

$$P = G + E$$

- \*  $G$  is the genotypic component
- \*  $E$  is the environmental component

$$P = G + E + G \cdot E$$

- \*  $G \cdot E$  is the interaction between the genotypic component and environmental component

- linkage = co-segregation = cosegregation
  - $\theta$  = recombination fraction: 1% recombination = 1 crossover per 100 meioses = 1 cM(centiMorgan) on genetic map
- statistical hypothesis testing for linkage mapping
  - statistical hypothesis testing for categorical trait in linkage mapping: PLA = parametric linkage analysis
    - \*  $H_0$ : no linkage  $\theta = 0.5$
    - \*  $H_1$ : linkage  $\theta < 0.5$
  - statistical hypothesis testing for quantitative trait in linkage mapping: VCLLA = variance component linkage analysis
    - \*  $H_0$ : no linkage  $\sigma_q^2 = 0$
    - \*  $H_1$ : linkage  $\sigma_q^2 > 0$
- study design
  - case control
  - trio
  - affected / discordant sib-pair
  - extended pedigree
- data format: linkage format
  - family-based
    - \* PID = pedigree Id
    - \* IID = individual Id
    - \* FID = father Id
    - \* MID = mother Id
    - \* gender
    - \* affection
    - \* marker
      - M1 = marker 1
      - M2 = marker 2
      - ...
- single major locus model
  - a two-allele  $A$  and  $a$  locus influences a dichotomous trait

- allele frequency
  - \*  $p = P(A)$
  - \*  $q = P(a) = 1 - p$
- penetrance
  - \*  $f_{AA} = P(\text{affected} \mid AA)$
  - \*  $f_{Aa} = P(\text{affected} \mid Aa) = f_{aA}$
  - \*  $f_{aa} = P(\text{affected} \mid aa)$
- disease mode of inheritance
  - \* dominant model
 
$$\begin{cases} f_{AA} = 1 \\ f_{Aa} = 1 \\ f_{aa} = 0 \end{cases}$$
  - \* recessive model
 
$$\begin{cases} f_{AA} = 0 \\ f_{Aa} = 0 \\ f_{aa} = 1 \end{cases}$$
  - \* additive model
 
$$\begin{cases} f_{AA} = 1 \\ f_{Aa} = \frac{1}{2} \\ f_{aa} = 0 \end{cases}$$
  - \* phenocopy model
    - $f_{aa} > 0$  perhaps due to environmental cause
  - \* liability model
    - e.g.  $f_{AA}, f_{Aa}, f_{aa}$  are age-dependent

#### 55.4.1 PLA = parametric linkage analysis

- LOD score

#### 55.4.2 VCLA = variance component linkage analysis

- allele-sharing: IBS and IBD
  - IBS = identity-by-state
  - IBD = identity-by-descent

### 55.5 association analysis

- LD = linkage disequilibrium
- genotype & allele frequency
  - diallelic marker
    - \*  $p_{AA}$
    - \*  $p_{Aa}$
    - \*  $p_{aA}$
    - \*  $p_{aa}$
    - \*  $p_A$
    - \*  $p_a$
- HWC = Hardy-Weinberg condition
  - HWE = Hardy-Weinberg equilibrium
  - HWD = Hardy-Weinberg disequilibrium
- gametic or haplotype frequency
  - LE = linkage equilibrium
  - LD = linkage disequilibrium



# Chapter 56

## multivariate normal distribution

**Definition 56.1.** probability density function (PDF) of normal distribution (= Gaussian distribution)

$$\mathcal{N}(x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

If a continuous random variable  $X$  follows a normal distribution with mean  $\mu$  and variance  $\sigma^2$

$$\begin{aligned} X &\sim n(\mu, \sigma^2) = \mathcal{N}(\mu, \sigma^2) \\ \Leftrightarrow f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}} = n(x | \mu, \sigma^2) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \mathcal{N}(x | \mu, \sigma^2) \end{aligned}$$

A random variable  $X$  can be standardized by subtracting the mean  $\mu$  and dividing by the standard deviation  $\sigma$ , resulting in the standardized random variable  $Z$

$$Z = \frac{X - \mu}{\sigma} \text{ or } z = \frac{x - \mu}{\sigma}$$

The standardized random variable  $Z$  follows the standard normal distribution

$$\begin{aligned} Z &\sim n(0, 1^2) = \mathcal{N}(0, 1^2) \\ \Leftrightarrow f_Z(z) &= \frac{e^{-\frac{1}{2}\left(\frac{z-0}{1}\right)^2}}{1 \cdot \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \\ &= n(z | 0, 1^2) = \mathcal{N}(z | 0, 1^2) \end{aligned}$$

To generalize from univariate random variables to multivariate random vectors, a random vector<sup>15</sup>

$$\mathbf{Z} = \langle Z_1, Z_2, \dots, Z_p \rangle = [Z_1 \ Z_2 \ \dots \ Z_p]^\top = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}$$

with  $p$  random variable components is said to follow the standard multivariate normal distribution if and only if its joint PDF is given by

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{\mathbf{z}^\top \mathbf{z}}{2}\right\} = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{\mathbf{z} \cdot \mathbf{z}}{2}\right\} \quad (56.1)$$

(56.1) can be rewritten as the following

$$\begin{aligned}
f_{\mathbf{Z}}(\mathbf{z}) &= \underbrace{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \cdots \frac{1}{\sqrt{2\pi}}}_{p \text{ times}} \exp \left\{ -\frac{z_1^2}{2} - \frac{z_2^2}{2} - \cdots - \frac{z_p^2}{2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_1^2}{2} \right\} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_2^2}{2} \right\} \cdots \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_p^2}{2} \right\} \\
&= f(z_1) f(z_2) \cdots f(z_p)
\end{aligned}$$

where

$$f_{Z_i}(z_i) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_i^2}{2} \right\} = f(z_i) \Rightarrow Z_i \sim \mathcal{N}(0, 1^2) = n(0, 1^2) \quad (56.2)$$

$$\begin{aligned}
f_{Z_i}(z_i) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{Z}}(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_p) dz_1 \cdots dz_{i-1} dz_{i+1} \cdots dz_p \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1) \cdots f(z_{i-1}) f(z_i) f(z_{i+1}) \cdots f(z_p) dz_1 \cdots dz_{i-1} dz_{i+1} \cdots dz_p \\
&= f(z_i) \int_{-\infty}^{\infty} f(z_1) dz_1 \cdots \int_{-\infty}^{\infty} f(z_{i-1}) dz_{i-1} \int_{-\infty}^{\infty} f(z_{i+1}) dz_{i+1} \cdots \int_{-\infty}^{\infty} f(z_p) dz_p \\
&= f(z_i) \stackrel{(56.2)}{=} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z_i^2}{2} \right\}
\end{aligned}$$

covariance matrix<sup>[16]</sup>

**Definition 56.2.** covariance matrix of a random vector<sup>8</sup>

$$\mathbf{C}[\mathbf{X}] = \text{Cov}[\mathbf{X}] = \mathbf{V}[\mathbf{X}] = \mathbf{E}[(\mathbf{X} - \mathbf{E}(\mathbf{X}))(\mathbf{X} - \mathbf{E}(\mathbf{X}))^T]$$

$$\mathbf{X} = \langle X_1, X_2, \dots, X_p \rangle = [X_1 \ X_2 \ \cdots \ X_p]^T = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$$

$$\mathbf{E}[\mathbf{X}] = \langle \mathbf{E}[X_1], \mathbf{E}[X_2], \dots, \mathbf{E}[X_p] \rangle = [\mathbf{E}[X_1] \ \mathbf{E}[X_2] \ \cdots \ \mathbf{E}[X_p]]^T = \begin{bmatrix} \mathbf{E}[X_1] \\ \mathbf{E}[X_2] \\ \vdots \\ \mathbf{E}[X_p] \end{bmatrix}$$

$$\mathbf{X} - \mathbf{E}[\mathbf{X}] = \begin{bmatrix} X_1 - \mathbf{E}[X_1] \\ X_2 - \mathbf{E}[X_2] \\ \vdots \\ X_p - \mathbf{E}[X_p] \end{bmatrix}$$

$$\begin{aligned}
&[\mathbf{X} - \mathbf{E}(\mathbf{X})][\mathbf{X} - \mathbf{E}(\mathbf{X})]^T \\
&= \begin{bmatrix} X_1 - \mathbf{E}[X_1] \\ X_2 - \mathbf{E}[X_2] \\ \vdots \\ X_p - \mathbf{E}[X_p] \end{bmatrix} [X_1 - \mathbf{E}[X_1] \ X_2 - \mathbf{E}[X_2] \ \cdots \ X_p - \mathbf{E}[X_p]] \\
&= \begin{bmatrix} (X_1 - \mathbf{E}[X_1])(X_1 - \mathbf{E}[X_1]) & (X_1 - \mathbf{E}[X_1])(X_2 - \mathbf{E}[X_2]) & \cdots & (X_1 - \mathbf{E}[X_1])(X_p - \mathbf{E}[X_p]) \\ (X_2 - \mathbf{E}[X_2])(X_1 - \mathbf{E}[X_1]) & (X_2 - \mathbf{E}[X_2])(X_2 - \mathbf{E}[X_2]) & \cdots & (X_2 - \mathbf{E}[X_2])(X_p - \mathbf{E}[X_p]) \\ \vdots & \vdots & \ddots & \vdots \\ (X_p - \mathbf{E}[X_p])(X_1 - \mathbf{E}[X_1]) & (X_p - \mathbf{E}[X_p])(X_2 - \mathbf{E}[X_2]) & \cdots & (X_p - \mathbf{E}[X_p])(X_p - \mathbf{E}[X_p]) \end{bmatrix} \\
&= \begin{bmatrix} (X_1 - \mathbf{E}[X_1])^2 & \cdots & (X_1 - \mathbf{E}[X_1])(X_p - \mathbf{E}[X_p]) \\ \vdots & \ddots & \vdots \\ (X_p - \mathbf{E}[X_p])(X_1 - \mathbf{E}[X_1]) & \cdots & (X_p - \mathbf{E}[X_p])^2 \end{bmatrix}
\end{aligned}$$

$$\mathbb{E}[(\mathbf{X} - \mathbb{E}(\mathbf{X}))(\mathbf{X} - \mathbb{E}(\mathbf{X}))^\top] \quad (56.3)$$

$$= \mathbb{E} \begin{bmatrix} (X_1 - \mathbb{E}[X_1])^2 & \cdots & (X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p]) \\ \vdots & \ddots & \vdots \\ (X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1]) & \cdots & (X_p - \mathbb{E}[X_p])^2 \end{bmatrix} \quad (56.4)$$

$$= \begin{bmatrix} \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] & \cdots & \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_p - \mathbb{E}[X_p])] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[(X_p - \mathbb{E}[X_p])(X_1 - \mathbb{E}[X_1])] & \cdots & \mathbb{E}[(X_p - \mathbb{E}[X_p])^2] \end{bmatrix} \quad (56.5)$$

$$= \begin{bmatrix} V(X_1, X_1) & \cdots & V(X_1, X_p) \\ \vdots & \ddots & \vdots \\ V(X_p, X_1) & \cdots & V(X_p, X_p) \end{bmatrix} = \begin{bmatrix} V(X_1, X_1) & V(X_1, X_2) & \cdots & V(X_1, X_p) \\ V(X_2, X_1) & V(X_2, X_2) & \cdots & V(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ V(X_p, X_1) & V(X_p, X_2) & \cdots & V(X_p, X_p) \end{bmatrix} \quad (56.6)$$

$$= \begin{bmatrix} V(X_1) & \cdots & V(X_1, X_p) \\ \vdots & \ddots & \vdots \\ V(X_p, X_1) & \cdots & V(X_p) \end{bmatrix} = \begin{bmatrix} V(X_1) & V(X_1, X_2) & \cdots & V(X_1, X_p) \\ V(X_2, X_1) & V(X_2) & \cdots & V(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ V(X_p, X_1) & V(X_p, X_2) & \cdots & V(X_p) \end{bmatrix} \quad (56.7)$$

$$= \begin{bmatrix} V(X_1) & \cdots & C(X_1, X_p) \\ \vdots & \ddots & \vdots \\ C(X_p, X_1) & \cdots & V(X_p) \end{bmatrix} = \begin{bmatrix} V(X_1) & C(X_1, X_2) & \cdots & C(X_1, X_p) \\ C(X_2, X_1) & V(X_2) & \cdots & C(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ C(X_p, X_1) & C(X_p, X_2) & \cdots & V(X_p) \end{bmatrix} \quad (56.8)$$

$$= \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_p^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} = [\sigma_{ij}]_{p \times p} = \Sigma \quad (56.9)$$

$$\mathbf{X} \sim \mathcal{D}(\boldsymbol{\mu}, \Sigma) = d(\boldsymbol{\mu}_{\mathbf{X}}, \Sigma_{\mathbf{X}}) = d(\mathbb{E}[\mathbf{X}], \mathbb{C}[\mathbf{X}]) = d(\mathbb{E}[\mathbf{X}], \mathbb{V}[\mathbf{X}])$$

$$\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}}) = n(\mathbb{E}[\mathbf{Z}], \mathbb{V}[\mathbf{Z}])$$

$$\mathbb{E}[\mathbf{Z}] = \begin{bmatrix} \mathbb{E}[Z_1] \\ \mathbb{E}[Z_2] \\ \vdots \\ \mathbb{E}[Z_p] \end{bmatrix} = [\mathbb{E}[Z_i]]_{p \times 1}$$

$$\Rightarrow \mathbb{E}[Z_i] = \int_{-\infty}^{\infty} z_i f_{Z_i}(z_i) dz_i \stackrel{(56.2)}{=} \int_{-\infty}^{\infty} z_i \frac{e^{-\frac{1}{2}z_i^2}}{\sqrt{2\pi}} dz_i = 0$$

$$\Rightarrow \mathbb{E}[\mathbf{Z}] = \mathbf{0}$$

$$\Rightarrow \mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}} = \mathbf{0}, \Sigma_{\mathbf{Z}}) = n(\mathbf{0}, \mathbb{V}[\mathbf{Z}])$$

$$\mathbb{V}(Z_i) = \int_{-\infty}^{\infty} (z_i - \mu_{Z_i})^2 f_{Z_i}(z_i) dz_i \stackrel{(56.2)}{=} \int_{-\infty}^{\infty} (z_i - 0)^2 \frac{e^{-\frac{1}{2}z_i^2}}{\sqrt{2\pi}} dz_i = 1$$

$$\mathbb{V}(Z_i, Z_j) \stackrel{i \neq j \Rightarrow Z_i, Z_j \text{ are independent}}{=} 0 \quad (56.10)$$

$$\begin{aligned} V[\mathbf{Z}] &= \begin{bmatrix} V(Z_1) & V(Z_1, Z_2) & \cdots & V(Z_1, Z_p) \\ V(Z_2, Z_1) & V(Z_2) & \cdots & V(Z_2, Z_p) \\ \vdots & \vdots & \ddots & \vdots \\ V(Z_p, Z_1) & V(Z_p, Z_2) & \cdots & V(Z_p) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{bmatrix} \stackrel{56.10}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_{p \times p} = I_p = I \end{aligned}$$

$$\mathbf{Z} \sim \mathcal{N}(\boldsymbol{\mu}_\mathbf{Z}, \Sigma_\mathbf{Z}) = n(E[\mathbf{Z}], V[\mathbf{Z}]) = \mathcal{N}(\mathbf{0}, I) \Leftrightarrow \begin{cases} \boldsymbol{\mu}_\mathbf{Z} = E[\mathbf{Z}] = \mathbf{0} = [0]_p = [0]_{p \times 1} \\ \Sigma_\mathbf{Z} = V[\mathbf{Z}] = I = I_p = I_{p \times p} \end{cases}$$

$$\begin{aligned} \mathbf{Z} &= \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \begin{bmatrix} \frac{X_1 - \mu_1}{\sigma_1} \\ \frac{X_2 - \mu_2}{\sigma_2} \\ \vdots \\ \frac{X_p - \mu_p}{\sigma_p} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_p - \mu_p \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_p} \end{bmatrix} \left( \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \right) = B^{-1}(\mathbf{X} - \boldsymbol{\mu}) \end{aligned}$$

$$\Rightarrow \mathbf{X} = B\mathbf{Z} + \boldsymbol{\mu}$$

$$\mathbf{X} = B\mathbf{Z} + \boldsymbol{\mu} = T(\mathbf{Z})$$

by  $V[A\mathbf{X} + \mathbf{b}] = A V[\mathbf{X}] A^\top$

$$\Sigma = \Sigma_\mathbf{X} = V[\mathbf{X}] = V[B\mathbf{Z} + \boldsymbol{\mu}] = BV[\mathbf{Z}]B^\top = BIB^\top = BB^\top \quad (56.11)$$

Consider two infinitesimal volumes of  $p$ -dimensional parallelepipeds in the different  $\mathbb{R}^p$  spaces<sup>16</sup>

$$V_x = [x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2] \times \cdots \times [x_p, x_p + dx_p]$$

and

$$V_z = [z_1, z_1 + dz_1] \times [z_2, z_2 + dz_2] \times \cdots \times [z_p, z_p + dz_p]$$

Their relationship under linear transformation is

$$\begin{aligned} V_x &= T(V_z) = [T(z_1), T(z_1) + T(dz_1)] \\ &\quad \times [T(z_2), T(z_2) + T(dz_2)] \\ &\quad \times \cdots \\ &\quad \times [T(z_p), T(z_p) + T(dz_p)] \end{aligned}$$

and

$$dx_i = \sum_j \frac{\partial x_i}{\partial z_j} dz_j$$

For examples in 2 dimension,

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

Two element infinitesimal one-directional vectors of  $\mathbf{Z}$  transformed into another space of  $\mathbf{X}$  are

$$T(dz_1) = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} dz_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} dz_1 \\ \frac{\partial x_2}{\partial z_1} dz_1 \end{bmatrix}$$

and

$$T(dz_2) = \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_2} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} 0 \\ dz_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial z_2} dz_2 \\ \frac{\partial x_2}{\partial z_2} dz_2 \end{bmatrix}$$

Their corresponding area(volume) in the space of  $\mathbf{X}$  is

$$\begin{aligned} \int_{A_x} dA_x &= \int_{A_x} dx_1 dx_2 = \int_{T(A_z)} dx_1 dx_2 \\ &= \int_{A_z} |[T(dz_1) \ T(dz_2)]| = \int_{A_z} \left| \begin{bmatrix} \frac{\partial x_1}{\partial z_1} dz_1 & \frac{\partial x_1}{\partial z_2} dz_2 \\ \frac{\partial x_2}{\partial z_1} dz_1 & \frac{\partial x_2}{\partial z_2} dz_2 \end{bmatrix} \right| \\ &= \int_{A_z} \left| \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{bmatrix} \right| dz_1 dz_2 = \int_{A_z} |J| dA_z \end{aligned}$$

To generalize for volumes in  $p$  dimension,

$$\begin{aligned} \int_{V_x} dV_x &= \int_{V_x} dx_1 dx_2 \cdots dx_p = \int_{T(V_z)} dx_1 dx_2 \cdots dx_p \\ &= \int_{A_z} |[T(dz_1) \ T(dz_2) \ \cdots \ T(dz_p)]| = \int_{V_z} \left| \left[ \frac{\partial x_i}{\partial z_j} dz_j \right]_{p \times p} \right| \\ &= \int_{V_z} \left| \left[ \frac{\partial x_i}{\partial z_j} \right]_{p \times p} \right| dz_1 dz_2 \cdots dz_p = \int_{V_z} |J| dV_z \end{aligned}$$

i.e.

$$\int_{V_x} dV_x = \int_{V_z} |J| dV_z \tag{56.12}$$

where  $J$  is a Jacobian matrix

$$J = \left[ \frac{\partial x_i}{\partial z_j} \right]_{p \times p} = \frac{\partial \mathbf{x}}{\partial \mathbf{z}}$$

or  $|J|$  is a Jacobian determinant or simply Jacobian

$$|J| = \left| \frac{\partial x_i}{\partial z_j} \right|_{p \times p} = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right|$$

The probability of the same event should be invariant under transformation.

$$\begin{aligned}
\int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_{\mathbf{x}} &= \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dx_1 dx_2 \cdots dx_p \\
&= \int_{T(V_z)} f_{\mathbf{X}}(\mathbf{x}) dx_1 dx_2 \cdots dx_p \\
&= \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_{\mathbf{z}} = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dz_1 dz_2 \cdots dz_p
\end{aligned}$$

i.e.

$$\int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_{\mathbf{x}} = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_{\mathbf{z}} \quad (56.13)$$

$$\begin{cases} \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_{\mathbf{x}} = \int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_{\mathbf{z}} & 56.13 \\ \int_{V_x} dV_{\mathbf{x}} = \int_{V_z} |J| dV_{\mathbf{z}} & 56.12 \end{cases}$$

$$\begin{aligned}
\mathbf{Z} &= B^{-1}(\mathbf{X} - \boldsymbol{\mu}) \\
\mathbf{z} &= B^{-1}(\mathbf{x} - \boldsymbol{\mu}) \\
\mathbf{X} &= B\mathbf{Z} + \boldsymbol{\mu} \\
\mathbf{x} &= B\mathbf{z} + \boldsymbol{\mu}
\end{aligned} \quad (56.14)$$

$$\begin{aligned}
J &= \left[ \frac{\partial x_i}{\partial z_j} \right]_{p \times p} = \frac{\partial \mathbf{x}}{\partial \mathbf{z}} = B \\
|J| &= \left| \frac{\partial x_i}{\partial z_j} \right|_{p \times p} = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right| = |B|
\end{aligned} \quad (56.15)$$

$$\begin{aligned}
\int_{V_z} f_{\mathbf{Z}}(\mathbf{z}) dV_{\mathbf{z}} &\stackrel{56.13}{=} \int_{V_x} f_{\mathbf{X}}(\mathbf{x}) dV_{\mathbf{x}} = \int_{V_x} dV_{\mathbf{x}} f_{\mathbf{X}}(\mathbf{x}) \\
&\stackrel{56.12}{=} \int_{V_z} |J| dV_{\mathbf{z}} f_{\mathbf{X}}(\mathbf{x}(z)) = \int_{V_z} f_{\mathbf{X}}(\mathbf{x}(z)) |J| dV_{\mathbf{z}} \\
f_{\mathbf{Z}}(\mathbf{z}) &\stackrel{\ddagger}{=} f_{\mathbf{X}}(\mathbf{x}(z)) |J| \\
f_{\mathbf{X}}(\mathbf{x}(z)) &\stackrel{\ddagger}{=} |J|^{-1} f_{\mathbf{Z}}(\mathbf{z}) \stackrel{56.1}{=} |J|^{-1} \frac{1}{(2\pi)^{p/2}} \exp \left\{ \frac{-\mathbf{z}^\top \mathbf{z}}{2} \right\} \\
f_{\mathbf{X}}(\mathbf{x}) &\stackrel{\ddagger}{=} |J|^{-1} f_{\mathbf{Z}}(\mathbf{z}(x)) \stackrel{56.15, 56.14}{=} |B|^{-1} f_{\mathbf{Z}}(B^{-1}(\mathbf{x} - \boldsymbol{\mu})) \\
&= |B|^{-1} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} [B^{-1}(\mathbf{x} - \boldsymbol{\mu})]^\top [B^{-1}(\mathbf{x} - \boldsymbol{\mu})] \right\} \\
&= |B|^{-1/2} |B|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (B^{-1})^\top B^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\
&= |B|^{-1/2} |B^\top|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (B^\top)^{-1} B^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\
&= |BB^\top|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (BB^\top)^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\
&\stackrel{56.11}{=} |\Sigma|^{-1/2} (2\pi)^{-p/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \\
&= (|\Sigma| (2\pi)^p)^{-1/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}
\end{aligned}$$

**Definition 56.3.** probability density function (PDF) of multivariate normal distribution (= multivariate Gaussian distribution)

$$\mathcal{N}(\mathbf{x} | \boldsymbol{\mu}, \Sigma) = f_{\mathbf{X}}(\mathbf{x}) = (|\Sigma| (2\pi)^p)^{-1/2} \exp \left\{ \frac{-1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

**Definition 56.4.** correlation coefficient

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2}\sqrt{\sigma_j^2}} = \frac{\sigma_{ij}}{\sigma_i\sigma_j} = \frac{V(X_i, X_j)}{\sqrt{V(X_i)}\sqrt{V(X_j)}} = R(X_i, X_j)$$

## 56.1 bivariate normal distribution

$p = 2$  is the case of bivariate normal distribution

$$\Sigma = [\sigma_{ij}]_{2 \times 2} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}$$

$$\rho_{12} = \rho = \rho_{21}$$

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2(1 - \rho_{12}\rho_{21}) = \sigma_1^2\sigma_2^2(1 - \rho^2)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} \Sigma^{-1} &= \frac{1}{|\Sigma|} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{\sigma_1^2\sigma_2^2(1 - \rho^2)} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_2\sigma_1\rho & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{(1 - \rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathcal{N}\left(\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}\right) \\ = \left(|\Sigma|(2\pi)^{p=2}\right)^{-1/2} \exp\left\{\frac{-1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \\ = \left(\sigma_1^2\sigma_2^2(1 - \rho^2)(2\pi)^2\right)^{-1/2} \exp\left\{\frac{-1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}\right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{\frac{-1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \frac{1}{(1 - \rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}\right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{\frac{-1}{2(1 - \rho^2)} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}\right\} \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{\frac{-1}{2(1 - \rho^2)} \left[ \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1}\right) \left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right]\right\} \end{aligned}$$

**Definition 56.5.** probability density function (PDF) of bivariate normal distribution (= bivariate Gaussian distribution)

$$\begin{aligned} \mathcal{N}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_2^2 \end{bmatrix}\right) \\ = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{\frac{-1}{2(1 - \rho^2)} \left[ \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1}\right) \left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right]\right\} \end{aligned}$$



# Chapter 57

## logic

### 57.1 Shai Ben-David

<https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1p0eE5LEtv3N>

LEAN<sup>[79]</sup>



# **Chapter 58**

## **algorithm**

### **58.1 Chen, Vivian**

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUMcBaEuGyWbBXBn21seO3vA>

### **58.2 The Bubble Sort Curve**

[https://www.youtube.com/watch?v=Gm8v\\_MR7TGk](https://www.youtube.com/watch?v=Gm8v_MR7TGk)

<https://linesthatconnect.github.io/blog/a-rigorous-derivation-of-the-bubble-sort-curve/>



# Chapter 59

## computer graphics

### 59.1 Cem Yuksel

#### 59.1.1 introduction

[https://www.youtube.com/playlist?list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW\\_Hh](https://www.youtube.com/playlist?list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh)

#### 59.1.1.1 2D transformation

[https://www.youtube.com/watch?v=EKN7dTJ4ep8&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW\\_Hh&index=6](https://www.youtube.com/watch?v=EKN7dTJ4ep8&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh&index=6)

##### 59.1.1.1.1 translation

$$\mathbf{p}' = \mathbf{p} + t \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \end{bmatrix}$$

##### 59.1.1.1.2 scale

$$\mathbf{p}' = s\mathbf{p} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = s \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} sp_x \\ sp_y \end{bmatrix}$$

##### 59.1.1.1.3 non-uniform scale

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} s_x p_x \\ s_y p_y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = S\mathbf{p}$$

##### 59.1.1.1.4 rotation

$$\mathbf{p} = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}} = p_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = I\mathbf{p}$$

$$\mathbf{p}' = p_x \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + p_y \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_\theta \mathbf{p}$$

$$\mathbf{p}' = p_x \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix} + p_y \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_{-\theta} \mathbf{p}$$

##### 59.1.1.1.5 skew = rotation + non-uniform scale + rotation

$$back\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = R_{-\theta} S R_\theta \mathbf{p}$$

##### 59.1.1.1.6 any $2 \times 2$ matrix

$$\mathbf{p}' = M\mathbf{p}$$

SVD = singular value decomposition

$$M = U \Sigma V^\top \stackrel{\text{e.g.}}{=} R S R^\top = R_{-\theta} S R_\theta$$

any  $2 \times 2$  matrix + translation

$$\mathbf{p}' = M\mathbf{p} + \mathbf{t}$$

$$\mathbf{p}' = M_2(M_1\mathbf{p} + \mathbf{t}_1) + \mathbf{t}_2$$

#### 59.1.1.1.7 homogeneous coordinate

$$\mathbf{p}' = \mathbf{p} + \mathbf{t}$$

$$\mathbf{p}' = T\mathbf{p} = \mathbf{p} + \mathbf{t}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = T\mathbf{p}$$

$$\mathbf{p}' = M\mathbf{p}\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\mathbf{p}$$

#### 59.1.1.1.8 position vs. direction position vector: transformation affected by rotation and translation

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\mathbf{p}$$

direction vector: transformation affected by only rotation but not translation

$$\mathbf{d}' = \begin{bmatrix} p'_x \\ p'_y \\ 0 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} = M\mathbf{d}$$

#### 59.1.1.2 3D transformation

[https://www.youtube.com/watch?v=1z1S2kQKXD&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW\\_Hh&index=7](https://www.youtube.com/watch?v=1z1S2kQKXD&list=PLplnkTzzqsZTfYh4UbhLGpI5kGd5oW_Hh&index=7)

#### 59.1.1.2.1 homogeneous coordinate affine transformation

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M\mathbf{p}$$

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = M\mathbf{p}$$

#### 59.1.1.2.2 scale

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = S\mathbf{p}$$

#### 59.1.1.2.3 translation

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = T\mathbf{p}$$

### 59.1.2 interactive

<https://www.youtube.com/playlist?list=PLplnkTzzqsZS3R5DjmCQsqpu43oS9CFN>



# Chapter 60

## autoregression in time series

張翔老師. 2015. “ARMA Part1.” <https://www.youtube.com/watch?v=G-0dR57W-fo>.

張翔老師. 2015. “ARMA Part2.” <https://www.youtube.com/watch?v=fQaZzO7E6FE>.

張翔老師. 2015. “ARMA Part3.” <https://www.youtube.com/watch?v=Ocw4NXoO8Xo>.

time series [data]

$$\dots, Y_{t-2}, Y_{t-1}, Y_t, Y_{t+1}, Y_{t+2}, \dots$$

- lag  
– 1<sup>st</sup> lag = lag 1

$$Y_{t-1}$$

- $k^{\text{th}}$  lag = lag  $k$

$$Y_{t-k}$$

1<sup>st</sup> difference

$$\Delta Y_t = Y_t - Y_{t-1}$$

approximation for RoR = rate of return

$$\begin{aligned} \Delta \ln Y_t &= \ln Y_t - \ln Y_{t-1} = \ln \frac{Y_t}{Y_{t-1}} = \ln \left( 1 + \frac{Y_t - Y_{t-1}}{Y_{t-1}} \right) \\ &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} + O \left( \left( \frac{Y_t - Y_{t-1}}{Y_{t-1}} \right)^2 \right) \\ &\approx \frac{Y_t - Y_{t-1}}{Y_{t-1}} = \text{RoR} \end{aligned}$$

$$\begin{cases} \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\ \ln(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \end{cases}$$

$$\begin{aligned} \frac{1}{1+t} &= 1 - t + t^2 - t^3 + \dots \\ \ln(1+x) &= \int_0^x \frac{1}{1+t} dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

**Definition 60.1.** autocorrelation = serial correlation

$$\exists t_1 \neq t_2 [V(Y_{t_1}, Y_{t_2}) \neq 0]$$

$$\begin{cases} \mathbb{E}(Y_t) \triangleq \mu_t \\ \text{V}(Y_t) \triangleq \sigma_t^2 \end{cases}$$

**Definition 60.2.**  $k^{\text{th}}$  order autocovariance

$$\text{V}(Y_t, Y_{t-k}) \triangleq \gamma(t, k)$$

$$\sigma_t^2 \triangleq \text{V}(Y_t) = \text{V}(Y_t, Y_t) = \text{V}(Y_t, Y_{t-0}) = \gamma(t, k=0) = \gamma(t, 0)$$

**Definition 60.3.**  $k^{\text{th}}$  order autocorrelation

$$\rho_{t,t-k} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_{tt}}\sqrt{\sigma_{t-k,t-k}}} = \frac{\sigma_{t,t-k}}{\sqrt{\sigma_t^2}\sqrt{\sigma_{t-k}^2}} = \frac{\sigma_{t,t-k}}{\sigma_t\sigma_{t-k}} = \frac{\text{V}(Y_t, Y_{t-k})}{\sqrt{\text{V}(Y_t)}\sqrt{\text{V}(Y_{t-k})}} \triangleq \text{R}(Y_t, Y_{t-k}) \triangleq \rho(t, k)$$

**Definition 60.4.** stationary time series

$$\begin{cases} \mathbb{E}(Y_t) \triangleq \mu_t = \mu & (1) \text{ independent of } t \\ \text{V}(Y_t) \triangleq \sigma_t^2 = \sigma^2 < \infty & (2) \text{ independent of } t \\ \text{V}(Y_t, Y_{t-k}) \triangleq \gamma(t, k) = \gamma(k) & (3) \text{ independent of } t \end{cases}$$

properties

$$\begin{cases} \text{V}(Y_t) \triangleq \sigma_t^2 = \gamma(t, k=0) = \gamma(k=0) \stackrel{(3)}{=} \gamma(0) \stackrel{(2)}{=} \gamma_0 \stackrel{(2)}{=} \sigma^2 & (4) \Rightarrow \text{V}(Y_{t-k}) = \gamma(0) \\ \text{R}(Y_t, Y_{t-k}) \triangleq \rho(t, k) \triangleq \frac{\text{V}(Y_t, Y_{t-k})}{\sqrt{\text{V}(Y_t)}\sqrt{\text{V}(Y_{t-k})}} \\ = \frac{\gamma(t, k)}{\sqrt{\gamma(0)}\sqrt{\gamma(0)}} \stackrel{(3)}{=} \frac{\gamma(k)}{\gamma(0)} \stackrel{(4)}{=} \frac{\gamma(k)}{\gamma_0} \stackrel{(2)}{=} \frac{\gamma_k}{\gamma_0} \triangleq \rho(k) \triangleq \rho_k & (5) \\ \gamma(k) = \gamma(-k) & (6) \Rightarrow \rho(k) = \rho(-k) \end{cases}$$

$$\gamma(k) \stackrel{(3)}{=} \text{V}(Y_t, Y_{t-k}) = \text{V}(Y_{t-k}, Y_t) = \text{V}(Y_{t'}, Y_{t'+k}) = \text{V}(Y_{t'}, Y_{t'-(k)}) \stackrel{(3)}{=} \gamma(-k) \Rightarrow (6)$$

point estimation

$$\begin{cases} \widehat{\mathbb{E}}(Y_t) \triangleq \bar{Y} \triangleq \widehat{\mu} = \frac{1}{T} \sum_{t=1}^T Y_t & \rightarrow \mathbb{E}(Y_t) = \mu \\ \widehat{\text{V}}(Y_t) \triangleq \widehat{\gamma}_0 = \frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2 & \rightarrow \text{V}(Y_t) = \sigma^2 = \gamma_0 \\ \widehat{\text{V}}(Y_t, Y_{t-k}) \triangleq \widehat{\gamma}_k \\ = \frac{1}{T} \sum_{t=k+1}^T (Y_t - \bar{Y})(Y_{t-k} - \bar{Y}) = \frac{1}{T} \sum_{t=1}^{T-k} (Y_t - \bar{Y})(Y_{t+k} - \bar{Y}) & \rightarrow \text{V}(Y_t, Y_{t-k}) = \gamma_k \\ \frac{\widehat{\text{V}}(Y_t, Y_{t-k})}{\sqrt{\widehat{\text{V}}(Y_t)}\sqrt{\widehat{\text{V}}(Y_{t-k})}} \triangleq \widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0} = \frac{\frac{1}{T} \sum_{t=k+1}^T (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2} & \rightarrow \text{R}(Y_t, Y_{t-k}) = \rho_k \end{cases}$$

$$Y_1 Y_{1+k} + Y_2 Y_{2+k} + \dots + Y_t Y_{t+k} + \dots + Y_{T-k} Y_T$$

1<sup>st</sup>-order autocorrelation estimation

$$\widehat{\rho}_1 = \frac{\widehat{\gamma}_1}{\widehat{\gamma}_0} = \frac{\frac{1}{T} \sum_{t=1+1=2}^T (Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y})^2}$$

## 60.1 AR(1) = 1<sup>st</sup>-order autoregressive model = first-order autoregressive model

張翔老師. 2015. “ARMA Part3.” <https://www.youtube.com/watch?v=Ocw4NXoO8Xo>.

**Definition 60.5.** AR(1) = 1<sup>st</sup>-order autoregressive model\$

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 Y_{t-1} + \varepsilon_t \\ &= \beta_0 + \beta_1 (\beta_0 + \beta_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \beta_0 (1 + \beta_1) + \beta_1^2 Y_{t-2} + \varepsilon_t + \beta_1 \varepsilon_{t-1} \\ &= \beta_0 (1 + \beta_1) + \beta_1^2 (\beta_0 + \beta_1 Y_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \beta_1 \varepsilon_{t-1} \\ &= \beta_0 (1 + \beta_1 + \beta_1^2) + \beta_1^3 Y_{t-3} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_1^2 \varepsilon_{t-2} \\ &\quad \vdots \\ &= \beta_0 (1 + \beta_1 + \beta_1^2 + \cdots) + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_1^2 \varepsilon_{t-2} + \cdots \\ &\stackrel{|\beta_1| < 1}{=} \frac{\beta_0}{1 - \beta_1} + \sum_{k=0}^{\infty} \beta_1^k \varepsilon_{t-k} \end{aligned}$$

$$\left\{ \begin{array}{l} \mu = E(Y_t) = E\left(\frac{\beta_0}{1 - \beta_1} + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}\right) \\ = \frac{\beta_0}{1 - \beta_1} + \sum_{i=0}^{\infty} \beta_1^i E(\varepsilon_{t-i}) \stackrel{\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)}{=} \frac{\beta_0}{1 - \beta_1} \quad \mu = \frac{\beta_0}{1 - \beta_1} \\ \gamma_0 = \sigma^2 = V(Y_t) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}\right) = V\left(\sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}\right) \\ = \sum_{i=0}^{\infty} (\beta_1^i)^2 V(\varepsilon_{t-i}) \stackrel{\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)}{=} \sum_{i=0}^{\infty} (\beta_1^2)^i \sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1 - \beta_1^2} \quad \gamma_0 = \sigma^2 = \frac{\sigma_\varepsilon^2}{1 - \beta_1^2} \\ \gamma_k = V(Y_t, Y_{t-k}) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-k-i}\right) \\ = V\left(\sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-k-i}\right) = \cdots \quad \gamma_k = \frac{\beta_1^k \sigma_\varepsilon^2}{1 - \beta_1^2} \\ \rho_k = R(Y_t, Y_{t-k}) = \frac{\gamma_k}{\gamma_0} = \frac{1 - \beta_1^2}{\sigma_\varepsilon^2} = \beta_1^k \quad \rho_k = \frac{\gamma_k}{\gamma_0} = \beta_1^k \end{array} \right.$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \beta_1^1 = \beta_1$$

$$\begin{aligned} \gamma_k &= V(Y_t, Y_{t-k}) = V\left(\mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}, \mu + \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-k-i}\right) \\ &= V\left(\sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-i}, \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-k-i}\right) = \cdots \end{aligned}$$

$$\mathbf{Y} = \langle Y_1, Y_2, \dots, Y_T \rangle = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_T \end{bmatrix}^T = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix}$$

AR(1) covariance matrix

by (56.3)

$$\begin{aligned}
V(\mathbf{Y}) &= E \left[ [\mathbf{Y} - E(\mathbf{Y})] [\mathbf{Y} - E(\mathbf{Y})]^T \right] \\
&= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_T^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1T} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{T1} & \sigma_{T2} & \cdots & \sigma_{TT} \end{bmatrix} = [\sigma_{ij}]_{T \times T} = \Sigma \\
&= \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\beta_1^1 \sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-1} \sigma_{\varepsilon}^2}{1-\beta_1^2} \\ \frac{\beta_1^1 \sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\beta_1^{T-2} \sigma_{\varepsilon}^2}{1-\beta_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_1^{T-1} \sigma_{\varepsilon}^2}{1-\beta_1^2} & \frac{\beta_1^{T-2} \sigma_{\varepsilon}^2}{1-\beta_1^2} & \cdots & \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} \end{bmatrix} \\
&= \frac{\sigma_{\varepsilon}^2}{1-\beta_1^2} \begin{bmatrix} 1 & \beta_1 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \cdots & \beta_1^{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{T-1} \\ \beta_1 & 1 & \beta_1 & \cdots & \beta_1^{T-2} \\ \beta_1^2 & \beta_1 & 1 & \cdots & \beta_1^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{T-1} & \beta_1^{T-2} & \beta_1^{T-3} & \cdots & 1 \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{T-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{T-1} & \rho_{T-2} & \rho_{T-3} & \cdots & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}
\end{aligned}$$

$$\mathbf{Y} \sim \mathcal{D}(\boldsymbol{\mu}, \Sigma) = d(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{Y}}) = d(E[\mathbf{Y}], V[\mathbf{Y}])$$

where

$$\Sigma = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{T-1} & \gamma_{T-2} & \cdots & \gamma_0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}$$

## 60.2 AR(2) = 2<sup>nd</sup>-order autoregressive model = second-order autoregressive model

**Definition 60.6.** AR(2) = 2<sup>nd</sup>-order autoregressive model

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim \mathcal{N}(0, \sigma_{\varepsilon}^2)$$

## 60.3 AR( $p$ ) = $p^{\text{th}}$ -order autoregressive model

$$Y_t = \beta_0 + \sum_{i=1}^p \beta_i Y_{t-i} + \varepsilon_t$$

## 60.4 MA( $q$ ) = $q^{\text{th}}$ -order moving-average model

$$Y_t = \mu + \varepsilon_t + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}$$

**60.5 ARMA( $p, q$ ) =  $p^{\text{th}}, q^{\text{th}}$ -order autoregressive-moving-average model**

$$Y_t = \varepsilon_t + \sum_{i=1}^q \alpha_i \varepsilon_{t-i} + \sum_{i=1}^p \beta_i Y_{t-i}$$



# Chapter 61

## conductivity

### 61.1 resistance

<https://www.bilibili.com/video/BV1tN41147SK>

### 61.2 semiconductor

<https://www.youtube.com/watch?v=-lHXZk5M6cI>



## Chapter 62

### LaTeX annotation by TikZ

```
knitr::opts_chunk$set(fig.pos = "H", out.extra = "")
```

$$\begin{aligned} f(x) &= x^x \\ \ln f(x) &= x \ln x \\ \frac{d}{dx} \ln f(x) &= \frac{d}{dx} [x \ln x] \\ \frac{f'(x)}{f(x)} &= [x \ln x]' \\ f'(x) &= f(x) [x \ln x]' = x^x [(x)' \ln x + x (\ln x)'] \\ &= x^x \left[ 1 \ln x + x \frac{1}{x} \right] = x^x [\ln x + 1] = x^x [1 + \ln x] \\ &= x^x + x^x \ln x \end{aligned}$$

Figure 62.1: test

```
\begin{tikzpicture}
\begin{axis}[
 axis x line = center,
 axis y line = center,
 xlabel = {x}, xlabel style =
 {\right},
 ylabel = {$y=f(\ln(x))$},
 ylabel style = {above},
 xmin = -1, xmax = 1,
 ymin = -0.5, ymax = 0.5,
 %hide obscured x ticks=false, % for
 % origin x tick label i.e. xtick =
 % {0}
 xtick= \emptyset,
 xticklabels= \emptyset,
 extra x ticks={0},
 ytick = \emptyset,
 yticklabels= \emptyset,
 x = 5cm, y = 5cm, % x y scaling
 grid = major,
 domain = -1:1,
 samples = 1000
]
\node at (axis cs: 0,0) {
 $\begin{aligned}
 & f(x) = x^x \\
 & \ln f(x) = x \ln x \\
 & \frac{d}{dx} \ln f(x) = \frac{d}{dx} [x \ln x] \\
 & \frac{f'(x)}{f(x)} = [x \ln x]' \\
 & f'(x) = f(x) [x \ln x]' = x^x [(x)' \ln x + x (\ln x)'] \\
 & = x^x \left[1 \ln x + x \frac{1}{x} \right] = x^x [\ln x + 1] = x^x [1 + \ln x] \\
 & = x^x + x^x \ln x
 \end{aligned}$};
\end{axis}
\end{tikzpicture}
```

Figure 62.2: test

$$\begin{aligned}
 f(x) &= x^x \\
 \ln f(x) &= x \ln x \\
 \frac{d}{dx} \ln f(x) &= \frac{d}{dx} [x \ln x] \\
 \frac{f'(x)}{f(x)} &= [x \ln x]' \\
 f'(x) &= f(x) [x \ln x]' = x^x [(x)' \ln x + x (\ln x)'] \\
 &= x^x \left[ 1 \ln x + x \frac{1}{x} \right] = x^x [\ln x + 1] = x^x [1 + \ln x] \\
 &= x^x + x^x \ln x
 \end{aligned}$$

<https://tex.stackexchange.com/questions/670153/how-to-annotate-calculations>

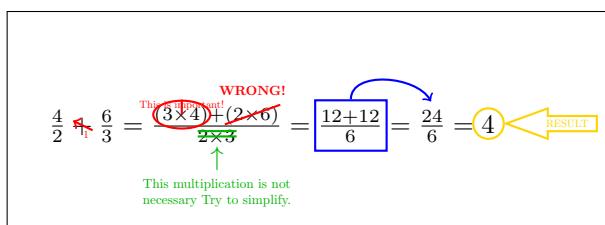


Figure 62.3: test

```

\begin{tikzpicture}
\begin{axis}[
 axis x line = center,
 axis y line = center,
 xlabel = {x}, xlabel style =
 → {right},
 ylabel = {$y=f\left(x\right)$}, ylabel style = {above},
 xmin = -0.7, xmax = 0.9,
 ymin = -0.3, ymax = 0.3,
 %hide obscured x ticks=false, % for
 → origin x tick label i.e. xtick =
 → {0}
 xtick= \emptyset,
 xticklabels= \emptyset,
 %extra x ticks={0},
 ytick = \emptyset,
 yticklabels= \emptyset,
 x = 5cm, y = 5cm, % x y scaling
 grid = major,
 domain = -1:1,
 %samples = 1000
]
\node at (axis cs:0,0) {
 → $\tikzmarknode{a1}{\frac{4}{2}}+\tikzmarknode{a2}{\frac{6}{3}}=$
 → $\frac{\tikzmarknode{b1}{(3\times 4)}}{4}+\tikzmarknode{b2}{(2\times 6)}$%
 → $6\}+\tikzmarknode{b3}{2\times 6}=$%
 → $3\}=\tikzmarknode{c1}{\frac{12+12}{6}}=\tikzmarknode{c2}{\frac{24}{6}}=\tikzmarknode{d1}{4}$;
\draw[red, thick, ->, shorten <=1em, ,
 → shorten >=1em](a2.south
 → east)--node[below, pos=.45,
 → scale=.4]{1}(a1.north west);
\node[draw=red, thick, ellipse, fit=(b1),
 → inner ysep=0, inner xsep=-1mm](ell){};
\draw[thick, red]
 → (ell)+++(0,.5)node[above, red,
 → scale=.4]{This is important!};
\draw[red, thick] (b2.south
 → west)--node[above=2mm, red,
 → scale=.5]{\bfseries WRONG!}(b2.north
 → east);
\draw[green!70!black, thick]
 → (b3.east)--node[below]{\uparrow}node[below=5mm,
 → align=left, green!70!black,
 → scale=.5]{This multiplication is
 → not\\necessary Try to
 → simplify.}(b3.west)
 → ([yshift=2pt]b3.east)--([yshift=2pt]b3.west);
\node[draw=blue, thick, fit=(c1), inner
 → xsep=1pt](box){};
\draw[blue, thick, ->, shorten >=3pt]
 → (box) to[out=90, in=90] (c2.north);
\node[draw=yellow!70!orange, thick,
 → circle, fit=(d1), inner
 → sep=1pt](cir){};
\node[draw=yellow!70!orange, thick,
 → text=yellow!70!orange, single arrow,
 → shape border rotate=180, anchor=west,
 → scale=.4, single arrow tip angle=40,
 → minimum height=30mm] at
 → (cir.east){\quad RESULT};
}
\end{tikzpicture}

```

**WRONG!**

$$\frac{4}{2} + \frac{6}{3} = \frac{(3 \times 4) + (2 \times 6)}{6} = \boxed{\frac{12+12}{6}} = \frac{24}{6} = 4$$

This multiplication is not necessary Try to simplify.

Figure 62.4: test

<https://tex.stackexchange.com/questions/494884/anchor-alignment-in-tikzmarknode>

$$\sum_{i=1}^3 \sum_{j=1}^4 a_{ij} = \sum_{j=1}^4 a_{1j} + \sum_{j=1}^4 a_{2j} + \sum_{j=1}^4 a_{3j}$$
$$= a_{11} + a_{12} + a_{13} + a_{14} + a_{21} + a_{22} + a_{23} + a_{24} + a_{31} + a_{32} + a_{33} + a_{34}$$

The diagram illustrates the summation of a 3x4 matrix. It shows the equation  $\sum_{i=1}^3 \sum_{j=1}^4 a_{ij} = \sum_{j=1}^4 a_{1j} + \sum_{j=1}^4 a_{2j} + \sum_{j=1}^4 a_{3j}$ . Below this, it is expanded to  $= a_{11} + a_{12} + a_{13} + a_{14} + a_{21} + a_{22} + a_{23} + a_{24} + a_{31} + a_{32} + a_{33} + a_{34}$ . The terms in the first row are highlighted in red, the second in green, and the third in blue. Arrows point from each row's sum to its respective row elements.

Figure 62.5: test

```

\tikzset{every tikzmarknode/.append
→ style={inner sep=3pt,rounded
corners}}

\begin{tikzpicture}
\begin{axis}[
 axis x line = center,
 axis y line = center,
 xlabel = {x}, xlabel style =
→ {right},
 ylabel = {$y=f\left(x\right)$}, ylabel style = {above},
 xmin = -1.5, xmax = 1.5,
 ymin = -0.4, ymax = 0.4,
 %hide obscured x ticks=false, % for
→ origin x tick label i.e. xtick =
→ {0}
 xtick= \emptyset,
 xticklabels= \emptyset,
 %extra x ticks={0},
 ytick = \emptyset,
 yticklabels= \emptyset,
 x = 5cm, y = 5cm, % x y scaling
 grid = major,
 domain = -1:1,
 %samples = 1000
]
\node at (axis cs: 0,0) {
 \begin{aligned}
 \sum\limits_{i=1}^3\sum\limits_{j=1}^4 a_{ij} &= \color{red}{\sum\limits_{j=1}^4 a_{1j}} + \color{green}{\sum\limits_{j=1}^4 a_{2j}} + \color{blue}{\sum\limits_{j=1}^4 a_{3j}} \\
 &= \color{red}{a_{11} + a_{12} + a_{13} + a_{14}} + \color{green}{a_{21} + a_{22} + a_{23} + a_{24}} + \color{blue}{a_{31} + a_{32} + a_{33} + a_{34}}
 \end{aligned}
}
\end{tikzpicture}

```

Figure 62.6: test

LaTeX macro: `annotate-equations`

```
\usepackage{annotate-equations}
```

---

[https://github.com/synercys/annotated\\_latex\\_equations](https://github.com/synercys/annotated_latex_equations)

[https://github.com/synercys/annotated\\_latex\\_equations/blob/main/example\\_prob.tex](https://github.com/synercys/annotated_latex_equations/blob/main/example_prob.tex)

## **Chapter 63**

### **Shannon sampling**

<https://www.youtube.com/watch?v=ePGDQpJAvjE>



## **Chapter 64**

### **brachistochrone**

[https://www.youtube.com/watch?v=2c\\_bdVC9KS8](https://www.youtube.com/watch?v=2c_bdVC9KS8)



# Chapter 65

## magnetic resonance

### 65.1 NMR

#### 65.1.1 Bloch equation

[https://en.wikipedia.org/wiki/Bloch\\_equations](https://en.wikipedia.org/wiki/Bloch_equations)

$$\dot{\mathbf{M}} = \mathbf{M} \times \gamma \mathbf{B} - \frac{\mathbf{M}_{xy}}{T_2} - \frac{\mathbf{M}_z - \mathbf{M}_{z0}}{T_1}$$

$$\begin{aligned}\dot{M}_x &= (\mathbf{M} \times \gamma \mathbf{B})_x - \frac{M_x}{T_2} \\ \dot{M}_y &= (\mathbf{M} \times \gamma \mathbf{B})_y - \frac{M_y}{T_2} \\ \dot{M}_z &= (\mathbf{M} \times \gamma \mathbf{B})_z - \frac{M_z - M_{z0}}{T_1}\end{aligned}$$

#### 65.1.2 Bloch-Torrey equation

Bloch-Torrey equation = Bloch equation + diffusion term

<https://journals.aps.org/pr/abstract/10.1103/PhysRev.104.563> <https://journals.aps.org/pr/pdf/10.1103/PhysRev.104.563>

[https://en.wikipedia.org/wiki/Diffusion\\_MRI#Magnetization\\_dynamics](https://en.wikipedia.org/wiki/Diffusion_MRI#Magnetization_dynamics)

<https://arxiv.org/abs/1608.02859> <https://arxiv.org/pdf/1608.02859>

$$\dot{\mathbf{M}} = \mathbf{M} \times \gamma \mathbf{B} - \frac{\mathbf{M}_{xy}}{T_2} - \frac{\mathbf{M}_z - \mathbf{M}_{z0}}{T_1} + \nabla \cdot D \nabla \mathbf{M}$$

$$\begin{aligned}\dot{M}_x &= (\mathbf{M} \times \gamma \mathbf{B})_x - \frac{M_x}{T_2} + \nabla \cdot D \nabla (M_x - M_{x0}) \\ \dot{M}_y &= (\mathbf{M} \times \gamma \mathbf{B})_y - \frac{M_y}{T_2} + \nabla \cdot D \nabla (M_y - M_{y0}) \\ \dot{M}_z &= (\mathbf{M} \times \gamma \mathbf{B})_z - \frac{M_z - M_{z0}}{T_1} + \nabla \cdot D \nabla (M_z - M_{z0})\end{aligned}$$

### 65.2 MRI

#### 65.2.1 Lin, Hsiu-Hau

Lin, Hsiu-Hau / Porcupine Lin / Hedgehog Note

<https://www.youtube.com/watch?v=Y-z-1XCu7fE&list=PLS0SUwlYe8czNqxfQq2XWeAHDqT8vYjmC&index=60>

#### 65.2.2 MRI: Physics and Image Creation

<https://www.youtube.com/playlist?list=PLRaiBmx1XQs5mrhyhzCmZWGHAG4bMILXj>

**65.2.3 Tseng, Wen-Yih Isaac**

<https://www.youtube.com/playlist?list=PLTpF-A8hKVUMRaGE0Zj4WCGJX9BZraFaU>

# Chapter 66

## OpenMMLab

MMLab

### 66.1 Tongji TommyZihao

<https://github.com/TommyZihao>

<https://space.bilibili.com/1900783/channel/series>

### 66.2 official

<https://github.com/open-mmlab>

<https://space.bilibili.com/1293512903/channel/series>



# **Chapter 67**

## **linear programming**

[https://www.youtube.com/watch?v=E72DWgKP\\_1Y](https://www.youtube.com/watch?v=E72DWgKP_1Y)

<https://www.bilibili.com/video/BV1KC411J7kU>

**67.1 simplex method**

**67.2 duality method**

<https://www.bilibili.com/video/BV1kT411b7Ph>

**67.3 integer linear programming**



# Chapter 68

## TaylorCatAlice physics

### 68.1 dimensional analysis

dimensional analysis = 量綱分析 = 因次分析

[https://www.bilibili.com/video/BV1314y1B7Hx<sup>17</sup>](https://www.bilibili.com/video/BV1314y1B7Hx) <https://www.books.com.tw/products/CN11371614>

### 68.2 classic mechanics

#### 68.2.1 Galileo

<https://www.bilibili.com/video/BV1C14y1B7Bv>

#### 68.2.2 Newton

<https://www.bilibili.com/video/BV1mh4y1c7QD>

#### 68.2.3 Lagrange

光力類比 = light-force analogy

<https://zhuanlan.zhihu.com/p/666330436>

<https://www.zhihu.com/question/26435474/answer/2723574018>

<https://www.youtube.com/watch?v=Uvx5OA605x4&t=5m14s>

高崇文: 墓碑不留白

<https://www.youtube.com/watch?v=9mmUpWw6x-w>

#### 68.2.4 Hamilton

### 68.3 electromagnetism

UdiProd: tensor

<https://www.youtube.com/watch?v=YxXyN2ifK8A&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=1>

<https://www.youtube.com/watch?v=A95jdIuUUW0&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=2>

<https://www.youtube.com/watch?v=51ARho2bvQY&list=PL2aHrV9pFqNTEMuDFre16Wx2SwBCNiR7j&index=3>

<https://zhuanlan.zhihu.com/p/680649946>

<https://www.bilibili.com/video/BV1PV411P7w4>

quaternion<sup>[34]</sup>

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643029>

<https://www.bilibili.com/video/BV1SN411b7cn>

<https://www.bilibili.com/video/BV1pp4y137e4>

<https://www.bilibili.com/video/BV19P411x7Nv>

<https://www.bilibili.com/video/BV1Nu4y1E78j>

## 68.4 relativity

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643042>

<https://www.bilibili.com/video/BV1sV4y1Y7fX>

<https://www.bilibili.com/video/BV1gP411s7PY>

## 68.5 quantum theory

### 68.5.1 old quantum theory

<https://www.youtube.com/watch?v=D2gS-nURAjE>

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643062>

<https://www.bilibili.com/video/BV1U84y1o7zk>

<https://www.bilibili.com/video/BV1Zh4y1S7AW>

### 68.5.2 black-body radiation

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643054>

<https://www.bilibili.com/video/BV1G14y1X7o4>

<https://www.bilibili.com/video/BV17u411G76U>

<https://www.bilibili.com/video/BV16h4y1k7qj>

<https://www.bilibili.com/video/BV1YM4y1W7XY>

### 68.5.3 quantum mechanics

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1643065>

#### 68.5.3.1 Heisenberg: matrix mechanics

<https://www.bilibili.com/video/BV1Rm4y1M7oL>

<https://www.bilibili.com/video/BV1mN4y197pY>

#### 68.5.3.2 Schrodinger: wave mechanics

<https://www.bilibili.com/video/BV1nV411F7es>

<https://www.bilibili.com/video/BV1gj411B7wz>

<https://www.bilibili.com/video/BV1kF41117cp>

<https://www.bilibili.com/video/BV1KK4y1c7S8>

<https://www.bilibili.com/video/BV18H4y1U7VN>

<https://www.bilibili.com/video/BV1uF4112785>

<https://www.bilibili.com/video/BV1zN411t7x1>

<https://www.bilibili.com/video/BV1Yc411r7sr>

<https://www.bilibili.com/video/BV1q5411v7p3>

<https://www.bilibili.com/video/BV1Xr421g7ki>

#### 68.5.3.2.0.1 matrix mechanics = wave mechanics <https://www.bilibili.com/video/BV1vy4y1c7cs>

<https://www.bilibili.com/video/BV1c5411z7Nc>

**68.5.3.3 Dirac: relativistic quantum mechanics**

<https://www.bilibili.com/video/BV1oB4y1d7JV>

<https://www.bilibili.com/video/BV1GC4y137Nh>

<https://www.bilibili.com/video/BV1y84y1U7Ps>

**68.5.3.3.1 spin <https://www.bilibili.com/video/BV1mv421i7v7>****68.5.3.4 other**

<https://www.bilibili.com/video/BV1qm4y1g7Km>

<https://www.bilibili.com/video/BV1ac411o7NJ>

**68.5.4 resistance and conductivity**

<https://space.bilibili.com/11008987/channel/collectiondetail?sid=1786884>

<https://www.bilibili.com/video/BV1tN41147SK>

<https://www.bilibili.com/video/BV1Cw411A75x>

<https://www.bilibili.com/video/BV1Hj411x7rg>

<https://www.bilibili.com/video/BV1BM411S7pM>

<https://www.bilibili.com/video/BV1mN4y1U7b2>

<https://www.bilibili.com/video/BV1Mw411P7Vt>



# Chapter 69

## fundamental theorem of algebra

### 69.1 Gauss

#### 69.1.1 FToA

FToA = fundamental theorem of algebra

cf. FToLA = fundamental theorem of linear algebra

<https://ccjou.wordpress.com/2009/03/23/%E7%BA%BF%E4%BB%A3%EF%BC%88%E4%BD%9C%EF%BC%89/>

cf. FToPF = fundamental theorem of prime factorization = fundamental theorem of arithmetic = prime factorization theorem  
= unique factorization theorem

[https://en.wikipedia.org/wiki/Fundamental\\_theorem\\_of\\_arithmetic](https://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic)

---

<https://www.bilibili.com/video/BV1pW4y1c7Zi>

<https://www.bilibili.com/video/BV1U7421K7gX>

#### 69.1.2 trial and error

<https://www.bilibili.com/video/BV1g94y1w7sy>

#### 69.1.3 heptadecagon

<https://www.bilibili.com/video/BV18V411Q7nD>

### 69.2 equation

#### 69.2.1 cubic equation

[https://en.wikipedia.org/wiki/Cubic\\_equation](https://en.wikipedia.org/wiki/Cubic_equation)

<https://www.bilibili.com/video/BV1nd4y1f7t8>

<https://www.bilibili.com/video/BV1Wh411c7jf>

#### 69.2.2 quintic equation

[https://en.wikipedia.org/wiki/Quintic\\_function](https://en.wikipedia.org/wiki/Quintic_function)

<https://www.bilibili.com/video/BV1ya4y1G79P>



# Chapter 70

## Basel problem

[https://en.wikipedia.org/wiki/Basel\\_problem](https://en.wikipedia.org/wiki/Basel_problem)

### 70.1 TaylorCatAlice

<https://www.bilibili.com/video/BV1wW421A75k>

### 70.2 3B1B

<https://www.youtube.com/watch?v=d-o3eB9sfsls>

### 70.3 Riemann hypothesis

[https://www.youtube.com/playlist?list=PL300nJOfVNKZH3H-2J\\_g1bZRYOx-dQUp3](https://www.youtube.com/playlist?list=PL300nJOfVNKZH3H-2J_g1bZRYOx-dQUp3)

<https://zhuanlan.zhihu.com/p/672704574>



# Chapter 71

## number theory

### 71.1 4 important theorems

<https://www.bilibili.com/video/BV1Nw4117757>

### 71.2 congruence modulo

<https://www.bilibili.com/video/BV18W4y1F7FU>



# Chapter 72

## spinor

### 72.1 EigenChris

spinor for beginner

[https://www.youtube.com/playlist?list=PLJHszsWbB6hoOo\\_wMb0b6T44KM\\_ABZtBs](https://www.youtube.com/playlist?list=PLJHszsWbB6hoOo_wMb0b6T44KM_ABZtBs)

<https://www.zhihu.com/question/554324716/answer/3101029714>

<https://zhuanlan.zhihu.com/p/341625428>

<https://www.cnblogs.com/quantum-condensed-matter-physics/category/1911234.html>

<https://www.cnblogs.com/quantum-condensed-matter-physics/p/14299801.html>

<https://www.cnblogs.com/quantum-condensed-matter-physics/p/14299854.html>

<https://arxiv.org/abs/1312.3824>

<https://arxiv.org/pdf/1312.3824>

### 72.2 Shinonome Masaki

東雲正樹：群論 (Group Theory) 終極速成 / 物理系零基礎火箭級 notes <https://zhuanlan.zhihu.com/p/294221308>

東雲正樹：群論 (Group Theory) 終極速成 / 群表示理論 <https://zhuanlan.zhihu.com/p/314567658>

東雲正樹：群論 (Group Theory) 終極速成 / 群表示論下的正交性與完備性 <https://zhuanlan.zhihu.com/p/337650698>

東雲正樹：群論 (Group Theory) 終極速成 / 李群 (Lie group) 的定義與常見李群 <https://zhuanlan.zhihu.com/p/340870993>

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 與 SO(3) 的夢幻聯動 <https://zhuanlan.zhihu.com/p/341625428>

東雲正樹：群論 (Group Theory) 終極速成 / SU(2) 的全體不可約表示與李群上的積分 <https://zhuanlan.zhihu.com/p/342592239>

東雲正樹：群論 (Group Theory) 終極速成 / 李群對應的李代數與可愛又迷人的伴隨表示 <https://zhuanlan.zhihu.com/p/348680652>

東雲正樹：群論 (Group Theory) 終極速成 / 淺談洛倫茲群 (Lorentz group) 與洛倫茲代數 <https://zhuanlan.zhihu.com/p/370828021>

東雲正樹：群論 (Group Theory) 終極速成 / 旋量空間與哈人的 Clifford 代數 <https://zhuanlan.zhihu.com/p/373020066>

<https://zhuanlan.zhihu.com/p/332873718>



# Chapter 73

## singular value decomposition

### 73.1 Visual Kernel

<https://www.youtube.com/playlist?list=PLWhu9osGd2dB9uMG5gKBArmk73oHUUQZS>

<https://www.youtube.com/watch?v=vSczTbgc8Rc&list=PLWhu9osGd2dB9uMG5gKBArmk73oHUUQZS&index=4>

<https://www.youtube.com/watch?v=vSczTbgc8Rc>

### 73.2 CCJou: LA Revelation

<https://ccjou.wordpress.com/專題探究/奇異值分解專題/>



# **Chapter 74**

## **support vector machine**

SVM = support vector machine

### **74.1 Visually Explained**

[https://www.youtube.com/playlist?list=PLqwozWPBo-FvuHWx3\\_aYwG2WVdbb-wC6q](https://www.youtube.com/playlist?list=PLqwozWPBo-FvuHWx3_aYwG2WVdbb-wC6q)

[https://www.youtube.com/watch?v=\\_YPScrkx28](https://www.youtube.com/watch?v=_YPScrkx28)



# **Chapter 75**

## **principal component analysis**

PCA = principal component analysis

### **75.1 Visually Explained**

<https://www.youtube.com/watch?v=FD4DeN81ODY&list=PLqwozWPBo-FtNyPKLDPTVDOHwK12QbVsM&index=4>

<https://www.youtube.com/watch?v=FD4DeN81ODY>

### **75.2 dimension reduction: PCA, tSNE, UMAP**



# **Chapter 76**

## **transformer**

### **76.1 Lee, Hung-Yi**

[https://www.youtube.com/watch?v=uhNsUCb2fJI&list=PLJV\\_el3uVTsPz6CTopeRp2L2t4aL\\_KgiI&index=11](https://www.youtube.com/watch?v=uhNsUCb2fJI&list=PLJV_el3uVTsPz6CTopeRp2L2t4aL_KgiI&index=11)  
<https://www.youtube.com/watch?v=uhNsUCb2fJI>

### **76.2 3B1B**

[https://www.youtube.com/playlist?list=PLZHQBObOWTQDNU6R1\\_67000Dx\\_ZCJB-3pi](https://www.youtube.com/playlist?list=PLZHQBObOWTQDNU6R1_67000Dx_ZCJB-3pi)  
[https://www.youtube.com/watch?v=wjZofJX0v4M&list=PLZHQBObOWTQDNU6R1\\_67000Dx\\_ZCJB-3pi&index=5](https://www.youtube.com/watch?v=wjZofJX0v4M&list=PLZHQBObOWTQDNU6R1_67000Dx_ZCJB-3pi&index=5)  
<https://www.youtube.com/watch?v=wjZofJX0v4M>  
[https://www.youtube.com/watch?v=eMlx5fFNoYc&list=PLZHQBObOWTQDNU6R1\\_67000Dx\\_ZCJB-3pi&index=6](https://www.youtube.com/watch?v=eMlx5fFNoYc&list=PLZHQBObOWTQDNU6R1_67000Dx_ZCJB-3pi&index=6)  
<https://www.youtube.com/watch?v=eMlx5fFNoYc>



# Chapter 77

## differential equation

DE = differential equation

<https://www.youtube.com/watch?v=0kY3Wpvutfs>

- guess: ansatz
- energy conservation
- series expansion
- Laplace transform
- Hamiltonian flow
- matrix exponential
- linear operator

### 77.1 ODE = ordinary DE = ordinary differential equation

<https://www.zhihu.com/question/33177784/answer/237992320>

<https://www.zhihu.com/question/572977895/answer/2814177886>

<https://zhuanlan.zhihu.com/p/666958247>

#### 77.1.1 Sturm-Liouville theory

<https://zhuanlan.zhihu.com/p/156941491>

[https://en.wikipedia.org/wiki/Sturm%20Liouville\\_theory](https://en.wikipedia.org/wiki/Sturm%20Liouville_theory)

### 77.2 PDE = partial DE = partial differential equation



# Chapter 78

## Shannon

### 78.1 entropy

<https://www.zhihu.com/question/27068465/answer/3318941468>



# Chapter 79

## LEAN

LΞĀN = LEAN

### 79.1 LEAN 4

[https://en.wikipedia.org/wiki/Lean\\_\(proof\\_assistant\)](https://en.wikipedia.org/wiki/Lean_(proof_assistant))

<https://leanprover-community.github.io/>

<https://lean-lang.org/>

#### 79.1.1 type theory

<https://www.bilibili.com/video/BV12m4y1x7CA>

#### 79.1.2 installation

<https://space.bilibili.com/520713026/channel/detail?sid=2671954>

## 79.2 ELAN

LEAN version control

## 79.3 MathLib = mathlib

library of LEAN theorems

<https://leanprover-community.github.io/mathlib-overview.html>

[https://leanprover-community.github.io/mathlib4\\_docs/](https://leanprover-community.github.io/mathlib4_docs/)

#### 79.3.1 yugu233 / Crisper : MathLibExplorer

<https://github.com/Crispher/MathlibExplorer>

<https://www.bilibili.com/video/BV1Hw4m1q7qP>

## 79.4 LEAN game server

<https://adam.math.hhu.de/#/>

#### 79.4.1 A LEAN Intro to Logic

<https://adam.math.hhu.de/#/g/trequetrum/lean4game-logic>

#### 79.4.2 1.2.1.2 Set Theory Game

<https://adam.math.hhu.de/#/g/djvelleman/stg4>

### 79.4.3 Natural Number Game

<https://adam.math.hhu.de/#/g/leanprover-community/NNG4>

#### 79.4.3.1 Natural Number Game in LEAN 4

<https://www.youtube.com/playlist?list=PLCaPonEDDnrkDQVfxWiYdF0tYOYRyLI3N>

[https://www.youtube.com/watch?v=u\\_0jW0lDoPw&list=PLCaPonEDDnrkDQVfxWiYdF0tYOYRyLI3N&t=8m42s](https://www.youtube.com/watch?v=u_0jW0lDoPw&list=PLCaPonEDDnrkDQVfxWiYdF0tYOYRyLI3N&t=8m42s)

## 79.5 LEAN 4 proof tutorial

<https://www.youtube.com/watch?v=7Ko3QZx4saU>

# Chapter 80

## recreational math

### 80.1 solved

#### 80.1.1 Monty Hall problem

[https://en.wikipedia.org/wiki/Monty\\_Hall\\_problem](https://en.wikipedia.org/wiki/Monty_Hall_problem)  
<https://www.youtube.com/watch?v=uNYVuk3JgQc>

#### 80.1.2 goat grazing problem

[https://en.wikipedia.org/wiki/Goat\\_grazing\\_problem](https://en.wikipedia.org/wiki/Goat_grazing_problem)

### 80.2 not yet solved

#### 80.2.1 moving sofa problem

[https://en.wikipedia.org/wiki/Moving\\_sofa\\_problem](https://en.wikipedia.org/wiki/Moving_sofa_problem)



# Chapter 81

## causal inference

- Brady Neal: causal inference
  - full lecture
  - short lecture



## Chapter 82

# Einstein-de Hass effect

<https://zh.wikipedia.org/wiki/愛因斯坦-德哈斯效應>

<https://en.wikipedia.org/wiki/Einstein>

<https://www.youtube.com/watch?v=BbEHApXFTQc>



# Chapter 83

## pythagorean theorem

### 83.1 2023 new proof with trigonometry

<https://www.youtube.com/watch?v=eFSvQHfrHQc>



# Chapter 84

## general relativity

### 84.1 EpicOrganism = AIRoswell = Pan, Yi-Wen<sup>1</sup>

<https://space.bilibili.com/14316464/video>

<https://www.bilibili.com/video/BV1T5411D7mS>

<https://www.bilibili.com/video/BV1QA4y1X7Xk>

<https://www.bilibili.com/video/BV1K34y1i75w>

<https://www.bilibili.com/video/BV1ZU411o7xL>

### 84.2 Elliot Schneider

- Elliot Schneider: Physics with Elliot

---

<sup>1</sup><https://www.linkedin.com/in/yiwen-pan-16a90076>



# Chapter 85

## LaTeX RegEx

```
\usepackage{xcolor}
\usepackage{expl3,xparse}

\ExplSyntaxOn
\NewDocumentCommand{\recolor}{m}
{
 \tl_set:Nn \l_tmpa_tl { #1 }
 \regex_replace_all:nnN { 2 } { \c{ensuremath}{\c{color}{red}{2}} } \l_tmpa_tl
 \tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff

\recolor{c^{2}=a^{2}+b^{2}}
```

<https://stackoverflow.com/questions/2116944/insert-programming-code-in-a-lyx-document>

<https://tex.stackexchange.com/questions/53260/lyx-is-ignoring-typewriter-font-setting-for-program-listings>

The LATEX3 Interfaces-20240220-including ExpL3,RegEx

<br>

# The LATEX3 Interfaces

The LATEX Project\*

Released 2024-02-20

## Abstract

This is the reference documentation for the `expl3` programming environment; see the matching `source3` PDF for the typeset sources. The `expl3` modules set up a naming scheme for LATEX commands, which allow the LATEX programmer to systematically name functions and variables, and specify the argument types of functions.

The `TeX` and  $\varepsilon$ -`TeX` primitives are all given a new name according to these conventions. However, in the main direct use of the primitives is not required or encouraged: the `expl3` modules define an independent low-level LATEX3 programming language.

The `expl3` modules are designed to be loaded on top of LATEX 2 $\varepsilon$ . With an up-to-date LATEX 2 $\varepsilon$  kernel, this material is loaded as part of the format. The fundamental programming code can also be loaded with other `TeX` formats, subject to restrictions on the full range of functionality.

---

\*E-mail: [latex-team@latex-project.org](mailto:latex-team@latex-project.org)

<br>

<https://tex.stackexchange.com/questions/534581/tex-compilation-after-regex-replace>

<https://www.overleaf.com/project/66512b6948077364474110de>

```
\documentclass[a4paper,11pt]{article}

\usepackage{expl3,xparse}
\usepackage{textcomp}

\ExplSyntaxOn
\NewDocumentCommand{\midarrow}{m}
{
 \tl_set:Nn \l_tmpa_tl { (#1) }
 \regex_replace_all:nnN { \, } { \c{textrightarrow} } \l_tmpa_tl
 \regex_replace_all:nnN { \; } { \; \c{hspace}\c{B}{ 1cm \c{E}\} \c{(} } \l_tmpa_tl
 \tl_use:N \l_tmpa_tl
}
\ExplSyntaxOff

\begin{document}
 \midarrow{a,A;b,B;c,C}
\end{document}
```



# **Chapter 86**

## **operator theory**

MathematicS: <https://www.bilibili.com/video/BV1Kf421X7X9>



# **Chapter 87**

## **physics monsters**

TaylorCatAlice: <https://www.bilibili.com/video/BV1jZ421p7MZ>



## **Chapter 88**

### **List of Theorem**



## Chapter 89

# Galois theory

<https://www.youtube.com/watch?v=BSHv9Elk1MU>

<https://www.youtube.com/watch?v=3imeTgGBaLc>

<https://www.youtube.com/watch?v=1EWUsef0iFs>

<https://www.youtube.com/watch?v=Ct2fyigNgPY>

Aleph 0 = Aleph Null

<https://www.youtube.com/watch?v=CwvuZ8aHyH4>

<https://www.youtube.com/watch?v=9aUsTlBjspE>

<https://www.youtube.com/watch?v=zCU9tZ2VkWc>

<https://www.bilibili.com/video/BV1vS411P7DV>



## **Chapter 90**

### **Sage**

Sage = SageMath = open-sourced Mathematica or Maple

[https://www.youtube.com/watch?v=4QZwb\\_0OOxY](https://www.youtube.com/watch?v=4QZwb_0OOxY)



# references

1. Xie, Y. *Bookdown: Authoring Books and Technical Documents with r Markdown*. (2023).
2. Xie, Y. *Dynamic Documents with R and Knitr*. (Chapman; Hall/CRC, Boca Raton, Florida, 2015).
3. Bookdown books on the web: Downloading and converting to pdf - R Markdown. *Posit Community* (2019).
4. ccjou. 二次型與正定矩陣. (2009).
5. Campbell, D. T. & Stanley, J. *Experimental and Quasi-Experimental Designs for Research*. (Cengage Learning, Belomt, CA, 1963).
6. 張翔. & 廖崇智. 提綱挈領學統計. (大碩教育, 2021).
7. Mittelhammer, R. C. *Mathematical Statistics for Economics and Business*. (Springer, New York Heidelberg Dordrecht London, 2013).
8. ccjou. 共變異數矩陣的性質. (2014).
9. Hu, F.-C. *Statistical Design and Analysis of Biomedical Research*. (International-Harvard (I-H) Statistical Consulting Company, Taipei, Taiwan, 2022).
10. Milliken, G. A. & Johnson, D. E. *Analysis of Messy Data Volume 1: Designed Experiments, Second Edition*. (Chapman; Hall/CRC, Boca Raton, Fla., 2004).
11. Buchanan, C. D. et al. Sorghum bicolor's Transcriptome Response to Dehydration, High Salinity and ABA. *Plant Molecular Biology* **58**, 699–720 (2005).
12. Casler, M. D., Vermerris, W. & Dixon, R. A. Replication Concepts for Bioenergy Research Experiments. *BioEnergy Research* **8**, 1–16 (2015).
13. Bernstein, M. A., King, K. F. & Zhou, X. J. *Handbook of MRI Pulse Sequences*. (Academic Press, Amsterdam ; Boston, 2004).
14. ccjou. 矩陣導數. (2013).
15. ccjou. 多變量常態分布. (2014).
16. ccjou. Jacobian 矩陣與行列式. (2012).
17. 孫博華. 量綱分析與Lie群. (高等教育出版社, 2016).