Numerics 1: HW 5

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Numerics 1: Homework 05

Setup

[59]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from IPython.display import display
import seaborn as sns
sns.set()
```

Problem 1

We will consider the system $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{A}$ and $\mathbf{b}$ defined in the code below, and we will examine various stationary iteration schemes in relation to this problem.

[60]:
```
A = np.array([[4,-1,0,-1,0,0],
              [-1,4,-1,0,-1,0],
              [0,-1,4,-1,0,-1],
              [-1,0,-1,4,-1,0],
              [0,-1,0,-1,4,-1],
              [0,0,-1,0,-1,4]])

b = np.array([2,1,2,2,1,2]).reshape((6,1))

print(A.shape)
print(b.shape)
```

```
(6, 6)
(6, 1)
```

All of the following stationary iterations will be using $\epsilon = 1E-7$, and some combination of the standard splitting which we compute below.

[61]:
```
eps = 1E-7 #Tolerance

L = np.tril(A, -1) #Lower triangular
U = np.triu(A, 1) #Upper triangular

D = np.tril(np.triu(A)) #Diagonal
```

a).

We use Gauss-Jacobi to approximate the solution. Gauss-Jacobi is given by the following iteration:

$$\mathbf{x}_{k+1} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})\mathbf{x}_k + \mathbf{D}^{-1}\mathbf{b}$$

[62]:
```
err = np.inf #Initial error
x_0 = np.zeros(b.shape) #Initial solution

#Pre-compute static matrices
D_inv = np.linalg.inv(D)
D_invL_U = D_inv@(L+U)
D_invb = D_inv@b

numI = 0 #Number of iterations

while(err >= eps):
    x_k = -D_invL_U@x_0 + D_invb

    #Compute error
    err = np.linalg.norm(x_k-x_0, ord=np.inf)/np.linalg.norm(x_k, ord=np.inf)␣
 ↪#Using 2-norm

    #Count iterations
    numI += 1

    x_0 = x_k
```

[63]:
```
print('Solution in %s iterations:\n\n x =' % numI, end=' ')
print(x_0, end='\n\n')
print('Final Error: %.7f' % np.linalg.norm(A@x_0-b))
```

```
Solution in 41 iterations:

 x = [[1.1666665 ]
 [1.20833311]
 [1.45833311]
 [1.45833311]
 [1.20833311]
 [1.1666665 ]]

Final Error: 0.0000006
```

b).

We use Gauss-Siedel to approximate the solution. Gauss-Seidel is given by the following iteration:

$$\mathbf{x}_{k+1} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U}\mathbf{x}_k + (\mathbf{D}+\mathbf{L})^{-1}\mathbf{b}$$

[64]:
```
err = np.inf #Initial error
x_0 = np.zeros(b.shape) #Initial solution

#Pre-compute static matrices
D_L_inv = np.linalg.inv(D+L)
D_L_invU = D_L_inv@U
D_L_invb = D_L_inv@b

numI = 0

while(err >= eps):
    x_k = -D_L_invU@x_0 + D_L_invb

    #Compute error
    err = np.linalg.norm(x_k-x_0, ord=np.inf)/np.linalg.norm(x_k, ord=np.inf)␣
 ↪#Using 2-norm

    #Count iterations
    numI += 1

    x_0 = x_k
```

[65]:
```
print('Solution in %s iterations:\n\n x =' % numI, end=' ')
print(x_0, end='\n\n')
print('Final Error: %.7f' % np.linalg.norm(A@x_0-b))
```

```
Solution in 23 iterations:

 x = [[1.16666658]
 [1.20833324]
 [1.45833325]
 [1.45833326]
 [1.20833327]
 [1.16666663]]

Final Error: 0.0000003
```

c).

We use SOR with $\omega = 1.6735$ to approximate the solution. SOR is given by the following iteration.

$$\mathbf{x}_{k+1} = (\mathbf{D} - \omega\mathbf{L})^{-1}[(1-\omega)\mathbf{D} + \omega\mathbf{U}]\mathbf{x}_k + \omega(\mathbf{D} - \omega\mathbf{L})^{-1}\mathbf{b}$$

[66]:
```
#Only defining this one as a function so we can look at changing omega.
def SOR(A, b, w):
    err = np.inf #Initial error
    x_0 = np.zeros(b.shape) #Initial solution

    #Pre-compute static matrices
    T = np.linalg.inv(D+w*L)
    Z = T@((1-w)*D - w*U)
    Tb = w*(T@b)

    numI = 0 #Number of iterations

    while(err >= eps):
        x_k = Z@x_0 + Tb

        #Compute error
        err = np.linalg.norm(x_k-x_0, np.inf)/np.linalg.norm(x_k, np.inf) #Using␣
 ↪2-norm

        #Count iterations
        numI += 1

        x_0 = x_k

    return x_0, numI

sol, numI = SOR(A, b, 1.6735)
```

[67]:
```
print('Solution in %s iterations:\n\n x =' % numI, end=' ')
print(sol, end='\n\n')
print('Final Error: %.7f' % np.linalg.norm(A@x_0-b))
```

```
Solution in 50 iterations:

 x = [[1.16666664]
 [1.20833334]
 [1.45833336]
 [1.45833327]
 [1.20833335]
 [1.16666666]]

Final Error: 0.0000003
```

d).

We can see that Gauss-Siedel has converged the fastest at 23 iterations – SOR and Gauss-Jacobi taking 50 and 41 iterations respectively. One should not always expect this to be the case. In fact, this runs counter to what one might assume as SOR is just a sped up version of Gauss-Siedel. Although, the important thing to note is that it is sped up for some choices of ω. Thus we expect other choices of ω to result in faster convergence for SOR compared to Gauss-Siedel.

e).

Letting $c = \rho(\mathbf{B})$ we look at the following error estimate.

$$||\mathbf{x}_{k+1} - \mathbf{x}|| \leq \frac{c}{1-c}||\mathbf{x}_{k+1} - \mathbf{x}_k||$$

With this error estimate we can then derive bounds for the last computed approximations in all three cases above. We note that $\mathbf{B}$ is the matrix in front of the $\mathbf{x}_k$ term in all of the iterations. We will also note that because we have convergence we have $0 < \rho(\mathbf{B}) < 1$. Given the general iteration scheme $\mathbf{x}_{k+1} = \mathbf{B}\mathbf{x}_k + \mathbf{Z}\mathbf{b}$ where $\mathbf{Z}$ is some matrix dependent on the problem (but will not matter here).

$$||\mathbf{x}_{k+1} - \mathbf{x}|| \leq \frac{c}{1-c}||\mathbf{x}_{k+1} - \mathbf{x}_k|| < ||\mathbf{x}_{k+1} - \mathbf{x}_k||$$

The above line following from the bounds on the spectral radius.

$$\cdots = ||\mathbf{B}\mathbf{x}_k + \mathbf{Z}\mathbf{b} - (\mathbf{B}\mathbf{x}_{k-1} + \mathbf{Z}\mathbf{b})|| = ||\mathbf{B}\mathbf{x}_k - \mathbf{B}\mathbf{x}_{k-1}||$$

$$\cdots \leq ||\mathbf{B}|| \cdot ||\mathbf{x}_k - \mathbf{x}_{k-1}||$$

Which follows from Cauchy-Schwarz.

Repeating this process $k-1$ more times:

$$||\mathbf{x}_{k+1} - \mathbf{x}|| \leq (||\mathbf{B}||)^k \cdot |||\mathbf{x}_1 - \mathbf{x}_0||$$

We can also use the fact that we have chosen $\mathbf{x}_0 = \mathbf{0}$ which further simplifies our bound.

$$\boxed{||\mathbf{x}_{k+1} - \mathbf{x}|| \leq (||\mathbf{B}||)^k \cdot ||\mathbf{Z}\mathbf{b}||}$$

f).

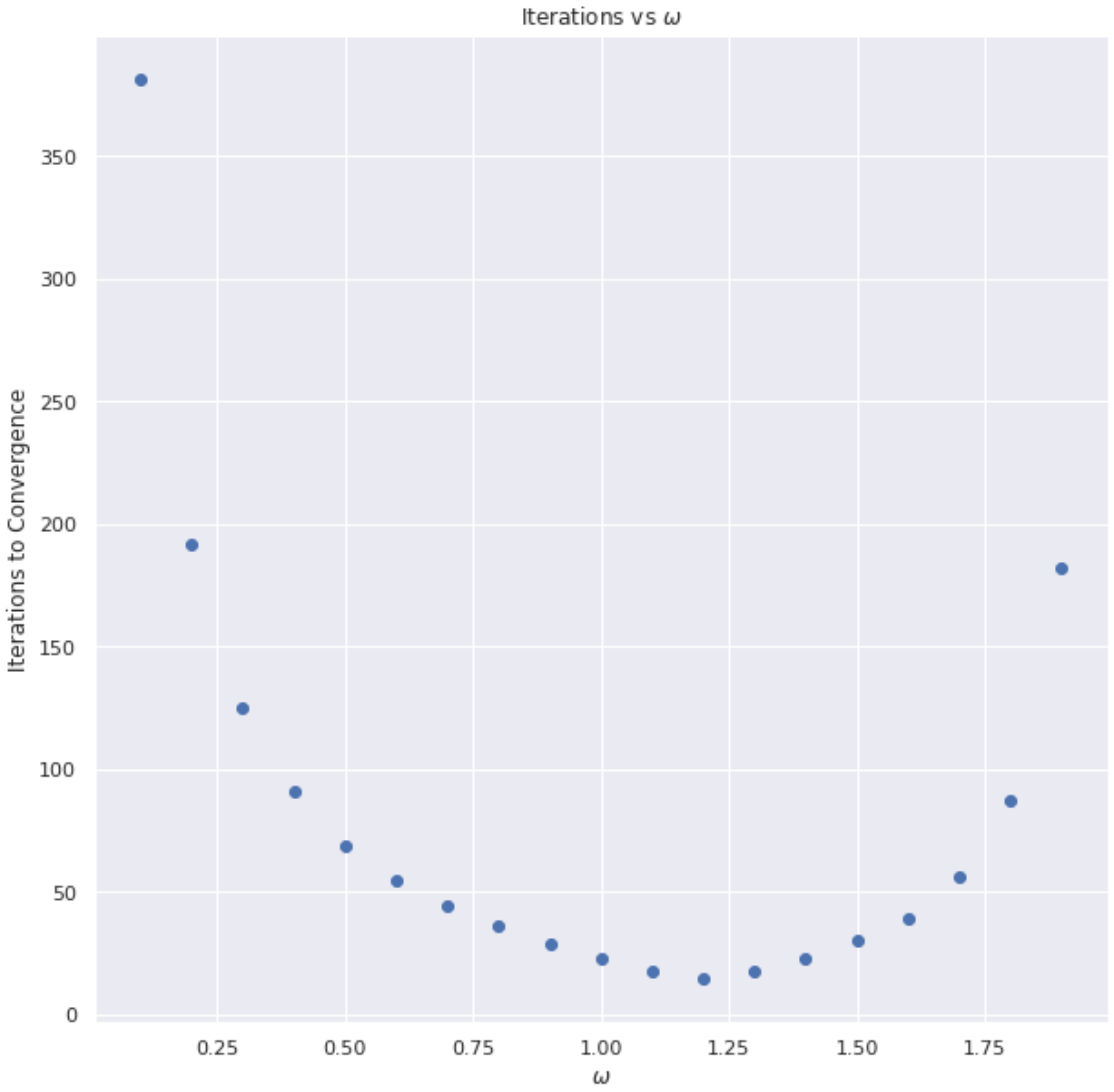
We can vary the ω parameter in SOR to see how that affects the solution and the convergence time. We will note that the iteration will converge for a SPD matrix with any initial guess if $\omega \in (0,2)$ (*Numerical Analysis 10e*). So we will look at a number of values for ω in this range and examine the number of iterations to converge.

[68]:
```
w_vec = np.arange(0.1,2,0.1)
I_vec = []
for w in w_vec:
    _, I = SOR(A, b, w)
    I_vec.append(I)
```

[69]:
```
fig, ax = plt.subplots(1,1,figsize=(10,10))
ax.scatter(w_vec, I_vec)

ax.set_title(r'Iterations vs $\omega$')
ax.set_ylabel('Iterations to Convergence')
ax.set_xlabel(r'$\omega$');
print('Minimum: %f, Iterations: %f' % (w_vec[np.argmin(I_vec)], min(I_vec)))
```

```
Minimum: 1.200000, Iterations: 15.000000
```



In the scatter plot above we see the number of iterations to convergence as a function of the parameter ω in SOR. For the omega values investigated here (and as can be seen in the figure) the minimum is achieved with $\omega = 1.2$, and converges in 15 iterations. Thus, we see that altering ω can have a dramatic effect on the rate of convergence in SOR.

Problem 2

We let $\mathbf{A}$ be a non-singular square matrix of order n, and $\mathbf{X}_0$ be an arbitrary square matrix of order n. Then define the following sequence of matrices:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{X}_k(\mathbf{I} - \mathbf{A}\mathbf{X}_k),\ k = 1, 2, ...$$

a).

We will show that $\lim_{k\to\infty} \mathbf{X}_k = \mathbf{A}^{-1}$ if and only if $\rho(\mathbf{I} - \mathbf{A}\mathbf{X}_0) < 1$.

Proof.
Let $\mathbf{A}$ be a non-singular matrix of order n, and $\mathbf{X}_0$ be an arbitrary matrix of order n. We begin by considering $\mathbf{I} - \mathbf{A}\mathbf{X}_{k+1}$.

$$\mathbf{I} - \mathbf{A}\mathbf{X}_{k+1} = \mathbf{I} - \mathbf{A}(\mathbf{X}_k + \mathbf{X}_k(\mathbf{I} - \mathbf{A}\mathbf{X}_k))$$
The above follows from plugging in the iteration scheme.
$$\cdots = \mathbf{I} - 2\mathbf{A}\mathbf{X}_k + (\mathbf{A}\mathbf{X}_k)^2 = (\mathbf{I} - \mathbf{A}\mathbf{X}_k)^2$$
We can then repeat the same process to yield...
$$\mathbf{I} - \mathbf{A}\mathbf{X}_{k+1} = ((\mathbf{I} - \mathbf{A}\mathbf{X}_{k-1})^2)^2$$
Continuing this process $k-1$ more times...
$$\implies \mathbf{I} - \mathbf{A}\mathbf{X}_{k+1} = (\mathbf{I} - \mathbf{A}\mathbf{X}_{k-1})^{2^{k+1}}$$

We will now note the following theorem (from class) for a square matrix $\mathbf{B}$:

$$\lim_{m\to\infty} \mathbf{B}^m = 0 \Leftrightarrow \rho(\mathbf{B}) < 1$$

Taking the limit of our expression defined above:

$$\lim_{k\to\infty} \mathbf{I} - \mathbf{A}\mathbf{X}_{k+1} = \lim_{k\to\infty} (\mathbf{I} - \mathbf{A}\mathbf{X}_{k-1})^{2^{k+1}}$$

If we assume that $\lim_{k\to\infty} \mathbf{X}_k = \mathbf{A}^{-1}$...

$$\implies 0 = \lim_{k\to\infty} (\mathbf{I} - \mathbf{A}\mathbf{X}_{k-1})^{2^{k+1}}$$

$$\therefore \rho(\mathbf{I} - \mathbf{A}\mathbf{X}_0) < 1$$

If we assume $\rho(\mathbf{I} - \mathbf{A}\mathbf{X}_0) < 1$...

$$\implies \lim_{k\to\infty} (\mathbf{I} - \mathbf{A}\mathbf{X}_{k-1})^{2^{k+1}} = 0$$
$$\implies 0 = \mathbf{I} - \mathbf{A}\mathbf{X}_\infty \implies \mathbf{A}^{-1} = \mathbf{X}_\infty$$
$$\therefore \mathbf{X}_\infty = \lim_{k\to\infty} \mathbf{X}_k = \mathbf{A}^-1$$

Having proved the both directions of the statement we can conclude that $\lim_{k\to\infty} \mathbf{X}_k = \mathbf{A}^{-1}$ if and only if $\rho(\mathbf{I} - \mathbf{A}\mathbf{X}_0) < 1$. ■

b).

We will use this iteration (defined above) to compute the inverse of $\mathbf{A}$ – where $\mathbf{A}$ and $\mathbf{X}_0$ are defined in the code below.

[70]:
```
A = np.array([[1,1],
              [1,2]])

X_0 = np.array([[1.9,-0.9],
                [-0.9,0.9]])

A_inv = np.array([[2,-1],
                  [-1,1]]) #True inverse
```

[71]:
```
maxI = 100 #Max iterations
tol = 1E-9 #Tolerance

X = X_0 #Just so we can have a version of X_0 unchanged
I = np.eye((2)) #Identity

for i in range(maxI):
    X_k = X + X@(I - A@X) #Iteration scheme

    if np.linalg.norm(X_k - X, ord=np.inf)/np.linalg.norm(X_k, ord=np.inf) < tol:
        X = X_k
        break

    X = X_k
```

[72]:
```
print('Calculated inverse in %s iterations:\n' % i)
print(X)
```

```
Calculated inverse in 4 iterations:

[[ 2. -1.]
 [-1.  1.]]
```

We are also interested in the cost for the iteration scheme examined above as compared to the cost of Gaussian Elimination. Gaussian Elimination is asymptotically $\mathcal{O}(n^3)$, and has a specific cost of $\frac{2}{3}n^3 + \mathcal{O}(n^2)$. We will calculate the cost of our iteration below:

$\mathbf{AX}_k$	n^3 multiplies and $n^2(n-1)$ adds
$\mathbf{I} - \mathbf{AX}_k$	n^2 adds
$\mathbf{X}_k(\mathbf{I} - \mathbf{AX}_k)$	n^3 multiplies and $n^2(n-1)$ adds
$\mathbf{X}_k + \mathbf{X}_k(\mathbf{I} - \mathbf{AX}_k)$	n^2 adds

Adding this all up we can see we have an asymptotic cost of $\mathcal{O}(n^3)$, but a specific cost of $4n^3 + \mathcal{O}(n^2)$. Thus we can see that the iteration scheme we have considered is asymptotically the same cost as Gaussian Elimination, but it has a larger constant. In reality they are the same cost.

Problem 3

We consider the following linear system where $a \in \mathcal{R}$:

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

Under certain conditions the system above can be solved with the following iterative method.

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix} \mathbf{x}_{k+1} = \begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix} \mathbf{x}_k + \omega \mathbf{b}$$

a).

We want to know for which values of a will the iteration converge, assuming $\omega = 1$. Plugging in this value for ω we get the following:

$$\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \mathbf{x}_{k+1} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{b}$$

If we let

$$\mathbf{A} = \begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}$$

Then our iteration equation is of the the following form:

$$(\mathbf{L} + \mathbf{D})\mathbf{x_{k+1}} = -\mathbf{U}\mathbf{x}_k + b$$

Which is clearly just Gauss-Siedel, so we will need $\rho(-(\mathbf{L}+\mathbf{D})^{-1}\mathbf{U}) < 1$.

$$-(\mathbf{L}+\mathbf{D})^{-1}\mathbf{U} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & a^2 \end{bmatrix}$$

Taking the determinant of the matrix above minus $\lambda\mathbf{I}$ we have the following characteristic polynomial.

$$(-\lambda)(a^2 - \lambda) = 0$$

This tells us our eigenvalues are 0 and a^2. This means we need $\boxed{|a| < 1}$ to ensure $\rho(-(\mathbf{L}+\mathbf{D})^{-1}\mathbf{U}) < 1$, giving us convergence of the iteration.

b).

For $a = 0.5$ we want to determine which $\omega \in \{0.8 : 0.1 : 1.3\}$ (that may not be proper set notation, but I think it makes sense), which minimizes the spectral radius of the following matrix.

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix}$$

[73]:
```
a = 0.5
w_vec = np.arange(0.8, 1.4, 0.1)

r_min = np.inf
w_min = None

for w in w_vec:
    #Calculate the matrix
    A = np.array([[1,0],[-w*a, 1]])
    B = np.array([[1-w, w*a],[0,1-w]])

    C = np.linalg.inv(A)@B

    e_max = max(np.abs(np.linalg.eigvals(C))) #Extract max magnitude eigenvalue

    if e_max<r_min:
        r_min = e_max
        w_min = w
```

[74]:
```
print('Minimum Spectral Radius: %f, Minimizing w: %f' % (r_min, w_min))
```

```
Minimum Spectral Radius: 0.100000, Minimizing w: 1.100000
```