HW06

March 17, 2021

[1]:
```
import numpy as np
import scipy.special as sps
import matplotlib.pyplot as plt
import seaborn as sns
sns.set()
```

Problem 1

We implement the Trapezoidal rule for numerically solving ODEs, and we increase its accuracy by implementing and applying Repeated Richardson Extraplolation. We then apply our methods to a specific example and examine the results.

We begin by defining a function below that will convert a higher-order linear ODE into a linear system of first-order ODEs.

[4]:
```
'''
convertToSys converts a higher order ODE into a system of first
order equations.

NOTE: One can generalize this to coupled sets of equations, but this
is not currently implemented

NOTE: We assume the form is y^(k) + f(t, y^(k-1), ..., y) = F(t)

Input:
    Y -> Function of t where element i is g(t)*y^(i) for some g
    F -> RHS of system
    order -> Order of the system
Output:
    func -> RHS of vector equation y'=f(t,y)
'''
def convertToSys(Y, F, order):
    def A(t):
        A = np.diag(np.ones(order-1), 1)
        A[order-1, :] = -Y(t)

        return A

    def b(t):
        b = np.zeros(order)
```

```
        b[order-1] = F(t)

        return b

    return (A, b)
```

Next, we define a function to solve the system of equations that is implicit in the Trapezoidal method.

[5]:
```
'''
impSolve will solve the system of equations required in a implicit
Trapezoidal ODE solver assuming a linear system of ODEs.

NOTE: We assume the form is A(t)*y + b(t) = f(t,y) = y'(t)

Input:
    A(t) -> See above
    b(t) -> See above
    y0 -> Past point
    t0 -> Past time point
    t1 -> Future time point
Output:
    y -> Next point
'''
def impSolve(A, b, y0, t0, t1):
    h = t1-t0

    LHS = np.eye(y0.shape[0]) - (h/2)*A(t1)
    RHS = y0 + h*((1/2)*A(t0)@y0 + b(t0) + b(t1))

    return np.linalg.solve(LHS, RHS)
```

Finally, we define our Trapezoidal method which takes in a linear system of first-order ODEs, an initial condition, a solution interval, a stepsize, and computes the solution.

[45]:
```
'''
odeTrap implements the Trapezoidal rule for solving first order
initial value ODE systems.

NOTE: Assumes vector input (i.e. >=1 dimensional system)
NOTE: Assumes linear system but not necessarily homogenous

Input:
    f -> RHS of the ODE. Either f(t,y) = A(t)*y + b(t) or f such that
        y_n+1 = f(t_n, y_n) (i.e. no longer implicit)
    y0 -> Initial condition
    interval -> Solution interval, [t0, T]
    h -> Stepsize (optional)
    last -> Return solution sequence or last iterate
Output:
```

```
    y -> Computed solution on interval
'''
def odeTrap(f, y0, interval, h=1e-2, last=False):
    if callable(f):
        future = lambda x : f(x[0], x[1], x[2])
    else:
        future = lambda x : impSolve(f[0], f[1], x[0], x[1], x[2])

    t = np.arange(interval[0], interval[1]+h, h)
    pts = len(t)

    y = np.zeros((y0.shape[0], pts))
    y[:,0] = y0

    for i in range(1, pts):
        y[:,i] = future((y[:,i-1], t[i-1], t[i]))

    if last:
        return y[0,-1]
    else:
        return y
```

We would like to test that our methods defined above are valid, so we consider the following cannonical test problem:

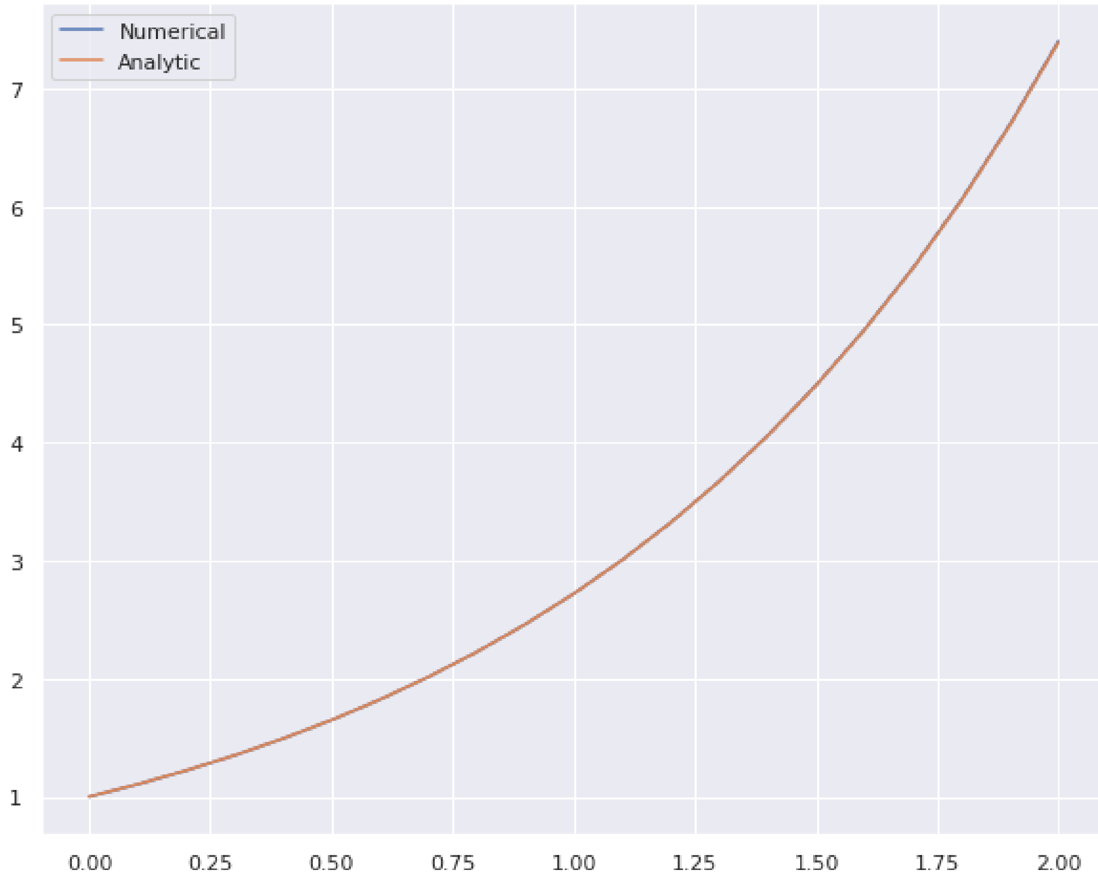
$$y'(t) = \lambda y(t), \, y(0) = y_0$$

Which has the solution $y(t) = y_0 e^{\lambda t}$. We define this below, (trivially) convert it to a system, and then solve it with our Trapezoidal method.

[46]:
```
a = 1
y0 = np.array([1])

test_Y = lambda t : np.array([-a])
test_F = lambda t: 0

f = convertToSys(test_Y, test_F, 1)

h = 1e-1
t = np.arange(0, 2+h, h)
y = odeTrap(f, y0, [0,2], h=h)
```

[47]:
```
fig, ax = plt.subplots(1,1,figsize=(10,8))
ax.plot(t, y[0,:])
ax.plot(t, np.exp(a*t))
ax.legend(['Numerical', 'Analytic']);
print('Total Error: {}'.format(np.linalg.norm(np.exp(a*t)-y[0,:])))
```

```
Total Error: 0.023543470568709382
```



Great! The numerically solution is barely even visible as it is covered by the analytic solution, but it is there. We see that our error is $\mathcal{O}(10^{-2})$ which is expected for a second-order method with stepsize 0.1, so everything is working well.

Lastly, we implement the Repeated Richardson Extrapolation for an arbitrary approximation method. In this case that will be our Trapezoidal method.

[48]:
```
'''
extraRich implements Repeated Richardson Extrapolation to improve
the accuracy of a given approximation method.

Input:
    F -> Approximation method
    kw -> Keyword arguments for F
    h0 -> Initial stepsize
    q -> Change in step (e.g h/q)
    p -> Form of the order (e.g. p_k = 2k)
    maxitr -> Maximum number of iterations
    tol -> Requested precision
'''
def extraRich(F, kw, h0, q, p, maxitr=20, tol=1e-6):
    A = np.zeros((maxitr, maxitr))
```

```
    A[0,0] = F(**kw, h=h0)

    for m in range(1, maxitr):
        A[m,0] = F(**kw, h=h0*(q**-m))

        for k in range(1, m+1):
            A[m,k] = A[m, k-1]+(A[m,k-1]-A[m-1,k-1])/(q**(p*k) - 1)

            if k<m and np.abs(A[m,k]-A[m-1,k])<tol:
                return A[m, k]+(A[m,k]-A[m-1,k])/(q**(p*(k+1)) - 1)

    print('Tolerance was not achieved.')

    return A[m,k]
```

Let's test the our Repeated Richardson on the same test problem we considered earlier. We note that for the Trapezoidal method we want $q, p = 2$.

[49]:
```
h0 = 1
kw = {'f':f, 'y0':y0, 'interval':[0,2], 'last':True}

y = extraRich(odeTrap, kw, h0, 2, 2)
```

[50]:
```
print(np.abs(y - y0*np.exp(a*2)))
```

```
[6.47419895e-11]
```

Yeah, it seems to be working as expected on the test problem.

Having implemented the necessary functionality we now consider the following second order ODE on $[0, 3\pi]$:

$$t^2y'' + ty' + (t^2 - 1)y = 0$$

With initial conditions $y(0)=0,:y'(0)=1/2 $

We can see that we will have some problems at $t = 0$, so we will make the change of variables $y(t) = tu(t)$. Doing this results in the following new ODE.

$$u'' + \frac{3u'}{t} + u = 0$$

We note that $u(0) = \frac{1}{2}$ and $u'(0) = 0$, with the former following directly from the Taylor expansion below:

$$y(t) = y(0) + ty'(0) + \frac{t^2}{2}y''(0) + \cdots = 0 + \frac{t}{2} + \frac{t^2}{2}y''(0) + \cdots$$

The second condition requires a bit more examination, but it is easy to show by first finding that $y''(0) = 0$. We note that using this change of variables we still have a problem at $t = 0$, but now we are able to evaluate $\frac{u'(t)}{t}|_{t=0}$.

$$\lim_{t\to 0} \frac{u'(t)}{t} = u''(0) = -\frac{1}{8}$$

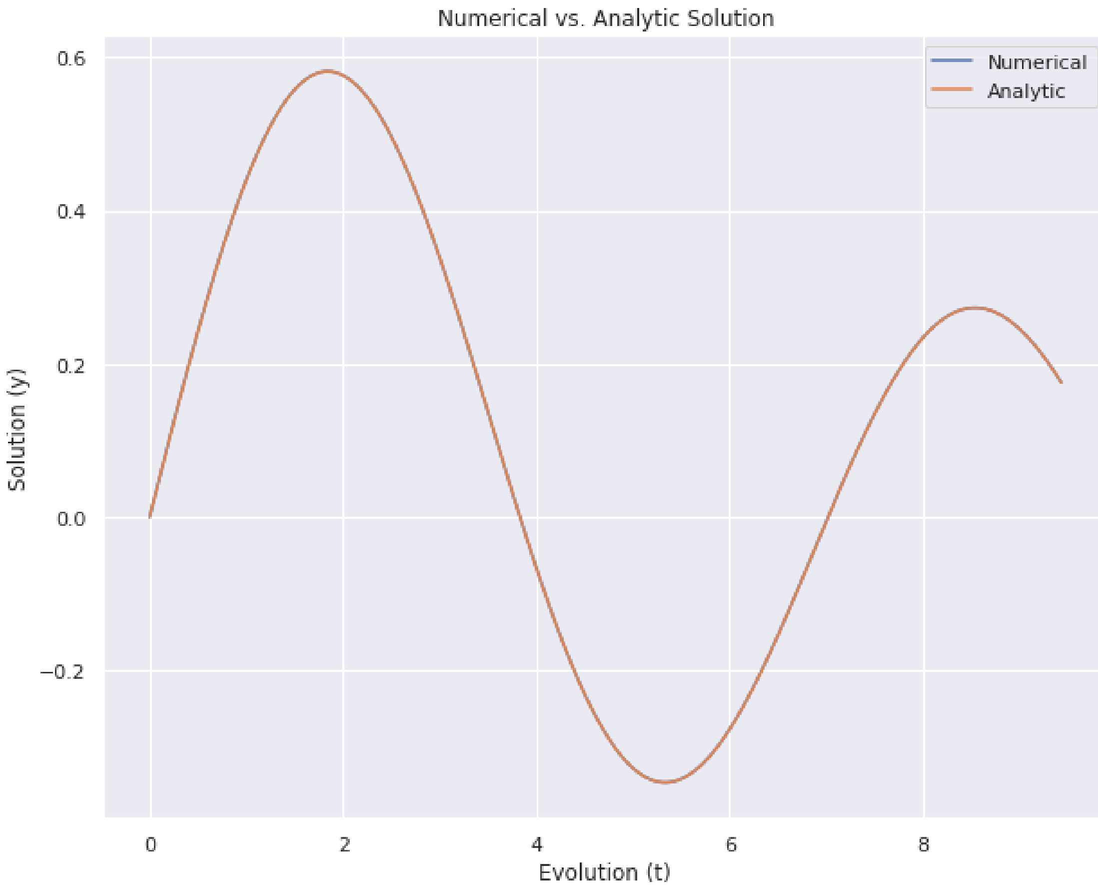
We then numerically define our system below and apply our solution method.

[54]:
```
def U(t):
    if t==0: return np.array([1, -1/8])
    else: return np.array([1, 3/t])

F = lambda t : 0

f = convertToSys(U, F, 2)
```

To analyze our solution we will compare it to the exact analytic solution which is the Bessel function of the first kind of order 1 ($J_1(t)$).

[55]:
```
h = 1e-2
t = np.arange(0, 3*np.pi+h, h)
u0 = np.array([1/2, 0])

u = odeTrap(f, u0, [0, 3*np.pi], h=h)
J = sps.jv(1, t)
```

[56]:
```
fig, ax = plt.subplots(1,1,figsize=(10,8))
ax.plot(t, u[0,:]*t)
ax.plot(t, J)

ax.set_title('Numerical vs. Analytic Solution')
ax.set_xlabel('Evolution (t)')
ax.set_ylabel('Solution (y)')
ax.legend(['Numerical', 'Analytic']);
```



Looks like everything went well. We will also consider computing the value $y(3\pi)$ using Repeated Richardson.

[94]:
```
h0 = 1
kw = {'f':f, 'y0':u0, 'interval':[0,3*np.pi], 'last':True}

u = extraRich(odeTrap, kw, h0, 2, 2, tol=1e-10)
```

[95]:
```
print(np.abs(3*np.pi*u-sps.jv(1, 3*np.pi)))
```

```
5.8455405465007715e-06
```

Not quite 10 significant digits, but hey…