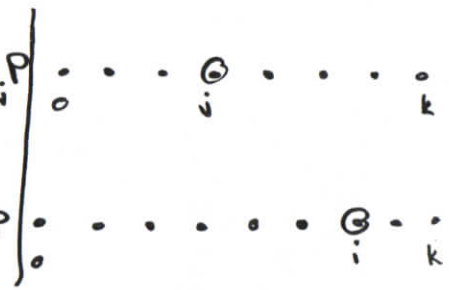
VARIATIONS on Thiel A

- Given $P_{0,1,\dots,j-1,j+1,\dots,k} = {}_jP$

and $P_{0,1,\dots,i-1,i+1,\dots,k} = {}_iP$



- Can form a new P that agrees at all points $0,1,\dots,k$.

$$\boxed{P = \frac{(x-x_j)}{(x_i-x_j)}\,{}_jP + \frac{(x-x_i)}{(x_j-x_i)}\,{}_iP}$$

if $x=x_i$ $\quad P = {}_jP$ which passes through x_i

if $x=x_j$ $\quad P = {}_iP$ " passes through x_j

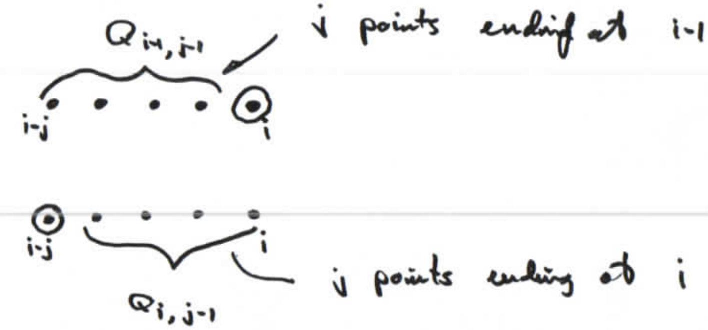
This is Theorem 3.5!

VARIATION B - Nevilles Method!

Define $Q_{i,j} = Q_{\text{last point},\ \text{degree}}$

$= P_{i-j,\ i-j+1,\ \dots,\ i-1,\ i}$

Now consider



Can combine these to get a higher degree = j ending at location i

$$\boxed{Q_{i,j} = \left(\frac{x-x_{i-j}}{x_i-x_{i-j}}\right) Q_{i,j-1} + \left(\frac{x-x_i}{x_{i-j}-x_i}\right) Q_{i-1,j-1}}$$

If $x = x_i$ $\quad Q_{ij} = Q_{i,j-1}\Big|_{x_i}$ ok

$x = x_{i-j}$ $\quad Q_{ij} = Q_{i-1,j-1}\Big|_{x_{i-j}}$ ok

all others are ok too since each $Q_{i-1,j-1}$ and $Q_{i,j-1}$ pass through them as well.

Specifically let $\tilde{x}$ be one of the common points, what is $Q_{ij}(\tilde{x})$?

$$Q_{ij} = \left(\frac{\tilde{x} - x_{i-j}}{x_i - x_{i-j}}\right)\tilde{Q} + \left(\frac{\tilde{x} - x_i}{x_{i-j} - x_i}\right)\tilde{Q}$$

$$= \frac{\tilde{x} - x_{i-j} - \tilde{x} + x_i}{x_i - x_{i-j}}\tilde{Q}$$

$$= \frac{x_i - x_{i-j}}{x_i - x_{i-j}}\tilde{Q}$$

$$= \tilde{Q} = f(\tilde{x}).$$

Neville's method.

(1)

$$Q_{i,j} = Q_{\text{last pt},\,\text{degree}} = P_{i-j,\,i-j+1,\,\ldots,\,i-1,\,i}$$

$\overbrace{\circ_{i-j} \quad \circ \quad \circ \quad \circ}^{Q_{i-1,\,j-1}} \quad \circ_{i}$ — j points ending at $i-1$

$\bullet \quad \underbrace{\bullet \quad \bullet \quad \bullet \quad \bullet}_{Q_{i,\,j-1}}$ — j points ending at pt i

Combine to get 1 higher degree poly.

$$Q_{i,j} = \left(\frac{x - x_{i-j}}{x_i - x_{i-j}}\right) Q_{i,j-1} + \left(\frac{x - x_i}{x_{i-j} - x_i}\right) Q_{i-1,j-1}$$

(2)

$x_0 = 1$	$y_0 = 0.7651977 = Q_{0,0}$				
$x_1 = 1.3$	$y_1 = 0.6200860 = Q_{1,0}$	$Q_{1,1} = 0.5233449$			
$x_2 = 1.6$	$y_2 = 0.4554022 = Q_{2,0}$	$Q_{2,1} = 0.5102968$	$Q_{2,2} = 0.5124715$		
$x_3 = 1.9$	$y_3 = 0.2818186 = Q_{3,0}$	$Q_{3,1} = 0.5132634$	$Q_{3,2} = 0.5112857$	$Q_{3,3} = 0.5118127$	
$x_4 = 2.2$	$y_4 = 0.1103623 = Q_{4,0}$	$Q_{4,1} = 0.5104270$	$Q_{4,2} = 0.5137361$	$Q_{4,3} = 0.5118302$	

$Q_{4,4} = 0.5118200$

Find value at $x = 1.5$

To find

$Q_{(),1}$ need two Q's

$$Q_{\underset{i}{1},\underset{j}{1}} = \left(\frac{x - x_0}{x_1 - x_0}\right) Q_{1,0} + \left(\frac{x - x_1}{x_0 - x_1}\right) Q_{0,0}$$

$$= \frac{1.5 - 1}{1.3 - 1}\, 0.6200860 + \frac{1.5 - 1.3}{1 - 1.3}\, 0.7651977 = 0.5233449$$

$$Q_{\underset{i}{2},\underset{j}{1}} = \frac{x - x_1}{x_2 - x_1} Q_{2,0} + \frac{x - x_2}{x_1 - x_2} Q_{1,0}$$

$$= \frac{\overset{1.5}{x} - 1.3}{1.6 - 1.3}\, 0.4554022 + \frac{\overset{1.5}{x} - 1.6}{1.3 - 1.6}\, 0.6200860 = 0.5102968$$

$$Q_{\underset{i}{3},\underset{j}{1}} = \frac{x - x_2}{x_3 - x_2} Q_{3,0} + \frac{x - x_3}{x_2 - x_3} Q_{2,0}$$

$$= \frac{1.5 - 1.6}{1.9 - 1.6}\, 0.2818186 + \frac{1.5 - 1.9}{1.6 - 1.9}\, 0.4554022 = 0.5132634$$

$$Q_{\underset{i}{4},\underset{j}{1}} = \frac{x - x_3}{x_4 - x_3} Q_{4,0} + \frac{x - x_4}{x_3 - x_4} Q_{3,0}$$

$$= \frac{1.5 - 1.9}{2.2 - 1.9}\, 0.1103623 + \frac{1.5 - 2.2}{1.9 - 2.2}\, 0.2818186 = 0.5104270$$

To find $Q_{(),2}$ need two previous $Q_{(),1}$'s.

$$Q_{\underset{i}{2},\underset{j}{2}} = \frac{x - x_0}{x_2 - x_0} Q_{2,1} + \frac{x - x_2}{x_0 - x_2} Q_{11}$$

$$= \frac{1.5 - 1}{1.6 - 1}\, 0.5102968 + \frac{1.5 - 1.6}{1 - 1.6}\, 0.5233449 = 0.5124715$$

$$Q_{\underset{i}{3},\underset{j}{2}} = \frac{x - x_1}{x_3 - x_1} Q_{3,1} + \frac{x - x_3}{x_1 - x_3} Q_{2,1}$$

$$= \frac{1.5 - 1.3}{1.9 - 1.3}\, 0.5132634 + \frac{1.5 - 1.9}{1.3 - 1.9}\, 0.5102968 = 0.5112857$$

$$Q_{\underset{i}{4},\underset{j}{2}} = \frac{x - x_2}{x_4 - x_2} Q_{4,1} + \frac{x - x_4}{x_2 - x_4} Q_{3,1}$$

$$= \frac{1.5 - 1.6}{2.2 - 1.6}\, 0.5104270 + \frac{1.5 - 2.2}{1.6 - 2.2}\, 0.5132634 = 0.5137361$$

To find $Q_{(),3}$ need 2 previous $Q_{(),2}$'s ⑤

$$Q_{3,3} = \frac{x - x_0}{x_3 - x_0} Q_{3,2} + \frac{x - x_3}{x_0 - x_3} Q_{2,2}$$

i " j "

$$= \frac{1.5 - 1}{1.9 - 1} 0.5112857 + \frac{1.5 - 1.9}{1 - 1.9} 0.5124715 = 0.5118127$$

$$Q_{4,3} = \frac{x - x_1}{x_4 - x_1} Q_{4,2} + \frac{x - x_4}{x_1 - x_4} Q_{3,2}$$

i " j "

$$= \frac{1.5 - 1.3}{2.2 - 1.3} 0.5137361 + \frac{1.5 - 2.2}{1.3 - 2.2} 0.5112857$$

$$= 0.5118302$$

To find $Q_{(),4}$ need 2 previous $Q_{(),3}$'s ⑥

$$Q_{4,4} = \frac{x - x_0}{x_4 - x_0} Q_{4,3} + \frac{x - x_4}{x_0 - x_4} Q_{3,3}$$

i " j "

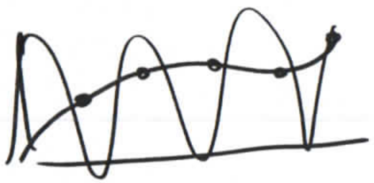
$$= \frac{1.5 - 1}{2.2 - 1} 0.5118302 + \frac{1.5 - 2.2}{1 - 2.2} 0.5118127$$

$$= 0.5118200$$

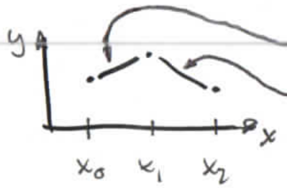
SPLINES

- Polynomials valid between the specified points
  $a = x_0 \ldots x_n = b$

- 1st order - straight line



Consider



$y_1 = m_1 x + b_1$

$y_2 = m_2 x + b_2$

4 unknowns
$m_1\ m_2$
$b_1\ b_2$

How many conditions

$y_1(x_0) = \ldots$ $\quad$ $y_2(x_1) = \ldots$

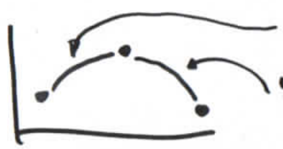
$y_1(x_1) = \ldots$ $\quad$ $y_2(x_2) = \ldots$

4!

Can find all the constants but can say nothing about slopes anywhere. In fact, the slope is discontinuous at x_1!

Try 2nd order - quadratic

Consider



$y_1 = a_2 x^2 + a_1 x + a_0$

$y_2 = b_2 x^2 + b_1 x + b_0$

6 unknowns.

Conditions

$y_1(x_0) = \ldots$ magnitude values match $\quad$ $y_2(x_1) = \ldots$

$y_1(x_1) = \ldots$ $\quad$ $y_2(x_2) = \ldots$

$y_1'(x_1) = y_2'(x_1)$ slopes match at x_1

5 so far

Could say something @ slope at 1 endpoint but not both!

Need 1 more constant to fix slopes at endpoints.

Try 3rd order.

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Now with $y_1 = a_3x^3 + a_2x^2 + a_1x + a_0$
$y_2 = b_3x^3 + b_2x^2 + b_1x + b_0$ } 8 unknowns.

Conditions

magnitudes { $y_1(x_0) = \dots$ $y_2(x_1) = \dots$
$y_1(x_1) = \dots$ $y_2(x_2) = \dots$

slopes { $y_1'(x_0) = \dots$ $y_2'(x_2) = \dots$
$y_1'(x_1) = y_2'(x_1)$

can now fix slopes at ends

2nd der. { $y_1''(x_1) = y_2''(x_1)$

8 unknowns + 8 conditions

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Ex Approx $f(x) = x^4$ on $-1 \le x \le 1$ with cubic spline on points $x_0 = -1$ $x_1 = 0$ $x_2 = 1$

Also, match slopes to $f'(x)$ at end points.

$P_0(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ $\quad -1 \le x \le 0$
$P_1(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ $\quad 0 \le x \le 1$

Conditions at $x = 0$;

$P_0(0) = P_1(0) = f(0) \Rightarrow \boxed{a_0 = 0, \; b_0 = 0}$

$P_0'(0) = P_1'(0) \Rightarrow \boxed{a_1 = b_1}$

$P_0''(0) = P_1''(0) \Rightarrow \boxed{a_2 = b_2}$

$P_0(-1) = f(-1) = 1 \Rightarrow -a_3 + a_2 - a_1 = 1$

$P_1(1) = f(1) = 1 \Rightarrow b_3 + \underset{a_2}{\underset{\shortparallel}{b_2}} + \underset{a_1}{\underset{\shortparallel}{b_1}} = 1$

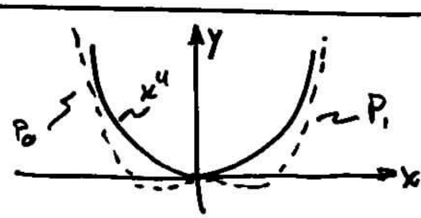
$P_0'(-1) = f'(-1) = -4 \Rightarrow 3a_3 - 2a_2 + a_1 = -4$

$P_1'(1) = f'(1) = 4 \Rightarrow 3b_3 + 2\underset{a_2}{\underset{\shortparallel}{b_2}} + \underset{a_1}{\underset{\shortparallel}{b_1}} = 4$

$$\begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & -2 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} b_3 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -4 \\ 4 \end{pmatrix}$$

$a_1 = 0 \quad a_2 = -1 \quad a_3 = -2 \quad b_3 = 2$

so
$$\boxed{\begin{array}{lll} P_0 = -2x^3 - x^2 & & -1 \le x \le 0 \\ P_1 = 2x^3 - x^2 & & 0 \le x \le 1 \end{array}}$$



Check $P_0(-1) = (-2)(-1) - 1 = 1 = f(-1)$

$$\left.\begin{array}{l} P_0(0) = 0 \\ P_1(0) = 0 \end{array}\right\} = f(0)$$

$P_1(1) = 2 \cdot 1 - 1 = 1 = f(1)$

$P_0' = -6x^2 - 2x$

$P_1' = 6x^2 - 2x \Rightarrow P_0'(0) = 0 = P_1'(0)$

$P_0'' = -12x - 2 \qquad P_0''(0) = -2 = P_1''(0)$

$P_1'' = 12x - 2$

$P_0'(-1) = -6 + 2 = -4 = f'(-1)$

$P_1'(1) = 6 - 2 = 4 = f'(1)$

So all 8 conditions are met.

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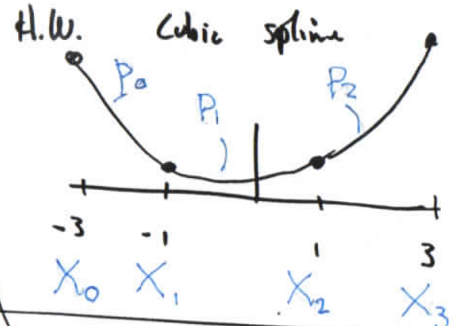

$f(x) = x^4$

match $f'(-3)$ & $f'(3)$

3 splines

$$P_0 = a_3x^3 + a_2x^2 + a_1x^1 + a_0$$
$$P_1 = b_3x^3 + b_2x^2 + b_1x^1 + b_0$$
$$P_2 = c_3x^3 + c_2x^2 + c_1x + c_0$$

12 unknowns

at x_1

$$P_0(x_1) = f(x_1)$$
$$P_1(x_1) = f(x_1)$$
$$P_0'(x_1) = P_1'(x_1)$$
$$P_0''(x_1) = P_1''(x_1)$$

similar at x_2

$$P_1(x_2) = P_2(x_2) = f(x_2)$$
$$P_1'(x_2) = P_2'(x_2)$$
$$P_1''(x_2) = P_2''(x_2)$$

at x_0

$$P_0(x_0) = f(x_0)$$
$$\left.\begin{array}{l} P_0'(x_0) = f'(x_0) \\ P_0''(x_0) = 0 \end{array}\right\} \text{Pick one}$$

at x_3

$$P_2(x_3) = f(x_3)$$
$$\left.\begin{array}{l} P_2'(x_3) = f'(x_3) \\ P_2''(x_3) = 0 \end{array}\right\} \text{Pick one}$$

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