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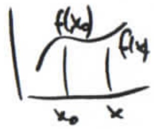
Approx Derivatives

Definition $f'(x) = \dfrac{f(x+h) - f(x)}{h}$ $\lim h \to 0$

Taylor series expansion

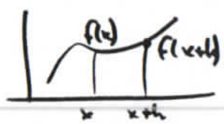


$$f(x) = f(x_0) + (x-x_0)\frac{f'(x_0)}{1!} + (x-x_0)^2\frac{f''(x_0)}{2!} + \cdots$$

or

$$\boxed{f(x+h) = f(x) + h\frac{f'(x)}{1!} + h^2\frac{f''(x)}{2!} + \cdots}$$

Most useful.



To calc first derivative, solve for $f'(x)$

$$f'(x) = \frac{f(x+h) - f(x) - \frac{h^2}{2}f''(x) - \cdots}{h}$$

$$= \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(x) + \cdots$$

$$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)}$$

Forward Diff $f'(x)$ $O(h)$

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Now try

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) \cdots$$

Solve for $f'(x) = \dfrac{f(x-h) - f(x) - \frac{h^2}{2}f''(x) + \cdots}{-h}$

$$= \frac{f(x) - f(x-h)}{h} + \frac{h}{2}f''(x) + \cdots$$

$$\boxed{f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)}$$

Backward Diff $f'(x)$ $O(h)$

Now subtract expansion for $f(x+h)$ and $f(x-h)$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + -\frac{h^3}{6}f'''(x) + \cdots$$

Subtract and get

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + \cdots$$

↑ solve

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$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(x) + \dots$$

or

$$\boxed{f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right] + O(h^2)}$$

Central Diff f' $O(h^2)$

Example $f(x) = e^x \Rightarrow f'(x) = e^x = f''(x)$

at $x=0$ $h = 0.1, 0.01$

Note: real value $= 1.0$

$$\left.\begin{array}{lll} \text{Forward} & f'(0) = \dfrac{e^{0.1} - e^0}{0.1} & = 1.051706 \\ \text{Back} & f'(0) = \dfrac{e^0 - e^{-0.1}}{0.1} & = 0.951626 \\ \text{Cent} & f'(0) = \dfrac{e^{0.1} - e^{-0.1}}{0.2} & = 1.001668 \end{array}\right\} h = 0.1$$

$$\left.\begin{array}{lll} \text{For} & f'(0) = \dfrac{e^{0.01} - e^0}{0.01} & = 1.005017 \\ \text{Back} & f'(0) = \dfrac{e^0 - e^{-0.01}}{0.01} & = 0.995017 \\ \text{Cent} & f'(0) = \dfrac{e^{0.01} - e^{-0.01}}{0.02} & = 1.000017 \end{array}\right\} h = 0.01$$

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Can calc. analytical error.

Recall $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

Forward $f'_{approx}(x) \approx \dfrac{e^{x+h} - e^x}{h}$

$$h f'_{approx}(x) = \left(1 + (x+h) + \frac{(x+h)^2}{2} + \frac{(x+h)^3}{6} + \dots\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

$$= 1 + (x+h) + \frac{x^2 + 2xh + h^2}{2} + \frac{x^3 + 3x^2h + 3xh^2 + h^3}{6} + \dots$$
$$-1 - x - \frac{x^2}{2} - \frac{x^3}{6} - \dots$$

$$= (h) + \left(xh + \frac{h^2}{2}\right) + \left(\frac{1}{2}x^2h + \frac{1}{2}xh^2 + \frac{h^3}{6}\right) + \dots$$

$$f'(x) = 1 + \left(x + \frac{h}{2}\right) + \left(\frac{x^2}{2} + \frac{xh}{2} + \frac{h^2}{6}\right) + \dots$$

$$f'_{approx}(x) \doteq \left(1 + x + \frac{x^2}{2}\right) + (1+x)\frac{h}{2} + \frac{h^2}{6} + \dots$$

But real derivative is $f'_{true} = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

So

$$f'_{approx}(x) - e^x = \left[-\frac{x^3}{6} + \dots\right] + (1+x)\frac{h}{2} + \frac{h^2}{6} + \dots$$

If $x = 0$

$$f'(x) - e^x \approx \frac{h}{2} + \frac{h^2}{6} + \dots$$

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If $h = 0.1$

$f'(0) - e^0 = 0.051667$ (error)

If $h = 0.01$

$f'(0) - e^0 = 0.005100$ (error)

$\Bigg\}$ Forward Diff y' $O(h)$

Look at central differences

$$2h f' = f(x+h) - f(x-h) = e^{x+h} - e^{x-h}$$

$$= 1 + (x+h) + \frac{(x+h)^2}{2} + \frac{(x+h)^3}{6} + \cdots - \left[1 + (x-h) + \frac{(x-h)^2}{2} + \frac{(x-h)^3}{6} + \cdots\right]$$

$$= 1 + x + h + \frac{x^2 + 2xh + h^2}{2} + \frac{x^3 + 3x^2h + 3xh^2 + h^3}{6} + \cdots$$

$$- \left[1 + x - h + \frac{x^2 - 2xh + h^2}{2} + \frac{x^3 - 3x^2h + 3xh^2 - h^3}{6} + \cdots\right]$$

$$= 2h + \frac{4xh}{2} + \frac{6x^2h + 2h^3}{6} + \cdots$$

$f'(x)_{approx} \approx 1 + x + \frac{x^2}{2} + \frac{h^2}{6} + \cdots$

Real derivative is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

So

$$f'_{approx} - f'_{exact} = \frac{h^2}{6} + \cdots - \frac{x^3}{6} + \cdots$$

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If $x = 0$ & $h = 0.1$

$f'_{approx} - f'_{exact} = \frac{0.1^2}{6} + \cdots \approx 0.001667$

$h = 0.01$

$= \frac{0.01^2}{6} = 0.000017$

$\Bigg\}$ Central Diff.

What about f'' expressions?

Look at

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \cdots$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \cdots$$

Add & get

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

$$f'' \approx \frac{f(x+h) - 2f(x) + f(x-h) + O(h^4)}{h^2}$$

$$\boxed{f''(x) \approx \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] + O(h^2)}$$ Central f'' $O(h^2)$

Example $f(x) = f'(x) = f''(x) = e^x$ $\quad x=0 \quad h = 0.1,\ 0.01$ (7)

$$f''(x) \approx \frac{1}{h^2}\left[e^{x+h} - 2e^{x} + e^{x-h} \right]$$

$$= \frac{e^{h} - 2 + e^{-h}}{h^2}$$

$$= \frac{e^{0.1} - 2 + e^{-0.1}}{(.1)^2} = 1.000834 \qquad h=0.1$$

$$= \frac{e^{0.01} - 2 + e^{-0.01}}{(.01)^2} = 1.000010 \qquad h=0.01$$

real value $f'' = e^x\big|_{x=0} = 1$

Using Lagrange Polynomials (8)

Approximate $f(x)$ with Lagrange Polynomials

using the points x_0 $\quad x_1 = x_0 + h$ $\quad x_2 = x_0 + 2h$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$+ \underbrace{\frac{(x-x_0)(x-x_1)(x-x_2)}{3\cancel{4}!} f^{(3)}(\xi(x))}_{\text{error } O(h^3) \text{ from } 3^{rd} \text{ order poly}}$$

$$f' = \frac{x-x_2 + x-x_1}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{x-x_2 + x-x_0}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{x-x_1 + x-x_0}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$+ \left(2^{nd} \text{ order Poly in } x\right) \frac{f^{4}(\xi(x))}{3\cancel{4}!} + \frac{(x-x_0)(x-x_1)(x-x_2)}{3\cancel{4}!} \frac{d}{dx} f^{(3)}(\xi(x))$$

Will be picking x as x_0, x_1 or x_2 so this $= 0$

and most of them, but not all

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$$\left[f'(x) = \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{2x - x_0 - x_2}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{2x - x_0 - x_1}{(x_2-x_0)(x_2-x_1)} f_2 + O(x-x_i)^2 \right]$$

A) Let $x = x_0$ $\quad x_1 = x_0 + h \quad x_2 = x_0 + 2h$

$$f'(x_0) = \frac{2x_0 - x_0 - h - x_0 - 2h}{(-h)(-2h)} f_0 + \frac{2x_0 - x_0 - x_0 - 2h}{(h)(-h)} f_1 + \frac{2x_0 - x_0 - x_0 - h}{(2h)(h)} f_2 + O(h^2)$$

$$= \frac{-3h}{2h^2} f_0 + \frac{-2h}{-h^2} f_1 + \frac{-h}{2h^2} f_2 + O(h^2)$$

$$f'(x_0) \simeq \frac{1}{h}\left[-\frac{3}{2} f_0 + 2 f_1 - \frac{1}{2} f_2 \right] + O(h^2)$$

B) Let $x = x_1$ $\quad x_1 = x_0 + h \quad x_2 = x_0 + 2h$

$$f'(x_1) = \frac{2x_0 + 2h - x_0 - h - x_0 - 2h}{(-h)(-2h)} f_0 + \frac{2x_0 + 2h - x_0 - x_0 - 2h}{h(-h)} f_1 + \frac{2x_0 + 2h - x_0 - x_0 - h}{2h \cdot h} f_2 + O(h^2)$$

$$= \frac{-h}{2h^2} f_0 + 0 f_1 + \frac{h}{2h^2} f_2 + O(h^2)$$

$$f'(x_1) \simeq \frac{1}{h}\left[-\frac{1}{2} f_0 + \frac{1}{2} f_2 \right] + O(h^2)$$

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C) $x = x_2$

$$f'(x_2) \simeq \frac{x_0 + 2h - x_0 - h}{(-h)(-2h)} f_0 + \frac{x_0 + 2h - x_0}{h(-h)} f_1 + \frac{2x_0 + 4h - x_0 - x_0 - h}{2h \cdot h} f_2 + O(h^2)$$

$$= \frac{h}{2h^2} f_0 - \frac{2h}{h^2} f_1 + \frac{3h}{2h^2} f_2 + O(h^2)$$

$$\left[f'(x_2) \simeq \frac{1}{h}\left[\frac{1}{2} f_0 - 2 f_1 + \frac{3}{2} f_2 \right] + O(h^2) \right]$$

Now change your reference position.

or relative to point of evaluation

$$\left\{ \begin{aligned} f'(x_0) &= \frac{1}{h}\left[-\frac{3}{2} f(x_0) + 2 f(x_0+h) - \frac{1}{2} f(x_0+2h) \right] && \text{Forward} \\ f'(x_0) &= \frac{1}{h}\left[-\frac{1}{2} f(x_0-h) + \frac{1}{2} f(x_0+h) \right] && \text{Central} \\ f'(x_0) &= \frac{1}{h}\left[\frac{1}{2} f(x_0-2h) - 2 f(x_0-h) + \frac{3}{2} f(x_0) \right] && \text{Backward} \end{aligned} \right.$$

These are all $O(h^2)$ accurate

Example $f(x) = e^{x}$ (11)

$h=0.1 \qquad x=0$

$$f'(0) = \frac{1}{0.1}\left[-\tfrac{3}{2}e^{0} + 2e^{0.1} - \tfrac{1}{2}e^{0.2}\right] = 0.996405$$

$$f'(0) = \frac{1}{0.1}\left[-\tfrac{1}{2}e^{-0.1} + \tfrac{1}{2}e^{0.1}\right] = 1.001668$$

$$f'(0) = \frac{1}{0.1}\left[\tfrac{1}{2}e^{-0.2} - 2e^{-0.1} + \tfrac{3}{2}e^{0}\right] = 0.996905$$

$h=0.01$

$$f'(0) = \frac{1}{0.01}\left[-\tfrac{3}{2}e^{0} + 2e^{0.01} - \tfrac{1}{2}e^{0.02}\right] = 0.999966$$

$$\frac{1}{0.01}\left[-\tfrac{1}{2}e^{-0.01} + \tfrac{1}{2}e^{0.01}\right] = 1.000017$$

$$\frac{1}{0.01}\left[\tfrac{1}{2}e^{-0.02} - 2e^{-0.01} + \tfrac{3}{2}e^{0}\right] = 0.999967$$

Can also use 5 point formulas (12)

Example in book

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right] + O(h^4)$$

central

Central

and

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right] + O(h^4)$$

Forward

EXAMPLE

Central with $x=0$ $h=0.1$

$$f'(0) = \frac{1}{12(0.1)}\left[e^{-0.2} - 8e^{-0.1} + 8e^{0.1} - e^{0.2}\right] = 0.999997$$

H.W. §4.1 1a 2a 3a 21

Show error is $O(h^2)$ for central diff $f'(x_0)$

H.W. $f = x^4$ on $-1 \le x \le 1$ $\quad x_0=-1 \quad x_1=0 \quad x_2=1$

~~Match~~ Match slopes at ends using $O(h)$ Forward & Back

$f' = \frac{f_1 - f_0}{h} \qquad f' = \frac{f_2 - f_1}{h}$

over

How about using $P'' \simeq f''$. Get

$$P'' \simeq f'' = \frac{2}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{2}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{2}{(x_2-x_0)(x_2-x_1)} f_2 + O(x-x_i)$$

$$= \frac{2}{(-h)(-2h)} f_0 + \frac{2}{(h)(-h)} f_1 + \frac{2}{(2h)(h)} f_2 + O(x-x_i)$$

$$f'' = \frac{1}{h^2} f_0 - \frac{2}{h^2} f_1 + \frac{1}{h^2} f_2 + O(h)$$

so

$$f''\Big|_{x_0} = f''\Big|_{x_1} = f''\Big|_{x_2} = \frac{1}{h^2}\left(f_0 - 2f_1 + f_2\right)$$

or shifting your reference frame

$$f''\Big|_{x} = \frac{1}{h^2}\Big(f(x) - 2f(x+h) + f(x+2h)\Big) \quad \text{forward}$$

$$= \frac{1}{h^2}\Big(f(x-h) - 2f(x) + f(x+h)\Big) \quad \text{central}$$

$$= \frac{1}{h^2}\Big(f(x-2h) - 2f(x-h) + f(x)\Big) \quad \text{backward}$$

all with $O(h)$ error

Recall that we arrived at the 2-pt <u>forward</u> <u>approx</u> to f' by writing

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \dots$$

$\uparrow$ and solving for this

But we could have said

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \dots \qquad (a)$$

$$f(x) = f(x) + 0 f' + 0 f'' + \dots \qquad (b)$$

and noted that (a) - (b) gives

$$f(x+h) - f(x) = \underset{\uparrow \text{solve}}{h f'(x)} + \frac{h^2}{2} f''(x) + \dots$$

so

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

which is the same result as before.

And to get the <u>central</u> difference we said

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + O(h^3) \qquad (a)$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + O(h^3) \qquad (b)$$

and noted that if you subtract them you get

$$f(x+h) - f(x-h) = \underset{\uparrow \text{ solve}}{2h f'(x)} + O(h^3)$$

so

$$f'(x) = \frac{f(x+h) - f(x+h)}{2h} + O(h^2)$$

But we could have said that

$$f(x+h) = \cdots \qquad (a)$$

$$f(x) = f(x) \qquad (c)$$

$$f(x-h) = \cdots \qquad (b)$$

and noted that $(a) + 0(c) + (-1)(b)$ gives

$$f(x+h) - f(x-h) = 2h f'(x) + O(h^3)$$

so $$f'(x) = \frac{f(x+h) - f(x+h)}{2h} + O(h^2)$$

which is still the same result as above.

In this context things like

$$f'(x) = \frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[f(x+h) - f(x) \right] \qquad \text{forward 2 pt}$$

$$= \frac{f(x+h) - f(x-h)}{2h} = \frac{1}{h}\left[f(x+h) + 0 \cdot f(x) - f(x-h) \right] \qquad \text{central 3 pt}$$

and

$$f'' = \frac{1}{h^2}\Big(f(x) - 2f(x+h) + f(x+2h) \Big) \qquad \text{3 pt forward}$$

$$= \frac{1}{h^2}\Big(f(x-h) - 2f(x) + f(x+h) \Big) \qquad \text{3 pt central}$$

all are just weighted averages of values of $f(x)$ at different locations.

So, $$f'(x) = a f(x+h) + b f(x) + c f(x-h)$$

how do you find a, b, & c without just "seeing it"?

Ex | Taylor Matching (17)

Lets try $f' = af(x+h) + bf(x)$

$$= a\left[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots\right]$$
$$+ b\left[f(x)\right]$$

Match coef on

$f(x)$ $\quad 0 = a + b \Rightarrow 0 = a + b$

$f'(x)$ $\quad 1 = ah + 0b \Rightarrow \frac{1}{h} = a$

$\left.\right\}$ $a = 1/h$, $b = -1/h$

$f''(x)$ $\quad 0 = a\frac{h^2}{2} \Rightarrow a \ne 0$ error.

error is of size $a\frac{h^2}{2} = \left(\frac{1}{h}\right)\frac{h^2}{2} = O(h)$

So

$$f'(x) = \left(\frac{1}{h}\right)f(x+h) + \left(\frac{-1}{h}\right)f(x) + O(h)$$

$$= \frac{f(x+h) - f(x)}{h} + O(h)$$

//

Ex | Taylor Series matching. (18)

Lets try $f'(x) = af(x+h) + bf(x) + cf(x-h)$

$$= a\left[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x) + \cdots\right]$$

$$+ b\left[f(x) + 0\right]$$

$$+ c\left[f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f''' + \frac{h^4}{24}f'''' + \cdots\right]$$

Match coef. on

$f(x)$ $\quad 0 = a + b + c \Rightarrow 0 = a + b + c$

~~error~~

$f'(x)$ $\quad 1 = ah + 0b - hc \Rightarrow \frac{1}{h} = a - c$

$f''(x)$ $\quad 0 = \frac{a}{2}h^2 + 0b + \frac{c}{2}h^2 \Rightarrow 0 = a + c$

$\left.\right\}$ $a = \frac{2}{h}$, $b = 0$, $c = \frac{-2}{h}$

$f'''(x)$ $\quad 0 = \frac{a}{6}h^3 + 0b - \frac{c}{6}h^3 \Rightarrow 0 = a - c$ ~~ok~~ error

$f''''(x)$ $\quad 0 =$ error is size $\frac{1}{6}ah^3 = \frac{1}{6}\left(\frac{2}{h}\right)h^3 = O(h^2)$

or $\frac{1}{6}ch^3 = \frac{1}{6}\left(\frac{-2}{h}\right)h^3 = O(h^2)$

$$\therefore \quad f'(x) = \frac{2}{h}f(x+h) + 0f(x) - \frac{2}{h}f(x-h) + O(h^2)$$

$$= \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

//

$$f''(x) = a f(x-h) + b f(x) + c f(x+h)$$

$$= a\left[f - hf' + \frac{h^2}{2}f'' - \frac{h^3}{6}f''' + \cdots\right]$$

$$+ b\left[f \qquad\qquad\qquad\qquad\right]$$

$$+ c\left[f + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \cdots\right]$$

f $\quad 0 = a + b + c \qquad a = c = 1/h^2$

f' $\quad 0 = -ah + ch \qquad b = -2a = -2/h^2$

f'' $\quad 1 = \frac{h^2}{2}(a + c)$

f''' $\quad 0 = \frac{h^3}{6}(c - a)$ ok

f'''' $\quad 0 = \frac{h^4}{24}(a + c)$ error $\quad |\text{error}| = \frac{h^4}{24} \cdot 2\frac{1}{h^2} = O(h^2)$

$$f''(x) = \frac{1}{h^2}\left[f(x-h) - 2f(x) + f(x+h)\right] + O(h^2)$$

$$f''(x) = a f(x) + b f(x+h) + c f(x+2h)$$

$$= a\left[f \qquad\qquad\qquad\qquad\right]$$

$$+ b\left[f + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \frac{h^4}{24}f'''' + \right]$$

$$+ c\left[f + 2hf' + \frac{(2h)^2}{2}f'' + \frac{(2h)^3}{6}f''' + \frac{(2h)^4}{24}f'''' + \cdots\right]$$

f $\quad 0 = a + b + c \qquad a = c = 1/h^2$

f' $\quad 0 = bh + c2h \qquad b = -2/h^2$

f'' $\quad 1 = \frac{h^2}{2}b + \frac{4h^2}{2}c$

f''' $\quad 0 = \frac{h^3}{6}b + \frac{8h^3}{6}c$ error $\quad |\text{error}| = h^3 \cdot \frac{1}{h^2} = O(h)$

so

$$f''(x) = \frac{1}{h^2}\left[f(x) - 2f(x+h) + f(x+2h)\right] + O(h)$$