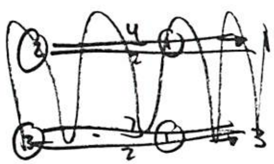
Gaussian Elimination

1

$$-\frac{4}{2}\times \quad + \to \quad \begin{pmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & \cancel{8}\ 12 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 3 \end{pmatrix}$$



$$-\frac{4}{2}(1) + (2) \to (2)$$

$$-\frac{8}{2}(1) + (3) \to (3)$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 4 \\ 0 & -1 & 0 & 6 \\ 0 & - & -- & -- \end{array}\right)$$

Pivot element

$$E_j - \frac{a_{ji}}{a_{ii}} E_i \to E_j$$

$$(j) - \frac{a_{ji}}{a_{ii}} (i) \to (j)$$

Gaussian Elimination

2

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & & a_{1n} & b_1 \\ 0 & a_{22} & & & \vdots & b_2 \\ \vdots & & a_{33} & & \vdots & b_3 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} & b_n \end{array}\right)$$

Now solve for:

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n} x_n}{a_{n-1\,n-1}}$$

$$\vdots$$

$$\boxed{x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}}$$

Back Substitution

If you continued the elimination process would get to the point

$$\left[\begin{array}{cccc|c} a_{11} & & & 0 & b_1 \\ & a_{22} & & & \vdots \\ & & \ddots & & \\ 0 & & & a_{nn} & b_n \end{array}\right]$$

$x_n = b_n / a_{nn}$

$\vdots$

$$\boxed{x_i = b_i / a_{ii}}$$

<u>Gauss-Jordan Method</u>.

Is ok but requires more calculations!

$\boxed{\text{Round-off errors}}$

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}$$

can get $\quad \tilde{\underline{x}} = \begin{pmatrix} -7.2 \\ 14.6 \\ -2.5 \\ 3.1 \end{pmatrix} \quad A\tilde{\underline{x}} = \begin{pmatrix} 31.9 \\ 23.1 \\ 32.9 \\ 31.1 \end{pmatrix}$

but $\quad \tilde{\underline{x}} = \begin{pmatrix} 0.18 \\ 2.36 \\ 0.65 \\ 1.21 \end{pmatrix} \quad A\tilde{\underline{x}} = \begin{pmatrix} 31.99 \\ 23.01 \\ 32.99 \\ 31.01 \end{pmatrix}$

Real solution $\quad \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$!

Can define <u>Residual</u> $\quad \underline{r} = \underline{b} - A\tilde{\underline{x}}$

$\underline{r} = \underline{0} = \underline{b} - \underline{A}\,\underline{x}_{exact}$

Ill-Conditioned

well | Ill

$$\left.\begin{aligned} 0.9999x - 1.0001y &= 1 \\ x - y &= 1 \end{aligned}\right\} \quad \begin{aligned} x &= 0.5 \\ y &= -0.5 \end{aligned}$$

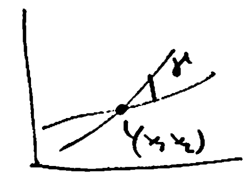
But

$$\left.\begin{aligned} 0.9999x - 1.0001y &= 1 \\ x - y &= 1+\varepsilon \end{aligned}\right\} \quad \begin{aligned} x &= 0.5 + 5000\varepsilon \\ y &= -0.5 + 4999.5\varepsilon \end{aligned}$$

So small change in ε greatly changes solution

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \underset{\sim}{A}\,\underset{\sim}{x} = \underline{b}$$

Is intersection between two lines



turns out that

$$\tan\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}a_{21} + a_{12}a_{22}}$$

So if $\gamma \to 0$ $\left(\frac{\text{numerator}}{\text{denom}}\right) \to 0 \Rightarrow$ numerator $\to 0$

but numerator is $\det A$.

So apparently ill-conditioned ~ close to singular!

singular $\Rightarrow$ lines are close to colinear
— i.e. linearly dependent equations

- If the RHS values came from a formula just use more significant figure

- If RHS was experimental data — trouble will not get good solution.

- Symptoms of ill conditioning

  - $|\det A| << |a_{ij}\ max|$ or $|b_i\ max|$

  - Poor approximate solutions will give small residuals

  - Elements of $\underline{A}^{-1}$ are large compared to elements of $\underline{A}$.

- Well - Conditioned
  - $|$diagonal element$| >> |$off diagonal element$|$
  - elements of A are $\frac{1}{10} \sim 10$ and elements of $A^{-1} \sim O(1)$

What to do before / during using Gauss-Elimination

- 1) Reformulate problem if necessary!

ADVANCE

1) Want largest coeff in rows to be of comparable magnitude. So multiply through by appropriate constants

ADVANCE

2) Rearrange equations by rows and colums (switch order of variables) to place largest element on diagonal.

In progress

3) Re-order rows to see that a_{ii} is the largest of a_{ij} for $j>i$. If not reorder.

(If you get all zeros, you do not have a unique solution!)

Iterative Technique

Gauss-Seidel Method.

— Same comments apply to pre-processing of A.

— $\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 \; a_{n2}x_2 \ldots a_{nn}x_n = b_n \end{array} \right\}$ system of linear eqns.

Solve for x_i:

$$x_1 = \frac{b_1 - a_{12}x_2 - \cdots a_{1n}x_n}{a_{11}}$$

$$\vdots$$

$$x_i = \frac{b_i - a_{i1}x_1 - \cdots a_{i\,i-1}x_{i-1} - a_{i\,i+1}x_{i+1} \cdots a_{in}x_n}{a_{ii}}$$

$$\vdots$$

$$x_n = \frac{b_i - a_{n1}x_1 - a_{n2}x_2 \cdots a_{n\,n-1}x_{n-1}}{a_{nn}}$$

$$x_i = b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j \Big/ a_{ii}$$

Can start with first approx

$$\left(\begin{array}{l} x_1 = b_1/a_{11} \\ \vdots \\ x_i = b_i/a_{ii} \\ \vdots \\ x_n = b_n/a_{nn} \end{array} \right)$$

and put int RHS to predict new x_i

$$\boxed{x_i^* = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j}{a_{ii}}}$$

Can improve iterates by using good values for some of the x_i's.

$$\boxed{x_i^* = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^* - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}}$$ Gauss - Seidel Method.

Residuals

$\underline{r} = \underline{b} - \underline{A}\,\underline{x}$ $\qquad$ $\underline{r} = 0 = \underline{b} - \underline{A}\,\underline{x}_c$

So $r_i = b_i - \sum_{j=1}^{n} a_{ij} x_j$

$$= b_i - \left[\sum_{j=1}^{i-1} a_{ij} x_j + a_{ii} x_i + \sum_{j=i+1}^{n} a_{ij} x_j \right]$$

can also be x_j^*

$$= b_i - a_{ii} x_i - \left[\sum_{j=1}^{i-1} a_{ij} x_j^* + \sum_{j=i+1}^{n} a_{ij} x_j \right]$$

But recall Gauss-Seidel $x_i^* = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^* - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$

so

$r_i = a_{ii} x_i^* - a_{ii} x_i$

$x_i^* - x_i = \dfrac{r_i}{a_{ii}}$

$$\boxed{x_i^* = x_i + \frac{r_i}{a_{ii}}}$$

when $\boxed{r_i = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^* - \sum_{j=i}^{n} a_{ij} x_j}$ (use best info possible)

Can also be used as an iteration scheme.

Goal is to reduce the Residual vector.

ie as $\underline{r} \to 0$ then $\underline{x}$ converges.

Modification - Relaxation Methods.

$$x_i^* = x_i + \omega \frac{r_i}{a_{ii}}$$

ω is the relaxation parameter.

$0<\omega<1$ under-relaxation 13

$1<\omega$ over-relaxation

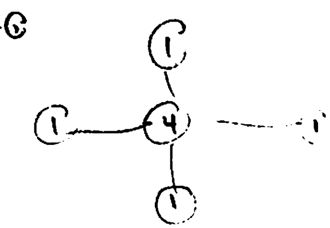
- Some systems do not converge (quickly) using Gauss Seidel but will do so for $0<\omega<1$.

- When using systems to solve PDEs can use over-relaxation to speed convergence.

$$\left.\begin{array}{l}\dfrac{\partial^2 f}{\partial x^2}=0 \\[2ex] \dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0\end{array}\right\}\text{ Boundary Value Problems.}$$

$$\frac{1}{h^2}\left[f_{-1,j}-2f_{0,j}+f_{1,j}\right] \qquad \frac{1}{h^2}\left[f_{i,-1}-2f_{i,0}+f_{i,1}\right]$$

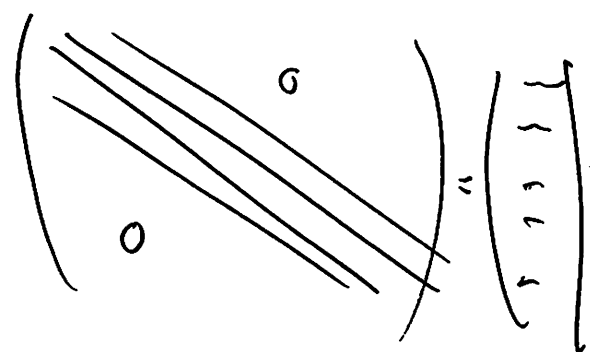
1 − (−2) − 1



stencil

Get big system of equations for values of f_{ij}. Often quickly solved using iterative techniques. 14

Will get systems with diagonals that are zero,



END