Back to Newton's method... looking for $f(x)=0$ (1)

Make T.S. expansion for $f(x)$ with center x_0 ...

$$f(x) = f(x_0) + (x-x_0)\frac{f'(x_0)}{1!} + (x-x_0)^2\frac{f''(x_0)}{2!} + \dots$$

Truncate here

so

$$f(x) \approx f(x_0) + (x-x_0)\frac{f'(x_0)}{1!}$$

solve for this x...

$$x = x_0 + \frac{f(x) - f(x_0)}{f'(x_0)}$$

let $x \approx p$ ($f(p)=0$).

$$p \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

an approximation to p. Call it p_1

Recall x_0 was an initial guess for p

or

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$ Same Newton's Method

warning !!!

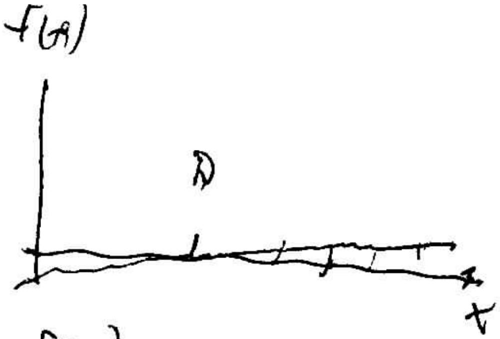
Problem with Newton's mtd...

$$\underset{\text{new value}}{P_{n+1}} = \underset{\text{old value}}{P_n} - \underbrace{\frac{f(P_n)}{f'(P_n)}}_{\text{"small" correction}}$$

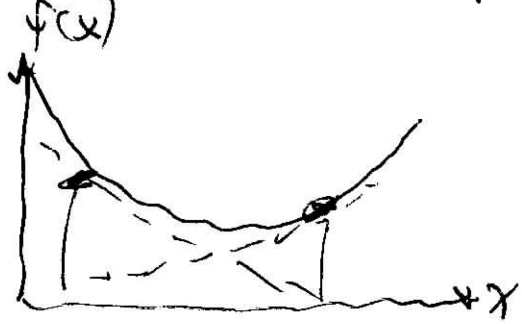
"Breaks" if...

1) $f'(P_n) = 0$ Trouble!

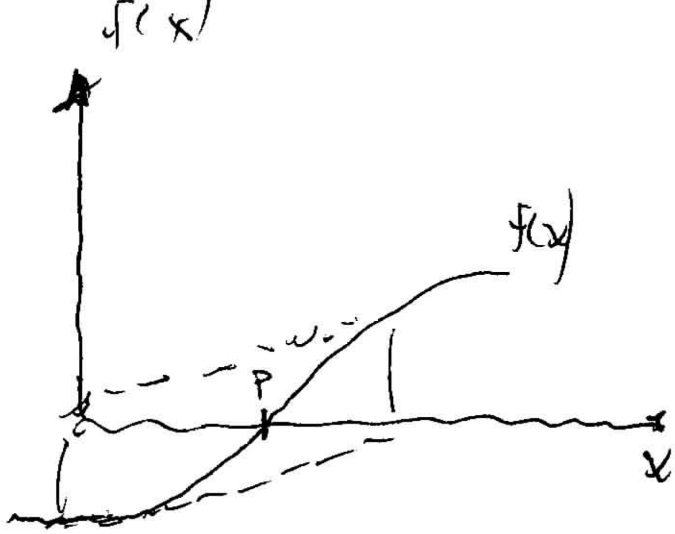
2) if $f' \to 0$ slow convergence near P



3) Local min



4) ?



Inflection pt.
(but simple root)

Convergence (when to stop?)

P_0

P_1

P_2

$\vdots$

P_n

P_{n+1}

$\vdots$

(1) $|P_{n+1} - P_n| \stackrel{?}{=} 0$ Bad ~~idea~~ idea

$< \varepsilon$ better idea (absolute)

(2) $|f(P_{n+1})| < \varepsilon$

(3) $\left|\dfrac{P_{n+1} - P_n}{P_n}\right| < \varepsilon$ (rel)

(4) Pull the plug!

Rate of convergence

Fix. Pt. Iter. $\quad x = g(x)$ $\quad$ let error

$$e_n \equiv r - x_n$$

$$\uparrow \qquad \uparrow$$

$$g(r) \qquad g(x_{n-1})$$

Make T.S for $g(r)$ with center of x_{n-1}

$$g(r) = g(x_{n-1}) + \underbrace{(r - x_{n-1})g'(x_{n-1}) + \frac{(r-x_{n-1})^2}{2} g''(x_{n-1}) + \cdots}$$

Taylor error!

$$= (r - x_{n-1})\, g'(\xi)$$

where $x_{n-1} \le \xi < r$

$\uparrow$ mystery location

So $e_n = r - x_n = g(r) - g(x_{n-1})$

$$= (r - x_{n-1})\, g'(\xi)$$

$$= e_{n-1} \cdot g'(\xi)$$

$$e_n \approx g'(r)\, e_{n-1}$$

new error $\sim g'(r)$ old error $\quad$ linear conv.

or $\boxed{|(r - x_n) \sim g'(r)(r - x_{n-1})|}$

So for $\boxed{\text{fixed Pt. Iteration}}$ we have

$$r - x_{n+2} \approx g'(r)(r - x_{n+1})$$

and

$$r - x_{n+1} \approx g'(r)(r - x_n)$$

$\therefore$

$$\frac{(r - x_{n+2})}{(r - x_{n+1})} \approx \frac{(r - x_{n+1})}{r - x_n} \quad \text{or} \quad \boxed{\frac{e_{n+2}}{e_{n+1}} \approx \frac{e_{n+1}}{e_n}}$$

We will exploit this property soon. Stay tuned!

What about Conv. of Newton's mtd?

T.S. of $f(x)$ with center x_{n-1} ... (truncate of order 2)

$$f(x) = f(x_{n-1}) + (x - x_{n-1}) f'(x_{n-1}) + \frac{(x-x_{n-1})^2}{2!} f''(\xi)$$

Now evaluate at $x = r$

$$\boxed{1} \quad \underbrace{f(r)}_{=0} = f(x_{n-1}) + (r - x_{n-1}) f'(x_{n-1}) + \frac{(r-x_{n-1})^2}{2} f''(\xi)$$

But Newton's used a linear approx.

$$\boxed{2} \quad \underbrace{L(x_n)}_{} = f(x_{n-1}) + (x_n - x_{n-1}) f'(x_{n-1})$$

we set $L(x_n) = 0$

Now subtract $\boxed{1} - \boxed{2}$

$$0 = (r - x_n) f'(x_{n-1}) + \frac{(r - x_{n-1})^2}{2} f''(\xi)$$

$$\approx e_n f'(r) + e_{n-1}^2 f''(r)$$

or $$\boxed{e_n \approx \frac{f''(r)}{2 f'(r)} e_{n-1}^2}$$ Quad. Conv for Newtons mtd.

If $\lim_{n \to \infty} \frac{e_{n+1}}{e_n^{\alpha}} = \lambda$ we call

α the order of conv.

λ is the (constant) asymptotic error c

So Fix. Point Iter ~ order 1

Newtons ~ order 2

observations (F.P.I required $|g'_{\text{near } r}| < 1$

Newtons fails if $f'_{\text{near } r}$ ~ too small

and it works best for "straight" functions