Direct Method of Finding

1

$$\int_{-1}^{1} f(x)\,dx = a f(-1) + b f(0) + c f(1)$$

Just start substituting in the terms $1, x, x^2, \ldots$ till you get an error.

$$\int_{-1}^{1} 1\,dx = a + b + c = x\Big|_{-1}^{1} = 2 \qquad \boxed{a+b+c=2}$$

$$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 = a\cdot(-1) + b\cdot 0 + c\cdot 1 \qquad \boxed{-a+c=0}$$

$$\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = a\cdot 1 + b\cdot 0 + c\cdot 1 \qquad \boxed{a+c=\frac{2}{3}}$$

$$\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0 = a\cdot(-1) + b\cdot 0 + c\cdot 1 \qquad \boxed{-a+c=0} \text{ ok}$$

$$\int_{-1}^{1} x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5} = a\cdot 1 + b\cdot 0 + c\cdot 1 \qquad \boxed{a+c=\frac{2}{5}}$$

Error

$$\boxed{a = c = \frac{1}{3} \qquad b = \frac{4}{3}}$$

2

So $$\boxed{\int_{-1}^{1} f(x)\,dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) + O(h^5)}$$

So far this was the same as Taylor Comparison and Polynomial Integration.

But is much more general.

Consider $$\int_{-1}^{1} f(x)\,dx = a f(-\tfrac{2}{3}) + b f(0) + c f(\tfrac{2}{3})$$

$$\int_{-1}^{1} 1\,dx = 2 = a\cdot 1 + b\cdot 1 + c\cdot 1 \qquad \boxed{a+b+c=2}$$

$$\int_{-1}^{1} x\,dx = 0 = a(-\tfrac{2}{3}) + b\cdot 0 + c(\tfrac{2}{3}) \qquad \boxed{-a+c=0}$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = a\left(-\frac{2}{3}\right)^2 + b\cdot 0 + c\left(\frac{2}{3}\right)^2 \qquad \boxed{a+c=\frac{3}{2}}$$

$$\int_{-1}^{1} x^3\,dx = 0 = a\left(-\frac{2}{3}\right)^3 + b\cdot 0 + c\left(\tfrac{2}{3}\right)^3 \qquad -a+c=0 \quad \text{ok}$$

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5} = a\left(-\frac{2}{3}\right)^4 + b\cdot 0 + c\left(-\frac{2}{3}\right)^4 \qquad a+c = \frac{2}{5}\left(\frac{3}{2}\right)^4$$

Error

$$\boxed{a = c = \frac{3}{4} \qquad b = \frac{2}{4}}$$

3

So $\int_{-1}^{1} f(x)\,dx = \frac{3}{4} f(-\tfrac{2}{3}) + \frac{2}{4} f(0) + \frac{3}{4} f(\tfrac{2}{3}) + O(h^5)$

Thus don't actually need to use $f(-1)$ $f(0)$ $f(1)$ values to do the integration

Can now get more general.

Consider $\int_{-1}^{1} f(x) \sin\frac{\pi}{2}x \, dx = a f(-1) + b f(0) + c f(1)$

$\int_{-1}^{1} 1 \cdot \sin\frac{\pi}{2}x \, dx = 0 = a+b+c$

$\int_{-1}^{1} x \sin\frac{\pi}{2}x \, dx = \frac{8}{\pi^2} = -a+c$

$\int_{-1}^{1} x^2 \sin\frac{\pi}{2}x \, dx = 0 = a+c$ $\quad -a = c = \frac{4}{\pi^2} \quad b=0$

$$\boxed{\int_{-1}^{1} f(x) \sin\frac{\pi}{2}x \, dx = \frac{4}{\pi^2}\left(-f(-1) + f(1)\right)}$$

4

H.W. Find $\int_0^1 f(x)\,dx = a f(\tfrac{1}{3}) + b f(\tfrac{2}{3})$ using direct method.

Note: the transformation

$t = \frac{2x-a-b}{b-a}$ transforms

$a \le x \le b$ into $-1 \le t \le 1$

so $\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{t(b-a)+a+b}{2}\right) \cdot \left(\frac{b-a}{2}\right) dt$

Can thus convert the above integrals to the useful form $\int_{x_0}^{x_1} f(x)\,dx = \cdots$

Gaussian Quadrature

5

Can write $\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} c_i f(x_i)$

Will now not only allow c_is to be variable but also the x_is.

Goal is to pick these to satisfy (exactly) the highest degree. We have $2n$ parameters so can get $\boxed{\text{degree} = 2n-1}$

Example from book $\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$

pick $c_1, c_2, x_1 + x_2$ to exactly satisfy $f(x) = 1, x, x^2, x^3$

$2(n=2)-1 = 3$

6

Thus

$$\left.\begin{aligned}\int_{-1}^{1} 1\,dx &= 2 = c_1\cdot 1 + c_2\cdot 1\\ \int_{-1}^{1} x\,dx &= 0 = c_1\cdot x_1 + c_2\cdot x_2\\ \int_{-1}^{1} x^2\,dx &= \tfrac{2}{3} = c_1\cdot x_1^2 + c_2\cdot x_2^2\\ \int_{-1}^{1} x^3\,dx &= 0 = c_1 x_1^3 + c_2 x_2^3\end{aligned}\right\}\quad \begin{aligned}c_1 &= c_2 = 1\\ x_1 &= -\sqrt{3}/3\\ x_2 &= \sqrt{3}/3\end{aligned}$$

So $\boxed{\int_{-1}^{1} f(x)dx \simeq f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)}$

If you do this for $n = 2, 3, 4, \dots$

$\uparrow$ just did this

will find out that the locations of $x_1, x_2, \dots x_n$ are the roots of the $\boxed{\text{Legendre Polynomials}}$

They are a set of <u>orthogonal</u> <u>functions</u> (7)

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = x^2 - \frac{1}{3}$

$P_3(x) = x^3 - \frac{3}{5}x$

$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

$\vdots$

n	root, x_i	resulting coefficients c_i
2	$\pm 1/\sqrt{3}$	1
3	$0, \pm\sqrt{3/5}$	$\frac{8}{9}, \frac{5}{9}$ (respectively)
4	$\pm\sqrt{\frac{15-\sqrt{120}}{35}}$	0.6521451549
	$\pm\sqrt{\frac{15+\sqrt{120}}{35}}$	0.3478548451

(8)

EXAMPLE $\int_0^1 e^{-x^2}\,dx$ with $n=3$

Use $t = \frac{2x-a-b}{b-a} = \frac{2x-0-1}{1-0} = \frac{2x-1}{1}$

$x = \frac{t+1}{2}$

So

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{2}\int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)^2}\,dt$$

$$\approx \frac{1}{2}\left[\frac{5}{9}f\left(t=-\sqrt{3/5}\right) + \frac{8}{9}f(t=0) + \frac{5}{9}f\left(t=+\sqrt{3/5}\right)\right]$$

$$\approx \frac{1}{2}\left[\frac{5}{9}e^{-\frac{1}{4}\left(1-\sqrt{3/5}\right)^2} + \frac{8}{9}e^{-\frac{1}{4}(1)^2} + \frac{5}{9}e^{-\frac{1}{4}\left(1+\sqrt{3/5}\right)^2}\right]$$

$\approx 0.746\,815$ (0.746 824 to 6D)

Almost as good as using Simpsons with $2n=10$ for which you get 0.746 825 but this is a <u>lot</u> <u>less</u> <u>work</u>!

Gauss Quad. with n+1 points

$$\int_a^b f(x)dx$$

$$E_n = \frac{(b-a)^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} f^{(2n+2)}(\xi)$$

$$\int_{-1}^{1} \phi(u)du = \frac{2}{b-a}\int_a^b f(x)dx$$

$$E_n = \frac{2^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} \phi^{(2n+2)}(\xi)$$

Advantage = good accuracy

Disadvantage = uneven spacing

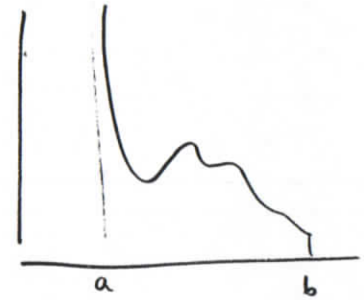
- no problem if you know $f(x)$
- trouble if you have tables – interpolation ruins accuracy gains.

H.W.	set 4.7	1 - 6	$n=2$
		2 - 6	$n=3$

Improper Integrals

Consider the integration of functions with a singularity at $x=a$ of the form

$$f(x) = \frac{g(x)}{(x-a)^p}$$



Want $\int_a^b f(x)\,dx$

Can only evaluate if $\boxed{0<p<1}$.

Several ways to do this. Book's method:

Approx $g(x)$ with n^{th} order Taylor Polynomial.

$$g(x) \approx P_n(x) = g(a) + (x-a)g'(a) + \frac{(x-a)^2}{2}g''(a) + \frac{(x-a)^3}{6}g'''(a) + \frac{(x-a)^4}{24}g''''(a) \cdots \frac{(x-a)^n}{n!}g^{(n)}(a)$$

Now consider

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx = \int_a^b \frac{g(x)-P_n(x)+P_n(x)}{(x-a)^p}\,dx$$

$$= \int_a^b \frac{g(x)-P_n(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_n(x)}{(x-a)^p}\,dx$$

integral of small deviations

Can be calculated. Worst term has $\int_a^b \frac{c}{(x-a)^p}\,dx$

Need $\int_a^b \frac{dx}{(x-a)^p} = \frac{(b-a)^{1-p}}{1-p}$

Use Simpsons Rule to calculate this. Its values can be calculated but need to define its value at $x=a$ to be $=0$.

$$G(x) = \begin{cases} \dfrac{g(x)-P_n(x)}{(x-a)^p} & a < x \le b \\ 0 & \text{if } x = a \end{cases}$$

— Can pick as many or as few points as you like. Can actually use any method of choice for integrating $G(x)$.

— In the book they use $P_4(x)$ and Simpsons Method $\frac{1}{3}$

EXAMPLE $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ $\quad p=\frac{1}{2} \quad a=0$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

So

$$\int_0^1 \frac{P_4(x)}{\sqrt{x}} = \int_0^1 \left(\frac{1}{\sqrt{x}} + \sqrt{x} + \frac{1}{2}x^{3/2} + \frac{1}{6}x^{5/2} + \frac{1}{24}x^{7/2}\right)dx$$

$$= \frac{(1-0)^{1-\frac{1}{2}}}{(1-\frac{1}{2})} + \left(\frac{2}{3}x^{3/2} + \frac{1}{2}\frac{2}{5}x^{5/2} + \frac{1}{6}\frac{2}{7}x^{7/2} + \frac{1}{24}\frac{2}{9}x^{9/2}\right)\Big|_0^1$$

$$= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{12\cdot 9}$$

$$\approx 2.923\,5450$$

Now need to evaluate other integral

$$\frac{e^x - P_4(x)}{\sqrt{x}} = G(x)$$

Will use Simpson's Rule with $n=4$, $h=0.25$

i	x_i	$G(x_i)$
0	0	0 (by definition)
1	0.25	0.000 0170
2	0.5	0.000 4013
3	0.75	0.002 6026
4	1.0	0.009 9485

$$\int_0^1 G(x)\,dx = \frac{h}{3}\left[G_0 + 4G_1 + 2G_2 + 4G_3 + G_4\right]$$

$$= \frac{0.25}{3}\left[0 + 4(0.000\,0170 + 0.002\,6026) + 2(0.000\,4013) + 0.009\,9485\right]$$

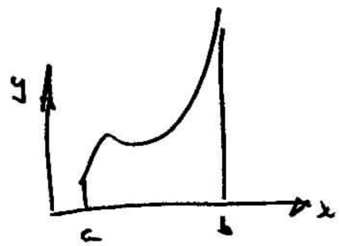
$$\simeq 0.001\,7691$$

$$\therefore \int_0^1 \frac{e^x}{\sqrt{x}}\,dx \simeq 2.923\,5450 + 0.001\,7691$$

$$\simeq 2.925\,3141$$

Other techniques

— Singularity on right:

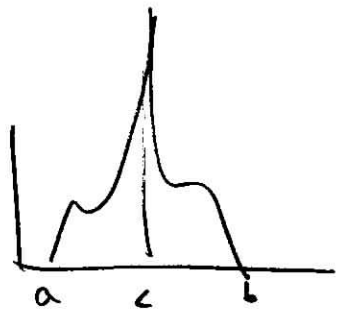


1) Expand Taylor Series about point $x=b$.

2) Let $z=-x \Rightarrow dz=-dx$

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-z)\,dz$$

— Singularity in Middle at $x=c$



Break into two integrals

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and use above methods.

15

Finally $\int_a^\infty f(x)dx$ is

handled by $t = \frac{1}{x}$ $\Rightarrow$ $dt = \frac{-1}{x^2}dx$ $= -t^2 dx$

$dx = \frac{-1}{t^2}dt$

$$\int_a^\infty f(x)dx = \int_{t=1/a}^{t=0} f\left(\frac{1}{t}\right)\left(\frac{-1}{t^2}\right)dt = \int_0^{1/a} \frac{f(1/t)}{t^2}dt$$

Is now in form that we might handle.

Need $(x-a)^p$ $0<p<1$.

- Side ways integration!

Also good for ODE's.

Monday & 4pm