Accelerating Conv. (from fix. pt. mtd)

We had a seq. $\{P_n\}_{n=0}^{\infty}$ That conv. to root $= P$.

Is there a way to speed it up.

Yes, Aitken's $\boxed{\Delta^2 \text{ method}}$.

What we know about F.P. is

$$\frac{e_{n+1}}{e_n} \approx \frac{e_{n+2}}{e_{n+1}} \approx g'(P) \text{ or } \frac{P_{n+1}-P}{P_n - P} \approx \frac{P_{n+2}-P}{P_{n+1}-P} \approx g'(P)$$

Cross mult. to get

$$P_{n+1}^2 - 2P\,P_{n+1} + \underline{\underline{P^2}} = P_n P_{n+2} - P\,P_{n+2} - P P_n + \underline{\underline{P^2}}$$

cancel

Solve linear eqation for P...

$$P = \frac{P_{n+2}P_n - P_{n+1}^2}{P_{n+2} - 2P_{n+1} + P_n} = \cdots = P_n - \frac{(P_{n+1}-P_n)^2}{(P_{n+2}-2P_{n+1}+P_n)} = \hat{P}_n$$

Use this to form a new series $\{\hat{P}_n\}_{n=0}^{\infty}$ that converges faster than $\{P_n\}_{n=0}^{\infty}$ in the sense

that $\lim_{n\to\infty} \frac{\hat{P}_n - P}{P_n - P} = 0$.

In practice... Pick P_0
Calc $P_1 = g(P_0)$
$P_2 = g(P_1)$
use these to form $\hat{P}_0$

Calc $P_3 = g(P_2) \rightarrow \hat{P}_1$ using $P_1 \; P_2 \; P_3$

Calc $P_4 = g(P_3) \rightarrow \hat{P}_2$ using $P_2 \; P_3 \; P_4$

These converge faster than these

Why not take advantage?

Steffensen's Mtd

Pick P_0
$P_1 = g(P_0)$
$P_2 = g(P_1)$
calc. what you might call $\hat{P}_0$

Now let $P_0 = \hat{P}_0$
$P_1 = g(P_0)$
$P_2 = g(P_1)$
calc a new $\hat{P}_0$

Again $P_0 = \hat{P}_0$
$P_1 = g(P_0)$
$P_2 = g(P_1)$
calc a new $\hat{P}_0$

What if $P_{n+2} - 2P_{n+1} + P_n = 0$?

You should simply use the last known P_2 as
The new P_0. No harm done. Then continue.

On to a new topic. (Seems odd, but stay tuned!)
Finding roots of $f(x) = ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ (Do you know the song?)

$\boxed{\text{Assume for now: } b > 0}$

Consider what happens if $b^2 \gg 4ac$, then

$$x = \frac{-b \pm \left(|b| + \text{small noise}\right)}{2a}.$$ If you calc. the following

$\underbrace{x_{\oplus} \text{ "small" root}}_{\text{SIN!}}$ $\underbrace{x_{\ominus} \text{ "large" root}}_{\text{Pure of heart...}}$

SIN!
Never subtract...

Pure of heart...
No problem

What to do?

Lots of Thought...

Fix it by multi. by $\dfrac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}$

then

$$x = \left(\frac{-b \pm \sqrt{b^2-4ac}}{2}\right)\left(\frac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}\right) = \dots$$

$$= \frac{b^2 - (b^2-4ac)}{2a(-b \mp \sqrt{b^2-4ac})} = \frac{2c}{-b \mp \sqrt{b^2-4ac}} = \frac{2c}{-b + \sqrt{b^2+\text{noise}}}$$

don't forget $b>0$

Now consider the ⊕ case which involves only the addition of $-b - \sqrt{b^2-4ac}$ and you no longer have the round off problem.

But the ⊖ case does have round-off errors,

So... for the small root use $\boxed{x = \dfrac{-2c}{b + \sqrt{b^2-4ac}}}$ College

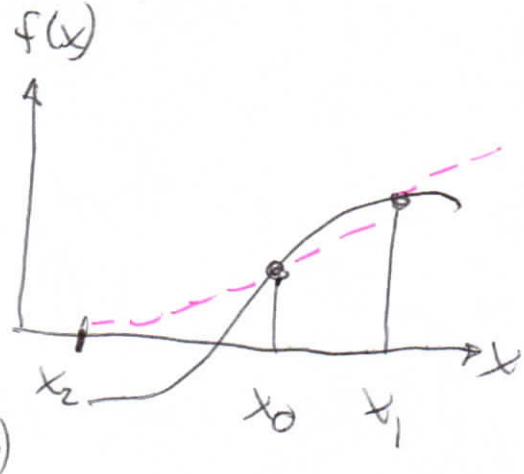
and the large root $\boxed{x = \dfrac{-b - \sqrt{b^2-4ac}}{2a}}$ Jr. High School

The point is...

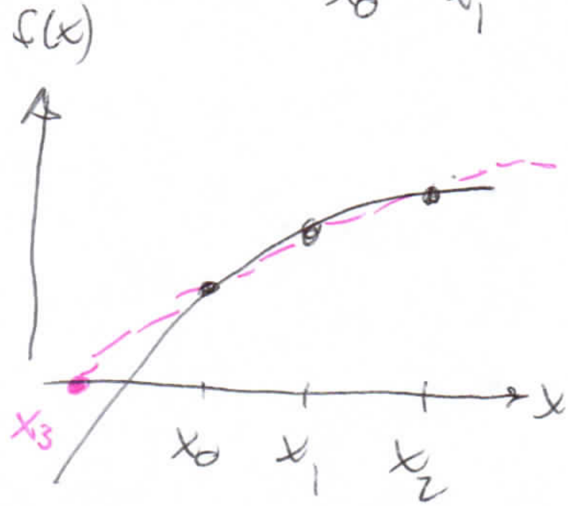
use either $b + \sqrt{b^2-4ac}$ or $-b - \sqrt{b^2-4ac}$!

Well that was fun, but who cares? (5)

Recall the <u>Secant method</u>



Would it not be better to try a parab.?



Let's try using a 2nd order Poly of the form

$$S(x) = a(x-x_2)^2 + b(x-x_2) + c$$

which is actually a T.S., second order trunc, center of x_2.

Now fit this the points $(x_0, f(x_0))$ $(x_1, f(x_1))$ $(x_2\, f_2)$

$$(x_0, f_0) \rightarrow f_0 = a(x_0 - x_2)^2 + b(x_0 - x_2) + c$$

$$(x_1, f_1) \rightarrow f_1 = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$

$$(x_2, f_2) \rightarrow f_2 = 0 \quad + 0 \quad + c$$

which leads to The following linear system...

$$\begin{pmatrix} (x_0-x_2)^2 & (x_0-x_2) & 1 \\ (x_1-x_2)^2 & (x_1-x_2) & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$$

Solve with your method of choice to get

$a = \ldots \quad b = \ldots \quad c = \ldots$ But clearly $c = f_2$.

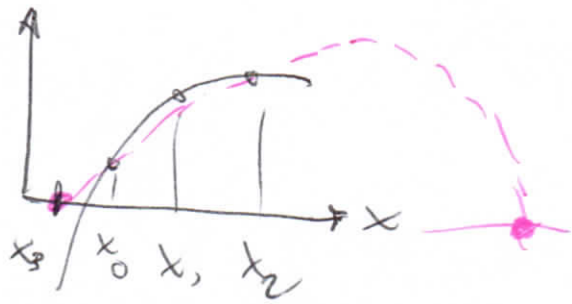
So we are really down to the system

$$\left.\begin{pmatrix} (x_0-x_2)^2 & (x_0-x_2) \\ (x_1-x_2)^2 & (x_1-x_2) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_0 - f_2 \\ f_1 - f_2 \end{pmatrix}\right\}$$ Solve for values of $a + b$

So now we have our parabola

And we want the roots of

$S(x_3) = 0$. That is from



$a(x_3-x_2)^2 + b(x_3-x_2) + c = 0$ solve for (x_3-x_2)

$$x_3 - x_2 = \underbrace{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}_{\text{Jr. High}} \text{ or } = \underbrace{\frac{2c}{-b \mp \sqrt{b^2-4ac}}}_{\text{college level}}$$

Which root do want?

How far from x_2?

I think the small root! So

$$(x_3 - x_2) = -\frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}}$$

think about this a bit...

or

$$x_3 = x_2 - \ldots$$

The general procedure is

$P_0 \; P_1 \; P_2 \rightarrow P_3$ using $a(x-x_2)^2 + b(x-x_2) + c$

drop P_0...

$P_1 \; P_2 \; P_3 \rightarrow P_4$ using $a(x-x_3)^2 + b(x-x_3) + c$

drop P_1

$P_2 \; P_3 \; P_4 \rightarrow P_5$ using $a(x-x_4)^2 + b(x-x_4) + c$

Eval. of Polynomials

$$P_n(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^n$$

How many mult. & how many (+, or -)

But consider the following calc...

$$\underbrace{x a_n + a_{n-1}}$$

$$\underbrace{x(x a_n + a_{n-1}) + a_{n-2}}$$

$$x\left[\text{sorry!}\ x(x a_n + a_{n-1}) + a_{n-2}\right] + a_{n-3}$$

$\vdots$ keep on going!

Now how many ⊗ and how many ⊕?

$$P(x) = x(x\{\cdots x[x(x a_n + a_{n-1}) + a_{n-2}] + a_{n-3} \cdots\} + a_0$$

Can we mechanize this?

of course! Otherwise why would I ask? Consider

$$\boxed{\begin{aligned} &b_{n+1} = 0 \\ &b_k = a_k + x b_{k+1} \quad \text{for } k = n, n-1, n-2, \ldots, 1, 0 \end{aligned}}$$

then $\boxed{P(x) = \sum_{k=0}^{n} a_k x^k = b_0}$

so...

$$b_n = a_n + \underbrace{b_{n+1}}_{\equiv 0} x = a_n$$

$$b_{n-1} = a_{n-1} + b_n x = a_{n-1} + a_n x$$

$$b_{n-2} = a_{n-2} + b_{n-1} x = a_{n-2} + (a_{n-1} + a_n x)x$$

$$\vdots$$

$$b_0 = a_0 + b_1 x = a_0 + \left(a_1 + \left\{ \cdots a_{n-2} + (a_{n-1} + a_n x)x \cdots \right\} x \right) x$$

It look worse than it is, but b_0 is what you want at least once you know what x is!

Ex! S'pose we want $P(x) = 2x^3 + 5x^2 + x + 3$ @ $x = 4$

Straight forward way is

$$P(4) = \underset{a_3}{2}(4)^3 + \underset{a_2}{5}(4)^2 + \underset{a_1 = 1}{(4)} + \underset{a_0 = 3}{3} = 215$$

Note that $n = 3$

check the number of $\times$ and $+$

But we could also say

$$b_4 = 0$$

$$b_3 = a_3 + b_4 \cdot (x = 4) = 2 + 0 \cdot 4 = 2$$

$$b_2 = a_2 + b_3 x = 5 + 2 \cdot 4 = 13$$

$$b_1 = a_1 + b_2 x = 1 + 13 \cdot 4 = 53$$

$$b_0 = a_0 + b_1 x = 3 + 53 \cdot 4 = 215$$

Now count the # of $\times$ and $+$

//

If you calc. $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$

divided by $(x-a)$ you get something like

$$\frac{P_n}{x-a} = \underbrace{P_{n-1}(x)}_{\substack{\text{Poly of} \\ \text{degree} \\ n-1}} + \underbrace{\left(\frac{R}{x-a}\right)}_{\text{remainder}}$$

Ex $\dfrac{P_4(x)}{(x-2)} = \dfrac{x^4 + 6x^2 - 7x + 8}{(x-2)}$ you could do this

$$\begin{array}{r|l}
 & \quad x^3 \quad + 2x^2 \quad + 10x \quad + 13 \\
\hline
(x-2) & x^4 + 0x^3 + 6x^2 - 7x + 8 \\
 & \underline{x^4 - 2x^3} \\
 & \qquad 2x^3 + 6x^2 \\
 & \qquad \underline{2x^3 - 4x^2} \\
 & \qquad\qquad 10x^2 - 7x \\
 & \qquad\qquad \underline{10x^2 - 20x} \\
 & \qquad\qquad\qquad 13x + 8 \\
 & \qquad\qquad\qquad \underline{13x - 26} \\
 & \qquad\qquad\qquad\qquad 34 = R
\end{array}$$

In other words

$$\frac{P_4(x)}{x-2} = \left(x^3 + 2x^2 + 10x + 13\right) + \frac{34}{x-2}$$

the remainder

Well, that was fun, but so what? This is the same as

$$P(x) = (x-2)(x^3 + 2x^2 + 10x + 13) + 34$$

left overs from remainder

But note that $P(2) = 34$!

To generalize... $P(x) = (x-z)Q + R$

Q = quotient
R = remainder

clearly $P(x=z) = R$

So now consider

$$P'(x) = Q(x) + (x-z)Q'(x)$$ and evaluate at $x=z$

$$P'(z) = Q(z)$$ so now we know that $P(z) = R$
$P'(z) = Q(z)$

Where are we going with this?

Well let's write $Q(x)$ in terms of a new quotient and remainder,

First, $\boxed{\text{change notation}}$...

$$P(x) = (x-z)\underline{Q_0(x)} + R_0$$ and then write

$$Q_0(x) = (x-z)Q_1(x) + R_1$$ so ...

$$P(x) = (x-z)\underbrace{\left[(x-z)Q_1(x) + R_1\right]}_{Q_0(x)} + R_0$$

$$= (x-z)^2 Q_1(x) + (x-z)\underbrace{R_1}_{P'(z)} + \underbrace{R_0}_{P(z)}$$

$P'(z)$!!! check it out! Do it... I mean it

If you proceed with

$Q_1(x) = (x-z)Q_2(x) + R_2$ then

$$P(x) = \underbrace{(x-z)^3 Q_3(x) + (x-z)^2 Q_2(x) + (x-z)Q_1(x) + Q_0}_{\text{So what is this?}}$$

Try and generalize this to see what

$Q_0(z), Q_1(z), Q_2(z), Q_3(z)$ are.

More to the point... $R_0 = P(z)$

$R_1 = P'(z)$

$R_2 = P''(z)/2!$

So then R_k = what?

So what have we actually built? Now what?