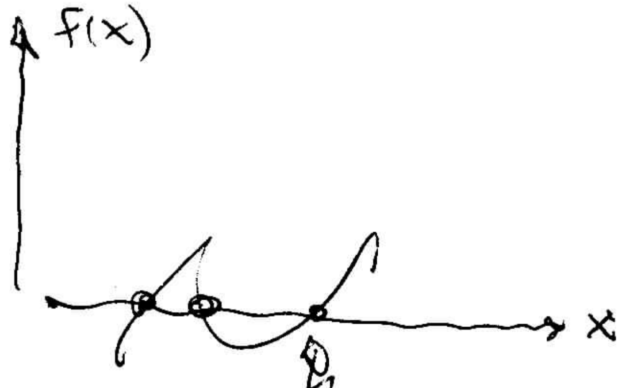
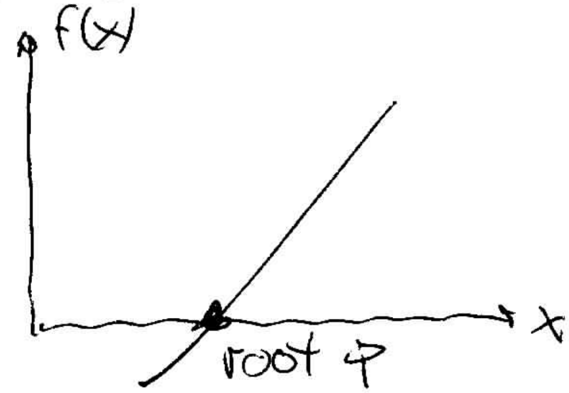
Root finding (zeros) of $f(x)=0$



Several methods...

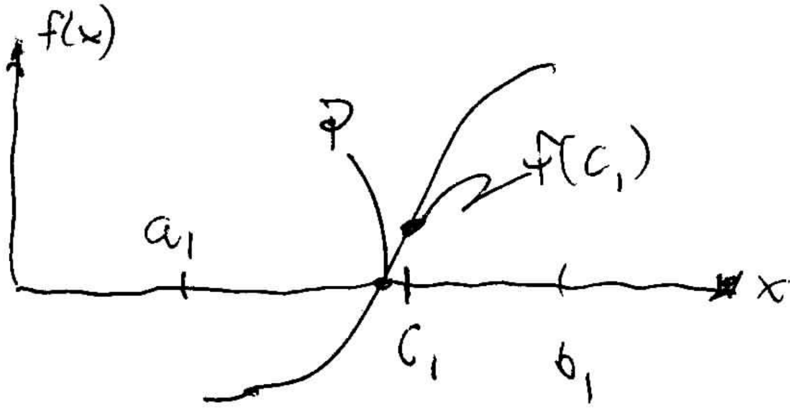
- Bisection
- Fixed point iter.
- Newton's method (Newton-Raphson)
- Secant
- False position

Bisection



- Easy method
- Always converges to a root
- Might take longer to do so

you need to trap a root with 2 estimates

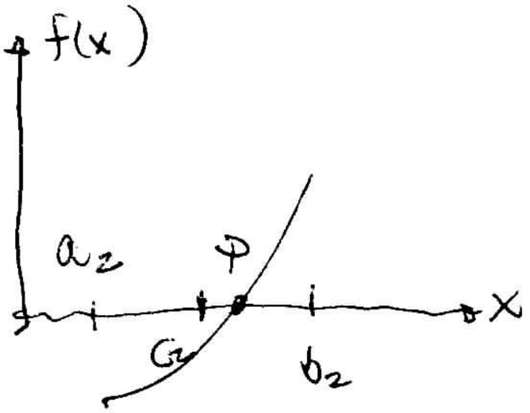


Check for sign change in f(x)!

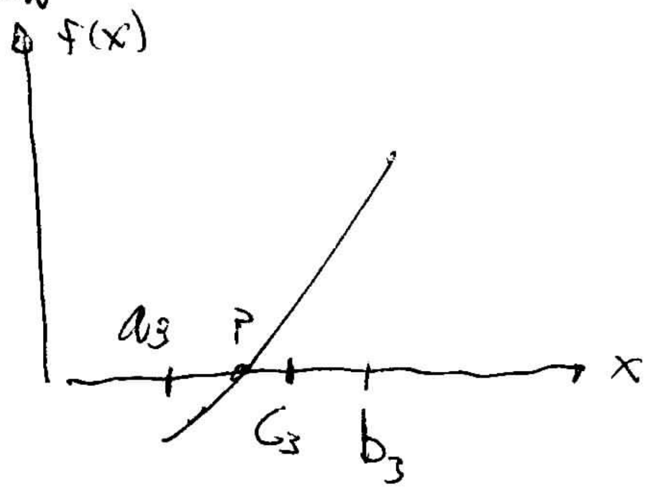
Perhaps $f(a_1)\, f(b_1) < 0$?

- Then calc $c_1 = \frac{a_1 + b_1}{2}$ (your new estimate)
- Calc $f(c_1)$
- Get rid of interval that does not contain p
- Check $f(a_1)\, f(c_1)$ > 0
  < 0
  $= 0$

We now have



then



and so on...

The new root estimate C_n is $\boxed{C_n = \frac{a_n + b_n}{2}}$

So now far away from p are you?

$$|C_n - p| \leq \frac{b_1 - a_1}{2^n} \quad \text{for } n \geq 1$$

Bisection only uses $f(x)$ values but nothing about the shape of $f(x)$.

Fixed Point Iter.

Start with $f(x)$ and we want $f(p) = 0$

Some algebra... rewrite $f(x)$ as $x = g(x)$

and then we want $\boxed{p = g(p)}$.

Works if $|g'(x)| \leq k < 1$

Smaller k gives faster conv.

- Convert to $x = g(x)$
- Pick $x = p_0$
- $p_1 = g(p_0)$
  $p_2 = g(p_1)$
  $p_3 = g(p_2)$
  $\vdots$

in general

$$p_{n+1} = g(p_n)$$

Study p_{n+1} values!!!

Question... seq. of P_n's, converge or not?

Ex) S'pose $f(x) = x + e^x = 0$ (is there even a p?)

Pick an x and solve for it...

1) $x = \underbrace{-e^x}_{g(x)}$ Pick $p_0 = -1$

then $p_1 = g(p_0) = -0.3679...$

$p_2 = g(p_1) = -0.6922...$

$p_3 = g(p_2) = -0.5005...$

$\vdots$

$p_? = -0.5671...$

2) $x = \underbrace{\ln(-x)}_{\text{new } g(x)}$ Pick $p_0 = -1$ (bad pick)

nice try

$p_0 = -0.99$

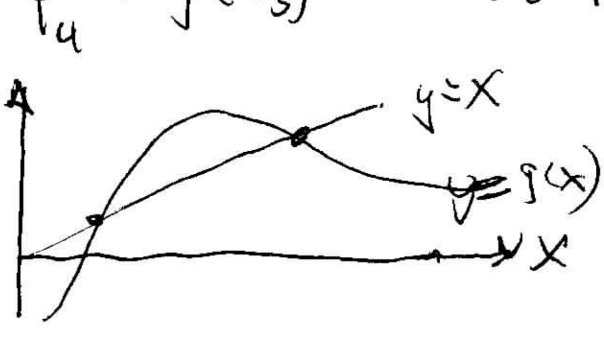
$p_1 = g(p_0) = -0.0100$

$p_2 = g(p_1) = -4.600$

$p_3 = g(p_2) = 1.526$

$p_4 = g(p_3)$ is undef.!

Note that we want



Ex) Fixed pt. iter example

$$f(x) = x^2 - 2x - 3 = 0$$
$$= (x+1)(x-3)$$ so roots are $-1, 3$

There are 3 obvious $x = g(x)$ arrangements

a) $x = g_1(x) = \sqrt{2x+3}$

Try

$x_0 = 4$

$x_1 = 3.31662$

$x_2 = 3.10375$

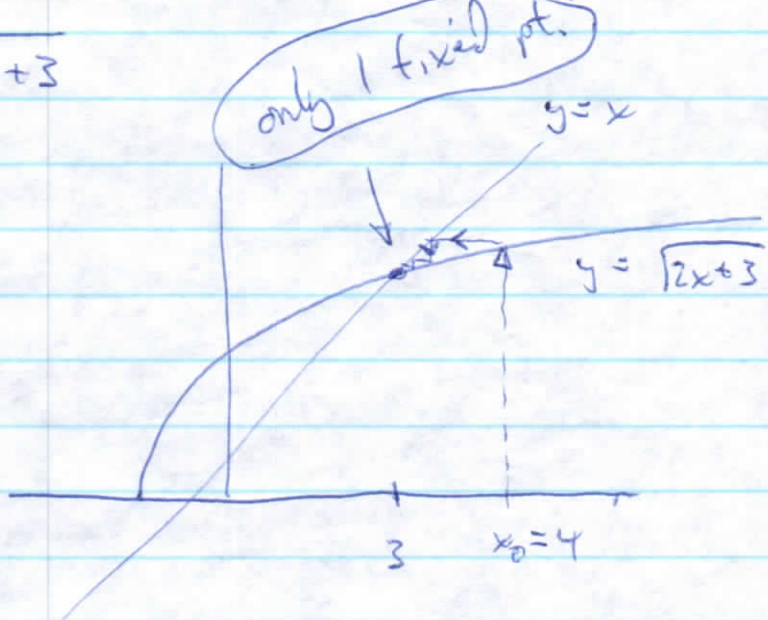
$x_3 = 3.03439$

$x_4 = 3.01144$

$x_5 = 3.00381$

$\vdots$

$x_\infty = 3$



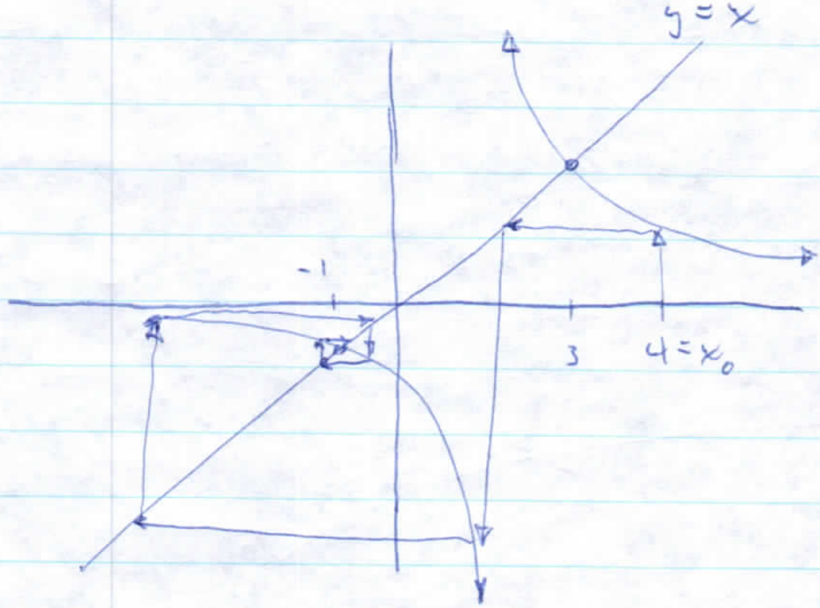
note that $g' = \dfrac{1}{2\sqrt{2x+3}}$

so near root $|g'|\Big|_{x \simeq 3} = \left|\dfrac{1}{2\sqrt{9}}\right| = \dfrac{1}{6} < 1$

so a fixed pt. exists

b) a second choice would be $x = g_2(x) = \frac{3}{x-2}$

$x_0 = 4$
$x_1 = 1.5$
$x_2 = -6$
$x_3 = -0.375$
$x_4 = -1.263158$
$x_5 = -0.919355$
$x_6 = -1.02762$
$x_7 = -0.990876$
$x_8 = -1.00305$
$\vdots$
$x_\infty = -1$



There are 2 fixed pts. now.

$$g' = \frac{-3}{(x-2)^2}$$

- Note near $x = -1$ $\quad |g'| = \left|\frac{3}{9}\right| = \frac{1}{3} < 1$

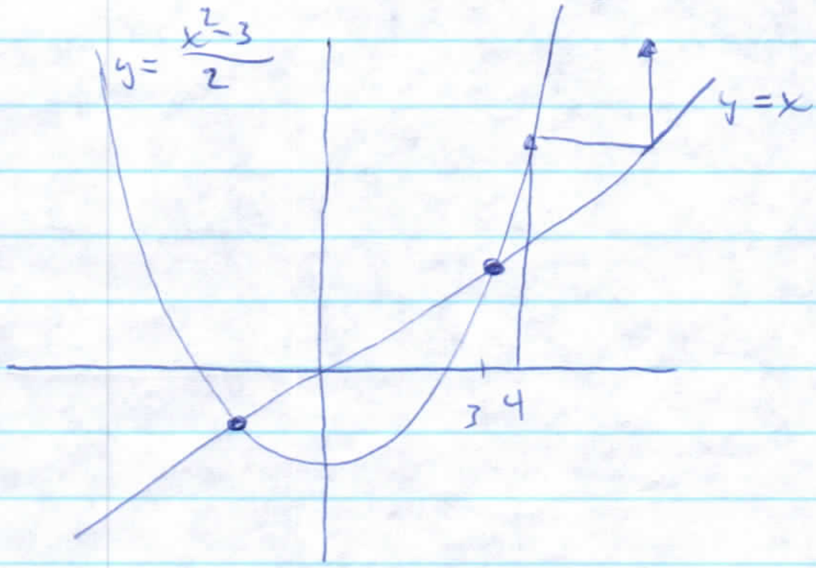
  so will converge

- Near $x = 3$ $\quad |g'| = \left|\frac{3}{1}\right| = 3 > 1$

  so not guaranteed to converge.

c) The last obvious choice is $x = g_3(x) = \dfrac{x^2-3}{2}$

$x_0 = 4$
$x_1 = 6.5$
$x_2 = 19.625$
$x_3 = 191.070$
$\vdots$
Diverges



2 fixed points

$g' = x$

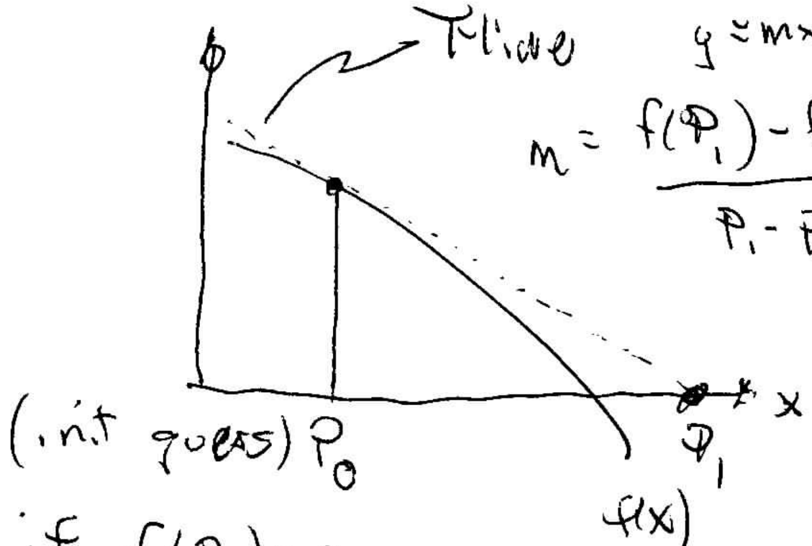
So near $x = -1$ $\quad |g'| = 1$

and near $x = 3$ $\quad |g'| = 3$

neither $|g'|$ values is < 1 on an interval surrounding the fixed pts so neither is guaranteed to converge. In fact they both diverge ~~converge~~.

Newton - Raphson Method

i) Calc I version



$y = mx + b$

$$m = \frac{f(P_1) - f(P_0)}{P_1 - P_0} = f'(P_0)$$

Solve for P_1 if $f(P_1) = 0$ ...

$$P_1 = P_0 + \frac{f(P_1) - f(P_0)}{f'(P_0)} = P_0 - \frac{f(P_0)}{f'(P_0)}$$

In general

$$\boxed{P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}} \ *$$

But what if you don't know $f'(x)$?

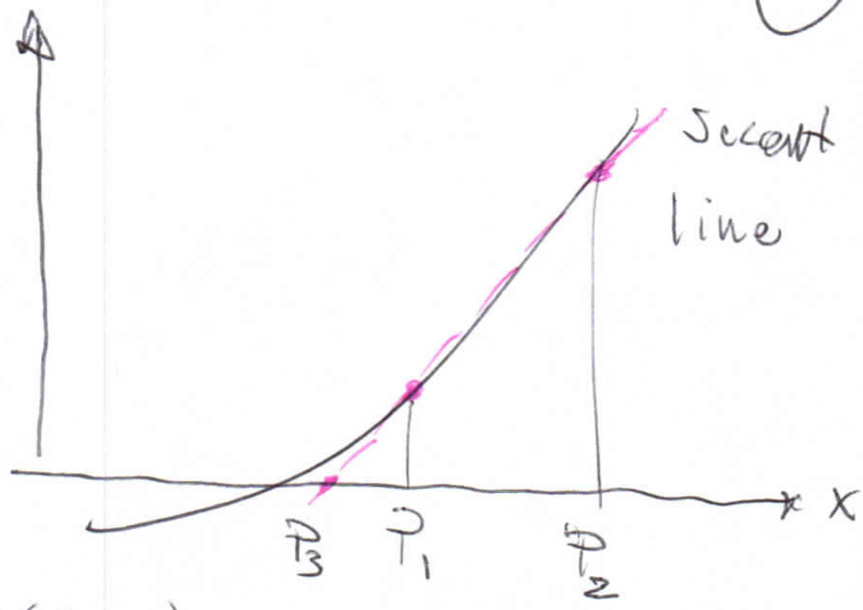
Leads to...
Secant
False pos.

Use two old (or initial guesses!) and

$$f'(P_n) \approx \frac{f(P_n) - f(P_{n-1})}{P_n - P_{n-1}} \quad *$$

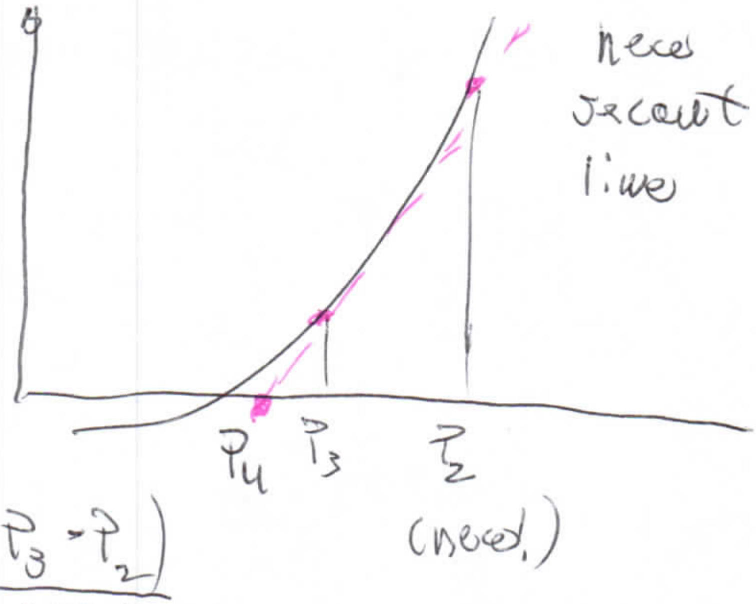
$$\boxed{P_{n+1} = P_n - \frac{f(P_n)(P_n - P_{n-1})}{f(P_n) - f(P_{n-1})}}$$

Secant Mtd.



Calc
$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)}$$

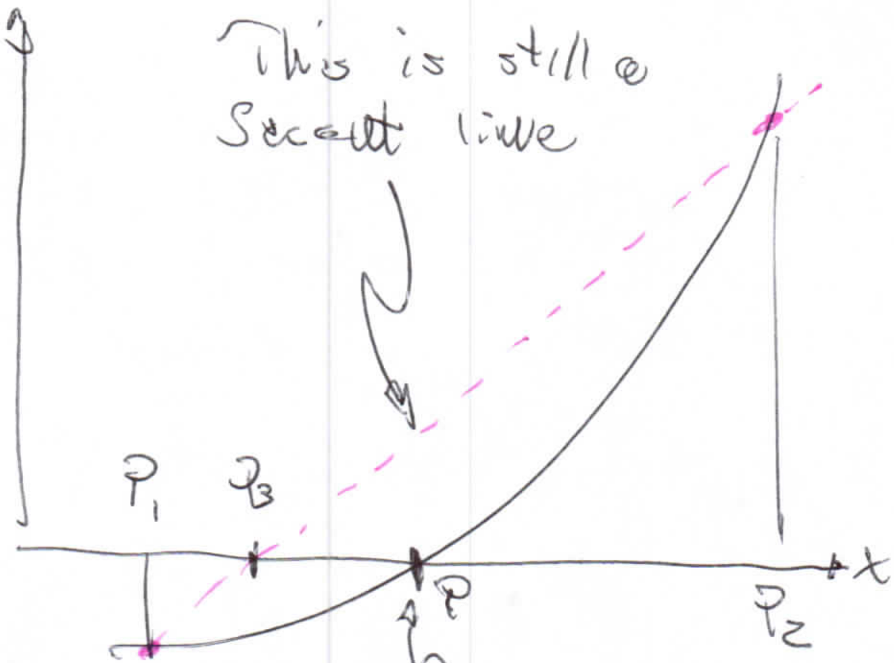
Now let p_3 & p_1 become your new p_1 & p_2!



$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)}$$

- Close to speed of Newton
- Approx of $f'(x)$
- No need to bracket p
- Might diverge!

what about root trap from Bisection...



- Speed of Secant Method
- But will converge to P

False Position

Initial thoughts on convergence...

p_0 or c_1 When do you stop?

p_1 c_2

p_2 c_3

p_3 c_4

p_4 $\vdots$

$\vdots$

First we need to discuss sins...

S'pose $a = 1.234$ $a+b = 2.467 \times 10^0$ Is OKAY

$b = 1.233$

but $a-b = 1.000 \times 10^{-3}$ Not cool!!!

But recall Secant & False Position... Yikes!
This was SIN #1.

AND now for SIN #2...

AND SIN #3... (related to CONVERGENCE!)