So we now have a way to calculate $P(z_n)$ & $P'(z_n)$

thus

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)} \quad \text{(Newton's mtd)}$$

Is there an easier method to calc $P(z_n)$ & $P'(z_n)$?

Let $P(x) = (x-z)\,Q_0(z) + R_0$

$$= (x-z)\left[b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1\right] + b_0$$

mult. out and compare to

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$= \left(b_n x^n + b_{n-1} x^{n-1} + \dots + b_2 x^2 + b_1 x\right)$$

$$- z\left(b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1\right) + b_0$$

x^n $\quad b_n = a_n$ $\qquad\qquad\qquad b_n = a_n$

x^{n-1} $\quad b_{n-1} - z b_n = a_{n-1}$ $\qquad b_{n-1} = a_{n-1} + z b_n$

$\vdots$ $\qquad\qquad\qquad\qquad\qquad\qquad \vdots$

x^1 $\quad b_1 - z b_2 = a_1$ $\qquad\qquad b_1 = a_1 + z b_2$

x^0 $\quad b_0 - z b_1 = a_0$ $\qquad\qquad b_0 = a_0 + z b_1$ $\qquad$ So...

$$\boxed{\text{let } b_{n+1} = 0 \quad \text{then } b_k = a_k + z\, b_{k+1} \quad \text{for } k = n, n-1, \dots, 1, 0}$$

We now have $b_0 = R_0 = P(z)$!!!

Now, how to find $Q_0(x) = (x-z)Q_1(x) + R_1$

$$Q_0(x) = (x-z)\left[c_n x^{n-2} + c_{n-1}x^{n-3} + \cdots + c_3 x + c_2\right] + c_1$$

$$= \left[b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1\right]$$ compare these

so

$$Q_0(x) = \left(c_n x^{n-1} + c_{n-1}x^{n-2} + \cdots + c_3 x^3 + c_2 x\right)$$
$$- z\left(c_n x^{n-2} + c_{n-1}x^{n-3} + \cdots + c_3 x + c_2\right) + c_1$$

x^{n-1}	$c_n = b_n$	$c_n = b_n$
x^{n-2}	$c_{n-1} - z c_n = b_{n-1}$	$c_{n-1} = b_{n-1} + z c_n$
$\vdots$		$\vdots$
x^1	$c_2 - z c_3 = b_2$	$c_2 = b_2 + z c_3$
x^0	$c_1 - z c_2 = b_1$	$c_1 = b_1 + z c_2$ or...

Let $c_{n+1} = 0$ then $c_k = b_k + z c_{k+1}$
for $k = n, n-1, n-2, \ldots, 2, 1$ no $k=0$

We want $c_1 = R_1 = P'(z)$

Finally $b_0 = R_0 = P(z)$
$c_1 = R_1 = P'(z)$

or

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)} = z_n - \left(\frac{b_0}{c_1}\right)_n$$

Yes!

So, ...

	$b_{n+1}=0$	$c_{n+1}=0$
a_n	$b_n = a_n + z\cdot 0$	$c_n = b_n + z\cdot 0$
a_{n-1}	$b_{n-1} = a_{n-1} + z b_n$	$c_{n-1} = b_{n-1} + z c_n$
a_{n-2}	$b_{n-2} = a_{n-2} + z b_{n-1}$	$c_{n-2} = b_{n-2} + z c_{n-1}$
$\vdots$		
a_1	$b_1 = a_1 + z a_2$	$c_1 = b_1 + z c_2$
a_0	$b_0 = a_0 + z a_1$	X

no x^3 term

Ex/ If $P(x) = 2x^4 - 3x^2 + 3x - 4$ & Initial root guess $x_0 = -2$

	$b_5 = 0$	$c_5 = 0$
$a_4 = 2$	$b_4 = 2 + (-2)\cdot 1 = 2$	$c_4 = 2 + (-2)\cdot 0 = 2$
$a_3 = 0$	$b_3 = 0 + (-2)\cdot 2 = -4$	$c_3 = -4 + (-2)(2) = -8$
$a_2 = -3$	$b_2 = -3 + (-2)(-4) = 5$	$c_2 = 5 + (-2)(-8) = 21$
$a_1 = 3$	$b_1 = 3 + (-2)(5) = -7$	$c_1 = -7 + (-2)(21) = -49$
$a_0 = -4$	$b_0 = -4 + (-2)(-7) = 10$	X

$P(-2)$ $\qquad$ $P'(-2)$

So next root estimate is

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = -2 - \left(\frac{10}{-49}\right) \approx -1.796$$

Now start over!

ok, so you finally converge to a root,
Let's call it r_1. So if we write

$$P_n(x) \doteq (x-r_1)\hat{Q}_1 + R_1 \leftarrow \text{This will be } 0!$$

$$\doteq (x-r_1)\hat{Q}_1(x) \leftarrow \text{degree } n-1$$

Now go root finding $\hat{Q}_1(x)$ ... find any one of its roots. Call it r_2. Then

$$\hat{Q}_1 = (x-r_2)\hat{Q}_2 + R_2 \leftarrow \text{This again } 0!$$

$$\doteq (x-r_2)\hat{Q}_2(x) \leftarrow \text{degree } n-2$$

... and so on. When you get down to a quad. Just use the obvious. You now have

r_1 pretty much ok

r_2

r_3 — root deflation !!!

⋮

r_{n-1}

r_n probably not so good

More later

On a side note...

$\boxed{\text{2nd order}}$ $P_2(x) = a_2 x^2 + a_1 x + a_0 = 0 \quad = x^2 + a_1 x + a_0$

a_2 always set to 1 $\quad = (x - r_1)(x - r_2)$

so

$P_2(x) = x^2 - x r_2 - x r_1 + r_1 r_2$

$= x^2 - (r_1 + r_2)x + r_1 r_2$

$\boxed{\text{3rd order}}$ $P_3(x) = (x - r_1)(x - r_2)(x - r_3)$

$= \dots$ just mult it out...

$= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x$

$- r_1 r_2 r_3$

So...

$P_n(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

$a_{n-1} = -$ sum of all roots

$a_{n-2} = +$ sum of all roots taken 2 at a time

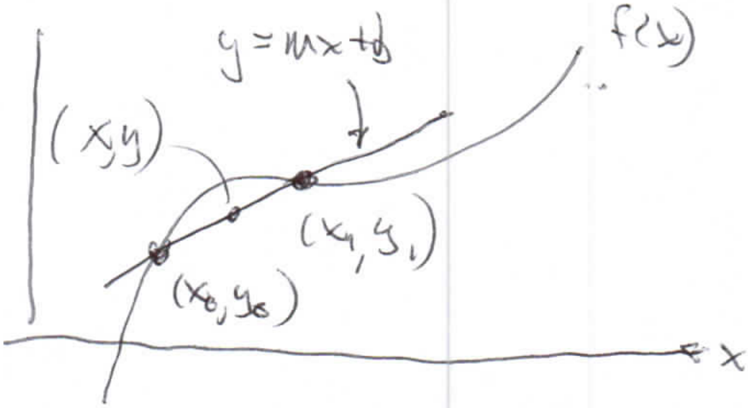
$a_{n-3} = -$ sum of all roots taken 3 at a time

$\vdots$

$a_0 = (-1)^n$ sum of all roots taken n at a time
(aka prod. of all roots!)

Interpolation (Lagrange Poly.)

To put a straight line through a function...



The slope $m = \dfrac{y - y_0}{x - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0}$ Now solve for y...

$$y = y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right)(y_1 - y_0)$$ put over com. denom....

$$= \frac{y_0(x_1 - x_0) + (x - x_0)(y_1 - y_0)}{x_1 - x_0} = \ldots$$

$$= \underbrace{\left(\frac{x - x_1}{x_0 - x_1}\right)}_{L_0(x)} y_0 + \underbrace{\left(\frac{x - x_0}{x_1 - x_0}\right)}_{L_1(x)} y_1$$

Note that $L_0(x_0) = 1$ $\quad L_1(x_0) = 1$ $\quad$ Hence we see that

$L_0(x_1) = 0$ $\quad L_1(x_1) = 1$

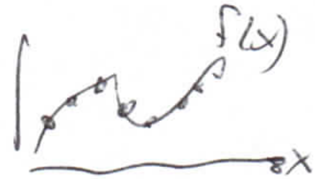
$$y = L_0(x) \cdot y_0 + L_1(x) y_1 \quad \text{so}$$

$$y(x_0) = \cancelto{1}{L_0(x_0)}\, y_0 + \cancelto{0}{L_1(x_0)}\, y_1 = y_0 \quad \text{as desired}$$

$$y(x_1) = \cancelto{0}{L_0(x_1)}\, y_0 + \cancelto{1}{L_1(x_1)}\, y_1 = y_1$$

In general, try to pass an n^{th} degree poly through

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots (x_k, y_k) \dots (x_n, y_n)$



Consider

no $(x-x_k)$ term

$$\ell(x) = (x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)$$

Observe that $\ell = 0$ $\forall$ $x = x_i$ $i = 0, 1, \dots, n$ except $x = x_k$

$$\ell \neq 0 \quad \text{for} \quad x = x_k$$

don't care about the value, it just $\neq 0$

Let's normalize this with

$$L_k = \frac{\ell(x)}{\ell(x_k)} = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Note that

$$\begin{cases} L_k = 0 & \text{if } x_i \neq x_k \text{ for } i = 0, \dots, n \text{ (but not } i = k) \\ L_k = 1 & \text{if } x_i = x_k \end{cases}$$

So can make a poly. That passes through the (x_k, y_k)

$$\boxed{P_k(x) = L_k \cdot y_k}$$ such that

$$P_k(x_k) = L_k(x_k)\, y_k = 1 \cdot y_k = y_k$$

and $\underset{i \neq k}{P_k(x_i)} = \underset{i \neq k}{L_k(x_i)}\, y_k = 0 \cdot y_k = 0$

Taking a linear combination of these allows one to pass through all the points. Build one of the P_k's for each point.

$$P(x) = \sum_{k=0}^{n} P_k(x) = \sum_{k=0}^{n} L_k \cdot y_k = \sum_{k=0}^{n} L_k \cdot f(x_k)$$

where $$L_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i \neq k}}^{n} \left(\frac{x-x_i}{x_k-x_i} \right) \qquad *$$

Got it? Simple, right? Lets try one...

Ex] Consider the points

x	$f(x)$
$x_0 = -1$	$1 = y_0$
$x_1 = 0$	$1 = y_1$
$x_2 = 1$	$1 = y_2$
~~$x_3 = 2$~~	$-5 = y_3$

Let's use x_0, x_1, x_2 to find a 2nd order poly..

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot f_2$$

$$= \frac{x(x-1)}{(-1)(-2)} \cdot 1 + \frac{(x+1)(x-1)}{(1)(-1)} \cdot 1 + \frac{(x+1)(x)}{(2)(1)} \cdot 1$$

$$= \frac{1}{2}(x)(x-1) - (x^2-1) + \frac{1}{2}(x)(x+1)$$

$$= \cdots = 0x^2 + 0x + 1$$, as expected.

Ok, so now let's use all five points.

Now we have four poly. each of degree 3.

Also, it is now not so obvious what the answer is.

$$P = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

$$= \frac{(x-0)(x-1)(x-2)}{(-1)(-2)(-3)} \cdot 1 + \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} \cdot 1$$

$$+ \frac{(x+1)(x-0)(x-2)}{(2)(1)(-1)} \cdot 1 + \frac{(x+1)(x-0)(x-1)}{(3)(2)(1)} \cdot (-5)$$

$= \ldots$ algebra $\ldots = -x^3 + x + 1$

But lets check...

if $P(x) = -x^3 + x + 1$

$P(-1) = 1 - 1 + 1 = 0$

$P(0) = 0$

$P(1) = -1 + 1 + 1 = 1$

$P(2) = -8 + 2 + 1 = -5$

ok, I think we did it!

Some final comments on Lag. Poly.

- Can use any number of points for x as long as $x_0 < x < x_n$ Don't extrapolate! For interpolation only.
- You dont actually have to write out $P(x)$! If all you want is $P(\alpha)$ do this back at * in a program (or by hand).
- And because somebody always wants to know The error for $P(x)$ of degree n is

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Whew!