Taylor Series

1

- $f(x+dx) = f(x) + \frac{dx}{1!} f'(x) + \frac{(dx)^2}{2!} f''(x) + \dots$ expanded about x

So $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$

from $f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots$ expanded about x_0

- Now consider

$$f(x+dx, y+dy, z+dz) = f(x,y,z) + \left[dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y} + dz \frac{\partial f}{\partial z} \right]$$

$$+ \frac{1}{2}\left[(dx)^2 \frac{\partial^2 f}{\partial x^2} + (dy)^2 \frac{\partial^2 f}{\partial y^2} + (dz)^2 \frac{\partial^2 f}{\partial z^2} + 2\, dx\, dy \frac{\partial^2 f}{\partial x \partial y} + 2\, dx\, dz \frac{\partial^2 f}{\partial x \partial z} + 2\, dy\, dz \frac{\partial^2 f}{\partial y \partial z} \right]$$

$+ \dots$

Can be written in notational form

$$\boxed{f(x+dx, y+dy, z+dz) = \Big[\underbrace{e^{\left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z}\right)}}_{\text{operator!}} \Big] f(x,y,z)}$$

2

Can also be written as

$$\boxed{f(x+dx, y+dy, z+dz) = f + df + \frac{1}{2!} d^2 f + \dots + \frac{1}{n!} d^n f + \dots}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2} (dx)^2 + \frac{\partial^2 f}{\partial y^2} (dy)^2 + \frac{\partial^2 f}{\partial z^2} (dz)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx\, dy + 2 \frac{\partial^2 f}{\partial x \partial z} dx\, dz + 2 \frac{\partial^2 f}{\partial y \partial z} dy\, dz$$

$\vdots$

$$\boxed{d^n f = \underbrace{\left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^n}_{\text{operator!}} f(x,y,z)}$$

1st order ODE - Initial Value

(3)

$y' = f(x,y) \qquad y(a) = y_a \qquad a \le x \le b$

Eulers Method.

Divide $[a,b]$ into equal parts

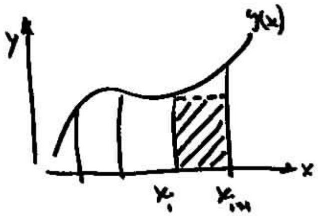
$$h = \frac{b-a}{n} = \text{step size}$$

x_i mesh points uniformly spaced

$$x_i = a + ih \qquad i = 0, \dots, n$$

Look at $\dfrac{dy}{dx} = f(x, y(x))$

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_i}^{x_{i+1}} f(x, y(x))\,dx$$



$$y(x_{i+1}) - y(x_i) \approx f(x_i, y(x_i)) \cdot h$$

$$\boxed{y_{i+1} \approx y_i + f(x_i, y_i) \cdot h}$$

(4)

Can also derive from Taylor expansion of

$$y(x+h) = y(x) + \frac{h}{1!} y'(x) + \frac{h^2}{2!} y''(x) + \cdots$$

$\to$ Truncate

$$\cong y(x) + h\, y'(x)$$

$$\approx y(x) + h \cdot f(x,y)$$

or $\boxed{y(x_{i+1}) \approx y(x_i) + h \cdot f(x_i, y_i)}$

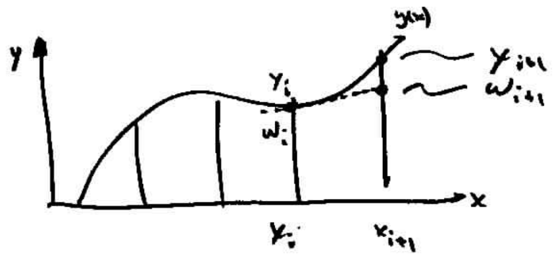
Will see the notation $w_i \approx y_i$ (approx) in book.

So Eulers Method is written

$$\boxed{w_{i+1} = w_i + h\, f(x_i, w_i)} \qquad i = 0, 1, 2, \dots N-1$$

(5)

It also has the usual geometric interpretation



$$\frac{w_{i+1} - w_i}{x_{i+1} - x_i} = f(x_i, w_i) = \text{slope at } x_i!$$

so

$$w_{i+1} = w_i + (x_{i+1} - x_i) \cdot f(x_i, w_i)$$

$$\boxed{w_{i+1} = w_i + h \cdot f(x_i, w_i)}$$

This is not really used in practice!

(6)

Ex1 $y' = y + x \quad y(0) = 0$

Find $y(x)$ for $0 \le x \le 1 \quad h = 0.2$

Recall $y_{i+1} = y_i + h \cdot f(x_i, y_i)$

i	x_i	y_i	y_{real}
0	0	0 ← given I.C.	0
1	0.2	$0 + 0.2(0+0) = 0$	0.0214
2	0.4	$0 + 0.2(0+0.2) = 0.04$	0.0918
3	0.6	$0.04 + 0.2(0.04+0.4) = 0.1280$	0.2221
4	0.8	$0.1280 + 0.2(0.1280+0.6) = 0.2736$	0.4255
5	1.0	$0.2736 + 0.2(0.2736+0.8) = 0.4883$	0.7183

error $O(h)$ $h = 0.2$!

Note: real soln is $y = e^x - x - 1$

Average Meth

Ex/ $y' = y + x \qquad y(0) = 0$

x	$y_{average}$	y_{exact}	y_{RK-2} (mid pt mtd)	RK-4
0	0	0	~~0.02~~ 0	0
0.2	0.02	0.0214	0.0200	0.0214
0.4	0.0884	0.0918	0.0884	0.0918
0.6	0.2158	0.2221	0.2158	0.2221
0.8	0.4153	0.4255	0.4153	0.4255
1.0	0.7027	0.7183	0.7027	0.7183

$$y_{i+1} = y_i + \frac{h}{2}\left[m\Big|_{Left} + m\Big|_{Right} \right]$$ corrected

where

$$m_{Left} = f(x_i, y_i)$$

$$m_{right} = f(x_{i+1}, \underbrace{y_i + hf(x_i, y_i)}_{\text{pred. } y_{i+1}})$$

7

Higher order Taylor series methods.

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \cdots + \frac{h^n}{n!} y_i^{(n)}$$

Can calculate $y_i' = f(x_i, y_i)$

thus

$$y_i'' = f'(x_i, y_i)$$

$$y_i''' = f''(x_i, y_i)$$

$$\vdots$$

$$y_i^{(n)} = f^{(n-1)}(x_i, y_i)$$

Then

$$y_{i+1} = y_i + h \cdot f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + \frac{h^3}{6} f''(x_i, y_i) + \cdots \frac{h^n}{n!} f^{(n-1)}(x_i, y_i)$$

$$= y_i + h \underbrace{\left[f(x_i, y_i) + \frac{h}{2} f'(x_i, y_i) + \frac{h^2}{6} f''(x_i, y_i) + \cdots \right]}_{T^{(n)}(x_i, y_i)}$$

— end at $\frac{h^{n-1}}{n!} f^{(n-1)}(x_i, w_i)$

$$\boxed{w_{i+1} = w_i + h T^{(n)}(x_i, w_i) \qquad i = 0, 1, \ldots, N-1}$$

n^{th} order Taylor Series

Total Derivatives

8

So Eulers method is order $n=1$

ie

$$w_{i+1} = w_i + h T^{(1)}(x_i, w_i)$$

$$= w_i + h[f(x_i, w_i)] \qquad \text{truncation error } O(h)$$

In text they work out an example with

Example $y' = y - t^2 + 1 \qquad 0 \le t \le 2 \qquad y(0) = \frac{1}{2} \qquad \boxed{n=4}$

$$y' = f(t, y) = y - t^2 + 1$$

$$y'' = \frac{df}{dt} = y' - 2t = (y - t^2 + 1) - 2t$$

$$= y - t^2 - 2t + 1$$

$$y''' = f'' = y' - 2t - 2$$

$$= (y - t^2 + 1) - 2t - 2 = y - t^2 - 2t - 1$$

$$y^{(4)} = f''' = y' - 2t - 2 = (y - t^2 + 1) - 2t - 2$$

$$= y - t^2 - 2t - 1$$

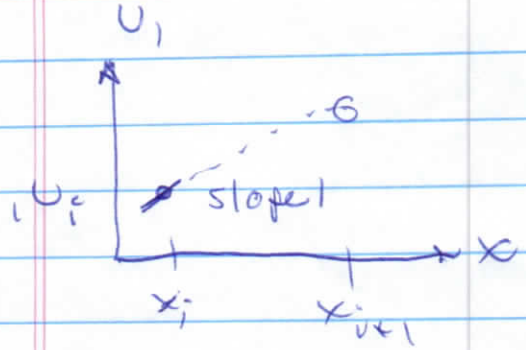
Eulers Mtd

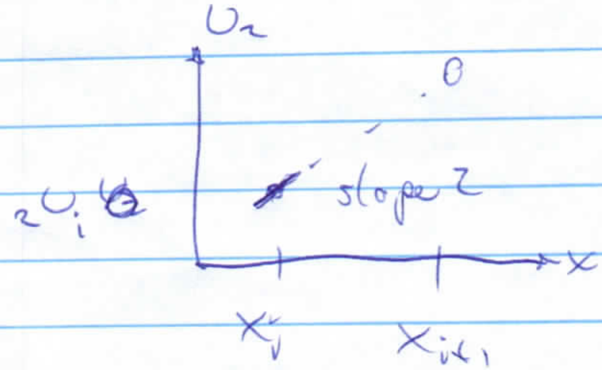
- If

$$y_{i+1} = y_i + h f(x_i, y_i) \quad \text{for} \quad y' = f(x, y)$$

- But for

$$u_1' = f_1(x, u_1, u_2)$$
$$u_2' = f_2(x, u_1, u_2)$$





$$_1u_{i+1} = {}_1u_i + h \underbrace{f_1(x_i, {}_1u_i, {}_2u_i)}_{\text{slope 1}}$$

$$_2u_{i+1} = {}_2u_i + h \underbrace{f_2(x_i, {}_1u_i, {}_2u_i)}_{\text{slope 2}}$$

- Extend to 3 coupled

$$_1u' = f_1(x, {}_1u, {}_2u, {}_3u)$$
$$_2u' = f_2(x, {}_1u, {}_2u, {}_3u)$$
$$_3u' = f_3(x, {}_1u, {}_2u, {}_3u)$$

$$_1u_{i+1} = {}_1u_i + \underbrace{h f_1(x_i, {}_1u_i, {}_2u_i, {}_3u_i)}_{{}_1k}$$

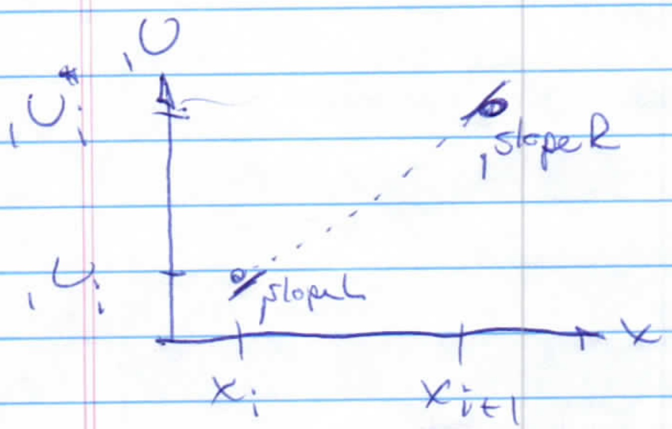
$$_2u_{i+1} = \ldots$$

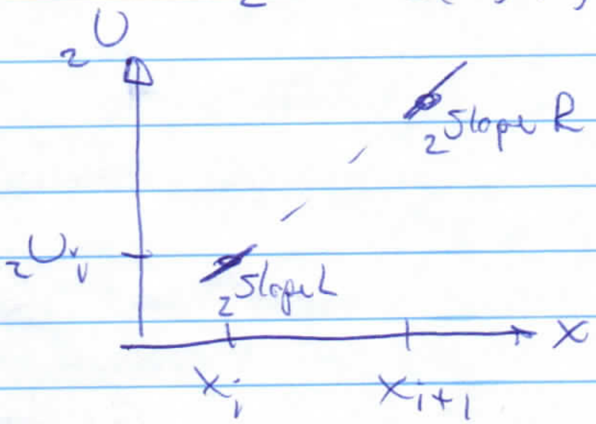
$$_3u_{i+1} = {}_3u_i + \underbrace{h f_3(x_i, {}_1u_i, {}_2u_i, {}_3u_i)}_{{}_3k}$$

AVG Mthd.

How about Avg. mthd.

$${}_1U' = f_1(x, {}_1U, {}_2U)$$
$${}_2U' = f_2(x, {}_1U, {}_2U)$$





$$\cancel{{}_1U_{i+1} = {}_1U_i + \frac{h}{2}\left[f(x_i, {}_1U_i, {}_2U_i)\ {}_1\text{slope}L_i + {}_1\text{slope}R_i \right]}$$

$$\begin{cases} {}_1k_1 = h\ {}_1\text{slope}L_i = h f_1(x_i, {}_1U_i, {}_2U_i) \\ {}_2k_1 = h\ {}_1\text{slope}L_i = h f_2(x_i, {}_1U_i, {}_2U_i) \end{cases}$$

$$\begin{cases} {}_1k_2 = h\,\text{slope} = h f_1(x_{i+1}, {}_1U_i + {}_1k_1, {}_2U_i + {}_2k_1) \\ {}_2k_2 = h f_2(x_{i+1}, {}_1U_i + {}_1k_1, {}_2U_i + {}_2k_1) \end{cases}$$

then

$$\begin{cases} {}_1U_{i+1} = {}_1U_i + \frac{1}{2}\left[{}_1k_1 + {}_1k_2 \right] \\ {}_2U_{i+1} = {}_2U_i + \frac{1}{2}\left[{}_2k_1 + {}_2k_2 \right] \end{cases}$$

So $w_{i+1} = w_i + h f(x_i,w_i) + \frac{h^2}{2} f'(x_i,w_i) + \frac{h^3}{6} f''(x_i,w_i)$

$$+ \frac{h^4}{24} f'''(x_i,w_i)$$

$$= w_i + h\left[w_i - t_i^2 + 1\right] + \frac{h^2}{2}\left[w_i - t_i^2 - 2t_i + 1\right]$$
$$+ \frac{h^3}{6}\left[w_i - t_i^2 - 2t_i - 1\right] + \frac{h^4}{24}\left[w_i - t_i^2 - 2t_i - 1\right]$$

$$= w_i + \Big[\left(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) w_i - \left(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) t_i^2$$
$$- 2t_i\left(\frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) + 1\left(h + \frac{h^2}{2} - \frac{h^3}{6} - \frac{h^4}{24}\right)\Big]$$

Can then substitute for h

<u>Truncation</u> <u>error</u> is $O(h^n)$ from

$$w_{i+1} = \underbrace{w_i + h f(x_i,w_i)}_{\text{order 1}} + \underbrace{\frac{h^2}{2} f'(x_i,w_i)}_{\text{order 2}} + \cdots \underbrace{\frac{h^n}{n!} f^{(n-1)}(x_i,w_i)}_{\text{order } n}$$

$$\tau_{i+1} = \frac{(y_{i+1} - y_i) - h\,\phi(x_i, y_i)}{h}$$

Runge-Kutta Methods

- Previous methods require the calculation of several derivatives. R-K methods use the evaluation of $f(x,y)$ at several locations to get high accuracy.

- Recall in numerical integration the Taylor series expansion and Gaussian quadrature, we used things like

$$\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2)$$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and compare coefficients for c, x, x^2, x^3 terms to find c_1, c_2, x_1, x_2

- R-K does something similar.

Consider

$$w_{i+1} = w_i + h \cdot T^{(n)}(x_i, w_i) = w_i + h\left[a \cdot f(x_i+\alpha,\ w_i+\beta)\right]$$

Consider $n=2$;

$$T^{(2)}(x,y) = f(x,y) + \frac{h}{2} f'(x,y)$$

$$= f(x,y) + \frac{h}{2}\left[\frac{\partial f}{\partial x}(x,y) \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y}(x,y) \cdot \frac{dy}{dx} \right]$$

$$= f(x,y) + \frac{h}{2}\left[f_x(x,y) \cdot \frac{dx}{dx} + f_y(x,y) \cdot f(x,y) \right]$$

and

$$a \cdot f(x+\alpha, y+\beta) = a\left[f(x,y) + \alpha f_x + \beta f_y \right] + \cdots$$

compare and get $a = 1$

$$a\alpha = \frac{h}{2} \ \Rightarrow\ \alpha = \frac{h}{2}$$

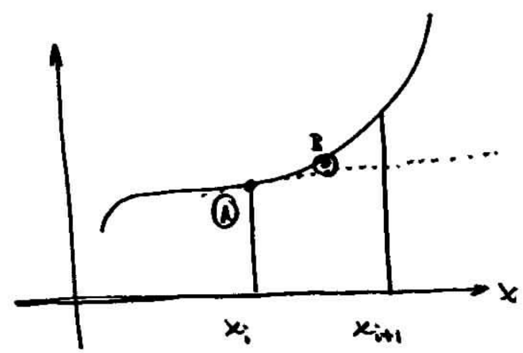
$$a\beta = \frac{h}{2} f \ \Rightarrow\ \beta = \frac{h}{2} f(x,y)$$

So

$$w_{i+1} = w_i + h \cdot \left[1 \cdot f\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2} f\right) \right]$$

$$= w_i + h \cdot f\left(x_i + \frac{h}{2},\ w_i + \frac{h}{2} f(x_i, w_i) \right)$$

- 2nd order R-K method
- known as Midpoint Method.
- Good to $O(h^2)$



Euler Method uses slope at A

R-K uses slope at B

(12 A)

Midpoint Mtd $\quad y' = x+y \quad y(0)=0 \quad h = 0.2$

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2},\ y_i + \frac{h}{2}f(x_i, y_i)\right)$$

$$= y_i + h\left[\left(x_i + \frac{h}{2}\right) + \left(y_i + \frac{h}{2}(x_i + y_i)\right)\right]$$

$$y_{i+1} = y_i\left(1 + h + \frac{h^2}{2}\right) + x_i\left(h + \frac{h^2}{2}\right) + \frac{h^2}{2}$$

$$y_{i+1} = y_i(1.22) + x_i(0.22) + 0.02$$

x_i	y_i (MIDPOINT MTD)		$y_{exact} = e^x - x - 1$
0	0		0
0.2	0(1.22) + 0 (0.22) + 0.02 =	0.02	0.02140276
0.4	0.02(1.22) + 0.2(0.22) + 0.02 = 0.0884		0.09182470
0.6	0.0884(1.22) + 0.4(0.22) + 0.02 = 0.215848		0.22211880
0.8	0.215848 (1.22) 0.6 (0.22) + 0.02 = 0.41533456		0.42554093
1.0	0.41533456(1.22) + 0.8(0.22) + 0.02 = 0.70270816		0.71828183

Now match $T^{(n)}$ up to order $n=4$.

6/28/99

Ex

4th adr R-K

$y' = y + x \qquad y(0) = 0 \qquad h = 0.2 \qquad 0 \le x \le 1$

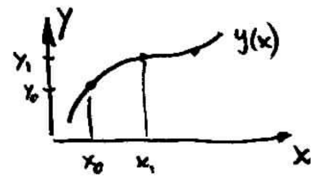
$n=4$

$$\boxed{\begin{aligned}
k_1 &= h\, f(x_i, y_i) \\
k_2 &= h\, f\left(x_i + \tfrac{h}{2},\ y_i + \tfrac{k_1}{2}\right) \\
k_3 &= h\, f\left(x_i + \tfrac{h}{2},\ y_i + \tfrac{k_2}{2}\right) \\
k_4 &= h\, f(x_{i+1},\ y_i + k_3) \\
&= h\, f(x_i + h,\ y_i + k_3) \\
y_{i+1} &= y_i + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}}$$

$\sim O(h^4)$

General ODE

$y' = f(x,y) \qquad y(0) = y_0$



$k_1 = h(x_i + y_i)$

$k_2 = h\left(x_i + \frac{h}{2} + y_i + \frac{k_1}{2}\right) = h\left(x_i + \frac{h}{2} + y_i + \frac{h}{2}(x_i + y_i)\right)$

$= h\left(\left(1 + \frac{h}{2}\right)x_i + \left(1 + \frac{h}{2}\right)y_i + \frac{h}{2}\right)$

$k_3 = h\left(x_i + \frac{h}{2} + y_i + \frac{k_2}{2}\right) = h\left(x_i + \frac{h}{2} + y_i + \frac{h}{2}\left(\left(1 + \frac{h}{2}\right)x_i + \left(1 + \frac{h}{2}\right)y_i + \frac{h}{2}\right)\right)$

$k_4 = \dots$

If $0 \le x \le 1 \qquad h = 0.2$

will get

$k_1 = 0.2(x_i + y_i)$

$k_2 = 0.22(x_i + y_i) + 0.02$

$k_3 = 0.222(x_i + y_i) + 0.022$

$k_4 = 0.2444(x_i + y_i) + 0.0444$

$y_{i+1} = y_i + 0.2214(x_i + y_i) + 0.0214$

n	x	y_i	y_{exact}
0	0	→ 0	0.
1	0.2	0.021 400	0.0214
2	0.4	0.091 818	0.0918
3	0.6	0.222 107	0.2221
4	0.8	0.425 521	0.4255
5	1.0	0.718 251	0.7183

Exact Soln is

$y = e^x - x - 1$

Can extend previous methods to this system of first order equations.

R-K is generally the easiest + most popular.

Consider the system

$$y_1' = f_1(x, y_1, y_2, y_3)$$
$$y_2' = f_2(x, y_1, y_2, y_3)$$
$$y_3' = f_3(x, y_1, y_2, y_3)$$

Calculate

$$k_{1,1} = h f_1(x_j, y_{1,j}, y_{2,j}, y_{3,j})$$ j ~ "time" index

$$k_{1,2} = h f_2(x_j, y_{1,j}, y_{2,j}, y_{3,j})$$

$$k_{1,3} = h f_3(x_j, y_{1,j}, y_{2,j}, y_{3,j})$$

$$k_{2,1} = h f_1\left(x_j + \frac{h}{2}, y_{1,j} + \frac{k_{1,1}}{2}, y_{2,j} + \frac{k_{1,2}}{2}, y_{3,j} + \frac{k_{1,3}}{2}\right)$$

$$k_{2,2} = h f_2\left(x_j + \frac{h}{2}, y_{1,j} + \frac{k_{1,1}}{2}, y_{2,j} + \frac{k_{1,2}}{2}, y_{3,j} + \frac{k_{1,3}}{2}\right)$$

$$k_{2,3} = h f_3(\quad " \quad)$$

$$k_{3,1} = h f_1\left(x_j + \frac{h}{2}, y_{1,j} + \frac{k_{2,1}}{2}, y_{2,j} + \frac{k_{2,2}}{2}, y_{3,j} + \frac{k_{2,3}}{2}\right)$$

$$k_{3,2} = h f_2(\quad " \quad)$$

$$k_{3,3} = h f_3(\quad " \quad)$$

$$k_{4,1} = h f_1(x_j + h, y_{1,j} + k_{3,1}, y_{2,j} + k_{3,2}, y_{3,j} + k_{3,3})$$

$$k_{4,2} = h f_2(\quad " \quad)$$

$$k_{4,3} = h f_3(\quad " \quad)$$

Finally,

$$y_{1,j+1} = y_{1,j} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$$

$$y_{2,j+1} = y_{2,j} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})$$

$$y_{3,j+1} = y_{3,j} + \frac{1}{6}(k_{1,3} + 2k_{2,3} + 2k_{3,3} + k_{4,3})$$

Do these in the same order as listed!

H.W. p301 1a)

Reduction of Higher Order to 1st order System

$y^{(2)} = f(x, y, y')$

Let

$$y = u_1$$
$$y' = u_1' = u_2$$
$$y'' = u_2' = f(x, u_1, u_2)$$

In general

$y^{(m)} = f(x, y, y', \ldots, y^{(m-1)})$

$$y = u_1$$
$$y' = u_1' = u_2$$
$$y'' = u_2' = u_3$$
$$y''' = u_3' = u_4$$
$$\vdots$$
$$y^{(m-2)} = u_{m-2}' = u_{m-1}$$
$$y^{(m-1)} = u_{m-1}' = u_m$$
$$y^{(m)} = u_m' = f(x, y, \ldots y^{(m-1)})$$
$$= f(x, u_1, u_2, u_3, \ldots, u^{(m-1)})$$

Could even have coupled higher order systems:

$$y'' = f(x, y, y', w, w')$$
$$w'' = g(x, y, y', w, w')$$

$$y = u_1$$
$$y' = u_1' = u_2$$
$$y'' = u_2' = f(x, y, y', w, w') = f(x, u_1, u_2, u_3, u_4)$$
$$w = u_3$$
$$w' = u_3' = u_4$$
$$w'' = u_4' = g(x, y, y', w, w') = g(x, u_1, u_2, u_3, u_4)$$

System is finally

$$\boxed{\begin{aligned} u_1' &= u_2 \\ u_2' &= f(x, u_1, u_2, u_3, u_4) \\ u_3' &= u_4 \\ u_4' &= g(x, u_1, u_2, u_3, u_4) \end{aligned}}$$

END