## Multi-Step Methods

Will look for scheme that uses info from more than 1 point in the past. Should increase accuracy.

### Adams-Bashforth (3-step)

Alternate approach from that in the text.

$$y_{i+1} = y_i + h\,\phi(x_i, y_i)$$

$$= y_i + h\left[ a f(x_i, y_i) + b f(x_{i-1}, y_{i-1}) + c f(x_{i-2}, y_{i-2}) \right]$$

Via Taylor series

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \cdots$$

$$f(x_{i-1}, y_{i-1}) = f(x_i, y_i) + \frac{(-h)}{1} f'(x_i, y_i) + \frac{(-h)^2}{2} f''(x_i, y_i) + \frac{(-h)^3}{6} f'''(x_i, y_i) \cdots$$

$$= y_i' - h y_i'' + \frac{h^2}{2} y_i''' - \frac{h^3}{6} y_i'''' + \cdots$$

$$f(x_{i-2}, y_{i-2}) = f(x_i, y_i) + \frac{(-2h)}{1} f'(x_i, y_i) + \frac{(-2h)^2}{2} f''(x_i, y_i) + \frac{(-2h)^3}{6} f'''(x_i, y_i)$$

$$= y_i' - 2h y_i'' + 2h^2 y_i''' - \frac{4}{3} h^3 y_i'''' + \cdots$$



So LHS = RHS

$$y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \cdots$$

$$= y_i + ha\left[ y_i' \right] + hb\left[ y_i' - h y_i'' + \frac{h^2}{2} y_i''' - \frac{h^3}{6} y_i'''' + \cdots \right]$$

$$+ hc\left[ y_i' - 2h y_i'' + 2h^2 y_i''' - \frac{4}{3} h^3 y_i'''' + \cdots \right]$$

1 $\quad y_i = y_i$

$h$ $\quad y_i' = a y_i' + b y_i' + c y_i' \qquad 1 = a + b + c$

$h^2$ $\quad \frac{1}{2} y_i'' = -b y_i'' - 2c y_i'' \qquad \frac{1}{2} = -b - 2c$

$h^3$ $\quad \frac{1}{6} y_i''' = \frac{b}{2} y_i''' + 2c y_i''' \qquad \frac{1}{6} = \frac{b}{2} + 2c$

$$a = \frac{23}{12} \quad b = \frac{-16}{12} \quad c = \frac{5}{12}$$

$h^4$ $\quad +\frac{1}{24} = \frac{b}{6} + c\frac{4}{3}$ error.

So

$$\boxed{y_{i+1} = y_i + \frac{h}{12}\left[ 23 f(x_i, y_i) + (-16) f(x_{i-1}, y_{i-1}) + 5 f(x_{i-2}, y_{i-2}) \right]}$$

## Local Truncation error

General scheme $y_{i+1} = y_i + h\phi(x_i, y_i)$

Truncation error $\boxed{T_{i+1}(h) = \dfrac{y_{i+1} - y_i - h\phi(x_i, y_i)}{h}}$

So error is $T_{i+1}(h) = \dfrac{O(h^4)}{h} = O(h^3)$

For Adams-Bashford 3-step

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Adams-Moulton 2-step

$$y_{i+1} = y_i + h\left[a f(x_{i+1}, y_{i+1}) + b f(x_i, y_i) + c f(x_{i-1}, y_{i-1})\right]$$

with get

$$y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \dots$$

$$= y_i + ah\left[f(x_i,y_i) + h f'(x_i,y_i) + \frac{h^2}{2} f''(x_i,y_i) + \frac{h^3}{6} f'''(x_i,y_i) + \dots\right]$$

$$+ bh f(x_i,y_i) + ch\left[f(x_i,y_i) - h f'(x_i,y_i) + \frac{h^2}{2} f''(x_i,y_i) - \frac{h^3}{6} f'''(x_i,y_i) + \dots\right]$$

$$= y_i + ah\left[y_i' + h y_i'' + \frac{h^2}{2} y_i''' + \frac{h^3}{6} y_i'''' + \dots\right]$$

$$+ bh y_i' + ch\left[y_i' - h y_i'' + \frac{h^2}{2} y_i''' - \frac{h^3}{6} y_i'''' + \dots\right]$$

(1) $y_i = y_i$

(h) $y_i' = a y_i' + b y_i' + c y_i'$

($h^2$) $\frac{1}{2} y_i'' = a y_i'' - c y_i''$

($h^3$) $\frac{1}{6} y_i''' = \frac{a}{2} y_i''' + \frac{c}{2} y_i'''$

($h^4$) $\frac{1}{24} y_i'''' = \frac{a}{6} y_i'''' - \frac{c}{6} y_i''''$ error

$$\left.\begin{aligned} 1 &= a + b + c \\ \tfrac{1}{2} &= a - c \\ \tfrac{1}{6} &= \tfrac{a}{2} + \tfrac{c}{2} \end{aligned}\right\} \quad \begin{aligned} a &= 5/12 \\ b &= 8/12 \\ c &= -1/12 \end{aligned}$$

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so

$$\boxed{y_{i+1} = y_i + \frac{h}{12}\left[5 f(x_{i+1}, y_{i+1}) + 8 f(x_i, y_i) - 1 f(x_{i-1}, y_{i-1})\right]}$$

Truncation error

$$\tau_{i+1}(h) = \frac{O(h^4)}{h} = O(h^3) \leftarrow$$

Corresponding A-B 2-step is

$$y_{i+1} = y_i + \frac{h}{2}\left[3 f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right] \qquad O(h^2)$$

Notice that Adams-Bashforth uses only <u>old</u> points

and Adams-Moulton uses <u>old</u> + <u>new</u> points

A-B is an <u>explicit</u> method

A-M <u>implicit</u> method.

— Generally $y_{i+1} = y_i + h\left[a f(x_{i+1}, y_{i+1}) + \dots\right]$

may actually be difficult to solve this for $y_{i+1}$!

— Explicit methods are generally less accurate than implicit methods for the same # of old points used.

Ex) $y' = y + x \qquad y(0) = 0 \qquad 0 \le x \le 1$

A-B 2step $\quad y_{i+1} = y_i + \frac{h}{2}\left[3\underbrace{f(x_i, y_i)}_{= y_i'} - \underbrace{f(x_{i-1}, y_{i-1})}_{= y_{i-1}'}\right]$

$$= y_i + \frac{h}{2}\left[3(y_i + x_i) - (y_{i-1} + x_{i-1})\right]$$

$$\boxed{y_{i+1} = \left(1 + \frac{3}{2}h\right)y_i + \left(\frac{3}{2}h\right)x_i - \left(\frac{h}{2}\right)(y_{i-1} + x_{i-1})}$$

$\left.\begin{matrix} x_0 & y_0 \\ x_1 & y_1 \end{matrix}\right\}$ seed values

$i = 1, \dots N-1$



A-M 2step $\quad y_{i+1} = y_i + \frac{h}{12}\left[5f(x_{i+1}, y_{i+1}) + 8f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right]$

$$= y_i + \frac{5}{12}h(y_{i+1} + x_{i+1}) + \frac{8}{12}h(y_i + x_i) - \frac{h}{12}(y_{i-1} + x_{i-1})$$

$$= \left(\frac{5}{12}h\right)y_{i+1} + \left(\frac{5}{12}h\right)x_{i+1} + \left(1 + \frac{8}{12}h\right)y_i + \left(\frac{8}{12}h\right)x_i - \frac{h}{12}(y_{i-1} + x_{i-1})$$

$$\left(1 - \frac{5}{12}h\right)y_{i+1} = \left(\frac{5}{12}h\right)x_{i+1} + \left(1 + \frac{8}{12}h\right)y_i + \left(\frac{8}{12}h\right)x_i - \frac{h}{12}(y_{i-1} + x_{i-1})$$

$$\boxed{y_{i+1} = \frac{\cdots}{\left(1 - \frac{5}{12}h\right)}}$$

$\left.\begin{matrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & \end{matrix}\right\}$ seed values

$i = 1, \dots N-1$

Generally use R-4 method to generate the seed values to use the multi-step techniques.

Adams - Bashforth

2-step $y_{i+1} = y_i + \frac{h}{2}\left[3f(x_i,y_i) - f(x_{i-1},y_{i-1})\right] + O(h^2)$

3-step $y_{i+1} = y_i + \frac{h}{12}\left[23f(x_i,y_i) - 16 f_{i-1} + 5 f_{i-2}\right] + O(h^3)$

4-step $y_{i+1} = y_i + \frac{h}{24}\left[55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right] + O(h^4)$

5-step $y_{i+1} = y_i + \frac{h}{720}\left[1901 f_i - 2774 f_{i-1} + 2616 f_{i-2} - 1274 f_{i-3} + 251 f_{i-4}\right] + O(h^5)$

Adams - Moulton

2-step $y_{i+1} = y_i + \frac{h}{12}\left[5 f_{i+1} + 8 f_i - f_{i-1}\right] + O(h^3)$

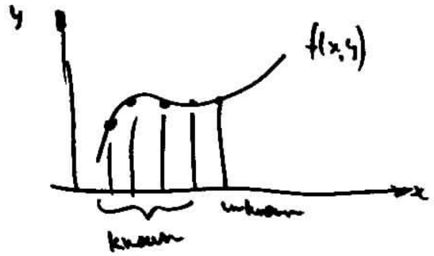
3-step $y_{i+1} = y_i + \frac{h}{24}\left[9 f_{i+1} + 19 f_i - 5 f_{i-1} + f_{i-2}\right] + O(h^4)$

4-step $y_{i+1} = y_i + \frac{h}{720}\left[251 f_{i+1} + 646 f_i - 264 f_{i-1} + 106 f_{i-2} - 19 f_{i-3}\right] + O(h^5)$



The Multi step methods can also be derived by integrating polynomial approximations

$$y' = f(x,y) \Rightarrow y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} f(x,y)\,dx$$




Use poly through all the points (plus the unknown) implicit

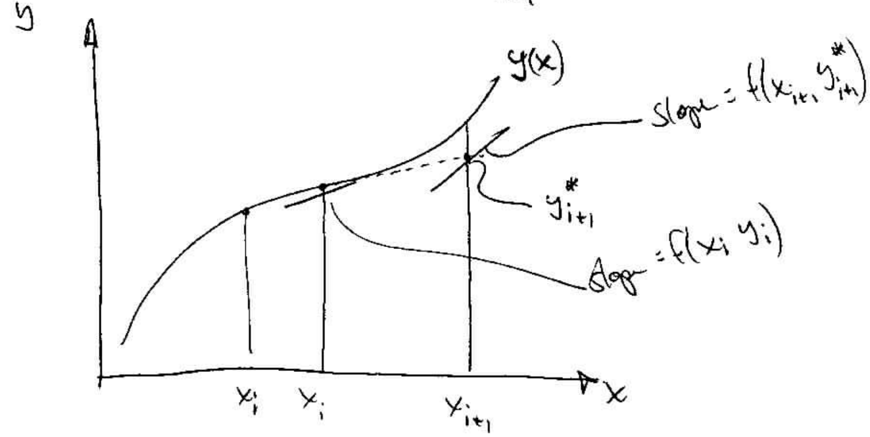
Can actually use any poly to do this - SEE TEXT.

Predictor-Corrector Mtd    Improved Euler Mtd

Euler $y^*_{i+1} = y_i + hf(x_i, y_i)$ Project + predict $y^*_{i+1}$

$$y_{i+1} = y_i + \underbrace{\frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})\right]}$$

Averages the slope at $x_i$ and $x_{i+1}$ to correct $y_{i+1}$ value.






Ex

$y' = x + y$    $y(0) = 0$    $h = 0.2$    Pred-Corr Ex

$$y^*_{i+1} = y_i + hf(x_i, y_i) = y_i + h(x_i + y_i) \quad \text{Predictor}$$

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})\right] \quad \text{Corrector}$$

$$= y_i + \frac{h}{2}\left[(x_i + y_i) + (x_{i+1} + \underbrace{y^*_{i+1}}_{y_i + h(x_i + y_i)})\right]$$

$\vdots$

$$y_{i+1} = y_i\left(1 + h + \frac{h^2}{2}\right) + x_i\left(\frac{h}{2} + \frac{h^2}{2}\right) + \frac{h}{2}(x_{i+1})$$

$$= y_i(1.22) + x_i(0.12) + (0.1)x_{i+1}$$

| $x_i$ | $y_i$ (Improved Euler) | $y_{exact}$ |
|---|---|---|
| 0 | 0 | 0 |
| 0.2 | 1.22(0) + 0.12(0) + 0.1(0.2) = 0.02 | 0.021403 |
| 0.4 | 1.22(0.02) + 0.12(0.2) + 0.1(0.4) = 0.0884 | 0.091825 |
| 0.6 | 1.22(0.0884) + 0.12(0.4) + 0.1(0.6) = 0.215848 | 0.222119 |
| 0.8 | 1.22(0.215848) + 0.12(0.6) + 0.1(0.8) = 0.415335 | 0.425541 |
| 1.0 | 1.22(0.415335) + 0.12(0.8) + 0.1(1.0) = 0.702708 | 0.718282 |

## Predictor Corrector

— First compute $y^*_{i+1} = y_i + h f(x_i, y_i)$ ← Euler

then use $y_{i+1} = y_i + \frac{h}{2}\underbrace{\left( f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})\right)}_{\text{average the slopes at two locations}}$

IS IMPROVED Euler Method (-Cauchy)

- $y^*_{i+1}$ is a prediction
- next step is a correction using the predicted info.

— Another approach is to use an explicit to get a prediction and an implicit to correct.

A-B 4-step | $y^*_{4p} = y_3 + \frac{h}{24}\left[55 f(x_3, y_3) - 59 f(x_2, y_2) + 37 f(x_1, y_1) - 9 f(x_0, y_0)\right]$

A-M 3-step | $y_4 = y_3 + \frac{h}{24}\left(9 f(x_4, y^*_4) + 19 f(x_3, y_3) - 5 f(x_2, y_2) + f(x_1, y_1)\right]$



— A-B 4-step, A-M 3-step } both $O(h^4)$ so accuracy of equations is matched!

— Also by using $y^*_{i+1}$ from an explicit and putting it into an implicit, eliminates need to solve an implicit equation!

— This is the general way of using implicit schemes.

— An alternate approach to solving the implicit equation is to use fixed point iteration

$$y_4^{(i+1)} = y_3 + \frac{h}{24}\left[9 f(x_4, y_4^{(i)}) + 19 f(x_3, y_3) - 5 f(x_2, y_2) + f(x_1, y_1)\right]$$

iterate

EX How to analytically get seed values. 6/29/94

$$\boxed{\begin{matrix} y' = x+y \\ y(0)=0 \end{matrix}}$$

Soln is $y = e^x - x - 1$

Generate a Series

$y' = x+y = f(x,y) \quad \Rightarrow y'(0) = 0 + y(0) = 0$

$y'' = 1 + y'$
$= 1 + x + y \quad \Rightarrow y''(0) = 1 + 0 + y(0) = 1$

$y''' = 1 + y'$
$= 1 + x + y \quad \Rightarrow y'''(0) = 1 + 0 + y(0) = 1$

$\vdots \qquad \vdots$

Now $y(x) = y(0) + x \cdot y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \cdots$

$= 0 + x \cdot 0 + \frac{x^2}{2} \cdot 1 + \frac{x^3}{6} \cdot 1 + \cdots$

$$\boxed{y(x) = \frac{x^2}{2} + \frac{x^3}{6} + \cdots}$$

Check by using known solution

$y = e^x - x - 1 = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - x - 1$

$\boxed{y = \frac{x^2}{2} + \frac{x^3}{6} + \cdots}$ ok



Approx. the ODE.

S'pose $y' = y + x \approx y$ for small $x$, ie $y \gg x$

$\frac{y'}{y} = 1 \quad \Rightarrow \ln y = x + c$

$y = \cancel{c} e^x \qquad y(0) = c \cdot 1 = 0$
$\Rightarrow c = 0$

only trivial soln!

can't be right.

Now try $y' \approx x$ ie $x \gg y$

so $y = \frac{x^2}{2} + \cancel{c} \qquad y(0) = 0 = \cancel{c}$

$y = \frac{x^2}{2}$

Is $x \gg y = \frac{x^2}{2}$ yes for $x \ll 1$ ie $x$ small.

So $\boxed{y \cong \frac{x^2}{2}}$ seems ok.

ON HW. from last time, use $y = \frac{x^2}{2}$ to generate any seed values.

END

## RK-4 method for systems

Given the coupled system

$$
\begin{aligned}
y_1' &= f_1(x,\ y_1,\ y_2,\ y_3) && \text{subject to} && y_1(x_0) = y_{1,0}\\
y_2' &= f_2( \qquad\qquad\qquad ) && \text{subject to} && y_2(x_0) = y_{2,0}\\
y_3' &= f_3( \qquad\qquad\qquad ) && \text{subject to} && y_3(x_0) = y_{3,0}
\end{aligned}
$$

we first calculate

$$
\begin{aligned}
k_{1,1} &= h \cdot f_1\,(x_j,\ y_{1,j},\ y_{2,j},\ y_{3,j}\,) && \text{where} && y_{1,j} = y_1(x_j)\\
k_{1,2} &= h \cdot f_2\,( \qquad\qquad\qquad\qquad ) && \text{where} && y_{2,j} = y_2(x_j)\\
k_{1,3} &= h \cdot f_3\,( \qquad\qquad\qquad\qquad ) && \text{where} && y_{3,j} = y_3(x_j)
\end{aligned}
$$

then calculate

$$
\begin{aligned}
k_{2,1} &= h \cdot f_1\left(x_j + \frac{h}{2},\ y_{1,j} + \frac{k_{1,1}}{2},\ y_{2,j} + \frac{k_{1,2}}{2},\ y_{3,j} + \frac{k_{1,3}}{2}\right)\\
k_{2,2} &= h \cdot f_2\left( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)\\
k_{2,3} &= h \cdot f_3\left( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)
\end{aligned}
$$

then

$$
\begin{aligned}
k_{3,1} &= h \cdot f_1\left(x_j + \frac{h}{2},\ y_{1,j} + \frac{k_{2,1}}{2},\ y_{2,j} + \frac{k_{2,2}}{2},\ y_{3,j} + \frac{k_{2,3}}{2}\right)\\
k_{3,2} &= h \cdot f_2\left( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)\\
k_{3,3} &= h \cdot f_3\left( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)
\end{aligned}
$$

then

$$
\begin{aligned}
k_{4,1} &= h \cdot f_1\,(x_j + h,\ y_{1,j} + k_{3,1},\ y_{2,j} + k_{3,2},\ y_{3,j} + k_{3,3}\,)\\
k_{4,2} &= h \cdot f_2\,( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad )\\
k_{4,3} &= h \cdot f_3\,( \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad )
\end{aligned}
$$

and finally, we calculate

$$
\begin{aligned}
y_{1,j+1} &= y_{1,j} + \frac{1}{6} \cdot (k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\,)\\
y_{2,j+1} &= y_{2,j} + \frac{1}{6} \cdot (k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\,)\\
y_{3,j+1} &= y_{3,j} + \frac{1}{6} \cdot (k_{1,3} + 2k_{2,3} + 2k_{3,3} + k_{4,3}\,)
\end{aligned}
$$

to give us the new values for $y_1$, $y_2$ and $y_3$ corresponding to $x_{i+1}$.

When solving the initial value problem $y' = f(x,y)$ with the initial condition $yx_0) = y_0$, we can use the following multi-point methods:

Adams–Bashforth (explicit method)

- $y_{i+1} = y_i + \frac{h}{2}\left[3f_i - f_{i-1}\right] + O(h^2)$ 2-step
- $y_{i+1} = y_i + \frac{h}{12}\left[23f_i - 16f_{i-1} + 5f_{i-2}\right] + O(h^3)$ 3-step
- $y_{i+1} = y_i + \frac{h}{24}\left[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right] + O(h^4)$ 4-step
- $y_{i+1} = y_i + \frac{h}{720}\left[1901f_i - 2774f_{i-1} + 2616f_{i-2} - 1274f_{i-3} + 251f_{i-4}\right] + O(h^5)$ 5-step

Adams–Moulton (implicit method)

- $y_{i+1} = y_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right] + O(h^3)$ 2-step
- $y_{i+1} = y_i + \frac{h}{24}\left[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right] + O(h^4)$ 3-step
- $y_{i+1} = y_i + \frac{h}{720}\left[251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}\right] + O(h^5)$ 4-step

where $f_i = f(x_i, y_i)$.