In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from tqdm import tqdm
```

Convex Optimization - Homework 3

Note : The present notebook can be found here : https://github.com/RSLLES/ConvexOptHW3

$$\text{minimize w.r.t. } w \quad \frac{1}{2}||Xw - y||_2^2 + \lambda||w||_1 \qquad \text{(LASSO)}$$

Question 1 : Derive the dual problem of LASSO and format it as a general Quadratic Problem

One get by reformulating (LASSO) :

$$\begin{array}{ll} \text{minimize w.r.t. } w, z & \frac{1}{2}||z||_2^2 + \lambda||w||_1 \\ \text{subject to} & z = Xw - y \end{array}$$

Let $\mathcal{L}$ be the lagrangien of the latter problem :

$$\mathcal{L}(w, z, \eta) = \frac{1}{2}||z||_2^2 + \lambda||w||_1 + {}^t\eta(Xw - y - z)$$

$$\mathcal{L}(w, z, \eta) = \underbrace{\frac{1}{2}||z||_2^2 - {}^t\eta z}_{f(z)} + \underbrace{\lambda||w||_1 + {}^t\eta Xw}_{g(w)} - {}^t\eta y$$

The next step is to minimize $\mathcal{L}$ with respect to w and z. To do so, one needs to minimize both f and g defined above.

First, $\nabla f(z) = z - \eta$, so $\nabla f(z) = 0 \Rightarrow z = \eta$ and $f(\eta) = \frac{-1}{2}||\eta||_2^2$. Hence, $\inf_z f(z) = \frac{-1}{2}||\eta||_2^2$.

Second, $g(w) = \lambda||w||_1 + {}^t\eta Xw = -[-{}^t\eta Xw - \lambda||w||_1]$. Hence :

$$\inf_w g(w) = -\lambda \sup_w\{-\frac{1}{\lambda}{}^t\eta Xw - ||w||_1\}$$

According to the second homework, the conjuguate of $||.||_1$ is :

$$y \mapsto \sup_x\{{}^tyx - ||x||_1\} = \begin{cases} 0 & \text{if} \quad ||y||_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

The latter function evaluted with $y = -\frac{1}{\lambda}{}^tX\eta$ gives exactly $\inf_w g(w)$:

$$\inf_w g(w) = \begin{cases} 0 & \text{if} \quad ||{}^tX\eta||_\infty \leq \lambda \\ -\infty & \text{otherwise} \end{cases}$$

All in all :

$$\inf_{w,z} \mathcal{L}(w, z, \eta) = \begin{cases} \frac{-1}{2}||\eta||_2^2 - {}^t\eta y & \text{if} \quad ||{}^tX\eta||_\infty \leq \lambda \\ -\infty & \text{otherwise} \end{cases}$$

So the dual of (LASSO) is :

$$\begin{array}{ll} \text{maximize w.r.t. } \eta & \frac{-1}{2}||\eta||_2^2 - {}^ty\eta \\ \text{subject to} & ||{}^tX\eta||_\infty \leq \lambda \end{array}$$

And renaming η for v, multiplying by -1 to transform the maximization problem into a minimization problem, and reformulating $||^t X\eta||_\infty \leq \lambda$ as ${}^t X\eta \preceq b$ and $-{}^t X\eta \preceq b$ with $b = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \lambda$:

$$\begin{array}{ll} \text{minimize w.r.t. } v & \frac{1}{2} {}^t vv + {}^t yv \\ \text{subject to} & {}^t Xv \preceq b \\ & -{}^t Xv \preceq b \end{array}$$

And therefore :

$$\begin{array}{ll} \text{minimize w.r.t. } v & {}^t vQv + {}^t pv \\ \text{subject to} & Av \preceq b \end{array} \quad \text{with} \quad \begin{cases} Q = \frac{1}{2} I \in \mathbb{R}^{n \times n} \\ p = y \in \mathbb{R}^n \\ A = \begin{pmatrix} {}^t X \\ -{}^t X \end{pmatrix} \in \mathbb{R}^{2d \times n} \\ b = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \lambda \in \mathbb{R}^{2d} \end{cases} \qquad (QP)$$

Question 2 : Implement the barrier method to solve QP

Let $m = 2d$.

The barrier method reformulates the inequality constraints using logarithmic barrier functions :

$$\text{minimize w.r.t. } v \quad t \left[{}^t vQv + {}^t pv \right] - \sum_{i=1}^{m} \log(b_i - {}^t a_i v) \qquad (CP)$$

where $b = \begin{pmatrix} b_1 \\ \dots \\ b_m \end{pmatrix} \in \mathbb{R}^d$ and $A = \begin{pmatrix} {}^t a_1 \\ \dots \\ {}^t a_m \end{pmatrix} \in \mathbb{R}^{d \times n}$

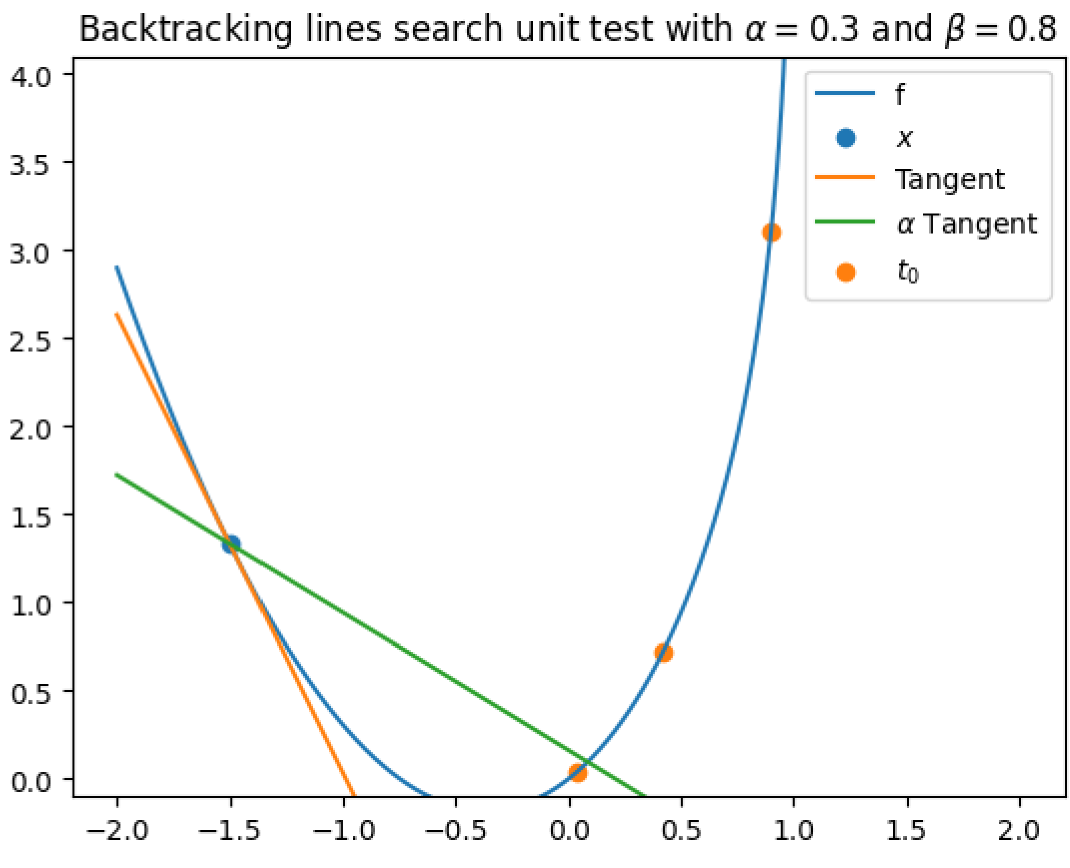
Step 1 : write `centering_step (Q, p, A, b, t, v0, eps)` to solve (CP).

As there is no equality constraint, the easiest option is to use the regular Method.

In [ ]:
```
def backtracking_line_search(f, grad_f_x, x, delta_x, alpha, beta, t0=1):
    assert 0 < alpha and alpha < .5
    assert 0 < beta and beta < 1
    fx = f(x)
    assert not np.isnan(fx)

    t = [t0]
    slope = alpha * np.dot(grad_f_x, delta_x)

    while True:
        y = f(x + t[-1] * delta_x)
        if not np.isnan(y):
            if f(x + t[-1] * delta_x) <= fx + t[-1] * slope:
                return np.array(t)
        t.append( beta * t[-1] )

### Unit test ###
def unit():
    alpha, beta = 0.3, 0.8
    plt.title(rf"Backtracking lines search unit test with $\alpha={alpha}$ and $\beta={beta}$
    T = np.linspace(-2, 2, 1000)
```

```
    # f
    f = lambda x : np.square(x) - np.log(1 - x)
    plt.plot(T, f(T), label="f")

    # x
    x = -1.5
    plt.scatter([x],[f(x)], label="$x$")

    # Tangents
    grad_f_x = 2*x + 1/(1 - x)
    tg = lambda t : f(x) + grad_f_x * (t - x)
    tg_alpha = lambda t : f(x) + alpha * grad_f_x * (t - x)
    plt.plot(T, tg(T), label="Tangent")
    plt.plot(T, tg_alpha(T), label=r"$\alpha$ Tangent")

    # t0
    delta_x = 1
    t0 = backtracking_line_search(
        f=f,
        grad_f_x=grad_f_x,
        x=x,
        delta_x=delta_x,
        alpha=alpha,
        beta=beta,
        t0=3
    )
    plt.scatter(x + delta_x * t0, f(x + delta_x * t0), label="$t_0$")

    # Plot
    plt.legend()
    plt.ylim([-0.1, 4.1])
    plt.show()

unit()
```

```
C:\Users\Romain\AppData\Local\Temp\ipykernel_4840\2032140524.py:24: RuntimeWarning: invalid v
alue encountered in log
  f = lambda x : np.square(x) - np.log(1 - x)
```



As this function will be extensively used, here is a optimized version which does not create a list at each call, leading to a general increase in performances :

In [ ]:
```
def faster_backtracking_line_search(f, grad_f_x, x, delta_x, alpha, beta, t0=1):
    assert 0 < alpha and alpha < .5
    assert 0 < beta and beta < 1
    fx = f(x)
    assert not np.isnan(fx)

    t = t0
    slope = alpha * np.dot(grad_f_x, delta_x)

    while True:
        y = f(x + t * delta_x)
        if not np.isnan(y):
            if f(x + t * delta_x) <= fx + t * slope:
                return t
        t = beta * t

def test_backtracking_function(backtracking_function, N = 10000):
    alpha, beta = 0.3, 0.8
    f = lambda x : np.square(x) - np.log(1 - x)
    x = -1.5
    grad_f_x = 2*x + 1/(1 - x)
    delta_x = 1
    t0 = 3
    for _ in range(N):
        backtracking_function(
            f=f,
            grad_f_x=grad_f_x,
            x=x,
            delta_x=delta_x,
            alpha=alpha,
            beta=beta,
        )
```

In [ ]:
```
%%time
test_backtracking_function(backtracking_function=backtracking_line_search)
```

```
CPU times: total: 93.8 ms
Wall time: 94.4 ms
```

In [ ]:
```
%%time
test_backtracking_function(backtracking_function=faster_backtracking_line_search)
```

```
CPU times: total: 93.8 ms
Wall time: 89.7 ms
```

Let f be :

$$f : v \mapsto t\underbrace{\left[{}^t v Q v + {}^t p v\right]}_{f_0(v)} - \sum_{i=1}^{d} \underbrace{\log(b_i - {}^t a_i v)}_{-\phi(v)} = t f_0(v) + \sum_{i=1}^{d} \phi_i(v)$$

In [ ]:
```
def build_f0(Q, p):
    return lambda u : np.dot(Q @ u + p, u)

def build_f(Q, p, A, b, t):
    f0 = build_f0(Q=Q, p=p)
    return lambda u : t * f0(u) - np.sum( np.log( b - A @ u ) )
```

Then its gradient and hessian are :

$$\nabla f = t\nabla f_0 + \sum_{i=1}^{d} \nabla\phi \quad ; \quad \nabla^2 f = t\nabla^2 f_0 + \sum_{i=1}^{d} \nabla^2\phi$$

f_0 is just a quadratic function, so its gradient and hessian are straight forward to compute (Q is assumed symmetric positive semi definite):

$$\nabla f_0(v) = 2Qv + p \quad ; \quad \nabla^2 f_0(v) = 2Q$$

In [ ]:
```
def f0_derivatives(Q, p, v):
    return 2 * Q @ v + p, 2 * Q
```

ϕ_i is the log barrier function with affine constraints :

$$\phi_i(v) = -\log\left(b_i - {}^t a_i v\right)$$

$$\nabla\phi_i(v) = \frac{1}{b_i - {}^t a_i v} a_i$$

$$\nabla^2\phi_i(v) = \frac{1}{(b_i - {}^t a_i v)^2} a_i {}^t a_i = \nabla\phi_i(v)^t \nabla\phi_i(v)$$

To compute those efficiently, it is better to limit the amount of for loop and to use tensor operations.

First, `affine` $= \begin{pmatrix} b_1 - {}^t a_1 v \\ \dots \\ b_d - {}^t a_d v \end{pmatrix} = b - Av$ given that $A = \begin{pmatrix} {}^t a_1 \\ \dots \\ {}^t a_d \end{pmatrix}$.

Using numpy's elementwise inversion, it is easy to define `1/affine` $= \begin{pmatrix} \frac{1}{b_1 - {}^t a_1 v} \\ \dots \\ \frac{1}{b_d - {}^t a_d v} \end{pmatrix}$

Then `grad` $= \left(\nabla\phi_1(v) \quad \dots \quad \nabla\phi_d(v) \right) = \left(a_1 \frac{1}{b_1 - {}^t a_1 v} \quad \dots \quad a_n \frac{1}{b_d - {}^t a_d v} \right)$, this is a columnwise multiplication (or multipliction by a diagonal matrix), so $\nabla\phi_i(v) = {}^t A \,\mathrm{diag}\begin{pmatrix} \frac{1}{b_1 - {}^t a_1 v} \\ \dots \\ \frac{1}{b_d - {}^t a_d v} \end{pmatrix}$

And finally $\nabla^2\phi_i(v) = \nabla\phi_i(v)^t \nabla\phi_i(v)$, this is a batchify outer product, hence `hess = einsum("nd,md->mnd", grad, grad)` (`m` and `n` has no importance has the hessian is symmetric).

And given the sum in the final expression, there is left to sum over the d-dimensional axis for both gradients and hessians.

In [ ]:
```
def phi_derivatives(A, b, v):
    affine = b - A @ v
    grad = A.T @ np.diag(1/affine)
    hess = np.einsum("nd,md->mnd", grad, grad)
    return np.sum(grad, axis=-1), np.sum(hess, axis=-1)

### Unit test ###
def unit():
    d=2
    n=3
    A = np.arange(n*d).reshape(n,d).T
    b = np.arange(n*d + 1, n*d + 1 + d)
    v = np.linspace(0, 1, n)

    return A, b, v, *phi_derivatives(A=A, b=b, v=v)
```

```python
# This result is indeed the one I found by hand calculation
unit()
```

Out[ ]:
```
(array([[0, 2, 4],
        [1, 3, 5]]),
 array([7, 8]),
 array([0. , 0.5, 1. ]),
 array([0.66666667, 3.        , 5.33333333]),
 array([[ 0.44444444,  1.33333333,  2.22222222],
        [ 1.33333333,  5.        ,  8.66666667],
        [ 2.22222222,  8.66666667, 15.11111111]]))
```

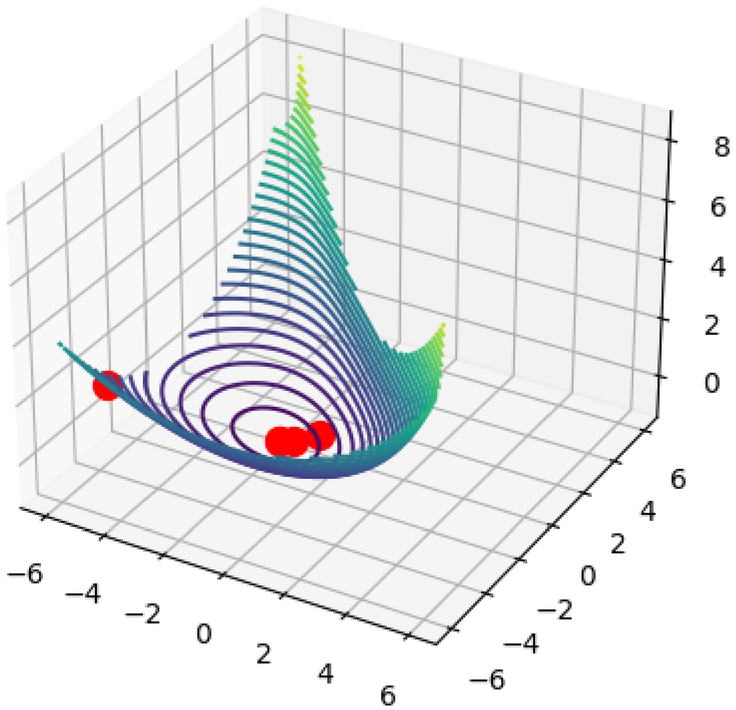
This leads to the folowing implementation of the newton method :

In [ ]:
```python
def centering_step(Q, p, A, b, t, v0, eps=1e-7, alpha=0.1, beta=0.9):
    # v0 must be in the domain of the log,
    # ie a feasible point in the original problem
    assert np.all(b - A @ v0 > 0)
    v = [v0]
    f = build_f(Q=Q, p=p, A=A, b=b, t=t)


    while True:
        if len(v) % 50 == 0:
            print(f"Iteration {len(v)} : 0.5*squared_decrement={squared_decrement}")
        phi_grad, phi_hessian = phi_derivatives(A=A, b=b, v=v[-1])
        f0_grad, f0_hessian = f0_derivatives(Q=Q, p=p, v=v[-1])
        grad = t * f0_grad + phi_grad
        hessian = t * f0_hessian + phi_hessian

        delta_v = np.linalg.solve(hessian, -grad)
        squared_decrement = np.dot(grad, -delta_v)

        if 0.5*squared_decrement <= eps: return v
        s = faster_backtracking_line_search(
            f=f, x=v[-1],
            grad_f_x=grad,
            delta_x=delta_v,
            alpha=alpha, beta=beta
        )

        v.append( v[-1] + s * delta_v )
```

In [ ]:
```python
### Unit Test ###
def unit():
    N = 100
    Q = np.diag((1,2))
    p = -2*np.arange(1, 2)
    A = np.ones(shape=(1,2))
    b = np.array([1])
    t = 0.05

    f = build_f(Q=Q, p=p, A=A, b=b, t=t)
    X, Y = np.linspace(-6, 6, N), np.linspace(-6, 6, N)
    Z = np.zeros((N, N))

    for j, y in enumerate(X):
        for i, x in enumerate(Y):
            v = np.array((x,y))
            Z[j,i] = f(v)


    fig = plt.figure()
    ax = plt.axes(projection='3d')
```

```python
    X, Y = np.meshgrid(X, Y)
    ax.contour3D(X, Y, Z, 30, cmap='viridis')

    V = centering_step(
        Q=Q,p=p,
        A=A, b=b,
        t=t,
        v0=-5*np.ones(shape=(2,))
    )

    for v in V:
        ax.scatter([v[0]], [v[1]], [f(v)], s=100, c='red')

    print(f"v* = {V[-1]}, f(v*) = {f(V[-1]):0.5f}")
    # ax.view_init(90, 0)
    fig.show()

unit()
```

```
C:\Users\Romain\AppData\Local\Temp\ipykernel_4840\3029965375.py:6: RuntimeWarning: invalid va
lue encountered in log
  return lambda u : t * f0(u) - np.sum( np.log( b - A @ u ) )
C:\Users\Romain\AppData\Local\Temp\ipykernel_4840\1343674766.py:37: UserWarning: Matplotlib i
s currently using module://matplotlib_inline.backend_inline, which is a non-GUI backend, so c
annot show the figure.
  fig.show()
v* = [-1.7536703  -0.87683515], f(v*) = -0.79567
```



Step 2 : write `barr_method (Q, p, A, b, v0, eps)` to solve (QP).

In [ ]:
```python
def barr_method (Q, p, A, b, v0, mu, t0=1, eps=1e-6, alpha=0.3, beta=0.8):
    v, t = [v0], t0
    m = b.shape[0]

    for _ in tqdm(range( int(np.log(m/(eps*t0))/np.log(mu)) + 1), desc=f"Barrier method for m
        v += centering_step(
            Q=Q, p=p,
            A=A, b=b,
            t=t, v0=v[-1],
            eps=eps,
            alpha=alpha, beta=beta
        )
```

```
        t = t*mu

    return v
```

Question 3 : Test your function on randomly generated matrices X and observations y

In [ ]:
```
def Generate_Data (n, d, noise_std=0.5):
    X = 2*np.random.rand(n, d) - 1
    w = 2*np.random.rand(d) - 1
    y = X @ w + np.random.normal(scale=noise_std, size=(n,))
    return X, y, w
```

`format_problem` is a function formating (LASSO) dual problem as a Quadratic Programm. As a reminder :

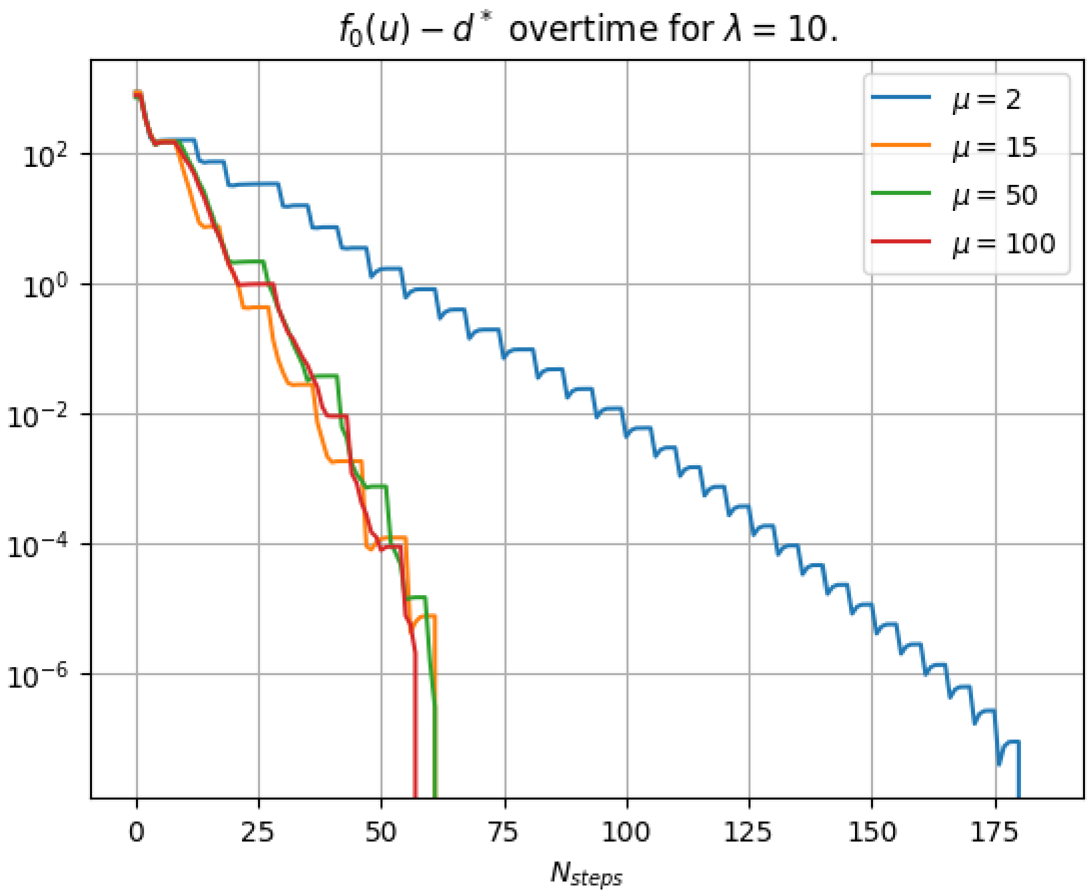
$$\begin{array}{lll} \text{minimize w.r.t. } v & {}^t vQv + {}^t pv \\ \text{subject to} & Av \preceq b \end{array} \quad \text{with} \quad \begin{cases} Q = \frac{1}{2} I \in \mathbb{R}^{n \times n} \\ p = y \in \mathbb{R}^n \\ A = \begin{pmatrix} {}^t X \\ -{}^t X \end{pmatrix} \in \mathbb{R}^{2d \times n} \\ b = \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \lambda \in \mathbb{R}^{2d} \end{cases} \qquad (QP)$$

Given that $\lambda > 0$, $v_0 = 0$ is always a feasible point.

In [ ]:
```
def format_problem(X, y, lambd):
    # Returns Q, p, A, b, v0 feasible (= 0)
    n, d = X.shape
    return 0.5 * np.eye(n), y, np.concatenate((X.T, -X.T), axis=0) , np.ones(shape=(2*d,)) *
```

In [ ]:
```
def test_method(n, d, lambd, list_mu):
    ax = plt.gca()
    for mu in list_mu:
        X, y, _ = Generate_Data (n=n, d=d)
        Q, p, A, b, v0 = format_problem(X=X, y=y, lambd=lambd)
        v = barr_method (Q=Q, p=p, A=A, b=b, v0=v0, mu=mu, eps=1e-6)

        f0 = build_f0(Q=Q, p=p)
        v_star = v[-1]
        d_star = f0(v_star)

        ax.semilogy(np.arange(len(v)), [f0(u) - d_star for u in v], label=fr"$\mu={mu}$")

    ax.set_title(rf"$f_0(u) - d^*$ overtime for $\lambda = {lambd}$.")
    ax.grid(True, which="both")
    ax.set_xlabel(r"$N_{steps}$")
    plt.legend()
    plt.show()



test_method(n=100, d=1000, lambd=10, list_mu=[2, 15, 50, 100])
```

```
Barrier method for mu = 2:   0%|          | 0/31 [00:00<?, ?it/s]C:\Users\Romain\AppData\Local\Temp\ipykernel_4840\3029965375.py:6: RuntimeWarning: invalid value encountered in log
  return lambda u : t * f0(u) - np.sum( np.log( b - A @ u ) )
Barrier method for mu = 2: 100%|██████████| 31/31 [00:18<00:00,  1.71it/s]
Barrier method for mu = 15: 100%|██████████| 8/8 [00:06<00:00,  1.23it/s]
Barrier method for mu = 50: 100%|██████████| 6/6 [00:06<00:00,  1.09s/it]
Barrier method for mu = 100: 100%|██████████| 5/5 [00:06<00:00,  1.25s/it]
```



This matches the theoretical results. Now it is time to retrieve w^* from the solution of the dual.

The original problem (LASSO) is strictly feasible (there is no inequality constraints and the objective function is defined on $\mathbb{R}^d$) and the problem is convex, hence **Slater's conditions** are gathered and **strong duality holds**.

It means that the KKT's conditions are respected. Therefore, the lagrangien at the optimum $\mathcal{L}(w^*, z^*, v^*)$ is minimal.

As a reminder :

$$\mathcal{L}(w, z, v) = \underbrace{\frac{1}{2}||z||_2^2 - {}^t v z}_{f(z)} + \underbrace{\lambda||w||_1 + {}^t v X w}_{g(w)} - {}^t v y$$

First, $\mathcal{L}$ is differentiable with respect to z, and $\nabla_z \mathcal{L}(z^*, w^*, v^*) = 0 \Leftrightarrow z^* = v^*$. z is by construction equal to $Xw - y$, hence $Xw^* = y + v^*$.

This equation is equivalent to ${}^t X X w^* = {}^t X (y + v^*)$. But $X \in \mathbb{R}^{n \times d}$, hence X is full rank (and the equation can be solve) only if $n \geq d$. But LASSO is designed to tackle this problem in a high dimensional setting, when $d \gg n$, so this alone is not enough to solve in this setting.

$\mathcal{L}$ is not differentiable with respect to w due to the absolute value function. However, the l_1-norm admits subderivatives. Hence $\mathcal{L}$ has subderivatives and the relaxed KKT's condition states that at least one of those subderivative is 0 at the optimum.

A subderivative of $||.||_1$ is a function that maps any x in $\R^d$ to a vector $y \in \R^d$ such that for $1 \leq i \leq n$, $y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ -1 & \text{if } x_i < 0 \\ \alpha & \text{if } x_i = 0 \end{cases}$ for a given $\alpha \in [-1, 1]$. This function is note $\nabla^{(\alpha)}||.||_1$.

Let use the KKT conditions and note $\nabla_w \mathcal{L}$ a subderivative of $\mathcal{L}$ with respect to w, defined with an $\alpha \in [-1, 1]$ such that $\nabla_w \mathcal{L}(z^*, w^*, v^*) = 0$.

This means that $\lambda \nabla^{(\alpha)}||w^*||_1 + {}^tXv^* = 0$, which is equivalent to :

$$\nabla^{(\alpha)}||w^*||_1 = -\frac{1}{\lambda}{}^tXv^*$$

Then if $w_i^* \neq 0$, then $|[\frac{1}{\lambda}{}^tXv^*]_i| = 1$, the contraposition being $|[\frac{1}{\lambda}{}^tXv^*]_i| < 1 \Rightarrow w_i^* = 0$ (Indeed, the dual's inequality constraint imposes $||\frac{1}{\lambda}{}^tXv^*||_1 \preceq 1$).

This criteria helps to find the 0 of w, and if it sets enough 0, then the original system might become solvable.

Here is an implementation of this method to find w. Note that the checked condition is relaxed with a parameter $\epsilon = 10^{-6}$, because the barrier method introduces a slight offset. It becomes $|[\frac{1}{\lambda}{}^tXv^*]_i| < 1 - \epsilon \Rightarrow w_i^* = 0$.

In [ ]:
```python
def retrieve_w_from_v(v, X, y, lambd, eps= 1e-6):
    n, d = X.shape
    w = np.zeros((d,))
    crit = X.T @ v
    non_zeros = np.abs(crit) >= lambd - eps
    print(f"Found {d - np.sum(non_zeros)} zeros in w for n = {X.shape[0]} and d = {w.shape[0]

    shorten_X = X[:, non_zeros]
    w[non_zeros] = np.linalg.solve(shorten_X.T @ shorten_X, shorten_X.T @ (y + v))
    return w

def unit(n, d, lambd, list_mu):
    for mu in list_mu:
        X, y, _ = Generate_Data (n=n, d=d)
        Q, p, A, b, v0 = format_problem(X=X, y=y, lambd=lambd)
        v = barr_method (Q=Q, p=p, A=A, b=b, v0=v0, mu=mu, eps=1e-6)

        f0 = build_f0(Q=Q, p=p)
        v_star = v[-1]
        d_star = f0(v_star)

        w_star = retrieve_w_from_v(v=v_star, X=X, y=y, lambd=lambd)
        p_star = np.dot(X @ w_star - y, X @ w_star - y) + lambd * np.sum(np.abs(w_star))

        print(f"d_star = {-d_star:0.1f}, p_star = {p_star:0.1f}, dual_gap = {(p_star + d_star

unit(n=100, d=1000, lambd=10, list_mu=[2, 15, 50, 100, 500, 1000])
```

```
Barrier method for mu = 2:   0%|          | 0/31 [00:00<?, ?it/s]C:\Users\Romain\AppData\Loca
l\Temp\ipykernel_4840\3029965375.py:6: RuntimeWarning: invalid value encountered in log
  return lambda u : t * f0(u) - np.sum( np.log( b - A @ u ) )
Barrier method for mu = 2: 100%|██████████| 31/31 [00:17<00:00,  1.81it/s]
Found 907 zeros in w for n = 100 and d = 1000.
d_star = 859.5, p_star = 909.0, dual_gap = 49.54, relative dual gap = 5.4%
Barrier method for mu = 15: 100%|██████████| 8/8 [00:07<00:00,  1.09it/s]
Found 908 zeros in w for n = 100 and d = 1000.
d_star = 810.6, p_star = 854.5, dual_gap = 43.84, relative dual gap = 5.1%
Barrier method for mu = 50: 100%|██████████| 6/6 [00:06<00:00,  1.05s/it]
```

```
Found 912 zeros in w for n = 100 and d = 1000.
d_star = 827.4, p_star = 875.9, dual_gap = 48.55, relative dual gap = 5.5%
Barrier method for mu = 100: 100%|██████████| 5/5 [00:06<00:00,  1.37s/it]
Found 908 zeros in w for n = 100 and d = 1000.
d_star = 990.7, p_star = 1044.5, dual_gap = 53.85, relative dual gap = 5.2%
Barrier method for mu = 500: 100%|██████████| 4/4 [00:08<00:00,  2.04s/it]
Found 907 zeros in w for n = 100 and d = 1000.
d_star = 798.0, p_star = 846.2, dual_gap = 48.21, relative dual gap = 5.7%
Barrier method for mu = 1000: 100%|██████████| 4/4 [00:08<00:00,  2.05s/it]
Found 909 zeros in w for n = 100 and d = 1000.
d_star = 829.4, p_star = 875.9, dual_gap = 46.53, relative dual gap = 5.3%
```

It looks like a higher μ does not change the performances of the found w. Given that it takes fewer iterations to converge, it looks reasonnable to take a high value for μ.