Section 2 (Cont'd)
The Chain Rule and "Special Functions"

In the last lecture, we considered some basic mathematical properties of the derivative. We also considered the exponential function and its inverse function, the natural logarithm. In this section we extend our analysis of the derivative further by considering the Chain Rule of Differentiation and the derivatives of some well kown but "special" functions.

"A neat trick"
Before we consider this chain rule and these special functions, a word on the general derivative of any power of x. We alluded in the lectures as to how this is done but here we state explicitly the method used to calculate these derivatives.

1. $\dfrac{d}{dx}x^n \quad \forall\, n\in\mathbb{Z}\setminus\{0\}$

   Here $f(x)=x^n$ which is smooth for all x_0 in the domain of $f(x)$. Then

$$\begin{aligned}
\frac{d}{dx}\left(x^n\right) &= \lim_{\Delta x\to 0}\frac{\left(x+\Delta x\right)^n-\left(x^n\right)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\left(x^n+n\,x^{n-1}\Delta x+\ldots+n\,x(\Delta x)^{n-1}+(\Delta x)^n\right)-\left(x^n\right)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\left(n\,x^{n-1}\Delta x+\ldots+n\,x(\Delta x)^{n-1}+(\Delta x)^n\right)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(n\,x^{n-1}+O\left(\Delta x\right)\right)\\
&= n\,x^{n-1} \quad \forall\, n\in\mathbb{Z}\setminus\{0\}
\end{aligned}$$

   The notation $O\left(\Delta x\right)$ indicates all **<u>O</u>**ther terms that have Δx and all its subsequent powers present: i.e . in this case

$$\begin{aligned}
O\left(\Delta x\right)=\frac{n(n-1)}{2}x^{n-2}\Delta x+\frac{n(n-1)(n-2)}{6}x^{n-3}(\Delta x)^2+\ldots\\
+n\,x(\Delta x)^{n-2}+(\Delta x)^{n-1}
\end{aligned}$$

2. $\frac{d}{dx} x^{1/n} \qquad \forall n \in \mathbb{Z} \setminus \{0\}$

Here $f(x) = x^{1/n}$ which is smooth for all x_0 in the domain of $f(x)$. Then

$$\frac{d}{dx}\left(x^{1/n}\right) \quad = \quad \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/n} - \left(x^{1/n}\right)}{\Delta x}$$

We cannot subtract the elements on the numerator as it stands. If the numerator was

$$(x+\Delta x)^1 - (x^1) = \Delta x$$

for example we could subtract them. The relationship between the above and the original numerator is

$$(x+\Delta x)^{1/n} - x^{1/n} \text{ is to } (x+\Delta x) - x \text{ as } (x-y) \text{ is to } x^n - y^n$$

We have the following relationship between $(x-y)$ and $x^n - y^n$:

$$(x-y)\left(x^{n-1} + x^{n-2} y + \ldots + x y^{n-2} + y^{n-1}\right) = x^n - y^n$$

Therefore

$$\left((x+\Delta x)^{1/n} - x^{1/n}\right)\left((x+\Delta x)^{\frac{n-1}{n}} + (x+\Delta x)^{\frac{n-2}{n}} x^{\frac{1}{n}} + \ldots + (x+\Delta x)^{\frac{1}{n}} x^{\frac{n-2}{n}} + x^{\frac{n-1}{n}}\right)$$
$$= (x+\Delta x)^{\frac{n}{n}} - x^{\frac{n}{n}} = (x+\Delta x) - x = \Delta x$$

and, on substitution back into our derivative, we get

$$\begin{aligned}
\frac{d}{dx}\left(x^{1/n}\right) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/n} - \left(x^{1/n}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left((x+\Delta x)^{1/n} - \left(x^{1/n}\right)\right)\left((x+\Delta x)^{n-1/n} + \ldots + \left(x^{n-1/n}\right)\right)}{\Delta x\left((x+\Delta x)^{n-1/n} + \ldots + \left(x^{n-1/n}\right)\right)} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x\left((x+\Delta x)^{n-1/n} + \ldots + \left(x^{n-1/n}\right)\right)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left((x+\Delta x)^{n-1/n} + \ldots + \left(x^{n-1/n}\right)\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{n-1/n} + \ldots + \left(x^{n-1/n}\right)} \\
&= \frac{1}{n} x^{\frac{n-1}{n}} \qquad \forall n \in \mathbb{Z} \setminus \{0\}
\end{aligned}$$

From 1. and 2. above it follows that

$$\frac{d}{dx}\left(x^{n}\right)=n\,x^{n-1}\quad \forall\, n\in\mathbb{R}\setminus\{0\}$$

Exercise:

Use the method discussed above to find the derivatives of the following functions with respect to *x*:

1. $f(x)=x^{4}$
2. $f(x)=x^{1/5}$

Some "Special Functions"

By special functions, we mean functions whose names belies the complexity of their nature; e.g. The trigonometric functions.

Before we consider their derivatives we need to consider some properties of the functions in the limit as $\Delta x\rightarrow 0$.

$$\lim_{\Delta x\rightarrow 0}\sin(\Delta x)=\Delta x$$

$$\lim_{\Delta x\rightarrow 0}\cos(\Delta x)=1$$

$$\lim_{\Delta x\rightarrow 0}\tan(\Delta x)=\Delta x$$

Now to the derivatives:

$f(x)=\cos(x)$

Then

$$\frac{d}{dx}\cos(x)\ =\ \lim_{\Delta x\rightarrow 0}\frac{\cos(x+\Delta x)-\cos(x)}{\Delta x}$$

Using the identity (from Pg 9 of the Log Tables)

$$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$$

we get

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \lim_{\Delta x\rightarrow 0}\frac{\left(\cos(x)\cos(\Delta x)-\sin(x)\sin(\Delta x)\right)-\cos(x)}{\Delta x}\\ &= \lim_{\Delta x\rightarrow 0}\left(\cos(x)\left(\frac{\cos(\Delta x)-1}{\Delta x}\right)-\sin(x)\left(\frac{\sin(\Delta x)}{\Delta x}\right)\right)\\ &= -\sin(x)\end{aligned}$$

where we have used

$$\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1$$
$$\lim_{\Delta x \to 0} \cos(\Delta x) - 1 = 0$$

to arrive at the final answer.

$f(x) = \sin(x)$

Then

$$\frac{d}{dx}\sin(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Using the identity (from Pg 9 of the Log Tables)

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

we get

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{\Delta x \to 0} \frac{(\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x)) - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\sin(x)\left(\frac{\cos(\Delta x) - 1}{\Delta x} \right) + \cos(x)\left(\frac{\sin(\Delta x)}{\Delta x} \right) \right) \\
&= \cos(x)
\end{aligned}$$

where, again, we have used

$$\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1$$
$$\lim_{\Delta x \to 0} \cos(\Delta x) - 1 = 0$$

to arrive at the final answer.

$f(x) = \tan(x)$

Two methods for dealing with this function:

Method 1:

$$\frac{d}{dx}\tan(x) = \lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan(x)}{\Delta x}$$

Using the identity (from Pg 9 of the Log Tables)

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

we get

$$
\begin{aligned}
\frac{d}{dx}\tan(x) &= \lim_{\Delta x\to 0}\frac{\left(\dfrac{\tan(x)+\tan(\Delta x)}{1-\tan(x)\tan(\Delta x)}\right)-\tan(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\left(\frac{\tan(x)+\tan(\Delta x)-\tan(x)+\tan^2(x)\tan(\Delta x)}{(1-\tan(x)\tan(\Delta x))\Delta x}\right) \\
&= \lim_{\Delta x\to 0}\left(\frac{\tan(\Delta x)}{\Delta x}\frac{(1+\tan^2(x))}{(1-\tan(x)\tan(\Delta x))}\right) \\
&= 1+\tan^2(x)=\sec^2(x)
\end{aligned}
$$

where, again, we have used

$$
\lim_{\Delta x\to 0}\frac{\tan(\Delta x)}{\Delta x}=1
$$

$$
\lim_{\Delta x\to 0} 1-\tan(x)\tan(\Delta x)=1
$$

to arrive at the final answer.

Method 2:

We know that $\tan(x)=\dfrac{\sin(x)}{\cos(x)}\equiv\dfrac{f(x)}{g(x)}$. We have a quotient and can use the quotient formula:

i.e.

$$
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}
$$

We know from above that

$$
\frac{d}{dx}f(x)=\frac{d}{dx}\sin(x)=\cos(x)
$$

$$
\frac{d}{dx}g(x)=\frac{d}{dx}\cos(x)=-\sin(x)
$$

and so

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2} \\
&= \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)}
\end{aligned}
$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$
$$= \frac{1}{\cos^2(x)}$$
$$= \sec^2(x)$$

where we have used the identities

$$\cos^2(A)+\sin^2(A)=1$$
$$\frac{1}{\cos(A)}=\sec(A)$$

As you can appreciate, method 2 is far easier and it is by this and similar methods that we actually calculate derivatives.

Chain Rule of Differentiation

We have already discussed compound functions in previous sections. Here we consider the dericatives of such functions; namely $\frac{d}{dx}f(u(x))$. To motivate the rule to follow, we consider products of fractions:

$$\frac{a}{b} = \frac{a}{c}\times\frac{c}{b} = \frac{a}{c}\times\frac{c}{d}\times\frac{d}{b} = \frac{a}{c}\times\frac{c}{d}\times\ldots\times\frac{e}{b}$$

So the left most fraction can be constructed by a product of related fractions whose numerators are cancelled by denominators leaving the final fraction.

In an analogous manner we introduce the chain rule: i.e.

$$\frac{d}{dx}f(u(x))=\left[\frac{d}{du}f(u)\right]_{u=u(x)}\cdot\frac{d}{dx}u(x)$$

or less formally with the *x* dependence removed

$$\frac{df}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$$

where you can "imagine" that the *du*'s cancel out to leave *df/dx* on each side of the "equal to" symbol.

N.B.

Under no circumstances should you treat the differential operator acting on a function $\frac{df}{dx}$ as a fraction; i.e. df divided by dx. The analogy with fractions is to help you visualise the operation of the chain rule not to cast the differential operator as a fraction.

It is best to illustrate by example and so we will consider some examples.

Examples:

1. $f(u(x))=\sqrt{3x^2-2x+4}$

We can differentiate $3x^2-2x+4$ using our established rules and so we set $u(x)=3x^2-2x+4$.

Then $f(u(x))=\sqrt{u(x)}$ which we can differentiate with respect to $u(x)$.

The chain rule proceeds thus:

- Let $u=3x^2-2x+4 \Rightarrow f(u)=\sqrt{u}$. Then
$$u=3x^2-2x+4 \Rightarrow f(u)=\sqrt{u}$$
and the chain rule progresses
$$\frac{df}{dx}=\frac{df}{du}\times\frac{du}{dx}=\frac{d}{du}\sqrt{u}\times\frac{d}{dx}u$$
- Now substitute for u in the second derivative and evaluate the first
$$\frac{df}{dx}=\frac{1}{2}u^{-1/2}\times\frac{d}{dx}(3x^2-2x+4)$$
- Now substitute $3x^2-2x+4$ for u in the first derivative and evaluate the second
$$\frac{df}{dx}=\frac{1}{2}(3x^2-2x+4)^{-1/2}\times(6x-2)$$
$$\Rightarrow\frac{df}{dx}=\frac{1}{2}\frac{6x-2}{\sqrt{3x^2-2x+4}}$$

2. $f(u(x))=e^{-4\cos(x)}$

We can differentiate $-4\cos(x)$ with respect to x and so we set $u(x)\equiv u=-4\cos(x)$.

Then $f(u(x))\equiv f(u)=\exp(u)=e^{u}$ which we can differentiate w.r.t u.

Thus

$$\frac{df}{dx}=\frac{df}{du}\times\frac{du}{dx}=\frac{d}{du}e^{u}\times\frac{d}{dx}u$$

and, evaluating the first and substituting $-4\cos(x)$ for u in the second derivative, we get

$$\frac{df}{dx}=e^{u}\times\frac{d}{dx}(-4\cos(x))$$

Now evaluating the second derivative and substituting for u in the first yields

$$\frac{df}{dx}=-4e^{-4\cos(x)}\times(-\sin(x))$$

$$\Rightarrow\frac{df}{dx}=4\sin(x)e^{-4\cos(x)}$$

We can summarise the operation of the chain rule via the following "rules":

1. Find the largest (most extensive) function $u(x)$ that when substituted into f yields $f(u(x))\equiv f(u)$.
2. Check for any additional "chains" in the complete derivative: i.e. Is there a need to further split the function into $f(v(u(x)))\equiv f(v(u))\ldots$?
3. Perform the operation of the derivative by differentiating each "link" in the "chain" and multiplying the result:

$$\frac{df}{dx}=\frac{df}{du}\times\frac{du}{dx} \quad \text{or} \quad \frac{df}{dx}=\underbrace{\underbrace{\frac{df}{du}}_{\text{Link}}\times\frac{d}{dv}u\times\ldots\times\frac{d}{dx}w}_{\text{Chain}}$$

4. Substitute the various functions back in for u, v, etc. as required to complete the overall calculation and simplify.