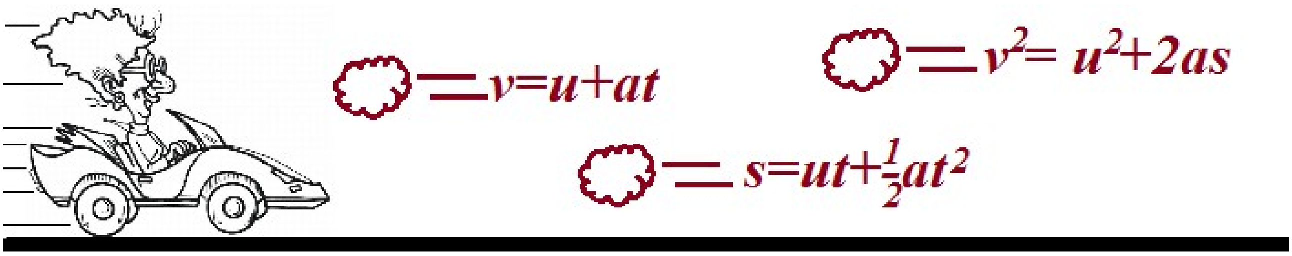
Section 6 Cont'd
Application to Linear Motion and Projectiles

In this final section on calculus, we'll show how concepts like velocity, acceleration, the linear equations of motion, and projectile motion can be derived from both differential and integral calculus considerations. First we consider what the relationship is between the derivative and velocity and acceleration. Then the linear equations of motion will be derived and examples of objects obeying these equations given. Then a discussion of motion under the influence of gravity will be proffered before moving on to consider the equations of motion for 2D projectiles.

Linear Motion



Consider a car/vehicle/ etc. moving along a horizontal road with negligible friction from the air or the road. Then we can define the following:

Average velocity, $\bar{v}$:

This is equal to the total distance travelled, s, divided by the total elapsed time takento complete the journey, $t_{elapsed}$: i.e.

$$\bar{v} := \frac{\text{Total distance travelled}}{\text{Total Elapsed time}} = \frac{s}{t_{elapsed}}. \quad \text{Units } \frac{m}{s}$$

Then, if it takes 1 hour ($1h$) to travel $100km$, the average velocity is

$$\bar{v} := \frac{\text{Total distance travelled}}{\text{Total Elapsed time}} = \frac{100}{1}\frac{km}{h}$$

In m/s this equates to approximately $28m/s$.[1]

Instantaneous velocity, v:

This is the measured velocity at a specific instance (or the very small distance travelled Δx over an infinitesimal interval of time Δt). Therefore

$$v := \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{d}{dt} x. \qquad \text{Units } \frac{m}{s}$$

1 1 km = 1000m and 1 hour = 3600s. Then 100km/h = 100000m/3600s = 100/3.6 m/s ~28m/s

Initial velocity, *u*:

This is the measured velocity at the instant the motion comes under scrutiny. If the object is initially at rest then u=0m/s.

Final velocity, *v:*

This is the measured velocity at the instant the motion ceases to be of interest; i.e.after the period of time has elapsed or a displacement has been covered, etc. If the object finally comes to rest then v=0m/s.

Acceleration, *a*:

This is defined to be the time rate of change of velocity: i.e. the derivative of instantaneous velocity w.r.t time

$$a := \frac{d}{dt} v = \frac{d^2}{dt^2} x. \qquad \text{Units } \frac{m}{s^2}$$

Displacement (Distance Travelled), *s*:

This is defined to be the distance travelled in the elapsed time while starting at an initial velocity, *u*, under a constant accleeration, *a*.

Then we can derive the equations of motion for such a system under the influence of a constant accleleration:

Linear Equations of Motion.

Given s, t and a with $\frac{d}{dt} v(t) = a$. Velocity $v(t)$ has the initial and final velocities of u and v respectively: i.e.

$$v(0) = u \text{ and } v(t_{elapsed}) = v$$

Then

$$\begin{aligned} \frac{d}{dt} v(t) = a \Rightarrow \int_0^t \frac{d}{dt} v(t)\, dt &= \int_0^t a\, dt \\ \Rightarrow \int_{v(0)}^{v(t)} dv &= \int_0^t a\, dt \\ \Rightarrow v(t) - v(0) &= a\, t \\ \Rightarrow v - u &= a\, t \\ \Rightarrow v &= u + a\, t. \end{aligned}$$

Now integrate this noting that $\frac{d}{dt} v(t) = a$ where after a time t the displacement is s.

$$\frac{d}{dt} x(t) = v(t) \Rightarrow \int_o^s dx = \int_0^t u + a\, t\, dt$$

$$\Rightarrow s-0=ut+\frac{1}{2}at^2$$

$$s=ut+\frac{1}{2}at^2$$

Back to our original equation:

$$\frac{d}{dt}v(t)=a\Rightarrow v\frac{d}{dt}v(t)=av=a\frac{d}{dt}x(t)$$

$$\Rightarrow\int_u^v v\,dv=a\int_0^s dx$$

$$\frac{1}{2}\left(v^2-u^2\right)=as$$

$$v^2=u^2+2as$$

These constitute the three equations of linear motion.

Example:

Consider a vehicle initially at rest that is put under the influence of a constant accleleration of 3m/s^2.

1. What is the final velocity after 10 seconds and what distance has the vehicle travelled?
2. If it is then subject to a decleration of $8\ \text{m/s}^2$, what distance does the vehicle travel before coming to rest?

Here a=3m/s^2, u=0, t=10s.

1. Velocity and distance travelled after t=10s.

Using

$$v=u+at$$

$$\Rightarrow v=0+3\times10\,m/s$$

$$\Rightarrow v=30\,m/s\,(=108\,km/h).$$

Distance travelled is given from

$$s=ut+\frac{1}{2}at^2$$

and thus

$$s=0\times10+\frac{1}{2}3(10^2)$$

$$\Rightarrow s=150\,m$$

2. Distance travelled under constant deceleration of $8m/s^2$.
The vehicle is now subject to a deceleration of 8m/s^2. Using

$$v^2 = u^2 + 2as$$

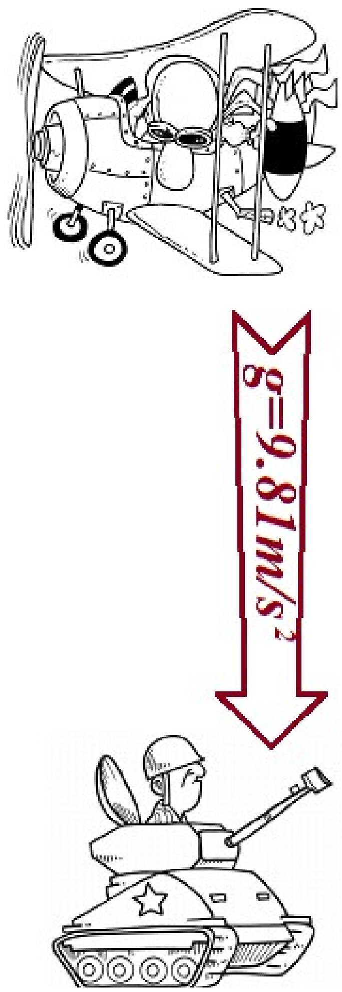
with[2]

$$v = 0\, m/s\,,\, a = -8\text{m}/s^{2,} \text{ and } u = 30\, m/s$$

then

$$0^2 = 30^2 + 2(-8)s$$
$$\Rightarrow 16s = 900$$
$$\Rightarrow s = \frac{225}{4} m = 56.25\text{m}$$

Motion under the influence of gravity, *g*.



Now we turn our attention to a problem that any pilot will attest to; motion involving the acceleration due to the influence of the Earth's mass; i.e. gravity.

At sea level on the Earth's surface, it is accepted that the acceleration due to gravity, denoted *g*, has the value 9.81 m/s^2.

As the cartoon to the left depicts, it always points downwards towards the Earth's surface. Therefore equations of motion involving gravity hves differing forms for upward cases and downwards cases.

The linear equations of motion are applicable here as we'll assume the motion is restricted to an altitude where the change in the value of *g* is negligible.

Then the acceleration/deceleration is constant for these simple problems and the three equations for linear motion can be used.

2 Note that a deceleration is essentially a braking phenomenon and as such is negative w.r.t. the direction of motion.

Free-falling objects:
Here g acts as an acceleration, a=g, and so the three linear equations of motion become

$$v = u + g t$$
$$s = u t + \frac{1}{2} g t^2$$
$$v^2 = u^2 + 2 g s$$

Objects propelled vertically upwards:
These objects are effectively undergoing a two stage motion;

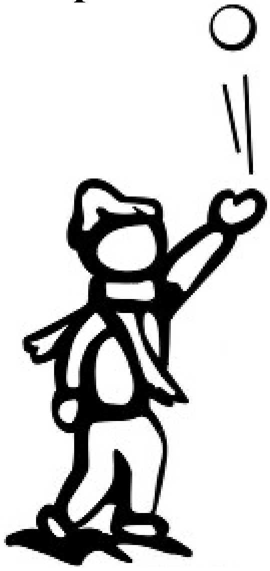
- the motion upwards when g is acting against the motion as a deceleration
- the motion downards when the objects is essentially free falling from a maximum height (with an inital velocity of zero at this maximum height).

For the former the following hold: (a=-g)

$$v = u - g t$$
$$s = u t - \frac{1}{2} g t^2$$
$$v^2 = u^2 - 2 g s$$

For the latter use the equations as for the free-falling objects.

Example:



A snowball is thrown vertically upwards with an initial velocity of 10m/s.

Then, using the equations of motion discussed, answer the following:

1. What is the maximum height acheived by this snowball if it was 1.87m above the ground when it was realeased?

2. How long does it take to get to maximum height?

3. How long does it take to hit the ground?

Take g=9/81m/s^2 and that the hand was 1.9m above ground level when the snowball was released.

N.B.

Note the fact that when the ball leaves the thrower's hand the following holds:

$$a=-g=-9.81\,m/s^2 \quad u=10\,m/s \quad t_{max}= \text{ time to maximum height}$$

$$v=0 \text{ when } t=t_{max}$$

1. Maximum Height: h_{max}

At maximum height, the upwards velocity is zero as the object is momentarily stationary before beginning its descent under g. Then using

$$v^2=u^2+2as$$

with

$$v=0, a=-g=-9.81\,m/s^2, u=10\,m/s \text{ and } s=h_{max}$$

we get

$$0=(10^2)-2g\,h_{max}$$

$$\Rightarrow h_{max}=\frac{u^2}{2g}=\frac{100}{19.62}\simeq 5.1\text{m}$$

Time to maximum height: t_{max}

Taking the first equation of motion

$$v=u+at$$

and using the values above we get

$$0=u-g\,t_{max}$$

$$\Rightarrow t_{max}=\frac{u}{g}=\frac{10}{9.81}\simeq 1.02\text{s}$$

Time of flight: t_{flight}

The time to get from the thrower's hand back down to the same height this hand was raised to when the snowball was released is

$$t_{flight}=2t_{max}$$

$$\Rightarrow t_{flight}=2.04\text{s}$$

It must still fall through the 1.9m the hand was at before release and this time needs to be added to the time of flight above:

$t_{hand}=$ time to ground from the height the hand is at using $s=ut+\frac{1}{2}at^2$

Then

$$1.9=10\,t_{hand}+\frac{1}{2}9.81\times t_{hand}^{2}$$

Solving for t_{hand} we get

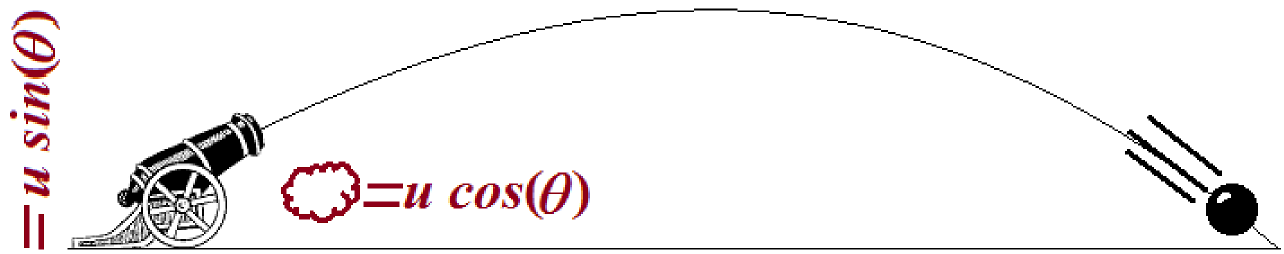
$$t_{hand}=-10\pm\sqrt{100+2\times9.81\times1.9}=-10\pm11.72\,s$$

$$\Rightarrow t_{hand}=\underbrace{-21.72\,s}_{\text{RUBBISH!}}\ \text{or}\ t_{hand}=\underbrace{1.72\,s}_{\text{CORRECT!}}$$

Total time of flight is then

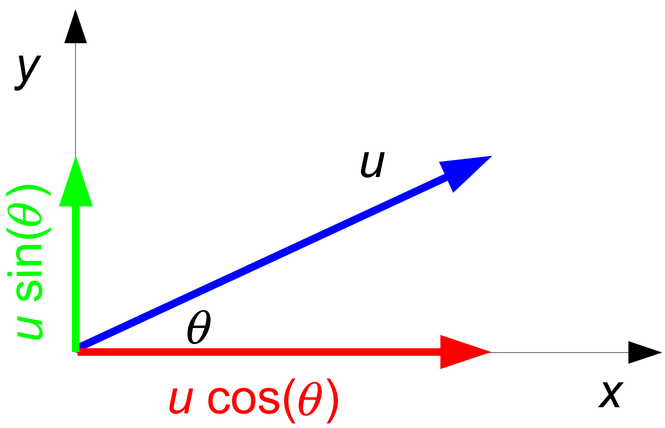
$$t_{flight}+t_{hand}=2.04+1.72\text{s}=3.76\,s$$

Projectile Motion in 2D



This motion has two independent parts (components) to it; a horizontal component that has no acceleration/deceleration acting upon it and a vertical motion similar to that just discussed. You first task in tackling these problems is to correctly split the motion into its component parts. Once this has been acheived, the two are solved independently of each other with only the time to max height t_{max} being carried from one to the other.

To see how this is accomplished, we need to recall how a vector is split into two components; one parallel to the x axis (horizontal component) ans the other to the $y$axis (vertical component). For a known initial velocity of u m/s, these components would be defined



Then we consider the vertical components in the same way as for the vertically thrown ball previously and the horizontal components as for the car moving horizontally under no applied acceleration.
Vertically:
The initial velocity is the vertical component of u: i.e. $u \sin(\theta)$.
Then time to max height, $t_{\max}$ is given by

$$t_{max} = \frac{u}{g} \sin(\theta)$$

The max height, $h_{\max}$, is given by

$$h_{max} = u \sin(\theta) t_{max} - \frac{1}{2} g t_{max}^2 = \frac{u^2}{g} \sin^2(\theta) - \frac{1}{2} \frac{u^2}{g} \sin^2(\theta)$$

$$\Rightarrow h_{max} = \frac{1}{2} \frac{u^2}{g} \sin^2(\theta)$$

The time of flight, t_{flight}, is given by

$$t_{flight} = 2 t_{max} = 2 \frac{u}{g} \sin(\theta)$$

Horizontally:
The initial velocity is the horizontal component of u: i.e. $u \cos(\theta)$.
Then the range, s, is given by

$$s = u \cos(\theta) t_{flight} = 2 \frac{u^2}{g} \cos(\theta) \sin(\theta)$$

$$\Rightarrow s = \frac{u^2}{g} 2 \cos(\theta) \sin(\theta)$$

$$\Rightarrow s = \frac{u^2}{g} \sin(2\theta)$$

This latter expression is a maximum when $\theta = \pi / 4$.

Proof:
Maximise s, with respect to θ: i.e. $s_{\max}$

$$\frac{d}{d\theta} s_{max} = 0$$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{u^2}{g}\sin(2\theta)\right) = 0$$

$$\Rightarrow \frac{u^2}{g}\frac{d}{d\theta}\sin(2\theta) = 0$$

$$\Rightarrow 2\cos(2\theta) = 0 \Leftrightarrow \theta = \frac{\pi}{4}$$

$$\Rightarrow s_{max} = \frac{u^2}{g}$$

This is an important result as you'll see in a future experiment next year.

Example:

A cannon fires a projectile with an initial velocity of 300 m/s at an angle of π/6. Using the equations of motion for projectiles determine:

1. The maximum height reached by the projectile.
2. The flight time of the projectile
3. The range of the projectile.
4. The maximum range of the projectile

Given $u = 300\, m/s, \theta = \pi/6 (= 30^o)$ and we know that $g = 9.81\, m/s^2$. Then

1. The max height, h_{max}, is given by

$$h_{max} = \frac{1}{2}\frac{u^2}{g}\sin^2(\theta)$$

$$\Rightarrow h_{max} = \frac{1}{2}\frac{300^2}{9.81}\sin^2(\pi/6)$$

$$\Rightarrow h_{max} = \frac{1}{8}\frac{90000}{9.81}$$

$$\Rightarrow h_{max} \simeq 1147\text{m}$$

2. The time of flight, t_{flight}, is given by

$$t_{flight} = 2\, t_{max} = 2\frac{u}{g}\sin(\theta)$$

$$\Rightarrow t_{flight} = 2\frac{300}{9.81}\sin(\pi/6)$$
$$\Rightarrow t_{flight} = \frac{300}{9.81}$$
$$\Rightarrow t_{flight} \simeq 30.6\text{s}$$

3. The range, *s*, is given by

$$s = \frac{u^2}{g}\sin(2\theta)$$
$$\Rightarrow s = \frac{300^2}{9.81}\sin(\pi/3)$$
$$\Rightarrow s = \frac{90000}{9.81}\frac{\sqrt{3}}{2}$$
$$\Rightarrow s \simeq 7945\, m$$

4. The maximum range $s_{\max}$ is given by

$$s_{max} = \frac{u^2}{g}$$
$$\Rightarrow s_{max} = \frac{300^2}{9.81}$$
$$\Rightarrow s_{max} = \frac{90000}{9.81}$$
$$\Rightarrow s_{max} \simeq 9174\, m$$