Section 3
Taylor Series

In this section we continue with our analysis of the derivative by looking at Taylor and MacLaurin Series and what they mean for special functions such as the exponential function, the trigonometric functions, and such like. As such we need to revise what is meant by a series, convergence and divergence criteria, and polynomials.

Series

A finite series of real numbers is the sum of the first *n* numbers of a sequence[1] of real numbers: i.e.

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n \qquad \forall\, a_i \in \mathbb{R}\,;\, 1 \le i \le n$$

It is inconvenient to have to re-write this out repreatedly and so we use the summation or "sigma" notation

$$S_n = \sum_{i=1}^{n} a_i \qquad \forall\, a_i \in \mathbb{R}$$

The large "zig-zag" symbol before a_i is the Greek uppercase S or Sigma. The suffix below the symbol gives us the index letter (in our case *i*) along with the starting value for *i* (1 in our example above). The suffix is the value *i* assumes at the end of the sequence (*n* in our example). For an infinite sequence this *n* would be replaced with the infinity symbol, ∞.

A finite series of a infnite series is often referred to as a partial sum of the infinite series (or just partial sum of a series).

Convergent Series

A series S is said to be convergent if the sequence of its partial sums S_n is convergent to some finite value $L \in \mathbb{R}$: i.e.

$$S = \sum_{i=1}^{\infty} a_i \text{ is convergent } \Leftrightarrow (S_n) = (S_1, S_2, \ldots, S_n) \to L \quad \forall\, L \in \mathbb{R}$$

Consider the following example:

1 A sequence of real numbers is an ordered list of related numbers. For example the sequence of positive even numbers is the infinite list $(2, 4, 6, 8, 10, 12, \ldots)$. A finite sequence is a sequence that stops after the required number of elements is reached. An infinite sequence has either no upper or lower bound or neither.

Example

Is the series $S=\sum_{k=1}^{\infty}\frac{1}{k^2}$ convergent?

Let's examine the sequence of partial sums, S_n

$$S_n=\sum_{k=1}^{n}\frac{1}{k^2}$$

$$S_1=\sum_{k=1}^{1}\frac{1}{k^2}=1$$

$$S_2=\sum_{k=1}^{2}\frac{1}{k^2}=1+\frac{1}{4}=1.25$$

$$S_3=\sum_{k=1}^{3}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}\simeq 1.36111$$

$$S_4=\sum_{k=1}^{4}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\simeq 1.423611$$

$$S_5=\sum_{k=1}^{5}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}\simeq 1.463611$$

$$\ldots\ldots\ldots$$

$$S_{40}=\sum_{k=1}^{40}\frac{1}{k^2}=1+\frac{1}{4}+\ldots+\frac{1}{1600}\simeq 1.62024\ldots$$

Graphing S_n vs *n* we get the following

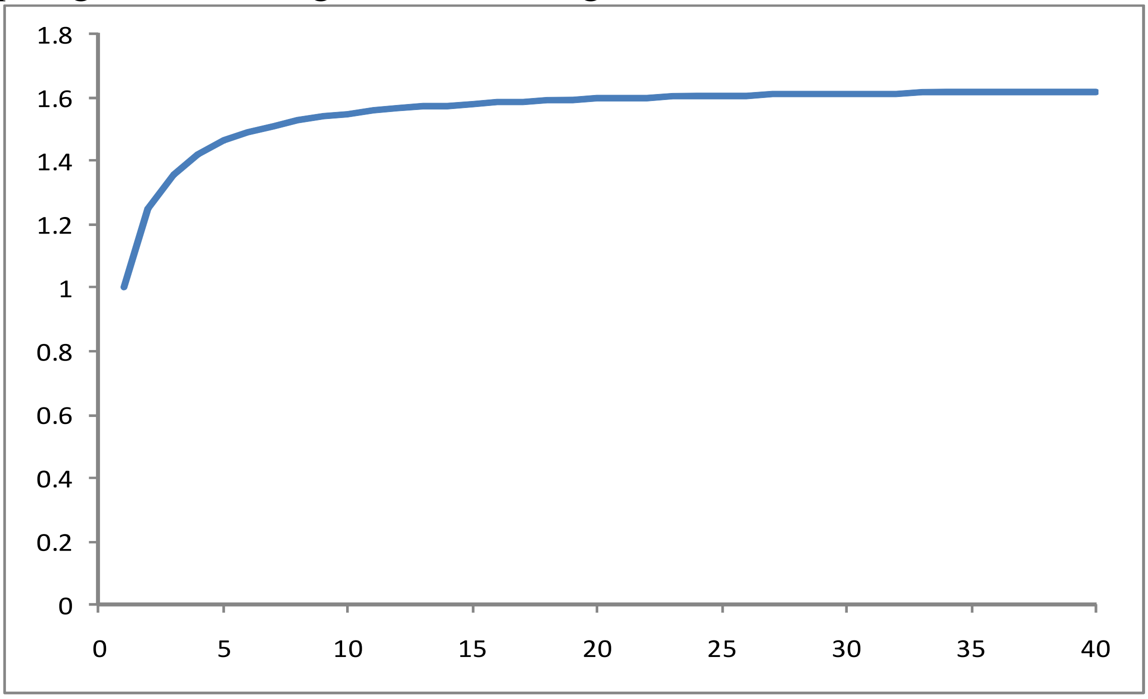

Figure: The sequence of S_n for $n\in(1,40)$

Obviously this is convergent and so we say the series is convergent.

N.B.
In fact as a result of this, any series of the form

$$S=\sum_{k=1}^{\infty}\frac{1}{k^m} \quad m\geq 2$$

is convergent. Can you reason why this is true?

Divergent Series

A series S is said to be divergent if the sequence of its partial sums S_n does not converge to some finite value $L\in\mathbb{R}$: i.e.

$$S=\sum_{i=1}^{\infty} a_i \text{ is divergent} \Leftrightarrow (S_n)=(S_1, S_{2,}\ldots, S_n)\rightarrow\pm\infty$$

Consider the following example:

Example

Is the series $S=\sum_{k=1}^{\infty}\frac{1}{k}$ convergent?

Let's examine the sequence of partial sums, S_n

$$S_n=\sum_{k=1}^{n}\frac{1}{k}$$

$$S_1=\sum_{k=1}^{1}\frac{1}{k}=1$$

$$S_2=\sum_{k=1}^{2}\frac{1}{k}=1+\frac{1}{2}=1.5$$

$$S_3=\sum_{k=1}^{3}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}\simeq 1.83333$$

$$S_4=\sum_{k=1}^{4}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\simeq 2.08333$$

$$\ldots\ldots\ldots$$

$$S_{40}=\sum_{k=1}^{40}\frac{1}{k}=1+\frac{1}{4}+\ldots+\frac{1}{40}\simeq 4.278543$$

Not very obviously convergent so let's graph it as we did above

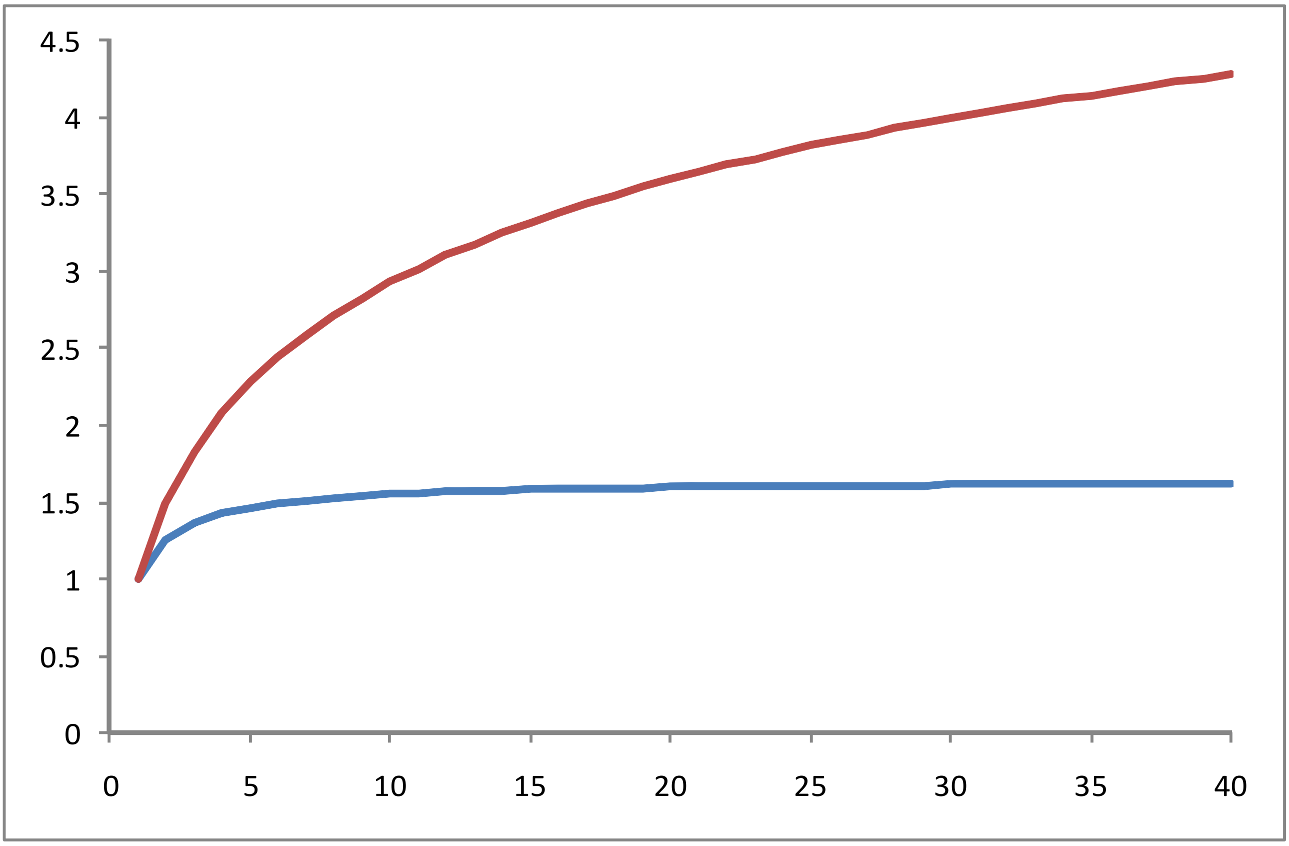

Figure: The sequence of S_n for $S=\sum_{k=1}^{\infty}\frac{1}{k}$ (red) and $S=\sum_{k=1}^{\infty}\frac{1}{k}$ (blue).

Again, it is bending over and is not obviously diverging but is it convergent? To answer this we consider the series itself:

$$S=\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{4}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}+\dots$$

Therefore

$$S=\sum_{k=1}^{\infty}\frac{1}{k}>1+\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}+\dots\rightarrow\infty$$

is a divergent series. Surprising and unpredictable result until we analyse the elementsas we did above.

Polynomial of order *n*

A polynomial of order n of a real variable x is defined to be the finite series

$$P_n(x)=\sum_{k=0}^{n}a_n x^n=a_0+a_1x+\dots+a_nx^n \quad \forall a_k\in\mathbb{R};\ \ 0\le k\le n$$

Let us imagine this polynomial's derivatives were known to us but the individual $a_k{}'s$ are not. This is actually quite common for “special” functions and hence the need for us to consider this.
Given $P_n(x)$ we can say that

$$P_n(0)=a_0$$

where we have set x to 0.

Now do we find a_1?
Well it must involve differentiation; otherwise why include it here? Well

$$\frac{d}{dx}P_n(x)=\sum_{k=1}^{n} n\,a_n x^{n-1}=a_1+2\,a_2 x+\ldots+n\,a_n x^{n-1}$$

and so

$$\frac{d}{dx}P_n(0)=a_1$$

Similarly

$$\frac{d^2}{dx^2}P_n(x)=\sum_{k=2}^{n} n(n-1)\,a_n x^{n-2}=2\,a_2+3\times 2\,a_3 x+\ldots+n(n-1)\,a_n x^{n-2}$$

and therefore

$$\frac{d^2}{dx^2}P_n(0)=2\,a_2\Rightarrow\frac{1}{2}\frac{d^2}{dx^2}P_n(0)=a_2$$

Proceeding in the same way

$$\frac{d^3}{dx^3}P_n(0)=3\times 2\,a_3\Rightarrow\frac{1}{3!}\frac{d^3}{dx^3}P_n(0)=a_3$$

$$\frac{d^4}{dx^4}P_n(0)=4\times 3\times 2\,a_4\Rightarrow\frac{1}{4!}\frac{d^4}{dx^4}P_n(0)=a_4$$

$$\ldots\ldots\ldots$$

$$\frac{d^k}{dx^k}P_n(0)=k\times\ldots\times 3\times 2\,a_k\Rightarrow\frac{1}{k!}\frac{d^k}{dx^k}P_n(0)=a_k$$

So why bother with this process?
There is a theorem that states that any differentiable function admits a power series expansion; e.g.

$$e^x=a_0+a_1 x+a_2 x^2+a_3 x^3+\ldots$$

What interests us, both from an analysis and mathematical view, is what are the coefficients a_k for such functions as e^x and are the series finite or not?
Let us consider e^x.
If the series representing e^x is finite then we find

$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$$
$$\frac{d}{dx} e^x = a_1 + 2 a_2 x + 3 a_3 x^2 + \ldots + n a_n x^{n-1} \neq e^x$$

It could be argued that the coefficients a_k are such that the above does hold.

True! However the last term in the original e^x would have to be zero as there is no x^n term in the derivative. So $a_n = 0$ but that makes the $n a_n x^{n-1} = 0$ in the derivative.

Then $a_{n-1} = 0$ in the original e^x series. However that makes the $(n-1) a_{n-1} x^{n-2} = 0$ in the derivative. And so on until we find the entire series has equality with its derivative if and only if they are both zero: RUBBISH!

Then for

$$\frac{d}{dx} e^x = e^x \;\Leftrightarrow\; e^x = \sum_{k=0}^{\infty} a_k x^k$$

Having established that the exponential function has an infinite series representation we need to determine what form the a_k coefficients take.

We have, from above, the relationship

$$\frac{1}{k!} \frac{d^k}{dx^k} P_n(0) = a_k$$

and so for an infinite polynomial we drop the subscript and get

$$\frac{1}{k!} \frac{d^k}{dx^k} P(0) = a_k$$

For the exponential function $p(x) = e^x$ and thus

$$\frac{1}{k!}\frac{d^k}{dx^k}P(0)=a_k=\frac{1}{k!}\left|\frac{d^k}{dx^k}e^x\right|_{x=0}$$

As e^x is invariant under the differential operator, then all derivatives are equal to e^x. Then

$$a_k=\frac{1}{k!}\left|\underbrace{\frac{d^k}{dx^k}e^x}_{e^x}\right|_{x=0}=\frac{1}{k!}\left|e^x\right|_{x=0}=\frac{1}{k!}$$

and the exponential function has series representation

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

MacLaurin Series

Let $f(x)$ be a differentiable function whose higher derivatives are themselves differentiable for all x in the neighbourhood of 0. Then we can represent $f(x)$ by the series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left|\frac{d^k}{dx^k}f(x)\right|_{x=0}$$

called the MacLaurin series of $f(x)$.

Notation

For convenience we write higher order derivatives

$$\frac{d^k}{dx^k}f(x) \text{ as } f^{(k)}(x) \text{ and } \left|\frac{d^k}{dx^k}f(x)\right|_{x=0} \text{ as } f^{(k)}(0)$$

Then the MacLaurin series for $f(x)$ can be written as

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

Examples

Find the MacLaurin series of $f(x)=\cos(x)$.

We know that

$$f^{(0)}(0)=f(0)=\cos(0)=1$$

For the derivatives we have

$$f^{(1)}(x)=\frac{d}{dx}\cos(x)=-\sin(x)$$
$$f^{(2)}(x)=\frac{d}{dx}(-\sin(x))=-\cos(x)$$
$$f^{(3)}(x)=\frac{d}{dx}(-\cos(x))=\sin(x)$$
$$f^{(4)}(x)=\frac{d}{dx}(\sin(x))=\cos(x)=f(x)$$

Then $f(x)=\cos(x)$ is cyclic of period 4 under differentiation; i.e.

$$\begin{array}{ccccccccc}
f(x) & = & f^{(4)}(x) & = & f^{(8)}(x) & =\ldots= & f^{(4m)}(x) & =\ldots= & \cos(x) \\
f^{(1)}(x) & = & f^{(5)}(x) & = & f^{(9)}(x) & =\ldots= & f^{(4m+1)}(x) & =\ldots= & -\sin(x) \\
f^{(2)}(x) & = & f^{(6)}(x) & = & f^{(10)}(x) & =\ldots= & f^{(4m+2)}(x) & =\ldots= & -\cos(x) \\
f^{(3)}(x) & = & f^{(7)}(x) & = & f^{(11)}(x) & =\ldots= & f^{(4m+3)}(x) & =\ldots= & \sin(x)
\end{array}$$

The derivatives evaluated at $x=0$, $f^{(k)}(0)$, oscillates between three values

$$\begin{array}{ccccccccc}
f(0) & = & f^{(4)}(0) & = & f^{(8)}(0) & =\ldots= & f^{(4m)}(0) & =\ldots= & 1 \\
f^{(1)}(0) & = & f^{(5)}(0) & = & f^{(9)}(0) & =\ldots= & f^{(4m+1)}(0) & =\ldots= & 0 \\
f^{(2)}(0) & = & f^{(6)}(0) & = & f^{(10)}(0) & =\ldots= & f^{(4m+2)}(0) & =\ldots= & -1 \\
f^{(3)}(0) & = & f^{(7)}(0) & = & f^{(11)}(0) & =\ldots= & f^{(4m+3)}(0) & =\ldots= & 0
\end{array}$$

It is apparent from above that only the even terms survive:

$$\begin{array}{rcl}
f(x) & = & \cos(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \\
 & = & 1+0\times x+(-1)\frac{x^2}{2}-0\times\frac{x^3}{3!}+\frac{x^4}{4!}+0\times\frac{x^5}{5!}+(-1)\frac{x^6}{6!}-0\times\frac{x^7}{7!}+\ldots \\
 & = & 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots+(-1)^k\frac{x^{2k}}{(2k)!} \\
 & = & \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}
\end{array}$$

We now have the series representation for the cosine function:

$$f(x)=\cos(x)=\sum_{k=0}^{\infty}\frac{(-1)^{(k)}}{(2k)!}x^{2k}$$

In actuality, this is how the cosine of a value x is determined by your electronic calculator and by the log tables.

Exercise

Show that the MacLaurin series of $f(x)=\sin(x)$ is given by

$$f(x)=\sin(x)=\sum_{k=0}^{\infty}\frac{(-1)^{(k)}}{(2k+1)!}x^{2k+1}$$

Taylor Series

Let $f:U\subseteq\mathbb{R}\rightarrow V\subseteq\mathbb{R}$ be a differentiable function whose higher derivatives are themselves differentiable for all *x* in the neighbourhood of some $x_0\in U$.

Then we can represent *f*(*x*) by the series

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{k}(x_0)}{k!}(x-x_0)^{k}$$

called the Taylor series of *f*(*x*) about x_0 .

Therefore the MacLaurin series is a special case of the Taylor series; i.e. The MacLaurin series is the Taylor series at $x_0=0$.

Why bother with this more general method?

There are functions that do not have a finite, or possibly defined, value at $x_0=0$ but are differentiable at other values of *x*.

A common example of such a function is the natural logarithm, $\ln|x|$. This function does not exist at $x=0$ but is real for all positive real *x*. With this in mind we can determine the series representation of $\ln|x|$ at some other x_0 ; say $x_0=1$.

Then, with $f(x)=\ln|x|$

$$f^{(1)}(x)=\frac{d}{dx}\ln|x|=\frac{1}{x}$$

$$f^{(2)}(x)=\frac{d}{dx}\frac{1}{x}=-\frac{1}{x^2}$$

$$f^{(3)}(x)=\frac{d}{dx}\left(-\frac{1}{x}\right)=\frac{2}{x^3}$$

$$f^{(4)}(x)=\frac{d}{dx}\left(\frac{2}{x}\right)=-\frac{3!}{x^4}$$

$$\cdots\cdots\cdots$$

$$f^{(k)}(x)=(-1)^{k-1}\frac{(k-1)!}{x^k} \quad \forall k\geq 1$$

Then we can evaluate $f^{(k)}(x_0)$ when $x_0=1$, $f(1)=\ln|1|=0$

$$f^{(1)}(1)=\frac{1}{1}=1$$

$$f^{(2)}(1)=-\frac{1}{1^2}=-1$$

$$f^{(3)}(1)=\frac{2}{1^3}=2$$

$$f^{(4)}(1)=-\frac{3!}{1^4}=-3!$$

$$\cdots\cdots\cdots$$

$$f^{(k)}(1)=(-1)^{k-1}\frac{(k-1)!}{1^k}=(-1)^{k-1}(k-1)! \quad \forall k\geq 1$$

Then the series evaluates as

$$\begin{aligned} f(x) &= \ln|x| = \sum_{k=0}^{\infty}\frac{f^k(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k \\ &= 0+(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+(-1)^{k-1}\frac{(x-1)^k}{k}+\cdots \end{aligned}$$

Exercise

Find the Taylor Series of the following functions aboout the specified x_0:

1. $f(x)=\ln|x|$
2. $f(x)=e^{3x} \quad x_0=1$
3. $f(x)=\ln|1+x| \quad x_0=2$

In the next section we'll consider the theorem associated with Taylor Series and some new “Special Functions”.