Section 4 (Cont'd)
The inverse of a Matrix

In this section we consider the inverse of a square matrix of real numbers. We will showhow it is related both tothe determinant and the adjoint of the originating square matrix, A. We will also consider an important use of the inverse; that of solving systems of linear equations.

The Inverse of a square Matrix

The inverse of a square matrix, A, of real numbers is the matrix, denoted A^{-1}, that satisfies the following:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

i.e.

the product of the inverse matrix with its originating matrix is the identity matrix which is the matrix analogy of 1. Then for any non-zero real number

$$\alpha \cdot \alpha^{-1} = \alpha \cdot \frac{1}{\alpha} = 1$$

and so we can think of this as analogous but not, in any form, equivalent to division[1].

Let A be a square $n \times n$ matrix of real numbers with determinant , $|A|$, and adjoint, A^*. Then, from our previous section on the adjoint, we know that

$$A \cdot A^* = \begin{pmatrix} |A| & 0 & \cdots & 0 & 0 \\ 0 & |A| & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| & 0 \\ 0 & 0 & \cdots & 0 & |A| \end{pmatrix}$$

This resulting square matrix is just

$$A \cdot A^* = \begin{pmatrix} |A| & 0 & \cdots & 0 & 0 \\ 0 & |A| & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| & 0 \\ 0 & 0 & \cdots & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = |A| \cdot I$$

1 It is important that you do not confuse this inverse operation with division. Itis more appropriate to think of it in terms of the inverse function we discussed in earlier sections.

and so we can say that

$$A \cdot A^* = |A| I$$
$$\Rightarrow A \cdot \frac{1}{|A|} A^* = I = A \cdot A^{-1}$$

Therefore

$$A^{-1} = \frac{1}{|A|} A^*$$

and the inverse is defined.

N.B.
From the definition above, we can see that singular matrices (ones whose determinants are zero) do not have inverse matrices because $\frac{\mathbf{1}}{|A|}$ is not defined:

$$\text{Let } A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \quad \Rightarrow \exists A^{-1} \;\Leftrightarrow |A| \neq 0$$

The Inverse of a 3×3 Matrix
Then, for the 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we have, given $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} adj(A) \equiv \frac{1}{|A|} A^* = \frac{1}{|A|} \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix}$$

N.B.
It should now be apparent as to why we think of the determinant as a magnitude. For matrices with zero determinant, there is no inverse just as in real numbers there is no inverse to 0.

Some Notable Inverses

We consider here some inverses of note; specifically diagonal matrices, the identity matrix, triangular matrices, and the zero matrix.

1. **Diagonal Matrices**

   In the last section we showed that the determinant of a diagonal matrix was the product of all the principle diagonal elements. Therefore it follows that for the diagonal matrix not to be singular, all the principle diagonal elements must be non-zero: i.e. For

$$A=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

   the determinant is:

$$|A|=a_{11}a_{22}a_{33}\cdots a_{nn}=\prod_{k=1}^{n} a_{kk}\neq 0 \Leftrightarrow a_{kk}\neq 0 \quad 1\leq k\leq n$$

   Example:

   Let A be the 3×3 diagonal matrix

$$A=\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

   Then

$$|A|=\begin{vmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix}=-24 \text{ and } A^{*}=\begin{pmatrix} 12 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -6 \end{pmatrix}\Rightarrow A^{-1}=-\frac{1}{24}\begin{pmatrix} 12 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

   as expected.

   N.B.

   Note the structure of the diagonal matrices' adjoint. For all diagonal matrices, their adjoint is itself diagonal.

2. **Identity matrix**

   Just as the inverse of 1 is 1, the inverse of the $n\times n$ identity matrix, I, is itself: i.e.

$$I^{-1}\cdot I=I\Rightarrow I^{-1}=I$$

3. **Zero Matrix**

The zero matrix O, as an $n \times n$ diagonal matrix with all zero elements on the principle diagonal, has zero determinant and thus has no inverse:

$$|O| = 0 \Rightarrow O^{-1} \text{ does not exist}$$

4. **Triangular Matrices**

For any $n \times n$ Lower triangular matrix, *L*,

$$|L| = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

it follows that all diagonal elements must be non-zero for the inverse to exist.

Consider the 3×3 Lower (or Left) Triangular matrix, *L*:

$$L = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then, the inverse is defined, given that $|L| = a_{11} a_{22} a_{33} \neq 0$,

$$L^{-1} = \frac{1}{|L|} \begin{pmatrix} a_{22} a_{33} & 0 & 0 \\ -a_{21} a_{33} & a_{11} a_{33} & 0 \\ a_{21} a_{32} - a_{31} a_{22} & -a_{11} a_{32} & a_{11} a_{22} \end{pmatrix}$$

itself a lower diagonal matrix.

Exercise

Show that

$$L \cdot L^{-1} = L^{-1} \cdot L = I$$

For the Upper triangular matrix, U, we have the a similar result as for the Lower triangular matrix. For clarity we'll consider the 3×3 case:

Given U

$$U = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

we find the inverse is defined, given that $|U| = a_{11}a_{22}a_{33} \neq 0$,

$$U^{-1} = \frac{1}{|U|}\begin{pmatrix} a_{22}a_{33} & -a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ 0 & a_{11}a_{33} & -a_{11}a_{23} \\ 0 & 0 & a_{11}a_{22} \end{pmatrix}$$

itself an upper diagonal matrix.

Exercise
Show that

$$U \cdot U^{-1} = U^{-1} \cdot U = I$$

Solution of Systems of Linear Equations
Consider the following matrix product

$$A \cdot X = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where the $n \times 1$ "column matrix", *X*, has variables as elements. Then

$$A \cdot X = \begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{matrix}$$

which is a system of linear equations (or simultaneous equations). If we set each equation equal to some constant $b_k \; 1 \leq k \leq n$ then

$$A \cdot X = \begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{matrix} = B$$

In other words, any system of linear equations can be expressed in the matrix form

$$A\cdot X=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}=B$$

Once we have established such a relationship and assuming A has a non-zero determinant then

$$A^{-1}\cdot(A\cdot X)=A^{-1}\cdot B$$
$$\underbrace{A^{-1}\cdot A}_{I}\cdot X=A^{-1}\cdot B$$
$$I\cdot X=A^{-1}\cdot B$$
$$\Rightarrow X=A^{-1}\cdot B \quad \Leftrightarrow |A|\neq 0$$

The solution to the system of linear equations is the product of the inverse matrix A^{-1} with B.

N.B.

In order for the above to hold the matrix A must be square; i.e. as many equations as unknown variables. Otherwise this method cannot be applied.

Example:

Solve the following system of linear equations:

$$2x+3y-4z=1$$
$$x-y+2z=0$$
$$-3x+2y-z=-2$$

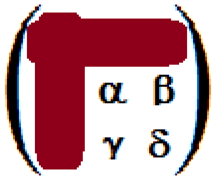
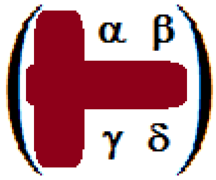
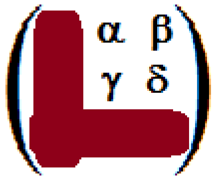
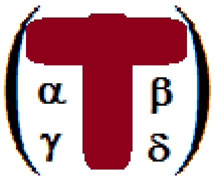
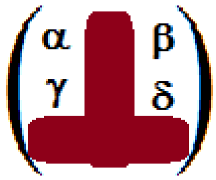
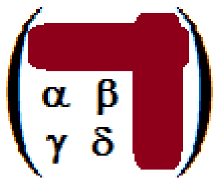
Then re-writing the system in matrix form $A\cdot X=B$

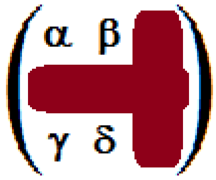
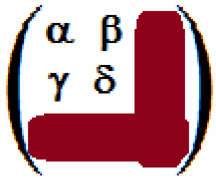
$$A\cdot X=\begin{pmatrix} 2 & 3 & -4 \\ 1 & -1 & 2 \\ -3 & 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}=B$$

For this to have a solution we require A^{-1} to exist:i.e. $|A|\neq 0$

$$|A|=2(1-4)+3(-6+1)-4(2-3)=-6-15+4=-17\neq 0$$
$$\Rightarrow \exists\, A^{-1}$$

Next we calculate the adjoint of A

$\Delta_{11}=(-1)^2\begin{vmatrix}-1 & 2\\ 2 & -1\end{vmatrix}=(-1)^2(1-4)=(1)(-3)=-3$	
$\Delta_{12}=(-1)^3\begin{vmatrix}3 & -4\\ 2 & -1\end{vmatrix}=(-1)^3(-3+8)=(-1)(5)=-5$	
$\Delta_{13}=(-1)^4\begin{vmatrix}3 & -4\\ -1 & 2\end{vmatrix}=(-1)^4(6-4)=(1)(2)=2$	
$\Delta_{21}=(-1)^3\begin{vmatrix}1 & 2\\ -3 & -1\end{vmatrix}=(-1)^3(-1+6)=(-1)(5)=-5$	
$\Delta_{22}=(-1)^4\begin{vmatrix}2 & -4\\ -3 & -1\end{vmatrix}=(-1)^4(-2-12)=(1)(-14)=-14$	
$\Delta_{23}=(-1)^5\begin{vmatrix}2 & -4\\ 1 & 2\end{vmatrix}=(-1)^5(4+4)=(-1)(8)=-8$	
$\Delta_{31}=(-1)^{3+1}\begin{vmatrix}1 & -1\\ -3 & 2\end{vmatrix}=(-1)^4(2-3)=(1)(-1)=-1$	

$\Delta_{32}=(-1)^{5}\begin{vmatrix}2 & 3\\-3 & 2\end{vmatrix}=(-1)^{5}(4+9)=(-1)(13)=-13$	
$\Delta_{33}=(-1)^{6}\begin{vmatrix}2 & 3\\1 & -1\end{vmatrix}=(-1)^{6}(-2-3)=(1)(-5)=-5$	

The adjoint is thus

$$adj(A)\equiv A^{*}=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}=\begin{pmatrix}-3 & -5 & 2\\ -5 & -14 & -8\\ -1 & -13 & -5\end{pmatrix}$$

The inverse of A is

$$A^{-1}=-\frac{1}{17}\begin{pmatrix}-3 & -5 & 2\\ -5 & -14 & -8\\ -1 & -13 & -5\end{pmatrix}$$

The solution to the system of equations is then

$$X=A^{-1}\cdot B=-\frac{1}{17}\begin{pmatrix}-3 & -5 & 2\\ -5 & -14 & -8\\ -1 & -13 & -5\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}=-\frac{1}{17}\begin{pmatrix}-7\\ 11\\ 9\end{pmatrix}$$

Then

$$X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-\frac{1}{17}\begin{pmatrix}-7\\ 11\\ 9\end{pmatrix}$$

or

$$x=\frac{7}{17},\ y=-\frac{11}{17},\ \text{and}\ z=-\frac{9}{17}$$

is the solution of the set of linear equations above.

VERIFY:

To verify the solution above, substitute each of the values in for x, y, and z in the original system and check if it tallies with the values stored in B.

$$2x+3y-4z=\frac{1}{17}(2\times7+3\times(-11)-4\times(-9))=\frac{17}{17}=1$$

$$x-y+2z=\frac{1}{17}(7-(-11)+2(-9))=\frac{0}{17}=0$$

$$-3x+2y-z=\frac{1}{17}(-3\times7+2\times(-11)-1\times(-9))=-\frac{34}{17}=-2$$

The solution is correct.

This finishes your initial foray into linear algebra. Next we tackle vectors.