Section 7
Roots of Complex Numbers

In this section we'll extend our analysis of complex numbers, defined in the previous section, to consider the problem of finding their roots. We will extend the result of DeMoivre's Theorem to facilitate this analysis and also consider common issues with hand-held calculators that need to be highlighted in order to prevent calculation errors from occurring.

DeMoivre's Theorem and the Roots of Complex Numbers

Let $z \in \mathbb{C}$ be any complex number and let n be an integer. There are n n^{th} roots and the m^{th} n^{th} root is defined

$$z_m^{1/n} = (a+bj)^{1/n} = \sqrt[n]{a+bj}$$

$$z_m^{1/n} = |z|^{1/n}\left(\cos\left(\frac{\theta+2m\pi}{n}\right)+j\sin\left(\frac{\theta+2m\pi}{n}\right)\right)$$

where $m=0,1,\ldots,n-1$.

You may ask why doesn't this formula have the same form as that specified by DeMoivre's Theorem? In other words, the arguments of the sine and cosine are just divided by n. Why do we need the extra $2m\pi$ term?

The answer to this is related to the periodic nature of both sine and cosine: i.e.

$$\sin(\theta+2m\pi)=\sin(\theta) \qquad \cos(\theta+2m\pi)=\cos(\theta)$$

where $m \in \mathbb{Z}$.

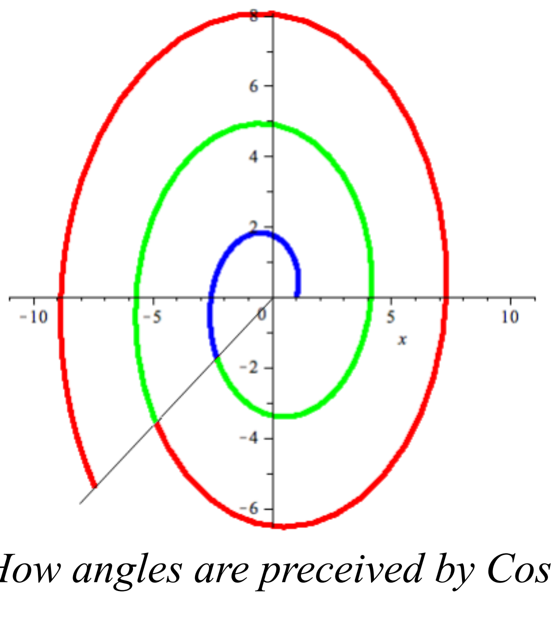

Figure: How angles are preceived by Cos and Sin.

Consider the figure above.

- The Blue part of the spiral represents an angle, say θ, being described about the origin that is less than 2π. This is the normal situation for the trigonometric functions.
- The Green and Blue arcs taken together now represent the original angle, θ, plus an additional full revolution $(\theta+2\pi)$. To sin and cos, the additional 2π (in Green) is stripped away and only the surplus θ (in Blue) remaining is operated upon.
- The total spiral (Red, Green and Blue arcs together) represents the original angle θ plus two full revolutions $(\theta+4\pi)$. Again sin and cos strip these away and operate only on the surplus angle θ (in Blue):

i.e.

$$\sin(\theta+4\pi)=\sin(\theta+2\pi)=\sin(\theta)$$
$$\cos(\theta+4\pi)=\cos(\theta+2\pi)=\cos(\theta)$$

Then, if we divide this argument, θ, by $n\in\mathbb{Z}$, we are left with just one angle: θ/n. Yet there are many more angles which when multiplied by n effectively yield θ from the sin and cos point of view.

We'll illustrate this by example.
Let $\theta=\pi/6$. Let n=3. Then, from the cosine and sine persective

$$\begin{array}{rll}
\theta=\dfrac{\pi}{6} & \Rightarrow\dfrac{\theta}{n}=\dfrac{\pi}{18} & (<2\pi)\\
\theta+2\pi=\dfrac{13\pi}{6} & \Rightarrow\dfrac{\theta}{n}=\dfrac{13\pi}{18} & (<2\pi)\\
\theta+4\pi=\dfrac{25\pi}{6} & \Rightarrow\dfrac{\theta}{n}=\dfrac{25\pi}{18} & (<2\pi)\\
\theta+6\pi=\dfrac{37\pi}{6} & \Rightarrow\dfrac{\theta}{n}=\dfrac{37\pi}{18}=\dfrac{\pi}{18}+2\pi\equiv\dfrac{\pi}{18} & (>2\pi)
\end{array}$$

so that when we add 6π onto θ we start cycling through the previous angles; therefore we stop at $(\theta+4\pi)$. That leaves three roots; one at $\pi/18$, one at $13\pi/18$, and one at $25\pi/18$

In general we have to ask: Is there a value of *m* at which, from the cosine and sine perspectives, the angles become degenerate (start to repeat)?

In other words, what is the smallest non-zero value of *m* for which

$$\sin\left(\frac{\theta+2m\pi}{n}\right)=\sin\left(\frac{\theta}{n}\right) \qquad \cos\left(\frac{\theta+2m\pi}{n}\right)=\cos\left(\frac{\theta}{n}\right)$$

The solution is simple: if $m=0,1,\ldots,n-1$ then the n angles will be unique.

Let's illustrate using some examples:

Example 1: Cube roots of unity

Let $z=1+0j$. Then it has the polar form

$$z=1+0j=\cos(0)+j\sin(0)$$

where $|z|=\sqrt{1^2+0^2}=1$ and $\theta=\tan^{-1}(0/1)=0$.

The cube roots of unity are the three complex numbers that satisfy the equation

$$z^3=1+0j$$

i.e. the three complex numbers

$$z=\sqrt[3]{1+0j}=\sqrt[3]{\cos(0)+j\sin(0)}$$

Using the formula above we need to evaluate

$$z_m=\cos\left(\frac{0+2m\pi}{3}\right)+j\sin\left(\frac{0+2m\pi}{3}\right) \qquad m=0,1,2$$

m=0:

$$z_0=\cos\left(\frac{0}{3}\right)+j\sin\left(\frac{0}{3}\right)=1+0j=1\in\mathbb{R}$$

m=1:

$$z_1=\cos\left(\frac{2\pi}{3}\right)+j\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+j\frac{\sqrt{3}}{2}$$

m=2:

$$z_2=\cos\left(\frac{4\pi}{3}\right)+j\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-j\frac{\sqrt{3}}{2}=\bar{z}_1$$

By convention we set $\omega=z_1$ and $\omega^2=z_2$. The cube roots of unity are then

$$\{1,\omega,\omega^2\}=\left\{1,-\frac{1}{2}+j\frac{\sqrt{3}}{2},-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right\}$$

which is as we should expect as these form the solution set of a cubic, and from the work of Girard, Descartes, and Gauss, we expect there to be as many solutions as the order of the polynomial; three in this example.

If we restricted our analysis to the real numbers then we would have only one solution which would be counter-intuitive.

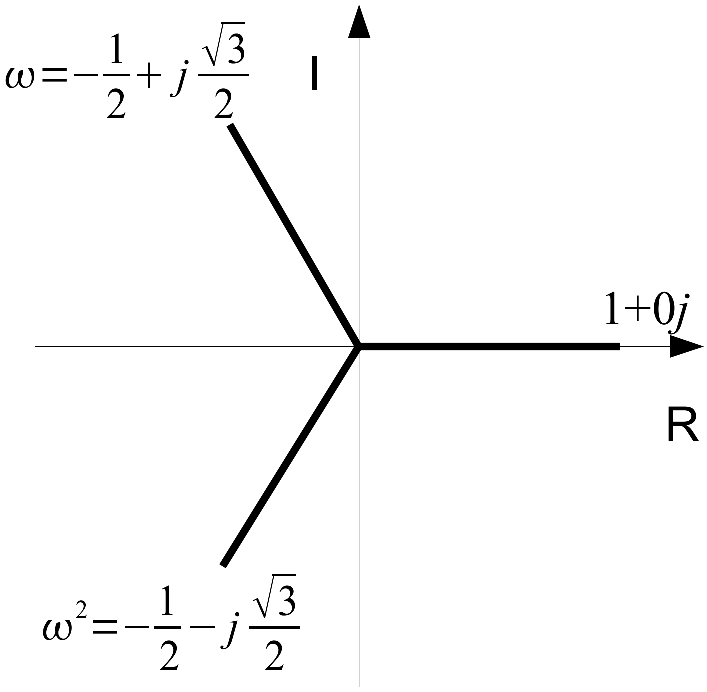

Figure: The cubes roots of unity.

Exercise:
Show that, for the cube roots of unity,
(a) $\omega^3=1$
(b) $\omega^6=1$
(c) $\omega^2=\bar{\omega}$

Let's take another less seminal example.

Example 2:
Let $z_o=-\sqrt{3}+j$. Find all z that satisfy

$$z^4=z_o=-\sqrt{3}+j$$

First express z_o in polar form

$$z_o=-\sqrt{3}+j=2\left(\cos\left(\frac{5\pi}{6}\right)+j\sin\left(\frac{5\pi}{6}\right)\right)$$

where $|z_o|=\sqrt{\sqrt{3}^2+1^2}=\sqrt{4}=2$ and $\theta=\tan^{-1}\left(1/\left(-\sqrt{3}\right)\right)=5\pi/6$.

The solution to the above is to find the four complex numbers that satisfy the equation

$$z^4 = z_o$$

i.e. the four complex numbers

$$z = \sqrt[4]{z_o} = \sqrt[4]{-\sqrt{3}+j} = \left(2^{1/4}\right)\sqrt[4]{\cos\left(\frac{5\pi}{6}\right)+j\sin\left(\frac{5\pi}{6}\right)}$$

Using the formula above we need to evaluate

$$z_m = \left(2^{1/4}\right)\cos\left(\frac{5\pi/6+2m\pi}{4}\right)+j\sin\left(\frac{5\pi/6+2m\pi}{4}\right) \qquad m=0,1,2,3$$

m=0:

$$z_{m=0} = 2^{1/4}\left(\cos\left(\frac{5\pi}{24}\right)+j\sin\left(\frac{5\pi}{24}\right)\right)$$

m=1:

$$z_{m=1} = 2^{1/4}\left(\cos\left(\frac{17\pi}{24}\right)+j\sin\left(\frac{17\pi}{24}\right)\right)$$

m=2:

$$z_{m=2} = 2^{1/4}\left(\cos\left(\frac{29\pi}{24}\right)+j\sin\left(\frac{29\pi}{24}\right)\right)$$

m=3:

$$z_{m=3} = 2^{1/4}\left(\cos\left(\frac{41\pi}{24}\right)+j\sin\left(\frac{41\pi}{24}\right)\right)$$

So these four complex numbers

$$\left\{z_{m=0}, z_{m=1}, z_{m=2}, z_{m=3}\right\}$$

share the property that, when they are raised to the fourth power, the resulting complex number is the original $z_o = -\sqrt{3}+j$ as required and that they form a complete solution; i.e. any other solution is contructed from these four.

This ends our all too brief introduction to complex numbers. The concepts introduced here will be of benefit next year when we deal with quaternions.