Section 1 (Cont'd)
Graph of a Function

To date we have defined functions, variables, constants, and mathematical operators. We have considered composite functions, 1-1 functions, and the inverse of a function. What we aim to tackle here are the concepts behind the graph of a function and some metrics associated with graphs in general.

What is the graph of a function?

The graph of a function, $f: U \subseteq \mathbb{R} \rightarrow V \subseteq \mathbb{R}$, is defined:

$$graph(f) := \{(x, y) | y = f(x) \forall x \in U \subseteq \mathbb{R}\}$$

i.e.

The graph of *f*(*x*) is the set, declared by the {} brackets, of points, (*x*,*y*), such that $y = f(x)$ for any value of *x* in the domain, *U*, of *f*.

As *x* can assume any value in *U* then we potentially have an uncountable set of points for the graph of *f*. In practice, it is not feasible to consider all points in an interval but to sample the function at appropriate values of *x*. By convention we sample first at integer values of *x* in *U*, then half-integer values, then quarter integer values, etc.

To determine the corresponding *f*(*x*) values we construct a table whose columns correspond to the *x*, the component parts of *f*(*x*), and finally *f*(*x*) itself. The rows correspond to the various integer, half-integer, etc. values of *x*.

Example:

Construct a table for the function $f(x) = x^2 - 5x + 3$ on $[-1,5]$.

We know that the component parts of *f*(*x*) are $x^2, -5x,$ and 3. Therefore our table has the columns

x	x^2	$-5x$	3	$f(x)$

We consider the graphs of *f*(*x*) for the integers only, the integers and half-integers only, etc. The various sets will be colour coded for comparison. In the table overleaf we choose black for the integer only, red for the half-integers, and finally blue quarter integers. The corresponding graphs will use the same colours.

x	x^2	$-5\,x$	3	$f(x)$
-1	1	5	3	9
-0.75	**0.5625**	**3.75**	**3**	**7.3125**
-0.5	**0.25**	**2.5**	**3**	**5.75**
-0.25	**0.0625**	**1.25**	**3**	**4.3125**
0	0	0	3	3
0.25	**0.0625**	**-1.25**	**3**	**1.8125**
0.5	**0.25**	**-2.5**	**3**	**0.75**
0.75	**0.5625**	**-3.75**	**3**	**-0.1875**
1	1	-5	3	-1
1.25	**1.5625**	**-6.25**	**3**	**-1.6875**
1.5	**2.25**	**-7.5**	**3**	**-2.25**
1.75	**3.0625**	**-8.75**	**3**	**-2.6875**
2	4	-10	3	-3
2.25	**5.0625**	**-11.25**	**3**	**-3.1875**
2.5	**6.25**	**-12.5**	**3**	**-3.25**
2.75	**7.5625**	**-13.75**	**3**	**-3.1875**
3	9	-15	3	-3
3.25	**10.5625**	**-16.25**	**3**	**-2.6875**
3.5	**12.25**	**-17.5**	**3**	**-2.25**
3.75	**14.0625**	**-18.75**	**3**	**-1.6875**
4	16	-20	3	-1
4.25	**18.0625**	**-21.25**	**3**	**-0.1875**
4.5	**20.25**	**-22.5**	**3**	**0.75**
4.75	**22.5625**	**-23.75**	**3**	**1.8125**
5	25	-25	3	3

Table: The tabulated $f(x)$ for integer, half-integer, and quarter-integer values of x.

The graph of the points, given by $(x, f(x))$, is presented overleaf.

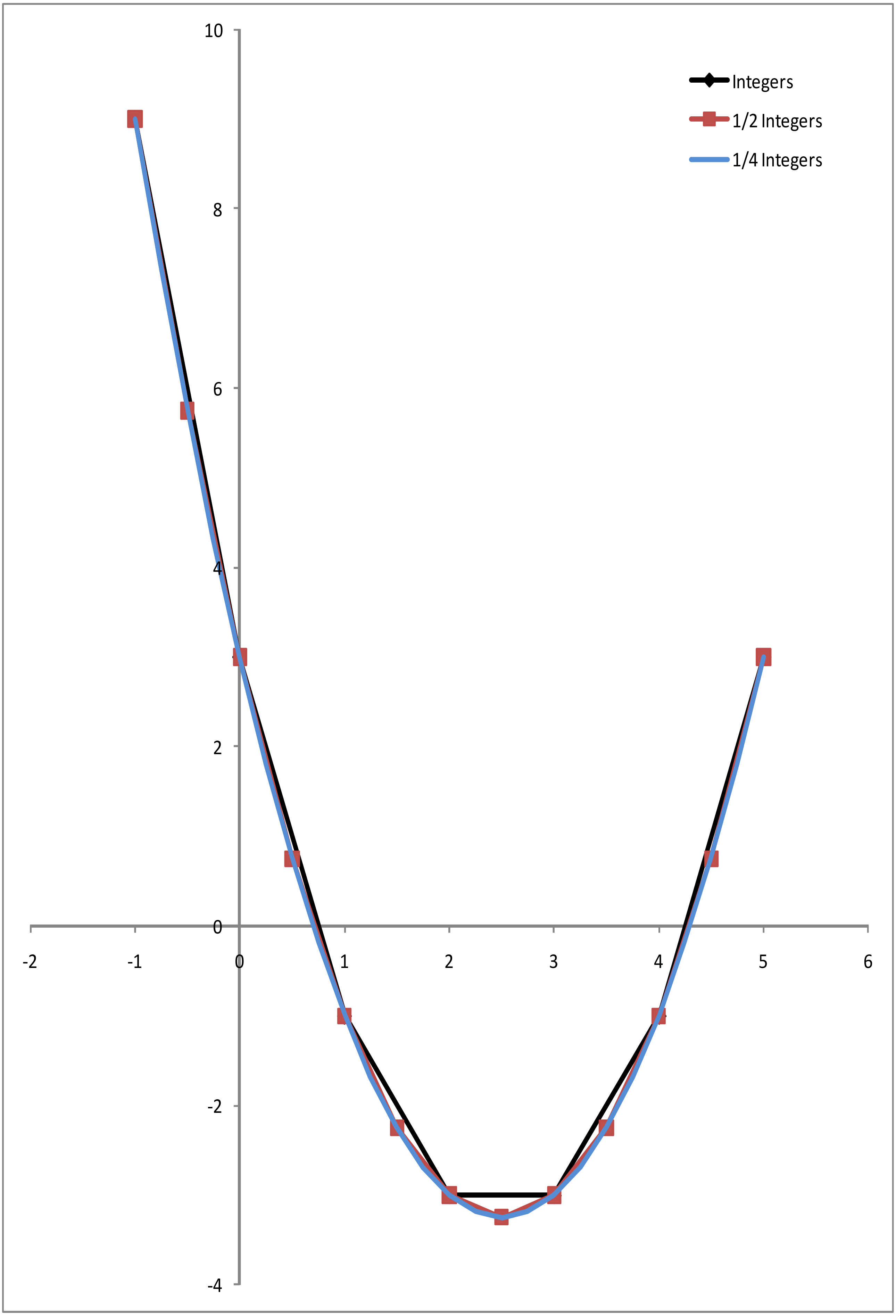

Figure: The graph of $f(x)$ for the tabulated values in the previous table.

The need for more points to adequately plot the function is evident about the “turning point” of the function. The original integer graph misses the point completely whereas the half-integer and quarter-integer graphs not only catch the point but show the function's behaviour more completely. The enlergement of part of the graph given below clearly shows this.

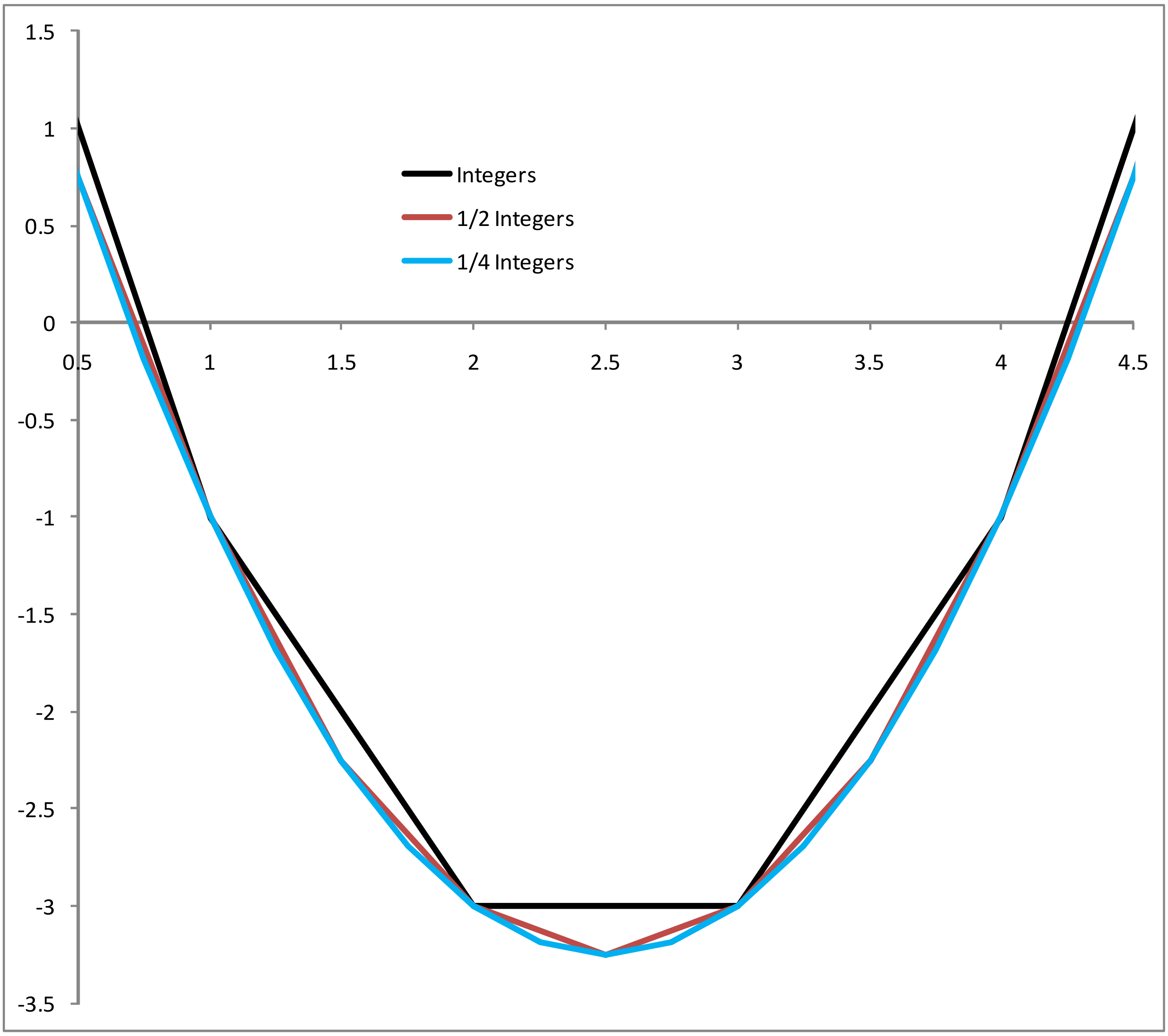

Figure: Enlargement about the “turning point” of $f(x)$

So that's how we generate the data for a graph. Now, how do we graph the function in the table given previously to produce the two figures above?

First, some notation and rules of convention:

1. The horizontal axis is the x-axis. It is along this axis that the interval $[a,b]$ over which the function will be plotted lies.
2. The vertical axis is the y-axis. It is along this axis that the function's values will be plotted via the identity $y=f(x)$.

3. The graph will consist of a finite set of 2D points, or couples; these are paired values enclosed in parentheses () with the independent variable first and the dependent variable second. In our graphs they are thus represented by $\{x, f(x)\}$ or $\{x, y\}$. These point are drawn with a crosshair and may be joined by line segments to the adjacent points.
4. Scaling of the graph to the page is a skill that should be practiced by the student. Typical graph paper (A4 size) has eight large squares across and 12 down each subdivided into 100 (10x10) smaller squares. To maximise the graph will require the student to scale the *x* interval to the eight horizontal boxes and the function's range to the twelve vertical boxes.
5. Always draw the graph in pencil to faciltate erasure of incorrectly placed points.
6. Title the graph and label the axes (*x* for the horizontal and *y* or *f*(*x*) for the vertical).

Below is presented an ideal graph page (A4) with eight horizontal and twelve vertical large boxes. The vertical has been compressed for clarity. The function here is such that the interval [*a*,*b*] is mapped to [*c*,*d*] which has been scaled so that the graph is maximised. A sample point (x_1, y_1) is included for completeness.

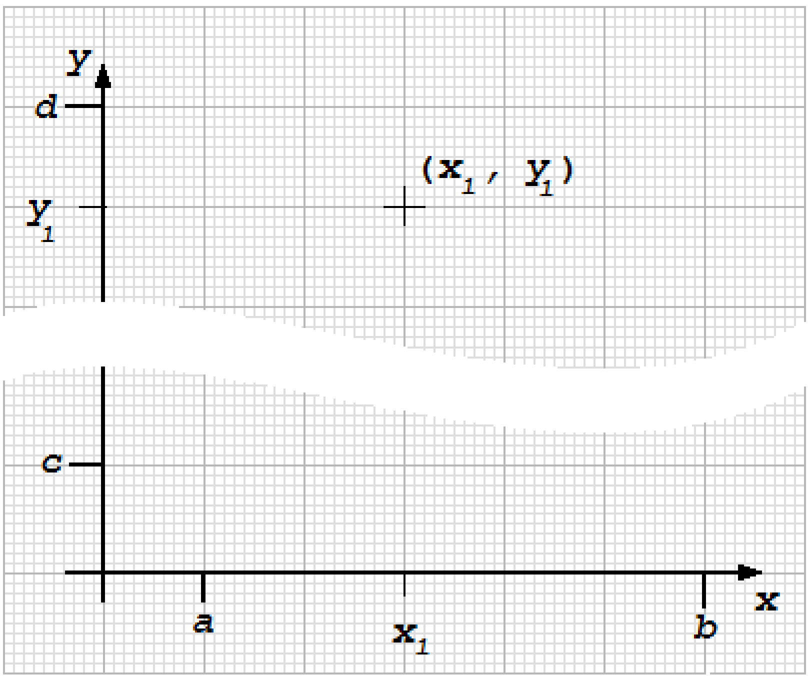

Figure: An example of a graph on A4 graphpaper

Exercise

For each of the following functions on their given intervals;

- construct an appropriate table for graphing the function,
- find a suitable scale for the graph paper,
- graph the function

(a) $f(x)=x^2+4x-6$ on $[-6,2]$

(b) $f(x)=2x^2-3x-4$ on $[-1,3]$

(c) $f(x)=x^3-3x^2+3x-4$ on $[-2,4]$

Geometric Measures

The following measures or metrics are important in the analysis of graphs and functions:

Slope

The slope of the line segment between two points (x_1, y_1) and (x_2, y_2) is defined to be the change in the *y* divided by the change in *x*. Slope is usually denoted by *m* and so

$$m \underset{\substack{\uparrow \\ \text{DEFN}}}{:=} \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

N.B.

(a) The symbol Δ (pronounced Delta) is the Greek Uppercase *d* and is used to indicate difference; i.e. *d* for difference or the subtraction operation between two mathematical objects. So whenever two mathematical objects (be they functions, variables, etc.) are differenced it is the convention to denote this by the symbol Δ.

(b) The slope can take on any real value. The following are extremes

(i) If the line segment is horizontal then $y_2=y_1 \Rightarrow \delta y=0$ and so $m=0$.

(ii) If the line segment is tending vertical with both $\Delta x>0$ and $\Delta y>0$ or both $\Delta x<0$ and $\Delta y<0$ then *m* is tending to infinity

(iii) If the line segment is tending vertical with either $\Delta x<0$ or $\Delta y<0$ but not both then *m* is tending to minus infinity.

Then $-\infty<m<\infty$.

For a linear function, the slope is constant regardless of the two points chosen. For any other function, the slope is dynamic and changes depending on the choice of points. We'll put this latter property to great use later.

Exercise:

(i) For the function defined in (a) above calculate the slope between the following pairs of points:

 (a) (-2,-10) and (0,-6)

 (b) (-5,-1) and (2,2)

(ii) Repeat for the function in (b) but with the following pairs of points:

 (a) (-1,1) and (1,-5)

 (b) (0,-4) and (3,5)

(iii) And again for function in (c) between the pairs of points

 (a) (-2,-30) and (4,24)

 (b) (-1,-11) and (2,-2)

Tangent Line

The tangent line to the curve that is the graph of a function $f(x)$ at a point (x_1, y_1) is a line that *touches* the curve at (x_1, y_1) but does not intersect it. It may intersect at some other point but not at the point it is tangent to.

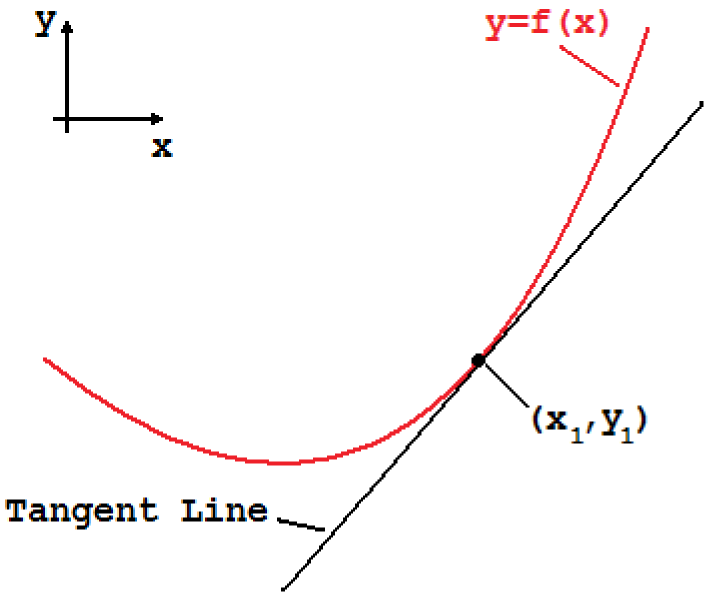

Figure: Schematic of the tangent line to a curve at a given point.

Why bother with this line?

As we'll see now the slope of this line can be used to locate turning points.

Turning Points

A turning point or extremum is the point on a curve where the curve reverses direction. When a curve reverses direction the slope changes sign and so we have a means of finding the associated turning point. If the slope changes sign it must cross zero and so the point where the slope is zero is a turning point.

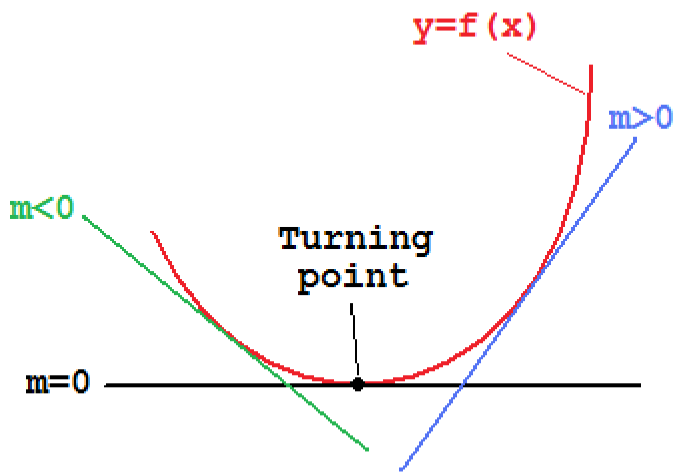

Figure: How the tangent line can locate extrema

We classify turning point of $f: U \subseteq \mathbb{R} \rightarrow V \subseteq \mathbb{R}$ as follows:

1. **Global Maximum**: A point (x_1, y_1) on the graph of $f(x)$ is a global maximum if and only if $y_1 \geq f(x) \forall x \in U \subseteq \mathbb{R}$. More formally y_1 is the *supremum* of the set of all values $f(x)$.
2. **Global Minimum**: A point (x_1, y_1) is a global minimum if and only if $y_1 \leq f(x) \forall x \in U \subseteq \mathbb{R}$. More formally y_1 is the *infemum* of the set of all values $f(x)$.
3. **Local Maximum**: A point (x_1, y_1) is a local maximum if and only if $y_1 \geq f(x) \forall x \in [\alpha, \beta] \subset U$.
4. **Local Minimum**: A point (x_1, y_1) is a local maximum if and only if $y_1 \leq f(x) \forall x \in [\alpha, \beta] \subset U$.

Exercise

For the functions defined previously in (a), (b), and (c):

(i) find all the turning points from their graphs

(ii) determine if they are local or global maxima or minima.