Section 1 (Cont'd)
Functions

We previously defined a function to be a relationship establishing a correspondence between two sets of numbers. We can think of a function in terms of a recipe; the ingredients being the variables and constants, the method being the use of mathematical operators to combine the variables and constants to form the final result.

What is a mathematical operator?
A mathematical operator takes variables, constants, functions, etc. as its inputs and outputs a predetermined combination of these inputs if appropriate. The basic operators are +, -, *, /, ^. We will illustrate with examples below:

Example 1
Consider the addition operation. Its operator is +. Thus 2+2 is a mathematical operation using +. The resulting equation would then be

$$2+2=4.$$

Example 2
Consider a succession of operations (composite operation). Take the operators +, *, /. A composite operation using these could be 5*6+2/8 which yields the equation

$$5*6+2/8=30\tfrac{1}{4}.$$

not

$$5*6+2/8=5$$

The second equation is incorrect because 6 is not added to 2 first before the division by 8 or multiplication by 5. There is a hierarchy of precedence among mathematical operations and the ordering is as follows:

Order of Precedence	Operator
Highest	() Parentheses
|	^ Power Indices
|	* or / Multiplication or Division
| ∨	+ or - Addition or Subtraction
Lowest	∧, ∨, ¬ Logical AND, OR, NOT

Examples of Precedence

Determine whether each of the following expressions is true or false if $x=1, y=2,$ and $z=3$:

(i) $3x^2-2y+5\geq z+1$

(ii) $3-y/4+5/z<2x+1$

(iii) $2y-1/(4+z)-2x>z-1$

We'll leave it as an exercise for you to work out the particulars but the answers are (i) True, (ii) False, and (iii) False.

A function takes such operations, generalises it with appropriate positioning of variables/constants and, possibly, specifies a valid set of values that can be input (domain) and output (range):

$$f:\underbrace{U\subseteq\mathbb{R}}_{\text{Domain}}\rightarrow\underbrace{V\subseteq R}_{\text{Range}}:x\rightarrow\underbrace{\dots\dots\dots\dots\dots}_{\text{Explicit Defn of f(x)}}$$

or shorthand

$$f(x)=\dots\dots\dots$$

N.B.

The letters *f*, *g*, and *h* are typically used for generic functions.
The letters *x*, *y*, and *z* are typically used for variables.

Using this notation, some examples of functions are

(i) $f(x) = 2x - 7$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

The graph of $2x - 7$ in the interval [-3,11] is shown below:

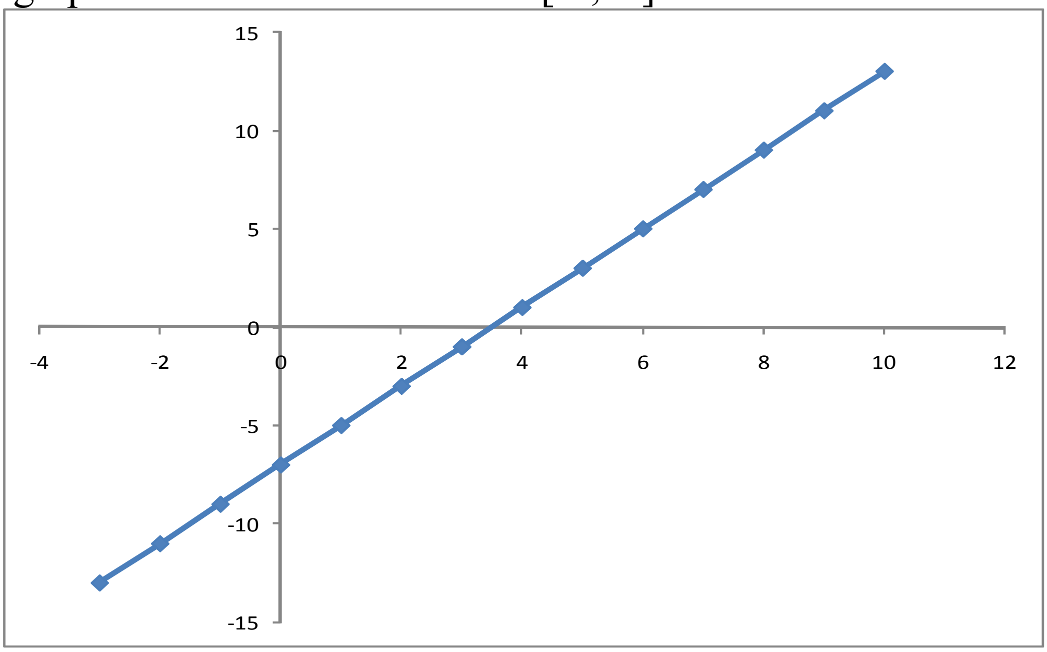


Figure: Graph of $f(x) = 2x-7$ on [-3,11]

This type of function is called a *Linear Function* because its graph is a straight line; i.e. The domain and range are related by a linear correspondence.

We'll consider the concepts behind graphing a function soon, but for now we'll mention two properties of this function:

(a) It crosses the horizontal axis at (3½, 0).

(b) It has a constant slope of 2.

How we arrived at these will become apparent later.

The generic form of such a linear function is

$$f(x)=ax+b \quad \forall a,b\in\mathbb{R}; a\neq 0$$

(ii) $f(x)=2x^2-4x+1$

Domain: $\mathbb{R}$

Range: $[-1,\infty)$

The graph of $2x^2-4x+1$ in the interval [-2,4½] is shown below:

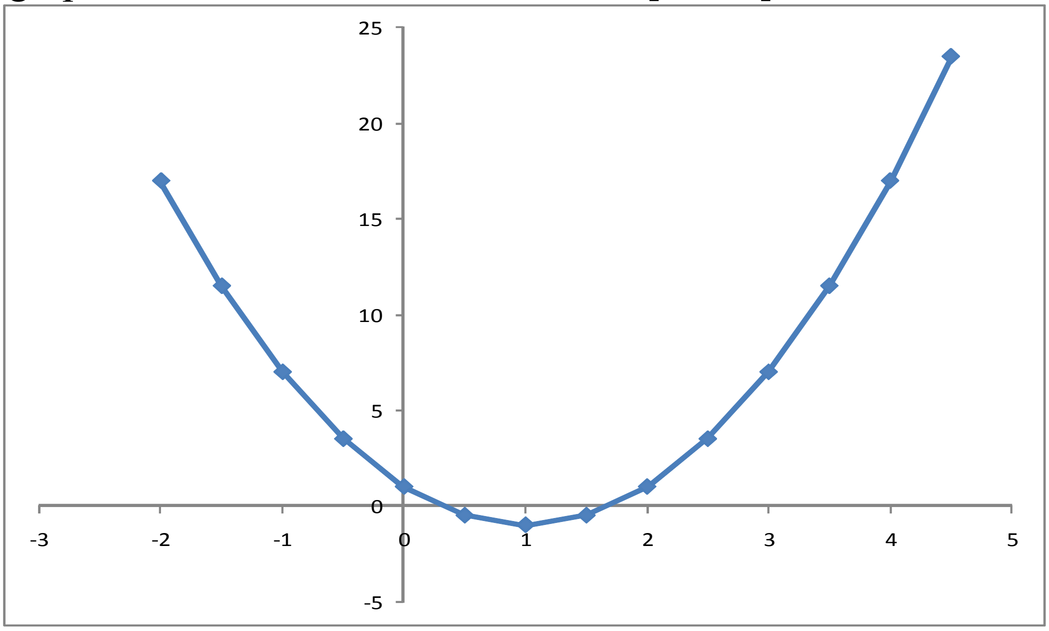


Figure: The graph of $f(x)=2x^2-4x+1$ on [-2, 4½]

The graph of this function is a curve that intersects the horizontal axis in two places determined by setting $f(x)=0$.

Why? More on this later.

One other important fact about this function is that its graph has a turning point. Estimate where this is from the graph above.

More on such points later.

The generic form of such a function, termed *Quadratic Function*, is

$$f(x)=ax^2+bx+c \quad \forall a,b,c\in\mathbb{R}; a\neq 0$$

(iii) $f(x)=\frac{x}{x^2+1}$

Domain: ℝ

Range: $[-1/2, 1/2]$

The graph of $x/(x^2+1)$ in the interval [-5,5] is shown below:

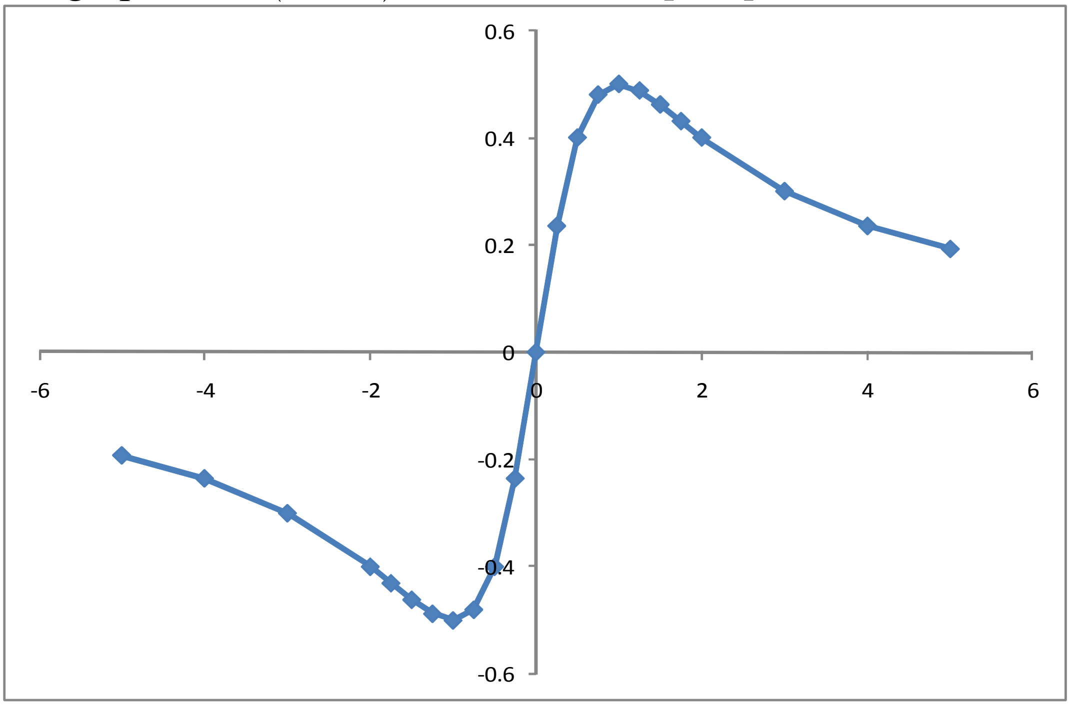


Figure: The $f(x)=x/(x^2+1)$ graph of on [-5, 5]

The graph of this function is a non-trivial curve that intersects the horizontal axis at the origin (0,0) determined by setting $f(x)=0$.
As before, more on this later.
One other important fact about this function is that its graph has two turning points both of which are extrema: one is the global minimum, the other the global maximum. Estimate where this is from the graph above.
More on extrema later.

Composite functions

For relatively uncomplicated functions, such as those above, the analysis of their behaviour is fairly straightforward:

e.g. analyse the behaviour of $f(x)=ax+b$ in order to answer questions about specific functions such as $f(x)=2x-7$, etc.

However more complicated functions (composite functions) may be considered to be made up or composed of less complex functions. To illustrate this consider the function:

$$f(x)=\sqrt{x^2+1}$$

As it stands it is too complex to determine its domain and range without exhaustive calculation. However we could decompose the function into two "constituent" functions:

$$\text{Let } h(x)=x^2+1 \text{ and } g(x)=\sqrt{x}$$
$$\Rightarrow g\circ h(x)=f(x)=\sqrt{x^2+1}$$

The latter equation declares that the function $g(x)$ is implemented *after* $h(x)$ and the inputs to $g(x)$ are the outputs of $h(x)$. The symbol $\circ$ indicates *after.*

Then

(i) $g\circ h(x)$ $\Rightarrow$ the operations of $g(x)$ are performed *after* those of $h(x)$.

(ii) the domain of $h(x)$ is the same as $f(x)$: i.e U.

(iii) the range of $h(x)$, W, becomes the domain of $g(x)$.

(iv) the range of $g(x)$ is the range of $f(x)$: i.e V.

$$U \xrightarrow{\quad f(x) \quad} V$$

$$U \xrightarrow{\quad h(x) \quad} W \xrightarrow{\quad g(x) \quad} V$$

For our example above, we determine the domain and range of $f(x)$ as follows:

Domain of $f(x)$: U

The domain of $h(x)$ is $\mathbb{R}$ because any real number can be squared and have 1 added to it. Therefore as the domain of $f(x)$ is the same as $h(x)$ it follows $U=\mathbb{R}$

Range of $f(x)$: V

The range of $g(x)$ is $[1,\infty)$. This is because its domain, which is the range (W) of $h(x)$, is restricted to all real numbers greater than or equal to one: $W=[1,\infty)$. The square root without the $\pm$ indicates only positive roots and thus the range of $g(x)$ is lower bounded by 1 and has no finite upper bound: i.e. $[1,\infty)$. As the range of $f(x)$ is the same as $h(x)$ it follows $V=[1,\infty)$.

Exercise:

Determine the domain and range of each of the following functions:

(i) $f(x)=1/(x^2+1)$

(ii) $f(x)=3x/\sqrt{x^2+1}$

Occasionally you may be required to operate on the range of a given function with a second function. If the former is denoted by $h(x)$ and the latter by $g(x)$ then the composite operation is

$$f(x)=g\circ h(x)=g(h(x))$$

Note the notation $g(h(x))$. This states explicitly that the argument (or input) to $g(x)$ is $h(x)$ (or the output from $h(x)$): i.e. You evaluate $h(x)$ fro a given value of x and then pass this result as the argument to $g(x)$.

For example consider the functions

$$h(x)=x^2-2x-3$$
$$g(x)=2x+1$$

then

$$f(x)=g\circ h(x)=g(h(x))=2h(x)+1=2(x^2-2x-3)+1$$
$$\Rightarrow f(x)=2x^2-4x-5$$

Exercise:

Express $f(x)=g\circ h(x)$ explicitly as a function of x for each of the following:

(iii) $g(x)=x^3+2x^2-3 \quad h(x)=3x-5$

(iv) $g(x)=3x-x^2 \quad h(x)=4\mathrm{x}^2-2x+5$

One to One (1-1)

Consider some interval of the domain of a given function, *f*. Let *x*, the variable associated with *f*, be in this interval. Then *f* is said to be 1-1 (one to one) in this interval if for all *x* in this interval there exists aa associated unique *f*(*x*), called the image of *x* under *f*. Conversely, it can be said that for every *f*(*x*) there is an associated unique *x* if *x* is in the interval.

Exercise

(i) All linear functions are 1-1. Why?

(ii) What about the other graphed functions given above?
If the function is not 1-1 on its entire domain then find closed intervals for which the functions are 1-1.

N.B

A function that is not 1-1 is many to one. Can you think of any such functions?

Why is 1-1 important?

We stated in our defintion of 1-1 that a converse definition existed for 1-1 functions; i.e. For every value of the function there is a corresponding unique value of the variable.

Then if $f(x)$ is 1-1 then $\exists g(x)$ such that $g \circ f(x) = x$; i.e. We can construct for every 1-1 function *f*(x) a corresponding function *g*(x) that returns the x we originally input to *f*(x). We'll illustrate this by example:

Example

Let

$$f(x) = x^2 \quad \forall x \in \mathbb{R}^+$$

then

$$g \circ f(x) = x \Leftrightarrow g(x) = \sqrt{x}.$$

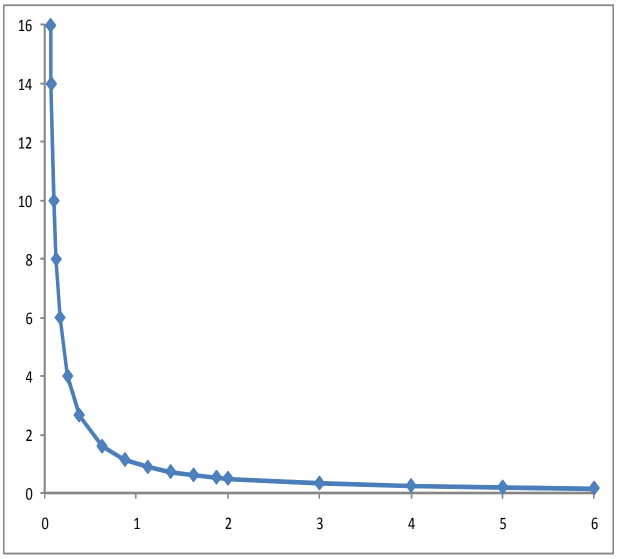
Thus

$$g \circ f(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = x \Leftrightarrow x \in \mathbb{R}^+$$

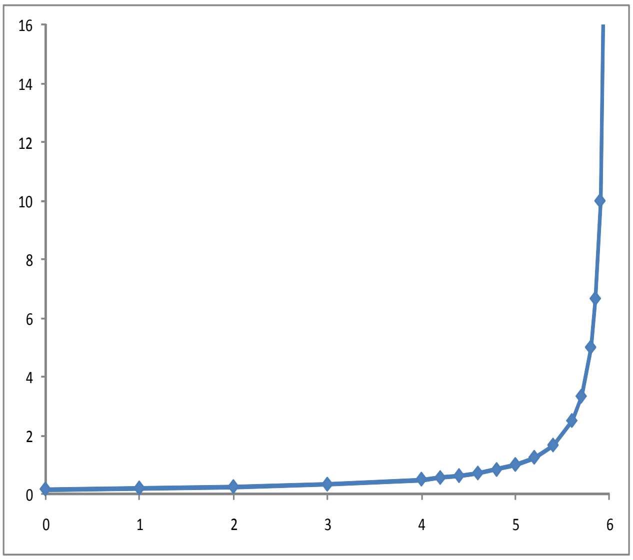
In such cases as that of the example above, the function *g*(x) is given a special name; one that links it explicitly to *f*(x). We call *g*(x) the inverse function of *f*(x) and denote it $f^{-1}(x)$.

N.B.

A function that is 1-1 is also said to be monotonic. A monotonic function can be monotonically increasing or monotonically decreasing. Such a function has no turning points and has a non-zero finite slope. What constitutes a turning point and how slope is measued will be examined in the next section entitled *The graph of a function*.



Monotonically decreasing



Monotonically increasing