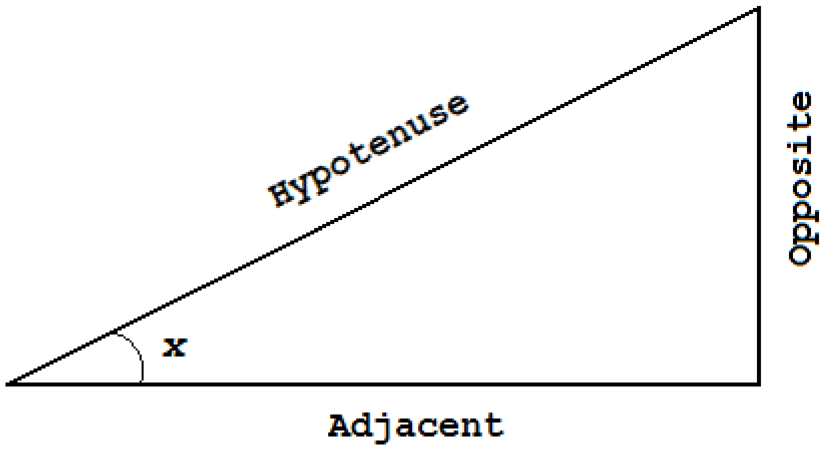
Section 1 (Cont'd)
Trigonometric Functions

The area of mathematics dealing with triangles is called trigonometry from *tri* (three), *gon* (angle) and *metric* (measure). In dealing with such geometric objects three mathematical functions were defined, each dealing with a unique relationship between a chosen angle and two sides of the triangle. These three functions have many important properties and we'll consider some of them in this section.

What are these three mathematical functions?
The three functions are called the *sine* function, sin(x) the *cosine* function, cos(x), and the *tangent* function, tan(x). Their definitions from the right angled triangle are



$$\sin(x) \underset{\mathrm{DEF}^{\mathrm{N}}}{\underset{\uparrow}{:=}} \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(x) \underset{\mathrm{DEF}^{\mathrm{N}}}{\underset{\uparrow}{:=}} \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(x) \underset{\mathrm{DEF}^{\mathrm{N}}}{\underset{\uparrow}{:=}} \frac{\text{opposite}}{\text{adjacent}}$$

For a triangle, the angle x is permitted to vary from 0 to 180 degrees. From a mathematical perspective the use of degrees as an angular measure is both unwieldy and irreconcilable. We need to introduce a proper measure for angles that is complementary to our defintions of mathematical functions. This natural metric for angles is called the *radian* measure and is defined thus.

The Radian

The radian measure is a natural metric for angles in that it uses properties assocaited with the circle in its defintion. You are acquainted with the relationship between the length of a circle's circumference (length of its boundary) and its radius (distance from the circle's centre to the boundary): i.e if the radius is *r*, then the circumference has length

$$\text{Length of circumference} = 2\pi r$$

Then it follows that there are 2π arcs of length equal to the radius, *r*, in a full revolution of a circle; i.e. 360 degrees.

If we denote by θ the angle subtended by the arc of length r then it follows that there are 2π of these θ in 360 degrees (or a full revolution).

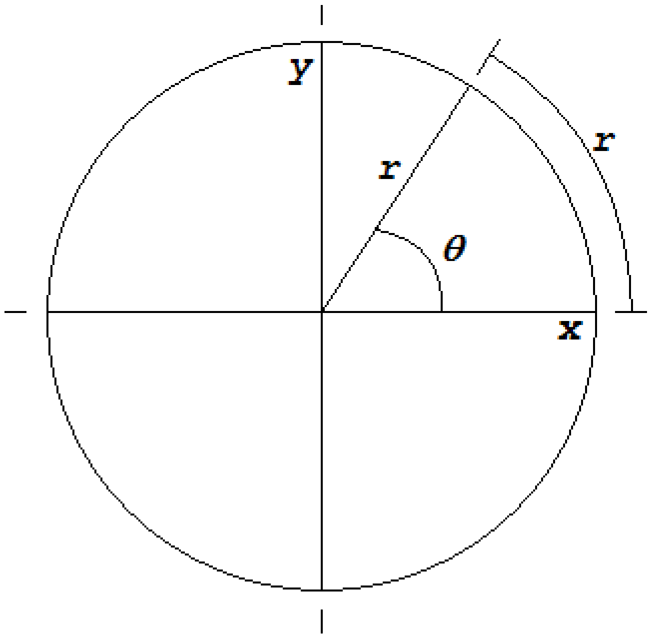


Figure: Radian Measure in a Circle.

Then the following table relates the radian to degrees fro commonly used angles:

Degrees	Radians		Degrees	Radians
0	0		60	$\pi/3$
30	$\pi/6$		90	$\pi/2$
45	$\pi/4$		180	π

Table: Commonly used Angles in Degrees and Radians

Why bother with such a measure?
When we deal with trigonometric functions in mathematical analysis it is treated a any other function. Therefore operations such as addition, multiplication, indices, and the calculus operations to follow will be applied to these functions just as they would be to the functions encountered so far. As the variables passed to such functions must be not depend on an arbitrary metric, it follows that only metrics that occur naturally from the properties of the function under analysis should be used. This, for circles and trigonometric functions, is the radian measure.

Formal function defintions of $\sin(x)$ and $\cos(x)$:
The sine function is thus defined in terms of fundamental operations on *x*,

$$\sin:\mathbb{R}\to[-1,1]:x\to\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

The cosine

$$\cos:\mathbb{R}\to[-1,1]:x\to\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

The tangent is defined as the quotient

$$\tan:\mathbb{R}\backslash\{(2n+1)\frac{\pi}{2}\}\to\mathbb{R}:x\to\frac{\sin(x)}{\cos(x)}\quad\forall n\in\mathbb{Z}$$

How do we know these are the correct definitions?
We'll show later using Taylor Series how the functional forms are found. For now we'll graph each function for a set domain and see how they behave.

N.B.

(a) Every even number is divisible by 2 and so we can define an even number as follows:

$$x \text{ is an even number } \Leftrightarrow x=2n\quad\forall n\in\mathbb{Z}$$

(b) Every odd number leaves one over on division by 2 and so we can define an odd number as follows:

$$x \text{ is an odd number } \Leftrightarrow x=2n+1\quad\forall n\in\mathbb{Z}$$

The graphs of $\sin(x)$, $\cos(x)$, and $\tan(x)$
The combined graphs of sin(*x*), cos(*x*), and tan(*x*) are shown overleaf. It is evident from the forms the graphs take that there is something interesting happening. Can you see what it is?

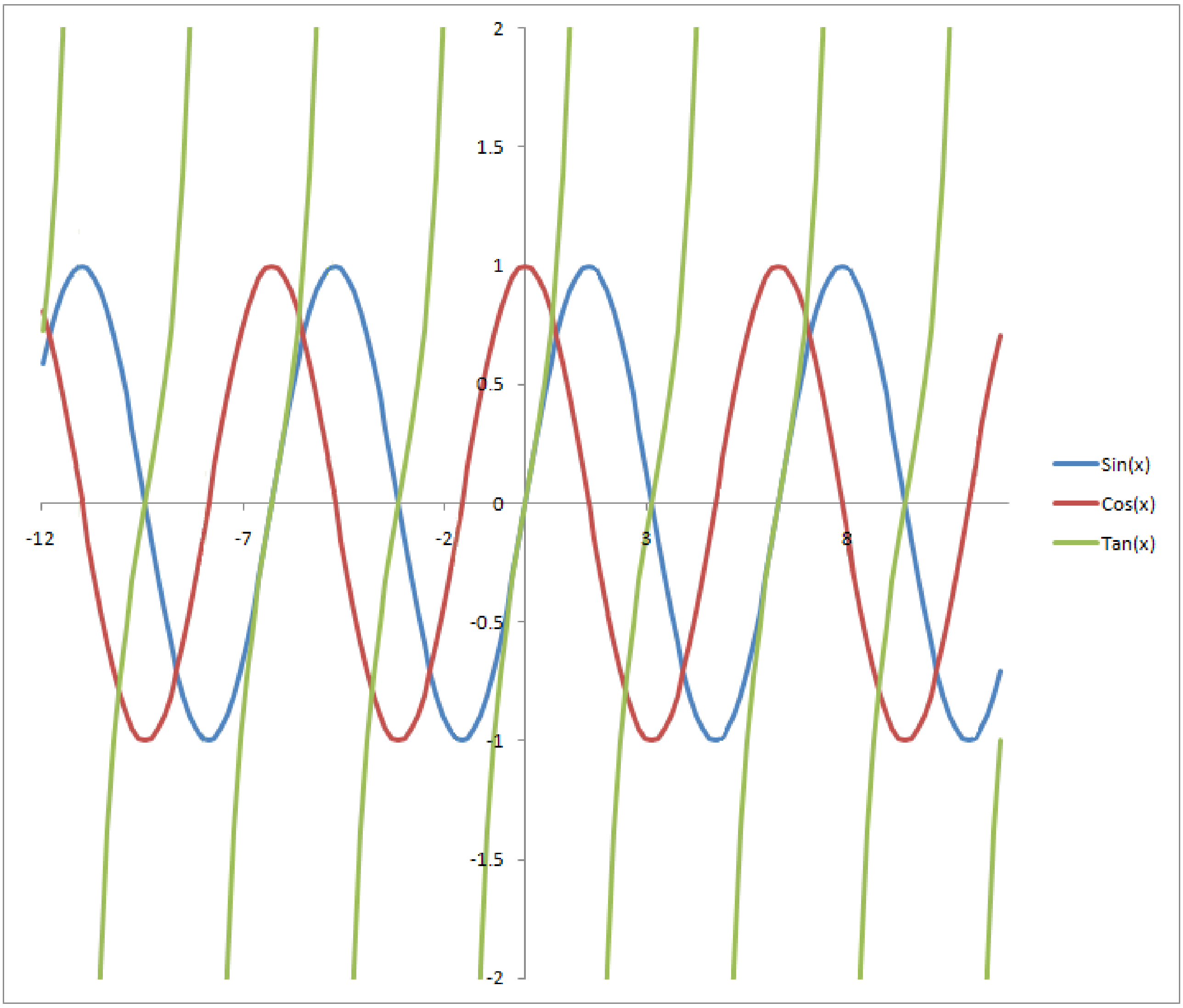


Figure: Combined graphs of sin(x), cos(x), and tan(x) on [-12,12]

The blue graph shows sin(x), the red graph cos(x) and the green tan(x). Some notable properties should be evident:

1. Both sin(x) and cos(x) have the same domain, namely $\mathbb{R}$ and the same range [-1,1].
2. tan(x) has singularities at every odd multiple of $\pi/2$; hence the omission of the set of odd multiples of $\pi/2$ from $\mathbb{R}$ in its domain.
3. All three functions are *periodic*. Sin(x) and cos(x) are both periodic in 2π whereas tan(x) is periodic in π.
4. The graphs of sin(x) and cos(x) are identical except for a phase shift of $\pi/2$; i.e.

$$\cos(x+\pi/2)=\sin(x)$$

Periodic Functions

A function $f: U \rightarrow V$ is *periodic* if and only if there exists a compact interval $[a, b] \subset U$ and $X = b - a$ such that

$$f(x+X) = f(x) \quad \forall x \in [a, b]$$

i.e.

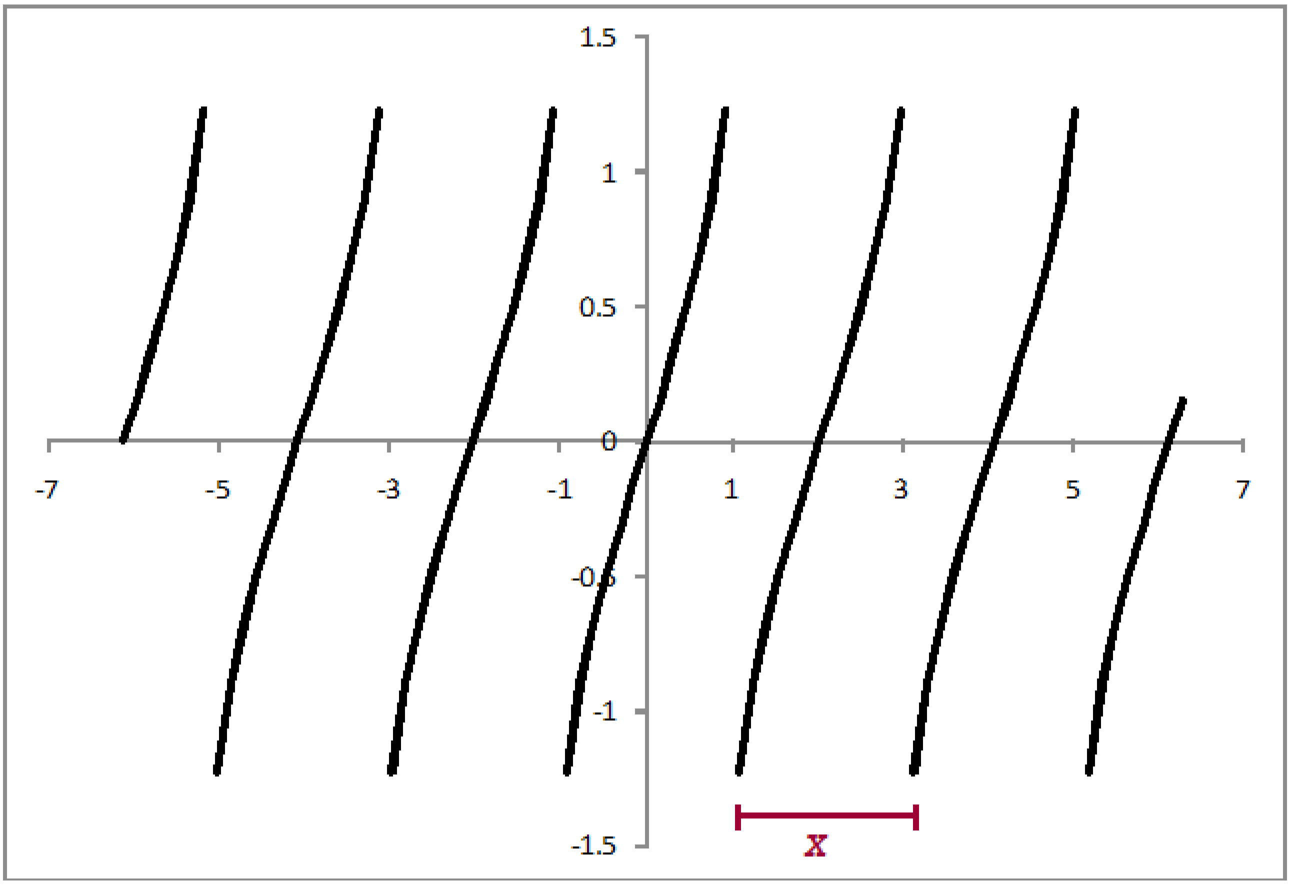


Figure: Periodic function of period *X*.

Exercise:

1. Using the methods introduced for graphing functions, show that the periodicity of sin(x) and cos(x) is 2π.
2. Do the same for tan(x) and show it has a periodicity of π.

Periodic functions do not have inverse functions as such. Due to the repetitive nature of a periodic function, any function that purports to be its inverse has no prior knowledge of which interval the periodic function originated in.

To illustrate this consider the following example with $f(x) = \sin(x)$.

If $x = \frac{7}{3}\pi$ then

$$f(x) = \sin(x) = \frac{\sqrt{3}}{2}$$

If the inverse function $f^{-1}(x)=\sin^{-1}(x)$ existed for all x in the domain of sin(x) then we should get

$$f^{-1}(f(x))=\sin^{-1}(\sin(x))=x \quad \forall x\in\mathbb{R}$$

However,

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\neq\frac{7}{3}\pi$$

Why?

Simply put

$$\sin\left(\frac{\pi}{3}\right)=\sin\left(\frac{7\pi}{3}\right)=\ldots=\sin\left(\frac{\pi}{3}\pm2n\pi\right)=\frac{\sqrt{3}}{2} \quad \forall n\in\mathbb{Z}$$

and so we (without prior knowledge) don't know which x value to choose.

How do we proceed?

We can define an inverse function for a periodic function if we limit the domain of the original function to an interval that is no larger than a single periodic interval. Then, if the interval is chosen properly, the function will not be periodic and the inverse will uniquely return the correct x passed into the original function.

So we have the following property for periodic functions:

If $f(x)$ is periodic with period X then an inverse function $f^{-1}(x)$ may exist with

$$f^{-1}(f(x))=x \quad \forall x\in[\alpha,\beta] \;\; \beta-\alpha\leq X$$

Inverse Trigonometric functions

For the sine function we define the inverse sine function

$$\sin^{-1}:[-1,1]\rightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \text{ such that } \sin^{-1}(\sin(x))=x \quad \forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

For the cosine function we define the inverse cosine function

$$\cos^{-1}:[-1,1]\rightarrow[0,\pi] \text{ such that } \cos^{-1}(\cos(x))=x \quad \forall x\in[0,\pi]$$

For the tangent function we define the inverse tangent function

$$\tan^{-1}:\mathbb{R}\rightarrow[-\frac{\pi}{2},\frac{\pi}{2}] \text{ such that } \tan^{-1}(\tan(x))=x \quad \forall x\in[-\frac{\pi}{2},\frac{\pi}{2}]$$

We can see from the ranges of the above functions that the effective domains for x in order for it to be invertible are halved.

Exercise:
Why is this the case?

The graphs of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are shown below. The blue graph is of the inverse sine function, $\sin^{-1}(x)$, while the red is that of the inverse cosine function, $\cos^{-1}(x)$.

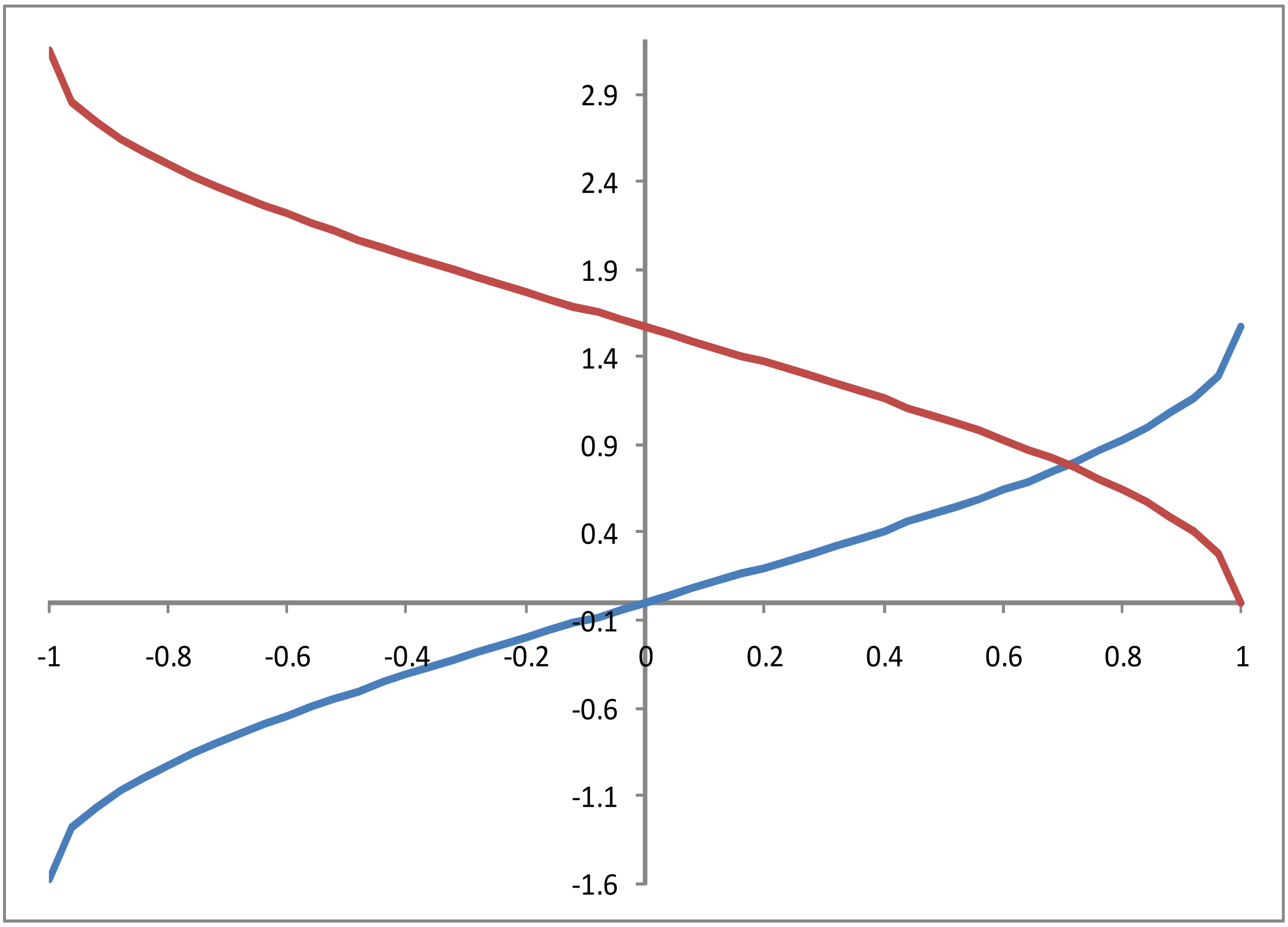


Figure: Graphs of $\sin^{-1}(x)$ and $\cos^{-1}(x)$

The inverse tangent function, $\tan^{-1}(x)$, is shown overleaf. Unlike the sine and cosine functions, the tangent function has the entire set $\mathbb{R}$ for its range and hence the domain of $\tan^{-1}(x)$ is $\mathbb{R}$. The range is restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ brought about by the periodic singularities at odd multiples of $\pi/2$.

There is one aspect of trigonometric functions we have yet to explore; the relationship between the circle of radius r centered at (0,0) in the xy plane. This is one of the most important applied relationships for trigonometric functions.

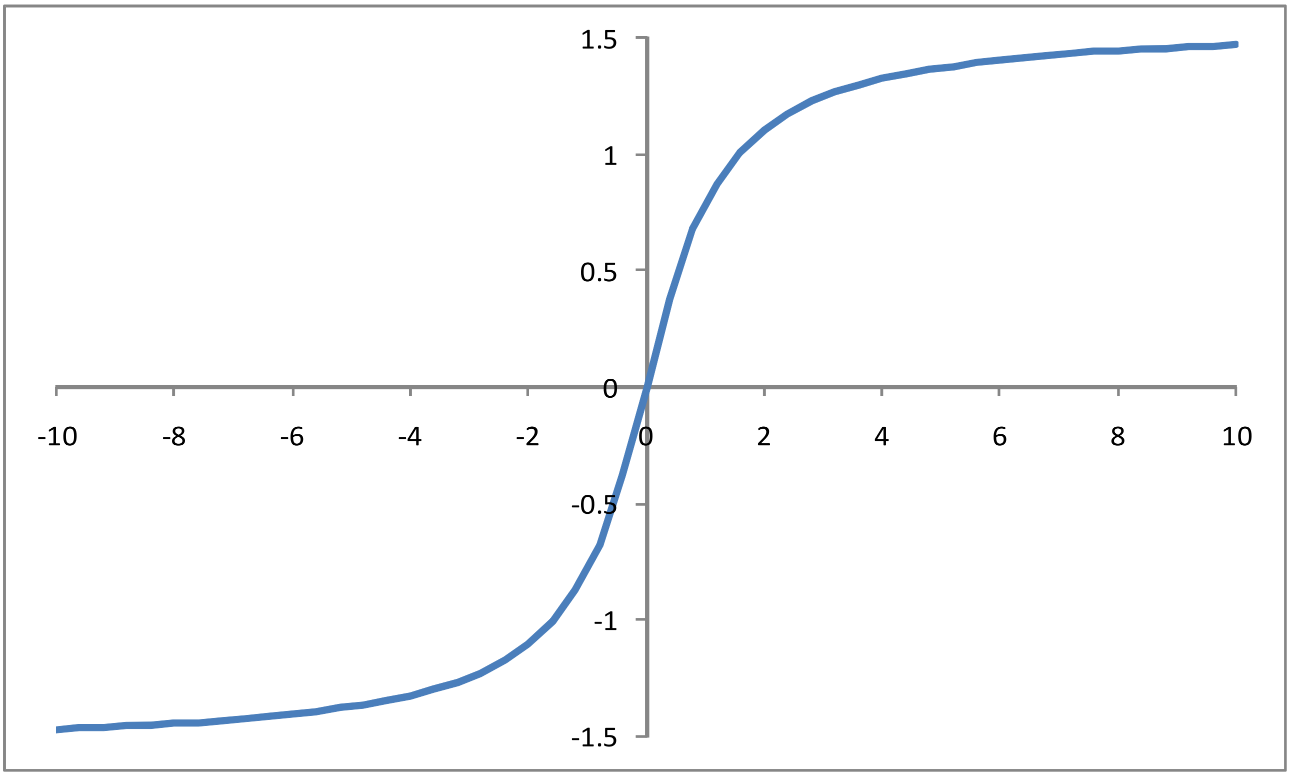


Figure: The inverse tangent function.

The circle and the trigonometric functions

Consider the construction below:

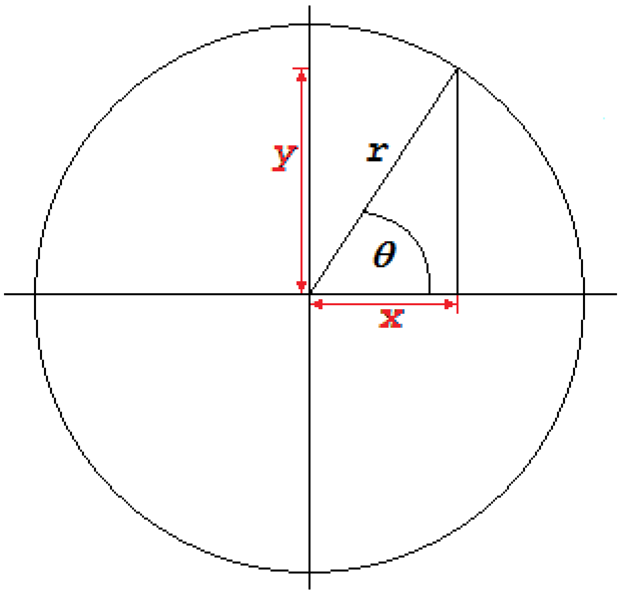


Figure: sin(x), cos(x), and tan(x) in the circle of radius r.

Using the original definitions of sin(x), cos(x), and tan(x) we have the following identities

$$\sin(\theta)=\frac{y}{r}, \ \cos(\theta)=\frac{x}{r}, \text{ and } \tan(\theta)=\frac{y}{x}$$

Then

$$y = r\sin(\theta)$$
$$x = r\cos(\theta)$$
$$m = \frac{\Delta y}{\Delta x} \equiv \frac{y}{x} = \tan(\theta)$$

The first two form the basis of a circular (or polar) coordinate system and is used extensively in complex analysis and in the parameterisation of circular paths often used in computer graphics. The third relates the slope of the radius (line segment from the origin (0,0) to a point on the circumference) to the tangent of the subtended angle, θ.

Some final formulae and remarks

The formula of a circle centered at (0,0) of radius *r* is

$$x^2 + y^2 = r^2$$

Furthermore, from the definitions of the sin(x) and cos(x) we have

$$\cos^2(x) + \sin^2(x) = 1$$

probably the most useful of trigonometric properties.

On Page 9 of the Logarithm Tables we have some other properties such as

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(-A) = -\sin(A)$$
$$\cos(-A) = \cos(A)$$
$$1 + \tan^2(A) = \sec^2(A) \quad \text{where} \quad \sec(A) = \frac{1}{\cos(A)}$$
$$1 + \cot^2(A) = \csc^2(A) \quad \text{where} \quad \cot(A) = \frac{1}{\tan(A)} \quad \text{and} \quad \csc(A) = \frac{1}{\sin(A)}$$

N.B.

1. sec = secant.
2. csc = cosecant (complementary of secant)
3. cot = cotangent (complementary of tangent)