<u>Section 2 (Cont'd)</u>
<u>Mathematical Properties of the Derivative</u>

In the last lecture, we calculated the derivative of a few simple functions. In these notes we'll examine the mathematical prioperties of this differential operator *d*/*dx* and show how more complicated functions can be operated upon via techniques developed in this section.

Mathematical Properties

Before we consider more complex examples, let us consider some basic mathematical properties of the differential operator.

1. Addition of functions

   Let $f(x)$ and $g(x)$ be two functions that are smooth at some x_0 common to the domains of both functions. Then

$$\begin{aligned}
\frac{d}{dx}\left(f(x)+g(x)\right) &= \lim_{\Delta x\to 0}\frac{\left(f(x+\Delta x)+g(x+\Delta x)\right)-\left(f(x)+g(x)\right)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{\left(f(x+\Delta x)-f(x)\right)+\left(g(x+\Delta x)-g(x)\right)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right) \\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} \\
&= \frac{d}{dx}f(x)+\frac{d}{dx}g(x)
\end{aligned}$$

   The derivative of a sum of functions is the sum of the derivatives. This can be generalised for any number of functions being summed; e.g. 3, 4, 5, ...

2. Scalar multiplication of functions

   Let $f(x)$ be a smooth functions at some x_0 in its domain. Let α be any real number. Then

$$\begin{aligned}
\frac{d}{dx}\left(\alpha f(x)\right) &= \lim_{\Delta x\to 0}\frac{\alpha f(x+\Delta x)-\alpha f(x)}{\Delta x} \\
&= \alpha\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}
\end{aligned}$$

Therefore

$$\frac{d}{dx}(\alpha f(x))=\alpha\frac{d}{dx}f(x)$$

From 1. and 2. above it follows that

$$\frac{d}{dx}(\alpha f(x)+\beta g(x))=\alpha\frac{d}{dx}f(x)+\beta\frac{d}{dx}g(x)\quad\forall\alpha,\beta\in\mathbb{R}$$

3. Products of functions (Product Rule)

Let $f(x)$ and $g(x)$ be two functions that are smooth at some x_0 common to the domains of both functions. Then the following holds

$$f(x+\Delta x)\equiv f(x)+\Delta f(x)$$
$$g(x+\Delta x)\equiv g(x)+\Delta g(x)$$

in the neighbourhood of x_0.

Furthermore for smooth functions that are finite in the neighbourhood of x_0 we have the following

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}\Delta g=0$$

Now to the derivative

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)g(x+\Delta x))-(f(x)g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(f(x)+\Delta f(x))(g(x)+\Delta g(x))-(f(x)g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{g(x)\Delta f(x)+f(x)\Delta g(x)+\Delta f(x)\Delta g(x)}{\Delta x}\right)\\
&= \lim_{\Delta x\to 0}g(x)\frac{\Delta f(x)}{\Delta x}+\lim_{\Delta x\to 0}f(x)\frac{\Delta g(x)}{\Delta x}\\
&\qquad +\lim_{\Delta x\to 0}\Delta f(x)\frac{\Delta g(x)}{\Delta x}\\
&= g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x)
\end{aligned}$$

Without the *x* dependence this yields

$$\frac{d}{dx}(f\,g)=g\frac{d\,f}{dx}+f\frac{d\,g}{dx}$$

This is the product formula for the derivative of two differentiable functions of a real variable, x.

4. Quotient of two functions (Quotient Rule)

Let $f(x)$ and $g(x)$ be two functions that are smooth at some x_0 common to the domains of both functions. Then, using the properties outlined in 3. above, the derivative is determined as follows:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \lim_{\Delta x\to 0}\frac{\left(\frac{f(x+\Delta x)}{g(x+\Delta x)}\right)-\left(\frac{f(x)}{g(x)}\right)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\frac{f(x)+\Delta f(x)}{g(x)+\Delta g(x)}-\frac{f(x)}{g(x)}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{g(x)(f(x)+\Delta f(x))-f(x)(g(x)+\Delta g(x))}{g(x)(g(x)+\Delta g(x))\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\left(\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)(g(x)+\Delta g(x))\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\left(\frac{g(x)\frac{\Delta f(x)}{\Delta x}-f(x)\frac{\Delta g(x)}{\Delta x}}{g(x)(g(x)+\Delta g(x))}\right)\\
&= \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}
\end{aligned}$$

Again, as with the formula for products, removing the x dependence we arrive at the more commonly quoted quotient formula

$$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$$

Such properties are important in finding the derivatives of more complicated compound functions. We have one more “rule” to consider but, before we do, it is appropriate now to introduce the most important number in calculus; the universal mathematical constant, e.

The Number *e*

Consider the derivative of the function

$$f(x)=a^x \quad \forall a \in \mathbb{R}\setminus\{0\}$$

Then, using our method for evaluating the derivative, we find

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{\Delta x \to 0} \frac{a^{x+\Delta x}-a^x}{\Delta x} \\ &= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x}-1}{\Delta x}\end{aligned}$$

We're still no closer to an answer we can make sense out of. What does this limit

$$\lim_{\Delta x \to 0} \frac{a^{\Delta x}-1}{\Delta x}$$

evaluate to? Is it a mathematical function of some kind? What do we do? To answer these and other questions (yet to be posed) we consider two examples[1] of $f(x)=a^x$; $f(x)=2^x$ and $f(x)=3^x$.

Δx	$\frac{2^{\Delta x}-1}{\Delta x}$		Δx	$\frac{3^{\Delta x}-1}{\Delta x}$
1	1		1	2
0.1	0.71774		0.1	1.16123
0.01	0.69556		0.01	1.10467
0.001	0.69339		0.001	1.09922
0.0001	0.69317		0.0001	1.09867
0.00001	0.69315		0.00001	1.09862

So,

$$\begin{aligned}\frac{d}{dx}2^x &\simeq 0.6931\times 2^x < 2^x \\ \frac{d}{dx}3^x &\simeq 1.0986\times 3^x > 3^x\end{aligned}$$

1 Why these two values are chosen will become clear. It is a case of knowing the solution before we start so we can make the appropriate choice.

This is begging the question: Is there a number, α, such that

$$\frac{d}{dx}\alpha^x = \alpha^x \quad \forall\, x \in \mathbb{R}$$

and is this α between 2 and 3 as alluded to above?
The answer to both is YES!
The number α that satisfies the above is called the exponent, *e*. *e* lies between 2 and 3 and (like many mathematical constant) has no exact decimal representation. However to a reasonable accuracy we define a decimal approximation to *e*

$$e \simeq 2.718282\ldots\ldots$$

Definition
The exponential number, *e*, is defined thus:

$$\frac{d}{dx}\alpha^x = \alpha^x \quad \forall\, x \in \mathbb{R} \quad \Leftrightarrow \quad \alpha = e$$

i.e.
e is the real number between 2 and 3 that satisfies

$$\frac{d}{dx}e^x = e^x \quad \forall\, x \in \mathbb{R}$$

Now to the $\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$ part. We know that when $a = e$ we have 1. For any other value the value is different and so this amounts to a function:

$$f : (0, \infty) \to \mathbb{R} : x \to f(x) = \lim_{\Delta x \to 0} \frac{x^{\Delta x} - 1}{\Delta x}$$

To see how this function changes for different *x* works let us tabulate this function for $x = e, e^2, e^3,$ and e^4.

x	e	e^2	e^3	e^4
1	1.71828183	6.38905610	19.08553692	53.59815003
0.1	1.05170918	2.21402758	3.49858808	4.91824698
0.01	1.00501671	2.02013400	3.04545340	4.08107742
0.001	1.00050017	2.00200133	3.00450450	4.00801068
0.0001	1.00005000	2.00020001	3.00045005	4.00080011
0.00001	1.00000500	2.00002000	3.00004500	4.00008000
0.000001	1.00000050	2.00000200	3.00000450	4.00000800

From this table we can infer that

$$\begin{aligned}
f(e) &= \lim_{\Delta x \to 0} \frac{e^{\Delta x}-1}{\Delta x} &= 1 \\
f(e^2) &= \lim_{\Delta x \to 0} \frac{e^{2\Delta x}-1}{\Delta x} &= 2 \\
f(e^3) &= \lim_{\Delta x \to 0} \frac{e^{3\Delta x}-1}{\Delta x} &= 3 \\
f(e^4) &= \lim_{\Delta x \to 0} \frac{e^{4\Delta x}-1}{\Delta x} &= 4
\end{aligned}$$

and thus

$$f(e^n) = \lim_{\Delta x \to 0} \frac{e^{n\Delta x}-1}{\Delta x} = n \quad \forall n \in \mathbb{R}$$

So we have a function that, for whatever positive real number, *x*, we pass into it will return the power that *e* of needs to be raised by to get this *x*: i.e.

$$\text{Let } x = e^k \quad \forall k \in \mathbb{R} \Rightarrow f(x) = f(e^k) = k$$

There is a special name for this function; it is the *Natural Logarithm,* ln|*x*|.

Natural Logarithm, ln|*x*|

The natural logarithm is the function that satisfies the following relationship:

$$\ln|e^x| = e^{\ln|x|} = x \quad \forall x \in (0, \infty)$$

Like all logarithms it returns the power its base is raised by to get a number passed to it. For example, the standard base 10 logarithm returns the power 10 is raised by to arrive at the number passed to it: e.g.

$$\log_{10} 100 = 2 \quad \text{because } 10^2 = 100$$

$$\log_{10} 25 = 1.39794... \quad \text{because } 10^{1.39794...} = 25$$

Another representation of $\ln|x|$ is $\log_e|x|$.

N.B.

1. Like all logarithms, $\ln|x|$ cannot have negative arguments passed to it and return a real value. To this end we specify the absolute value of *x* in the argument of $\ln|x|$; i.e. we use a vertical line | each side of *x* instead of round brackets () to highlight the positive value of *x*.
2. *x* cannot be equal to 0 either as this returns a not-defined number result.

To finish this discussion, we'll sum up with the following observations:

- The exponential function is defined $f(x)=e^x$ or $f(x)=\exp(x)$
- The natural logarithm is defined $f(x)=\ln|x|$ or $f(x)=\log_e|x|$
- $\frac{d}{dx}e^x \quad = \quad e^x \quad \forall x \in \mathbb{R}$
- $\ln|e^x| \quad = \quad e^{\ln|x|} \quad = \quad x \quad \forall x \in (0,\infty)$

Graphs of $f(x)=e^x$ and $f(x)=\ln|x|$

Below we show the graphs of $f(x)=e^x$ and $f(x)=\ln|x|$

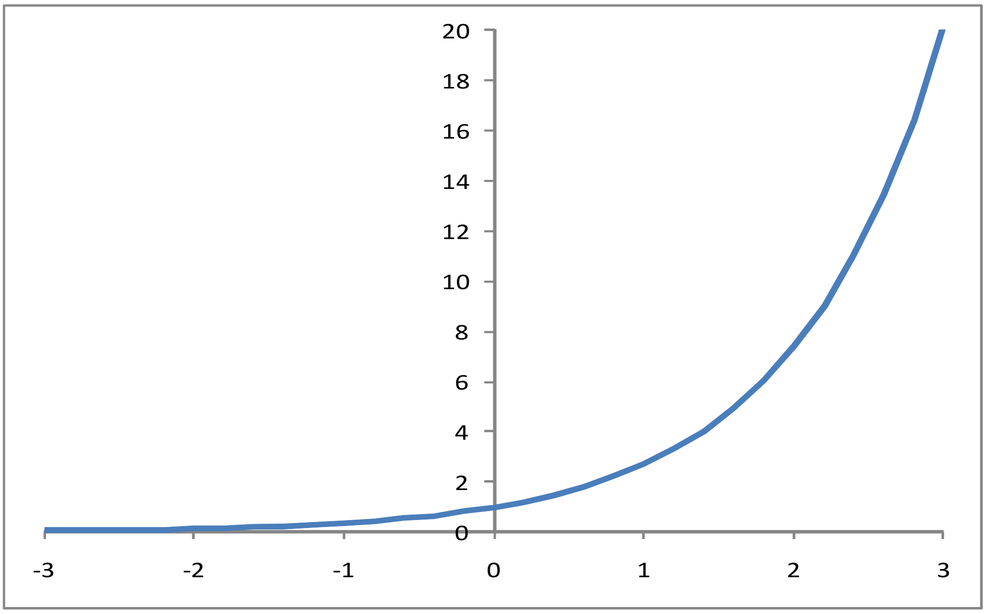

Figure: Graph of $f(x)=e^x$ on $[-3,3]$

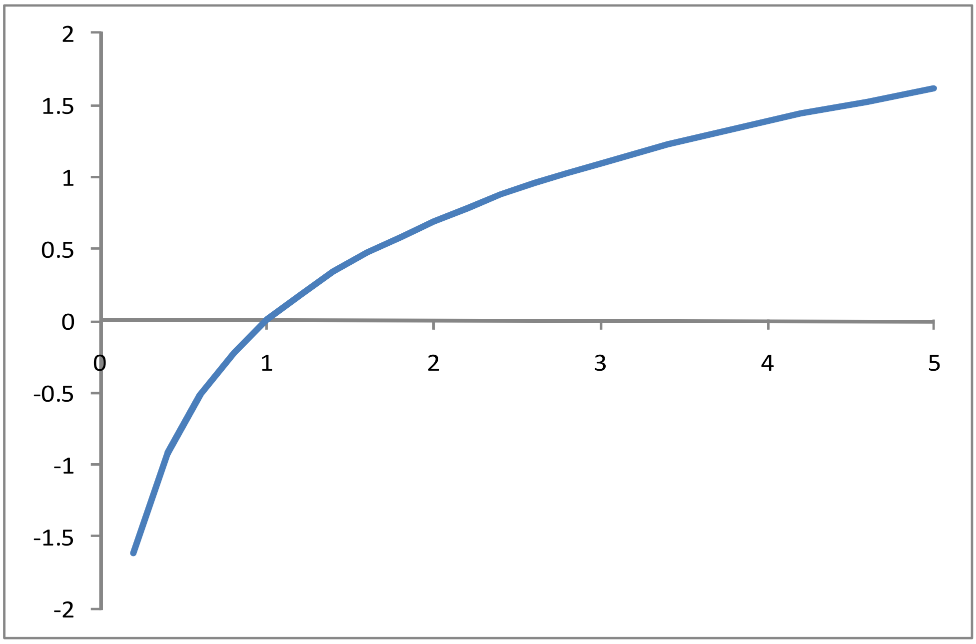

Figure: Graph of $f(x)=\ln|x|$ on $[0,5]$