Section 6 Cont'd
Integrals of Rational Functions using Partial Fractions

In this section we continue our analysis of integrals involving rational functions. We apply the method of partial fractions and show how it can be used to simplify greatly the integration of certain forms of rational functions. We consider issues involved with using such techniques and the limitations of this approach for integrating rational functions in general.

The Integral involving type 1 partial fractions

The result of the decomposition of some rational function

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_mx^m}$$

is the partial fraction representation

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_mx^m}=\frac{A_1}{(x-\alpha_1)}+\ldots+\frac{A_m}{(x-\alpha_m)}$$

assuming the denominator's m^{th} order polynomial can be expressed as a product of m unique factors. Applying the representation above yields

$$\int h(x)dx=\int\frac{f(x)}{g(x)}dx=\int\left(\frac{A_1}{(x-\alpha_1)}+\ldots+\frac{A_m}{(x-\alpha_m)}\right)dx$$

Now, from integration by substitution, we have the following result

$$\int\frac{\alpha}{(x-\beta)}dx=\alpha\ln|x-\beta|+C$$

Then

$$\int h(x)dx=\int\frac{f(x)}{g(x)}dx=\sum_{i=1}^{m}\int\frac{A_i}{(x-\alpha_i)}dx=\sum_{i=1}^{m}A_i\ln|x-\alpha_i|+C$$

From our knowledge of logarithms we know

$$\alpha\ln|x|=\ln|x^\alpha|$$

and

$$\ln|x|+\ln|y|=\ln|xy|$$

Therefore

$$\int h(x)dx=\sum_{i=1}^{m}A_i\ln|x-\alpha_i|+C=\ln\left|(x-\alpha_1)^{A_1}\times\ldots\times(x-\alpha_m)^{A_m}\right|+C$$

Example:
Evaluate

$$\int \frac{x-3}{(x-1)(x+2)(x+1)}\,dx$$

Using the partial fractions method we know

$$\int \frac{x-3}{(x-1)(x+2)(x+1)}\,dx = \int \left(\frac{A_1}{x-1} + \frac{A_2}{x+2} + \frac{A_3}{x+1} \right) dx$$

where A_1, A_2, A_3 need to be determined. Then

$$\begin{aligned} \frac{x-3}{(x-1)(x+2)(x+1)} &= \frac{A_1}{x-1} + \frac{A_2}{x+2} + \frac{A_3}{x+1} \\ &= \frac{A_1(x+2)(x+1) + A_2(x-1)(x+1) + A_3(x-1)(x+2)}{(x-1)(x+2)(x+1)} \end{aligned}$$

The denominators are equal so we can equate the numerators

$$\begin{aligned} x-3 &= A_1(x+2)(x+1) + A_2(x-1)(x+1) + A_3(x-1)(x+2) \\ &= (A_1 + A_2 + A_3)x^2 + (3A_1 + A_3)x + (2A_1 - A_2 - 2A_3) \end{aligned}$$

Equating like with like results in the system of linear equations

$$\begin{aligned} x^2: \quad A_1 + A_2 + A_3 &= 0 \\ x^1: \quad 3A_1 + A_3 &= 1 \\ x^0: \quad 2A_1 - A_2 - 2A_3 &= -3 \end{aligned}$$

which has solution

$$A_1 = -\frac{1}{3},\ A_2 = -\frac{5}{3},\ A_3 = 2$$

and so

$$\begin{aligned} \int \frac{x-3}{(x-1)(x+2)(x+1)}\,dx &= \int \left(-\frac{1}{3}\frac{1}{x-1} - \frac{5}{3}\frac{1}{x+2} + 2\frac{1}{x+1} \right) dx \\ &= -\frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{5}{3}\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x+1}\,dx \\ &= -\frac{1}{3}\ln|x-1| - \frac{5}{3}\ln|x+2| + 2\ln|x+1| + C \end{aligned}$$

which, technically, is correct. If we continue to the final form we get

$$\int \frac{x-3}{(x-1)(x+2)(x-1)}\,dx = \ln\left| (x-1)^{-\frac{1}{3}} (x+2)^{-\frac{5}{3}} (x+1)^2 \right| + C$$

The Integral involving type 2 partial fractions

The result of the decomposition of the target rational function

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_mx^m}$$

is the partial fraction representation

$$h(x)=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_mx^m}=\left(\frac{A_1}{(x-\alpha_1)}+\ldots+\frac{A_p}{(x-\alpha_1)^p}\right)+\frac{B}{(x-\alpha_2)}+\ldots+\frac{C}{(x-\alpha_k)}$$

assuming the denominator's m^{th} order polynomial can be expressed as a product of k unique factors with some factors raised to a power greater than 1[1]. This representation applied to the integral yields

$$\int h(x)dx=\int\left(\left(\frac{A_1}{(x-\alpha_1)}+\ldots+\frac{A_p}{(x-\alpha_1)^p}\right)+\frac{B}{(x-\alpha_2)}+\ldots+\frac{C}{(x-\alpha_k)}\right)dx$$

Now, from integration by substitution, we have the following results:

$$\int\frac{\alpha}{(x-\beta)}dx=\alpha\ln|x-\beta|+C$$

$$\int\frac{\alpha}{(x-\beta)^p}dx=\frac{\alpha}{1-p}(x-\beta)^{1-p}+C \quad p>1$$

Then

$$\int h(x)dx=\sum_{k=1}^{p}\int\frac{A_k}{(x-\alpha_1)^k}dx+\int\frac{B}{x-\alpha_2}dx+\ldots+\int\frac{C}{x-\alpha_k}dx$$

yielding

$$\int h(x)dx=\sum_{k=2}^{p}\frac{A_k}{1-k}\frac{1}{(x-\alpha_1)^{k+1}}+A_1\ln|x-\alpha_1|+B\ln|x-\alpha_2|+\ldots+C\ln|x-\alpha_k|+C$$

The latter part is equivalent to the type 1 partial fraction we already dealt with above and can be replaced by

$$\ln\left|(x-\alpha_1)^{A_1}\times(x-\alpha_2)^B\times\ldots\times(x-\alpha_k)^C\right|+C$$

Unfortunately the first part is in its simplest form as shown above. Then

$$\int h(x)dx=\sum_{k=2}^{p}\frac{A_k}{1-k}\frac{1}{(x-\alpha_1)^{k+1}}+\ln\left|(x-\alpha_1)^{A_1}\times(x-\alpha_2)^B\times\ldots(x-\alpha_k)^C\right|+C$$

1 Of course, from the representation shown above, we've restricted the number of such factors (factors with powers greater than 1) to one, for clarity otherwise the equation would become too unwieldy and act to confuse rather than clarify.

Example:
Evaluate

$$\int \frac{x^2-2x+1}{(x-2)^2(x+1)}\,dx$$

Using the partial fractions method we know

$$\int \frac{x^2-2x+1}{(x-2)^2(x+1)}\,dx=\int\left(\frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{B}{x+1}\right)dx$$

where A_1, A_2, B need to be determined. Then

$$\begin{aligned}\frac{x^2-2x+1}{(x-2)^2(x+1)} &= \frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{B}{x+1}\\ &= \frac{A_1(x-2)(x+1)+A_2(x+1)+B(x-2)^2}{(x-2)^2(x+1)}\end{aligned}$$

The denominators are equal so we can equate the numerators

$$\begin{aligned}x^2-2x+1 &= A_1(x-2)(x+1)+A_2(x+1)+B(x-2)^2\\ &= (A_1+B)x^2+(-A_1+A_2-4B)x+(-2A_1+A_2+4B)\end{aligned}$$

Equating like with like results in the system of linear equations

$$\begin{array}{lrcr} x^2: & A_1+B & = & 1\\ x^1: & -A_1+A_2-4B & = & -2\\ x^0: & -2A_1+A_2+4B & = & 1\end{array}$$

which has solution

$$A_1=\frac{5}{9},\ A_2=\frac{1}{3},\ B=\frac{4}{9}$$

and so

$$\begin{aligned}\int \frac{x^2-2x+1}{(x-2)^2(x+1)}\,dx &= \int\left(\frac{5}{9}\frac{1}{x-2}+\frac{1}{3}\frac{1}{(x-2)^2}+\frac{4}{9}\frac{1}{x+1}\right)dx\\ &= \frac{5}{9}\int\frac{1}{x-2}\,dx+\frac{1}{3}\int\frac{1}{(x+2)^2}\,dx+\frac{4}{9}\int\frac{1}{x+1}\,dx\\ &= \frac{5}{9}\ln|x-2|-\frac{1}{3}\frac{1}{x+2}+\frac{4}{9}\ln|x+1|+C\end{aligned}$$

The final form is then

$$\int\frac{x-3}{(x-1)(x+2)(x-1)}\,dx=-\frac{1}{3}\frac{1}{x+2}+\ln\left|(x-2)^{\frac{5}{9}}(x+1)^{\frac{4}{9}}\right|+C$$

N.B.
It should be noted here that the final step in each of the examples above is optional. It is not necessary to proceed with that level of simplification in an examination process as the intricacies of such methods can be subtle to say the least.

This method is very restrictive; not only in the sense of being restricted to rational functions of linear factor terms but also in the sense of the restrictions we have placed on such functions. In general partial fractions can be applied to any rational function once you know how to handle irreducable[2] higher order polynomials in the integral. Such methods are well beyond the scope of this course and so will not even be illustrated.

This ends our brief introduction to the solution of integrals of rational functions of *x*. In the next section we'll return to the origins of the definite integral to motivate the integral equations governing the determination of areas bounded between two curves and volumes of rotation. The latter being of significance as it prepares the way for the more involved multi-variable calculus next year.

2 Irreducable here implies the polynomials in question have no real factors and cannot be put into a factored or reduced form.