Geometric Interpretation
of the 2x2 Determinant

Consider the 2x2 square matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Its determinant is defined (from page 20 of your maths tables)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

So we now ask: What does this mean/signify?
Consider each row of the original matrix

$$\begin{array}{ll} \text{row 1 :} & a \quad b \\ \text{row 2 :} & c \quad d \end{array}$$

and, on graph paper, draw a line from the origin (0,0) to points represented by these rows; (a,b) for row 1 and (c,d) for row 2. These two lines from two sides of the parallelogram shown in figure 1 below.

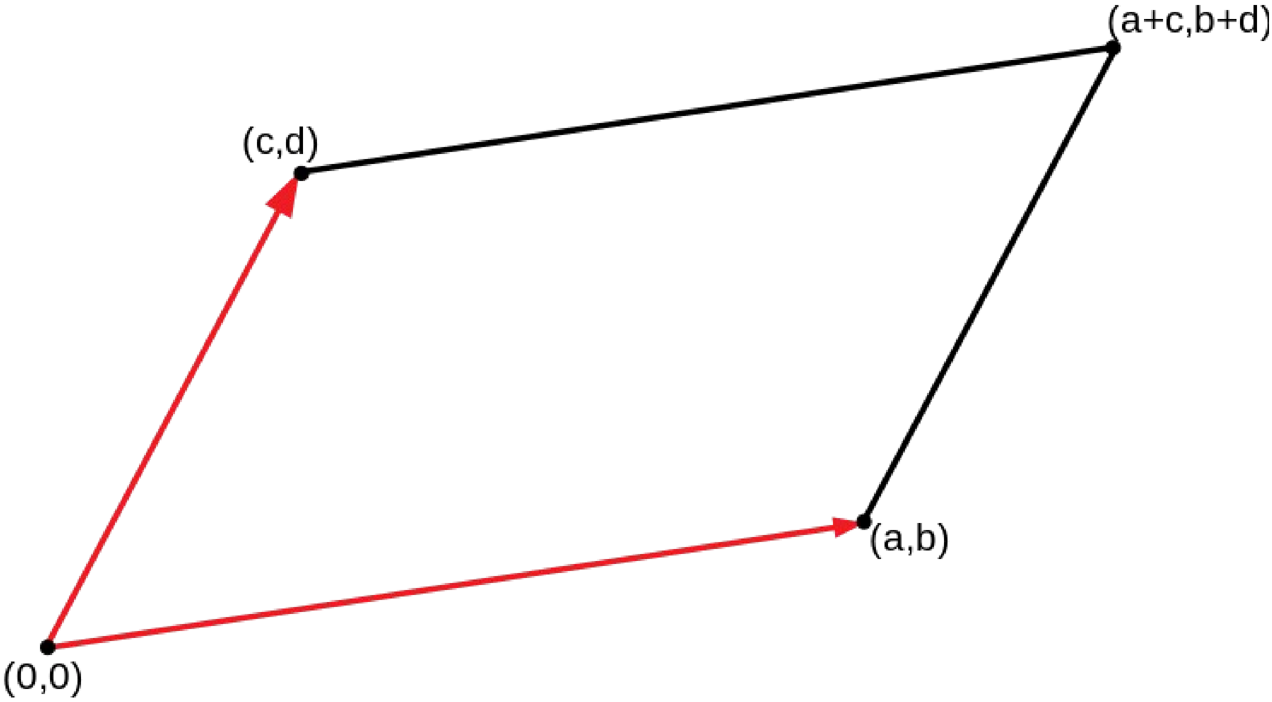

Figure 1: Parallelogram formed from rows of a 2x2 matrix

We want to determine the area of this parallelogram using the points in figure 1. To proceed it will be necessary to add to the geometic picture of figure 1. This is shown in figure 2 overleaf.

The orginal parallelogram is still evident from the thicker lines. The augmentations to the parallelogram in figure 1 include:

- A vertical line drawn from (0,0) to (0,d) and a horizontal line from (0,d) to (c,d). The area encompassed is coloured in green.

- A horizontal line drawn from (0,0) to (a,0). The line from (a,b) to (a+c,b+d) is extended to cut this horizontal line with the resulting area coloured in yellow.
- A vertical is dropped from (a,b) to (a,0) with the small area now bounded coloured in red.
- A vertical is drawn from (a,b) to (a,b+d) and a hroizontal from (a,b+d) to (a+c,b+d). The bounded area above the original parallelogram is coloured purple.
- Finally, a horizontal is drawn from (c,d) to (a+c,d) with a vertical from (a+c,b+d) to (a+c,d).
- Equivalent areas are coloured the same.

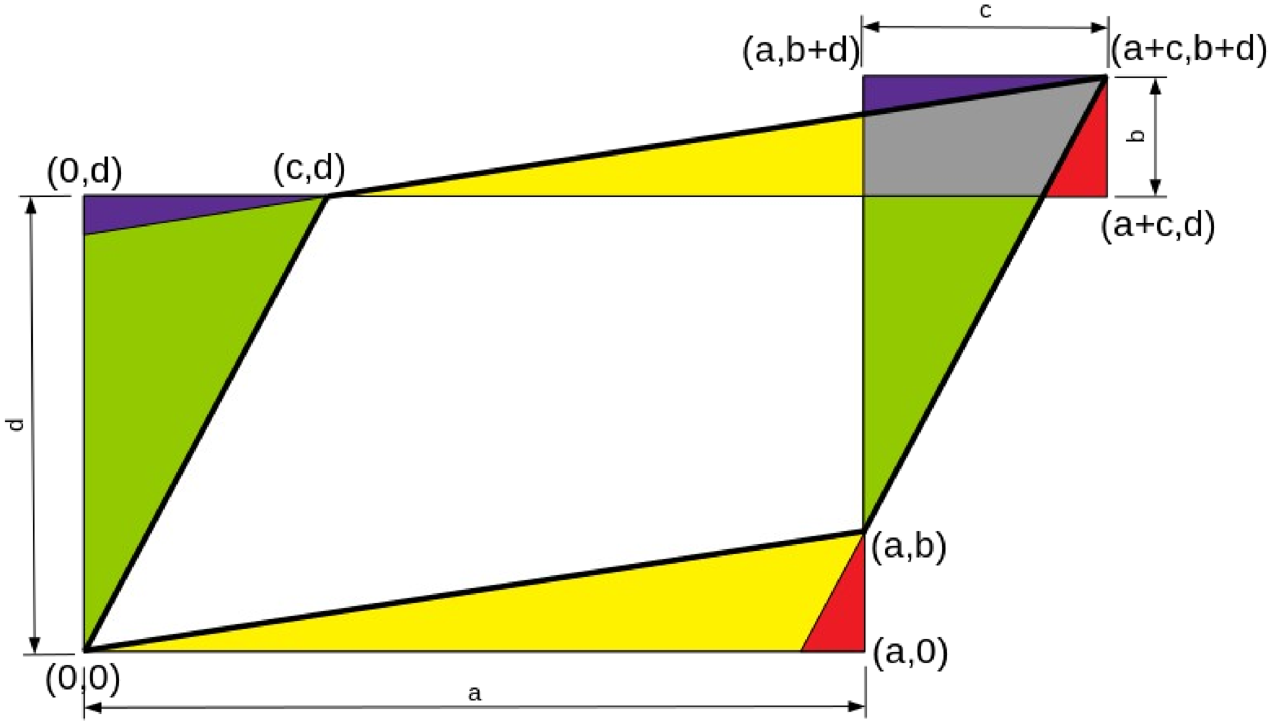

Figure 2: The geometric construction for the determinant

Then, the area of the original parallelogram is equal to the area of the rectangle of sides a and d less the area of the smaller rectangle of sides b and c.
This can be seen from the construction above where:

1. The yellow triangle on the bottom is taken from the yellow and grey areas on the top.
2. The green triangle on the left is taken from the green and grey areas on the right (Note: we have included the grey area twice so we must subtract this at the end).
3. The purple and red areas must be added to complete the rectangle of sides a and d and hence must (along with one of the grey areas) be subtracted from the final figure.
4. The grey, purple, and red areas to be subtracted complete the smaller rectangle of sides b and c.
5. Therefore the area of the parallelogram is the area of the larger rectangle less the area of the smaller; i.e. ad-bc.

Therefore the determinant of a 2x2 matrix, A, is the area of the rectangle spanned by the line segments formed from the rows of A.

It follows then that if the determinant is zero the line segments are collinear and effectively the 2x2 matrix is one dimensional.

The extension to 3D is valid; i.e. the volume of the 3D parallelepiped

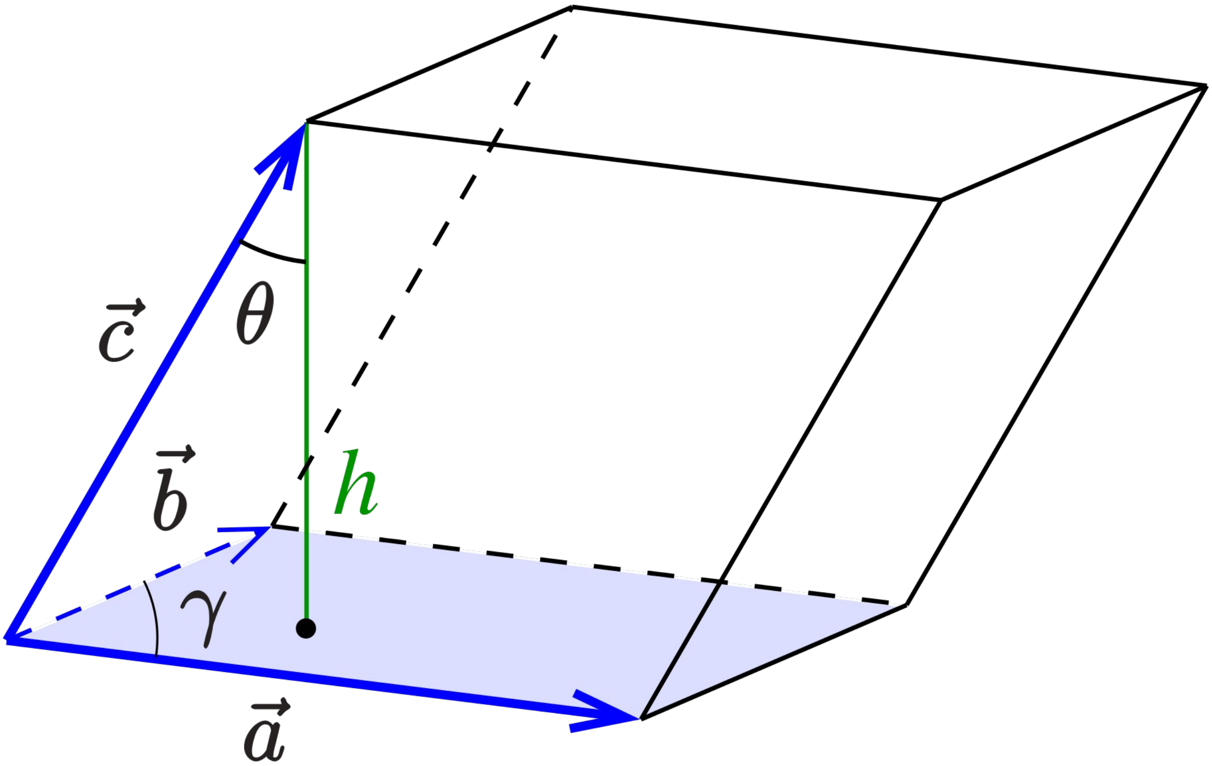

Figure 3: The parallelepiped

Here

$$\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

and

$$Volume=\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}=a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-c_2a_3)+c_1(a_2b_3-b_2a_3)$$