Section 6 (cont'd)
Integration by Parts and Integration by Substitution.

In this section we continue our analysis of the integral and its properties by considering how two techniques developed for the derivative (chain rule and product rule) can be used to develop corresponding techniques for the integral; namely Integration by Substitution and Integration by Parts. Then we introduce the idea of rules of thumb and use them with those integral techniques we've introduced to date to tackle problems you may otherwise have difficulties with.

Integration by Parts

To introduce this technique we need to remind you of the product rule of differentiation[1]. Let $F(x)$ and $G(x)$ be two differentiable functions that are primitives of two further continuous functions $f(x)$ and $g(x)$ respectively. Then

$$\begin{aligned} \frac{d}{dx}\left(F(x)G(x)\right) &= F(x)\frac{d}{dx}G(x)+G(x)\frac{d}{dx}F(x) \\ &= F(x)g(x)+G(x)f(x) \\ \Rightarrow \int \frac{d}{dx}\left(F(x)G(x)\right)dx &= \int \left(F(x)g(x)+G(x)f(x)\right)dx \\ \Rightarrow F(x)G(x) &= \int F(x)g(x)dx+\int G(x)f(x)dx \end{aligned}$$

and re-arranging we get

$$\int F(x)g(x)dx=F(x)G(x)-\int G(x)f(x)dx$$

yielding the Integration by parts formula

$$\int u\,dv=uv-\int v\,du$$

where we have replaced

$$F(x)\rightarrow u\,,G(x)\rightarrow v$$

$$f(x)dx=\frac{d}{dx}F(x)dx\rightarrow du$$

$$g(x)dx=\frac{d}{dx}G(x)dx\rightarrow dv$$

1 As mentioned frequently in class, the integral operation is dependent on the existence of the Primitive; i.e. on the pre-existence of a function that when differentiated gives you the function to be integrated. Therefore, in all methods we introduce, the pre-existence of a technique/method/rule that, under differentiation, becomes our starting point for integration is essential. Hence Product Rule gives Integration by Parts. Chain rule gives Integration by Substitution, etc.

When would you use such a formula?

If you are required to integrate the product of two functions then this technique (being derived from the product rule of differentiation) is useful. It is particularly useful when one of the functions simplifies under differentiation as you'll see below. The principle reason for using this, and similar methods, is to simplify the calculation involved in determining the integral. There are some issues that need to be considered. We'll show you how these crop up by taking a few examples:

Examples

1. Evaluate $\int 3\,x\,e^{x}\,dx$.

   We can start by removing the scalar constant 3 from the integral:

$$\int 3\,x\,e^{x}\,dx = 3\int x\,e^{x}\,dx$$

   Now we need to choose which will be our *u* and which will be the *dv*. We proceed by choosing[2]

$$u = x \;\Rightarrow \frac{d}{dx}u = 1 \;\overset{!}{\Rightarrow} du \equiv dx$$

$$dv \overset{!}{\equiv} e^{x}\,dx \;\Rightarrow \frac{d}{dx}v = e^{x} \;\Rightarrow v = e^{x}$$

   which yields

$$\int 3\,x\,e^{x}\,dx = 3\left(\underbrace{x\,e^{x}}_{uv} - \underbrace{\int e^{x}\,dx}_{\int v\,du}\right)$$

$$\Rightarrow \int 3\,x\,e^{x}\,dx = 3\left(x\,e^{x} - e^{x}\right) + C$$

   The integral has been evaluated with the result quoted in terms of elementary functions.

   Now a thought: What if you chose the *u* and *dv* differently?
   This is not as strange as it might seem. After all, you had two choices for *u,* so why let *u*=*x*?
   To see why we'll consider the other available choice of *u* and *dv*.

2 The reasons for above choice will become apparent in a moment when we swap the *u* and *dv* around and apply the formula again. This will lead to the first of our "rules of thumb" for integration. The exclamation marks ! above are highlighting two technically incorrect equivalences but ones which, if handled correctly, give the right result. Be careful!

So, let

$$u = e^x \quad \Rightarrow \frac{d}{dx} u = e^x \quad \overset{!}{\Rightarrow} du \equiv e^x\, dx$$

$$dv \overset{!}{\equiv} x\, dx \quad \Rightarrow \frac{d}{dx} v = x \quad \Rightarrow v = \frac{x^2}{2}$$

This now yields

$$\int 3\, x\, e^x\, dx = 3\left(\underbrace{\frac{x^2}{2} e^x}_{uv} - \underbrace{\int \frac{x^2}{2} e^x\, dx}_{\int v\, du} \right)$$

The second integral on the right hand side (RHS) is more complicated than the original. This method does not appear to be helping to solve our integral.

So what went wrong?

By setting $dv = xdx$ we are assigning a primitive of $x^2/2$ to x; a function that is more complicated than the original. Therefore the resulting integral on the RHS was more complicated than the original we were asked to evaluate negating the use of the technique which is to simplify the process of evaluating the integral.

What do we do?

You saw from the first attempt that the integral resulting from the application of this method was simplified. This is due to the appropriate choice of functions in the integral for u and dv. This leads nicely into our rules of thumb.

Rules of Thumb

Our first rule of thumb for integration is:

When taking the integral of a product of two functions, always choose u to be the function that simplifies under differentiation.

Our second rule of thumb is:

If neither function simplifies under d/dx then repeat the integration by parts as many times as is equal to the LCM[3] of the periodicities of the functions under d/dx.

This second rule sounds complex but is quite straightforward. The periodicity of a function under d/dx is the number of times it has to be differentiated until it becomes its original self again functionally speaking:

3 LCM – Lowest Common Multiple: i.e. the LCM of 4 and 3 is 12. The LCM of 6 and 4 is 12.etc.

e.g. Sine and cosine are both periodic with period 4 under *d*/*dx*

$$\frac{d^4}{dx^4}\cos(x) \equiv \cos(x).$$

However, functionally they are of period 2 because

$$\frac{d^2}{dx^2}\cos(x) \equiv -\cos(x).$$

Then, for sine and cosine, only two applications of the integral are required. We'll consider this in the next example

2. Evaluate $\int e^x \sin(x)\, dx$.

   Neither sin(*x*) nor e^x simplify under the *d*/*dx* operator. Therefore the choice of *u* and *dv* becomes arbitrary:

   We proceed by choosing

$$u=\sin(x) \;\Rightarrow \frac{d}{dx}u=\cos(x) \;\overset{!}{\Rightarrow} du \equiv \cos(x)\, dx$$

$$dv \overset{!}{\equiv} e^x\, dx \;\Rightarrow \frac{d}{dx}v=e^x \;\Rightarrow v=e^x$$

   which yields

$$\int e^x \sin(x)\, dx = \underbrace{e^x \sin(x)}_{uv} - \underbrace{\int e^x \cos(x)\, dx}_{\int v\, du}$$

$$\Rightarrow \int \sin(x) e^x\, dx = \sin(x) e^x - \int \cos(x) e^x\, dx$$

   The integral

$$\int \cos(x) e^x\, dx$$

   on the RHS needs to be evaluated using the same process with the exception that

$$u=\cos(x) \;\Rightarrow \frac{d}{dx}u=-\sin(x) \;\overset{!}{\Rightarrow} du \equiv -\sin(x)\, dx$$

$$dv \overset{!}{\equiv} e^x\, dx \;\Rightarrow \frac{d}{dx}v=e^x \;\Rightarrow v=e^x$$

   Then

$$\int \cos(x) e^x\, dx = \cos(x) e^x + \int e^x \sin(x)\, dx$$

   Replacing the $\int \cos(x) e^x\, dx$ in the first application of this process we get

$$\int \sin(x)e^x\,dx = \sin(x)e^x - \underbrace{\cos(x)e^x - \int e^x \sin(x)\,dx}_{\int \cos(x)e^x\,dx}$$

The integral remaining on the RHS is the same as that on the LHS and so we re-write above as

$$2\int \sin(x)e^x\,dx = \sin(x)e^x - \cos(x)e^x$$

and the final expression is that below:

$$\int \sin(x)e^x\,dx = \frac{1}{2}\left(\sin(x)e^x - \cos(x)e^x\right) + C$$

In the tutorial sheet to follow, you'll be given ample opportunity to practice.

Integration by Substitution

Now to integration by substitution. As we've already stated, this technique derives its existence from the Chain Rule of Differentiation. From the chain rule (where the primitive $F(x)$ has derivative $f(x)$)

$$\frac{d}{dx}F\left(u(x)\right) = \left[\frac{d}{du}F(u)\right]_{u=u(x)} \frac{d}{dx}u(x)$$

we can derive

$$\int \frac{d}{dx}F\left(u(x)\right)dx = \int \left[\frac{d}{du}F(u)\right]_{u=u(x)} \frac{d}{dx}u(x)\,dx$$

$$\Rightarrow F(u(x)) = \int f(u)\,|_{u=u(x)}\,\frac{d}{dx}u(x)\,dx$$

$$\Rightarrow F(u(x)) = \left[\int f(u)\,du\right]_{u=u(x)}$$

In other words, if the function, $f(x)$, being integrated has as its argument another function, $u(x)$, whose derivative[4] is present in the integral (and multiplying $f(x)$) then we can substitute u for $u(x)$ and integrate just f with respect to (w.r.t) u. Once integrated, we can substitute $u(x)$ for u and simplify if required.

Again, it helps to have examples so let's consider a few more.

4 The presence of any multiple of the derivative is acceptable. Once the derivative is present to within a multiplicative constant, this method can be applied: e.g if $u(x)=3x^2$ and $7x$ was present in the integral then we use the above method because $7x = (7/6)\,du/dx$.

Examples:

1. Evaluate $\int 2e^{3x}\,dx$

   Here the function $f(u(x))$ is e^{3x} and the argument is $3x$. If we let

$$u = 3x \;\Rightarrow \frac{d}{dx}u = 3 \;\Rightarrow 2\,dx \overset{!}{\equiv} \frac{2}{3}\,du$$

$$\Rightarrow \int 2e^{3x}\,dx = \left[\frac{2}{3}\int e^{u}\,du\right]_{u=3x}$$

   The RHS integral can be evaluated as if it was

$$\int e^{x}\,dx$$

   because the variable in the integral is immaterial (could have used y or w or ξ etc. instead of u). Then

$$\int e^{u}\,du = e^{u} + C$$

   and the integral becomes

$$\int 2e^{3x} = \frac{2}{3}\left[e^{u}\right]_{u=3x} + C = \frac{2}{3}e^{3x} + C$$

   Let's consider another.

2. Evaluate $\int 3x\cos\left(2x^2\right)dx$

   At first glance you may be tempted to use integration by parts because you have two functions present; one of which ($3x$) simplifies under differentiation. What prevents us using this method is that $\cos(2x^2)$ alone has no primitive and so we cannot integrate it. So we're left with integration by substitution. Now to the problem at hand.

   What do we choose u to be in this integral?
   The argument of the cosine gives us the clue. The derivative of $2x^2$ is $4x$; which is a constant times $3x$, the second function in the integral. Therefore we can set $u = 2x^2$ in the integral and the calculation proceeds thus:

$$u = 2x^2 \;\Rightarrow \frac{d}{dx}u = 4x \;\Rightarrow 3x\,dx \overset{!}{\equiv} \frac{3}{4}\,du$$

$$\Rightarrow \int 3x\cos\left(2x^2\right)dx = \left[\frac{3}{4}\int \cos(u)\,du\right]_{u=2x^2}$$

$$\Rightarrow \int 3x\cos\left(2x^2\right)dx = \left[\frac{3}{4}\sin(u)\right]_{u=2x^2} + C$$

$$\Rightarrow \int 3x\cos\left(2x^2\right)dx = \frac{3}{4}\sin\left(2x^2\right) + C$$

As you practice with the tutorials, you'll become familiar with the methods described above and when to use them effectively.

A word of caution (a caveat as it were):
With all novel techniques, it is understandable if you, the student, feels initially uncertain of where, when, and how to use them. This is only natural and shouldn't be seen as an insurmountable problem. If it was such then no progress would ever be made. You should practice with the supplied tutorials, past exams papers, and reference books in the library. The more you practice, the more proficient you will become at using these methods.

In our next section we'll tackle partial fractions and how they can be used to solve integrals involving rational functions.