# Section 6 Cont'd
# Ordinary Differential Equations

In this section we progress our analysis of integration by considering the problem of finding solutions to equations that include derivatives; these equations are called Ordinary Differential Equations (O.D.E.'s). We restrict our attention to first order O.D.E.'s. We start with separable equations, then proceed through homogeneous forms to end up finally with linear first order O.D.E's. We derive/motivate the general solutions for such equations and show how some can be used to solve some simple physics problems.

**What is an O.D.E.?**

As you are now familiar with functions and their representations, we can state that an equation relates a dependent variable , say $y(x)$, to a series of mathematical operations of a independent variable, $x$: e.g.

$$y(x)=a\,x^2+b\,x+c \quad \forall\, a,b,c,\in\mathbb{R};\, a\neq 0$$

Then in a similar way, a First Order Ordinary Differential Equation (O.D.E.), is an equation that contains the first derivative of a dependent variable, again $y(x)$, with respect to (w.r.t.) some independent variable, $x$; e.g.

$$\frac{d}{dx}y(x)+p(x)\,y(x)=q(x)$$

where $p(x)$ and $q(x)$ are, preferably, real integrable functions of $x$. Our purpose here, is to show how some forms of first order O.D.E.'s can be solved.

**Separable First Order Ordinary Differential Equations**

We start with the simplest non-trivial[1] case: that of separable equations. A separable O.D.E. is an equation that can be written in the form

$$\frac{d}{dx}y(x)=\frac{A(x)}{B(y)} \quad \Leftrightarrow \quad B(y)\neq 0$$

Then, if we multiply both sides by $B(y)$ and integrate w.r.t. $x$, we get

$$\int B(y)\frac{d}{dx}y(x)\,dx=\int A(x)\,dx$$

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1 A trivial case would be $\frac{d}{dx}y(x)=\alpha \quad \forall\,\alpha\in\mathbb{R}$. Here $y(x)=\alpha\,x+C \quad \alpha, C\in\mathbb{R}$

Then, the solution is obtained from

$$\int B(y)\,dy = \int A(x)\,dx$$

Of course, it has solution if and only if both of the two integrals are actually integrable. Let's illustrate with an example:

**Example**:
Find the solution of the separable O.D.E.

$$\frac{d}{dx}y(x) = \frac{e^{3x}}{y^2}$$

Then, here,

$$B(y) = y^2 \text{ and } A(x) = e^{3x}$$

and the solution is obtained from

$$\int B(y)\,dy = \int A(x)\,dx$$
$$\Rightarrow \int y^2\,dy = \int e^{3x}\,dx$$
$$\Rightarrow \frac{y^3}{3} = \frac{1}{3}e^{3x} + C$$
$$\Rightarrow y^3 = e^{3x} + C$$
$$\Rightarrow y(x) = \left(e^{3x} + C\right)^{1/3}$$
$$\Rightarrow y(x) = \sqrt[3]{e^{3x} + C} \quad \forall\, C \in \mathbb{R}$$

Let's verify:

$$y(x) = \sqrt[3]{e^{3x} + C}$$

$$\Rightarrow \frac{d}{dx}y(x) = \frac{3e^{3x}}{3}\left(e^{3x} + C\right)^{-2/3} = \frac{e^{3x}}{\left(e^{3x} + C\right)^{2/3}} = \frac{e^{3x}}{\left(\sqrt[3]{e^{3x} + C}\right)^2} = \frac{e^{3x}}{\left(y(x)\right)^2}$$

as expected. Now onto the next least complicated form; homogeneous first order O.D.E.'s.

**Homogeneous First Order Ordinary Differential Equations**
If the O.D.E under analysis can be written in the form

$$\frac{d}{dx}y(x) = f\left(\frac{y}{x}\right)$$

then it can be solved by using a method similar to integration by substitution; namely by substituting

$$u=\frac{y}{x}$$

we can solve the system as follows:

$$\frac{d}{dx}y(x)=f\left(\frac{y}{x}\right)$$

$$\Rightarrow\frac{d}{dx}(u\,x)=f(u)\quad\Leftrightarrow\quad u=\frac{y}{x}\;\Rightarrow\;y=u\,x$$

Then

$$x\frac{d}{dx}u+u\frac{d}{dx}x=f(u)$$

$$\Rightarrow x\frac{d\,u}{d\,x}+u=f(x)$$

$$\Rightarrow\frac{d\,u}{d\,x}=\left(\frac{1}{x}\right)(f(u)-u)$$

$$\Rightarrow\int\frac{1}{f(u)-u}\frac{d\,u}{d\,x}dx=\int\frac{1}{x}dx$$

$$\Rightarrow\int\frac{1}{f(u)-u}du=\ln|x|+C$$

Again only solvable if the LHS integral can be evaluated.

**Example:**
Find the solution of

$$\frac{d}{dx}y(x)=\frac{y^2}{x^2}-2$$

Here

$$f(u)=u^2-2\;\Leftrightarrow\;u=\frac{y}{x}$$

then

$$\int\frac{1}{f(u)-u}du=\int\frac{1}{u^2-u-2}du=\int\frac{1}{(u-2)(u+1)}du$$

Using the partial fractions method discussed in an earlier section, where

$$\int\frac{1}{(u-2)(u+1)}du=\int\frac{A_1}{u-2}du+\int\frac{A_2}{u+1}du$$

we get

$$A_1=1/3\;\text{ and }\;A_2=-1/3$$

and therefore

$$\int \frac{1}{f(u)-u}\,du=\int \frac{1}{(u-2)(u+1)}\,du=\frac{1}{3}\int \frac{1}{u-2}\,du-\frac{1}{3}\int \frac{1}{u+1}\,du$$

$$\Rightarrow \int \frac{1}{f(u)-u}\,du=\frac{1}{3}\ln|u-2|-\frac{1}{3}\ln|u+1|$$

$$\Rightarrow \int \frac{1}{f(u)-u}\,du=\frac{1}{3}\ln\left|\frac{u-2}{u+1}\right|=\ln|x|+C=\ln|\alpha x|$$

where $\ln|\alpha|=C$. Then the solution to the ODE is obtained thus

$$\ln\left|\frac{u-2}{u+1}\right|=3\ln|\alpha x|=\ln|\alpha^3 x^3|$$

$$\Rightarrow \frac{u-2}{u+1}=(\alpha x)^3$$

Substituting $u=\frac{y}{x}$ we get

$$\frac{y-2x}{y+x}=(\alpha x)^3$$

$$\Rightarrow y(x)=x\left(\frac{2+(\alpha x)^3}{1-(\alpha x)^3}\right)$$

Pretty messy! Anyhow, not to worry. You will not be asked to solve these in any exam question. The purpose of this section is to show you how such complicated equations can be tackled.

**Linear First Order Ordinary Differential Equations**

Finally to the most complicated form of first order O.D.E. we'll consider this year: the Linear First Order O.D.E. We define the first order linear Ordinary Differential Equation (O.D.E.) to be any equation of the form

$$\frac{d}{dx}y(x)+p(x)y(x)=q(x)$$

where $p(x)$ and $q(x)$ are functions of $x$. We seek a general solution for $y(x)$ for those values of $x$ for which $y(x)$ is non-zero; otherwise the solution is trivial being just the primitive of $q(x)$. To solve this equation we make use of the following identity

$$\frac{d}{dx}e^{\int p(x)dx}=p(x)e^{\int p(x)dx}$$

Then it follows

$$\frac{d}{dx}y(x)e^{\int p(x)dx} \underset{\substack{\uparrow \\ \text{product rule}}}{=} \left(\frac{d}{dx}y(x)+p(x)y(x)\right)e^{\int p(x)dx}$$

The expression in the round brackets on the RHS of this equation is equal to the LHS of the original O.D.E. which is equal to $q(x)$. Thus we replace the expression in the brackets with $q(x)$ to get

$$\frac{d}{dx}y(x)e^{\int p(x)dx} \underset{\substack{\uparrow \\ \text{product rule}}}{=} \left(\frac{d}{dx}y(x)+p(x)y(x)\right)e^{\int p(x)dx}=q(x)e^{\int p(x)dx}$$

$$\Rightarrow y(x)e^{\int p(x)dx}=\int q(x)e^{\int p(x)dx}dx+C$$

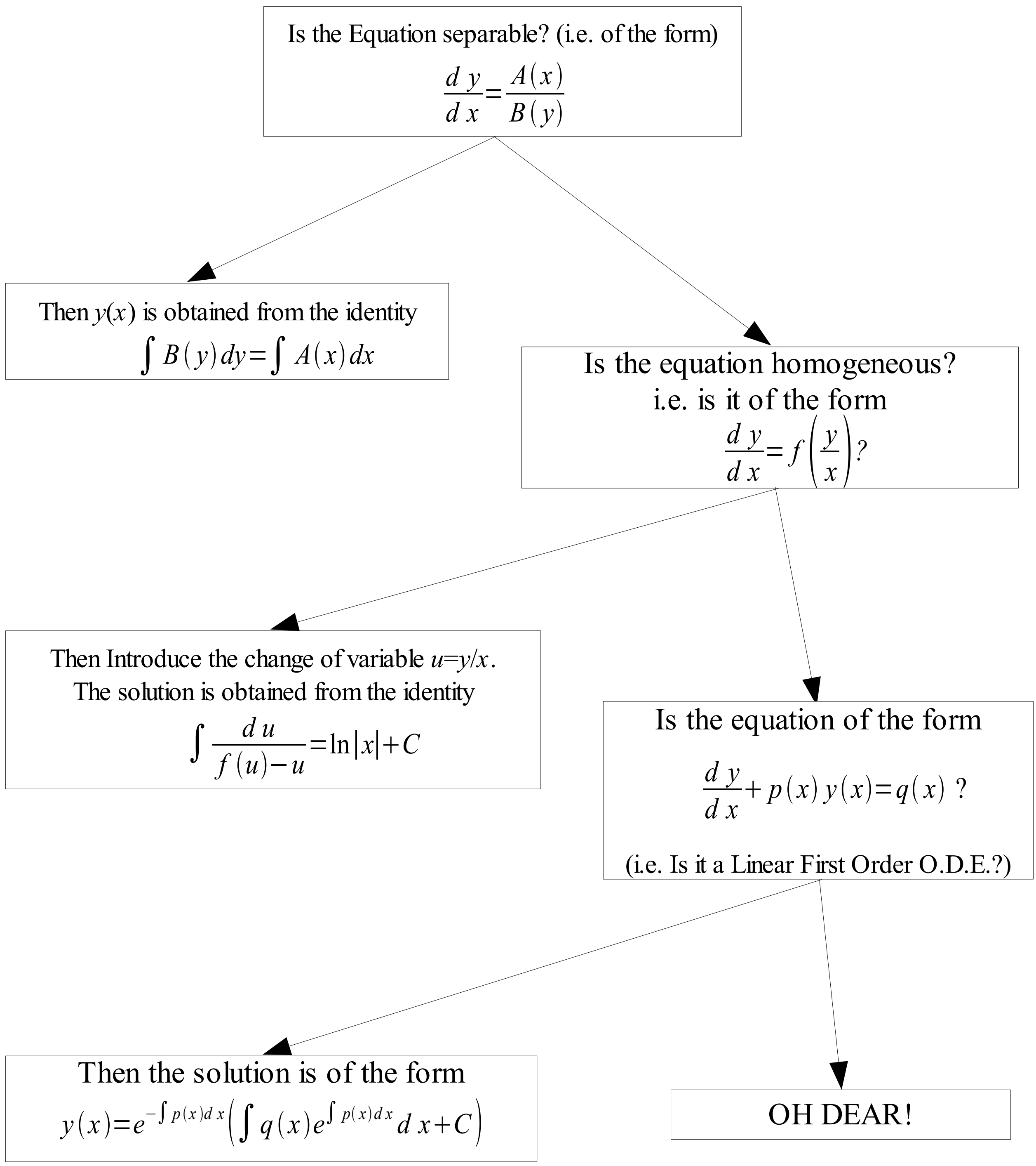
where $C$ is the constant of integration. The solution is then found

$$y(x)=e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx+C\right)$$

valid $\forall x\in\mathbb{R}\Leftrightarrow p(x)\neq 0$.

There will be times when $\int q(x)e^{\int p(x)dx}dx$ may not have an analytical solution either as an indefinite integral as above or as a definite integral. In such cases it becomes desirable to find a solution to such equations using numerical approximation techniques; this corresponds to the box with “OH DEAR!” in our summary of first order O.D.E.'s at the end of this section.

# A pictorial representation of some[2] common forms of First Order O.D.E.'s




2 We have omitted "Nearly Homogeneous" and "Exact" forms of first order O.D.E.'s as the former is a very restricted version of the homogeneous form and the latter requires partial derivatives which we won't be considering until next year.