Section 5 (Cont'd)
The Innerproduct and Crossproduct of vectors.

In this section we continue our analysis of vectors in 2 and 3 dimensional spaces. We consider the two products involving vectors with vectors. Their geometric interpretations will also be examined and how they can be used in both physical systems and computer graphics.

The Innerproduct (Dot Product) of two vectors.

Let $\vec{u}$ and $\vec{v}$ be any two vectors in 2D and let

$$\vec{u}=u_1\hat{i}+u_2\hat{j} \text{ and } \vec{v}=v_1\hat{i}+v_2\hat{j}$$

then, we define the innerproduct of $\vec{u}$ with $\vec{v}$

$$\begin{aligned}\langle\vec{u},\vec{v}\rangle &= u_1v_1+u_2v_2 = \sum_{i=1}^{2}u_iv_i \quad \in \mathbb{R}\\ &= v_1u_1+v_2u_2 = \langle\vec{v},\vec{u}\rangle\end{aligned}$$

Thus the innerproduct takes two vectors and returns a real number and NOT a vector. It is also commutative.

For 3D we have the following; given

$$\vec{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k} \text{ and } \vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$$

we define

$$\langle\vec{u},\vec{v}\rangle=u_1v_1+u_2v_2+u_3v_3=\sum_{i=1}^{3}u_iv_i=\langle\vec{v},\vec{u}\rangle$$

In general, for *n*-dimensional vectors (with the basis set $\{\hat{e}_1,\hat{e}_2,\ldots,\hat{e}_n\}$ for $\mathbb{R}^n$)

$$\vec{u}=u_1\hat{e}_1+u_2\hat{e}_2+\cdots+u_n\hat{e}_n \text{ and } \vec{v}=v_1\hat{e}_1+v_2\hat{e}_2+\cdots+v_n\hat{e}_n$$

then

$$\langle\vec{u},\vec{v}\rangle=u_1v_1+u_2v_2+\cdots+u_nv_n=\sum_{i=1}^{n}u_iv_i=\langle\vec{v},\vec{u}\rangle$$

Alternate form of the innerproduct

Normally we wouldn't deal with such potentially confusing forms as they tend to confuse more than illuminate. However, in this case, the alternate form helps to visualise, geometrically, what is resulting from the calculation of the innerproduct.

This form is geometric in nature and uses the vectors' magnitudes and the smaller angle between the vectors to calculate the innerproduct. The innerproduct in this form is written

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$$

where θ is the smaller angle between $\vec{u}$ and $\vec{v}$. To understand what this represents, graphically, consider the following construction below:

Let $\vec{u}$ be an arbitrary vector in 2D and let $\vec{u}_i$ and $\vec{u}_j$ be its component vectors along the horizontal and vertical respectively; i.e.

$$\vec{u}_i = u_1 \hat{i} \text{ and } \vec{u}_j = u_2 \hat{j} \text{ where } \vec{u} = u_1 \hat{i} + u_2 \hat{j}$$

or geometrically

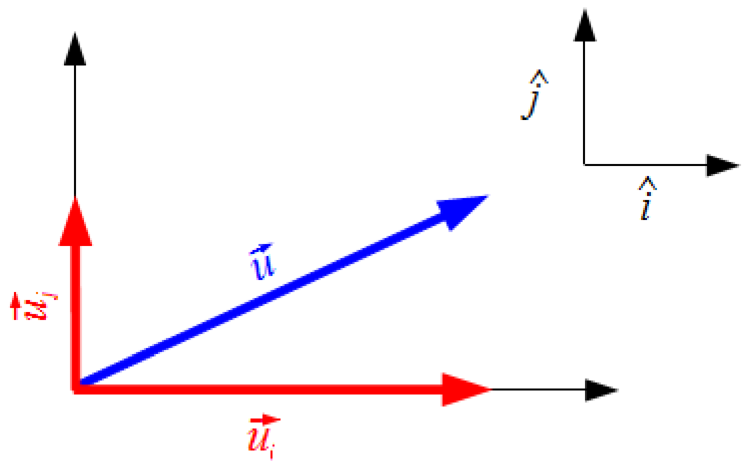


Figure: An arbitrary 2D vector $\vec{u}$ and its component vectors.

Let us consider this representation. It is quite straightforward to relate, say, the component vector $\vec{u}_i$ along the horizontal can to the vector $\vec{u}$ using trigononetric functions you are already acquainted with. To use these functions we only need to denote the angle the vector $\vec{u}$ makes with the horizontal (normally called θ (theta)).

Then we can easily show that the component vectors, $\vec{u}_i$ and $\vec{u}_j$, are given by

$$\vec{u}_i = u_1 \hat{i} = \|\vec{u}\| \cos(\theta) \hat{i}$$

and

$$\vec{u}_j = u_1 \hat{j} = \|\vec{u}\| \sin(\theta) \hat{j}$$

This is shown in the figure overleaf.

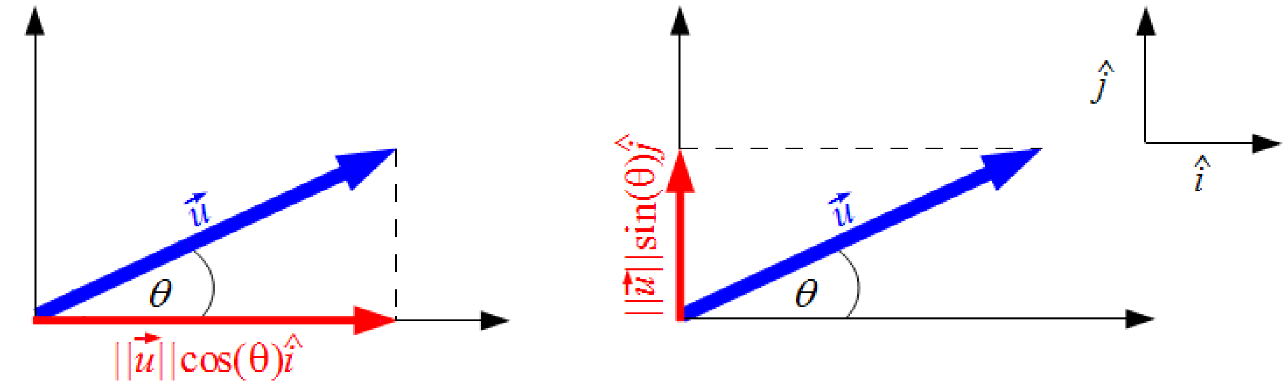


Figure: The component vectors expressed in terms of $\vec{u}$, its magnitude and the subtended angle θ

We can then consider $\vec{u}_i = u_1 \hat{i} = \|\vec{u}\|\cos(\theta)\hat{i}$ to the projection of $\vec{u}$ onto the horizontal axis. Similarly for $\vec{u}_j = u_1 \hat{j} = \|\vec{u}\|\sin(\theta)\hat{j}$ on the vertical axis. If we take the analogy that the vector is a stick driven into the ground at some arbitrary angle and the Sun is directly overhead, then the shadow cast on the ground by the stick would be $\vec{u}_i$:
i.e.

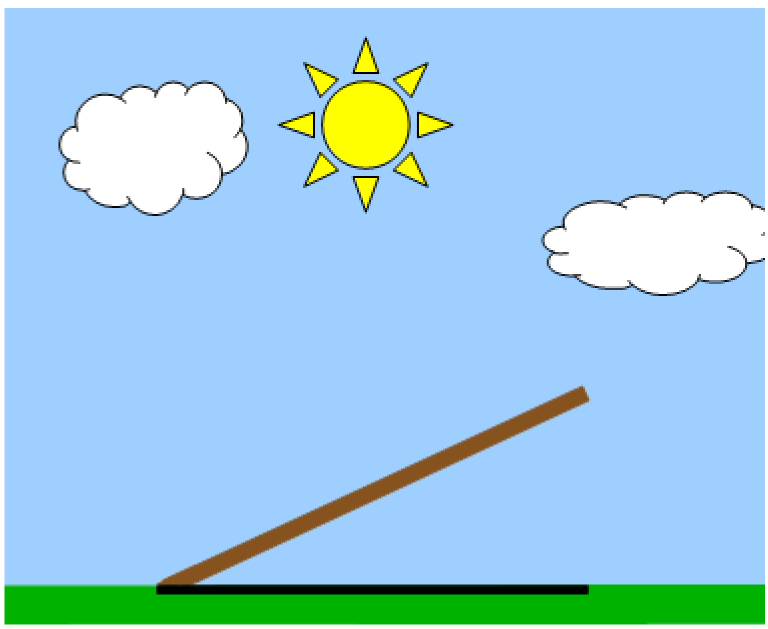


Figure: Physical analogy of vector projection onto horizontal axis.

How does all this relate to the innerproduct?
From the geometric version you can see that there is a direct relationship between the magnitude of the component projected onto the horizontal and the innerproduct equation:

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos(\theta) \text{ and } \|\vec{u}_i\| = \|u_1 \hat{i}\| = \|\vec{u}\| \cos(\theta) \underbrace{\|\hat{i}\|}_{=1}$$

Where does $\vec{v}$ fit into all this?

Essentially $\vec{v}$ is taking on the role of the horizontal axis. In effect, the innerproduct is projecting $\vec{u}$ onto $\vec{v}$ and then multiplying this projection by the magnitde of $\vec{v}$; i.e. it is determining the magnitude of the component of $\vec{u}$ that is parallel to $\vec{v}$ times the magnitude of $\vec{v}$. In physics we sometimes refer to this as the *interaction strength* of the two vectors:

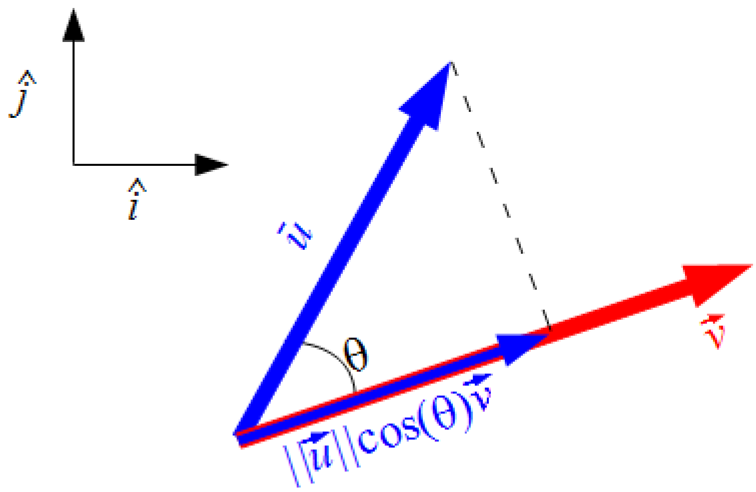


Figure: The projection of a vector $\vec{u}$ onto another vector $\vec{v}$.

N.B.

1. If the two vectors, $\vec{u}$ and $\vec{v}$, passed to the innerproduct, <,>, are parallel to each other then θ is 0 and the cosine of θ is 1. The innerproduct is thus maximised.
2. If the two vectors, $\vec{u}$ and $\vec{v}$, passed to the innerproduct, <,>, are perpendicular to each other then θ is $\pi/2$ and the cosine of θ is 0. The innerproduct is thus zero for mutually perpendicular vectors.
3. If the two vectors, $\vec{u}$ and $\vec{v}$, passed to the innerproduct, <,>, are anti-parallel then θ is π and the cosine of θ is -1. The innerproduct is thus minimised.

We will use the second of these properties in developing a test for orthogonality that has implications far beyond vectors.

Orthogonality test using the innerproduct.

We can use the innerproduct as a test for orthogonality between two vectors. We know that the innerproduct depends on the cosine of the smaller angle θ between the two vectors, where $0 \le \theta \le \pi$. Using property 2 in the previous *nota bene* we can design an orthogonality rule:

Let $\vec{u}$ and $\vec{v}$ be arbitrary vectors in $\mathbb{R}^n \ \forall n \in \mathbb{N}$. Then $\vec{u}$ and $\vec{v}$ are mutually orthogonal if and only if the innerproduct of $\vec{u}$ with $\vec{v}$, or vice versa, is zero: i.e.

$$\vec{u} \perp \vec{v} \Leftrightarrow \langle \vec{u}, \vec{v} \rangle = 0$$

Example:

Test $\{\hat{i}, \hat{j}, \hat{k}\}$ for orthogonality.

We know that

$$\hat{i} = \hat{i} + 0\hat{j} + 0\hat{k} \quad \hat{j} = 0\hat{i} + \hat{j} + 0\hat{k} \quad \hat{k} = 0\hat{i} + 0\hat{j} + \hat{k}$$

then

$$\begin{aligned}
\langle \hat{i}, \hat{j} \rangle &= 1\times 0 + 0\times 1 + 0\times 0 = 0 = \langle \hat{j}, \hat{i} \rangle & \Rightarrow \quad \hat{i} \perp \hat{j} \\
\langle \hat{i}, \hat{k} \rangle &= 1\times 0 + 0\times 0 + 0\times 1 = 0 = \langle \hat{k}, \hat{i} \rangle & \Rightarrow \quad \hat{i} \perp \hat{k} \\
\langle \hat{j}, \hat{k} \rangle &= 0\times 0 + 1\times 0 + 0\times 1 = 0 = \langle \hat{k}, \hat{j} \rangle & \Rightarrow \quad \hat{j} \perp \hat{k}
\end{aligned}$$

Therefore $\{\hat{i}, \hat{j}, \hat{k}\}$ form an orthogonal set of vectors.

Crossproduct of two vectors

Let $\vec{u}$ and $\vec{v}$ be arbitrary vectors in 3D: i.e.

$$\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k} \text{ and } \vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

Then we define the crossproduct of $\vec{u}$ with $\vec{v}$, $\vec{u} \times \vec{v}$, as the determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\hat{i} + (u_3 v_1 - u_1 v_3)\hat{j} + (u_1 v_2 - u_2 v_1)\hat{k}$$

Unlike the innerproduct, the crossproduct results in a vector and is not commutative: i.e. If we let $\vec{w} = \vec{u} \times \vec{v}$ then it is easy to show that $\vec{v} \times \vec{u} = -\vec{w}$.

Therefore

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$

What is the relationship between $\vec{w}=\vec{u}\times\vec{v}$ and both $\vec{u}$ and $\vec{v}$?

One of the more interesting results from the crossproduct is the relationship the resulting vector $\vec{w}=\vec{u}\times\vec{v}$ has with the original $\vec{u}$ and $\vec{v}$. To see this, let us consider the angle $\vec{w}$ makes with both $\vec{u}$ and $\vec{v}$. To determine this angle, we need to use the innerproduct.

Then, from $\langle\vec{w},\vec{u}\rangle$ and $\langle\vec{w},\vec{v}\rangle$, we can determine the angle between the vectors. Recall that both $\vec{u}$ and $\vec{v}$ are arbitrary so the smaller angle between them can be any angle from 0 to π.

Let's calculate these quantities:

$$\begin{aligned}
\vec{w}=w_1\hat{i}+w_2\hat{j}+w_3\hat{k} &= \vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\\
&= (u_2v_3-u_3v_2)\hat{i}+(u_3v_1-u_1v_3)\hat{j}+(u_1v_2-u_2v_1)\hat{k}
\end{aligned}$$

Therefore

$$\begin{aligned}
w_1\hat{i}&=(u_2v_3-u_3v_2)\hat{i}\\
w_2\hat{j}&=(u_3v_1-u_1v_3)\hat{j}\\
w_3\hat{k}&=(u_1v_2-u_2v_1)\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
\langle\vec{w},\vec{u}\rangle &= w_1u_1+w_2u_2+w_3u_3\\
&= (u_2v_3-u_3v_2)u_1+(u_3v_1-u_1v_3)u_2+(u_1v_2-u_2v_1)u_3\\
&= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_1u_2v_3+u_1u_3v_2-u_2u_3v_1\\
&= 0
\end{aligned}$$

Similarly

$$\begin{aligned}
\langle\vec{w},\vec{v}\rangle &= w_1v_1+w_2v_2+w_3v_3\\
&= (u_2v_3-u_3v_2)v_1+(u_3v_1-u_1v_3)v_2+(u_1v_2-u_2v_1)v_3\\
&= u_2v_1v_3-u_3v_1v_2+u_3v_1v_2-u_1v_2v_3+u_1v_2v_3-u_2v_1v_3\\
&= 0
\end{aligned}$$

This is a remarkable result. The innerproduct between the vector, $\vec{w}$, and the two vectors, $\vec{u}$ and $\vec{v}$, whose crossproduct resulted in $\vec{w}$ is always 0.

Therefore we make the following conclusion:

The vector resulting from the crossproduct of two originating vectors is always mutually orthogonal to these twoorginating vectors.

i.e.

$$\text{if } \vec{w}=\vec{u}\times\vec{v} \ \Rightarrow\langle\vec{w},\vec{u}\rangle=\langle\vec{w},\vec{v}\rangle=0$$
$$\Rightarrow\vec{w}\perp\vec{u} \text{ and } \vec{w}\perp\vec{v}$$

for all vectors in $\mathbb{R}^n \ \forall n\in\mathbb{N}$.

Geometric interpretation.

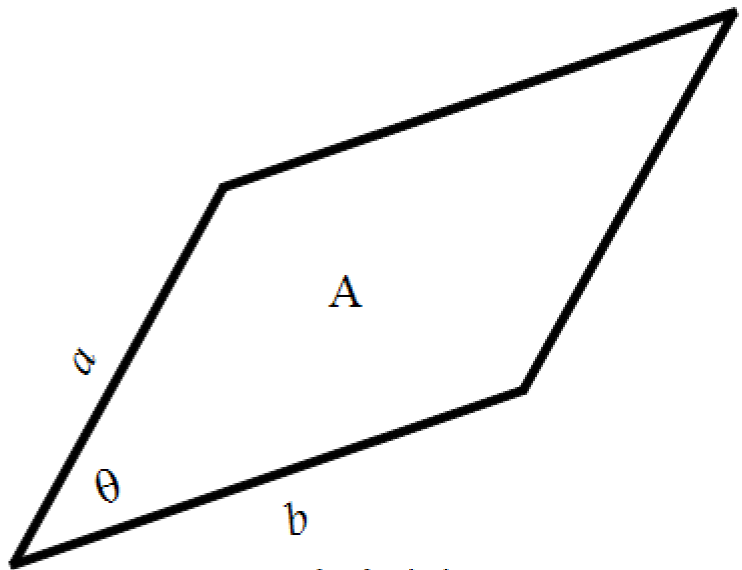
Just as for the innerproduct, there is an equivalent form for the crossproduct using trigonometric functions except this method results in the magnitude of the resulting vector. We saw, from our treatment of the innerproduct, how the cosine of the smaller angle θ gives rise to the projection of one vector on to the horizontal axis or another vector. The sine of that angle resulted in the perpendicular component; the part we're now interested in. Therefore our geometry based rule should have sine not cosine present.

Let $\vec{u}$ and $\vec{v}$ be any two vectors. Then

$$\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin(\theta)$$

So what is this?

From your log (maths) tables you'll find the following formula for the area of a parallelogram



$$A = ab\sin(\theta)$$

So the magnitude of the vector resulting from a crossproduct is equal to the magnitude of the area of the parallelogram spanned by the two vectors $\vec{u}$ and $\vec{v}$.

If, as before, we let $\vec{w}=\vec{u}\times\vec{v}$ then $\|\vec{w}\|=\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin(\theta)$ with $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$:

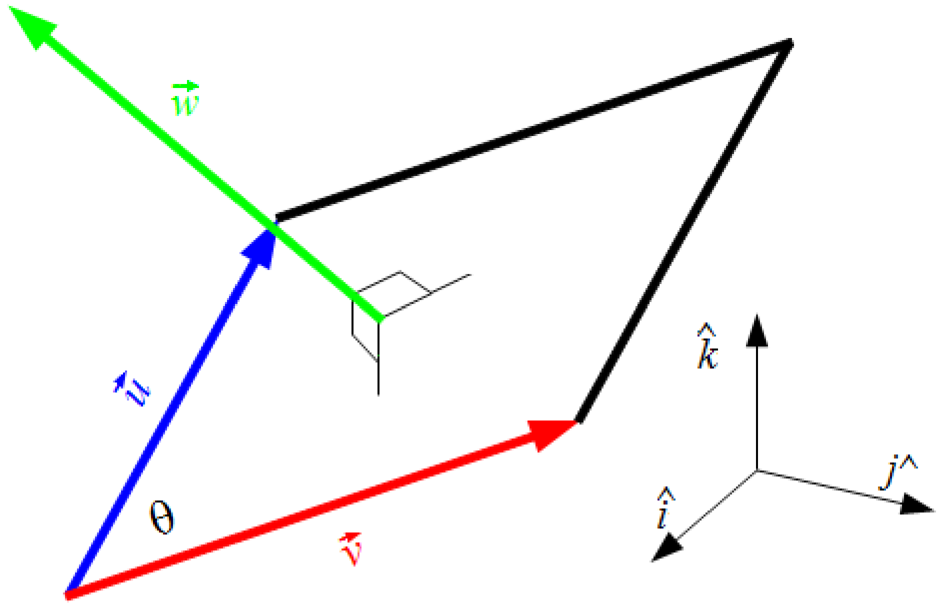


Figure: How the vector $\vec{w}=\vec{u}\times\vec{v}$ is oriented w.r.t spanning vectors $\vec{u}$ and $\vec{v}$

Example:

(a) Find a vector $\vec{w}$ that is mutually orthogonal to both

$$\vec{u}=3\hat{i}-2\hat{j}+\hat{k}$$
$$\vec{v}=-2\hat{i}+4\hat{j}-\hat{k}$$

(b) Show that the magnitude of the vector $\vec{w}$ is equal to the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Answer

(a) We know if we let $\vec{w}=\vec{u}\times\vec{v}$, then $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$. Similarly if we set $\vec{w}=\vec{v}\times\vec{u}$, then $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$. The difference between these two calculations is that one will be anti-parallel the other:

$$\vec{v}\times\vec{u}=-\vec{u}\times\vec{v}$$

For consistency with our approach so far, w'll choose $\vec{w}=\vec{u}\times\vec{v}$. Then

$$\vec{w}=(u_2v_3-u_3v_2)\hat{i}+(u_3v_1-u_1v_3)\hat{j}+(u_1v_2-u_2v_1)\hat{k}$$

which, using the values given above, becomes

$$\vec{w}=\left((-2)(-1)-(1)(4)\right)\hat{i}+\left((1)(-2)-(3)(-1)\right)\hat{j}+\left((3)(4)-(-2)(-2)\right)\hat{k}$$

$$\Rightarrow\vec{w}=-2\hat{i}+\hat{j}+8\hat{k}$$

Check $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\langle\vec{w},\vec{u}\rangle &= w_1u_1+w_2u_2+w_3u_3\\ &= (-2)(3)+(1)(-2)+(8)(1)\\ &= -6-2+8\\ &= 0\end{aligned}$$

Therefore $\vec{w}\perp\vec{u}$.

$$\begin{aligned}\langle\vec{w},\vec{v}\rangle &= w_1v_1+w_2v_2+w_3v_3\\ &= (-2)(-2)+(1)(4)+(8)(-1)\\ &= 4+4-8\\ &= 0\end{aligned}$$

Therefore $\vec{w}\perp\vec{v}$.

(b) The magnitude of the crossproduct vector can be determined using the formula $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin(\theta)$. We don't know the angle so we can find it using the innerproduct[1]:

$$\langle\vec{u},\vec{v}\rangle=\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

The innerproduct of $\vec{u}$ and $\vec{v}$ and their respective magnitudes are

$$\langle\vec{u},\vec{v}\rangle=(3)(-2)+(-2)(4)+(1)(-1)=-6-8-1=-15$$

$$\|\vec{u}\|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{14}$$

$$\|\vec{v}\|=\sqrt{(-2)^2+4^2+(-1)^2}=\sqrt{21}$$

and the angle is determined from

$$\cos(\theta)=\frac{\langle\vec{u},\vec{v}\rangle}{\|\vec{u}\|\|\vec{v}\|}=\frac{-15}{\sqrt{14}\sqrt{21}}$$

$$\Rightarrow\theta\simeq 2.63585585\,\text{Radians}\simeq 151.023^o$$

Now to the crossproduct. We use this angle to show that the length obtained using this method is equivalent to that using the determinant.

1 If we used the crossproduct to find the angle, we are using the result to verify the result which is a circular argument. By using the innerproduct we are verifying the veracity of the method used.

Using

$$||\vec{w}||=||\vec{u}\times\vec{v}||=||\vec{u}||||\vec{v}||\sin(\theta)$$

and the above caluclations we find

$$||\vec{u}||||\vec{v}||\sin(\theta)\simeq\sqrt{14}\sqrt{21}\sin(2.63585585)\simeq\sqrt{69}$$

From the result with the determinant we have

$$\vec{w}=-2\hat{\imath}+\hat{\jmath}+8\hat{k}$$
$$\Rightarrow||\vec{w}||=\sqrt{(-2)^2+1^2+8^2}=\sqrt{4+1+64}=\sqrt{69}$$

Verified as required: i.e.the area spanned by the two vectors $\vec{u}$ and $\vec{v}$ is equal in magnitude to the length of the vector $\vec{w}=\vec{u}\times\vec{v}$.

In the next section,w'll look at the scalar triple product and how it can be used to describe the interaction between a immersed planar surfaces with the flow they are immersed within; be they mathematical, electro-magnetic, hydrodynamical, aerodynamical, etc. type flows.