Section 4 (Cont'd)
Towards the inverse of a Matrix

In this section we consider those processes necessary to calculate the inverse of a square matrix of real numbers. We will introduce the determinant of such a matrix, the tranpose of a square matrix, and finally the adjoint (or adjugate) of a square matrix. The calculation of the adjoint (or adjugate) is by far the most complicated matrix process you'll encounter this year.

The Determinant of a Matrix

Let *A* be a square $n \times n$ matrix of real numbers, then the determinant of *A,* denoted $|A|$, is the the sum over the products of each of the first row elements, $a_{1\mathrm{j}}$, with the determinants of the $n-1 \times n-1$ matrix formed cyclically[1] when the 1st row and *j*-th column is removed. The $n-1 \times n-1$ determinants are similarly expressed in terms of the $n-2 \times n-2$ determinants, as on on down to the 2×2 determinant.

The Determinant of a 2×2 Matrix

The 2×2 determinant is defined thus:

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

The Determinant of a 3×3 Matrix

Then, for the 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Let's consider each of these 3 parts and show how they are constructed

1 With the removal of the *j*-th column the columns to the left of it are placed after the n-th column in the order 1,2,3,...,(*j*-1): i.e. the order of the columns would be *j*+1, *j*+2, ..., *n*-1, *n*, 1, 2, 3, ..., *j*-1

cyclically.

The first is the easiest. The determinant multiplying a_{11} is that of the 2×2 matrix formed when the 1st row and 1st column of A is removed: i.e.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Figure: The construction of the first term in the 3×3 determinant

The second term is constructed thus:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \left|\begin{array}{c|cc|c|} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{21} & a_{23} & a_{21} \\ a_{31} & a_{31} & a_{33} & a_{31} \end{array}\right. \Rightarrow \left|\begin{array}{cc:c|} a_{12} & a_{13} & \\ a_{21} & a_{23} & a_{21} \\ a_{31} & a_{33} & a_{31} \end{array}\right. \Rightarrow a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}$$

Figure: The construction of the second term in the 3×3 determinant

Finally the third term is constructed as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \left|\begin{array}{cc|c:cc|} a_{11} & a_{12} & a_{13} & & \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right. \Rightarrow \left|\begin{array}{c:cc|} a_{13} & & \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{array}\right. \Rightarrow a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Figure: The construction of the third term in the 3×3 determinant

The Determinant of a 4×4 Matrix

Then, for the 4×4 matrix,

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

we have, using this cyclic method,

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\Rightarrow |A|=a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}+a_{12}\begin{vmatrix} a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix}+a_{13}\begin{vmatrix} a_{24} & a_{21} & a_{22} \\ a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \end{vmatrix}+a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

As you can see it is getting very complicated very quickly. In fact the number of calculations scales as the factorial of the dimension: i.e. If we defined the 2×2 determinant to be proportional to 2 units, then a 3×3 would be proportional to 3 2×2 determinants and thus 3.2=3! units. The 4×4 has 4 3×3 determinants and would be scaling as 4.3.2 = 4! etc.

Example

Let A be the 3×3 matrix of real numbers

$$A=\begin{pmatrix} 3 & -7 & 2 \\ 0 & 4 & -5 \\ -1 & 5 & 6 \end{pmatrix}$$

then

$$|A|=3\begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix}+(-7)\begin{vmatrix} -5 & 0 \\ 6 & -1 \end{vmatrix}+2\begin{vmatrix} 0 & 4 \\ -1 & 5 \end{vmatrix}$$

$$\Rightarrow|A|=3(4\times6-(-5)\times5)+(-7)((-5)\times(-1)-0\times6)+2(0\times5-4\times(-1))$$

$$\Rightarrow|A|=3(24-(-25))-7(5-0)+2(0+4)$$

$$\Rightarrow|A|=147-35+8=120$$

N.B.

In effect the determinant is a magnitude of the matrix it is calculated from. You will see how this statement hold true when we consider both the inverse of the matrix and a few special (or notable) determinants now.

Some Notable Determinants

We consider here some determinants of note; specifically diagonal matrices, the identity matrix, triangular matrices, and the zero matrix.

1. **Diagonal Matrices**

   In the last section we defined the diagonal matrix to be one where the only non-zero elements were located on the principle diagonal: i.e.

$$A=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

   Then the determinant of this matrix is the product over all of the principle diagonal elements:

$$|A| = a_{11} a_{22} a_{33} \cdots a_{nn} = \prod_{k=1}^{n} a_{kk}$$

The symbol $\prod$ indicates *product* just as $\sum$ indicates *sum*. Just think of *p* for *pi* and *product* in the same way you think of *s* for *sigma* and *sum*.

Example:

Let A be the 3×3 diagonal matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} = \underbrace{3}_{a_{11}} \times \underbrace{(-2)}_{a_{22}} \times \underbrace{1}_{a_{33}} = -6$$

as expected.

2. **Identity matrix**

From examples done in class, you know that the $n \times n$ identity matrix, *I*, has the property that any square $n \times n$ matrix, *A*, multiplying it yields itself: i.e.

$$A \cdot I = I \cdot A = A$$

Using the result for diagonal matrices (of which *I* is one) then we have

$$|I| = a_{11} a_{22} a_{33} \cdots a_{nn} = 1 \times 1 \times 1 \times \cdots \times 1 = 1^n = 1$$

3. **Zero Matrix**

If we treat the zero matrix O, as an $n \times n$ diagonal matrix with all zero elements on the principle diagonal, then we get

$$|O| = a_{11} a_{22} a_{33} \cdots a_{nn} = 0 \times 0 \times 0 \times \cdots \times 0 = 0^n = 0$$

4. **Triangular Matrices**

Not as easy to treat but they yield an interesting result. Consider the 3×3 Lower (or Left) Triangular matrix, *L*:

$$L = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then the determinant is

$$|L|=\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix}+0\begin{vmatrix} 0 & a_{21} \\ a_{33} & a_{31} \end{vmatrix}+0\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|L|=a_{11}a_{22}a_{33}$$

exactly the result for a diagonal matrix. Similarly the 4×4 determinant will yield the same result; that its determinant is equal to the 4×4 diagonal matrix and so on[2].

Then we can state: For any $n\times n$ Lower triangular matrix, *L*,

$$|L|=\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

For and Upper triangular matrix, U, we have the following result for the 3×3 case: Given U

$$U=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

we find

$$|U|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix}+a_{12}\begin{vmatrix} a_{23} & 0 \\ a_{33} & 0 \end{vmatrix}+a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix}$$

$$|U|=a_{11}a_{22}a_{33}$$

the same result as for the Lower triangular matrix, *L*. IN a similar manner we find the $n\times n$ Upper triangular matrix, *U*, has determinant

$$|U|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\mathrm{n}} \\ 0 & a_{22} & \cdots & a_{2\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}=a_{11}a_{22}\cdots a_{nn}$$

These latter results explain why *L* and *U* type matrices are used

2 This is due to the fact that the higher order determinants are dependent on the lower orders and so what holds true for them can, and usually does, hold true for higher orders.

extensively when numerically attempting to solve large matrix problems on a computer: i.e. *LU* decomposition technique, etc.

N.B.
You'll be relieved when informed that only 3×3 determinants will be considered in our analysis this year. Next year, we consider more advanced techniques but for now we'll content ourselves with the 3×3 matrix case.

Singular Matrices
A singular matrix is one whose determinant is zero. 3×3 examples of such matrcies would be the 3×3 zero matrix

$$O=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

and the 3×3 matrix

$$A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

The former's determinant is obvious and one we have already determined. The latter, though, is not obvious so let's calculate its determinant:

$$|A|=\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}=1\begin{vmatrix}5&6\\8&9\end{vmatrix}+2\begin{vmatrix}6&4\\9&7\end{vmatrix}+3\begin{vmatrix}4&5\\7&8\end{vmatrix}$$
$$\Rightarrow|A|=(45-48)+2(42-36)+3(32-35)=-3+12-9=0$$

In general it can be quite difficult to find square matrices that are singular.

One simple manner is to:

- fix the elements that must remain fixed
- insert a variable, x, into those elements you can vary
- calculate the resulting determinant to obtain at most a cubic equation in x and solve for x.

Example:
Find the values of x for which the following matrix is non-singular.

$$A=\begin{pmatrix}6-x & -2 & 0\\ -4 & 1 & 8\\ 3 & 5 & -2\mathrm{x}\end{pmatrix}$$

Then the determinant of A is

$$|A|=\begin{vmatrix}6-x & -2 & 0\\ -4 & 1 & 8\\ 3 & 5 & -2x\end{vmatrix}=(6-x)\begin{vmatrix}1 & 8\\ 5 & -2x\end{vmatrix}-2\begin{vmatrix}8 & -4\\ -2x & 3\end{vmatrix}+0\begin{vmatrix}-4 & 1\\ 3 & 5\end{vmatrix}$$

$$\Rightarrow |A|=(6-x)(-2x-40)-2(24-8x)+0=2x^2+44x-288$$

For A to be singular this polynomial must equal zero:

$$|A|=2x^2+44x-288=0$$
$$\Rightarrow x^2+22x-144=0$$

Using the ubiquitous formula for the roots, x_r, of a quadratic ax^2+bx+c

$$x_r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

we get

$$x_r=\frac{-22\pm\sqrt{22^2-4(-144)}}{2}=\frac{-22\pm\sqrt{1060}}{2}=-11\pm\sqrt{265}$$

Then the determinant of A is zero (i.e. A is singular) whenever $x=-11+\sqrt{265}$ or $x=-11-\sqrt{265}$.

Then the matrix

$$A=\begin{pmatrix}6-x & -2 & 0\\ -4 & 1 & 8\\ 3 & 5 & -2\mathrm{x}\end{pmatrix}$$

is non-singular for all real x excluding $\{-11-\sqrt{265},-11+\sqrt{265}\}$: i.e.

$$|A|\neq 0 \quad \forall\, x\in\mathbb{R}\setminus\{-11-\sqrt{265},-11+\sqrt{265}\}$$

Now onto the most complicated process involved with matrices this year; the calculation of the adjoint (or adjugate) of a square matrix. In what is to follow, we'll unambiguously use the term adjoint though some texts use the more accurate and formal adjugate.

The Transpose of a Square Matrix

The transpose of a square $n \times n$ matrix, A, is that $n \times n$ matrix, A^T, with the property that the rows and column elements of A are interchanged. In other words, if $a_{ij} \in A$ then it is equal to the $a_{ji} \in A^T$.

Transpose of a 2×2 Matrix

Consider the 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The Transpose of A, denoted A^T, is defined to be

$$A^T = \begin{pmatrix} a_{11}^T & a_{12}^T \\ a_{21}^T & a_{22}^T \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

For example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Transpose of a 3×3 Matrix

Consider the 3×3 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The Transpose of A, denoted A^T, is defined to be

$$A^T = \begin{pmatrix} a_{11}^T & a_{12}^T & a_{13}^T \\ a_{21}^T & a_{22}^T & a_{23}^T \\ a_{31}^T & a_{32}^T & a_{33}^T \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

For example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

It should be obvious to you that the transpose of any diagonal matrix is itself and the transpose of L is U and vice versa.

The Adjoint of a Matrix

Simply put, the adjoint of a square $n \times n$ matrix, A, is that $n \times n$ matrix, $adj(A)$ or A^*, with the property that the product of A with A^* yields a diagonal matrix whose principle diagonal elements are all equal to the determinant of A: i.e.

$$A \cdot A^* = \begin{pmatrix} |A| & 0 & \cdots & 0 & 0 \\ 0 & |A| & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| & 0 \\ 0 & 0 & \cdots & 0 & |A| \end{pmatrix}$$

The process necessary to find the adjoint from a given matrix is involved. Due to the complexity of this task, we will, for clarity, restrict our analysis to the 2×2 and 3×3 square matrices.

Adjoint of a 2×2 Matrix

Consider the 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The adjoint of A, denoted $adj(A)$ or A^*, is defined to be

$$adj(A) \equiv A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Then

$$A \cdot adj(A) \equiv A \cdot A^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\Rightarrow A \cdot A^* = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & -a_{11}a_{12} + a_{12}a_{11} \\ a_{21}a_{22} - a_{22}a_{21} & -a_{21}a_{12} + a_{22}a_{11} \end{pmatrix}$$

$$\Rightarrow A \cdot A^* = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

$$\Rightarrow A \cdot A^* = \begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}$$

where we have used the aforementioned definition for the determinant of a 2×2 matrix to go from the second last to the last step above.

Exercise
Show that the determinant of the 2×2 adjoint, A^*, above is equal to the determinant of A itself: i.e.

$$|adj(A)|\equiv|A^*|=|A|$$

Exercise
Show that, for any matrix and its adjoint, the matrix product commutes: i.e.

$$A\cdot adj(A)\equiv A\cdot A^*=\begin{pmatrix}|A| & 0\\ 0 & |A|\end{pmatrix}=A^*\cdot A\equiv adj(A)\cdot A$$

Adjoint of a 3×3 Matrix
Consider the 3×3 matrix A

$$A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

The adjoint of A, denoted $adj(A)$ or A^*, is defined to be the matrix of cofactors

$$adj(A)\equiv A^*=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}$$

where each cofactor, Δ_{ij} with $1\leq i, j\leq3$, is defined

$$\Delta_{ij}=(-1)^{i+j}\left|\begin{matrix}2\times2 \text{ matrix formed by removing the}\\ i^{th} \text{ column and } j^{th} \text{ row of } A\end{matrix}\right|$$

This is a non-trivial calculation in that the row and column indices in the adjoint refer to the removal of the corresponding columns and rows in A.

For example:

$\boldsymbol{\Delta_{11}}$
To calculate Δ_{11} we need to remove the 1st row and 1st column of A and then calculate the determinant of the resulting 2×2 matrix. This is then multiplied by $(-1)^{1+1}=1$.

Δ_{12}

To calculate Δ_{12} we need to remove the 2nd row and 1st column of A (NOTE THE SWITCH OF ROWS AND COLUMNS like a transpose) and then calculate the determinant of the resulting 2×2 matrix. This is then multiplied by $(-1)^{1+2}=-1$.

Δ_{23}

To calculate Δ_{23} we need to remove the 3rd row and 2nd column of A and then calculate the determinant of the resulting 2×2 matrix. This is then multiplied by $(-1)^{2+3}=-1$.

N.B.

1. It is worth noting that the principle diagonal elements will never be confusing to calculate because both indices are equal.
2. $(-1)^{i+j}$ evaluates to +1 if $i+j$ is an even number and to -1 if $i+j$ is an odd number: i.e.

$$(-1)^n=1 \Leftrightarrow n\equiv 2\mathrm{m}\ \forall\ m\in\mathbb{N}$$
$$(-1)^n=-1 \Leftrightarrow n\equiv 2\mathrm{m}+1\ \forall\ m\in\mathbb{N}$$

3. Do not confuse the rows and columns otherwise the adjoint will only be correct for the principle diagonal elements.

As for the determinant and matrix products, it is easier to grasp this method when a few examples are considered.

Example 1

Calculate the adjoint of the 3×3

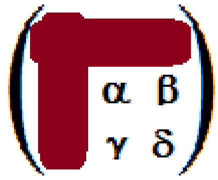
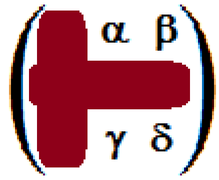
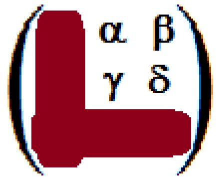
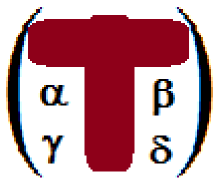
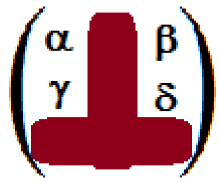
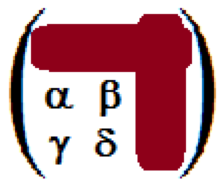
$$A=\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix}$$

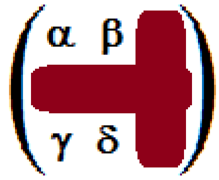
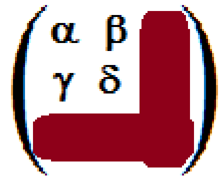
We define its adjoint by

$$adj(A)\equiv A^*=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}$$

and now we're left with the task of calculating the nine cofactors. We tabulate the cofactors overleaf.

The picture to the right shows the rows and columns removed from A leaving the four elements in the 2×2 for the determinant $\alpha\delta-\beta\gamma$.

$\Delta_{11}=(-1)^{1+1}\begin{vmatrix}5 & 8\\ 6 & 9\end{vmatrix}=(-1)^2(45-48)=(1)(-3)=-3$ In this calculation $\alpha=5$ $\beta=8$ $\gamma=6$ $\delta=9$	
$\Delta_{12}=(-1)^{1+2}\begin{vmatrix}4 & 7\\ 6 & 9\end{vmatrix}=(-1)^3(36-42)=(-1)(-6)=6$ In this calculation $\alpha=4$ $\beta=7$ $\gamma=6$ $\delta=9$	
$\Delta_{13}=(-1)^{1+3}\begin{vmatrix}4 & 7\\ 5 & 8\end{vmatrix}=(-1)^4(32-35)=(1)(-3)=-3$ In this calculation $\alpha=4$ $\beta=7$ $\gamma=5$ $\delta=8$	
$\Delta_{21}=(-1)^{2+1}\begin{vmatrix}2 & 8\\ 3 & 9\end{vmatrix}=(-1)^3(18-24)=(-1)(-6)=6$ In this calculation $\alpha=2$ $\beta=8$ $\gamma=3$ $\delta=9$	
$\Delta_{22}=(-1)^{2+2}\begin{vmatrix}1 & 7\\ 3 & 9\end{vmatrix}=(-1)^4(9-21)=(1)(-12)=-12$ In this calculation $\alpha=1$ $\beta=7$ $\gamma=3$ $\delta=9$	
$\Delta_{23}=(-1)^{2+3}\begin{vmatrix}1 & 7\\ 2 & 8\end{vmatrix}=(-1)^5(8-14)=(-1)(-6)=6$ In this calculation $\alpha=1$ $\beta=7$ $\gamma=2$ $\delta=8$	
$\Delta_{31}=(-1)^{3+1}\begin{vmatrix}2 & 5\\ 3 & 6\end{vmatrix}=(-1)^4(12-15)=(1)(-3)=-3$ In this calculation $\alpha=2$ $\beta=5$ $\gamma=3$ $\delta=6$	

$\Delta_{32}=(-1)^{3+2}\begin{vmatrix}1 & 4\\ 3 & 6\end{vmatrix}=(-1)^5(6-12)=(-1)(-6)=6$ In this calculation $\alpha=1\quad\beta=4\quad\gamma=3\quad\delta=6$	
$\Delta_{33}=(-1)^{3+3}\begin{vmatrix}1 & 4\\ 2 & 5\end{vmatrix}=(-1)^6(5-8)=(1)(-3)=-3$ In this calculation $\alpha=1\quad\beta=4\quad\gamma=2\quad\delta=5$	

The adjoint is thus

$$adj(A)\equiv A^*=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}=\begin{pmatrix}-3 & 6 & -3\\ 6 & -12 & 6\\ -3 & 6 & -3\end{pmatrix}$$

Exercise
Show that

$$A\cdot A^*=\begin{pmatrix}|A| & 0\\ 0 & |A|\end{pmatrix}=A^*\cdot A$$

and hence determine $|A|$.

Example 2
Calculate the adjoint of the 3×3

$$A=\begin{pmatrix}1 & -2 & 0\\ -9 & 7 & 4\\ 5 & -1 & 2\end{pmatrix}$$

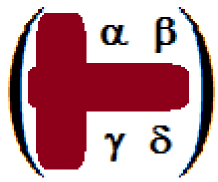
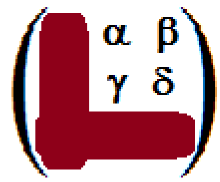
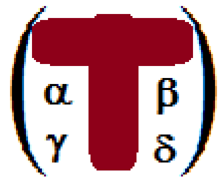
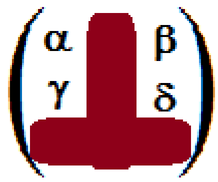
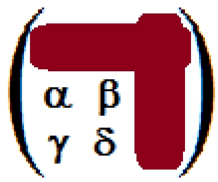
As before we define its adjoint by

$$adj(A)\equiv A^*=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}$$

and now calculate the nine cofactors. These are tabulated overleaf.

$$\Delta_{11} = (-1)^{1+1} \begin{vmatrix} 7 & 4 \\ -1 & 2 \end{vmatrix} = (-1)^2 (14 - (-4)) = (1)(18) = 18$$

In this calculation $\alpha = 7 \quad \beta = 4 \quad \gamma = -1 \quad \delta = 2$

$\Delta_{12}=(-1)^{1+2}\begin{vmatrix}-2 & 0\\ -1 & 2\end{vmatrix}=(-1)^3(-4-0)=(-1)(-4)=4$ In this calculation $\alpha=-2\ \ \beta=0\ \ \gamma=-1\ \ \delta=2$	
$\Delta_{13}=(-1)^{1+3}\begin{vmatrix}-2 & 0\\ 7 & 4\end{vmatrix}=(-1)^4(-8-0)=(1)(-8)=-8$ In this calculation $\alpha=-2\ \ \beta=0\ \ \gamma=7\ \ \delta=4$	
$\Delta_{21}=(-1)^{2+1}\begin{vmatrix}-9 & 4\\ 5 & 2\end{vmatrix}=(-1)^3(-18-20)=(-1)(-38)=38$ In this calculation $\alpha=-9\ \ \beta=4\ \ \gamma=5\ \ \delta=2$	
$\Delta_{22}=(-1)^{2+2}\begin{vmatrix}1 & 0\\ 5 & 2\end{vmatrix}=(-1)^4(2-0)=(1)(2)=2$ In this calculation $\alpha=1\ \ \beta=0\ \ \gamma=5\ \ \delta=2$	
$\Delta_{23}=(-1)^{2+3}\begin{vmatrix}1 & 0\\ -9 & 4\end{vmatrix}=(-1)^5(4-0)=(-1)(4)=-4$ In this calculation $\alpha=1\ \ \beta=0\ \ \gamma=-9\ \ \delta=4$	
$\Delta_{31}=(-1)^{3+1}\begin{vmatrix}-9 & 7\\ 5 & -1\end{vmatrix}=(-1)^4(9-35)=(1)(-26)=-26$ In this calculation $\alpha=-9\ \ \beta=7\ \ \gamma=5\ \ \delta=-1$	
$\Delta_{32}=(-1)^{3+2}\begin{vmatrix}1 & -2\\ 5 & -1\end{vmatrix}=(-1)^5(-1+10)=(-1)(9)=-9$ In this calculation $\alpha=1\ \ \beta=-2\ \ \gamma=5\ \ \delta=-1$	

$$\Delta_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\ -9 & 7\end{vmatrix}=(-1)^{6}(7-18)=(1)(-11)=-11$$

In this calculation $\alpha=1 \quad \beta=-2 \quad \gamma=-9 \quad \delta=7$

The adjoint is thus

$$adj(A)\equiv A^{*}=\begin{pmatrix}\Delta_{11} & \Delta_{12} & \Delta_{13}\\ \Delta_{21} & \Delta_{22} & \Delta_{23}\\ \Delta_{31} & \Delta_{32} & \Delta_{33}\end{pmatrix}=\begin{pmatrix}18 & 4 & -8\\ 38 & 2 & -4\\ -26 & -9 & -11\end{pmatrix}$$

We can easily show that

$$A\cdot A^{*}=\begin{pmatrix}1 & -2 & 0\\ -9 & 7 & 4\\ 5 & -1 & 2\end{pmatrix}\cdot\begin{pmatrix}18 & 4 & -8\\ 38 & 2 & -4\\ -26 & -9 & -11\end{pmatrix}=\begin{pmatrix}-58 & 0 & 0\\ 0 & -58 & 0\\ 0 & 0 & -58\end{pmatrix}$$

and that

$$A^{*}\cdot A=\begin{pmatrix}18 & 4 & -8\\ 38 & 2 & -4\\ -26 & -9 & -11\end{pmatrix}\cdot\begin{pmatrix}1 & -2 & 0\\ -9 & 7 & 4\\ 5 & -1 & 2\end{pmatrix}=\begin{pmatrix}-58 & 0 & 0\\ 0 & -58 & 0\\ 0 & 0 & -58\end{pmatrix}$$

and that

$$|A|=-58$$

In the next section we'll define the inverse of a square matrix and how it can be used to solve simultaneous equations; more formally known as solving systems of linear equations.