Section 2 (Cont'd)
General differentiation and Inverse Functions

In the last document, we considered the chain rule of differentiation and how to differentiate some well-known special functions. In this section we extend our analysis of the derivative further by considering inverse functions and, first, how to perform the differential operation without having to resort to ugly and clumsy limits. In fact, it is this latter method that is employed in general “day-to-day” differentiation.

Some Chain Rule Identities

Before we consider this new method of differentiation, let's derive some new identities for compound functions.

1. Generic $f(u(x))$ with $u(x)$ differentiable

   Then $\frac{d}{dx}u(x)$ exists and by the chain rule of differentiation

$$\frac{d}{dx}f(u(x)) = \left[\frac{d}{du}f(u)\right]_{u=u(x)}\frac{d}{dx}u(x)$$

2. $(u(x))^n$ with $u(x)$ differentiable $\forall\ n \in \mathbb{R}\setminus\{0\}$

   Then by the chain rule of differentiation

$$\frac{d}{dx}(u(x))^n = n(u(x))^{n-1}\frac{d}{dx}u(x)$$

3. $\cos(u(x))$ with $u(x)$ differentiable

   Then by the chain rule of differentiation

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))\frac{d}{dx}u(x)$$

4. $\sin(u(x))$ with $u(x)$ differentiable

   Then by the chain rule of differentiation

$$\frac{d}{dx}\sin(u(x)) = \cos(u(x))\frac{d}{dx}u(x)$$

5. $\tan(u(x))$ with $u(x)$ differentiable

   Then by the chain rule of differentiation

$$\frac{d}{dx}\tan(u(x)) = sec^2(u(x))\frac{d}{dx}u(x)$$

6. $\exp(u(x))$ or $e^{u(x)}$ with $u(x)$ differentiable

   Then by the chain rule of differentiation

$$\frac{d}{dx}\exp(u(x)) = \frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{d}{dx}u(x)$$

We will make good use of these and similar identities in the analysis to follow in this and later sections.

"Day-to-day" differentiation

Up to now we've performed the differential operation using the fundamental definition of the derivative; namely for a smooth function *f*(*x*)

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \forall\ x \in U \subseteq \mathbb{R}$$

where *U* is the domain of *f*(*x*). This is called the *Method of Differentiation by first principles*. This method is involved enough for simple functions but for more complicated expression, it becomes unwieldy.

So, do we have an alternative?

Well naturally we do otherwise we wouldn't be considering this issue.

So what is this alternative?

We have proved certain mathematical properties in previous documents such as the addition/subtraction of functions, scalar multiplication, product rule, and quotient rule. We recently tackled the chain rule. All these rules and properties combined allow us to decompose a compound function into manageable parts that can be differentiated using the log/mathematical tables.

Let's illustrate by example:

Example

Consider $f(x) = 3x^4 + \cos(2x) - e^{5x^2+4}$

Then using our previous method of first principles, we'd split the function into its three components, then represent the derivative of each component using the method above, and then attempt to calculate their limit as $\Delta x \to 0$.

Our method now replaces the limit calculation as follows:

- Using the mathematical properties introduced previously we find

$$\frac{d}{dx} f(x) = 3\underbrace{\frac{d}{dx} x^4}_{\text{part (i)}} + \underbrace{\frac{d}{dx} \cos(2x)}_{\text{part (ii)}} - \underbrace{\frac{d}{dx} e^{5x^2+4}}_{\text{part(iii)}}$$

- Now consider part (i) above. Using the identity

$$\frac{d}{dx}x^n = n\,x^{n-1}\ \forall\ n \in \mathbb{R}\setminus\{0\}$$

we find (with n=4)

$$3\frac{d}{dx}x^4 = 3\times 4\,x^3 = 12\,x^3$$

- Now consider part (ii) above. Using the identity

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))\frac{d}{dx}u(x)$$

with $u(x)$ differentiable then

$$\frac{d}{dx}\cos(2\,x) = -\sin(2\,x)\frac{d}{dx}2\,x = -2\sin(2\,x)$$

- Now consider part (iii) above. Using the identity

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{d}{dx}u(x)$$

with $u(x)$ differentiable then

$$\frac{d}{dx}e^{5x^2+4} = e^{5x^2+4}\frac{d}{dx}(5\,x^2+4) = 10\,x\,e^{5x^2+4}$$

- Then adding these three results gives the overall derivative

$$\begin{aligned}\frac{d}{dx}f(x) &= \underbrace{3\frac{d}{dx}x^4}_{\text{part (i)}} + \underbrace{\frac{d}{dx}\cos(2\,x)}_{\text{part (ii)}} - \underbrace{\frac{d}{dx}e^{5x^2+4}}_{\text{part(iii)}} \\ &= 12\,x^3 - 2\sin(2\,x) - (10\,x)e^{5x^2+4}\end{aligned}$$

Example

Let's consider a second example before we formulate the process of handling such functions.

Let $f(x) = \sin(2\,x)e^{\cos(x^2-5x+3)}$

This is a product of two functions that satisfy our chain rule identities; namely $\sin(2\,x)$ and $e^{\cos(x^2-5x+3)}$ as both $2\,x$ and $\cos(x^2-5\,x+3)$ are differentiable, the latter by application of the chain rule a second time.

- Then

$$f(x) = \underbrace{\sin(2\,x)}_{\text{u}} \times \underbrace{e^{\cos(x^2-5\,x+3)}}_{\text{v}}$$

and, using the product rule

$$\frac{d}{dx}f(x)=\frac{d}{dx}(\mathrm{u\,v})=\underbrace{\sin(2x)}_{\mathrm{u}}\underbrace{\frac{d}{dx}e^{\cos(x^2-5x+3)}}_{\frac{dv}{dx}}+\underbrace{e^{\cos(x^2-5x+3)}}_{\mathrm{v}}\underbrace{\frac{d}{dx}\sin(2\mathrm{x})}_{\frac{du}{dx}}$$

- Taking the derivative of $\sin(2x)$ and using
$$\frac{d}{dx}\sin(u(x))=\cos(u(x))\frac{d}{dx}u(x)$$
we get
$$\frac{du}{dx}=\frac{d}{dx}\sin(2x)=\cos(2x)\frac{d}{dx}2x=2\cos(2x)$$
- Taking the derivative of $e^{\cos(x^2-5\mathrm{x}+3)}$ and using
$$\frac{d}{dx}e^{u(x)}=e^{u(x)}\frac{d}{dx}u(x)$$
then
$$\frac{dv}{dx}=\frac{d}{dx}e^{\cos(x^2-5x+3)}=e^{\cos(x^2-5x+3)}\underbrace{\frac{d}{dx}\cos(x^2-5x+3)}_{*}$$
Now using the third chain rule identity * above becomes
$$\begin{aligned}\frac{d}{dx}\cos(x^2-5x+3) &= -\sin(x^2-5x+3)\frac{d}{dx}(x^2-5x+3)\\ &= -(2x-5)\sin(x^2-5x+3)\end{aligned}$$
and so
$$\frac{dv}{dx}=\frac{d}{dx}e^{\cos(x^2-5x+3)}=(2x-5)\sin(x^2-5x+3)e^{\cos(x^2-5x+3)}$$
- Then combining everything
$$\begin{aligned}\frac{d}{dx}f(x)=\underbrace{\sin(2x)}_{\mathrm{u}}&\underbrace{(2x-5)\sin(x^2-5x+3)e^{\cos(x^2-5x+3)}}_{\frac{dv}{dx}}\\ &+\underbrace{e^{\cos(x^2-5x+3)}}_{\mathrm{v}}\underbrace{2\cos(2\mathrm{x})}_{\frac{du}{dx}}\end{aligned}$$

This can be simplified by removing the common exponential term but expressed in this form is just as correct.

Exercise

Using the mathematical properties already discussed, the derivatives listed in your log/maths tables, the relevant differentiation rules, and the chain rule identities listed above, find the derivatives of each of the following functions:

1. $f(x)=\tan(2x^3)e^{5x-6}$
2. $f(x)=\dfrac{e^{\cos(2x)}}{\sin^2(2x)+\cos^2(2x)}$
3. $f(x)=\sin(\cos(\tan(2x-5)))$

Inverse Functions

Let $f:U\subseteq\mathbb{R}\to V\subseteq R$ have inverse function $f^{-1}:V\subseteq\mathbb{R}\to U\subseteq R$. Then, as we already know $f^{-1}\circ f(x)=x$ and $f\circ f^{-1}(x)=x$. Then we can define a second variable y such that

$$y=f^{-1}(x)\Leftrightarrow f(y)=x$$

We will use this identity to effect in dealing with the derivatives of inverse functions. The method can be summarised thus:

- Given $f^{-1}(x)$ we determine its derivative as follows
- Let $y=f^{-1}(x)$ then

$$\frac{d}{dx}y=\frac{d}{dx}f^{-1}(x)$$

- Then $x=f(y)$

$$\Rightarrow\frac{d}{dx}x=\frac{d}{dx}f(y)=\left|\frac{d}{dy}f(y)\right|_{y=f^{-1}(x)}\frac{d}{dx}y$$

$$\Rightarrow 1=\left|\frac{d}{dy}f(y)\right|_{y=f^{-1}(x)}\frac{d}{dx}f^{-1}(x)$$

$$\frac{d}{dx}f^{-1}(x)=\left|\frac{1}{\frac{d}{dy}f(y)}\right|_{y=f^{-1}(x)}$$

- The derivative on the right hand side (RHS) of this equation is evaluated on a function by function basis with the substitution made after the derivative is calculated.

We'll illustrate this with examples:

1. The natural logarithm, $\ln|x|$.
   This is the inverse function of the exponential function; i.e.

$$\ln|e^x|=e^{\ln|x|}=x$$

Then we calculate $\frac{d}{dx}\ln|x|$ as follows:
Let $y=\ln|x|$ then $e^y=x$ and

$$\frac{d}{dx}e^y=\frac{d}{dy}e^y\frac{d}{dx}y=\frac{d}{dx}x=1$$

Then as $\frac{d}{dy}e^y=e^y$ we find

$$\frac{d}{dy}e^y\frac{d}{dx}y=e^y\frac{d}{dx}y=1$$

Setting $y=\ln|x|$ we get

$$e^{\ln|x|}\frac{d}{dx}\ln|x|=1$$

$$\Rightarrow x\frac{d}{dx}\ln|x|=1$$

$$\frac{d}{dx}\ln|x|=\frac{1}{x} \quad \forall\ x>0$$

2. The inverse sine function, $\sin^{-1}(x)$.
We calculate $\frac{d}{dx}\sin^{-1}(x)$ as follows:
Let $y=\sin^{-1}(x)$ then $\sin(y)=x$ and

$$\frac{d}{dx}\sin(y)=\frac{d}{dy}\sin(y)\frac{d}{dx}y=\frac{d}{dx}x=1$$

Then as $\frac{d}{dy}\sin(y)=\cos(y)$ we find

$$\cos(y)\frac{d}{dx}y=1$$

Setting $y=\sin^{-1}(x)$ we get

$$\frac{d}{dx}y=\frac{1}{\cos(y)}$$

$$\Rightarrow\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\cos(\sin^{-1}(x))}=\frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}}$$

where we have used $\cos(A)=\sqrt{1-\sin^2(A)}$ obtained from

$$\cos^2(A)+\sin^2(A)=1$$

Then as $\sin(\sin^{-1}(x))=x \quad \forall\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\Rightarrow \frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$$

3. The inverse cosine function, $\cos^{-1}(x)$.

We calculate $\frac{d}{dx}\cos^{-1}(x)$ as follows:

Let $y=\cos^{-1}(x)$ then $\cos(y)=x$ and

$$\frac{d}{dx}\cos(y)=\frac{d}{dy}\cos(y)\frac{d}{dx}y=\frac{d}{dx}x=1$$

Then as $\frac{d}{dy}\cos(y)=-\sin(y)$ we find

$$-\sin(y)\frac{d}{dx}y=1$$

Setting $y=\cos^{-1}(x)$ we get

$$\frac{d}{dx}y=-\frac{1}{\sin(y)}$$

$$\Rightarrow \frac{d}{dx}\cos^{-1}(x)=-\frac{1}{\sin(\cos^{-1}(x))}=-\frac{1}{\sqrt{1-\cos^2(\cos^{-1}(x))}}$$

using $\sin(A)=\sqrt{1-\cos^2(A)}$. Then as

$$\cos(\cos^{-1}(x))=x \quad \forall\ x\in(0,\pi)$$

$$\Rightarrow \frac{d}{dx}\cos^{-1}(x)=-\frac{1}{\sqrt{1-x^2}}$$

4. The inverse tangent function, $\tan^{-1}(x)$.

We calculate $\frac{d}{dx}\tan^{-1}(x)$ as follows:

Let $y=\tan^{-1}(x)$ then $\tan(y)=x$ and

$$\frac{d}{dx}\tan(y)=\frac{d}{dy}\tan(y)\frac{d}{dx}y=\frac{d}{dx}x=1$$

Then as $\frac{d}{dy}\tan(y)=\sec^2(y)=1+\tan^2(y)$ we find

$$\left(1+\tan^2(y)\right)\frac{d}{dx}y=1$$

Setting $y=\tan^{-1}(x)$ we get

$$\frac{d}{dx}y=\frac{1}{1+\tan^2(y)}$$

$$\Rightarrow\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+\tan^2(\tan^{-1}(x))}=\frac{1}{1+x^2}$$

$$\Rightarrow\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$$

Implicit Differentiation

In our method for dealing with inverse functions above, it was convenient to introduce a second variable *y* and express the function in terms of both *x* and *y* where this second dependent variable *y* is implicit in the function; i.e. It is not expressed explicitly until we need to and then only to evaluate the derivative.

In mathematics in general, an implicit function is one in which the dependent variable has not been given "explicitly" in terms of the independent variable.

When a function *f* is given explicitly it provides a method (recipe) for determining how an output value of the function, *y*, is given in terms of the input value *x*: i.e. $y = f(x)$.

By contrast, the function is implicit if the value of *y* is obtained from *x* by solving an equation of the form:

$$R(x,y) = 0.$$

Then one of the two variables can be used to determine the other, but we are not given a formula that explicitly relates one variable in terms of the other. Implicit functions can often be useful in situations where it is inconvenient to solve explicitly an equation of the form $R(x,y) = 0$ for *y* in terms of *x*.

In many cases, it may be possible to rearrange this equation to obtain y as an explicit function $f(x)$. However it may not be desirable to do so since the f may have a formthat is more complicated than the form R takes.

In other situations, the equation $R(x,y) = 0$ may not define a function at all. Nevertheless, in many situations, it is still possible to work with implicit functions. Our task here is to analyse such functions using differentiation via implicit differentiation.

The method uses the chain rule in a similar manner to the way we employed the chain rule with inverse functions. Again, let's illustrate by example:

Examples:

1. Given $R(x,y)=4x+5y+5=0$ find an expression for dy/dx

$$\Rightarrow \frac{d}{dx}R(x,y)=4\frac{d}{dx}x+5\frac{d}{dx}y+5\frac{d}{dx}1=0$$

$$4+5\frac{d}{dx}y+0=0$$

$$\frac{d}{dx}y=-\frac{4}{5}$$

Of course you could have expressed the above as

$$y=-1-\frac{4}{5}x$$

and so

$$\frac{d}{dx}y=0-\frac{4}{5}\frac{d}{dx}x=-\frac{4}{5}$$

In the above example the linear relationship between x and y allows y to be explicitly given in terms of x and so it can be differentiated in a straightforward manner.

2. Let's look at a more complicated example where explicit functions do not exist.
Let $R(x,y)=2x^2y+7y^2-7=0$. An expression of this function's derivative is determined thus

$$\frac{d}{dx}R(x,y)=2\frac{d}{dx}(x^2y)+7\frac{d}{dx}(y^2)-7\frac{d}{dx}1=0$$

$$\Rightarrow 2\left(x^2\frac{d\,y}{dx}+y\frac{d}{dx}x^2\right)+7\frac{d}{dy}\left(y^2\right)\frac{d\,y}{dx}-0=0$$

$$\Rightarrow 2\left(x^2\frac{d\,y}{dx}+2\,x\,y\right)+14\,y\frac{d\,y}{dx}=0$$

$$\Rightarrow 2\,x^2\frac{d\,y}{dx}+4\,x\,y+14\,y\frac{d\,y}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2\,x\,y}{x^2+7\,y}$$

Exercise:

Use implicit differentiation to find an expression for $d\,y/dx$.

1. $R(x\,,y)=4\,x^3\,y^2+6\,x\,y^2=4$
2. $R(x\,,y)=x^3\,y-8\,y\,x^5+8\,y=0$
3. $R(x\,,y)=x\,y^4+2\,x^6\,y^2-7\,x^3=9$
4. $R(x\,,y)=e^{x^6\,y^2}-3\cos(x\,y^2)=0$