Section 3(Cont'd)
Taylor's Theorem

In this section we consider the implications of Taylor's theorem on the practical application of using a Taylor Series to approximate a function $f:U\subseteq\mathbb{R}\rightarrow V\subseteq\mathbb{R}$ about some chosen $x_0\in U$. We need to define the Taylor Polynomial of $f(x)$ first before we can state the theorem (without proof) and show how it can be used to approximate a function.

Taylor Polynomials

The n^th order Taylor Polynomial of a continuously differentiable function[1] is denoted $T_n(x)$ and is defined to be the finite polynomial

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_o)}{k!}\left(x-x_0\right)^k$$

for some suitably chosen $x_0\in U$ (the domain of f).

Taylor's Theorem

Let $f:U\subseteq\mathbb{R}\rightarrow V\subseteq\mathbb{R}$ be a continuously differentiable function of a real variable x at some $x_0\in U$ and for $x\in\left(a,b\right)\Leftrightarrow x_0=(a+b)/2$. Then $f(x)$ admits a Taylor series expansion about x_0

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_o)}{k!}\left(x-x_0\right)^k$$

Let $T_n(x)$ be the n^th order Taylor Polynomial of $f(x)$. Then we define the error in this polynomial, $E_n(x)$, to be

$$E_n=|f(x)-T_n(x)|=\frac{A_{n+1}}{(n+1)!}\left(x-x_0\right)^{n+1}$$

where

$$A_{n+1}=f^{(n+1)}(\xi)\quad \xi\in\left(x-x_0,x+x_0\right)$$

Corollary

Clearly this theorem has two important outcomes:

1. The more terms we include in (the higher the order n of) the Taylor Polynomial, $T_n(x)$, the smaller the error $E_n(x)$. In other words

1 A continuouslydifferentiable function, $f(x)$, is one whose first order and higher order derivtaives are finiteat some particular $x_0\in U$ where U is the domain of f.

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} \frac{A_{n+1}}{(n+1)!}\left(x-x_0\right)^{n+1} \sim \lim_{n\to\infty} \frac{1}{(n+1)!} = 0$$

2. The closer x is to x_0, the smaller the error. This is obvious as the difference term $\left(x-x_0\right)^{n+1} \to 0$ as x tends to x_0. Thus
$$\lim_{x\to x_0} E_n(x) = \lim_{x\to x_0} \frac{A_{n+1}}{(n+1)!}\left(x-x_0\right)^{n+1} \sim \lim_{\Delta x\to 0} \left(\Delta x\right)^{n+1} = 0$$
where $\Delta x = x - x_0$.

As with all our new concepts, it is easier to visualise the above results after considering a few examples.

Example 1: $f(x)=e^x$ about $x_0=0$

From the section on Taylor Series we know that the exponential function has the following Taylor Series expansion about $x_0=0$

$$f(x)=e^x=\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Therefore the first five Taylor Polynomials take on the forms

$$T_1(x)=\sum_{k=0}^{1} \frac{x^k}{k!}=1+x$$
$$T_2(x)=\sum_{k=0}^{2} \frac{x^k}{k!}=1+x+\frac{x^2}{2!}$$
$$T_3(x)=\sum_{k=0}^{3} \frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$
$$T_4(x)=\sum_{k=0}^{4} \frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$
$$T_5(x)=\sum_{k=0}^{5} \frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$$

Now that we have the first five Taylor Polynomials for this function, what remains is to show how accurate these functions are in the neighbourhood of the chosen $x_0=0$.

To this end we have chosen an interval of [-2,2] over which to evaluate the polynomials and the functions, estimate the errors, and to plot the resulting functions and errors to illustrate the efficacy of the theorem.

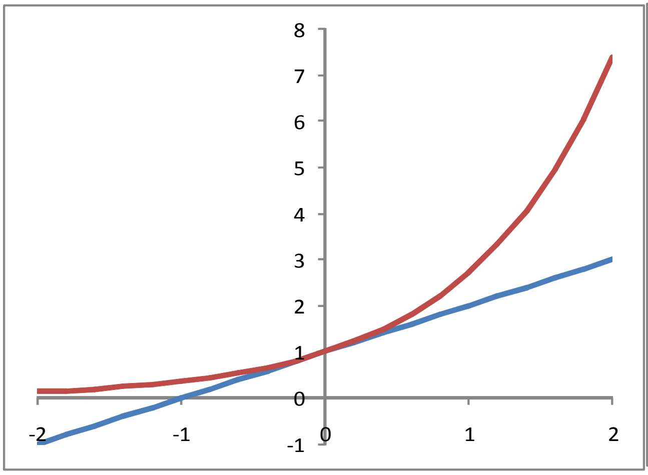
x	$T_1(x)$	$T_2(x)$	$T_3(x)$	$T_4(x)$	$T_5(x)$	e^x
-2	-1	1	-0.33333	0.33333	0.06667	0.13534
-1.8	-0.8	0.82	-0.15200	0.28540	0.12794	0.16530
-1.6	-0.6	0.68	-0.00267	0.27040	0.18302	0.20190
-1.4	-0.4	0.58	0.12267	0.28273	0.23791	0.24660
-1.2	-0.2	0.52	0.23200	0.31840	0.29766	0.30119
-1	0	0.5	0.33333	0.37500	0.36667	0.36788
-0.8	0.2	0.52	0.43467	0.45173	0.44900	0.44933
-0.6	0.4	0.58	0.54400	0.54940	0.54875	0.54881
-0.4	0.6	0.68	0.66933	0.67040	0.67031	0.67032
-0.2	0.8	0.82	0.81867	0.81873	0.81873	0.81873
0	1	1	1.00000	1.00000	1.00000	1.00000
0.2	1.2	1.22	1.22133	1.22140	1.22140	1.22140
0.4	1.4	1.48	1.49067	1.49173	1.49182	1.49182
0.6	1.6	1.78	1.81600	1.82140	1.82205	1.82212
0.8	1.8	2.12	2.20533	2.22240	2.22513	2.22554
1	2	2.5	2.66667	2.70833	2.71667	2.71828
1.2	2.2	2.92	3.20800	3.29440	3.31514	3.32012
1.4	2.4	3.38	3.83733	3.99740	4.04222	4.05520
1.6	2.6	3.88	4.56267	4.83573	4.92311	4.95303
1.8	2.8	4.42	5.39200	5.82940	5.98686	6.04965
2	3	5	6.33333	7.00000	7.26667	7.38906

Table: e^x vs $T_1(x), T_2(x), T_3(x), T_4(x), T_5(x)$ on $[-2,2]$

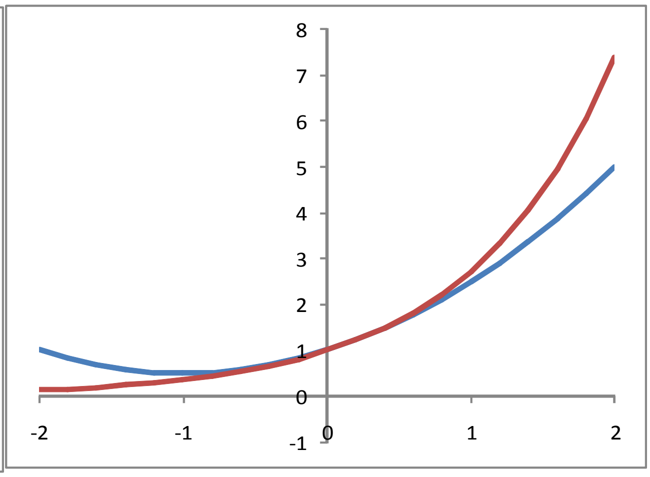
The colours in the table above are to highlight the accuracy of the polynomial estimates. The key is as follows:

- If the cell is green then the Taylor Polynomial value contained therein is within 0.01 in absolute value of the actual function.
- If the cell is highlighted in yellow then polynomial value is between 0.01 and 0.1 (again in absolute terms) of the actual function value.
- If highlighted in red then the polynomials value has a difference that

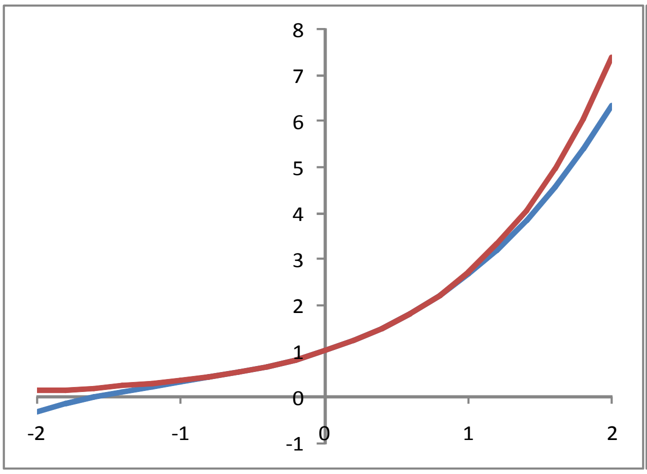
is greater than 0.1 in absolute terms from the actual function value. The choice of values is arbitrary and it could be argued that a percentage or ratio based scoring would be more appropriate; the above is not intended to be definitive but more heuristic.



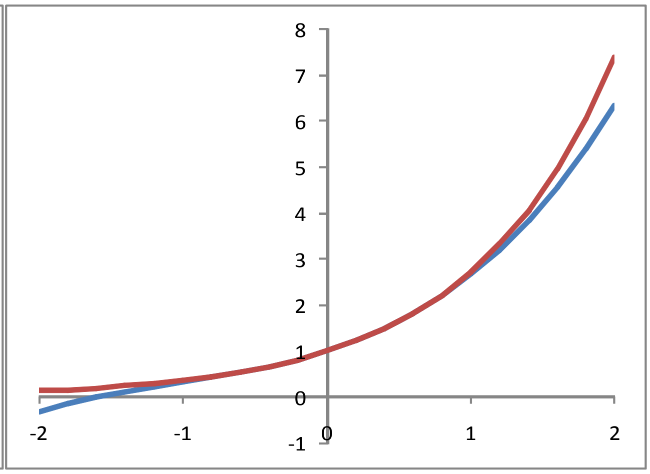
(a) e^x vs $T_1(x)$ on $[-2,2]$



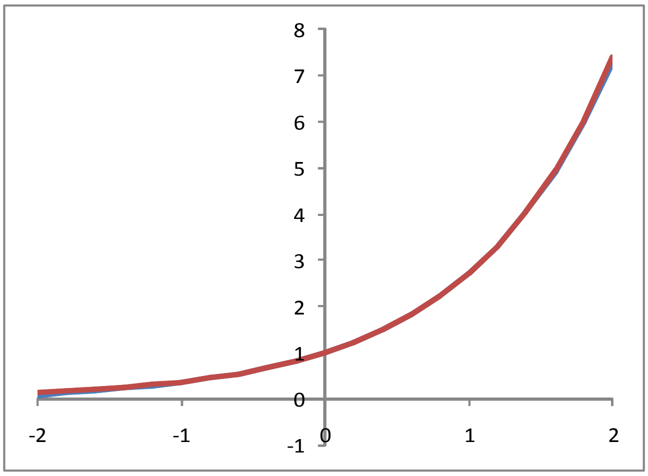
(b) e^x vs $T_2(x)$ on $[-2,2]$



(c) e^x vs $T_3(x)$ on $[-2,2]$



(d) e^x vs $T_4(x)$ on $[-2,2]$



(e) e^x vs $T_5(x)$ on $[-2,2]$

Table: Plots showing the differences between the first five Taylor Polynomials of e^x and e^x on $[-2,2]$

We tabulated above some graphs showing the differences between these first five Taylor Polynomials and the original e^x on $[-2,2]$. The corresponding errors, $E_n(x)=|e^x-T_n(x)|$, are shown collectively below:

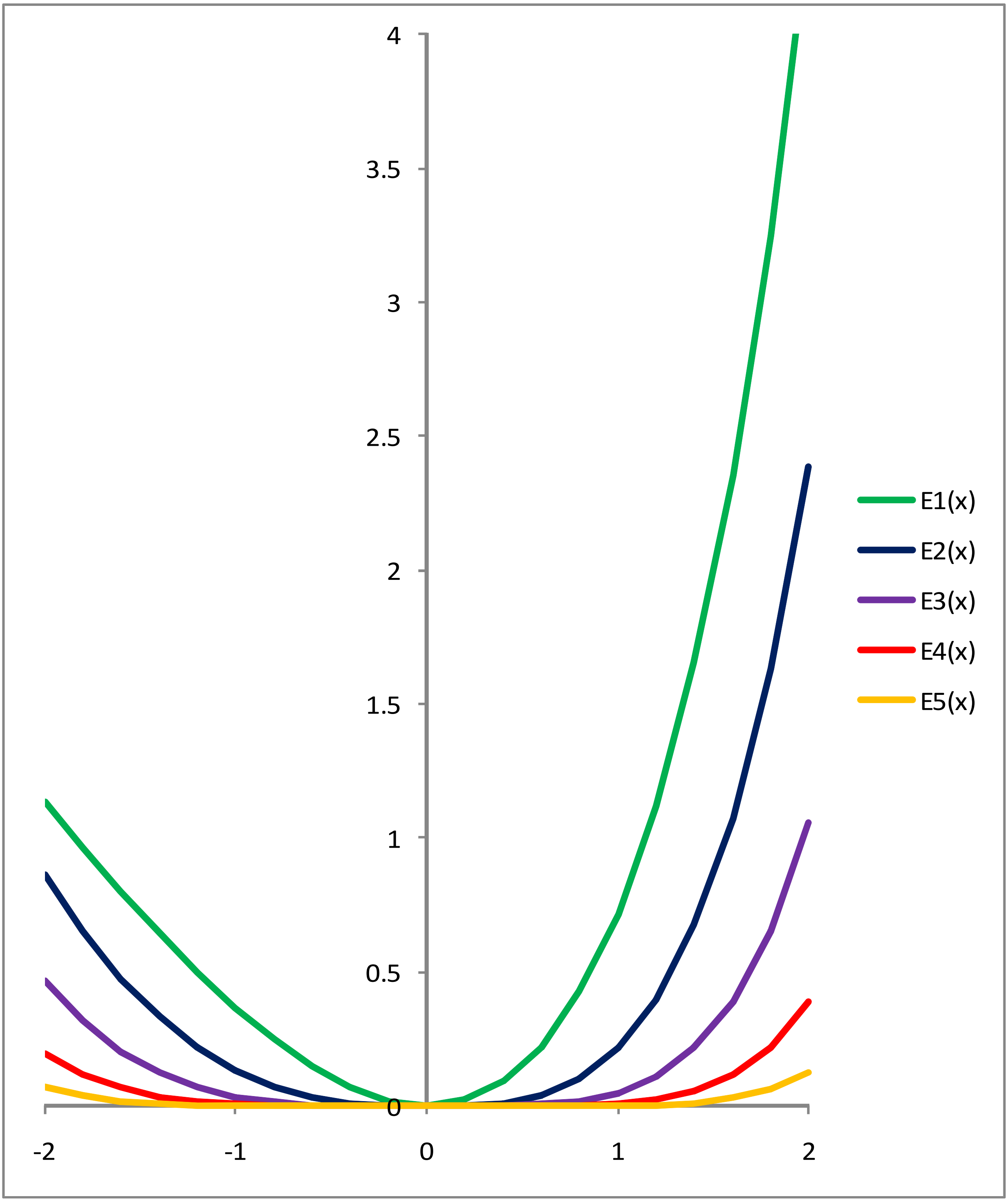


Figure: Errors, $E_n(x)$, in the Taylor Polynomial approximations, $T_n(x)$, to e^x on $[-2,2]$

It should be fairly clear that the error is significantly less for the higher order polynomials than for the lower order ones. This holds true for any function that admits a Taylor Series representation.

Example 2: $\boldsymbol{f(x)=\cos(x)}$ **about** $\boldsymbol{x_0=\pi/2}$

From the section on Taylor Series we know that the Taylor Series about $x_0=\pi/2$ is given by

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}}{k!}\left(x-\frac{\pi}{2}\right)^k$$

and that $\cos(x)$ is periodic under differentiation:

$$\begin{array}{ccccccccccc}
f(x) & = & f^{(4)}(x) & = & \dots & = & f^{(4n)}(x) & = & \dots & = & \cos(x)\\
f^{(1)}(x) & = & f^{(5)}(x) & = & \dots & = & f^{(4n+1)}(x) & = & \dots & = & -\sin(x)\\
f^{(2)}(x) & = & f^{(6)}(x) & = & \dots & = & f^{(4n+2)}(x) & = & \dots & = & -\cos(x)\\
f^{(3)}(x) & = & f^{(7)}(x) & = & \dots & = & f^{(4n+3)}(x) & = & \dots & = & \sin(x)
\end{array}$$

which evaluate at $x_0=\pi/2$ to

$$\begin{array}{ccccccccccc}
f(\pi/2) & = & f^{(4)}(\pi/2) & = & \dots & = & f^{(4n)}(\pi/2) & = & \dots & = & 0\\
f^{(1)}(\pi/2) & = & f^{(5)}(\pi/2) & = & \dots & = & f^{(4n+1)}(\pi/2) & = & \dots & = & -1\\
f^{(2)}(\pi/2) & = & f^{(6)}(\pi/2) & = & \dots & = & f^{(4n+2)}(\pi/2) & = & \dots & = & 0\\
f^{(3)}(\pi/2) & = & f^{(7)}(\pi/2) & = & \dots & = & f^{(4n+3)}(\pi/2) & = & \dots & = & 1
\end{array}$$

Therefore the first four Taylor Polynomials of odd order[2] take on the forms

$$T_1(x)=\sum_{k=0}^{1}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k=-\left(x-\frac{\pi}{2}\right)$$

$$T_3(x)=\sum_{k=0}^{3}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3$$

$$T_5(x)=\sum_{k=0}^{5}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{120}\left(x-\frac{\pi}{2}\right)^5$$

$$T_7(x)=\sum_{k=0}^{7}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{120}\left(x-\frac{\pi}{2}\right)^5+\frac{1}{5040}\left(x-\frac{\pi}{2}\right)^7$$

2 Why do we restrict the polynomials this way?

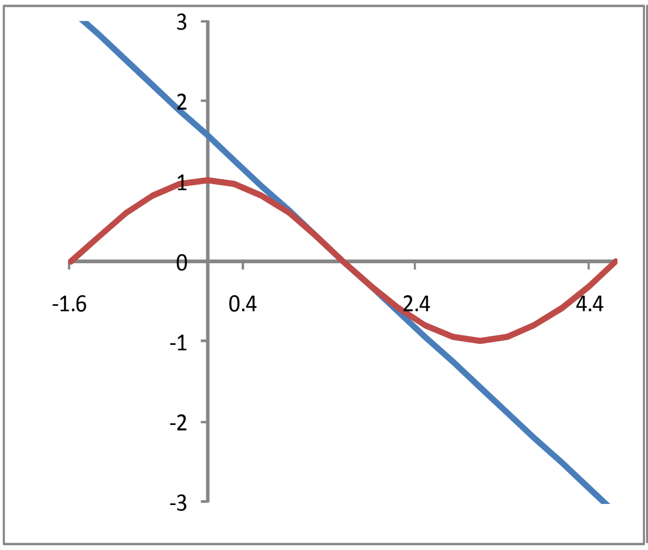
Now that we have the first four odd indexed Taylor Polynomials for this function, what remains is to show how accurate these functions are in the neighbourhood of the chosen $x_0 = \pi/2$.

To this end we have chosen an interval of $[-\pi/2, 3\pi/2]$ over which to evaluate the polynomials against the function and to estimate the errors.

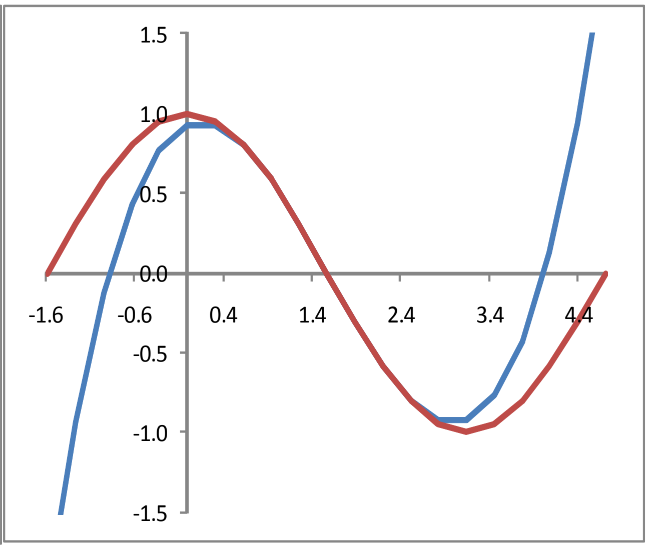
x	$T_1(x)$	$T_3(x)$	$T_5(x)$	$T_7(x)$	$\cos(x)$
-1.57079	3.14159	-2.02612	0.52404	-0.07522	0.00000
-1.25664	2.82743	-0.93983	0.56602	0.27939	0.30902
-0.94248	2.51327	-0.13259	0.70304	0.57737	0.58779
-0.62832	2.19912	0.42659	0.85520	0.80584	0.80902
-0.31416	1.88496	0.76873	0.96703	0.95025	0.95106
0.00000	1.57079	0.92483	1.00452	0.99984	1.00000
0.31416	1.25664	0.92590	0.95202	0.95104	0.95106
0.62832	0.94248	0.80295	0.80915	0.80902	0.80902
0.94248	0.62832	0.58698	0.58779	0.58779	0.58779
1.25664	0.31416	0.30899	0.30902	0.30902	0.30902
1.57079	0.00000	0.00000	0.00000	0.00000	0.00000
1.88496	-0.31416	-0.30899	-0.30902	-0.30902	-0.30902
2.19912	-0.62832	-0.58698	-0.58779	-0.58779	-0.58779
2.51327	-0.94248	-0.80295	-0.80915	-0.80902	-0.80902
2.82743	-1.25664	-0.92590	-0.95202	-0.95104	-0.95106
3.14159	-1.57079	-0.92483	-1.00452	-0.99984	-1.00000
3.45575	-1.88496	-0.76873	-0.96703	-0.95025	-0.95106
3.76991	-2.19911	-0.42659	-0.85520	-0.80584	-0.80902
4.08407	-2.51327	0.13259	-0.70304	-0.57737	-0.58779
4.39823	-2.82743	0.93983	-0.56602	-0.27939	-0.30902
4.71239	-3.14159	2.02612	-0.52404	0.07522	0.00000

Table: $\cos(x)$ vs $T_1(x), T_3(x), T_5(x), T_7(x)$ on $[-\pi/2, 3\pi/2]$

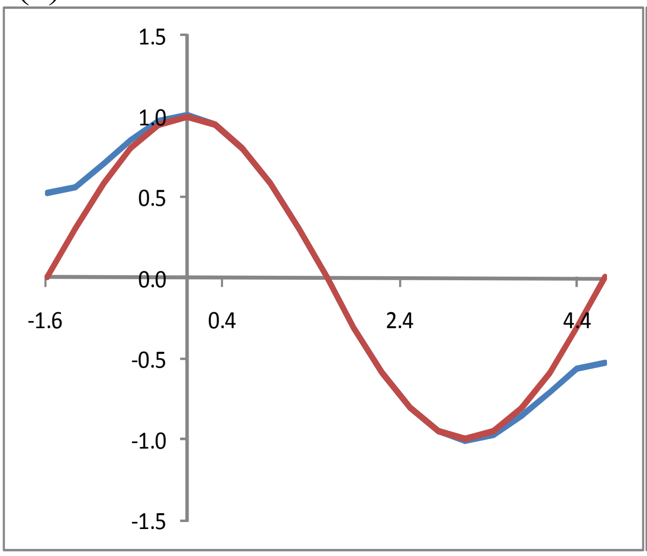
The colours in the table above are as before. Just as for the exponential function, we have tabulated graphs showing the differences between these Taylor Polynomials and the original $\cos(x)$ on $[-\pi/2, 3\pi/2]$.



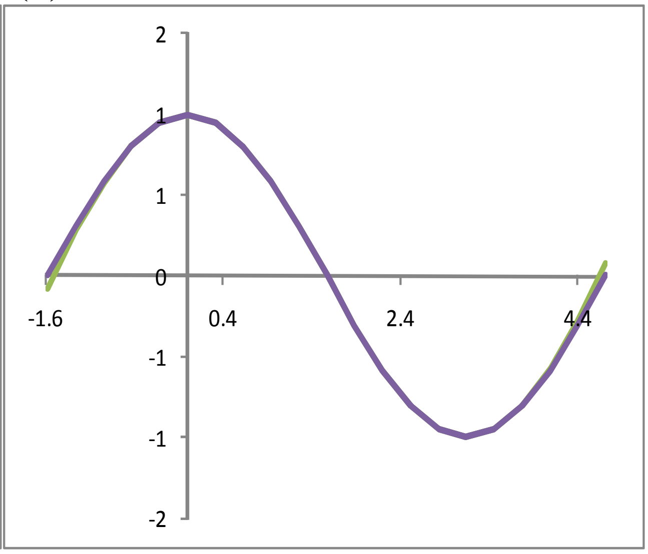
(a) $\cos(x)$ vs $T_1(x)$ on $[-\pi/2, 3\pi/2]$



(b) $\cos(x)$ vs $T_3(x)$ on $[-\pi/2, 3\pi/2]$



(c) $\cos(x)$ vs $T_5(x)$ on $[-\pi/2, 3\pi/2]$



(d) $\cos(x)$ vs $T_7(x)$ on $[-\pi/2, 3\pi/2]$

Table: Plots showing the differences between the first four odd Taylor Polynomials of $\cos(x)$ and $\cos(x)$ on $[-\pi/2, 3\pi/2]$

The corresponding errors, $E_n(x) = |\cos(x) - T_n(x)|$, are shown collectively overleaf below. The graph clearly shows that, as for the exponential function, the error in the Taylor Polynomial approximation decreases as the order of the polynomial increases. Furthermore, the graph shows that the approximations are very accurate in the neighbourhood of the choice of x_0 ($\pi/2$ in our example) and diverge at rates dependent on the order of the Taylor Polynomial.

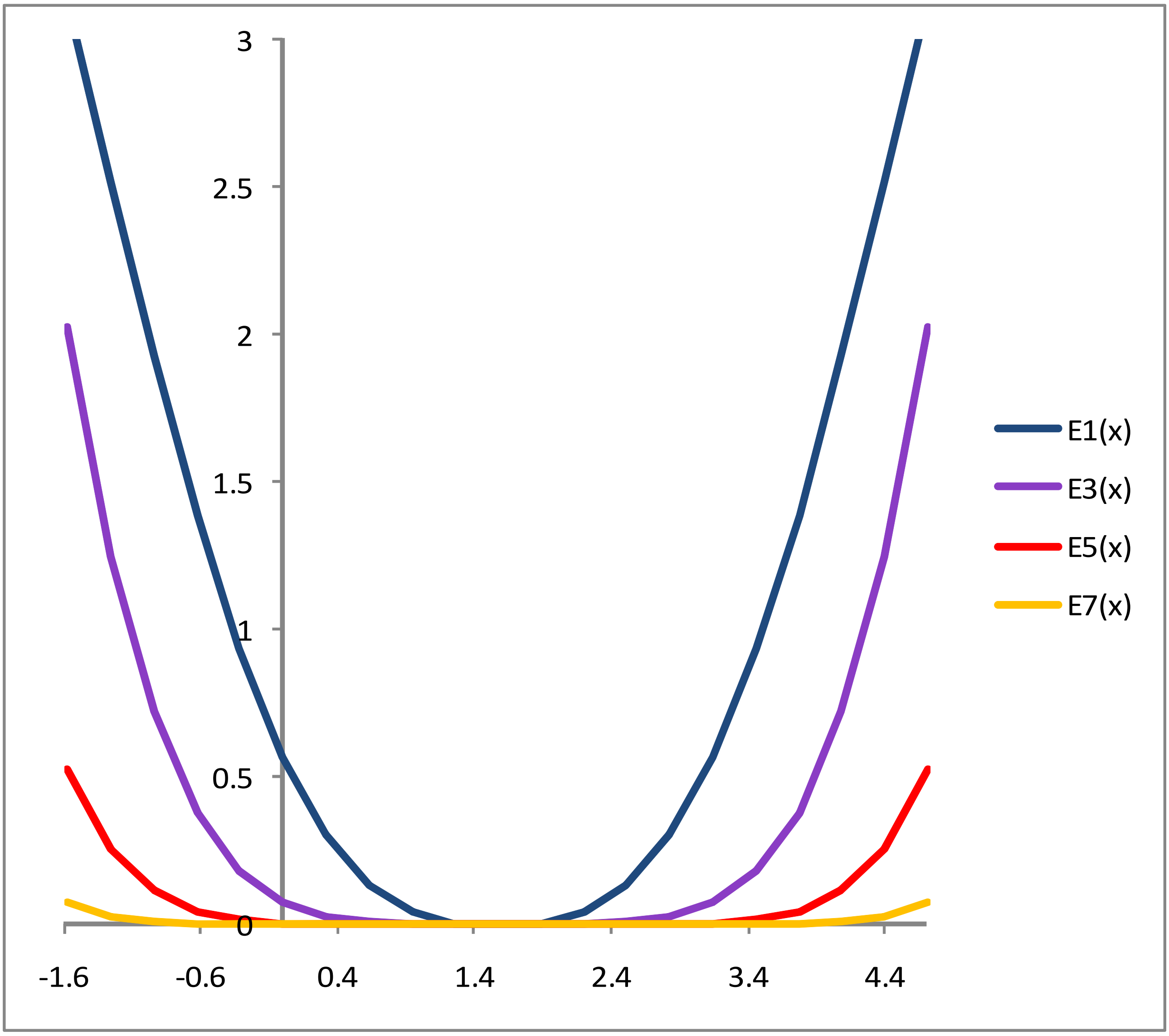


Figure: Errors, $E_n(x)$, in the Taylor Polynomial approximations, $T_n(x)$, to $\cos(x)$ on $[-\pi/2, 3\pi/2]$

Exercise:

Repeat the above analysis for the following functions about the points specified:

1. $f(x) = \ln|x|$ with $x_0 = 1$
2. $f(x) = \tan(x)$ with $x_0 = 0$
3. $f(x) = e^{3x}$ with $x_0 = 2$