Section 1 (Cont'd)
Limit of a Function, Continuity

We have almost completed our introduction to functions, their graphs, their properties, and their basic analysis. What we aim to tackle here are the concepts behind the limit of a function and what is meant when we say a function is continuous.

What is the limit of a function?
The above title is misleading in that it should more formally be written as the limit of a function as x tends to some chosen value x_o. So what does this phrase mean and what use is it to us in the analysis of a function's behaviour?
A limit is essentially a method of determining how the function being considered is behaving as the independent variable, x, is approaching (but not actually reaching) a chosen value, x_o. We denote the limit by lim with the value and variable fixed below the lim: i.e.
Given $f: U \subseteq \mathbb{R} \rightarrow V \subseteq \mathbb{R}$

$$\lim_{x \rightarrow x_0} f(x) := \text{The limit of } f(x) \text{ as } x \text{ approaches } x_0 \in U$$

It should be noted that the limit is determining the *trend* of $f(x)$ as its variable x *approaches* x_o and not necessarily what $f(x)$ is doing when $x=x_o$. This may appear at first glance to be a case of "hair splitting" but a few examples will hopefully clarify the situation.

Example:
Construct a table for the function $f(x)=1/x \ \ \forall x \in \mathbb{R} \setminus \{0\}$ to determine the trend of $f(x)$ as x approaches 0.

x	$1/x$		x	$1/x$
-1	-1		1	1
-0.1	-10		0.1	10
-0.01	-100		0.01	100
-0.001	-1000		0.001	1000
-0.0001	-10000		0.0001	10000

Table: $f(x)$ for negative and positive values of x as $x \rightarrow 0$.

We know that $f(x)$ is not defined when $x=0$ but we can examine its behaviour as x approaches 0. As x gets closer to 0 the function is tending to either a very large negative number, when approaching 0 from below, or a very large positive number, when approaching 0 from above. This gives rise to two limits;

- one for the negative x $\lim_{x \to x_0^-} f(x)$
- one for the positive x $\lim_{x \to x_0^+} f(x)$

They do not converge to the same value and therefore both limits have to be treated separately. We can see from the table above that

$$\lim_{x \to x_0^-} f(x) = -\infty \text{ and } \lim_{x \to x_0^+} f(x) = \infty$$

This is evident from the graph of $f(x)$ below.

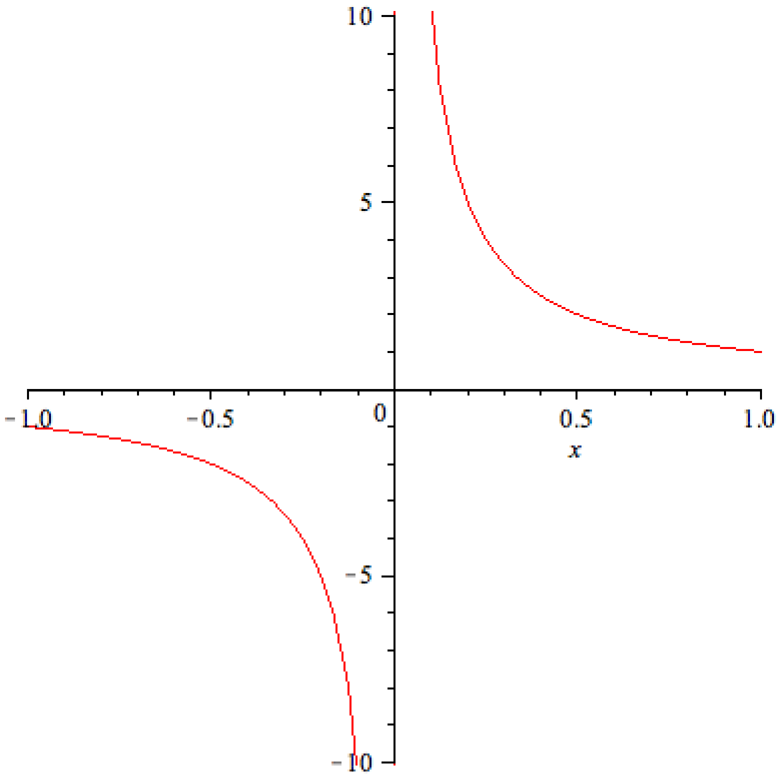

Figure: The graph of $f(x)=1/x$ in the neighbourhood of 0.

Two observations from the above graph:

1. The limits are not tending to the same value; they are diverging.
2. The graph is not defined at $x=0$. It cannot be said to have a single singularity because the direction the function takes to approach zero determines its divergence.

As we'll see later we have an additional name for such functions.

Example

Consider the function $f(x)=1/(x-1)^2 \quad \forall x \in \mathbb{R} \setminus \{1\}$. Determine the trend of $f(x)$ as x approaches 1.

x	$1/(x-1)^2$		x	$1/(x-1)^2$
0	1		2	1
0.9	100		1.1	100
0.99	10000		1.01	10000
0.999	1000000		1.001	1000000
0.9999	100000000		1.0001	100000000

Table: $f(x)$ for adjacent values of x as $x \to 1$.

Both limits $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ are tending to the same value; we can see this from the table above. The values of x each side of 1 are equal if they are equi-distant from 1. Then when x=1.01 and 0.99 the answers are the same because both are 0.01 from 1. This is evident from the graph below of $f(x)$ in [-1,3].

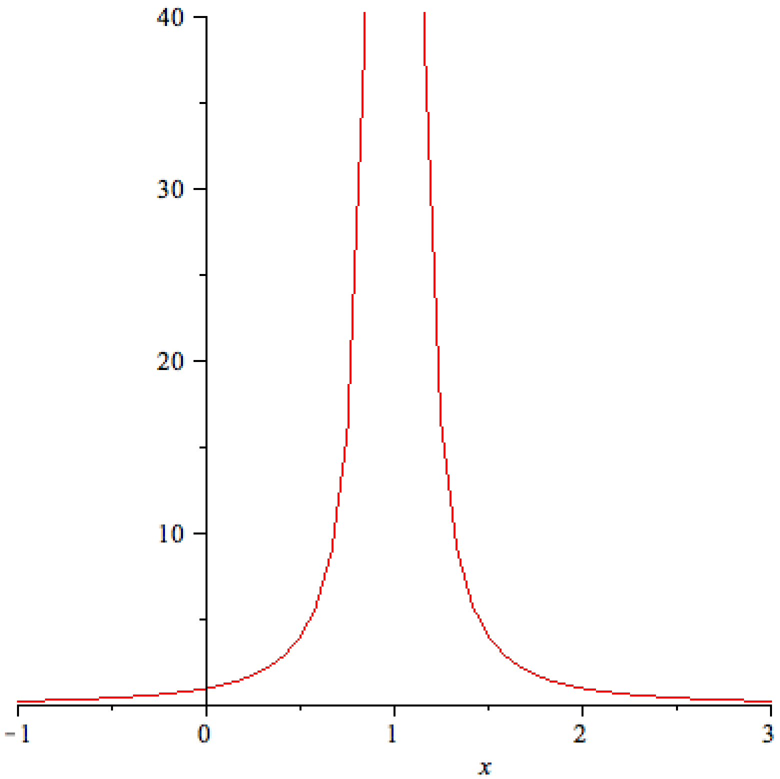

Figure: The function $f(x)=1/(x-1)^2$ on $[-1,3]$

There is a singularity at x=1; this is evident from the function on both sides of the singularity tending in the same direction. Sometimes such a singularity is called a pole for obvious reasons.

You can see from our approach to this problem (forgive the pun) that we determine the behaviour of the function, $f(x)$, under consideration by tabulating the values of $f(x)$ as x gets closer to the point of interest from both sides. This should clarify how the function approaches the limit under consideration.

Some mathematics texts list "Rules of Thumb" or give heuristic arguments on how to deal with limits in general. I find these confuse rather than clarify and so the only approach we'll consider here is the tabulation of x and $f(x)$ in the vicinity of the point x_o to be considered. We'll finish this section with a much quoted limit:

Example

Though we have not dealt with trigonomettric functions yet, we'll consider this function here because it has a very special property.

Calculate the following limit:

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

The figure below shows the function in the interval [-15,15]

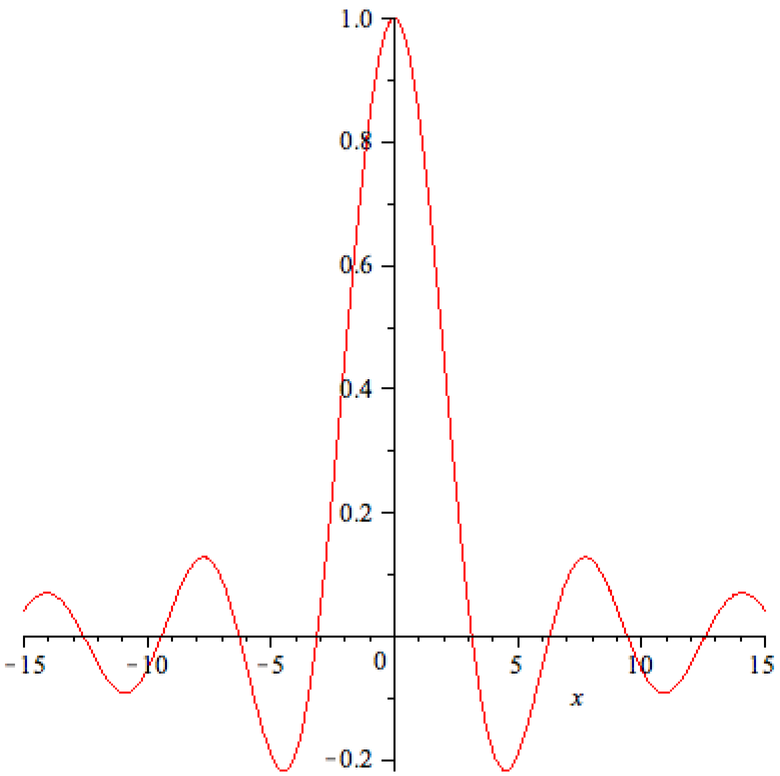

Figure: The graph of $f(x) = \frac{\sin(x)}{x}$ in the interval [-15,15]

The Table below shows the trend as $x \to 0$ more clearly.

x	**sin(x)/x**		**x**	**sin(x)/x**
-1	0.8414709...		1	0.8414709...
-0.1	0.9983341...		0.1	0.9983341...
-0.01	0.9999833...		0.01	0.9999833...
-0.001	0.9999998...		0.001	0.9999998...
-0.0001	0.999999998...		0.0001	0.999999998...

Table: $f(x)$ for negative and positive values of x as $x \to 0$.

It is obvious from the figure and table that the function is tending to a value of +1 and so from the trend we can state that the limit as $x \to 0$ is 1 for this function: i.e.

$$\lim_{x \to 0^-} \frac{\sin(x)}{x} = \lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

What about the actual value of 0?

$$f(0) = \frac{\sin(0)}{0} = \frac{0}{0} \text{ undefined}$$

So we have a dilemma:

- The function appears finite everywhere.
- We can plot it easily without resorting to singular scales.

HOWEVER we cannot assign the function a value when x=0. The point corresponding to x=0 is a singularity but a special one that doesn't advertise its existence with nearby divergent behaviour (as for the $1/(x-1)^2$ above). The graph (though appearing to be in one piece) is in fact broken into two parts; one for the positive x and one for the negative.

There is a specific name given to functions whose graphs cannot be drawn in one piece; discontinuous. Correspondingly those that can be drawn in one piece are termed continuous.

Continuity and Continuous Functions

A function $f : U \subseteq \mathbb{R} \to V \subseteq \mathbb{R}$ is deemed continuous at a point (x_o, $f(x_o)$) where $x_0 \in U$ if and only if the following holds:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

Simply put a function $f(x)$ is continuous at $x = x_0$ if the graph of $f(x)$ can be drawn in a single curve in the vicinity of the point $(x_o, f(x_o))$:

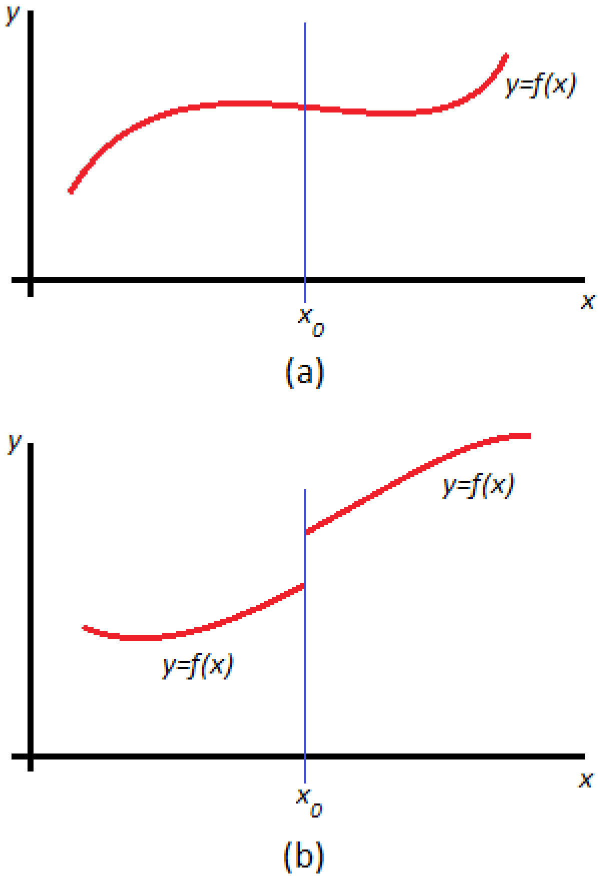

Figure: (a) A continuous function and (b) a discontinuous function at some $x_0 \in U$ where $f: U \rightarrow V$.

Functions that have singular points, such as those in the previous section, are naturally discontinuous. A function that is continuous and finite on an interval is a "nice" function. A function that has singularities, fails to remain finite over an interval, and behaves unpredictably is euphemistically termed "patholgoical".

Exercise:

Using the tools,methods, concepts, etc. covered to date, determine which of the following functions are continuous or discontinuous.

1. $f(x) = \dfrac{\sin(x)}{x}$ on (0,5]
2. $f(x) = \dfrac{2x-3}{x-4}$ on [1,6]
3. $f(x) = \dfrac{x^2-x-2}{x-2}$ on [-1,4]