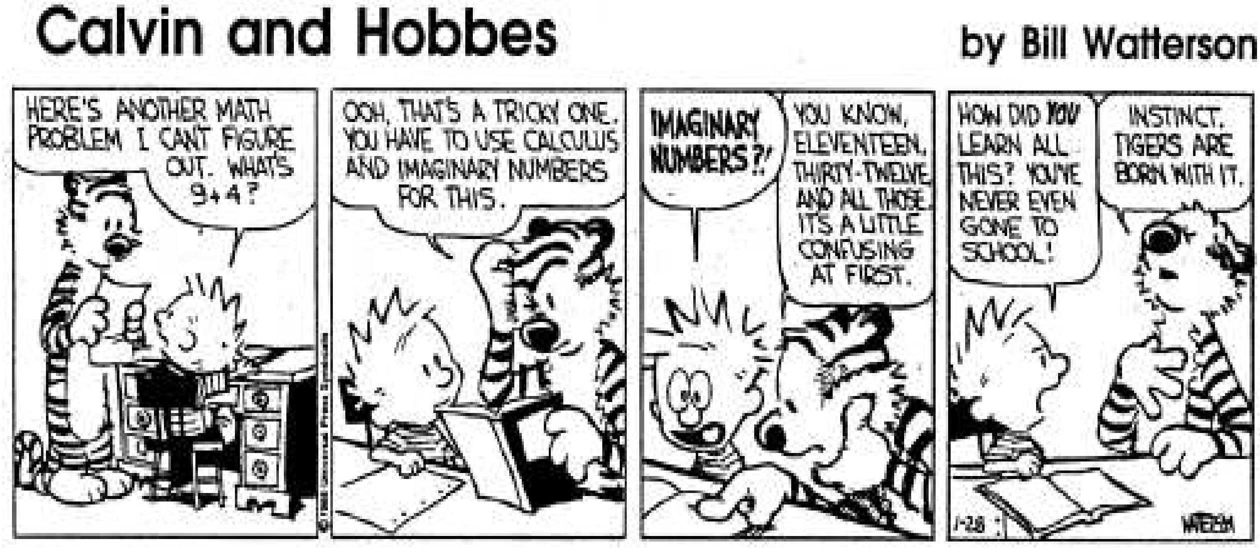
Section 7
An Introduction to Complex Numbers

In this section we'll consider that final set of numbers we defined in our first lectures; namely the set of Complex Numbers, $\mathbb{C}$. All complex number, z, are defined to be of the type

$$z = a + b\,j \quad \forall\, z \in \mathbb{C}$$

where a and b are real numbers and $j = \sqrt{-1}$. Whereas Natural numbers, Integers, Rationals, Irrationals, and Real numbers all have some meaning to you, it can be difficult to reconcile the need for such numerical entities as complex numbers. Indeed, the reasons for them existing in modern mathematics, at all, is an interesting tale but is rarely told.



To this end, we'll commence this section with a historical review of the people and ideas that ultimately led to the development and formal definition of Complex Numbers over a period of three hundred years.

Jerome Cardan (Girolamo Cardano) (1501-1576)

In 1545 Jerome Cardan, an Italian mathematician, physician, gambler, and philosopher published a book called *Ars Magna* (*The Great Art*). In it he described a procedure for solving, algebraically, cubic and quartic equations. He also posed an interesting problem in quadratics:

> *If some one says to you, divide 10 into two parts, one of which multiplied into the other shall produce...40, it is evident that this case or question is impossible. Nevertheless, we shall solve it in this fashion.*

In effect, Cardan applied the method of completing the square to

$$x + y = 10 \quad \text{and} \quad xy = 40$$

i.e.

$$y = 10 - x \quad \Rightarrow xy \equiv x(10 - x) = 40$$
$$\Rightarrow x^2 - 10x + 40 = 0$$

to get the zeroes,

$$5 + \sqrt{-15}, 5 - \sqrt{-15}$$

such that if

$$x = 5 + \sqrt{-15} \text{ and } y = 5 - \sqrt{-15}$$
$$\begin{aligned} \Rightarrow x + y &= 5 + \sqrt{-15} + 5 - \sqrt{-15} &= 10 \\ xy &= (5 + \sqrt{-15})(5 - \sqrt{-15}) &= 25 - (-15) = 40 \end{aligned}$$

After that, he concluded that the result was "*as subtle as it is useless*." Actually it was not from this problem that complex numbers were noted but from Cardan's own formula for solving cubic equations of the form

$$x^3 = \alpha x + \beta$$

given by[1]

$$x = \sqrt[3]{\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - \left(\frac{\alpha}{3}\right)^3}} + \sqrt[3]{\frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 - \left(\frac{\alpha}{3}\right)^3}}$$

When this was applied to the classic cubic

$$x^3 = 15x + 4,$$

the formula returns the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Cardan claimed that this general formula wasn't applicable for this equation (because of the presence of the $\sqrt{-121}$). It was clear that more analysis was required especially given that the cubic actually has three real roots: $x = 4$ and $x = -2 \pm \sqrt{3}$. So we take a leap of thirty years to Bombelli and his analysis of Cardan's problem.

Rafael Bombelli (1526-1572)

Almost thirty years after the *Ars Magna* was published, Bombelli justified the use of Cardan's formula by introducing numbers of the form

$$\alpha + \sqrt{-\beta} \quad \forall \alpha, \beta \in \mathbb{R}$$

and by applying the standard rules of algebra to them. In this case, Bombelli conjectured that, since the irrational conjugate pair

$$2 + \sqrt{-121}, 2 - \sqrt{-121}$$

1 An excellent treatment of the derivation of this can be found in "The Roots of Complex Numbers," Katz, Victor J., *Math Horizons*, Nov. 1995.

differed only in the sign before the square root, the same might be true of their cube roots. He set

$$\begin{aligned} 2+\sqrt{-121} &= \left(\alpha+\sqrt{-\beta}\right)^3 \\ 2-\sqrt{-121} &= \left(\alpha-\sqrt{-\beta}\right)^3 \end{aligned}$$

and went about solving for α and β. He got α = 2 and β = 1, thus showing that

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}=\left(2+\sqrt{-1}\right)+\left(2-\sqrt{-1}\right)=4$$

as desired. This method gave him a correct solution to the equation and convinced Bombelli that his ideas about complex numbers were valid. Thus, Bombelli laid the groundwork for future work on complex numbers and went on to develop some of the rules for complex numbers including problems involving addition and multiplication of complex numbers.

Post-Bombelli

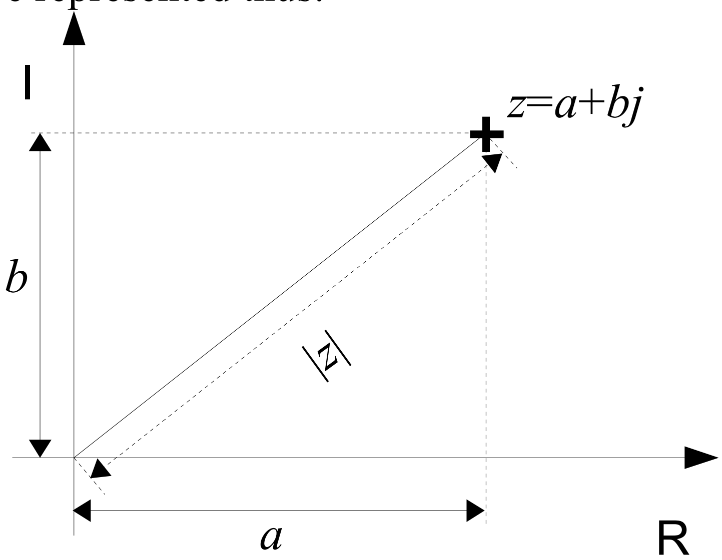
For years after Bombelli's work, many still considered complex numbers to be nothing more than an oddity. Notable others, however, used complex numbers extensively and as a consequence much was discovered.

- **Albert Girard (1595-1632)** suggested that an equation may have as many roots as its degree in 1620.
- **Rene Descartes (1596-1650)**, contributed the term "imaginary" to these numbers. He said that one can imagine that every equation has as many roots as its degree however real numbers may not correspond to all of these imagined roots.
- **Gottfried Wilhelm von Liebnitz (1646-1716)** spent much time applying the laws of algebra to complex numbers.
- Liebnitz and **Johann Bernoulli (1667-1748)** used imaginary numbers as integration aids.
- **Leonhard Euler (1707-1783)** was the first to denote the square root of -1 with the symbol *i*.
- **Johann Carl Friedrich Gauss (1777-1855)** published the first correct proof of the fundamental theorem of algebra in his doctoral thesis of 1797. By 1831 he published his work on the geometric representation of complex numbers as points in the plane.
- **William Rowan Hamilton (1805-1865)** expressed complex numbers as pairs of real numbers (a,b) in 1833.

Developments in complex numbers continued after this. Work on them led to the fundamental theorem of algebra and a branch of mathematics called complex function theory. They are used in quantum mechanics, electro-magnetism and electric circuitry. What many once believed impossible, ridiculous, and even fictitious, has become a reality.

Notation

1. We initially represent any complex number, z, by the pair
$$z=a+bj$$
where $a,b\in\mathbb{R};\ j=\sqrt{-1}$. We use *j* instead of the more usual *i* for the square root of -1 because engineers use *i* for representing *ac* current in electronics.
2. The real part of a complex number $z=a+bj$, denoted $\Re(z)$ is
$$\Re(z)=a$$
3. The imaginary part of a complex number $z=a+bj$, denoted $\Im(z)$ is
$$\Im(z)=b$$
4. We define the *complex conjugate*, $\bar{z}$, of $z=a+bj$;
$$\bar{z}=a-bj$$
5. The norm, magnitude or length, of a complex number, *z*, denoted $|z|$, is defined
$$|z|=\sqrt{a^2+b^2}$$
6. The complex number *z* can be represented graphically on a 2-D plane where the real part is measured along the horizontal axis and the imaginary part along the vertical axis. Then, for $z=a+bj$ where *a*>0 and *b*>0 would be represented thus:



Mathematical Properties

1. **Properties of *j***

   We have defined $j=\sqrt{-1}$. It has some very interesting properties.

   (a) $j^2=-1 \;\Rightarrow j^3=-j \;\Rightarrow j^4=1 \;\Rightarrow j^5=j$

   (b) $j^{4n+1}=j$ and $j^{4n}=1 \quad \forall\, n\in\mathbb{N}$

   (c) $\frac{1}{j}=\frac{1}{j}\frac{j}{j}=\frac{j}{j^2}=\frac{j}{(-1)}=-j$

2. **Addition**

   Let $z_1=a+b\,j$ and $z_2=c+d\,j$ be any two complex numbers where $a,b,c,$ and $d\in\mathbb{R}$. Then we define the sum of z_1 with z_2 to be the sum of the real parts and the sum of the imaginary parts:

$$z_1+z_2=\underbrace{(a+c)}_{\Re(z_1)+\Re(z_2)}+j\,\underbrace{(b+d)}_{\Im(z_1)+\Im(z_2)}$$

3. **Scalar Multiplication**

   Let $z_1=a+b\,j$ be any two complex number and let $\alpha\in\mathbb{R}$ be any real constant. Then we define the scalar product of z_1 with α to be

$$\alpha z_1=\alpha(a+b\,j)=\alpha a+\alpha b\,j$$

4. **Products of Complex Numbers**

   Let $z_1=a+b\,j$ and $z_2=c+d\,j$ be any two complex numbers where $a,b,c,$ and $d\in\mathbb{R}$. Then we define the product of z_1 with z_2 to be:

$$\begin{aligned} z_1 z_2 &= \underbrace{(a+b\,j)}_{z_1}\underbrace{(c+d\,j)}_{z_1} = a(c+d\,j)+b\,j(c+d\,j) \\ &= ac+ad\,j+bc\,j+bd\,j^2 \\ &= \underbrace{(ac-bd)}_{\Re(z_1 z_2)}+j\,\underbrace{(ad+bc)}_{\Im(z_1 z_2)} \end{aligned}$$

5. **Division of Complex Numbers**

   Let $z_1=a+b\,j$ and $z_2=c+d\,j$ be any two complex numbers where $a,b,c,$ and $d\in\mathbb{R}$. Then we define the division of z_1 by z_2 to be:

$$\frac{z_1}{z_2}=\frac{(a+b\,j)}{(c+d\,j)}=\frac{(a+b\,j)}{(c+d\,j)}\times\frac{(c-d\,j)}{(c-d\,j)}\equiv\frac{z_1}{z_2}\frac{\bar{z}_2}{\bar{z}_2}$$

$$\frac{z_1}{z_2} = \underbrace{\frac{(ac+bd)}{(c^2+d^2)}}_{\Re\left(\frac{z_1}{z_2}\right)} + j\underbrace{\frac{(bc-ad)}{(c^2+d^2)}}_{\Im\left(\frac{z_1}{z_2}\right)} \equiv \frac{z_1 \bar{z}_2}{|\bar{z}_2|^2}$$

6. **Complex Exponential Function**

We know that the exponential function is important both in calculus and algebra. We defined it in terms of a real variable, x, and showed that it has the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The complex exponential function is that function where the real argument, x, above is replaced[2] by $j\theta$. Then the Taylor series becomes

$$e^x = \sum_{k=0}^{\infty} j^k \frac{\theta^k}{k!} = \sum_{k=0}^{\infty} \left(j^{2k} \frac{\theta^{2k}}{2k!} + j^{2k+1} \frac{\theta^{2k+1}}{(2k+1)!} \right)$$

Now

$$j^2 = -1 \ \Rightarrow j^{2k} = (-1)^k \ \Rightarrow j^{2k+1} = (-1)^k j$$

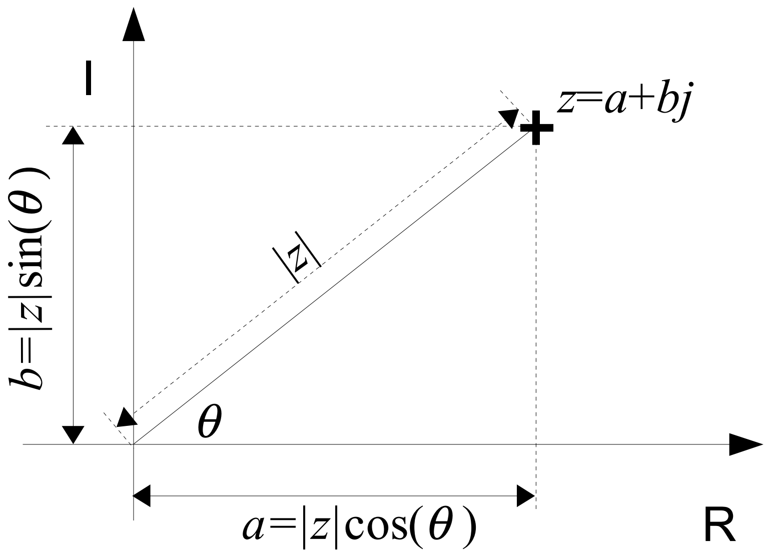
Then

$$e^{j\theta} = \sum_{k=0}^{\infty} j^k \frac{\theta^k}{k!} = \sum_{k=0}^{\infty} \left((-1)^k \frac{\theta^{2k}}{2k!} + j(-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \right)$$

$$\Rightarrow e^{j\theta} = \sum_{k=0}^{\infty} \left((-1)^k \frac{\theta^{2k}}{2k!} \right) + j \sum_{k=0}^{\infty} \left((-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \right)$$

$$\Rightarrow e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Consider the figure below



2 If we replace x by $z=a+bj$ then $e^x \to e^z = e^{a+bj} = e^a e^{bj}$. The first part is real. Only the second part with bj is complex. Hence we use the equivalent argument of $j\theta$ by convention.

Then

$$z=a+b\,j=\underbrace{|z|\cos(\theta)}_{a}+j\,\underbrace{|z|\sin(\theta)}_{b}$$

and so we have the *Polar Form* of a complex number. In a similar manner

$$z=a+b\,j=\underbrace{|z|\left(\cos(\theta)+j\sin(\theta)\right)}_{\text{polar form}}=\underbrace{|z|e^{j\theta}}_{\text{exponential form}}$$

referred to as the complex exponential form.

DeMoivre's Theorem

Let $z\in\mathbb{C}$ be any complex number. Then for any integer n

$$z^n=(a+b\,j)^n=|z|^n\left(\cos(n\theta)+j\sin(n\theta)\right)$$

Proof:

We know that every complex number can be represented in the complex exponential form:

$$z=a+b\,j=|z|e^{j\theta}$$

Then, for $n\in\mathbb{Z}$

$$z^n=(a+b\,j)^n=|z|^n\left(e^{j\theta}\right)^n=|z|^n e^{jn\theta}$$
$$\Rightarrow z^n=|z|^n\left(\cos(n\theta)+j\sin(n\theta)\right)$$

as required.

Example

Let

$$z=1+\sqrt{3}\,j$$

Then find the form of z^4.

Normally,

$$z^n=(a+b\,j)^n=a^n+n\,a^{n-1}(b\,j)+\frac{n(n-1)}{2}a^{n-2}(b\,j)^2+\ldots+n\,a(b\,j)^{n-1}+(b\,j)^n$$

from the binomial expansion of $(x+y)^n$. In this case

$$\begin{aligned} z^4 &= \left(1+\sqrt{3}\,j\right)^4=1+4\sqrt{3}\,j+6\left(\sqrt{3}\,j\right)^2+4\left(\sqrt{3}\,j\right)^3+\left(\sqrt{3}\,j\right)^4 \\ &= (-8)-8\sqrt{3}\,j \\ &= -8\left(1+\sqrt{3}\,j\right) \end{aligned}$$

I have omitted a considerable number of calculation steps to get to the final answer.

Let's compare this against the DeMoivre's Theorem approach. First convert to polar form.

$$z=1+\sqrt{3}\,j=|z|(\cos(\theta)+j\sin(\theta))$$

where

$$|z|=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2$$

and

$$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$$

Then

$$z=1+\sqrt{3}\,j=2(\cos(\pi/3)+j\sin(\pi/3))$$

From DeMoivre's Theorem

$$\begin{aligned} z^4 &= (1+\sqrt{3}\,j)^4 = 2^4(\cos(4\pi/3)+j\sin(4\pi/3)) \\ &= 16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right) \\ &= -8(1+\sqrt{3}) \end{aligned}$$

as expected.

Once expressed in Polar form, the calculation is fairly straightforward.

Exercise:

Use DeMoivre's theorem to evaluate each of the following:

1) z^2 where $z=1-j$
2) z^5 where $z=-\sqrt{3}-j$
3) z^3 where $z=j$

This finishes our brief introduction to Complex Numbers. In the next section, we'll consider an extension of DeMoivre's Theorem to the problem of finding roots of complex numbers; an appropriate subject given that it was the problem of root finding in cubic equations that gave birth to the area of Complex Analysis.