Section 1
Number Systems and Functions

The number systems used in Mathematics are (in order of complexity):

- $\mathbb{N}$ The Natural Numbers.
  This is the set of all positive whole numbers: i.e. { 1, 2, 3, ... }

- $\mathbb{Z}$ The Integers
  This is the set of all whole numbers, negative and positive, including 0: i.e. { ..., -4, -3, -2, -1, 0, 1, 2, 3, ... }

- $\mathbb{Q}$ The Rationals
  This is the set of all real numbers that can be written in the form a/b where a and b are both integers and b is not zero.
  e.g. -3, 7/10, 6/-5, 20, etc.

- $\mathbb{Q}^*$ The Irrationals
  This is the set of all real numbers that cannot be written in the form a/b where a and b are both integers with b being non-zero.
  e.g. π, e, $\sqrt{5}$, etc.

- $\mathbb{R}$ The Real Numbers
  This is the set of all numbers formed from the union of $\mathbb{Q}$ with $\mathbb{Q}^*$.

- $\mathbb{C}$ The Complex Numbers
  This is the set of all numbers that can be written as $a+bj$ where a and b are real numbers and $j=\sqrt{-1}$.

Before we proceed any further let us consider some notation that will be used to great advantage in our notes to follow:

NOTATION

- $\cup$ This denotes *Union* or "the union of two sets". The result of this is to form a new set where all the unique elements of the initial two sets are brought together.
  e.g. If $A=\{1,2,3\}$ and $B=\{3,4,5\}$ then $A \cup B=\{1,2,3,4,5\}$
- $\cap$ This denotes *Intersection* or "the intersection of two sets". The result of this is to form a new set where all the common

elements of the initial two sets are brought together.
e.g. If A={1,2,3} and B={3,4,5} then $A \cap B = \{3\}$

- $\setminus$ This denotes *Less* (or set minus). The result of this is to form a new set where all the elements of the first set that are not common to the second set are brought together.
e.g. If A={1,2,3} and B={3,4,5} then $A \setminus B = \{1,2\}$
- $\in$ This denotes *in* or "is an element of". This is a logical statement stating that a specified element is in fact an element of the considered set.
e.g. If A={1,2,3} then if is valid to say $1 \in A$
- $\notin$ This denotes *not in* or "is not an element of". This is the logical opposite of "is an element of".
e.g. If A={1,2,3} then if is valid to say $5 \notin A$
- $\Rightarrow$ This denotes *implies* or "it follows that".
e.g. If a=5, b=2, and $c=8 \Rightarrow c > a+b$.
- $\Leftrightarrow$ This denotes *if and only if* (or iff in some texts).
e.g. If a=5, b=2 $\Rightarrow c > a+b \Leftrightarrow c > 7$.
- $\subset$ This denotes *Subset* or "is a subset of a set". This is a logical statement declaring the former set is entirely contained within the latter: i.e. All the elements in the former are in the latter.
e.g. If A={1,2,3} and D={0,1,2,3,4,5,6} then it is valid to say $A \subset D$.
- $\not\subset$ This denotes *nsubset* or "is not a subset of a set". This is a logical statement declaring the former set has at least one element not contained within the latter.
e.g. If A={1,2,3} and D={0,2,4,6} then if is valid to say $A \not\subset D$.
- $\exists$ This denotes *exists* or "there exists". Consider the following statement and see if you can reason it out:
Let $a \in \mathbb{R}$ and let $b \in \mathbb{N}$. Then $\exists a < b$.
- $\forall$ This denotes *forall.*
Let $a \in \mathbb{R}$ and let $b \in \mathbb{R}$. Then $\exists a < b \forall b \in \mathbb{R}$.

That's enough notation for now. Considering our original number sets we have the following statements:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$$\mathbb{Q}^* \subset \mathbb{R} \subset \mathbb{C}$$

Furthermore

$$\mathbb{Q} \cup \mathbb{Q}^* = \mathbb{R}$$

Now let us consider the different types of intervals and their properties.

INTERVALS

The main types of interval (subsection) on the real number line are:

- Open Intervals which do not include the end points and are denoted by (a, b). Then $x \in (a, b) \Leftrightarrow a < x < b$.
- Closed Intervals are intervals that include the end points and are denoted [a, b]. Now $x \in [a, b] \Leftrightarrow a \leq x \leq b$. Closed intervals are sometiems called compact intervals.
- Semi-Open (Semi-Closed) Intervals are those where one boundary is open and the other is closed. Then
$$x \in [a, b) \Leftrightarrow a \leq x < b$$
$$x \in (a, b] \Leftrightarrow a < x \leq b$$
- Semi-Infinite Intervals are not strictly speaking a new classification being as they are just semi-open (or closed) intervals. Their uniqueness is in the use of infinity, ∞, for one boundary. Thus
$$x \in [a, \infty) \Leftrightarrow a \leq x$$
$$x \in (-\infty, b] \Leftrightarrow x \leq b$$

FUNCTIONS, VARIABLES & CONSTANTS

A *variable* is a mathematical object that usually represents a number and, in light of its name, can take on a range of values in a given set of operations subject to constraints either explicit or implicit in nature. For example if the values the variable could take in a given operation were to be restricted to some closed interval bounded below by a and above by b then we would impose the restriction $x \in [a, b]$.

A *constant* represents a fixed value in the chosen mathematical operation. The constant cannot change once the analysis has begun but can change from operation to operation.

For example, the linear equation
$$y = 2x - 3$$
has constants fixed at 2 and -3 but has introduced a variable x that can assume many values. y depends on x and is called a *dependent variable* whereas x depends on nothing in the equation and is called an *independent variable*. If we wanted to study all equations of the form above we can replace the 2 and -3 with generic constants a and b so that all relevant properties can be applied to any such equation of the form
$$y = ax + b$$
It is at this point that some studetns have difficulty with the concepts of

variables and constants. Remember a constant, even when represented in a generic fashion, can only assume a fixed value. A variable can assume any value subject to any relevant constraints.

A *function* is a relationship (usually a formula) by which a correspondence is established between two sets of numbers, the domain U and the range V, such that for each number in U there corresponds only one number in V. Think of a function as a recipe; the ingredients are the variables and constants, the tasks of mixing, sieving, etc. are the mathematical operations of multiplication, division, etc. The recipe, if followed successfully, will result in something edible while the function, if properly evaluated, will return a meaningful result.

Consider the linear equation above. The corresponding function is

$$f(x)=ax+b \quad \forall a,b \in \mathbb{R} \quad a\neq 0$$

or in more formal notation

$$f: U\in\mathbb{R} \to V\in\mathbb{R} : x \to ax+b$$

where $a,b\in\mathbb{R}$ and $a\neq 0$. This latter notation explicitly includes the domain U and range V of $f(x)$. Usually this is not required but some functions are sufficiently badly behaved to necessitate their inclusion. The variable x that we pass to the function is called the *argument* of f.

Examples:

1) Consider the real valued function $f(x)=\sqrt{x} \quad \forall x\in\mathbb{R}$.
The domain and range of a real valued function lie within $\mathbb{R}$ and so negative x are prohibited because $f(x)$ would then assume a complex value and lie within $\mathbb{C}$.
The domain of $f(x)$ above is the semi-infinite interval $[0,\infty)$ or $\mathbb{R}^+$
The range of $f(x)$ is the same interval $[0,\infty)$ because $\sqrt{x}$ refers by convention to the positive square roots of x. If both are required it is necessary to amend $f(x)$ to include this via

$$f(x)=\pm\sqrt{x} \quad \forall x\in\mathbb{R}$$

2) Try this one for size:

$$f(x)=\frac{x}{x^2+1} \quad \forall x\in\mathbb{R}$$

I'll leave it to you to show that the domain of this function is $\mathbb{R}$ but its range is [-½, ½]