Section 4
Matrices

In this section we introduce the concept of a matrix of real numbers[1]. We will consider the various forms such a matrix can take, the associated mathematical properties, how vectors can be defined in terms of matrices, and an application of matrices to the solution of a problem type you would have already encountered at second level.

What is a Matrix?

Simply put, a matrix is a 2-D collection of mathematical objects (most commonly numbers) that are formally ordered in rows and columns. By convention, a matrix is denoted by an uppercase letter; *A*, *B*, *C*, *D*, ...

The components making up the matrix are termed elements (such as in a set). However, unlike sets, in order to uniquely locate an element within the matrix, a pair of subscripts is employed; the first, denoted *i*, specifies the row the element is located within while the second, denoted *j*, specifies the column. Then, if we let *A* be our matrix of real numbers, the element of *A* in the $i^{\text{-th}}$ row and $j^{\text{-th}}$ column would be denoted a_{ij}.

The total number of rows and columns in a matrix determines its dimension. If a matrix *A* has *m* rows and *n* columns it is deemed to be a $m \times n$ matrix. An element of *A* can now be formally defined

$$a_{ij} \in A \iff 1 \leq i \leq m, 1 \leq j \leq n; \quad m, n \in \mathbb{N}$$

where the dimensions *m* and *n* must be natural numbers.

All well and good, but what does the matrix look like?

We represent a matrix as a grid enclosed within parentheses ():

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The first element of any matrix is a_{11} and the last (for an $m \times n$ matrix) is a_{mn}.

1 As you'll see from the definition, a matrix can contain any mathematical object so long as the appropriate rules, etc. are obeyed.

Example

Let A be the 2×3 matrix of real numbers

$$A = \begin{pmatrix} 3 & -7 & 2 \\ 0 & 4 & -5 \end{pmatrix}$$

then

$$a_{11} = 3$$
$$a_{22} = 4$$
$$a_{13} = 2$$
$$a_{32} \notin A$$

The last element is referencing a third row and second column in A which does not exist. Consequently a_{32} does not exist.

N.B.

Take care, when referencing elements in a matrix, not to confuse the order of the indices: ***ROWS FIRST COLUMNS SECOND***. Otherwise you may reference elements that are not in the matrix in question and consequently are non-existent.

Forms of a Matrix

Now to formally name the different forms of matrix commonly encountered:

1. **Square Matrix**
   Let A be a matrix of real numbers. Then A is a Square Matrix if and only if the number of rows, m, equals the number of columns, n. i.e. if and only if A is a $m \times m$ matrix. A is deemed **Rectangular** otherwise.
2. **Symmetric Matrix**
   Let A be a matrix of real numbers. Then A is a symmetric matrix if and only if A is square (i.e. a $m \times m$ matrix) and
   $$a_{ij} = a_{ji} \quad 1 \leq i, j \leq m \quad m \in \mathbb{N}$$
3. **Triangular Matrix**
   Let A be a matrix of real numbers. Then A is a triangular matrix if and only if all the elements above or below the **Principle Diagonal** of A are zero.

N.B.

The Principle Diagonal of a matrix, A, is the set of elements of A whose indices are equal: i.e.

Principle Diagonal of a $m \times m$ matrix $A = \{a_{11}, a_{22}, \ldots, a_{mm}\}$

Then

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1\,1} & a_{m-1\,2} & \ldots & a_{m-1\,m-1} & 0 \\ a_{m1} & a_{m2} & \ldots & a_{m-1\,m} & a_{mm} \end{pmatrix}$$

is called a lower triangular (or left triangular) matrix and

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\,m-1} & a_{1m} \\ 0 & a_{22} & \ldots & a_{2\,m-1} & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{m-1\,m-1} & a_{m-1\,m} \\ 0 & 0 & \ldots & 0 & a_{mm} \end{pmatrix}$$

is called a upper triangular (or right triangular) matrix.

4. **Diagonal Matrix**

   Let A be a matrix of real numbers. Then A is a diagonal matrix if and only if the principle diagonal elements contain the only non-zero elements in A with at least one element on this principle diagonal being non-zero: i.e.

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{mm} \end{pmatrix}$$

5. **Identity Matrix**

   The $m \times m$ Identity matrix, I, is the diagonal matrix whose principle diagonal elements are all 1: i.e.

$$I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

6. **Zero Matrix**
   The $m \times n$ zero matrix, *O*, is the matrix whose elements are all 0: i.e.

$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

7. **Column Matrix**
   Let *A* be a matrix of real numbers. Then *A* is a column matrix if and only if *A* has only one column; i.e. *A* is the $m \times 1$ matrix

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

   Such a matrix is often called an *m*-dimensional column **vector**. More on this later and in the next topic on vectors.

8. **Row Matrix**
   Let *A* be a matrix of real numbers. Then *A* is a row matrix if and only if *A* has only one row; i.e. *A* is the $1 \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}$$

   Such a matrix is often called an *n*-dimensional row **vector**. More on this later.

9. **Singular Matrix**
   Let *A* be a matrix of real numbers. Then *A* is singular if the inverse matrix of *A*, denoted A^{-1}, doesn't exist.
   Given that we haven't yet defined the Inverse of a matrix the above definition is incomplete. More on this later when we do define the inverse more formally.

Mathematical Properties of Real Matrices

We now define the fundamental mathematical properties for these matrices; including Addition, Scalar Multiplication, Matrix Multiplication, the Determinant of a Matrix, and the Inverse of a Matrix

Addition of Matrices

Let A and B be two matrix of real numbers. A third matrix C exists and is equal to the sum of A and B if and only if

- the dimension of A is equal to that for B.
- then elements c_{ij} of C are equal to the sum of the corresponding elements of A with B; i.e.

$$\exists C = A + B \iff c_{ij} = a_{ij} + b_{ij} \quad 1 \le i \le m, 1 \le j \le n$$

In such cases, the dimension of C will be equal to that of A and B.

Scalar Multiplication

Let A be a $m \times n$ matrix of real numbers and let α be any real number. Then αA is the $m \times n$ matrix whose elements are αa_{ij} : i.e.

$$\alpha A = \alpha \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{pmatrix} \quad \forall \alpha \in \mathbb{R}$$

As before, the combination of the sum of A with B after scalar multiplication by (-1) is equivalent to the difference of two matrices; i.e.

$$\begin{aligned} \exists D = A + (-1) B \quad &\Leftrightarrow d_{ij} = a_{ij} + (-1) b_{ij} \\ &\Rightarrow d_{ij} = a_{ij} - b_{ij} = A - B \end{aligned}$$

where $1 \le i \le m, 1 \le j \le n$.

Matrix Multiplication

Up to now the analysis of these matrices has been relatively straightforward. Now the true complexity of such objects becomes apparent when we attempt to multiply two matrices together.

Formally, the product of two A and B matrices is defined thus:

Let A be a $m \times p$ matrix and let B be a $p \times n$ matrix of real elements. Then there exists a $m \times n$ matrix $C = A \cdot B$ of real elements whose elements are defined

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ip} b_{pj}$$

where $1 \le i \le m, 1 \le j \le n$.

Not very nice. However we can visualise the product differently to make this calculation more tangible.

Consider the summation above. It should be apparent that every element in the $i^{\text{-th}}$ row of A is multiplied with every corresponding element in the $j^{\text{-th}}$ column of B. This gives rise to one of the constraints with this operation; namely that the first matrix, in our case A, must have the same number of columns as the second matrix, B, has rows in order for the matrix product $A.B$ to exist.

A pictorial respresentation of this same operation should help. In the figure below, we show how the $i^{\text{-th}}$ row in A and $j^{\text{-th}}$ column in B combine to calculate c_{ij}. This figure should also make the previous constraint on the dimension of A and B apparent:
If the number of columns in A did not match the number of rows in B then there would be elements in the matrix of larger dimension, either A or B, that would not have a corresponding element in the other matrix.

$$c_{ij} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p-1} & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p-1} & a_{2p} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip-1} & a_{ip} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp-1} & a_{mp} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n-1} & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n-1} & b_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ b_{p-11} & b_{p-12} & \cdots & b_{p-1j} & \cdots & b_{p-1n-1} & b_{p-1n} \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn-1} & b_{pn} \end{pmatrix}$$

Figure: The $i^{\text{-th}}$ row in A and $j^{\text{-th}}$ column in B that result in c_{ij}

We mentioned one constraint on the matrices for the product to exist. There are other consequences resulting from this operation that need to be considered:

1. If A and B are such that we can define a matrix $C = A \cdot B$, it does not follow that there exists another matrix $D = B \cdot A$. In fact, generally the existence of one product, $B \cdot A$ or $A \cdot B$, is not itself guaranteed until the number of columns in the first matrix matches the number of rows in the second.
2. If $C = A \cdot B$ does exist, it follows that, in order for $D = B \cdot A$ to exist, the dimensions of A and B must be $m \times n$ and $n \times m$ respectively. Then C will be a $m \times m$ matrix and D a $n \times n$ matrix.

3. From the second consequence above, it follows that in general matrix multiplication is NON-COMMUTATIVE. This means that, in general, $A\cdot B \neq B\cdot A$. In order for this to have equality we require that the dimension of C and D be to each other; i.e. that both A and B be square matrices of the same dimension. Even then we are not guaranteed of a commutative operation. Square matrices only guarantees that the resulting C and D matrices are both square and of the same dimension; nothing more.
4. In very rare circumstances, then, we have the commutation relationship that $A\cdot B = B\cdot A$. For these rare cases we can have a further refinement that
$$A\cdot B = B\cdot A = I$$
then B is said to be A's inverse matrix and vice versa. More on this later.

In conclusion we can summarize the operation in the flowchart overleaf with the following key:

- **A one-sided matrix product**
  This is one where either $A\cdot B$ or $B\cdot A$ exists but not both. This is naturally non-commuting.
- **An asymmetric non-commuting product**
  This is when the two matrices are such that both $A\cdot B$ and $B\cdot A$ exist but, due to the fact that the matrices are not square, the dimensions of the resulting $C = A\cdot B$ and $D = B\cdot A$ are different. Naturally they cannot be equal and so the product is non-commuting.
- **A symmetric non-commuting product**
  When the two matrices are square then both $C = A\cdot B$ and $D = B\cdot A$ exist and are of the same dimension; hence the term symmetric. They are not the same though, so the product is still non-commuting.
- **A commuting product**
  That rare case when the two matrices are square, both $C = A\cdot B$ and $D = B\cdot A$ exist and are equal. Then $A\cdot B = B\cdot A$; i.e. a commuting product.
- **A matrix inverse exists**
  The rarest case of all. The matrices are such that not only are they square and have a commuting product but they are related to each other in such a way that their product is the identity matrix.

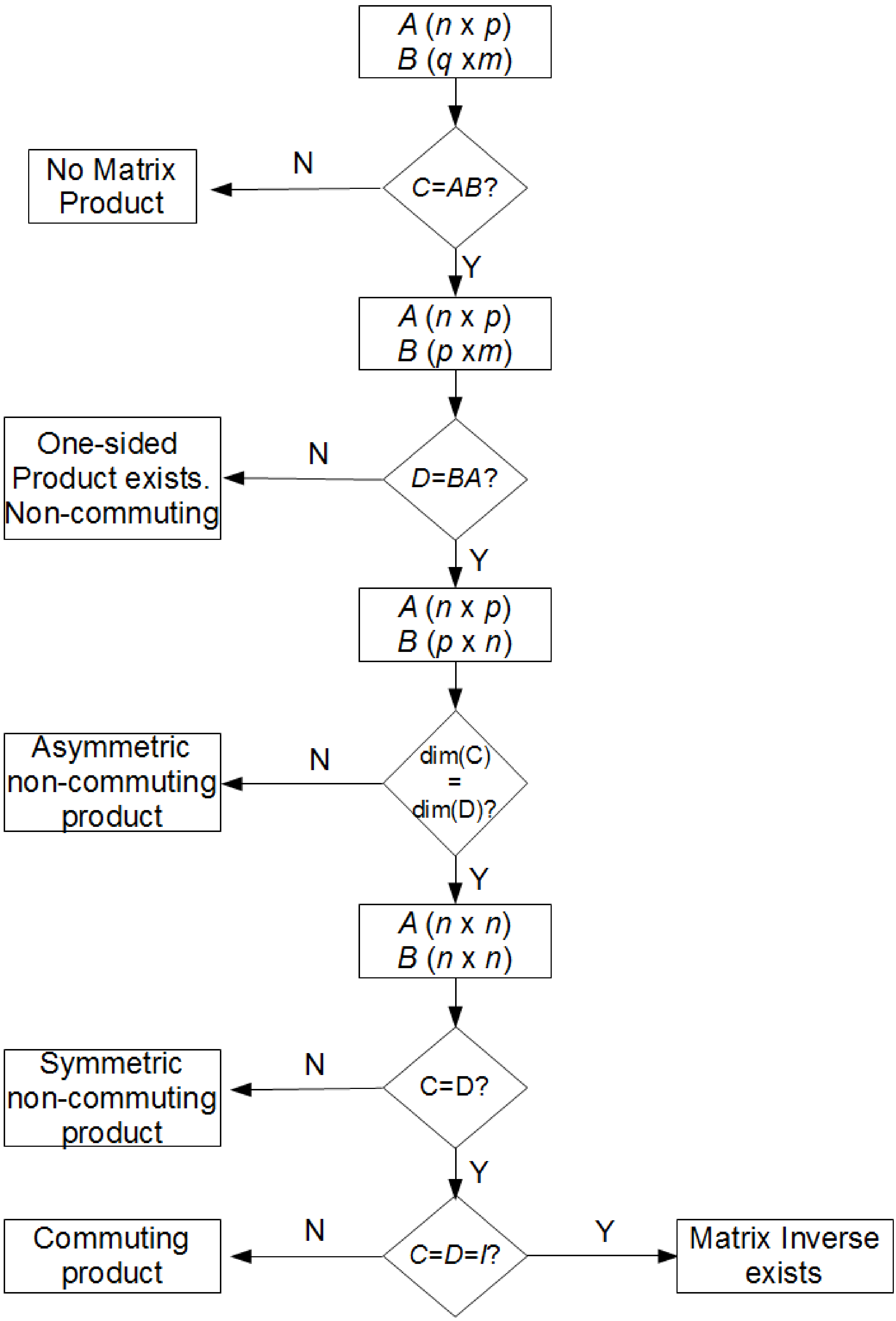

Figure: The options for the product of two matrices.

Example 1

Let $A=\begin{pmatrix} 1 & 4 & 7 & 0 \\ -2 & 7 & 6 & 9 \end{pmatrix}$ and $B=\begin{pmatrix} -1 & 7 & -9 & 0 \\ 0 & 2 & 3 & 1 \end{pmatrix}$ then

(i) $A+B=\begin{pmatrix} 1 & 4 & 7 & 0 \\ -2 & 7 & 6 & 9 \end{pmatrix}+\begin{pmatrix} -1 & 7 & -9 & 0 \\ 0 & 2 & 3 & 1 \end{pmatrix}=\begin{pmatrix} 1-1 & 4+7 & 7-9 & 0+0 \\ -2+0 & 7+2 & 6+3 & 9+1 \end{pmatrix}$

$$\Rightarrow A+B=\begin{pmatrix} 0 & 11 & -2 & 0 \\ -2 & 9 & 9 & 10 \end{pmatrix}$$

(ii) $3A=3\begin{pmatrix} 1 & 4 & 7 & 0 \\ -2 & 7 & 6 & 9 \end{pmatrix}=\begin{pmatrix} 3\times 1 & 3\times 4 & 3\times 7 & 3\times 0 \\ 3\times -2 & 3\times 7 & 3\times 6 & 3\times 9 \end{pmatrix}$

$$\Rightarrow 3A=\begin{pmatrix} 3 & 12 & 21 & 0 \\ -6 & 21 & 18 & 27 \end{pmatrix}$$

(iii) Neither $C=A\cdot B$ nor $D=B\cdot A$ exist. Why?

Example 2

Let $A=\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix}$ and $B=\begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$ then $\exists C=A\cdot B$ because A has dimension 3×3 and B has dimension 3×2 $\Rightarrow C$ has dimension 3×2.

Now to calculate C:

$$c_{ij}=\sum_{k=0}^{3} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}$$

where $1\le i\le 3, 1\le j\le 3$.

Then

$$c_{11}=\sum_{k=1}^{3} a_{1k}b_{k1}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}$$
$$\Rightarrow c_{11}=0\times -5+1\times -8+3\times 1=-5$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix}\bullet\begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

$$c_{12}=\sum_{k=1}^{3} a_{1k}b_{k2}=a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}$$
$$\Rightarrow c_{12}=0\times 6+1\times 2+3\times 4=14$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix}\bullet\begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

$$c_{21}=\sum_{k=1}^{3} a_{2k}b_{k1}=a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}$$
$$\Rightarrow c_{21}=-8\times -5+7\times -8+6\times 1=-10$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix}\bullet\begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

$$c_{22}=\sum_{k=1}^{3} a_{2k}b_{k2}=a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}$$
$$\Rightarrow c_{21}=-8\times 6+7\times 2+6\times 4=-10$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix} \bullet \begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

$$c_{31}=\sum_{k=1}^{3} a_{3k}b_{k1}=a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}$$
$$\Rightarrow c_{21}=9\times -5+2\times -8+(-4)\times 1=-65$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix} \bullet \begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

$$c_{32}=\sum_{k=1}^{3} a_{3k}b_{k2}=a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}$$
$$\Rightarrow c_{21}=9\times 6+2\times 2+(-4)\times 4=42$$

$$\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix} \bullet \begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}$$

Then we write

$$C=A\cdot B=\begin{pmatrix} 0 & 1 & 3 \\ -8 & 7 & 6 \\ 9 & 2 & -4 \end{pmatrix}\cdot\begin{pmatrix} -5 & 6 \\ -8 & 2 \\ 1 & 4 \end{pmatrix}=\begin{pmatrix} -5 & 14 \\ -10 & -10 \\ -65 & 42 \end{pmatrix}$$

Now onto the tutorial for some exercises.
In the next section on matrices we'll consider the inverse of a matrix, further properties of the product and the effective “magnitude” or “length” of a matrix.