Section 4 (Cont'd)
Alternate procedure for calculating the Adjoint of a 3x3 Matrix

Given the 3×3 matrix, A

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we are required to determine its adjoint; A^*

$$A^*=\begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix}$$

where the cofactors are defined

$$\Delta_{ij}=(-1)^{i+j}\times\begin{pmatrix}\text{The determinant of the } 2\times2 \text{ matrix formed by} \\ \text{removing the } i^{th} \text{ COLUMN and } j^{th} \text{ ROW in } A\end{pmatrix}$$

This latter definition can be handled all in one go as done in the lecture notes.

However, given the number of potential pitfalls and the significant probability that mistakes will occur, we introduce this method as an alternative.

It still includes all the stages of the former method but it separates them in an attempt to provide some clarity.

Existing Method

The former method determines the cofactors by performing three distinct tasks:

1. The mutliplier, $(-1)^{i+j}$, by 1 or -1 depending on the location of the cofactor in the adjoint.
2. The removal of the i^{th} column (not row) and j^{th} row (not column) of A (effectively working with the transpose of A; A^T.
3. The calculation of the resulting 2×2 determinant.

Alternate Method

Essentially these tasks can be performed separately as follows:

1. Take the transpose of A; A^T
2. Calculate the 2×2 determinants by removing the i^{th} row and j^{th} column of A^T, as you would expect when referencing the $i\,j^{th}$ element of a matrix, and put them in the corresponding *ij* location in the adjoint, A^*.
3. Multiply the resulting matrix elements by ± 1 according to the pattern laid out in the matrix composed of $(-1)^{i+j}$: i.e. for our 3 times 3 problem this matrix would have the form

$$\begin{pmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Let's illustrate this by example:

Example 1:

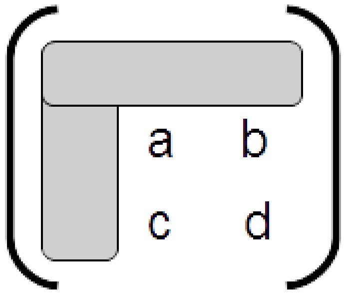
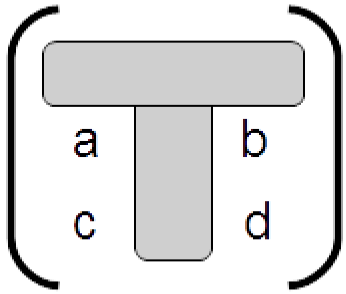
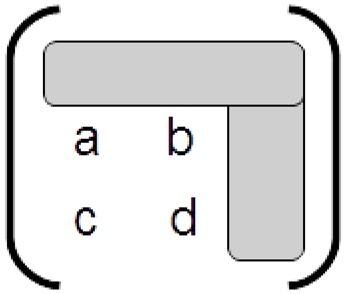
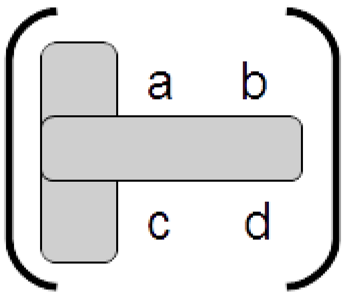
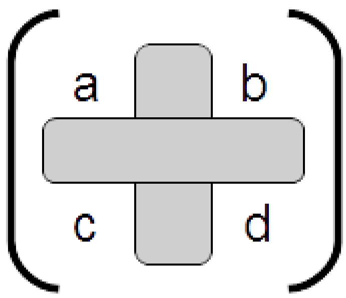
Find the adjoint of the 3×3 matrix, A:

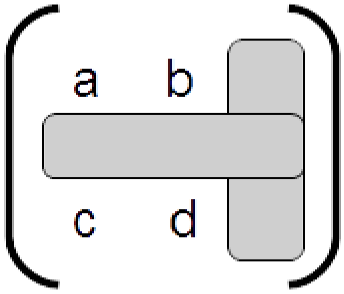
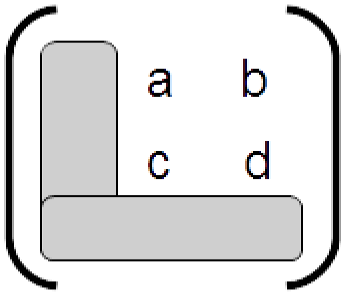
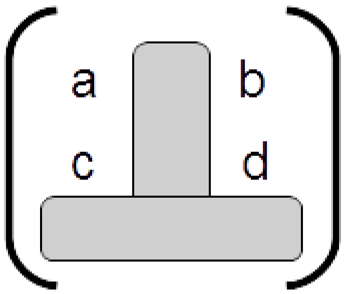
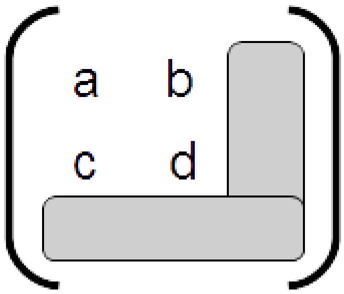
$$A=\begin{pmatrix} -3 & 2 & 6 \\ 0 & -4 & 8 \\ 5 & 1 & 7 \end{pmatrix}$$

Using the alternate method we first find the transpose of A:

$$A^T=\begin{pmatrix} -3 & 0 & 5 \\ 2 & -4 & 1 \\ 6 & 8 & 7 \end{pmatrix}$$

Now calculate the corresponding determinants making up the ajdoint's cofactors:

i	j	Determinant	Picture
1	1	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 8 & 7 \end{vmatrix} = (-4)(7)-(8)(1)$ $=-28-8=-36$	
1	2	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 6 & 7 \end{vmatrix} = (2)(7)-(6)(1)$ $=14-6=8$	
1	3	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix} = (2)(8)-(6)(-4)$ $=16-(-24)=40$	
2	1	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 5 \\ 8 & 7 \end{vmatrix} = (0)(7)-(8)(5)$ $=0-40=-40$	
2	2	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 6 & 7 \end{vmatrix} = (-3)(7)-(6)(5)$ $=-21-30=-51$	

2	3	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 6 & 8 \end{vmatrix} = (-3)(8)-(6)(0)$ $= -24-0 = -24$	
3	1	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 5 \\ -4 & 1 \end{vmatrix} = (0)(1)-(-4)(5)$ $= 0-(-20) = 20$	
3	2	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 2 & 1 \end{vmatrix} = (-3)(1)-(5)(2)$ $= -3-10 = -13$	
3	3	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 2 & -4 \end{vmatrix} = (-3)(-4)-(0)(2)$ $= 12-0 = 12$	

The “pre-adjoint”matrix resulting from this is

$$\begin{pmatrix} -36 & 8 & 40 \\ -40 & -51 & -24 \\ 20 & -13 & 12 \end{pmatrix}$$

We now multiply the elements by +-1 according to the pattern in the matrix

$$\begin{pmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -36 & 8 & 40 \\ -40 & -51 & -24 \\ 20 & -13 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} -36 & -8 & 40 \\ 40 & -51 & 24 \\ 20 & 13 & 12 \end{pmatrix}$$

The adjoint is thus

$$A^* = \begin{pmatrix} -36 & -8 & 40 \\ 40 & -51 & 24 \\ 20 & 13 & 12 \end{pmatrix}$$

VERIFICATION

We can verify if this is indeed the adjoint by multiuplying it into the original matrix, A; the resulting matrix should be

$$A^* \cdot A = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

So, for our example,

$$A^* \cdot A = \begin{pmatrix} -36 & -8 & 40 \\ 40 & -51 & 24 \\ 20 & 13 & 12 \end{pmatrix} \cdot \begin{pmatrix} -3 & 2 & 6 \\ 0 & -4 & 8 \\ 5 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (-36)(-3)+(-8)(0)+(40)(5) & (-36)(2)+(-8)(-4)+(40)(1) & (-36)(6)+(-8)(8)+(40)(7) \\ (40)(-3)+(-51)(0)+(24)(5) & (40)(2)+(-51)(-4)+(24)(1) & (40)(6)+(-51)(8)+(24)(7) \\ (20)(-3)+(13)(0)+(12)(5) & (20)(2)+(13)(-4)+(12)(1) & (20)(6)+(13)(8)+(12)(7) \end{pmatrix}$$

$$= \begin{pmatrix} 308 & 0 & 0 \\ 0 & 308 & 0 \\ 0 & 0 & 308 \end{pmatrix}$$

Therefore

$$A^* \cdot A = \begin{pmatrix} 308 & 0 & 0 \\ 0 & 308 & 0 \\ 0 & 0 & 308 \end{pmatrix}$$

The form of the matrix is correct; i.e. A 3×3 diagonal matrix whose principle diagonal elements are all equal to the matrix's determinant.

Exercise:
Determine if 308 is equal to the determinant of A?

Exercise:
Using the above method, calculate the adjoint of

$$A = \begin{pmatrix} 4 & -3 & 76 \\ 1 & 2 & 8 \\ 0 & -5 & 2 \end{pmatrix}$$

Determine the correctness of A^* by verifying as shown above.