Section 6 Cont'd
Rational Functions
and the Partial Fractions Representation

In this section we will start the process of finding solutions to integrals of rational functions. We consider the method of partial fractions and show how it can be used to simplify greatly specific forms of rational functions. We start by defining the common types of rational functions you'll encounter and then proceed to using various techniques to simplify them. In the next section we'll then progress to the actual integrating of such functions.

What is a Rational Function?

You may recall from our discussion of number systems at the beginning of this course that set of numbers called Rational Numbers, $\mathbb{Q}$, defined:

$$x \in \mathbb{Q} \Leftrightarrow x \equiv \frac{a}{b} \quad \forall a, b \in \mathbb{Z}$$

A rational function is similarly expressed but instead of *a* and *b* we have *f*(*x*) and *g*(*x*) two polynomials of *x*: i.e. *h*(*x*) is a rational function if and only if

$$h(x) = \frac{f(x)}{g(x)}$$

where

$$f(x) = a_0 + a_1 x + \ldots a_n x^n$$
$$g(x) = b_0 + b_1 x + \ldots b_m x^m$$

The order of a polynomial is equal to the highest power of *x* in the polynomial; then the orders of *f*(*x*) and *g*(*x*) above are *n* and *m* respectively. For all examples we'll consider, the order of *f*(*x*) will be less than that of *g*(*x*): i.e.

$$h(x) = \frac{f(x)}{g(x)} = \frac{a_0 + a_1 x + \ldots a_n x^n}{b_0 + b_1 x + \ldots b_m x^m} \quad \text{where } m > n; \;\; \forall m, n \in \mathbb{N}$$

Examples of this would be

1. $h(x) = \dfrac{5x - 7}{6x^3 + 4x^2 - 7\text{x} + 6}$
2. $h(x) = \dfrac{5x^4 + 7x^2 - 8}{x^9 + 8}$

3. $h(x)=\dfrac{x^2-4x+2}{(x-3)(x+1)(x-8)(x+4)}$

etc.

Partial Fractions

Partial fractions are a clever method of splitting high order polynomials, in the denominator of a rational function, into manageable pieces that can then be operated upon. This technique works by decomposing the target polynomial into lower order polynomials and factors, $(x-\alpha_i)$: i.e.

$$\begin{aligned} h(x) &= \frac{a_0+a_1x+\ldots a_nx^n}{b_0+b_1x+\ldots b_mx^m} \\ &= \frac{a_0+a_1x+\ldots a_nx^n}{(x-\alpha_1)^{m_1}\ldots(x-\alpha_k)^{m_k}(\beta_1x^2+\gamma_1x+\delta_1)^{p_1}\ldots(\beta_rx^2+\gamma_rx+\delta_r)^{p_r}} \end{aligned}$$

which can get very messy if some constraints are not imposed. In all examples we'll consider, the denominator will be decomposed into, at most, m linear factors $(x-\alpha_i)$ raised to various powers r_i. Let's consider the various possibilities given this general constraint.

Case 1:

Let

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots a_nx^n}{\underbrace{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)}_{\text{m-factors}}} \quad m>n; \quad \forall\, m,n\in\mathbb{N}$$

then

$$h(x)=\frac{A_1}{(x-\alpha_1)}+\frac{A_2}{(x-\alpha_2)}+\cdots+\frac{A_m}{(x-\alpha_m)}$$

This is the simplest case where all factors are linear and the resulting partial fractions are in their simplest form. Let's illustrate the process via some examples.

Example:

Express

$$h(x)=\frac{3x-1}{(x-1)(x+1)(x-2)}$$

in partial fraction form.

As the function is of the form

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots a_nx^n}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)}$$

it can thus be decomposed

$$h(x)=\frac{A_1}{(x-1)}+\frac{A_2}{(x+1)}+\frac{A_3}{(x-2)}$$

Then

$$h(x)=\frac{3x-1}{(x-1)(x+1)(x-2)}=\frac{A_1}{(x-1)}+\frac{A_2}{(x+1)}+\frac{A_3}{(x-2)}$$

Adding the fractions on the RHS together, we get

$$h(x)=\frac{A_1(x+1)(x-2)+A_2(x-1)(x-2)+A_3(x-1)(x+1)}{(x-1)(x+1)(x-2)}$$

$$=\frac{3x-1}{(x-1)(x+1)(x-2)}$$

The denominators in both equations are equal and so we equate the numerators. Before doing this multiply out the partial fraction numerator and group terms with similar powers of *x*. Then

$$3x-1=(A_1+A_2+A_3)x^2+(-A_1-3A_2)x+(-2A_1+2A_2-A_3)$$

Then equating like with like we get:

$$\begin{array}{rrcl} x^2: & 0 & = & A_1+A_2+A_3 \\ x^1: & 3 & = & -A_1-3A_2 \\ x^0: & -1 & = & -2A_1+2A_2-A_3 \end{array}$$

Solving this system of linear equations we get the solution

$$A_1=-1 \quad A_2=-2/3 \quad A_3=5/3$$

Then

$$h(x)=\frac{3x-1}{(x-1)(x+1)(x-2)}=-\frac{1}{(x-1)}-\frac{2}{3}\frac{1}{(x+1)}+\frac{5}{3}\frac{1}{(x-2)}$$

Example:

Express

$$h(x)=\frac{x^2-4x+2}{(x-3)(x+1)(x-8)(x+4)}$$

in partial fraction form.

The function is of the form

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1 x+\ldots a_n x^n}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)}$$

and can thus be decomposed

$$h(x)=\frac{A_1}{(x-\alpha_1)}+\frac{A_2}{(x-\alpha_2)}+\frac{A_3}{(x-\alpha_3)}+\frac{A_4}{(x-\alpha_4)}$$

where $\alpha_1=3, \alpha_2=-1, \alpha_3=8,$ and $\alpha_3=-4$. Then

$$h(x)=\frac{x^2-4x+2}{(x-3)(x+1)(x-8)(x+4)}=\frac{A_1}{(x-3)}+\frac{A_2}{(x+1)}+\frac{A_3}{(x-8)}+\frac{A_4}{(x+4)}$$

Then, adding the fractions on the RHS together, we get

$$\begin{aligned} h(x)=&\frac{A_1(x+1)(x-8)(x+4)+A_2(x-3)(x-8)(x+4)}{(x-3)(x+1)(x-8)(x+4)} \\ &+\frac{A_3(x-3)(x+1)(x+4)+A_4(x-3)(x+1)(x-8)}{(x-3)(x+1)(x-8)(x+4)} \\ &=\frac{x^2-4x+2}{(x-3)(x+1)(x-8)(x+4)} \end{aligned}$$

The denominators are equal in both equations and so, for equality, we equate the numerators. Before we do this we multiply out the partial fraction numerator and group like terms: i.e.

$$\begin{aligned} &(A_1+A_2+A_3+A_4)x^3+(-3A_1-7A_2+2A_3-10A_4)x^2 \\ &\quad+(-36A_1-20A_2-11A_3+14A_4)x+(-32A_1+96A_2-12A_3+24A_4) \end{aligned}$$

Then equating like with like we get:

$$\begin{aligned} x^3: &\quad 0 &=& \quad A_1+A_2+A_3+A_4 \\ x^2: &\quad 1 &=& \quad -3A_1-7A_2+2A_3-10A_4 \\ x: &\quad -4 &=& \quad -36a_1-20A_2-11A_3+14A_4 \\ x^0: &\quad 2 &=& \quad -32A_1+96A_2-12A_3+24A_4 \end{aligned}$$

yielding the solution

$$A_1=519/2520 \quad A_2=-293/1512 \quad A_3=11/945 \quad A_4=1/42$$

and therefore

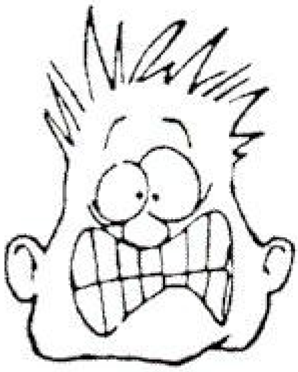
$$h(x)=\frac{x^2-4x+2}{(x-3)(x+1)(x-8)(x+4)}=\frac{519}{2520}\frac{1}{x-3}-\frac{293}{1512}\frac{1}{x+1}+\frac{11}{945}\frac{1}{x-8}+\frac{1}{42}\frac{1}{x+4}$$

Case 2:
Let

$$h(x)=\frac{f(x)}{g(x)}=\frac{a_0+a_1x+\ldots a_nx^n}{\underbrace{(x-\alpha_1)^{r_1}(x-\alpha_2)^{r_2}\cdots(x-\alpha_k)^{r_k}}_{\text{k-factors}}}\quad m=\sum_{j=1}^{k}r_j>n;\quad \forall\, m,n\in\mathbb{N}$$

then

$$h(x)=\left(\frac{A_1}{(x-\alpha_1)}+\frac{A_2}{(x-\alpha_1)^2}+\cdots+\frac{A_{r_1}}{(x-\alpha_1)^{r_1}}\right)+\left(\frac{B_1}{(x-\alpha_2)}+\cdots+\frac{B_{r_2}}{(x-\alpha_1)^{r_2}}\right)+$$
$$\cdots+\left(\frac{C_1}{(x-\alpha_k)}+\cdots+\frac{C_{r_k}}{(x-\alpha_1)^{r_k}}\right)$$



Looks horrible!
Unlike the graphic to the left, don't panic.
This is the generic description of what is going on. By its very nature it will be complicated.
However, for your examples and problems, the complexity is much reduced as, at most, only one or two factors will actually be raised to a power above 1. All other factors will be raised to the power of 1.

Let's illustrate the process via an example.

Example:
Express

$$h(x)=\frac{5x^2-2x+4}{(x-1)^2(x)(x+3)}$$

in partial fraction form.
Noting that it is of the form mentioned above, we express $h(x)$ as

$$h(x)=\frac{5x^2-2x+4}{(x-1)^2(x)(x+3)}=\left(\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}\right)+\frac{B}{x}+\frac{C}{x+3}$$

and then proceed as for the last case. Adding the fractions on the RHS

$$h(x)=\frac{A_1(x-1)(x)(x+3)+A_2(x)(x+3)+B(x-1)^2(x+3)+C(x)(x-1)^2}{(x-1)^2(x)(x+3)}$$
$$=\frac{5x^2-2x+4}{(x-1)^2(x)(x+3)}$$

The denominators in both equations are equal and so we equate the numerators. Before doing this multiply out the partial fraction numerator and group terms with similar powers of *x*. Then

$$5x^2-2x+4=\left(A_1+B+C\right)x^3+\left(2A_1+A_2+B-2C\right)x^2$$
$$+\left(-3A_1+3A_2-5B+C\right)x+\left(3B\right)$$

Then equating like with like we get:

$$\begin{aligned} x^3: &\quad 0 &=&\quad A_1+B+C \\ x^2: &\quad 5 &=&\quad 2A_1+A_2+B-2C \\ x^1: &\quad -2 &=&\quad -3A_1+3A_2-5B+C \\ x^0: &\quad 4 &=&\quad 3B \end{aligned}$$

Solving this system of linear equations we get the solution

$$A_1=-3/16 \quad A_2=7/4 \quad B=4/3 \quad C=-55/48$$

Then

$$h(x)=\frac{5x^2-2x+4}{(x-1)^2(x)(x+3)}=-\frac{3}{16}\frac{1}{(x-1)}+\frac{7}{4}\frac{1}{(x-1)^2}+\frac{4}{3}\frac{1}{x}-\frac{55}{48}\frac{1}{(x+3)}$$

From this decomposition we can hopefully integrate the remaining functions relatively easily.

So how do we integrate rational functions?

There are nearly as many methods for handling these functions' integrals as there are forms of rational functions. We'll restrict our attention to three distinct types:

- The numerator divides the denominator leading to a simpler form for the rational functions that may be integrated diectly or via another method. Given the possibilities for these functions, we'll restrict them to functions where the order *m* is not greater then *n*+2; i.e. *m*=*n*+1 or *m*=*n*+2.
- The denominator is a quadratic and the numerator is either a constant or a linear function of *x*.
- The denominator is either already factored or easily factored. Then we use partial fractions to enable the integral evaluation.

The first of these three relies on the other two for its solution and so we will concentrate on the second and third types and apply them as required to solve problems of the first type.

Integrals of Rational Functions with Quadratic Denominator

These functions are of the form

$$h(x)=\frac{a_1 x+a_0}{b_2 x^2+b_1 x+b_0} \quad a_0, a_1, b_0, b_1, b_2 \in \mathbb{N}; b_2 \neq 0$$

We can handle these functions as follows:

1. If the numerator is such that it is proportional to the derivative of the denominator, then we can use integration by substitution to solve the integral: i.e.

$$\int \frac{2ax+b}{ax^2+bx+c}dx; \quad a\neq 0$$

$$\text{Let } u=ax^2+bx+c \Rightarrow \frac{d}{dx}u=2ax+b$$

$$\Rightarrow \int \frac{2ax+b}{ax^2+bx+c}dx=\left[\int \frac{1}{u}du\right]_{u=ax^2+bx+c}$$

$$\Rightarrow \int \frac{2ax+b}{ax^2+bx+c}dx=\ln|u|\Big|_{u=ax^2+bx+c}+C$$

$$\Rightarrow \int \frac{2ax+b}{ax^2+bx+c}dx=\ln|ax^2+bx+c|+C$$

2. If the numerator is as above but the quadratic is raised to some power of *n*, then the following approach is used:

$$\int \frac{2ax+b}{\left(ax^2+bx+c\right)^n}dx; \quad a\neq 0; n\in\mathbb{N}$$

$$\text{Let } u=ax^2+bx+c \Rightarrow \frac{d}{dx}u=2ax+b$$

$$\Rightarrow \int \frac{2ax+b}{\left(ax^2+bx+c\right)^n}dx=\left[\int \frac{1}{u^n}du\right]_{u=ax^2+bx+c}$$

$$\Rightarrow \int \frac{2ax+b}{\left(ax^2+bx+c\right)^n}dx=\left[\frac{1}{1-n}u^{1-n}\right]_{u=ax^2+bx+c}+C$$

$$\Rightarrow \int \frac{2ax+b}{\left(ax^2+bx+c\right)^n}dx=\left(\frac{1}{1-n}\right)\frac{1}{\left(ax^2+bx+c\right)^{n-1}}+C$$

3. If the numerator is a constant the substitution approach above is useless. Instead a more involved approach is required that has its

origins in the ubiquitous "roots of a quadratic equation formula". Consider the integral:

$$\int \frac{1}{\left(a x^2+b x+c\right)^n} dx\,; \quad a\neq 0$$

We need to construct a perfect square in the denominator. To do this we divide out the quadratic coefficient, *a*, and construct the perfect square as follows:

$$\int \frac{1}{a x^2+b x+c} dx=\frac{1}{a}\int \frac{1}{x^2+\frac{b}{a}x+\frac{c}{a}} dx$$

$$\Rightarrow \int \frac{1}{a x^2+b x+c} dx=\frac{1}{a}\int \frac{1}{x^2+2\frac{b}{2a}x+\frac{c}{a}} dx$$

$$\Rightarrow \int \frac{1}{a x^2+b x+c} dx=\frac{1}{a}\int \frac{1}{x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}} dx$$

$$\Rightarrow \int \frac{1}{a x^2+b x+c} dx=\frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{4\mathrm{ac}-b^2}{4a^2}} dx$$

Believe it or not, we're nearly there. We make the following substitution

$$u=x+\frac{b}{2a} \quad \alpha=\frac{1}{2\mathrm{a}}\sqrt{4ac-b^2}$$

$$\Rightarrow \int \frac{1}{a x^2+b x+c} dx=\left(\frac{1}{a}\right)\left[\int \frac{1}{u^2+\alpha^2} du\right]_{u=x+b/2\mathrm{a}}$$

$$\Rightarrow \int \frac{1}{a x^2+b x+c} dx=\left(\frac{1}{a}\right)\left[\frac{1}{\alpha}\tan^{-1}\left(\frac{u}{\alpha}\right)\right]_{u=x+b/2\mathrm{a}}+C$$

Then substituting back for *u* and α yields the result:

$$\int \frac{1}{a x^2+b x+c} dx=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C$$

In the next section we'll apply the partial fractions method to the solution of the integral of more general rational functions.