Section 5
Vectors in 2D and 3D

In this section we introduce the concept of vectors in 2 and 3 dimensional spaces. We will define these spaces mathematically, describe the various types that will be of relevence, define mathematical properties of vectors and prepare the way for more complicated operations to follow in the next section.

What is a vector?

Simply put, a vector (or Euclidean Vector) is a directed line segment. It is a mathematical entity that has both magnitude (length of the line segment) and direction (hence directed).

Notation

By convention the letters u, v, and w are used in vector notation. Therefore if the letter u is being used for our vector then the vector is denoted $\vec{u}$; the right arrow $\rightarrow$ indicating that u is a vector.

Graphically a vector $\vec{u}$ in 2D would be represented as below

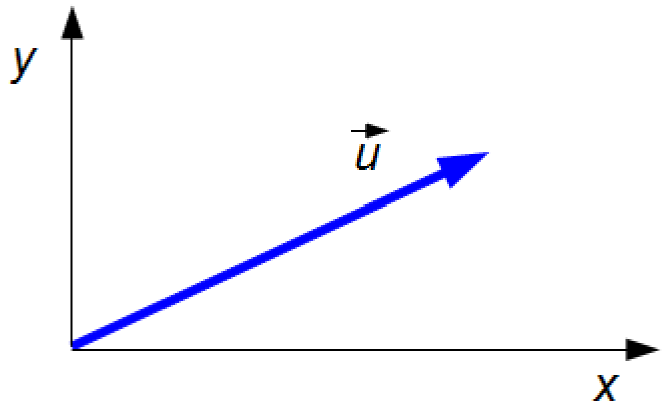

Figure: Graphical representation of a 2D vector $\vec{u}$.

The vector (in blue) is shown as a finite line segment which *starts* at the origin (0,0) and *ends* at some arbitrary point in the 2D plane that determines the specific vector.

How can a line segment have a start and end?

In itself, a line segment cannot but a vector, which is a *directed* line segment, can. The start (base) of a vector is that end that does not have the arrow head. Therefore, the vector's end is that which has the arrow head.

How do we assign quantities to such an entity?
Let's startr with the length or magnitude of the vector, $\vec{u}$, denoted $\|\vec{u}\|$. If we denote by (a,b) the point at the end of the vector then the magnitude (or norm) of the vector, $\|\vec{u}\|$, is simply the length of the line segment from the origin (0,0) to the endpoint (a,b):

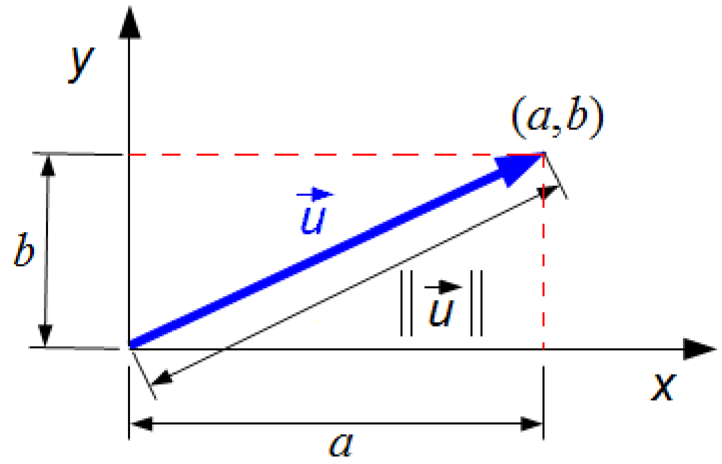

Figure: The magnitude (norm)of the vector $\vec{u}$, $\|\vec{u}\|$.

From your elementary geometry of the line and circle, you should recall that the length of a line segment between two points (x_1, y_1) and (x_2, y_2) on the plane is determined from the formula:

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For our vector the point (x_1, y_1) serves as the *start* and thus is the origin (0,0). The second point (x_2, y_2) equates to the *end* of the vector and is equal to (a, b). Then the above formula becomes:

$$\text{Length} = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

All well and good, but what if the vector started at a point other than the origin, (0,0)? To answer this and related questions we need to introduce a proper analytical approach to handling this entity: we need a convention whereby specific directions are defined analytically. To this end, we introduce the concepts of basis vectors and orthogonal bases.

Unit and Basis Vectors

A unit vector, $\hat{v}$, is a vector whose magnitude is 1. For such vectors, a special notation is used to distinguish them from normal vectors; i.e. the arrow $\rightarrow$ is replaced with the hat ^. Then any vector that has a hat instead of an arrow is a unit vector (or vector of unit length).

Basis Vectors
For higher dimensional spaces (≥ 2 Dimensions), there is an analytical requirement to express an arbitrary vector, $\vec{u}$, in terms of a fixed invariant set of unit vectors such that all vectors in this space are so expressed. This will permit differences, properties, and mathematical operations in general to be developed.

So, what does all this mean and how do we apply it to our 2D case?
Consider the 2D vector $\vec{u}$ in the last figure. We can see that $\vec{u}$ has part of itself (a *component*) parallel to the horizontal x axis; this is called the *horizontal* or *x-component*. Similarly, $\vec{u}$ has a *vertical* or *y-component* parallel to the vertical y axis. We have labelled these a and b respectively. In going from (0,0) to (a,b) along $\vec{u}$, we could just as easily have taken two equally valid equivalent paths:

1. Move from (0,0) to (a,0) horizontally and the from (a,0) to (a,b) vertically
2. Move from (0,0) to (0,b) vertically and the from (0,b) to (a,b) horizontally.

i.e.

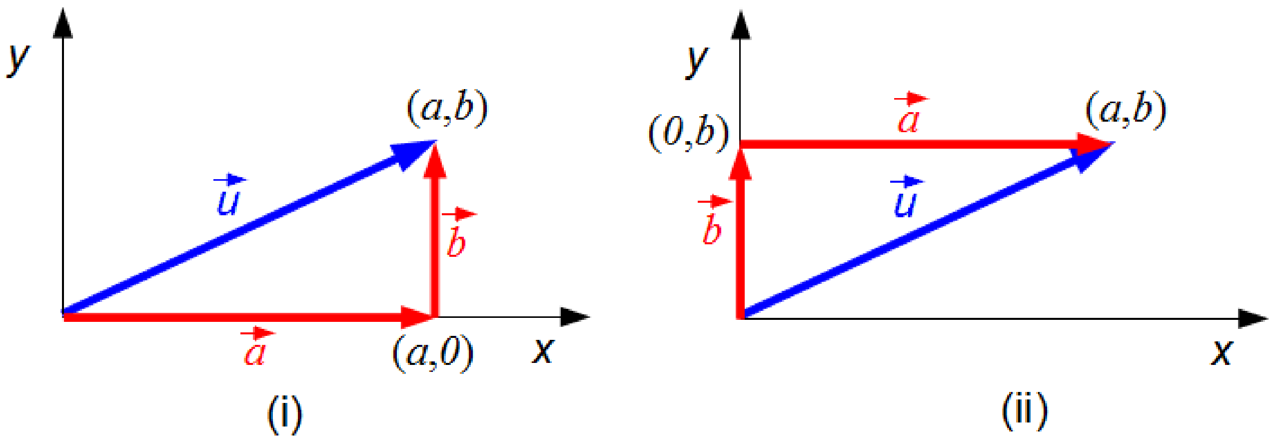

Figure: Two equivalent paths from (0,0) to (a,b).

N.B.

1. You can see from the figure above that we have chosen to denote the path from (0,0) to (a,0) by the vector $\vec{a}$ and that from (0,0) to (0,b) as $\vec{b}$.
2. The reason for doing so is quite simple. In going from (0,0) to (a,0), for example, we are moving a distance of length (magnitude) a in a

given direction (from (0,0) to (a,0)). So we have magnitude and direction; i.e. a vector.

3. For the path from (a,0) to (a,b) we are travelling a distance of b up the vertical axis: i.e. equivalent to the vector $\vec{b}$.[1]

So what can we take from the construction in the last figure?
We have de-constructed the arbitrary vector into two component vectors that are mutually perpendicular. We did this because we frequently express points, lines, geometric entities on the 2D plane in terms of two variables; namely x and y for the horizontal and vertical displacements respectively.

Can we do the same with vectors?
There is an equivalent method for vectors that allows us to express the horizontal and vertical components of a vector in terms of two special vectors. These are called **BASIS VECTORS** and they have the property that all vectors in the 2D plane can be expressed in terms of them. Now to their definitions:

Definition
For the 2D Euclidean Plane, we define the basis set, $\{\hat{i}, \hat{j}\}$, where

- $\hat{i}$ is the unit vector from (0,0) to (1,0); horizontal along the x-axis.
- $\hat{j}$ is the unit vector from (0,0) to (0,1); vertical along the y-axis.

Then any arbitrary vector $\vec{u}$ from (0,0) to (a,b) can be written as

$$\vec{u} = a\,\hat{i} + b\,\hat{j} \quad \forall\, a, b \in \mathbb{R}$$

Graphically this is represented below.

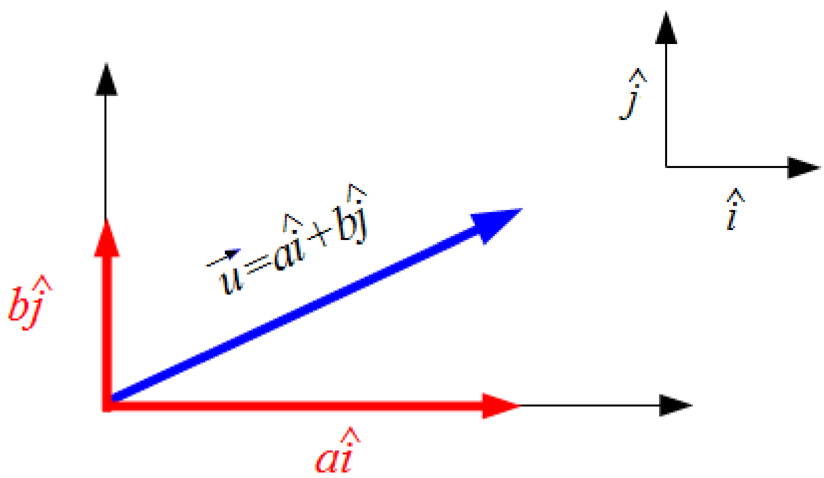

Figure: The conventional representation of a 2D Euclidean vector

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1 If $\vec{u}$ and $\vec{v}$ are two vectors that are mutually parallel and if $\|\vec{u}\| = \|\vec{v}\|$ then $\vec{u}$ and $\vec{v}$ are equal.

The small axes are drawn in the figure above to illustrate what directions the unit vectors forming the basis point to.

3D basis set

This process can be extended to 3D vectors with the added complexity that in 3 dimensional space your basis set must have 3 unit vectors. Two of these can, naturally, coincide with those in the 2D basis set but a third is necessary to completely describe the 3D volume.

So, if we use $\hat{i}$ for the *x*-direction and $\hat{j}$ for the *y*-direction, it makes sense to extend the basis by using $\hat{k}$ for the *z*-direction. For the 3D space the basis set would then be $\{\hat{i},\hat{j},\hat{k}\}$.

An arbitrary vector $\vec{u}$ pointing from the origin (0,0,0) to a point (*a*,*b*,*c*) would now have the representation

$$\vec{u}=a\,\hat{i}+b\,\hat{j}+c\,\hat{k} \quad \forall\, a,b,c\in\mathbb{R}$$

Graphically it would be represented as follows:

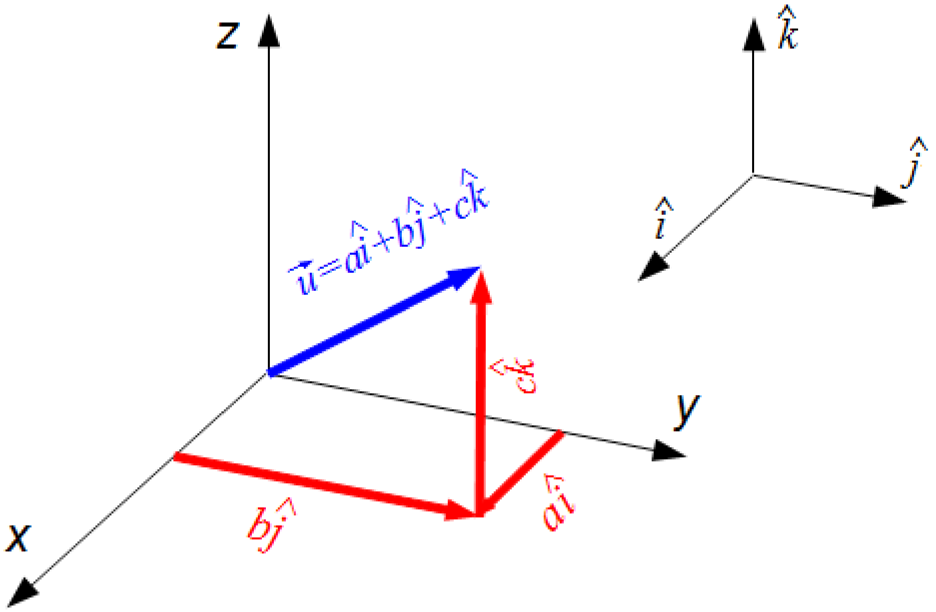

Figure: Graphical representation of an arbitary 3D vector $\vec{u}$

The magnitude of this vector is calculated using an extension to 3D of the same *length of a line segment* formula we used previously: i.e. the length of a line segment from (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\text{Length}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

For our vector this equates to the length from (0,0,0) to (a,b,c); i.e.

$$\text{Length} = \sqrt{(a-0)^2 + (b-0)^2 + (c-0)^2} = \sqrt{a^2+b^2+c^2}$$

Orthogonal and Orthonormal Bases

Two vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal if and only if they are mutually perpendicular; i.e. the lesser angle between them is $\pi/2$ radians. Then

- a basis set is orthogonal if and only if every element of the set is mutually perpendicular to every other element.
- A basis set is orthonormal if and only if it is both an orthogonal basis and every element in the basis is a unit vector.

For 2D the basis $\{\hat{i}, \hat{j}\}$ is orthonormal whereas the basis $\{3\hat{i}, -2\hat{j}\}$ is orthogonal.
For 3D the basis $\{\hat{i}, \hat{j}, \hat{k}\}$ is orthonormal while the basis $\{4\hat{i}, \pi\hat{j}, \sqrt{2}\hat{k}\}$ is orthogonal.

N.B.

By convention, when using the above i, j, k notation the k should always be last, the j should follow the i which should be first. However if any of the components are zero they can be omitted. However, for clarity, in all the examples we'll consider, all components will be included even those that are zero.
e.g.

$$\vec{u} = 4\hat{i} - 0\hat{j} + 3\hat{k} \text{ instead of } \vec{u} = 4\hat{i} + 3\hat{k}$$

Mathematical Properties

To continue our analysis of vectors let us consider some fundamental mathematical properties.

1. **Sum of two vectors**.
   Let $\vec{u}$ and $\vec{v}$ be any two arbitrary vectors in 2D and let
   $$\vec{u} = a\hat{i} + b\hat{j} \text{ and } \vec{v} = c\hat{i} + d\hat{j} \quad \forall a, b, c, d \in \mathbb{R}$$
   Then we define the sum of $\vec{u}$ and $\vec{v}$ to be
   $$\begin{aligned} \vec{u} + \vec{v} &= (a\hat{i} + b\hat{j}) + (c\hat{i} + d\hat{j}) \\ &= (a+c)\hat{i} + (b+d)\hat{j} \\ &= \vec{v} + \vec{u} \end{aligned}$$

i.e. you sum the components in each direction independently of each other. Furthermore, it is a commutative operation.
In 3D, it can be modified for

$$\vec{u}=a\,\hat{i}+b\,\hat{j}+c\,\hat{k} \text{ and } \vec{v}=\alpha\,\hat{i}+\beta\,\hat{j}+\gamma\,\hat{k} \quad \forall\, a,b,c,\alpha,\beta,\gamma\in\mathbb{R}$$

with

$$\begin{aligned}\vec{u}+\vec{v} &= \left(a\,\hat{i}+b\,\hat{j}+c\,\hat{k}\right)+\left(\alpha\,\hat{i}+\beta\,\hat{j}+\gamma\,\hat{k}\right)\\ &= (a+\alpha)\,\hat{i}+(b+\beta)\,\hat{j}+(c+\gamma)\,\hat{k}\\ &= \vec{v}+\vec{u}\end{aligned}$$

As before, you sum the corresponding components independently.

2. **Scalar Multiplication**
   Let $\vec{u}=a\,\hat{i}+b\,\hat{j}$ be any 2D vector and let $\alpha\in\mathbb{R}$ be any real constant. Then we define the scalar product

$$\begin{aligned}\alpha\vec{u} &= \alpha\left(a\,\hat{i}+b\,\hat{j}\right)\\ &= \left(\alpha a\,\hat{i}+\alpha b\,\hat{j}\right)\end{aligned}$$

   For $\alpha>0$ the resulting vector lies along the same direction as the original but is either shorter ($\alpha<1$) or longer ($\alpha>1$). If $\alpha<0$ the vector is in the opposite direction and can be shorter ($\alpha<1$) or longer ($\alpha>1$). $\alpha=1$ is the identity operation while $\alpha=-1$ results in a central symmetric operation.

There are more properties to list and we'll examine them in the next section. First let us consider these two graphically, then introduce a few "laws" and finally relate vectors to matrices.

The Parallelogram and Triangle Laws
These are actually equivalent laws and apply to the sum and difference of vectors. For clarity we'll restrict them to the 2D case.

The parallelogram law:
This law states:

> *If two vectors are represented by two adjacent sides of a parallelogram, then the diagonal of parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.*

A graphic representation of the parallelogram law and its interpretation in terms of the summation of two vectors $\vec{u}$ and $\vec{v}$ resulting in a new vector $\vec{w}$ is shown in the figure below:

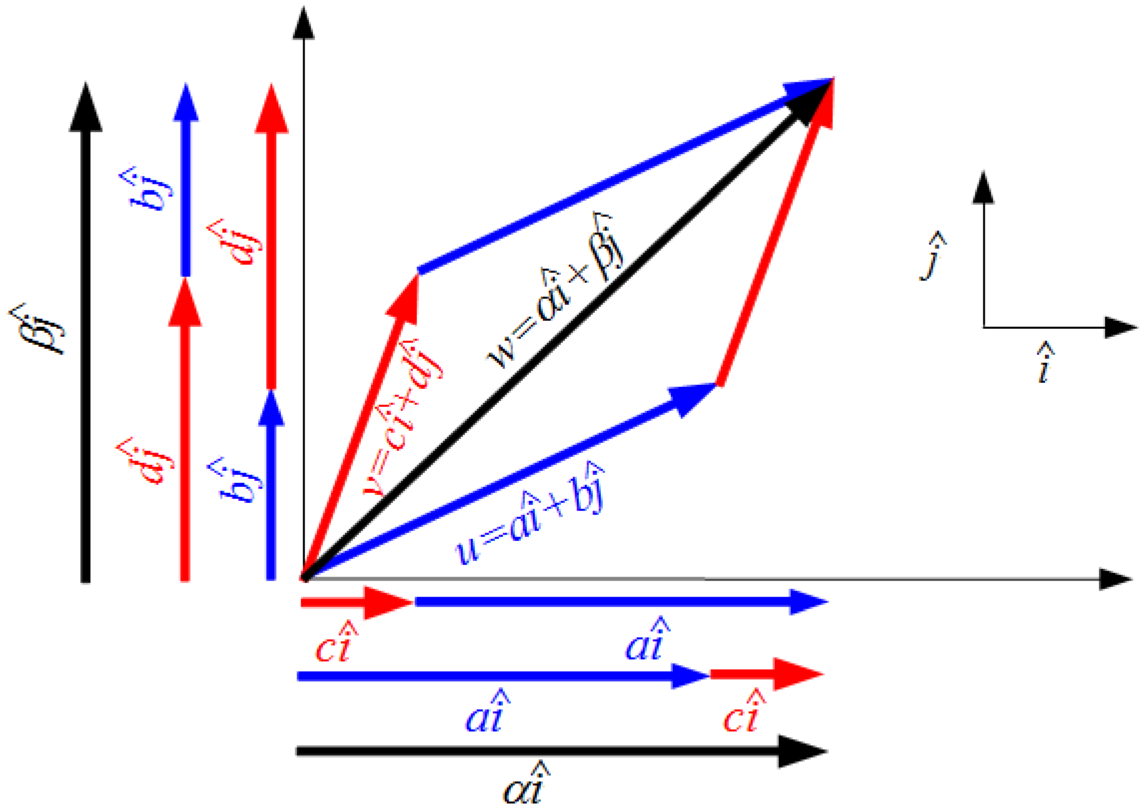

Figure: The sum of two vectors under the parallelogram law.

Then from this figure above we can deduce:

$$\begin{aligned} \vec{w} = \vec{u}+\vec{v} &= (a\,\hat{i}+b\,\hat{j})+(c\,\hat{i}+d\,\hat{j}) \\ &= (a+c)\,\hat{i}+(b+d)\,\hat{j} \\ &= \alpha\,\hat{i}+\beta\,\hat{j} \end{aligned}$$

Therefore

$$\begin{aligned} \alpha &= a+c \\ \beta &= b+d \end{aligned}$$

as illustrated above.

The Parallelogram law, as a matter of fact, is an alternate statement of triangle law of vector addition which we'll state now.

The Triangle Law

This law states:

> *If two vectors are represented by two sides of a triangle in sequence, then third closing side of the triangle, in the opposite direction of the sequence, represents the sum (or resultant) of the two vectors in both magnitude and direction.*

Here, the phrase "in sequence" means that the vectors are placed such that tail of a vector begins at the arrow head of the vector placed before it.

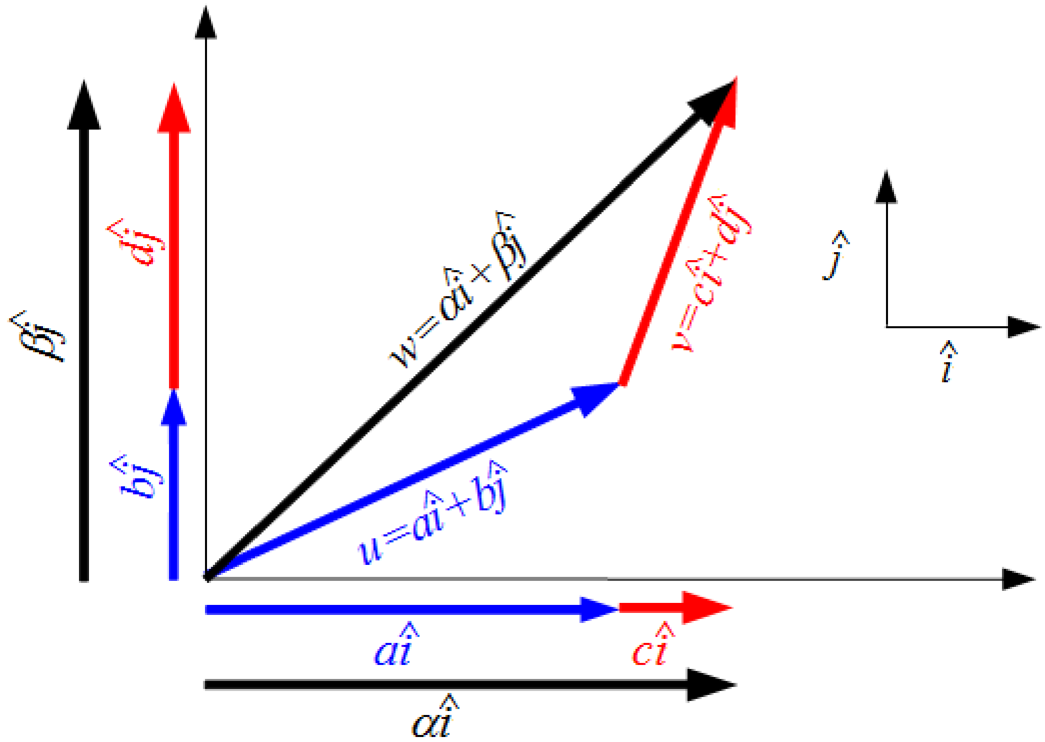

Figure: The triangle law applied to $\vec{u}+\vec{v}$

Note how in the figure caption we have specifically stated $\vec{u}+\vec{v}$ and not the sum of $\vec{u}$ and $\vec{v}$. For the triangle law, the order is important even if the result is the same as the sum is commutative. Nevertheless, as the law states, the second vector $\vec{v}$ (in red) is added to the end of the initial vector $\vec{u}$ (in blue) giving the resultant vector $\vec{w}=\vec{u}+\vec{v}$.

Then, as for the parallelogram law, we find

$$\begin{aligned}\vec{w} = \vec{u}+\vec{v} &= (a\hat{i}+b\hat{j})+(c\hat{i}+d\hat{j})\\ &= (a+c)\hat{i}+(b+d)\hat{j}\\ &= \alpha\hat{i}+\beta\hat{j}\end{aligned}$$

with

$$\begin{array}{rcl}\alpha & = & a+c \\ \beta & = & b+d\end{array}$$

as expected.

For the reverse order sum, $\vec{v}+\vec{u}$, we have the mirror of the above triangle through the vector $\vec{w}$. This is shown below. The resultant vector, $\vec{w}=\vec{v}+\vec{u}$, is identical to the result above thus confirming the commutative nature of the summation of two vectors.

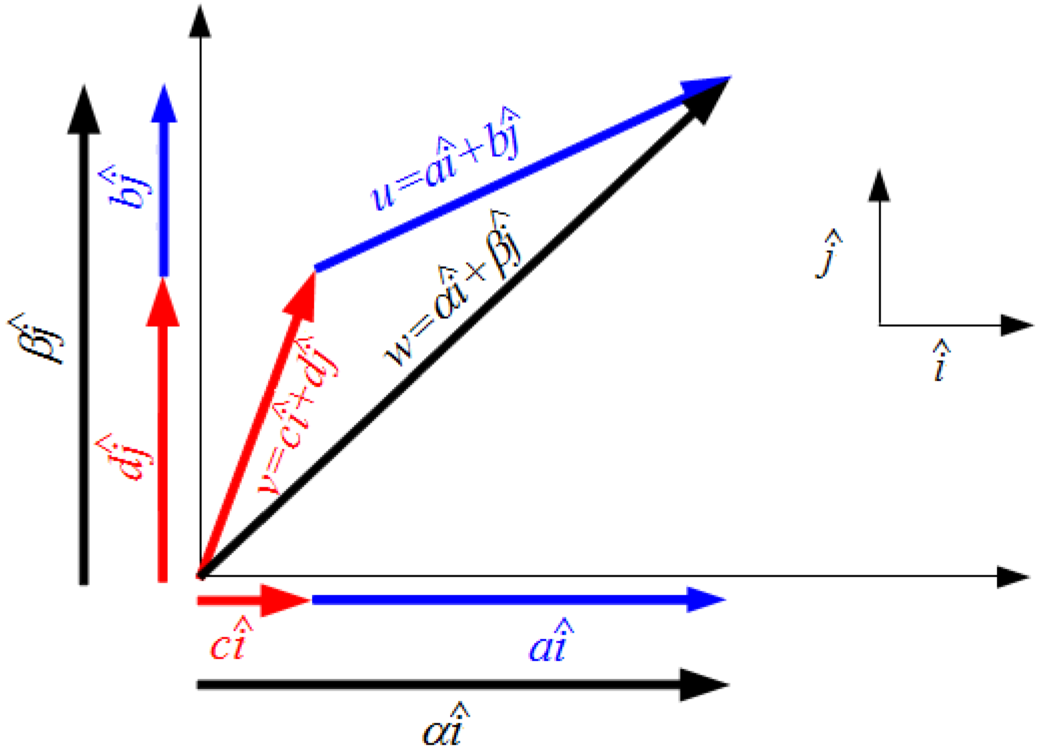

Figure: The triangle law for the sum $\vec{v}+\vec{u}$

So, in conclusion, the parallelogram law effectively includes both versions of the triangle law and doesn't care about order. The triangle laws do care about the order in the summation but, as it is commutative, the result is order independent.

Graphical Interpretation of Scalar multiplication

As we have already stated, the result of multiplying a vector by a real constant ($\alpha \in \mathbb{R}$) is to scale the vector; hence the term scalar multiplication.

If the scalar is negative it first turns the vector around through π radians and then scales it according to the magnitude of the scalar:

- If $|\alpha|<1$ then the vector is shrunk in length.
- If $|\alpha|>1$ the vector is stretched in length.

These concepts are shown below.

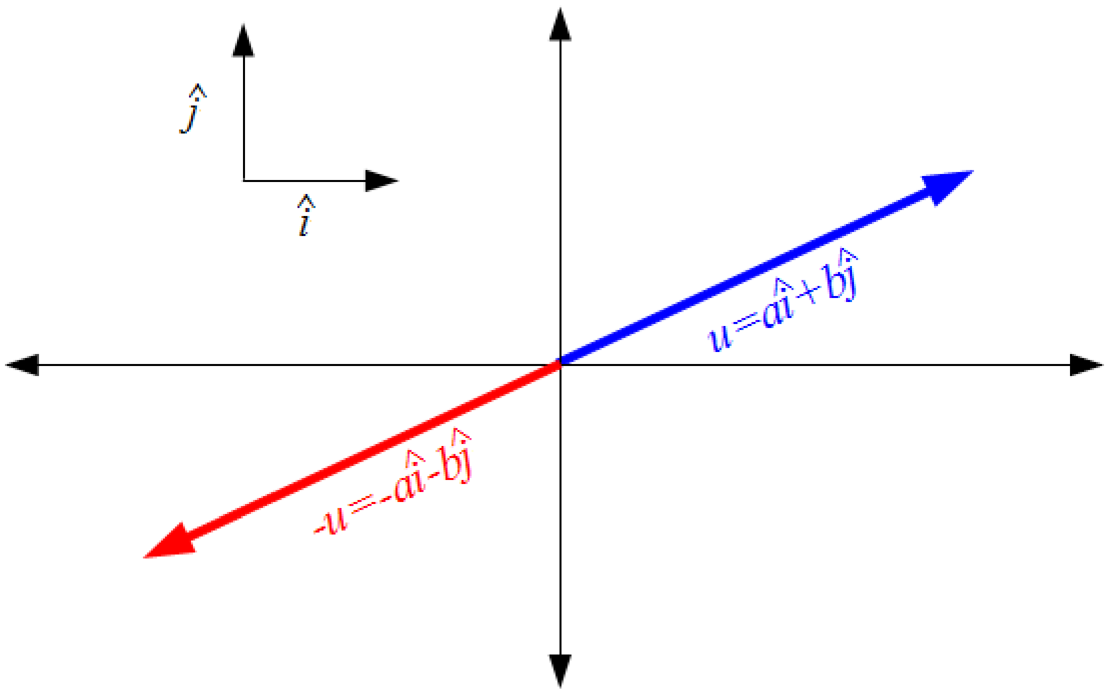

Figure: The result of multiplying $\vec{u}$ by -1.

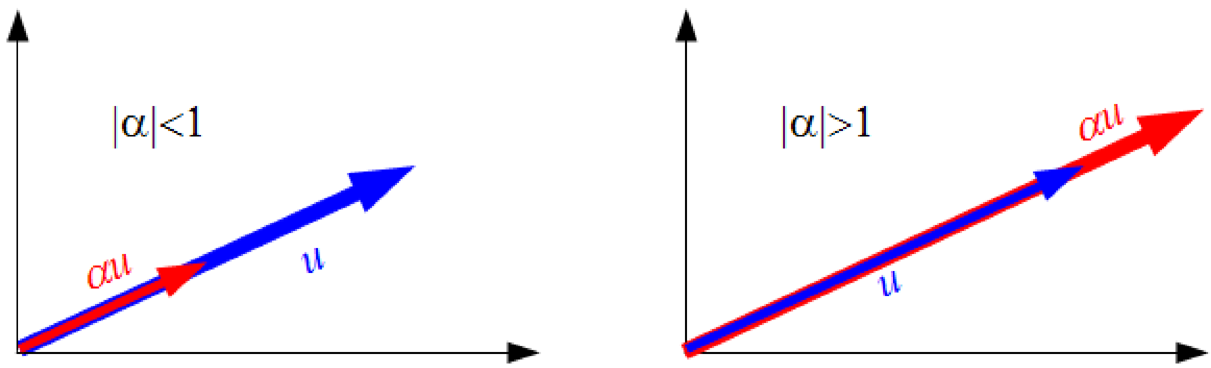

Figure: The shrinking and stretching effect of a scalar multipliation.

Now to the final concept to be introduced in this section. What follows is a notation convention that is widely used in mathematics; that of using matrix notation to represent a vector.

Vector Matrix Notation

To see how vectors are related to matrices, consider the method of using matrix inverses to solve systems of linear equations: i.e. to solve

$$\begin{aligned} a_{11}x+a_{12}y+a_{13}z &= b_1 \\ a_{21}x+a_{22}y+a_{23}z &= b_2 \\ a_{31}x+a_{32}y+a_{33}z &= b_3 \end{aligned}$$

we expressed this system in the form

$$A\cdot X=B$$

where

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Now let us examine this matrix *X*.

This column matrix has three elements coincidentally named *x*, *y*,and *z*. Have these anything to do with the 3 axes in the 3D Euclidean volume? Consider the constuction below. We are attempting to relate the 3×1 column matrix with components *X*, *Y*, and *Z* with the point (*X*,*Y*,*Z*)in the 3D volume.

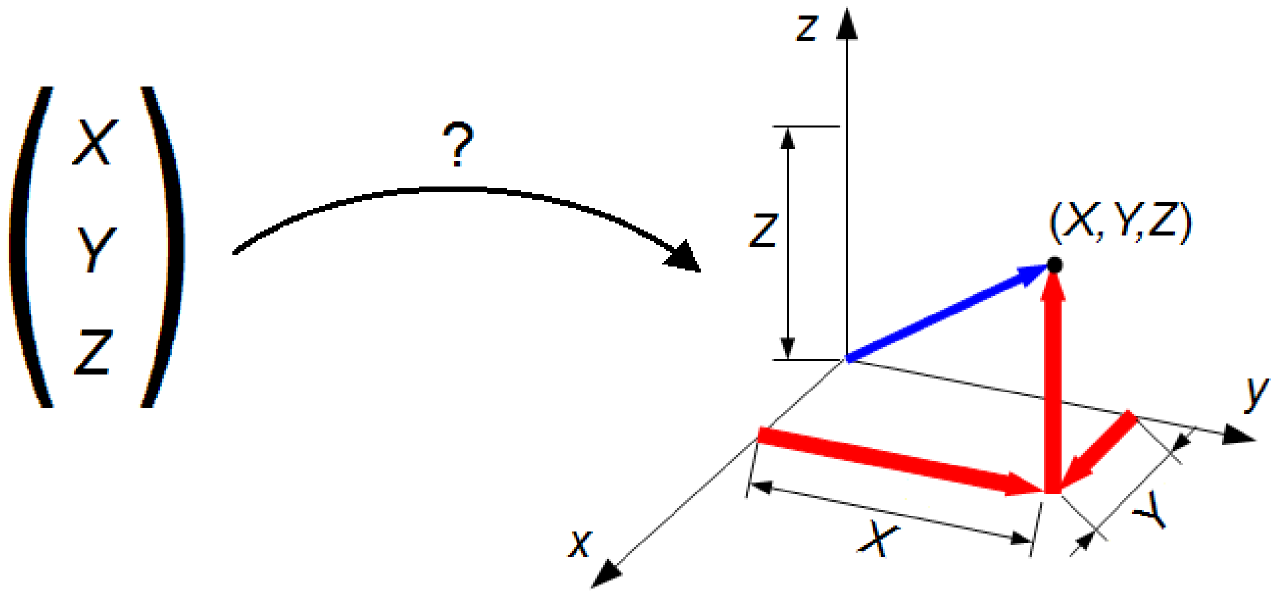

Figure: Correspondence between a 3×1 column matrix and a point in 3D space.

The correspondence is quite simple:

- The first row element in the matrix is equated to the first axis in the volume; i.e. the *x*-component.
- The second row element to the second axis; i.e. the *y*-component.
- The third row element to the third axis; i.e. the *z*-component.

By this method we can express vectors like

$$\vec{u}=a\hat{i}+b\hat{j}+c\hat{k} \text{ and } \vec{v}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k} \quad \forall a,b,c,\alpha,\beta,\gamma\in\mathbb{R}$$

as

$$\vec{u}=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \vec{v}=\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \forall a,b,c,\alpha,\beta,\gamma\in\mathbb{R}$$

where $u,v\in\mathbb{R}^3$.

Our basis $\{\hat{i},\hat{j}\}$ for 2D then becomes $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ while our basis $\{\hat{i},\hat{j},\hat{k}\}$ for 3D becomes $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Question:
If you combined the three 3×1 column matrices above into one 3×3 matrix, what matrix would it be?
Why do you think this is so?

In the next section we take the vectors to the next level by considering three products involving vectors; the innerproduct (or dotproduct), the crossproduct and the triple product.

We also consider the geometric interpretations of these products.