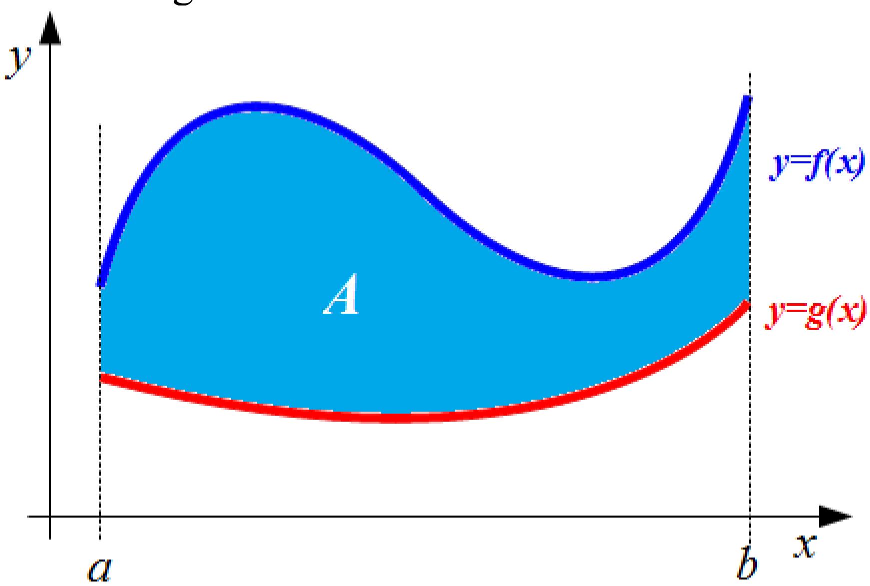
Section 6 Cont'd
Areas and Volumes of Rotation

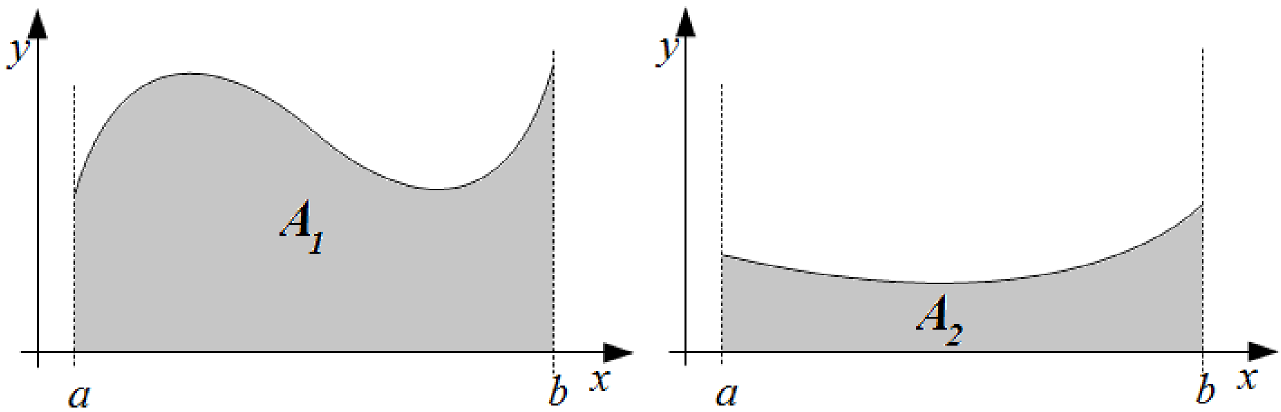
In this section we continue our analysis of integration by considering the problem of calculating areas and volumes using integrals. We return to the origins of the integral and re-examine the problem of areas under curves in order to apply the technique to areas bounded by two curves. Then, we consider the related problem of calculating volumes by considering the simplest form; the volume of rotation.

Areas bounded by two curves.

Consider the following construction:



These two functions $f(x)$ and $g(x)$ are continuous and finite on the compact interval [a,b] and, as such, from our introduction to the definite integral, they both bound areas between the x-axis and their respective curves; $y=f(x)$ and $y=g(x)$. We denote these areas by A_1 and A_2 respectively in the figures below:



with

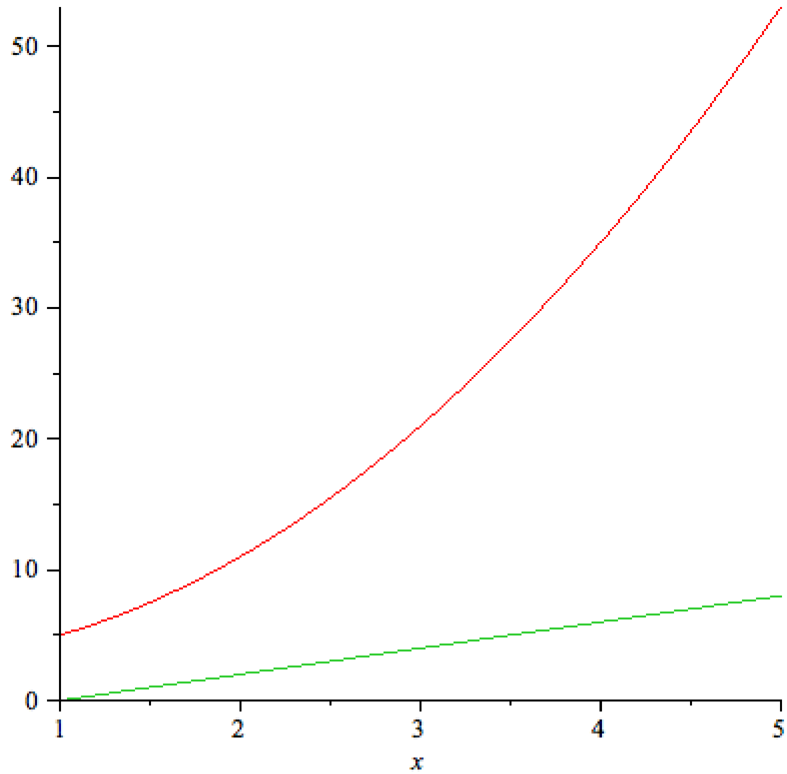
$$A_1 = \int_a^b f(x)\,dx \quad \text{and} \quad A_2 = \int_a^b g(x)\,dx$$

It follows, therefore, that the area A shaded in blue in the first figure is given by

$$A = |A_1 - A_2| = \left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| = \int_a^b |f(x) - g(x)|\,dx$$

where the absolute value guarantees that those areas, resulting from functions that are negative, are made positive.

Example:
Find the area under the curves $y = f(x) = 2x^2 + 3$ and $y = g(x) = 3x - 3$ on [1,4]. The functions are shown below:



As is obvious from the picture, the red curve $y = f(x) = 2x^2 + 3$ is always greater than the green curve $y = g(x) = 3x - 3$. The area between the curves is then

$$A = |A_1 - A_2| = \left| \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx \right| = \int_1^5 2x^2 + 3\,dx - \int_1^5 3\mathrm{x} - 3\,dx$$

Therefore

$$A=\left[\frac{2}{3}x^3+3x\right]_1^5-\left[\frac{3x}{2}-3x\right]_1^5$$

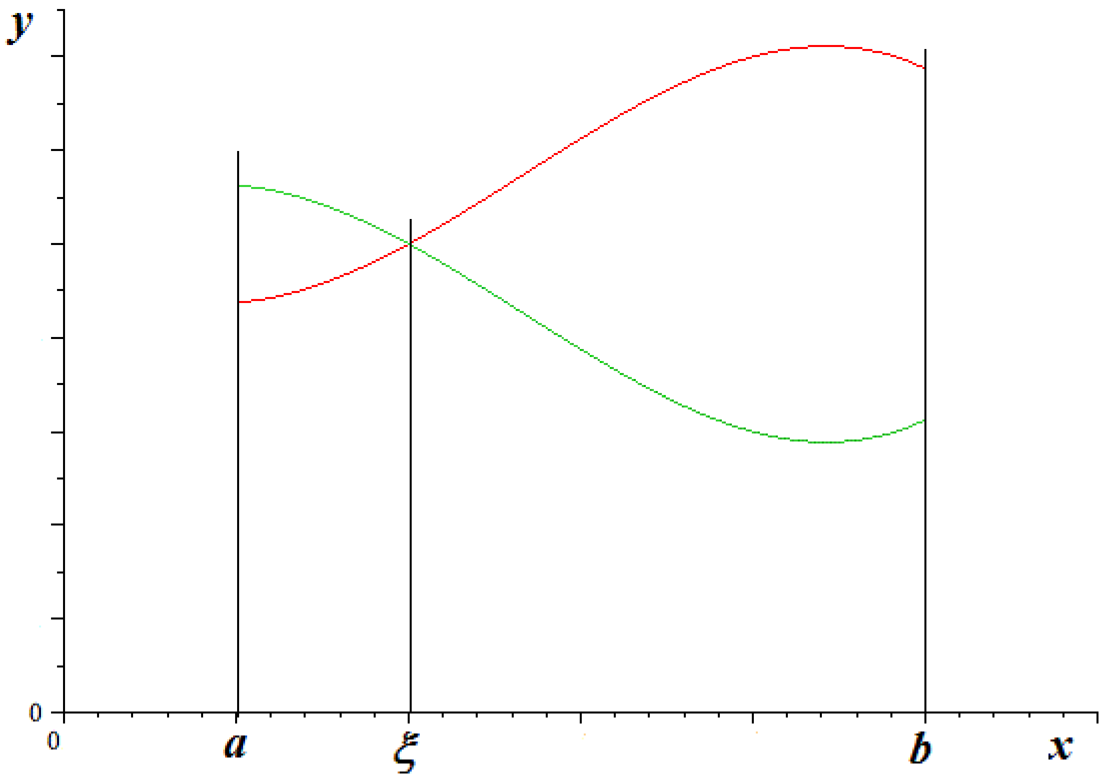
$$\Rightarrow A=\left[\frac{2}{3}(125-1)+3(5-1)\right]-\left[\frac{3}{2}(25-1)-3(5-1)\right]$$

$$\Rightarrow A=\frac{248}{3}-36=\frac{356}{3}\ \textit{square units}$$

This is relatively straightforward. If the functions are such that they intersect at some $\xi\in[a,b]$ then we have the situation where either

$$f(x)>g(x)\,\forall\,x\in[a,\xi) \text{ and } g(x)>f(x)\,\forall\,x\in(\xi,b]$$

or vice versa. This is shown below where the green curve is $f(x)$ and the red is $g(x)$:



Then the area contained between the curves on the compact interval $[a,b]$ is

$$A=\int_a^\xi (f(x)-g(x))\,dx+\int_\xi^b (g(x)-f(x))\,dx$$

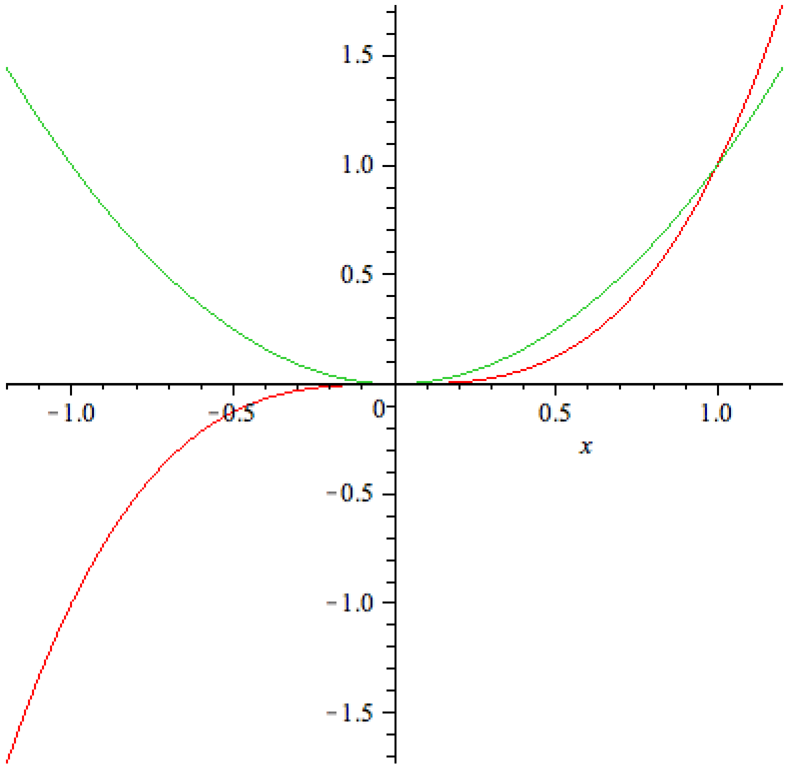
If the curves intersect at a set of values $\{\xi_1,\ldots,\xi_n\}$, then the area would be

$$A=\sum_{k=0}^{n}\int_{\xi_k}^{\xi_{k+1}}|f(x)-g(x)|\,dx$$

with $\xi_0=a$ and $\xi_{n+1}=b$. The absolute value |..| guarantees that each area

segment will be always positive.
Let us consider a related example:

Example:
Find the area bounded by the curves $y=x^2$ and $y=x^3$.
The first thing you should notice is no compact interval is specified. There is a reason for this. The question has asked you to determine the area BOUNDED by the curves and so you should find that there is a finite area contained by the two curves above. To illustrate this we graph the two functions in the interval [-1.2,1.2]:



The symmetric positive green curve should be familiar to you as $y=x^2$ with the red asymmetric curve being that for $y=x^3$. You can see that the two intersect at two points and bound a small area between (0,0) and (1,1). This was known to be there because of the points of intersection between $y=x^2$ and $y=x^3$:
At intersection

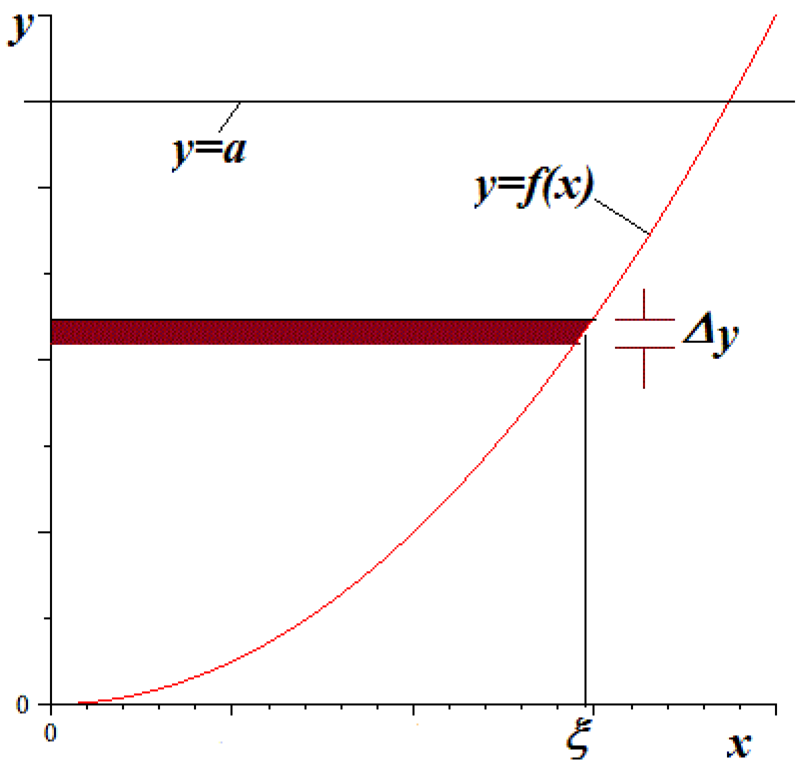
$$x^2=x^3$$
$$\Rightarrow x^2-x^3=0$$
$$\Rightarrow x^2(1-x)=0$$
$$\Rightarrow x=0 \text{ or } x=1$$

If x=0, then $y=x^2$ is at (0,0) and so is $y=x^3$. If x=1, then $y=x^2$ is at (1,1) and so is $y=x^3$.
Then the area is determined to be contained between the curves for $x\in[0,1]$:

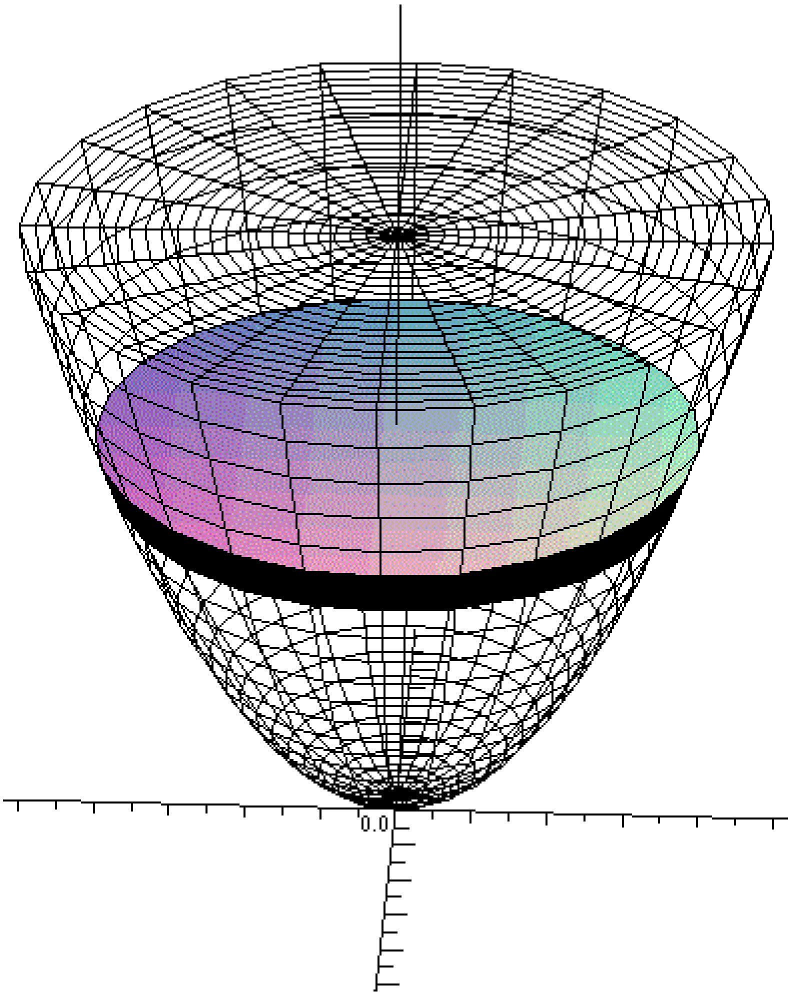
$$\begin{aligned} A=\int_0^1 (f(x)-g(x))\,dx &= \int_0^1 x^2-x^3\,dx \\ &= \left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1 \\ &= \frac{1}{3}-\frac{1}{4} \\ &= \frac{1}{12} \text{ square units} \end{aligned}$$

Volumes of Rotation (Disk-like slices)

The use of the integral to determine areas is well established. It might not be as obvious but integrals can be used to evaluate volumes too. We do this by rotating the area bounded by a curve (established through the integral) about its *y*-axis and show that the volume thus created can be calculated using the integral:
Consider the function below:



The area to be rotated about the *y*-axis is that bounded by the *y*-axis, the line *y*=*a*, and the curve *y*=*f*(*x*). The area of the red slice is approximately $\xi \Delta y$. If this is rotated about the *y*-axis then this slice traces out a disk-like cylindrical volume:



The volume of this disk is approximately the area of the disk times its thickness: i.e.

$$\Delta V \simeq \pi r^2 h = \pi \xi^2 \Delta y$$

from the volume of a cylinder. The total volume is then the sum of all such volumes as the radius ξ varies from 0 to the required height, *a* in this instance. The wireframe above shows the total volume to be calculated. Then, the total volume, *V*, is defined

$$V = \lim_{n \to \infty} \sum_{k=0}^{k=n} \pi \xi_k^2 \Delta y_k$$

where $\Delta y_k \equiv (a-0)/n = a/n$.

We know what this tends to in the above limit and so we make the following definition:

The volume of rotation, *V*, formed when the area bounded by the curves $y=f(x), y=a, x=0$ (the y-axis) is rotated about the *y*-axis is

$$V=\lim_{n\to\infty}\sum_{k=0}^{k=n}\pi\xi_k^2\Delta y_k=\int_0^a \pi x^2\,dy$$

which has a solution if and only if

$$\exists f^{-1}\text{ s.t. } f^{-1}\circ f(x)=x$$
$$\Rightarrow x^2\equiv\left(f^{-1}(y)\right)^2$$

Then we can write this volume as

$$V=\pi\int_0^a\left(f^{-1}(y)\right)^2dy$$

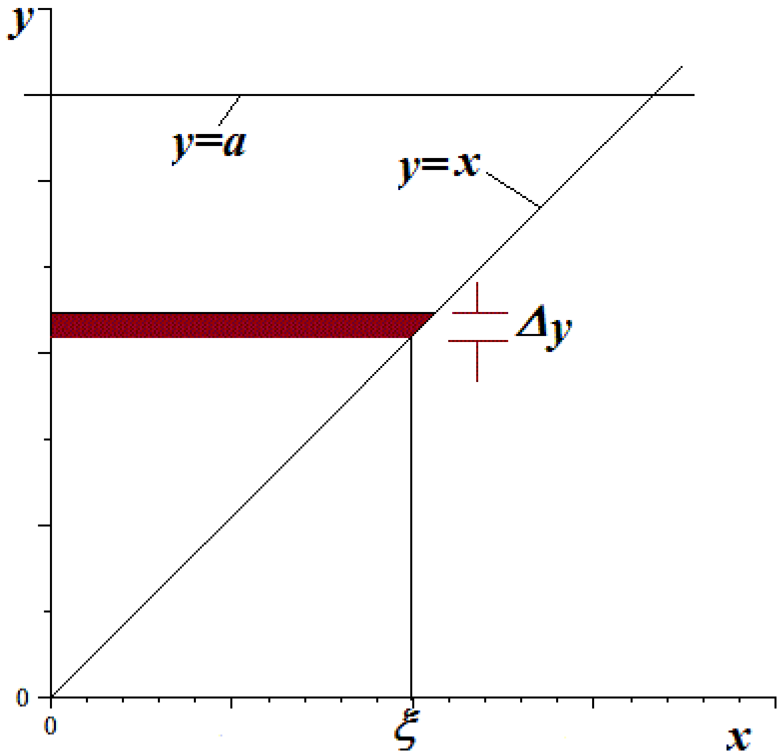
This is the formula for calculating the volume of rotation using disk-like slices.

Let's illustrate by example.

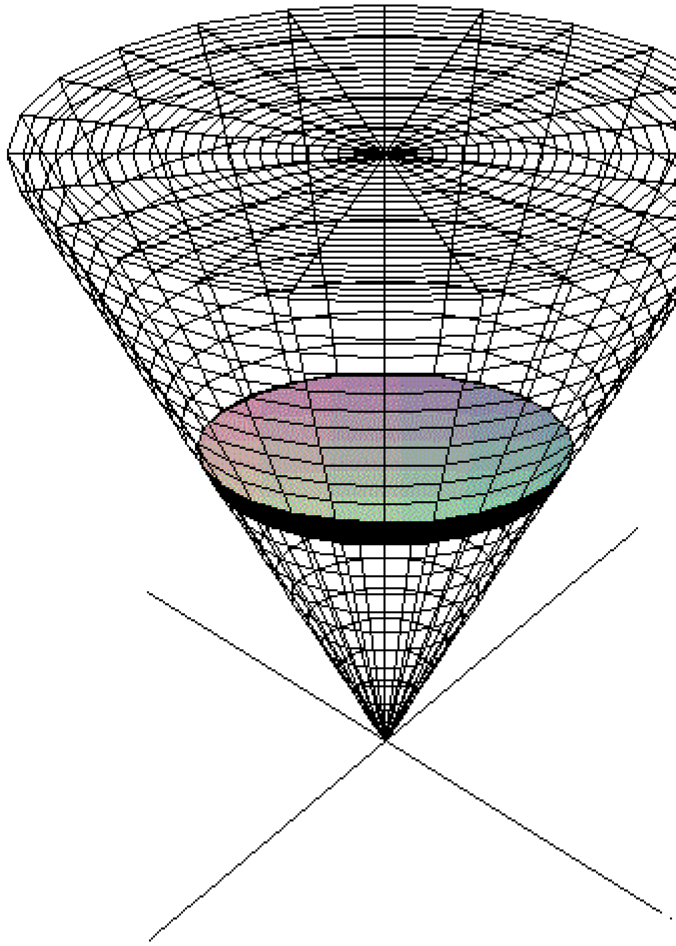
Example:

Calculate the volume of the region formed when the area bounded by the lines $y=r, x=0$ (the y-axis), and the curve $y=x$ is rotated about the *y*-axis.

The area is illustrated below:



From the rotation we get the conical volume



The area of the disk-like slice is

$$\Delta V = \pi \xi^2 \Delta y$$

As $\xi \equiv y$ on the curve $y = x$

$$\Rightarrow \Delta V = \pi y^2 \Delta y$$

Then, the total volume is

$$V = \pi \int_0^a \left(f^{-1}(y)\right)^2 dy = \pi \int_0^r y^2 dy$$

$$\Rightarrow V = \frac{\pi}{3}\left[y^3\right]_0^r = \frac{1}{3}\pi r^3$$

The volume of a cone is

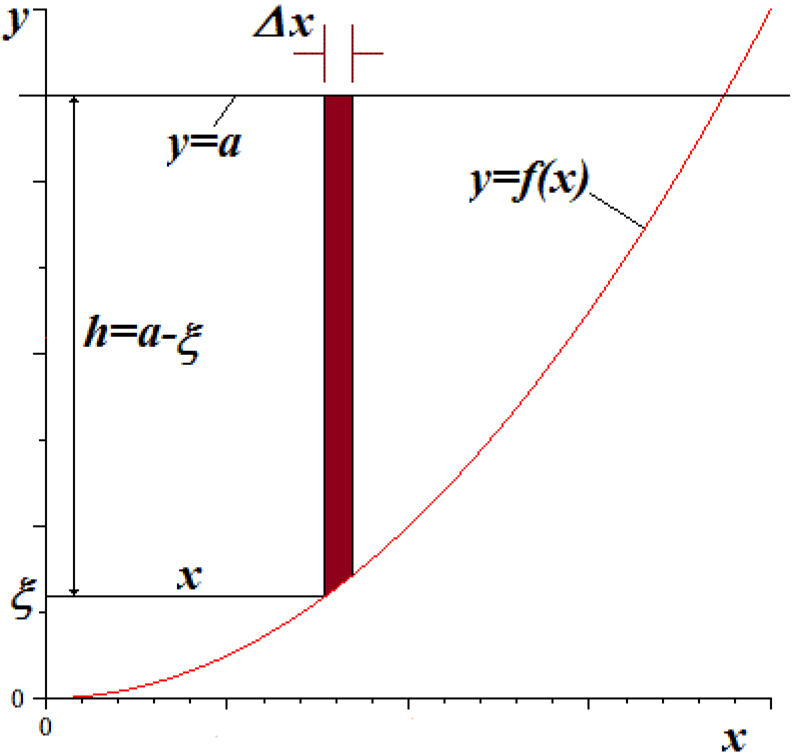
$$V_{cone} = \frac{1}{3}\pi r^2 h$$

but here height (y) equals width (x) through the curve $y=x$ and so

$$V = \frac{1}{3}\pi r^3 \equiv \frac{1}{3}\pi r^2 h$$

Exercise:
Repeat the above replacing the curve $y=\alpha x$ for $y=x$ and justify your answer with respect to the standard cone result above.

Volumes of Rotation (Cylindrical shells)
The above method is one of two equivalent methods for determining the volume of rotation. The other method uses cylindrical shells, not disk-like slices, to determine the volume: i.e. we start with the construction



The area of the slice is now $(a-\xi)\Delta x$. When this is rotated about the y-axis the vertical slice forms a cylindrical shell of volume

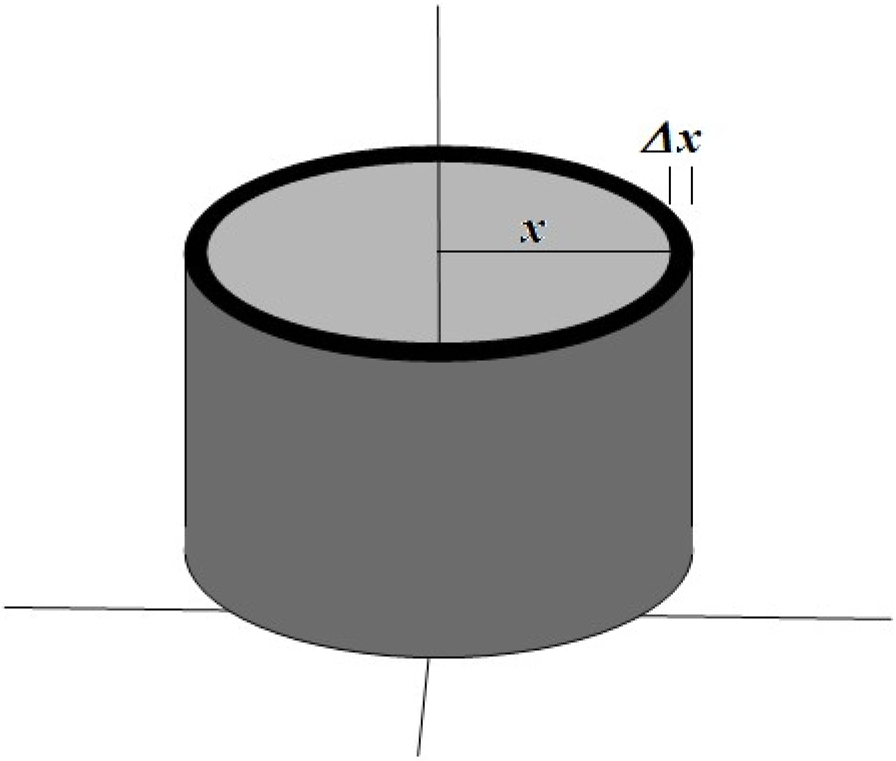
$$\Delta V_{shell} \simeq \pi\left((x+\Delta x)^2-(x)^2\right)h \simeq \pi 2 x h \Delta x$$

where $h \simeq (a-\xi)$ and $\xi = f(x) = y$. Then

$$\Delta V_{shell} \simeq \pi 2 x h \Delta x \simeq 2\pi x (a-f(x))\Delta x$$

Then as before we sum up all such cylindrical shells from $x=0$ to $x=f^{-1}(a)$ to get the total volume: i.e.
The volume of rotation, V, formed when the area bounded by the curves $y=f(x)$, $y=a$, $x=0$ (the y-axis) is rotated about the y-axis is

$$V = \lim_{n \to \infty} \sum_{k=0}^{k=n} 2\pi\, x_k (a - f(x_k)) \Delta x_k = \int_0^{f^{-1}(a)} 2\pi\, x (a - f(x))\, dx$$



We'll illustrate by example.

Example

Consider the example we had for the disk-like slices: i.e. to calculate the volume of the region formed when the area bounded by the lines $y = r$, $x = 0$ (the y-axis), and the curve $y = x$ is rotated about the y-axis.

Here $y = f(x) = x \Rightarrow f^{-1}(x) = x$

Then

$$V = \int_0^{f^{-1}(r)} 2\pi\, x (r - f(x))\, dx = 2\pi \int_0^{r} (r x - x^2)\, dx$$

$$\Rightarrow V = 2\pi \left[r \frac{x^2}{2} - \frac{x^3}{3} \right]_0^r = 2\pi \left[\frac{r^3}{6} \right] = \frac{\pi}{3} r^3$$

just as before. So both methods are consistent.

N.B.

To finish this section, a few comments:

- The cylindrical shell method above is typically more complicated to realise than the disk-like slice method. For this reason, if not otherwise instructed, it is probably best if the cylindrical shell method was seldom used in favour of the disk-like slice method.
- Many books, when stating the cylindrical shell method, will give the equation

$$V=\int_{0}^{f^{-1}(a)} 2\pi x\, f(x)\,dx$$

  Care should be taken when applying identities like this from reference books blindly because this form is applicable only when the area being considered for rotation has a flat base incident on the *x*-axis. Our problem has a curved base given by the function itself and as such needs to be modified. As all problems we'll consider will be of this form, we'll keep the form given in the notes above. Just keep in mind that various formula forms can exist for the same operation.
- Care should always be taken when attempting to calculate the volume. It is easy to forget a variable, parameter, etc. Draw a picture of the area to be rotated and put in all the necessary information to construct the volume of rotation. Use the images presented in this section as a guide of what to look for and what to include.
- You are measuring physical quantities. If the volume calculation yields a ludicrous result, then it is more likely to be down to a calculation error than a conceptual one. Check your answer against what you'd expect and if there is a significant difference, check over the calculations.

That finishes us with the area/volume determinations using integrals for this year. Next we will consider the inverse trigonometric functions and then Ordinary Differential Equations (O.D.E.'s).

