Bsc Software Design (Game Dev./Cloud Comp.) Year 1
Maths 1
Selected Taylor Series Solutions

Solution 1: $f(x)=\cos(x)$ **about** $x_0=\pi/2$

Part (a) Calculate the first six term in the Taylor Series
Given $f(x)=\cos(x)$ and asked to determine the first six terms in the Taylor series about $x_0=\pi/2$.
i.e.

$$f(x)=\cos(x)\simeq\sum_{k=0}^{5}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{where} \quad f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$$

Explicitly writing out the summation we have

$$f(x)=\cos(x)\simeq \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0+\frac{f^{(1)}(x_0)}{1!}(x-x_0)^1+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3$$
$$+\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4+\frac{f^{(5)}(x_0)}{5!}(x-x_0)^5$$

It is apparent that the $f^{(k)}(x)$ need to be evaluated for $0\le k\le 5$.

Then,

$k=0$ $\quad f^{(k)}(x)=f^{(0)}(x)=f(x)=\cos(x) \quad \Rightarrow f(x_0)=f(\pi/2)=\cos(\pi/2)=0$

$k=1$ $\quad f^{(k)}(x)=f^{(1)}(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\cos(x)=-\sin(x) \quad \Rightarrow f^{(1)}(x_0)=f^{(1)}(\pi/2)=-\sin(\pi/2)=-1$

$k=2$ $\quad f^{(k)}(x)=f^{(2)}(x)=\frac{d^2}{dx^2}f(x)=\frac{d}{dx}-\sin(x)=-\cos(x) \quad \Rightarrow f^{(2)}(x_0)=f^{(2)}(\pi/2)=-\cos(\pi/2)=0$

$k=3$ $\quad f^{(k)}(x)=f^{(3)}(x)=\frac{d^3}{dx^3}f(x)=\frac{d}{dx}-\cos(x)=\sin(x) \quad \Rightarrow f^{(3)}(x_0)=f^{(3)}(\pi/2)=\sin(\pi/2)=1$

$k=4$ $\quad f^{(k)}(x)=f^{(4)}(x)=\frac{d^4}{dx^4}f(x)=\frac{d}{dx}\sin(x)=\cos(x) \quad \Rightarrow f^{(4)}(x_0)=f^{(4)}(\pi/2)=\cos(\pi/2)=0$

$k=5$ $\quad f^{(k)}(x)=f^{(5)}(x)=\frac{d^5}{dx^5}f(x)=\frac{d}{dx}\cos(x)=-\sin(x) \quad \Rightarrow f^{(5)}(x_0)=f^{(5)}(\pi/2)=-\sin(\pi/2)=-1$

Substituting these $f^{(k)}(x_0)$ into our series above and replacing the x_0 with $\pi/2$ we get

$$f(x)=\cos(x)\simeq \frac{0}{1}\left(x-\frac{\pi}{2}\right)^0+\frac{-1}{1}\left(x-\frac{\pi}{2}\right)^1+\frac{0}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3+\frac{0}{24}\left(x-\frac{\pi}{2}\right)^4+\frac{-1}{120}\left(x-\frac{\pi}{2}\right)^5$$
$$= -\left(x-\frac{\pi}{2}\right)^1+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{120}\left(x-\frac{\pi}{2}\right)^5$$

This is the answer we seek. DO NOT EXPAND the $\left(x-\frac{\pi}{2}\right)^k$ terms.

This truncated Taylor series is often called a n[th] order Taylor Polynomial, $T_n(x)$. In our example n would be 5 and so

$$T_5(x)=-\left(x-\frac{\pi}{2}\right)^1+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{120}\left(x-\frac{\pi}{2}\right)^5$$

Part (b) Calculate the Error in $T_5(x)$ for $\cos(\pi/6)$ about $x_0=\pi/2$
The second part of this problem is to use this result to estimate the error in $T_5(x)$ for $\cos(\pi/6)$.

We do this by replacing the *x* in $T_5(x)$ with $\pi/2$ where $\pi=3.141592653587....$

Then

$$T_5\left(\frac{\pi}{6}\right)=-\left(\frac{\pi}{6}-\frac{\pi}{2}\right)^1+\frac{1}{6}\left(\frac{\pi}{6}-\frac{\pi}{2}\right)^3-\frac{1}{120}\left(\frac{\pi}{6}-\frac{\pi}{2}\right)^5\simeq 0.8663$$

To four significant digits after the decimal place

$$\cos(\pi/6)\simeq 0.8660$$

Our error $E_5(x)$ is the difference between the actual function, $\cos(x)$ and our approximation to this function $T_5(x)$:

$$E_5\left(\frac{\pi}{6}\right)=\left|\cos\left(\frac{\pi}{6}\right)-T_5\left(\frac{\pi}{6}\right)\right|=|0.8660-0.8663|=0.0003$$

Solution 2: $f(x)=\cos(x)$ **about** $x_0=0$

Calculate the first six term in the Taylor Series

Given $f(x)=\cos(x)$ and asked to determine the first six terms in the Taylor series about $x_0=0$. i.e.

$$f(x)=\cos(x)\simeq\sum_{k=0}^{5}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{where} \quad f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$$

Explicitly writing out the summation we have

$$\begin{aligned} f(x)=\cos(x)\simeq\ & \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0+\frac{f^{(1)}(x_0)}{1!}(x-x_0)^1+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 \\ & +\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4+\frac{f^{(5)}(x_0)}{5!}(x-x_0)^5 \end{aligned}$$

It is apparent that the $f^{(k)}(x)$ need to be evaluated for $0\le k\le 5$.

Then,

$k=0 \quad f^{(k)}(x)=f^{(0)}(x)=f(x)=\cos(x) \qquad \Rightarrow f(x_0)=f(0)=\cos(0)=1$

$k=1 \quad f^{(k)}(x)=f^{(1)}(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\cos(x)=-\sin(x) \qquad \Rightarrow f^{(1)}(x_0)=f^{(1)}(0)=-\sin(0)=0$

$k=2 \quad f^{(k)}(x)=f^{(2)}(x)=\frac{d^2}{dx^2}f(x)=\frac{d}{dx}-\sin(x)=-\cos(x) \qquad \Rightarrow f^{(2)}(x_0)=f^{(2)}(0)=-\cos(0)=-1$

$k=3 \quad f^{(k)}(x)=f^{(3)}(x)=\frac{d^3}{dx^3}f(x)=\frac{d}{dx}-\cos(x)=\sin(x) \qquad \Rightarrow f^{(3)}(x_0)=f^{(3)}(0)=\sin(0)=0$

$k=4 \quad f^{(k)}(x)=f^{(4)}(x)=\frac{d^4}{dx^4}f(x)=\frac{d}{dx}\sin(x)=\cos(x) \qquad \Rightarrow f^{(4)}(x_0)=f^{(4)}(0)=\cos(0)=1$

$k=5 \quad f^{(k)}(x)=f^{(5)}(x)=\frac{d^5}{dx^5}f(x)=\frac{d}{dx}\cos(x)=-\sin(x) \qquad \Rightarrow f^{(5)}(x_0)=f^{(5)}(0)=-\sin(0)=0$

Substituting these $f^{(k)}(x_0)$ into our series above and replacing the x_0 with 0 we get

$$\begin{aligned} f(x)=\cos(x)\simeq\ & \frac{1}{1}(x-0)^0+\frac{0}{1}(x-0)^1-\frac{1}{2}(x-0)^2+\frac{0}{6}(x-0)^3+\frac{1}{24}(x-0)^4+\frac{0}{120}(x-0)^5 \\ =\ & 1-\frac{1}{2}(x)^2+\frac{1}{24}(x)^4 \end{aligned}$$

This is the answer we seek.

This is the 5th order Taylor Polynomial, $T_5(x)$, for $f(x)=\cos(x)$ about $x_0=0$

$$T_5(x)=1-\frac{1}{2}(x)^2+\frac{1}{24}(x)^4$$

Exercise:

Estimate the Error, $E_5(x)$, in $T_5(x)$, for $\cos(\pi/6)$ about $x_0=0$.

Solution 3: $f(x)=e^x$ **about** $x_0=1$

Part (a) Calculate the first six term in the Taylor Series

Given $f(x)=e^x$ and asked to determine the first six terms in the Taylor series about $x_0=1$. i.e.

$$f(x)=e^x \simeq \sum_{k=0}^{5} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{where } f^{(k)}(x)=\frac{d^k}{dx^k} f(x)$$

Explicitly writing out the summation we have

$$f(x)=e^x \simeq \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0+\frac{f^{(1)}(x_0)}{1!}(x-x_0)^1+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3$$
$$+\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4+\frac{f^{(5)}(x_0)}{5!}(x-x_0)^5$$

It is apparent that the $f^{(k)}(x)$ need to be evaluated for $0 \leq k \leq 5$.

Then,

$k=0$ $\quad f^{(k)}(x)=f^{(0)}(x)=f(x)=e^x \quad \Rightarrow f(x_0)=f(1)=e^1=e$

$k=1$ $\quad f^{(k)}(x)=f^{(1)}(x)=\frac{d}{dx} f(x)=\frac{d}{dx} e^x=e^x \quad \Rightarrow f^{(1)}(x_0)=f^{(1)}(1)=e^1=e$

$k=2$ $\quad f^{(k)}(x)=f^{(2)}(x)=\frac{d^2}{dx^2} f(x)=\frac{d}{dx} e^x=e^x \quad \Rightarrow f^{(2)}(x_0)=f^{(2)}(1)=e^1=e$

$k=3$ $\quad f^{(k)}(x)=f^{(3)}(x)=\frac{d^3}{dx^3} f(x)=\frac{d}{dx} e^x=e^x \quad \Rightarrow f^{(3)}(x_0)=f^{(3)}(1)=e^1=e$

$k=4$ $\quad f^{(k)}(x)=f^{(4)}(x)=\frac{d^4}{dx^4} f(x)=\frac{d}{dx} e^x=e^x \quad \Rightarrow f^{(4)}(x_0)=f^{(4)}(1)=e^1=e$

$k=5$ $\quad f^{(k)}(x)=f^{(5)}(x)=\frac{d^5}{dx^5} f(x)=\frac{d}{dx} e^x=e^x \quad \Rightarrow f^{(5)}(x_0)=f^{(5)}(1)=e^1=e$

Substituting these $f^{(k)}(x_0)$ into our series above and replacing the x_0 with 1 we get

$$f(x)=e^x \simeq \frac{e}{1}(x-1)^0+\frac{e}{1}(x-1)^1+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3+\frac{e}{24}(x-1)^4+\frac{e}{120}(x-1)^5$$
$$= e\left(1+(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+\frac{1}{120}(x-1)^5\right)$$

This is the answer we seek. As before DO NOT EXPAND the $(x-1)^n$ terms.

This is the 5th order Taylor Polynomial, $T_5(x)$, for $f(x)=e^x$ about $x_0=1$

$$T_5(x)=e\left(1+(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+\frac{1}{120}(x-1)^5\right)$$

Part (b) : Calculate the Error in $T_5(x)$ **for** e^2 **about** $x_0=1$

The second part of this problem is to use this result to estimate the error in $T_5(x)$ for e^2 .

We do this by replacing the *x* in $T_5(x)$ with 2.

Then

$$\begin{aligned}
T_5(x) &= e\left(1+(2-1)+\frac{1}{2}(2-1)^2+\frac{1}{6}(2-1)^3+\frac{1}{24}(2-1)^4+\frac{1}{120}(2-1)^5\right) \\
&= e\left(1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}\right) \\
&\simeq 2.716666\,e \\
&\simeq 7.3847
\end{aligned}$$

To four significant digits after the decimal place

$$e^2 \simeq 7.3891$$

Our error $E_5(x)$ is the difference between the actual function, e^2 and our approximation to this function $T_5(x)$:

$$E_5(2) = \left|e^2 - T_5(2)\right| = |7.3891 - 7.3847| = 0.005$$

Solution 4: $f(x)=\ln|x|$ **about** $x_0=1$

Part (a) Calculate the first six term in the Taylor Series
Given $f(x)=\ln|x|$ and asked to determine the first six terms in the Taylor series about $x_0=1$. i.e.

$$f(x)=e^x \simeq \sum_{k=0}^{5} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{where} \quad f^{(k)}(x)=\frac{d^k}{d\,x^k}f(x)$$

Explicitly writing out the summation we have

$$\begin{aligned} f(x)=\ln|x| \simeq \quad & \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0+\frac{f^{(1)}(x_0)}{1!}(x-x_0)^1+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 \\ & +\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4+\frac{f^{(5)}(x_0)}{5!}(x-x_0)^5 \end{aligned}$$

It is apparent that the $f^{(k)}(x)$ need to be evaluated for $0 \le k \le 5$.

Then,

$k=0$ $\quad f^{(k)}(x)=f^{(0)}(x)=f(x)=\ln|x|$ $\quad \Rightarrow f(x_0)=f(1)=\ln|1|=0$

$k=1$ $\quad f^{(k)}(x)=f^{(1)}(x)=\dfrac{d}{dx}f(x)=\dfrac{d}{dx}\ln|x|=\dfrac{1}{x}$ $\quad \Rightarrow f^{(1)}(x_0)=f^{(1)}(1)=\dfrac{1}{1}=1$

$k=2$ $\quad f^{(k)}(x)=f^{(2)}(x)=\dfrac{d^2}{dx^2}f(x)=\dfrac{d}{dx}\dfrac{1}{x}=-\dfrac{1}{x^2}$ $\quad \Rightarrow f^{(2)}(x_0)=f^{(2)}(1)=-\dfrac{1}{1^2}=-1$

$k=3$ $\quad f^{(k)}(x)=f^{(3)}(x)=\dfrac{d^3}{dx^3}f(x)=\dfrac{d}{dx}\left(-\dfrac{1}{x^2}\right)=\dfrac{2}{x^3}$ $\quad \Rightarrow f^{(3)}(x_0)=f^{(3)}(1)=\dfrac{2}{1^3}=2$

$k=4$ $\quad f^{(k)}(x)=f^{(4)}(x)=\dfrac{d^4}{dx^4}f(x)=\dfrac{d}{dx}\dfrac{2}{x^3}=-\dfrac{6}{x^4}$ $\quad \Rightarrow f^{(4)}(x_0)=f^{(4)}(1)=-\dfrac{6}{1^4}=-6$

$k=5$ $\quad f^{(k)}(x)=f^{(5)}(x)=\dfrac{d^5}{dx^5}f(x)=\dfrac{d}{dx}\left(-\dfrac{6}{x^4}\right)=\dfrac{24}{x^5}$ $\quad \Rightarrow f^{(5)}(x_0)=f^{(5)}(1)=\dfrac{24}{1^5}=24$

Substituting these $f^{(k)}(x_0)$ into our series above and replacing the x_0 with 1 we get

$$\begin{aligned} f(x)=\ln|x| \simeq \quad & \frac{0}{1}(x-1)^0+\frac{1}{1}(x-1)^1-\frac{1}{2}(x-1)^2+\frac{2}{6}(x-1)^3-\frac{6}{24}(x-1)^4+\frac{24}{120}(x-1)^5 \\ = \quad & (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5 \end{aligned}$$

This is the answer we seek. As before DO NOT EXPAND the $(x-1)^n$ terms.

This is the 5th order Taylor Polynomial, $T_5(x)$, for $f(x)=\ln|x|$ about $x_0=1$

$$T_5(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5$$