Section 6 (cont'd)
The Definite Integral, The Indefinite Integral and Primitives

In this section we progress our analysis of the integral and its properties by considering first the Fundamental Theorem of Calculus and how we can use it to bypass the complicated method of the Riemann sum. Then we consider the basic properties of the integral (both definite and indefinite) and a discussion of primitives; what are they and how useful are they here?

Definite Integral

Recall in the last section we defined the definite integral of a finite and continuous function, $f(x)$, on a equi-partition $\mathscr{P}$ of [a,b] to be

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+k\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

where we have made the following substitution

$$x_k=x_0+k\,\Delta x=a+k\left(\frac{b-a}{n}\right)$$

$$\Delta x_k=\left(\frac{b-a}{n}\right)$$

This method is intensive at best and downright impossible at worst. There must be a better simpler method. Thankfully there is. To see how this method works, recall the final example we considered in the last section:

$$\int_0^X x\,dx=\frac{X^2}{2}$$

We know from differential calculus that

$$\frac{d}{dx}x^n=n\,x^{n-1}\;\forall\, n\in\mathbb{R}\setminus\{0\}$$

Then

$$\frac{d}{dx}x^2=2x$$

$$\Rightarrow\frac{d}{dx}\frac{x^2}{2}=\frac{2x}{2}=x$$

It follows that

$$\frac{d}{dx}\left(\int_0^x x\,dx\right)=\frac{d}{dx}\frac{x^2}{2}=x$$

i.e. the derivative is undoing what the integral does to the original function (in the same manner an inverse function undoes what its originating function does).

Then we can state the following:
If there exists a function, $F(x)$, such that the derivative of this function is equal to $f(x)$ then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \quad \Leftrightarrow \quad \frac{d}{dx}F(x) = f(x)$$

The problem is to find the function, $F(x)$, so that we can evaluate the integral.

Example:
Consider the following integral

$$\int_a^b x^n\,dx \quad \forall n \in \mathbb{N}$$

Then, using the fact that

$$\frac{d}{dx}x^k = k\,x^{k-1}$$

we set

$$F(x) = \alpha\,x^k \quad \forall \alpha \in \mathbb{R}$$

Then

$$\frac{d}{dx}F(x) = \frac{d}{dx}\alpha\,x^k = k\,\alpha\,x^{k-1} = f(x) = x^n$$

Therefore, equating like with like:

$$k\,\alpha\,x^{k-1} = x^n$$

$$\Rightarrow k = n+1 \text{ and } \alpha = \frac{1}{n+1}$$

$$\Rightarrow F(x) = \frac{1}{n+1}x^{n+1}$$

Then

$$\int_a^b x^n\,dx = \frac{1}{n+1}\left[x^{n+1}\right]_a^b \quad \forall n \in \mathbb{N}$$

So, without having to resort to nasty Riemann summations over equi-partitions, we have determined the integral of integer powers of x.

It is now time to give formality to the process we have employed above. This is done via the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function of $[a,b]$ and let $F(x)$ be such that

$$\frac{d}{dx}F(x)=f(x)$$

then $F(x)$ is the primitive[1] of $f(x)$ and the following holds

$$\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a)$$

Furthermore it follows that, on [a,b], because

$$\frac{d}{dx}F(x)=f(x)$$

then

$$\frac{d}{dx}\int f(x)\,dx=f(x)$$

where

$$\int f(x)\,dx=F(x)+C$$

is termed the indefinite integral

Similarly

$$\int \frac{d}{dx}F(x)\,dx=F(x)$$

This Theorem is truly remarkable. It states that the derivative and integral are inverse operations of each other and that the evaluation of the more complicated integrals may be performed using the less complicated derivative.

Therefore, when given a function, $f(x)$, to integrate you should ask yourself:

Is there another function which when differentiated gives me $f(x)$?

If the answer is Yes then that is your solution to within a constant of integration, C.

1 The term “Antiderivative” is often used in textbooks now but I prefer the term primitive. It implies the function has evolved from its earlier state to the later one and we use the “fingerprint” of this evolution to trace the function back to its origins. A personal choice and not compulsory; you can use “antiderivative” if it suits though the more correct term is “antederivative”; ante meaning “before”.

The Indefinite Integral

We introduced the term "indefinite integral" above but never elaborated. We will now address this very point. The indefinite integral is one where the limits (generically denoted *a* and *b*) are omitted.

Why?

Using primitives with the Fundamental Theorem allows you to circumvent the tedium of Riemann sums and determine, without any artificial limits and constraints, the relationship between a function, its primitive, the derivative operation, and that of the integral.

Remember that we motivated the existence of the Riemann Integral (Definite Integral) from the problem of evaluating areas under curves on finite closed intervals $[a,b]$. As a means of introduction, that approach works well. However, not all integration problems involve areas, volumes, hyper-volumes, etc.

We often think of differentiation as the *forward* operation and integration as a *backwards* operation in that integration brings us *back* to the primitive: i.e.

$$\begin{tikzcd} F(x)+C \arrow[rr, bend left, "d/dx"] & & f(x) \arrow[ll, bend left, "\int dx"] \end{tikzcd}$$

What's this *C* added to *F*(*x*) for?

The constant C (termed Constant of Integration) is added to the primitive of a function when evaluating indefinite integrals because the forward operation of differentiation zeroes any constants ($d/dx\ C=0$) and therefore, in the absence of any information to the contrary, to complete the solution we must incorporate the constant just in case it was initially present with the primitive.

When dealing with definite integrals,the limits remove the need for the constant as the difference between the limits removes it anyway: i.e.

$$\int_a^b f(x)\,dx=\left[F(x)+C\right]_a^b=F(b)+C-F(a)-C=F(b)-F(a)$$

Example:

You are required to evaluate the following:

$$\int \cos(x)\,dx$$

Is there a function which when differentiated results in cos(x)?

Checking the maths tables you find that

$$\frac{d}{dx}\sin(x)=\cos(x)$$

So, let $F(x) = \sin(x) + C$. Then

$$\frac{d}{dx}F(x)=\frac{d}{dx}(\sin(x)+C)=\cos(x)$$

From the Fundamental Theorem

$$\frac{d}{dx}\int f(x)\,dx=f(x)$$

$$\Rightarrow\frac{d}{dx}\int \cos(x)\,dx=\cos(x)=\frac{d}{dx}F(x)$$

$$\Rightarrow F(x)=\int \cos(x)\,dx$$

$$\Rightarrow \sin(x)+C=\int \cos(x)\,dx$$

This is the Method of Primitives approach to integration. Using this method, we can tabulate the following:

$F(x)$	$\frac{d}{dx}F(x)$	$f(x)$	$\int f(x)\,dx$
$x^n\ n\neq 0$	$nx^{n-1}\ n\neq 0$	$x^n\quad n\neq -1$	$\frac{x^{n+1}}{n+1}\quad n\neq -1$
e^x	e^x	e^x	e^x
$\cos(x)$	$-\sin(x)$	$\sin(x)$	$-\cos(x)$
$\sin(x)$	$\cos(x)$	$\cos(x)$	$\sin(x)$
$\tan(x)$	$\sec^2(x)$		
$\ln\lvert x\rvert$	$\frac{1}{x};x\neq 0$	$\frac{1}{x};x\neq 0$	$\ln\lvert x\rvert$
$\cosh(x)$	$\sinh(x)$	$\sinh(x)$	$\cosh(x)$
$\sinh(x)$	$\cosh(x)$	$\cosh(x)$	$\sinh(x)$

Now we can determine some mathematical properties of the integral and apply them.

Mathematical Properties

1. Integral of a sum

   Let $f(x)$ and $g(x)$ be two real continuous functions with primitives $F(x)$ and $G(x)$ respectively. Then

$$\int (f(x)+g(x))dx = F(x)+G(x)+C \quad \forall C \in \mathbb{R}$$

$$\Rightarrow \int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

   i.e. the integral of a sum is the sum of the integrals.

   The proof comes from the properties of the derivative. We know that

$$\frac{d}{dx}(F(x)+G(x)) = \frac{d}{dx}F(x)+\frac{d}{dx}G(x) = f(x)+g(x)$$

$$\Rightarrow \int \frac{d}{dx}(F(x)+G(x))dx = \int \frac{d}{dx}F(x)dx + \int \frac{d}{dx}G(x)dx = \int (f(x)+g(x))dx$$

$$\Rightarrow \int f(x)dx + \int g(x)dx = \int (f(x)+g(x))dx$$

   as required.

2. Scalar multiplication

   Let $f(x)$ be a real continuous function with primitive $F(x)$ and let $\alpha \in \mathbb{R}$ be any real constant. Then

$$\int \alpha f(x)dx = \alpha F(x)+C \quad \forall C \in \mathbb{R}$$

$$\Rightarrow \int \alpha f(x)dx = \alpha \int f(x)dx$$

   i.e. the constant α can be removed from within the integral as it doesn't depend on the variable of integration, x.

   Again the proof comes from the derivative.We know that

$$\frac{d}{dx}(\alpha F(x)) = \alpha \frac{d}{dx}F(x)$$

$$\Rightarrow \int \frac{d}{dx}(\alpha F(x))dx = \alpha \int \frac{d}{dx}F(x)dx$$

$$\Rightarrow \int \alpha f(x)dx = \alpha \int f(x)dx$$

   which result is what we wanted.

Then using 1. and 2. above we find that

$$\int (\alpha f(x)+\beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx$$

for real constants α and β.

In the next section we'll show how the chain and product rules of differentiation can lead to two new methods for solving both definite and indefinite integrals; Integration by Substitution and Integration by Parts respectively.

At this point you can now handle basic integration; i.e. integration of functions that have a determinable primitive. Such integrals are of limited use as the functions are too simple to use beyond a familiarisation exercise. The introduction of the two new methods in the next section will enable you to evaluate most of the integrals you'll encounter.

