Section 6
An Introduction to The Definite (Riemann) Integral

In this section we'll consider the problem of evaluating the area under the curve given by some arbitrary function, f, of a real variable x on some compact interval $[a,b]$of $\mathbb{R}$. IN order for the problem to make sense (from a real world perspective) we will limit $f(x)$ to be finite, continuous, and above all positive on $[a,b]$. Such a problem could look like

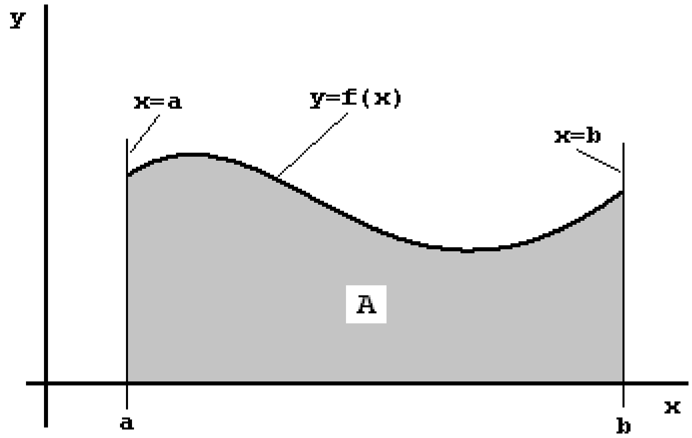


Figure 1: The area A under an arbitrary continuous finite function $f(x)$ on $[a,b]$.

The only way to calculate this area is to employ geometric shapes whose areas could be easily calculated and then fit these shapes to the area under the curve and estimate the area. If you used triangles then one possible configuration is

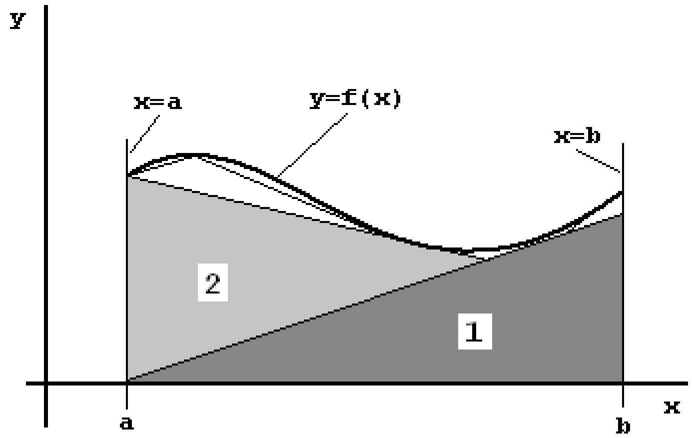


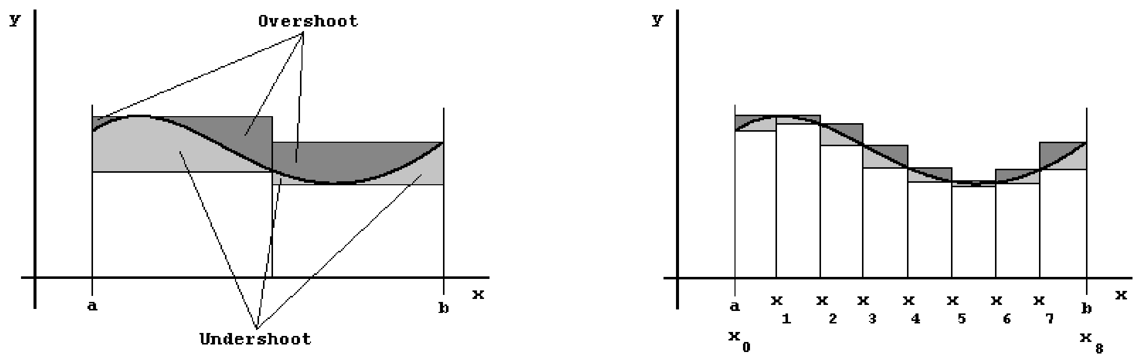
Figure 2: Using Triangular shapes to approximate the area A under the curve $y=f(x)$.

Calculating the area 1 is not difficult but the area denoted 2 and subsequent areas are more awkward and there is no obvious relationship between the triangular area in the above figure.

So how do we proceed?
What is required is a geometric shape whose area can be easily calculated and which has an obvious and straight forward functional relationship to its neighbours and to the region A whose area we want to calculate.

What do we know about this region, A, whose area we want to determine?
The region is bounded below by the x-axis, to the left by the line $x=a$, and to the right by $x=b$. These are all straight lines, two of which are parallel, and so a geometric shape with similar properties would appear to be the obvious choice. The only non-linear arbitrary boundary is the upper bound given by the curve $y=f(x)$ and, as we shall see later, tackling this is the essence of the Riemann Integral. Furthermore we can sub-divide the compact interval $[a,b]$ on the x-axis in a consistent manner using a process called partitioning. With all this in mind, the choice is obvious: ***The shape we choose is the rectangle***.

Its base is chosen to be a sub-interval of the interval $[a,b]$ with its height related to the function f in this interval. It is possible to estimate the area under this curve using a sum of these upright rectangles on the interval .



(a) (b)

Figure 3: (a) The area bounded by the function approximated by two rectangles of equal width but whose heights are proportional to $f(x)$. The over and under shoots are explained in the text below.
(b) The same area approximated by eight rectangles of equal width. The labelling and colouring are explained in the text below.

The figure above illustrates this concept when we use two and eight rectangles to estimate the area under the curve. The dark grey areas (labelled overshoot in Fig. 3(a)) correspond to the degree of overestimation of the area under the curve for that sub-interval of $[a,b]$ and occurs when the height of the rectangle is based on the largest value of $f(x)$ in that sub-interval. The undershoot is then the underestimation based on the smallest value of $f(x)$ in the sub-interval; this is coloured light grey in Fig. 3(a) above. As can be seen from Fig. 3(b) the use of more rectangles (resulting from a smaller subdivision of the compact interval) reduces the effects of under and over estimations. To mathematically describe this we need to define some new mathematical objects.

Partitions and Equi-Partitions of [*a*,*b*]

Definition:

We define a Partition P of order n on the compact interval $[a,b]$ to be the set of real numbers

$$P := \{x_k \,|\, x_0 = a \wedge x_n = b \wedge \Delta x_k = x_k - x_{k-1} > 0;\ \ 1 \leqslant k \leqslant n\}$$

It follows, then, that the interval is sub-divided into n parts that are not necessarily of equal width.

We can then refine this definition by defining an equi-partition $\mathscr{P}$ of order n on $[a,b]$ to be the set of n equally spaced numbers between a and b:

$$\mathscr{P} := \{x_k \,|\, x_0 = a \wedge x_n = b \wedge \Delta x_k = (b-a)/n > 0;\ \ 1 \leqslant k \leqslant n\}$$

where $\Delta x_n = \ldots = \Delta x_k = \ldots = \Delta x_0 = (b-a)/n$.

Typical examples of the two types of partition are shown below in Figure 4.

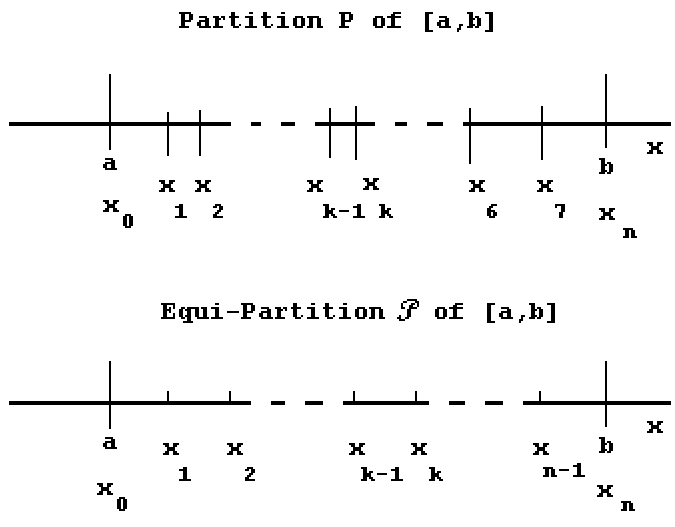


Figure 4: A typical partition P and equi-partition $\mathscr{P}$ of $[a,b]$.

Approximating the Area and the Riemann Sum

Having defined our partition of the interval, it remains for us to consistently determine the areas of the rectangles whose bases correspond to the partition's sub-intervals. The subsequent sum of these rectangular areas will give us our approximation to the area under the curve. The problem facing us now is, simply, how to use the curve $y=f(x)$ to determine these rectangular areas. To see how this can be tackled, consider the construction in Figure 5 below.

The upper and lower bounds on a rectangular area will be dependent on the largest and smallest heights respectively the rectangle can assume in this sub-interval; i.e. the largest and smallest values of $f(x)$ in this sub-interval.

Mathematically we define the largest value a function can assume in a given interval (or domain), $[x_{k-1}, x_k]$, to be the supremum (or sup) of the set

$$f(\xi_k)=\sup\{f(x)|x_{k-1}\leqslant x\leqslant x_k \wedge f(\xi_k)\geq f(x)\,\forall\, x\in[x_{k-1}, x_k]\}$$

Then the largest rectangular area in this sub-interval and, therefore, the upper bound on the actual area, A_k, under the function in this sub-interval is

$$f(\xi_k)\Delta x_k \underbrace{=}_{\substack{\text{on the}\\ \text{equi-partition}}} f(\xi_k)\left(\frac{b-a}{n}\right) \geq A_k$$

Similarly we define the smallest value a function can assume in a given interval to be the infemum (or inf) of the set

$$f(\zeta_k)=\inf\{f(x)|x_{k-1}\leqslant x\leqslant x_k \wedge f(\zeta_k)\geq f(x)\,\forall\, x\in[x_{k-1}, x_k]\}$$

Then the smallest rectangular area in this sub-interval and, therefore, the lower bound on the actual area, A_k, under the function in this sub-interval is

$$f(\zeta_k)\Delta x_k \underbrace{=}_{\substack{\text{on the}\\ \text{equi-partition}}} f(\zeta_k)\left(\frac{b-a}{n}\right) \leq A_k$$

For Figure 5 below, $f(\xi_k)=f(x_3)$ and $f(\zeta_k)=f(x_4)$ because $f(x)$ is monotonically decreasing in the chosen sub-interval.

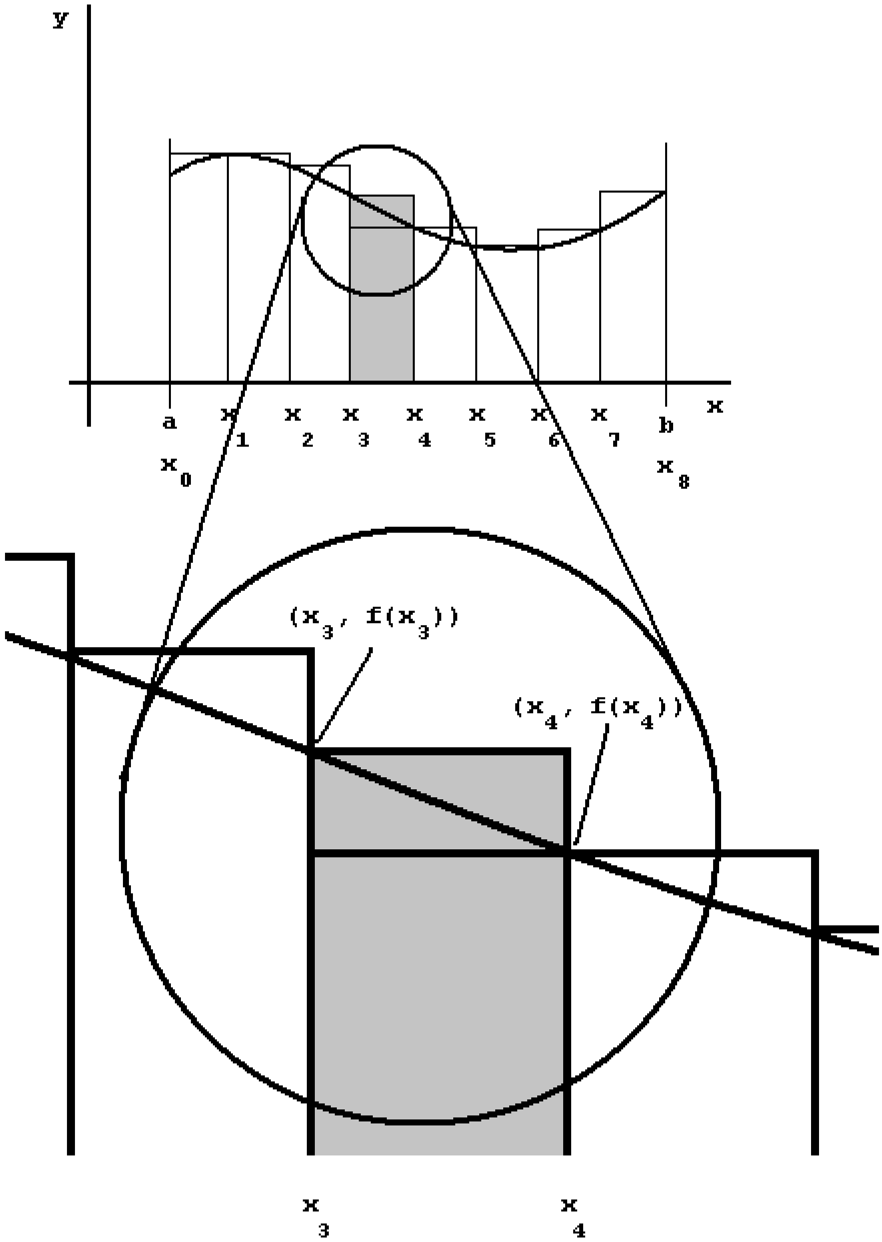


Figure 5: Enlargement of the top of a typical rectangular area under the curve $y = f(x)$ showing how, in this example, the upper and lower bounds on the rectangular area are dependent on the extreme left and right values of the sub-interval.

Then

$$f(x_4)\left(\frac{b-a}{n}\right) \leq A_k \leq f(x_3)\left(\frac{b-a}{n}\right)$$

The entire area, A, under the curve on $[a,b]$ is then bounded above and below according to

$$\sum_{k=1}^{n} f(\zeta_k)\left(\frac{b-a}{n}\right) \le \sum_{k=0}^{n} A_k = A \le \sum_{k=1}^{n} f(\xi_k)\left(\frac{b-a}{n}\right)$$

where ζ_k and ξ_k are in the inf and sup respectively of the set of $f(x)$ in the k-th sub-interval.

As the width of the sub-intervals contracts (i.e. as n increases in an unbounded fashion) the extreme values will converge to a common value. This is because $f(x)$ is both continuous and finite on the sub-interval. We can then define the upper and lower Riemann sums and give an expression for the area, A, under the curve $y=f(x)$ on $[a,b]$:

$$\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n} f(\zeta_k)\left(\frac{b-a}{n}\right)}_{\text{Lower Riemann Sum}} = \lim_{n\to\infty}\sum_{k=0}^{n} A_k = A = \underbrace{\lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\left(\frac{b-a}{n}\right)}_{\text{Upper Riemann Sum}}$$

And now to the purpose of this section.

Riemann (Definite) Integral
Definition

We define the Definite Integral of $f(x)$, continuous and finite, on $[a,b]$ to be

$$\int_a^b f(x)\,dx := \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = A$$

where $\Delta x=(b-a)/n$ and $x_k=x_0+k\Delta x=a+k(b-a)/n$ on the equi-partition $\mathscr{P}$ of $[a,b]$.

Examples:

1. Consider $f(x)=x$ on $[0,1]$.

Here $[a,b]=[0,1]\Rightarrow a=0$ and $b=1$. Define the equi-partition $\mathscr{P}$

$$\mathscr{P} := \{x_k \mid x_0=0 \wedge x_n=1 \wedge \Delta x_k=1/n>0;\ 1\leqslant k\leqslant n\}$$

Determine x_k for this interval and hence $f(x_k)$

$$x_k=x_0+k\Delta x=a+k(b-a)/n=0+k\frac{1}{n}=\frac{k}{n}$$

$$\Rightarrow f(x_k)=x_k=\frac{k}{n}$$

Then we can construct our integral:

$$\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} x_k \Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}\frac{1}{n}$$

$$\Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k$$

$$\Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty}\frac{1}{n^2}\frac{n}{2}(n+1)$$

$$\Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty}\frac{1}{2}+\frac{1}{2\mathrm{n}}$$

Therefore

$$\int_o^1 x\,dx = \frac{1}{2}$$

2. Consider $f(x)=x$ on $[0,X]$.

Here $[a,b]=[0,X]\Rightarrow a=0$ and $b=X$. Define the equi-partition $\mathscr{P}$

$$\mathscr{P} := \{x_k | x_0=0 \wedge x_n=X \wedge \Delta x_k = X/n>0;\ \ 1\leqslant k\leqslant n\}$$

Determine x_k for this interval and hence $f(x_k)$

$$x_k = x_0 + k\Delta x = a + k(b-a)/n = 0 + k\frac{X}{n} = k\frac{X}{n}$$

$$\Rightarrow f(x_k) = x_k = k\frac{X}{n}$$

Then we can construct our integral:

$$\int_0^X x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} x_k \Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} k\frac{X}{n}\frac{X}{n}$$

$$\Rightarrow \int_0^X x\,dx = \lim_{n\to\infty}\frac{X^2}{n^2}\sum_{k=1}^{n} k$$

$$\Rightarrow \int_0^X x\,dx = \lim_{n\to\infty}\frac{X^2}{n^2}\frac{n}{2}(n+1)$$

$$\Rightarrow \int_0^X x\,dx = \lim_{n\to\infty}\frac{X^2}{2}+\frac{X^2}{2\mathrm{n}}$$

Therefore

$$\int_0^X x\,dx = \frac{X^2}{2}$$

N.B.

Let $S_n = \sum_{k=1}^{n} k = 1+2+3+\ldots+n-1+n$ be the sum of the first n natural numbers. Then we can do the following:

$$\begin{array}{ccccccccccccc}
S_n & = & 1 & + & 2 & + & 3 & +\ldots+ & (n-2) & + & (n-1) & + & n \\
+S_n & = & n & + & (n-1) & + & (n-2) & +\ldots+ & 3 & + & 2 & + & 1 \\
\text{-----} & = & \text{-----} & \text{--} & \text{--------} & \text{--} & \text{--------} & \text{--}\ldots\text{--} & \text{--------} & \text{--} & \text{--------} & \text{--} & \text{-----} \\
2S_n & = & n+1 & + & n+1 & + & n+1 & +\ldots+ & n+1 & + & n+1 & + & n+1
\end{array}$$

Therefore

$$2S_n = \underbrace{(n+1)+\ldots+(n+1)}_{\text{n times}} = n(n+1)$$

$$\Rightarrow S_n = \frac{n}{2}(n+1)$$