Section 7
Quadrants, Angles, and Calculators

The purpose of this section is to highlight some issues that may occur when determining the polar form of a complex numbers; specifically the angle θ. We begin by considering the four quadrants of the complex plane and how the signs of the Real and Imaginary parts of any complex number, $z \in \mathbb{C}$, determine which quadrant it is in. Then we examine how shortcomings in the handheld calculator can attribute incorrect quadrants to such numbers before finishing with the corrections and methods needed to successfully handle angles in the polar form of a complex number.

Quadrants of the Complex Plane

Any Eulclidean plane can be subdivided into four quadrants located about the recti-linear origin, *O*, of the plane. The four quadrants for the complex plain are shown below.

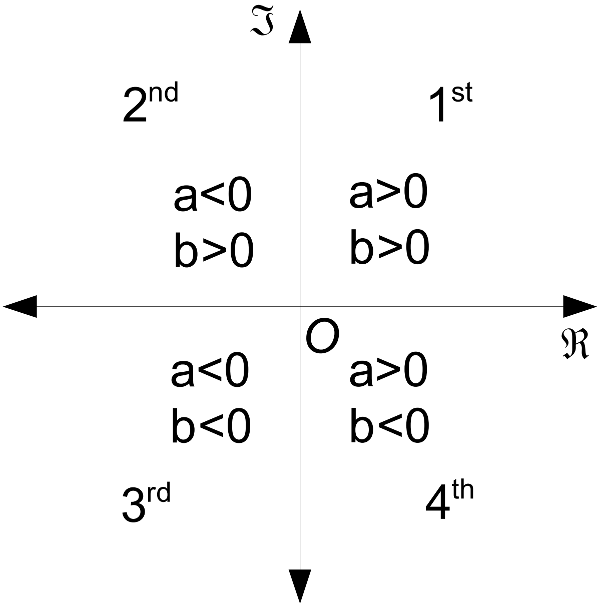

Figure: Four quadrants of the complex plain.

For any $z=a+bj \ \ \forall a,b \in \mathbb{R}; \ \ j=\sqrt{-1}$, we can use the signs of the Real and Imaginary parts of *z* to locate where on the plain the number is located: i.e.

$$\text{If } a<0 \text{ and } b>0 \Rightarrow z \in 2^{\text{nd}} \text{ quadrant}$$

The angle subtended from the positive $\Re$ axis to *z* depends on the quadrant z lies within;

$$\text{If } z \in 2^{\text{nd}} \text{ quadrant} \Rightarrow \frac{\pi}{2} \le \theta < \pi$$

So in determining the polar form of the complex number it is essential to know where on the complex plane the number is and in what quadrant it is located.

Quadrant	Angle bounds
1	$0 \le \theta < \frac{\pi}{2}$
2	$\frac{\pi}{2} \le \theta < \pi$
3	$\pi \le \theta < \frac{3\pi}{2}$
4	$\frac{3\pi}{2} \le \theta < 2\pi$

The table above associates with each quadrant an upper and lower bound on the angle, θ, so you can reconcile the angle you obtain from the calculations to convert a given z to polar form and the quadrant it should reside within.

Calculator Issues

Though convenient devices for evaluating arithmetic expressions quickly, many handheld calculators have to simple a set of algorithms to be effectively used for evaluating angles for complex numbers without some fundamental adjustments.

Much of the problems stem from the manner in which calculators deal with fractions. Consider the two complex numbers

$$z_2 = -\sqrt{3} + j \text{ and } z_2 = \sqrt{3} - j$$

From our analysis of the quadrants, it is clear that z_1 is in quadrant 2 and z_2 is in quadrant 4. If we wanted to express them in polar form we would determine θ via

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

which for our two numbers would be

$$\theta_1 = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6} \text{ and } \theta_2 = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{11\pi}{6}$$

However, our calculator would evalate the argument of the arctan (inverse tangent) first removing crucial information about which quadrant it is in; e.g. Consider our two complex numbers above. Then for z_1 in quadrant 2

$$\frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

because, from your Maths Tables,

$$\tan(-A) = -\tan(A)$$

This negative angle is in the 4th Quadrant and is incorrect by a factor of π radians;

$$(\theta_1)_{\text{CORRECT}} = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \underbrace{\pi}_{\text{Correction}} + \underbrace{\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)}_{\text{Calculator Angle}} = \pi + \left(\frac{-\pi}{6}\right) = \frac{5\pi}{6}$$

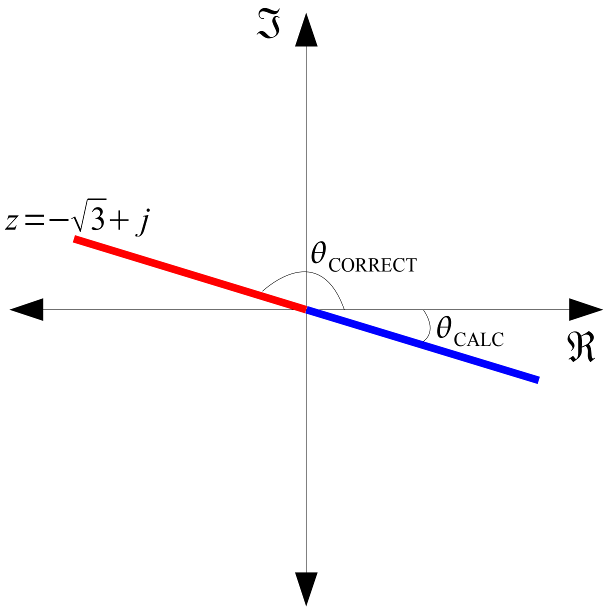

Figure: The correct z_1 in quadrant 2 (in Red) with corrected angle θ_{CORRECT} and the erroneous calculator angle θ_{CALC} (in Blue) for comparison.

Corrections

The corrections are tabulated below by quadrant.

Quadrant	**Calculator Angle**	**Corrected Angle**	**Correction**
1	θ	θ	None
2	$-\theta$	$-\theta+\pi$	Add π
3	θ	$\theta+\pi$	Add π
4	$-\theta$	$-\theta+2\pi$	Add 2π

Consider a few examples to illustrate this method of correction:

Examples

1. Let $z_2 = -1 - j$. Calculate the correct angle this complex number makes w.r.t. the positive $\Re$ axis.
   Here $z_2 = -1 - j = a + bj$ with both a and b being less than 0. Therefore, z_2 lies in the third quadrant and
   $$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$
   But this requires a correction as it is in the third quadrant and so the actual angle, θ_{CORRECT}, is
   $$\theta_{\text{CORRECT}} = \theta + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

2. Let $z_2 = 4 - 4j$. Calculate the correct angle this complex number makes w.r.t. the positive $\Re$ axis.
   Here $z_2 = 4 - 4j = a + bj$ with a positive a and negative b Therefore, z_2 lies in the fourth quadrant and
   $$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-4}{4}\right) = \tan^{-1}(-1) = \frac{-\pi}{4}$$
   But this requires a correction as it is in the fourth quadrant and so the actual angle, θ_{CORRECT}, is
   $$\theta_{\text{CORRECT}} = \theta + 2\pi = \frac{-\pi}{4} + 2\pi = \frac{7\pi}{4}$$

In all the questions you are likely to get, the calculated angle from the handheld calculator of θ will be present in the table on Pg 13 of yours maths tables; i.e.
$$\theta = \pm 0, \pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \pm\pi$$
with you making appropriate corrections as outlined above to correct for any calculator issues.