Homework 5: convergence of random variables

EE 325 (DD): Probability and Random Processes, Autumn 2016
Instructor: Animesh Kumar, EE, IIT Bombay

Instructions: Both the problems from Set-A has to be submitted by 5:00pm on 18/10/16 (Tuesday). Please follow the given instructions carefully.

If solutions from two or more students resemble or are copied from each other, all the concerned students will get -5 marks. You *must* write your own solution in your own words.

Each question should be submitted on a different sheet. Write your name+roll no. on each page of your submitted solution. Make a photocopy of your homework submission. Submit your Homework + photocopy (homework submission and its copy should not be stapled together) in the box kept in EE-Office for EE325-DD Homework. The box will become available on Friday morning. *If you have queries, then meet the instructor or the TA during office hours.*

Set-A

1. Let $\{X_1, X_2, X_3, \ldots\}$ be a sequence of *zero-mean* dependent random variables such that,

$$\mathrm{cov}(X_i, X_j) = \frac{1}{n^{|i-j|}}. \tag{1}$$

Notice that as $|i-j|$ increases, the covariance between X_i and X_j decreases. Is it true that $(S_n/n) \xrightarrow{\mathbb{P}} c$, where $S_n = X_1 + X_2 + \ldots + X_n$ and c is some constant? If yes, find the value of c.

2. Let $\{X_1, X_2, X_3, \ldots\}$ be an iid sequence of Unif$[0,1]$ random variables. Let $Y_n = n(1 - X_{(n)})$. Find if $Y_n \xrightarrow{d} Y$. If yes, find the cdf of the limit Y.

Set-B

1. Assume that $\{Y_n\}_{n\in\mathbb{N}}, \{Z_n\}_{n\in\mathbb{N}}$ are sequences of random variables such that $Y_n \xrightarrow{\mathbb{P}} Y$ and $Z_n \xrightarrow{\mathbb{P}} Z$. Show that $Y_n + Z_n \xrightarrow{\mathbb{P}} Y + Z$. (Hint: You may find the triangle inequality $|x+y| \leq |x| + |y|$ useful.)

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variable. Assume that $X_n \sim$ Poisson$(1/n)$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ and $nX_n \xrightarrow{\mathbb{P}} 0$.

3. Assuming that $Z_n \xrightarrow{\mathbb{P}} Z$, show that $Z_n \xrightarrow{d} Z$. (Hint: You need to show that $\mathbb{P}(Z_n \leq x) \to \mathbb{P}(Z \leq x)$ for all x where $F_Z(x)$ is continuous. If $F_Z(x)$ is continuous at x, then there is an interval $(x-\delta, x+\delta)$ in which $F_Z(x)$ is continuous. Further, $|Z_n - Z| \leq \epsilon$ with high probability. Connect these pieces with suitable inequalities to get the result.)

4. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables. Assume b to be a real number. Show that $X_n \xrightarrow{\mathcal{L}^2} b$ if and only if,

$$\lim_{n\to\infty} \mathbb{E}(X_n) = b \quad \text{and} \quad \lim_{n\to\infty} \mathrm{var}(X_n) = 0.$$

5. *(Typical sets)* Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables. Let $p(x), x = 0, 1$ be the pmf of X. Consider the typical set,

$$A_n(\epsilon) := \left\{ x_1^n : \left| -\frac{1}{n} \log_2(p(x_1^n)) - H_2(p) \right| \leq \epsilon \right\}.$$

(a) Show that for any fixed $\epsilon > 0$ and large enough n, $\mathbb{P}((X_1, X_2, \ldots, X_n) \in A_n(\epsilon)) \geq (1-\epsilon)$.

(b) Let $h_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$. Show that for any $(x_1, \ldots, x_n) \in A_n(\epsilon)$,

$$2^{-nh_2(p)-n\epsilon} \leq \mathbb{P}((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) \leq 2^{-nh_2(p)+n\epsilon}.$$

Thus, all typical set sequences have approximately the same probability of $\approx 2^{-nh_2(p)}$.

(c) Show that the number of typical sequences $|A_n(\epsilon)|$ satisfies the following inequality,

$$(1-\epsilon)2^{nh_2(p)-n\epsilon} \leq |A_n(\epsilon)| \leq 2^{nh_2(p)+n\epsilon}.$$

Thus about $2^{nh_2(p)}$ typical sequences are there and they require $nh_2(p)$ bits for representation. (Hint: use the Union bound.)

6. (Gaussian LTI filtering) Gaussian LTI filters are difficult to make. Assume that as a designer you have access to a large supply of an LTI filter having the following impulse response,

$$h(t) = 1 \text{ for } |t| \leq a, \qquad \text{and} \quad h(t) = 0 \text{ otherwise},$$

with the constant $a > 0$ as a tunable parameter. How will you realize an approximate LTI Gaussian filter from the supplied devices? You can use any amplitude gain element if necessary.