EE 308: Communication Systems

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Hilbert Transforms

- Hilbert transform: definition

$$\begin{aligned} H(f) &:= -j\,\mathrm{sgn}(f) \\ h(t) &= \frac{j}{\pi t} \\ \hat{x}(t) &= h(t) * x(t) \\ &= \int \frac{x(\tau)}{\pi(t-\tau)} d\tau \end{aligned}$$

$$\hat{\hat{x}}(t) = -x(t)$$

- Hilbert transform essentially shifts every frequency by $-\pi/2$.
- Can show that $\widehat{\cos(2\pi f_0 t)} = \sin 2\pi f_0 t$ and $\widehat{\sin(2\pi f_0 t)} = -\cos(2\pi f_0 t)$.
- Properties
  - Preserves energy/power
  - $x(t)$ and $\hat{x}(t)$ are orthogonal
  - $\widehat{m(t)c(t)} = m(t)\hat{c}(t)$ when spectra of $m(t)$ and $x(t)$ are non overlapping.

Hilbert Transforms

- Proof of Property 2:

$$\int x(t)\hat{x}(t)dt = \int X(f)X^*(f)df = (j\text{sgn}(f))|X(f)|^2 df$$

Last integrand is odd and is therefore zero.

- Proof of Property 3: Assume $|M(f)| = 0$ for $|f| < W$ and $|C(f)| = 0$ for$|f| > W$. Using the inverse FT representation, we can write

$$m(t)c(t) = \int\int M(f_1)C(f_2)e^{j2\pi(f_1+f_2)t}df_1 df_2$$

Using $\widehat{e^{j\pi f_0 t}} = -j\,[\text{sgn}(2\pi f_0 t)]\; e^{j2\pi f_0 t}$ we can write

$$\widehat{m(t)c(t)} = \int\int M(f_1)C(f_2)[-j\,\text{sgn}(f_1+f_2)]\; e^{j2\pi(f_1+f_2)t}df_1 df_2$$

Observing that $M(f_1) = 0$ for $|f_1| < W$ and $|f_2| > W$ we can write

$$\begin{aligned}\widehat{m(t)c(t)} \quad &= \quad \int_{f_1=-W}^{W}\int_{f_2=-\infty}^{-W} M(f_1)e^{j2\pi f_1 t}df_1\; C(f_2)[-j\text{sgn}(f_1+f_2)]\; e^{j2\pi f_2 t}df_2 \\ &\quad + \int_{f_1=-W}^{W}\int_{f_2=W}^{\infty} M(f_1)e^{j2\pi f_1 t}df_1\; C(f_2)[-j\text{sgn}(f_1+f_2)]\; e^{j2\pi f_2 t}df_2\end{aligned}$$

Properties of Hilbert Transforms

By substituting the values for $\mathrm{sgn}(f_1 + f_2)$ in the regions of integration, we can write the following.

$$
\begin{aligned}
\widehat{m(t)c(t)} &= \int_{f_1=-W}^{W} \int_{f_2=-\infty}^{-W} M(f_1) e^{j2\pi f_1 t} df_1 \; C(f_2)[-j(-1)] \; e^{j2\pi f_2 t} df_2 \\
&\quad + \int_{f_1=-W}^{W} \int_{f_2=W}^{\infty} M(f_1) e^{j2\pi f_1 t} df_1 \; C(f_2)[-j(+1)] \; e^{j2\pi f_2 t} df_2 \\
&= \left[\int_{f_1=-W}^{W} M(f_1) e^{j2\pi f_1 t} df_1 \right] \left[\int_{|f|>W} C(f_1)[-j\mathrm{sgn}(f_2)] \; df_2 \right]
\end{aligned}
$$

The limits of the last equality may be omitted. Observe that

$$
\hat{c}(t) = \int \widehat{C(f)\; e^{j2\pi ft}} \; df = \int C(f) \; \widehat{ej2\pi ft} \; df = \int C(f) \; [-j\mathrm{sgn}(f)] \; e^{j2\pi ft} \; df
$$

Thus

$$
\widehat{m(t)c(t)} = m(t)\hat{c}(t)
$$

Analytic Signals and Bandpass Signals

- Define the analytic signal $x_p(t) = x(t) + j\hat{x}(t)$
- Magnitude of $x_p(t)$ is called the *envelope* of a signal.
- The spectrum of $x_p(t)$ is given by

$$X_p(f) = \begin{cases} 2X(f) & |f| > 0 \\ 0 & |f| < 0 \end{cases}$$

- $x_p(t)$ is not real and hence $|X_p(f)|$ is not even.
- Let $x_b(t)$ be a real bandpass signal around frequency f_c. Express $x_p(t)$ as

$$x_p(t) = x_l(t)e^{j2\pi f_0 t}$$

where $x_l(t)$ is a complex low pass signal; can also be obtained as below.

$$\begin{aligned} x_p(t) &= x_l(t)e^{j2\pi f_c t} := x_b(t) + j\hat{x}_b(t) \\ x_b(t) &= \mathrm{Re}(x_l(t)e^{j2\pi f_c t}) \\ \hat{x}(t) &= \mathrm{Im}(x_l(t)e^{j2\pi f_c t}) \\ x_b(t) &= x_R(t)\cos(2\pi f_c t) - jx_I(t)\sin(2\pi f_c t) \\ x_l(t) &:= x_R(t) + j\sin x_I(t) \end{aligned}$$

$x_R(t)$ & $x_I(t)$ are, resp., *in-phase* and *quadrature* components of $x_l(t)$.

Analytic Signals and Bandpass Signals

$$
\begin{aligned}
\hat{x}_b(t) &= x_R(t)\sin(2\pi f_c t) + j x_I(t)\cos(2\pi f_c t) \\
|x_l(t)| = a(t) &= \sqrt{x_R^2(t) + x_I^2(t)} \\
\angle x_l(t) = \theta(t) &= \tan^{-1}\left(\frac{x_I(t)}{x_R(t)}\right) \\
x(t) &= \mathrm{Re}\left(x_l(t)e^{j2\pi f_c t}\right) \\
&= \mathrm{Re}\left(a(t)e^{j(2\pi f_c t + \theta(t))}\right) = a(t)\cos(2\pi f_c t + \theta(t))
\end{aligned}
$$

- Implementations to obtain $x_R(t)$ and $x_I(t)$ from bandpass $x_b(t)$

$$x_R(t) = x_b(t)\cos(2\pi f_c t) + \hat{x}_b(t)\sin 2\pi f_c t$$

- Alternately, write $x_R(t) = x_b(t)\cos(2\pi f_c t)$ and $x_I(t) = x_b(t)\sin 2(\pi f_c t)$ and pass through a LPF of suitable bandwidth.
- Given $x_R(t)$ and $x_I(t)$ we can obtain $x_b(t)$ as $x_R(t)\cos(2\pi f_c t) - x_I(t)\sin(2\pi f_c t)$.
- HW: What is the energy in $x_l(t)$ in terms of the energy in $x_l(t)$?

Analytic Signals and Bandpass Signals

- Freq domain relation between $x_b(t)$ and $x_l(t)$

$$
\begin{aligned}
x_b(t) &= \text{Re}\left(x_l(t)e^{j2\pi f_c t}\right) \\
X_b(f) &= \mathcal{F}\left(x_l(t)e^{j2\pi f_c t} + x_l^*(t)e^{-j2\pi f_c t}\right) \\
&= X_l(f+f_c) + X_l^*(-f-f_c)
\end{aligned}
$$

- Alternately, we can derive the above by realising that for $f > 0$, $X_b(f) = X_l(f - f_c)$ and since $x_b(t)$, is real $X_b(-f) = X^*(f)$

Bandpass Systems

- $H_b(f)$ is the impulse response of a bandpass system
- $H_l(f) = u(f+f_c)H(f+f_c)$. Then $H(f) = H_l(f-f_c) + H_l^*(-f-f_c)$ and $h_b(t) = \text{Re}\left[h_l(t)e^{j2\pi f_c t}\right]$
- Let $y_b(t) = h_b(t) * x_b(t)$. Since we can write

$$\begin{aligned} Y_b(f) &= X_b(f)H_b(f) \\ &= [S_l(f-f_c) + S^*(-f-f_c)][H_l(f-f_c) + H^*(-f-f_c)] \\ &\approx X_l(f-f_c)H_l(f-f_c) + X_l^*(-f-f_c)H_l^*(-f-f_c)] \\ &= X_l(f)H_l(f) \end{aligned}$$

- We use $X_l(f-f_c)H_l^*(-f-f_c) = 0$ for all f since $X_l(f-f_c), H_l(f-f_c) \approx 0$ for all $f \leq 0$
- We use $X_l^*(-f-f_c)H_l(f-f_c) = 0$ for all f since $X_l(f-f_c), H_l(f-f_c) \approx 0$ for all $f \leq 0$