```python
def main():
    Print_Function()
    (x, y, z) = xyz = symbols('x,y,z',real=True)
    (o3d, ex, ey, ez) = Ga.build('e_x e_y e_z', g=[1, 1, 1], coords=xyz)
    grad = o3d.grad
    (u, v) = uv = symbols('u,v',real=True)
    (g2d, eu, ev) = Ga.build('e_u e_v', coords=uv)
    grad_uv = g2d.grad
    v_xyz = o3d.mv('v','vector')
    A_xyz = o3d.mv('A','vector',f=True)
    A_uv = g2d.mv('A','vector',f=True)
    print '#3d orthogonal ($A$ is vector function)'
    print 'A =', A_xyz
    print '%A^{2} =', A_xyz * A_xyz
    print 'grad|A =', grad | A_xyz
    print 'grad*A =', grad * A_xyz
    print 'v|(grad*A) =',v_xyz|(grad*A_xyz)
    print '#2d general ($A$ is vector function)'
    print 'A =', A_uv
    print '%A^{2} =', A_uv * A_uv
    print 'grad|A =', grad_uv | A_uv
    print 'grad*A =', grad_uv * A_uv
    A = o3d.lt('A')
    print '#3d orthogonal ($A,\\;B$ are linear transformations)'
    print 'A =', A
    print '\\f{\\det}{A} =', A.det()
    print '\\overline{A} =', A.adj()
    print '\\f{\\Tr}{A} =', A.tr()
    print '\\f{A}{e_x^e_y} =', A(ex^ey)
    print '\\f{A}{e_x}^\\f{A}{e_y} =', A(ex)^A(ey)
    B = o3d.lt('B')
    print 'A + B =', A + B
    print 'AB =', A * B
    print 'A - B =', A - B
    print '#2d general ($A,\\;B$ are linear transformations)'
    A2d = g2d.lt('A')
    print 'A =', A2d
    print '\\f{\\det}{A} =', A2d.det()
    print '\\overline{A} =', A2d.adj()
    print '\\f{\\Tr}{A} =', A2d.tr()
    print '\\f{A}{e_u^e_v} =', A2d(eu^ev)
    print '\\f{A}{e_u}^\\f{A}{e_v} =', A2d(eu)^A2d(ev)
    B2d = g2d.lt('B')
    print 'B =', B2d
    print 'A + B =', A2d + B2d
    print 'AB =', A2d * B2d
    print 'A - B =', A2d - B2d
    a = g2d.mv('a','vector')
    b = g2d.mv('b','vector')
    print r'a|\f{\overline{A}}{b}-b|\f{\underline{A}}{a} =',((a|A2d.adj()(b))-(b|A2d(a))).simplify()
    m4d = Ga('e_t e_x e_y e_z', g=[1, -1, -1, -1],coords=symbols('t,x,y,z',real=True))
    T = m4d.lt('T')
    print 'g =', m4d.g
    print r'\underline{T} =',T
    print r'\overline{T} =',T.adj()
    print r'\f{\det}{\underline{T}} =',T.det()
    print r'\f{\mbox{tr}}{\underline{T}} =',T.tr()
    a = m4d.mv('a','vector')
    b = m4d.mv('b','vector')
```

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    print r'a|\f{\overline{T}}{b}-b|\f{\underline{T}}{a} =',((a|T.adj()(b))-(b|T(a))).simplify()
    coords = (r, th, phi) = symbols('r,theta,phi', real=True)
    (sp3d, er, eth, ephi) = Ga.build('e_r e_th e_ph', g=[1, r**2, r**2*sin(th)**2], coords=coords)
    grad = sp3d.grad
    sm_coords = (u, v) = symbols('u,v', real=True)
    smap = [1, u, v]  # Coordinate map for sphere of r = 1
    sph2d = sp3d.sm(smap,sm_coords,norm=True)
    (eu, ev) = sph2d.mv()
    grad_uv = sph2d.grad
    F = sph2d.mv('F','vector',f=True)
    f = sph2d.mv('f','scalar',f=True)
    print 'f =',f
    print 'grad*f =',grad_uv * f
    print 'F =',F
    print 'grad*F =',grad_uv * F
    tp = (th,phi) = symbols('theta,phi',real=True)
    smap = [sin(th)*cos(phi),sin(th)*sin(phi),cos(th)]
    sph2dr = o3d.sm(smap,tp,norm=True)
    (eth, ephi) = sph2dr.mv()
    grad_tp = sph2dr.grad
    F = sph2dr.mv('F','vector',f=True)
    f = sph2dr.mv('f','scalar',f=True)
    print 'f =',f
    print 'grad*f =',grad_tp * f
    print 'F =',F
    print 'grad*F =',grad_tp * F
    return
```

Code Output: 3d orthogonal (A is vector function)

$$A = A^{x} e_{x} + A^{y} e_{y} + A^{z} e_{z}$$

$$A^{2} = \left(A^{x}\right)^{2} + \left(A^{y}\right)^{2} + \left(A^{z}\right)^{2}$$

$$\boldsymbol{\nabla} \cdot A = \partial_{x} A^{x} + \partial_{y} A^{y} + \partial_{z} A^{z}$$

$$\boldsymbol{\nabla} A = \left(\partial_{x} A^{x} + \partial_{y} A^{y} + \partial_{z} A^{z}\right) + \left(- \partial_{y} A^{x} + \partial_{x} A^{y}\right) e_{x}\wedge e_{y} + \left(- \partial_{z} A^{x} + \partial_{x} A^{z}\right) e_{x}\wedge e_{z} + \left(- \partial_{z} A^{y} + \partial_{y} A^{z}\right) e_{y}\wedge e_{z}$$

$$v\cdot (\boldsymbol{\nabla} A) = \left(v^{y} \partial_{y} A^{x} - v^{y} \partial_{x} A^{y} + v^{z} \partial_{z} A^{x} - v^{z} \partial_{x} A^{z}\right) e_{x} + \left(- v^{x} \partial_{y} A^{x} + v^{x} \partial_{x} A^{y} + v^{z} \partial_{z} A^{y} - v^{z} \partial_{y} A^{z}\right) e_{y} + \left(- v^{x} \partial_{z} A^{x} + v^{x} \partial_{x} A^{z} - v^{y} \partial_{z} A^{y} + v^{y} \partial_{y} A^{z}\right) e_{z}$$

2d general (A is vector function)

$$A = A^{u} e_{u} + A^{v} e_{v}$$

$$A^{2} = \left(e_{u}\cdot e_{u}\right) \left(A^{u}\right)^{2} + 2 \left(e_{u}\cdot e_{v}\right) A^{u} A^{v} + \left(e_{v}\cdot e_{v}\right) \left(A^{v}\right)^{2}$$

$$\boldsymbol{\nabla} \cdot A = \partial_{u} A^{u} + \partial_{v} A^{v}$$

$$\boldsymbol{\nabla} A = \left(\partial_{u} A^{u} + \partial_{v} A^{v}\right) + \frac{- \left(e_{u}\cdot e_{u}\right) \partial_{v} A^{u} + \left(e_{u}\cdot e_{v}\right) \partial_{u} A^{u} - \left(e_{u}\cdot e_{v}\right) \partial_{v} A^{v} + \left(e_{v}\cdot e_{v}\right) \partial_{u} A^{v}}{\left(e_{u}\cdot e_{u}\right) \left(e_{v}\cdot e_{v}\right) - \left(e_{u}\cdot e_{v}\right)^{2}} e_{u}\wedge e_{v}$$

3d orthogonal ($A,\ B$ are linear transformations)

$$A = \left \{ \begin{array}{ll} L \left (e_{x} \right) = & A_{xx} e_{x} + A_{yx} e_{y} + A_{zx} e_{z} \\ L \left (e_{y} \right) = & A_{xy} e_{x} + A_{yy} e_{y} + A_{zy} e_{z} \\ L \left (e_{z} \right) = & A_{xz} e_{x} + A_{yz} e_{y} + A_{zz} e_{z} \end{array} \right \}$$

$$\det\left (A\right) = A_{xz} \left(A_{yx} A_{zy} - A_{yy} A_{zx}\right) - A_{yz} \left(A_{xx} A_{zy} - A_{xy} A_{zx}\right) + A_{zz} \left(A_{xx} A_{yy} - A_{xy} A_{yx}\right)$$

$$\overline{A} = \left \{ \begin{array}{ll} L \left (e_{x} \right) = & A_{xx} e_{x} + A_{xy} e_{y} + A_{xz} e_{z} \\ L \left (e_{y} \right) = & A_{yx} e_{x} + A_{yy} e_{y} + A_{yz} e_{z} \\ L \left (e_{z} \right) = & A_{zx} e_{x} + A_{zy} e_{y} + A_{zz} e_{z} \end{array} \right \}$$

$$\mathrm{Tr}\left (A\right) = A_{xx} + A_{yy} + A_{zz}$$

$$A \left (e_{x}\wedge e_{y}\right) = \left(A_{xx} A_{yy} - A_{xy} A_{yx}\right) e_{x}\wedge e_{y} + \left(A_{xx} A_{zy} - A_{xy} A_{zx}\right) e_{x}\wedge e_{z} + \left(A_{yx} A_{zy} - A_{yy} A_{zx}\right) e_{y}\wedge e_{z}$$

$$A\left (e_{x} \right) \wedge A\left (e_{y} \right) = \left (A_{xx} A_{yy} - A_{xy} A_{yx} \right) e_{x}\wedge e_{y} + \left (A_{xx} A_{zy} - A_{xy} A_{zx} \right) e_{x}\wedge e_{z} + \left (A_{yx} A_{zy} - A_{yy} A_{zx} \right) e_{y}\wedge e_{z}$$

$$A+B = \left \{ \begin{array}{ll} L \left (e_{x} \right) = & \left (A_{xx} + B_{xx} \right) e_{x} + \left (A_{yx} + B_{yx} \right) e_{y} + \left (A_{zx} + B_{zx} \right) e_{z} \\ L \left (e_{y} \right) = & \left (A_{xy} + B_{xy} \right) e_{x} + \left (A_{yy} + B_{yy} \right) e_{y} + \left (A_{zy} + B_{zy} \right) e_{z} \\ L \left (e_{z} \right) = & \left (A_{xz} + B_{xz} \right) e_{x} + \left (A_{yz} + B_{yz} \right) e_{y} + \left (A_{zz} + B_{zz} \right) e_{z} \end{array} \right \}$$

$$AB = \left \{ \begin{array}{ll} L \left (e_{x} \right) = & \left (A_{xx} B_{xx} + A_{xy} B_{yx} + A_{xz} B_{zx} \right) e_{x} + \left (A_{yx} B_{xx} + A_{yy} B_{yx} + A_{yz} B_{zx} \right) e_{y} + \left (A_{zx} B_{xx} + A_{zy} B_{yx} + A_{zz} B_{zx} \right) e_{z} \\ L \left (e_{y} \right) = & \left (A_{xx} B_{xy} + A_{xy} B_{yy} + A_{xz} B_{zy} \right) e_{x} + \left (A_{yx} B_{xy} + A_{yy} B_{yy} + A_{yz} B_{zy} \right) e_{y} + \left (A_{zx} B_{xy} + A_{zy} B_{yy} + A_{zz} B_{zy} \right) e_{z} \\ L \left (e_{z} \right) = & \left (A_{xx} B_{xz} + A_{xy} B_{yz} + A_{xz} B_{zz} \right) e_{x} + \left (A_{yx} B_{xz} + A_{yy} B_{yz} + A_{yz} B_{zz} \right) e_{y} + \left (A_{zx} B_{xz} + A_{zy} B_{yz} + A_{zz} B_{zz} \right) e_{z} \end{array} \right \}$$

$$A-B = \left \{ \begin{array}{ll} L \left (e_{x} \right) = & \left (A_{xx} - B_{xx} \right) e_{x} + \left (A_{yx} - B_{yx} \right) e_{y} + \left (A_{zx} - B_{zx} \right) e_{z} \\ L \left (e_{y} \right) = & \left (A_{xy} - B_{xy} \right) e_{x} + \left (A_{yy} - B_{yy} \right) e_{y} + \left (A_{zy} - B_{zy} \right) e_{z} \\ L \left (e_{z} \right) = & \left (A_{xz} - B_{xz} \right) e_{x} + \left (A_{yz} - B_{yz} \right) e_{y} + \left (A_{zz} - B_{zz} \right) e_{z} \end{array} \right \}$$

2d general ($A,\ B$ are linear transformations)

$$A = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & A_{uu} e_{u} + A_{vu} e_{v} \\ L \left (e_{v} \right) = & A_{uv} e_{u} + A_{vv} e_{v} \end{array} \right \}$$

$$\det\left (A \right) = A_{uu} A_{vv} - A_{uv} A_{vu}$$

$$\overline{A} = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & \frac{1}{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) - \left (e_{u}\cdot e_{v} \right) \wedge 2} \left (- \left (e_{u}\cdot e_{u} \right) \left (e_{u}\cdot e_{v} \right) A_{uv} + \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) A_{uu} - \left (e_{u}\cdot e_{v} \right) \wedge 2 A_{vv} + \left (e_{u}\cdot e_{v} \right) \left (e_{v}\cdot e_{v} \right) A_{vu} \right) e_{u} + \frac{1}{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) - \left (e_{u}\cdot e_{v} \right) \wedge 2} \left (\left (e_{u}\cdot e_{u} \right) \wedge 2 A_{uv} - \left (e_{u}\cdot e_{u} \right) \left (e_{u}\cdot e_{v} \right) A_{uu} + \left (e_{u}\cdot e_{u} \right) \left (e_{u}\cdot e_{v} \right) A_{vv} - \left (e_{u}\cdot e_{v} \right) \wedge 2 A_{vu} \right) e_{v} \\ L \left (e_{v} \right) = & \frac{1}{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) - \left (e_{u}\cdot e_{v} \right)^{2}} \left (- \left (e_{u}\cdot e_{v} \right)^{2} A_{uv} + \left (e_{u}\cdot e_{v} \right) \left (e_{v}\cdot e_{v} \right) A_{uu} - \left (e_{u}\cdot e_{v} \right) \left (e_{v}\cdot e_{v} \right) A_{vv} + \left (e_{v}\cdot e_{v} \right)^{2} A_{vu} \right) e_{u} + \frac{1}{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) - \left (e_{u}\cdot e_{v} \right)^{2}} \left (\left (e_{u}\cdot e_{u} \right) \left (e_{u}\cdot e_{v} \right) A_{uv} + \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) A_{vv} - \left (e_{u}\cdot e_{v} \right)^{2} A_{uu} - \left (e_{u}\cdot e_{v} \right) \left (e_{v}\cdot e_{v} \right) A_{vu} \right) e_{v} \end{array} \right \}$$

$$\mathrm{Tr}\left (A \right) = - \frac{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) A_{uu}}{- \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) + \left (e_{u}\cdot e_{v} \right)^{2}} - \frac{\left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) A_{vv}}{- \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) + \left (e_{u}\cdot e_{v} \right)^{2}} + \frac{\left (e_{u}\cdot e_{v} \right)^{2} A_{uu}}{- \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) + \left (e_{u}\cdot e_{v} \right)^{2}} + \frac{\left (e_{u}\cdot e_{v} \right)^{2} A_{vv}}{- \left (e_{u}\cdot e_{u} \right) \left (e_{v}\cdot e_{v} \right) + \left (e_{u}\cdot e_{v} \right)^{2}}$$

$$A\left (e_{u}\wedge e_{v} \right) = \left (A_{uu} A_{vv} - A_{uv} A_{vu} \right) e_{u}\wedge e_{v}$$

$$A\left (e_{u} \right) \wedge A\left (e_{v} \right) = \left (A_{uu} A_{vv} - A_{uv} A_{vu} \right) e_{u}\wedge e_{v}$$

$$B = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & B_{uu} e_{u} + B_{vu} e_{v} \\ L \left (e_{v} \right) = & B_{uv} e_{u} + B_{vv} e_{v} \end{array} \right \}$$

$$A+B = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & \left (A_{uu} + B_{uu} \right) e_{u} + \left (A_{vu} + B_{vu} \right) e_{v} \\ L \left (e_{v} \right) = & \left (A_{uv} + B_{uv} \right) e_{u} + \left (A_{vv} + B_{vv} \right) e_{v} \end{array} \right \}$$

$$AB = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & \left (A_{uu} B_{uu} + A_{uv} B_{vu} \right) e_{u} + \left (A_{vu} B_{uu} + A_{vv} B_{vu} \right) e_{v} \\ L \left (e_{v} \right) = & \left (A_{uu} B_{uv} + A_{uv} B_{vv} \right) e_{u} + \left (A_{vu} B_{uv} + A_{vv} B_{vv} \right) e_{v} \end{array} \right \}$$

$$A-B = \left \{ \begin{array}{ll} L \left (e_{u} \right) = & \left (A_{uu} - B_{uu} \right) e_{u} + \left (A_{vu} - B_{vu} \right) e_{v} \\ L \left (e_{v} \right) = & \left (A_{uv} - B_{uv} \right) e_{u} + \left (A_{vv} - B_{vv} \right) e_{v} \end{array} \right \}$$

$$a\cdot \overline{A}\left (b \right) - b\cdot \underline{A}\left (a \right) = 0$$

$$g = \left [\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$\underline{T} = \left \{ \begin{array}{ll} L \left (e_{t} \right) = & T_{tt} e_{t} + T_{xt} e_{x} + T_{yt} e_{y} + T_{zt} e_{z} \\ L \left (e_{x} \right) = & T_{tx} e_{t} + T_{xx} e_{x} + T_{yx} e_{y} + T_{zx} e_{z} \\ L \left (e_{y} \right) = & T_{ty} e_{t} + T_{xy} e_{x} + T_{yy} e_{y} + T_{zy} e_{z} \\ L \left (e_{z} \right) = & T_{tz} e_{t} + T_{xz} e_{x} + T_{yz} e_{y} + T_{zz} e_{z} \end{array} \right \}$$

$$\overline{T} = \left \{ \begin{array}{ll} L \left (e_{t} \right) = & T_{tt} e_{t} - T_{tx} e_{x} - T_{ty} e_{y} - T_{tz} e_{z} \\ L \left (e_{x} \right) = & - T_{xt} e_{t} + T_{xx} e_{x} + T_{xy} e_{y} + T_{xz} e_{z} \\ L \left (e_{y} \right) = & - T_{yt} e_{t} + T_{yx} e_{x} + T_{yy} e_{y} + T_{yz} e_{z} \\ L \left (e_{z} \right) = & - T_{zt} e_{t} + T_{zx} e_{x} + T_{zy} e_{y} + T_{zz} e_{z} \end{array} \right \}$$

$$\det\left (\underline{T} \right) = - T_{tz} \left (T_{xt} T_{yx} T_{zy} - T_{xt} T_{yy} T_{zx} - T_{xx} T_{yt} T_{zy} + T_{xx} T_{yy} T_{zt} + T_{xy} T_{yt} T_{zx} - T_{xy} T_{yx} T_{zt} \right) + T_{xz} \left (T_{tt} T_{yx} T_{zy} - T_{tt} T_{yy} T_{zx} - T_{tx} T_{yt} T_{zy} + T_{tx} T_{yy} T_{zt} + T_{ty} T_{yt} T_{zx} - T_{ty} T_{yx} T_{zt} \right) - T_{yz} \left (T_{tt} T_{xx} T_{zy} - T_{tt} T_{xy} T_{zx} - T_{tx} T_{xt} T_{zy} + T_{tx} T_{xy} T_{zt} + T_{ty} T_{xt} T_{zx} - T_{ty} T_{xx} T_{zt} \right) + T_{zz} \left (T_{tt} T_{xx} T_{yy} - T_{tt} T_{xy} T_{yx} - T_{tx} T_{xt} T_{yy} + T_{tx} T_{xy} T_{yt} + T_{ty} T_{xt} T_{yx} - T_{ty} T_{xx} T_{yt} \right)$$

$$\mathrm{tr}\left (\underline{T} \right) = T_{tt} + T_{xx} + T_{yy} + T_{zz}$$

$$a\cdot \overline{T}\left (b \right) - b\cdot \underline{T}\left (a \right) = 0$$

$$f = f$$

$$\boldsymbol{\nabla} f = \partial_{u} f e_{u} + \frac{\partial_{v} f}{\sin{\left (u \right)}} e_{v}$$

$$F = F^{u} e_{u} + F^{v} e_{v}$$

$$\boldsymbol{\nabla} F = \left(\frac{F^{u}}{\tan{(u)}} + \partial_{u} F^{u} + \frac{\partial_{v} F^{v}}{\sin{(u)}} \right) + \left(\frac{F^{v}}{\tan{(u)}} + \partial_{u} F^{v} - \frac{\partial_{v} F^{u}}{\sin{(u)}} \right) e_{u}\wedge e_{v}$$

$$f = f$$

$$\boldsymbol{\nabla} f = \partial_{\theta} f e_{\theta} + \frac{\partial_{\phi} f}{\sin{(\theta)}} e_{\phi}$$

$$F = F^{\theta} e_{\theta} + F^{\phi} e_{\phi}$$

$$\boldsymbol{\nabla} F = \left(\frac{F^{\theta}}{\tan{(\theta)}} + \partial_{\theta} F^{\theta} + \frac{\partial_{\phi} F^{\phi}}{\sin{(\theta)}} \right) + \left(\frac{F^{\phi}}{\tan{(\theta)}} + \partial_{\theta} F^{\phi} - \frac{\partial_{\phi} F^{\theta}}{\sin{(\theta)}} \right) e_{\theta}\wedge e_{\phi}$$