```
def basic_multivector_operations_3D():
    Print_Function()
    g3d = Ga('e*x|y|z')
    (ex,ey,ez) = g3d.mv()
    A = g3d.mv('A','mv')
    A.Fmt(1,'A')
    A.Fmt(2,'A')
    A.Fmt(3,'A')
    A.even().Fmt(1,'%A_{+}')
    A.odd().Fmt(1,'%A_{-}')
    X = g3d.mv('X','vector')
    Y = g3d.mv('Y','vector')
    print 'g_{ij} = ',g3d.g
    X.Fmt(1,'X')
    Y.Fmt(1,'Y')
    (X*Y).Fmt(2,'X*Y')
    (X^Y).Fmt(2,'X^Y')
    (X|Y).Fmt(2,'X|Y')
    return
```

Code Output:

$$A = A + A^{x} e_{x} + A^{y} e_{y} + A^{z} e_{z} + A^{xy} e_{x}\wedge e_{y} + A^{xz} e_{x}\wedge e_{z} + A^{yz} e_{y}\wedge e_{z} + A^{xyz} e_{x}\wedge e_{y}\wedge e_{z}$$

$$\begin{aligned} A = & A \\ & + A^{x} e_{x} + A^{y} e_{y} + A^{z} e_{z} \\ & + A^{xy} e_{x}\wedge e_{y} + A^{xz} e_{x}\wedge e_{z} + A^{yz} e_{y}\wedge e_{z} \\ & + A^{xyz} e_{x}\wedge e_{y}\wedge e_{z} \end{aligned}$$

$$\begin{aligned} A = & A \\ & + A^{x} e_{x} \\ & + A^{y} e_{y} \\ & + A^{z} e_{z} \\ & + A^{xy} e_{x}\wedge e_{y} \\ & + A^{xz} e_{x}\wedge e_{z} \\ & + A^{yz} e_{y}\wedge e_{z} \\ & + A^{xyz} e_{x}\wedge e_{y}\wedge e_{z} \end{aligned}$$

$$A_{+} = A + A^{xy} e_{x}\wedge e_{y} + A^{xz} e_{x}\wedge e_{z} + A^{yz} e_{y}\wedge e_{z}$$

$$A_{-} = A^{x} e_{x} + A^{y} e_{y} + A^{z} e_{z} + A^{xyz} e_{x}\wedge e_{y}\wedge e_{z}$$

$$g_{ij} = \left[\begin{array}{ccc} (e_{x}\cdot e_{x}) & (e_{x}\cdot e_{y}) & (e_{x}\cdot e_{z}) \\ (e_{x}\cdot e_{y}) & (e_{y}\cdot e_{y}) & (e_{y}\cdot e_{z}) \\ (e_{x}\cdot e_{z}) & (e_{y}\cdot e_{z}) & (e_{z}\cdot e_{z}) \end{array}\right]$$

$$X = X^{x} e_{x} + X^{y} e_{y} + X^{z} e_{z}$$

$$Y = Y^{x} e_{x} + Y^{y} e_{y} + Y^{z} e_{z}$$

$$\begin{aligned} XY = & \left(X^{x} Y^{x} (e_{x}\cdot e_{x}) + X^{x} Y^{y} (e_{x}\cdot e_{y}) + X^{x} Y^{z} (e_{x}\cdot e_{z}) + X^{y} Y^{x} (e_{x}\cdot e_{y}) + X^{y} Y^{y} (e_{y}\cdot e_{y}) + X^{y} Y^{z} (e_{y}\cdot e_{z}) + X^{z} Y^{x} (e_{x}\cdot e_{z}) + X^{z} Y^{y} (e_{y}\cdot e_{z}) + X^{z} Y^{z} (e_{z}\cdot e_{z})\right) \\ & + \left(X^{x} Y^{y} - X^{y} Y^{x}\right) e_{x}\wedge e_{y} + \left(X^{x} Y^{z} - X^{z} Y^{x}\right) e_{x}\wedge e_{z} + \left(X^{y} Y^{z} - X^{z} Y^{y}\right) e_{y}\wedge e_{z} \end{aligned}$$

$$X\wedge Y = \left(X^{x} Y^{y} - X^{y} Y^{x}\right) e_{x}\wedge e_{y} + \left(X^{x} Y^{z} - X^{z} Y^{x}\right) e_{x}\wedge e_{z} + \left(X^{y} Y^{z} - X^{z} Y^{y}\right) e_{y}\wedge e_{z}$$

$$X\cdot Y = X^{x} Y^{x} (e_{x}\cdot e_{x}) + X^{x} Y^{y} (e_{x}\cdot e_{y}) + X^{x} Y^{z} (e_{x}\cdot e_{z}) + X^{y} Y^{x} (e_{x}\cdot e_{y}) + X^{y} Y^{y} (e_{y}\cdot e_{y}) + X^{y} Y^{z} (e_{y}\cdot e_{z}) + X^{z} Y^{x} (e_{x}\cdot e_{z}) + X^{z} Y^{y} (e_{y}\cdot e_{z}) + X^{z} Y^{z} (e_{z}\cdot e_{z})$$

```python
def basic_multivector_operations_2D():
    Print_Function()
    g2d = Ga('e*x|y')
    (ex,ey) = g2d.mv()
    print 'g_{ij} =',g2d.g
    X = g2d.mv('X','vector')
    A = g2d.mv('A','spinor')
    X.Fmt(1,'X')
    A.Fmt(1,'A')
    (X|A).Fmt(2,'X|A')
    (X<A).Fmt(2,'X<A')
    (A>X).Fmt(2,'A>X')
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} (e_x \cdot e_x) & (e_x \cdot e_y) \\ (e_x \cdot e_y) & (e_y \cdot e_y) \end{bmatrix}$$

$$X = X^x e_x + X^y e_y$$

$$A = A + A^{xy} e_x \wedge e_y$$

$$X \cdot A = -A^{xy} \left(X^x (e_x \cdot e_y) + X^y (e_y \cdot e_y) \right) e_x + A^{xy} \left(X^x (e_x \cdot e_x) + X^y (e_x \cdot e_y) \right) e_y$$

$$X \rfloor A = \left(AX^x - A^{xy} X^x (e_x \cdot e_y) - A^{xy} X^y (e_y \cdot e_y) \right) e_x + \left(AX^y + A^{xy} X^x (e_x \cdot e_x) + A^{xy} X^y (e_x \cdot e_y) \right) e_y$$

$$A \lfloor X = \left(AX^x + A^{xy} X^x (e_x \cdot e_y) + A^{xy} X^y (e_y \cdot e_y) \right) e_x + \left(AX^y - A^{xy} X^x (e_x \cdot e_x) - A^{xy} X^y (e_x \cdot e_y) \right) e_y$$

```python
def basic_multivector_operations_2D_orthogonal():
    Print_Function()
    o2d = Ga('e*x|y',g=[1,1])
    (ex,ey) = o2d.mv()
    print 'g_{ii} =',o2d.g
    X = o2d.mv('X','vector')
    A = o2d.mv('A','spinor')
    X.Fmt(1,'X')
    A.Fmt(1,'A')
    (X*A).Fmt(2,'X*A')
    (X|A).Fmt(2,'X|A')
    (X<A).Fmt(2,'X<A')
    (X>A).Fmt(2,'X>A')
    (A*X).Fmt(2,'A*X')
    (A|X).Fmt(2,'A|X')
    (A<X).Fmt(2,'A<X')
    (A>X).Fmt(2,'A>X')
    return
```

Code Output:

$$g_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = X^x e_x + X^y e_y$$

$$A = A + A^{xy} e_x \wedge e_y$$

$$XA = \left(AX^x - A^{xy} X^y \right) e_x + \left(AX^y + A^{xy} X^x \right) e_y$$

$$X \cdot A = -A^{xy} X^y e_x + A^{xy} X^x e_y$$

$$X \rfloor A = \left(AX^x - A^{xy} X^y \right) e_x + \left(AX^y + A^{xy} X^x \right) e_y$$

$$X \lfloor A = AX^x e_x + AX^y e_y$$

$$AX = \left(AX^x + A^{xy} X^y \right) e_x + \left(AX^y - A^{xy} X^x \right) e_y$$

$$A\cdot X = A^{xy}X^{y}e_{x} - A^{xy}X^{x}e_{y}$$

$$A\rfloor X = AX^{x}e_{x} + AX^{y}e_{y}$$

$$A\lfloor X = \left(AX^{x} + A^{xy}X^{y}\right)e_{x} + \left(AX^{y} - A^{xy}X^{x}\right)e_{y}$$

```
def check_generalized_BAC_CAB_formulas():
    Print_Function()
    g4d = Ga('a b c d')
    (a,b,c,d) = g4d.mv()
    print 'g_{ij} =',g4d.g
    print '\\bm{a|(b*c)} =',a|(b*c)
    print '\\bm{a|(b^c)} =',a|(b^c)
    print '\\bm{a|(b^c^d)} =',a|(b^c^d)
    print '\\bm{a|(b^c)+c|(a^b)+b|(c^a)} =',(a|(b^c))+(c|(a^b))+(b|(c^a))
    print '\\bm{a*(b^c)-b*(a^c)+c*(a^b)} =',a*(b^c)-b*(a^c)+c*(a^b)
    print '\\bm{a*(b^c^d)-b*(a^c^d)+c*(a^b^d)-d*(a^b^c)} =',a*(b^c^d)-b*(a^c^d)+c*(a^b^d)-d*(a^b^c)
    print '\\bm{(a^b)|(c^d)} =',(a^b)|(c^d)
    print '\\bm{((a^b)|c)|d} =',((a^b)|c)|d
    print '\\bm{(a^b)\\times (c^d)} =',Com(a^b,c^d)
    return
```

Code Output:

$$g_{ij} = \left[\begin{array}{cccc}(a\cdot a) & (a\cdot b) & (a\cdot c) & (a\cdot d)\\ (a\cdot b) & (b\cdot b) & (b\cdot c) & (b\cdot d)\\ (a\cdot c) & (b\cdot c) & (c\cdot c) & (c\cdot d)\\ (a\cdot d) & (b\cdot d) & (c\cdot d) & (d\cdot d)\end{array}\right]$$

$$\boldsymbol{a\cdot (bc)} = -\left(a\cdot c\right)b + \left(a\cdot b\right)c$$

$$\boldsymbol{a\cdot (b\wedge c)} = -\left(a\cdot c\right)b + \left(a\cdot b\right)c$$

$$\boldsymbol{a\cdot (b\wedge c\wedge d)} = \left(a\cdot d\right)b\wedge c - \left(a\cdot c\right)b\wedge d + \left(a\cdot b\right)c\wedge d$$

$$\boldsymbol{a\cdot (b\wedge c)+c\cdot (a\wedge b)+b\cdot (c\wedge a)} = 0$$

$$\boldsymbol{a(b\wedge c)-b(a\wedge c)+c(a\wedge b)} = 3a\wedge b\wedge c$$

$$\boldsymbol{a(b\wedge c\wedge d)-b(a\wedge c\wedge d)+c(a\wedge b\wedge d)-d(a\wedge b\wedge c)} = 4a\wedge b\wedge c\wedge d$$

$$\boldsymbol{(a\wedge b)\cdot (c\wedge d)} = -\left(a\cdot c\right)\left(b\cdot d\right) + \left(a\cdot d\right)\left(b\cdot c\right)$$

$$\boldsymbol{((a\wedge b)\cdot c)\cdot d} = -\left(a\cdot c\right)\left(b\cdot d\right) + \left(a\cdot d\right)\left(b\cdot c\right)$$

$$\boldsymbol{(a\wedge b)\times (c\wedge d)} = -\left(b\cdot d\right)a\wedge c + \left(b\cdot c\right)a\wedge d + \left(a\cdot d\right)b\wedge c - \left(a\cdot c\right)b\wedge d$$

```
def rounding_numerical_components():
    Print_Function()
    o3d = Ga('e_x e_y e_z',g=[1,1,1])
    (ex,ey,ez) = o3d.mv()
    X = 1.2*ex+2.34*ey+0.555*ez
    Y = 0.333*ex+4*ey+5.3*ez
    print 'X =',X
    print 'Nga(X,2) =',Nga(X,2)
    print 'X*Y =',X*Y
    print 'Nga(X*Y,2) =',Nga(X*Y,2)
    return
```

Code Output:

$$X = 1\cdot 2e_{x} + 2\cdot 34e_{y} + 0\cdot 555e_{z}$$

$$Nga(X,2) = 1\cdot 2e_{x} + 2\cdot 3e_{y} + 0\cdot 55e_{z}$$

$$XY = 12\cdot 7011 + 4\cdot 02078e_{x}\wedge e_{y} + 6\cdot 175185e_{x}\wedge e_{z} + 10\cdot 182e_{y}\wedge e_{z}$$

$$Nga(XY,2) = 13\cdot 0 + 4\cdot 0e_{x}\wedge e_{y} + 6\cdot 2e_{x}\wedge e_{z} + 10\cdot 0e_{y}\wedge e_{z}$$

```
def derivatives_in_rectangular_coordinates():
    Print_Function()
    X = (x,y,z) = symbols('x y z')
    o3d = Ga('e_x e_y e_z',g=[1,1,1],coords=X)
    (ex,ey,ez) = o3d.mv()
    grad = o3d.grad
    f = o3d.mv('f','scalar',f=True)
    A = o3d.mv('A','vector',f=True)
    B = o3d.mv('B','bivector',f=True)
    C = o3d.mv('C','mv')
    print 'f =',f
    print 'A =',A
    print 'B =',B
    print 'C =',C
    print 'grad*f =',grad*f
    print 'grad|A =',grad|A
    print 'grad*A =',grad*A
    print '-I*(grad^A) =',-o3d.i*(grad^A)
    print 'grad*B =',grad*B
    print 'grad^B =',grad^B
    print 'grad|B =',grad|B
    return
```

Code Output:

$$f = f$$

$$A = A^{x}e_{x} + A^{y}e_{y} + A^{z}e_{z}$$

$$B = B^{xy}e_{x}\wedge e_{y} + B^{xz}e_{x}\wedge e_{z} + B^{yz}e_{y}\wedge e_{z}$$

$$C = C + C^{x}e_{x} + C^{y}e_{y} + C^{z}e_{z} + C^{xy}e_{x}\wedge e_{y} + C^{xz}e_{x}\wedge e_{z} + C^{yz}e_{y}\wedge e_{z} + C^{xyz}e_{x}\wedge e_{y}\wedge e_{z}$$

$$\boldsymbol{\nabla} f = \partial_{x} f e_{x} + \partial_{y} f e_{y} + \partial_{z} f e_{z}$$

$$\boldsymbol{\nabla}\cdot A = \partial_{x} A^{x} + \partial_{y} A^{y} + \partial_{z} A^{z}$$

$$\boldsymbol{\nabla} A = \left (\partial_{x} A^{x} + \partial_{y} A^{y} + \partial_{z} A^{z}\right) + \left (- \partial_{y} A^{x} + \partial_{x} A^{y}\right) e_{x}\wedge e_{y} + \left (- \partial_{z} A^{x} + \partial_{x} A^{z}\right) e_{x}\wedge e_{z} + \left (- \partial_{z} A^{y} + \partial_{y} A^{z}\right) e_{y}\wedge e_{z}$$

$$-I(\boldsymbol{\nabla}\wedge A) = \left (- \partial_{z} A^{y} + \partial_{y} A^{z}\right) e_{x} + \left (\partial_{z} A^{x} - \partial_{x} A^{z}\right) e_{y} + \left (- \partial_{y} A^{x} + \partial_{x} A^{y}\right) e_{z}$$

$$\boldsymbol{\nabla} B = \left (- \partial_{y} B^{xy} - \partial_{z} B^{xz}\right) e_{x} + \left (\partial_{x} B^{xy} - \partial_{z} B^{yz}\right) e_{y} + \left (\partial_{x} B^{xz} + \partial_{y} B^{yz}\right) e_{z} + \left (\partial_{z} B^{xy} - \partial_{y} B^{xz} + \partial_{x} B^{yz}\right) e_{x}\wedge e_{y}\wedge e_{z}$$

$$\boldsymbol{\nabla}\wedge B = \left (\partial_{z} B^{xy} - \partial_{y} B^{xz} + \partial_{x} B^{yz}\right) e_{x}\wedge e_{y}\wedge e_{z}$$

$$\boldsymbol{\nabla}\cdot B = \left (- \partial_{y} B^{xy} - \partial_{z} B^{xz}\right) e_{x} + \left (\partial_{x} B^{xy} - \partial_{z} B^{yz}\right) e_{y} + \left (\partial_{x} B^{xz} + \partial_{y} B^{yz}\right) e_{z}$$

```
def derivatives_in_spherical_coordinates():
    Print_Function()
    X = (r,th,phi) = symbols('r theta phi')
    s3d = Ga('e_r e_theta e_phi',g=[1,r**2,r**2*sin(th)**2],coords=X,norm=True)
    (er,eth,ephi) = s3d.mv()
    grad = s3d.grad
    f = s3d.mv('f','scalar',f=True)
    A = s3d.mv('A','vector',f=True)
    B = s3d.mv('B','bivector',f=True)
    print 'f =',f
    print 'A =',A
    print 'B =',B
    print 'grad*f =',grad*f
    print 'grad|A =',grad|A
    print '-I*(grad^A) =',(-s3d.i*(grad^A)).simplify()
    print 'grad^B =',grad^B
```

Code Output:

$$f = f$$

$$A = A^{r} e_{r} + A^{\theta} e_{\theta} + A^{\phi} e_{\phi}$$

$$B = B^{r\theta} e_{r}\wedge e_{\theta} + B^{r\phi} e_{r}\wedge e_{\phi} + B^{\phi\phi} e_{\theta}\wedge e_{\phi}$$

$$\boldsymbol{\nabla} f = \partial_{r} f e_{r} + \frac{1}{r} \partial_{\theta} f e_{\theta} + \frac{\partial_{\phi} f}{r \sin{\left (\theta \right)}} e_{\phi}$$

$$\boldsymbol{\nabla} \cdot A = \frac{1}{r} \left(r \partial_{r} A^{r} + 2 A^{r} + \frac{A^{\theta}}{\tan{\left (\theta \right)}} + \partial_{\theta} A^{\theta} + \frac{\partial_{\phi} A^{\phi}}{\sin{\left (\theta \right)}}\right)$$

$$-I (\boldsymbol{\nabla} \wedge A) = \frac{1}{r} \left(\frac{A^{\phi}}{\tan{\left (\theta \right)}} + \partial_{\theta} A^{\phi} - \frac{\partial_{\phi} A^{\theta}}{\sin{\left (\theta \right)}}\right) e_{r} + \frac{1}{r} \left(- r \partial_{r} A^{\phi} - A^{\phi} + \frac{\partial_{\phi} A^{r}}{\sin{\left (\theta \right)}}\right) e_{\theta} + \frac{1}{r} \left(r \partial_{r} A^{\theta} + A^{\theta} - \partial_{\theta} A^{r}\right) e_{\phi}$$

$$\boldsymbol{\nabla} \wedge B = \frac{1}{r} \left(r \partial_{r} B^{\phi\phi} - \frac{B^{r\phi}}{\tan{\left (\theta \right)}} + 2 B^{\phi\phi} - \partial_{\theta} B^{r\phi} + \frac{\partial_{\phi} B^{r\theta}}{\sin{\left (\theta \right)}}\right) e_{r}\wedge e_{\theta}\wedge e_{\phi}$$

```
def noneuclidian_distance_calculation():
    Print_Function()
    from sympy import solve,sqrt
    g = '0 # #,# 0 #,# # 1'
    nel = Ga('X Y e',g=g)
    (X,Y,e) = nel.mv()
    print 'g_{ij} =',nel.g
    print '%(X\\W Y)^{2} =',(X^Y)*(X^Y)
    L = X^Y^e
    B = L*e # D&L 10.152
    Bsq = (B*B).scalar()
    print '#%L = X\\W Y\\W e \\text{ is a non-euclidian line}'
    print 'B = L*e =',B
    BeBr =B*e*B.rev()
    print '%BeB^{\\dagger} =',BeBr
    print '%B^{2} =',B*B
    print '%L^{2} =',L*L # D&L 10.153
    (s,c,Binv,M,S,C,alpha) = symbols('s c (1/B) M S C alpha')
    XdotY = nel.g[0,1]
    Xdote = nel.g[0,2]
    Ydote = nel.g[1,2]
    Bhat = Binv*B # D&L 10.154
    R = c+s*Bhat # Rotor R = exp(alpha*Bhat/2)
    print '#%s = \\f{\\sinh}{\\alpha/2} \\text{ and } c = \\f{\\cosh}{\\alpha/2}'
    print '%e^{\\alpha B/{2\\abs{B}}} =',R
    Z = R*X*R.rev() # D&L 10.155
    Z.obj = expand(Z.obj)
    Z.obj = Z.obj.collect([Binv,s,c,XdotY])
    Z.Fmt(3,'%RXR^{\\dagger}')
    W = Z|Y # Extract scalar part of multivector
    # From this point forward all calculations are with sympy scalars
    #print '#Objective is to determine value of C = cosh(alpha) such that W = 0'
    W = W.scalar()
    print '%W = Z\\cdot Y =',W
    W = expand(W)
    W = simplify(W)
    W = W.collect([s*Binv])
    M = 1/Bsq
    W = W.subs(Binv**2,M)
    W = simplify(W)
    Bmag = sqrt(XdotY**2-2*XdotY*Xdote*Ydote)
    W = W.collect([Binv*c*s,XdotY])
    #Double angle substitutions
```

```
    W = W.subs(2*XdotY**2-4*XdotY*Xdote*Ydote,2/(Binv**2))
    W = W.subs(2*c*s,S)
    W = W.subs(c**2,(C+1)/2)
    W = W.subs(s**2,(C-1)/2)
    W = simplify(W)
    W = W.subs(1/Binv,Bmag)
    W = expand(W)
    print '#%S = \\f{\\sinh}{\\alpha} \\text{ and } C = \\f{\\cosh}{\\alpha}'
    print 'W =',W
    Wd = collect(W,[C,S],exact=True,evaluate=False)
    Wd_1 = Wd[one]
    Wd_C = Wd[C]
    Wd_S = Wd[S]
    print '%\\text{Scalar Coefficient} =',Wd_1
    print '%\\text{Cosh Coefficient} =',Wd_C
    print '%\\text{Sinh Coefficient} =',Wd_S
    print '%\\abs{B} =',Bmag
    Wd_1 = Wd_1.subs(Bmag,1/Binv)
    Wd_C = Wd_C.subs(Bmag,1/Binv)
    Wd_S = Wd_S.subs(Bmag,1/Binv)
    lhs = Wd_1+Wd_C*C
    rhs = -Wd_S*S
    lhs = lhs**2
    rhs = rhs**2
    W = expand(lhs-rhs)
    W = expand(W.subs(1/Binv**2,Bmag**2))
    W = expand(W.subs(S**2,C**2-1))
    W = W.collect([C,C**2],evaluate=False)
    a = simplify(W[C**2])
    b = simplify(W[C])
    c = simplify(W[one])
    print '#%\\text{Require } aC^{2}+bC+c = 0'
    print 'a =',a
    print 'b =',b
    print 'c =',c
    x = Symbol('x')
    C =  solve(a*x**2+b*x+c,x)[0]
    print '%b^{2}-4ac =',simplify(b**2-4*a*c)
    print '%\\f{\\cosh}{\\alpha} = C = -b/(2a) =',expand(simplify(expand(C)))
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} 0 & (X\cdot Y) & (X\cdot e) \\ (X\cdot Y) & 0 & (Y\cdot e) \\ (X\cdot e) & (Y\cdot e) & 1 \end{bmatrix}$$

$(X\wedge Y)^2 = (X\cdot Y)^2$

$L = X\wedge Y\wedge e$ is a non-euclidian line

$B = Le = X\wedge Y - (Y\cdot e)\,X\wedge e + (X\cdot e)\,Y\wedge e$

$BeB^{\dagger} = (X\cdot Y)\,(-(X\cdot Y) + 2\,(X\cdot e)\,(Y\cdot e))\,e$

$B^2 = (X\cdot Y)\,((X\cdot Y) - 2\,(X\cdot e)\,(Y\cdot e))$

$L^2 = (X\cdot Y)\,((X\cdot Y) - 2\,(X\cdot e)\,(Y\cdot e))$

$s = \sinh(\alpha/2)$ and $c = \cosh(\alpha/2)$

$e^{\alpha B/2|B|} = c + (1/B)sX\wedge Y - (1/B)\,(Y\cdot e)\,sX\wedge e + (1/B)\,(X\cdot e)\,sY\wedge e$

$$RXR^{\dagger} = \left((1/B)^2 (X\cdot Y)^2 s^2 - 2(1/B)^2 (X\cdot Y)(X\cdot e)(Y\cdot e) s^2 + 2(1/B)(X\cdot Y) cs - 2(1/B)(X\cdot e)(Y\cdot e) cs + c^2 \right) X$$
$$+ 2(1/B)(X\cdot e)^2 csY$$
$$+ 2(1/B)(X\cdot Y)(X\cdot e) s (-(1/B)(X\cdot Y) s + 2(1/B)(X\cdot e)(Y\cdot e) s - c) e$$

$$W = Z\cdot Y = (1/B)^2 (X\cdot Y)^3 s^2 - 4(1/B)^2 (X\cdot Y)^2 (X\cdot e)(Y\cdot e) s^2 + 4(1/B)^2 (X\cdot Y)(X\cdot e)^2 (Y\cdot e)^2 s^2 + 2(1/B)(X\cdot Y)^2 cs - 4(1/B)(X\cdot Y)(X\cdot e)(Y\cdot e) cs + (X\cdot Y) c^2$$

$S = \sinh(\alpha)$ and $C = \cosh(\alpha)$

$$W = (1/B)C(X\cdot Y)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)} - (1/B)C(X\cdot e)(Y\cdot e)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)} + (1/B)(X\cdot e)(Y\cdot e)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)} + S\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)}$$

$$\text{Scalar Coefficient} = (1/B)(X\cdot e)(Y\cdot e)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)}$$

$$\text{Cosh Coefficient} = (1/B)(X\cdot Y)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)} - (1/B)(X\cdot e)(Y\cdot e)\sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)}$$

$$\text{Sinh Coefficient} = \sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)}$$

$$|B| = \sqrt{(X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e)}$$

Require $aC^2 + bC + c = 0$

$$a = (X\cdot e)^2 (Y\cdot e)^2$$

$$b = 2(X\cdot e)(Y\cdot e)((X\cdot Y) - (X\cdot e)(Y\cdot e))$$

$$c = (X\cdot Y)^2 - 2(X\cdot Y)(X\cdot e)(Y\cdot e) + (X\cdot e)^2 (Y\cdot e)^2$$

$$b^2 - 4ac = 0$$

$$\cosh(\alpha) = C = -b/(2a) = -\frac{(X\cdot Y)}{(X\cdot e)(Y\cdot e)} + 1$$

```
def conformal_representations_of_circles_lines_spheres_and_planes():
    Print_Function()
    global n,nbar
    g = '1 0 0 0 0,0 1 0 0 0,0 0 1 0 0,0 0 0 0 2,0 0 0 2 0'
    c3d = Ga('e_1 e_2 e_3 n \\bar{n}',g=g)
    (e1,e2,e3,n,nbar) = c3d.mv()
    print 'g_{ij} =',c3d.g
    e = n+nbar
    #conformal representation of points
    A = make_vector(e1, ga=c3d)    # point a = (1,0,0)  A = F(a)
    B = make_vector(e2, ga=c3d)    # point b = (0,1,0)  B = F(b)
    C = make_vector(-e1, ga=c3d)   # point c = (-1,0,0) C = F(c)
    D = make_vector(e3, ga=c3d)    # point d = (0,0,1)  D = F(d)
    X = make_vector('x',3, ga=c3d)
    print 'F(a) =',A
    print 'F(b) =',B
    print 'F(c) =',C
    print 'F(d) =',D
    print 'F(x) =',X
    print '#a = e1, b = e2, c = -e1, and d = e3'
    print '#A = F(a) = 1/2*(a*a*n+2*a-nbar), etc.'
    print '#Circle through a, b, and c'
    print 'Circle: A^B^C^X = 0 =',(A^B^C^X)
    print '#Line through a and b'
    print 'Line  : A^B^n^X = 0 =',(A^B^n^X)
    print '#Sphere through a, b, c, and d'
    print 'Sphere: A^B^C^D^X = 0 =',(((A^B)^C)^D)^X
    print '#Plane through a, b, and d'
    print 'Plane : A^B^n^D^X = 0 =',(A^B^n^D^X)
    L = (A^B^e)^X
    L.Fmt(3,'Hyperbolic\\;\\; Circle: (A^B^e)^X = 0')
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$F(a) = e_1 + \frac{1}{2}n - \frac{1}{2}\bar{n}$$

$$F(b) = e_2 + \frac{1}{2}n - \frac{1}{2}\bar{n}$$

$$F(c) = -1e_1 + \frac{1}{2}n - \frac{1}{2}\bar{n}$$

$$F(d) = e_3 + \frac{1}{2}n - \frac{1}{2}\bar{n}$$

$$F(x) = x_1e_1 + x_2e_2 + x_3e_3 + \left(\frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2 + \frac{1}{2}(x_3)^2\right)n - \frac{1}{2}\bar{n}$$

a = e1, b = e2, c = -e1, and d = e3 A = F(a) = 1/2*(a*a*n+2*a-nbar), etc. Circle through a, b, and c

$$Circle : A \wedge B \wedge C \wedge X = 0 = -x_3e_1 \wedge e_2 \wedge e_3 \wedge n + x_3e_1 \wedge e_2 \wedge e_3 \wedge \bar{n} + \left(\frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2 + \frac{1}{2}(x_3)^2 - \frac{1}{2}\right)e_1 \wedge e_2 \wedge n \wedge \bar{n}$$

Line through a and b

$$Line : A \wedge B \wedge n \wedge X = 0 = -x_3e_1 \wedge e_2 \wedge e_3 \wedge n + \left(\frac{x_1}{2} + \frac{x_2}{2} - \frac{1}{2}\right)e_1 \wedge e_2 \wedge n \wedge \bar{n} + \frac{x_3}{2}e_1 \wedge e_3 \wedge n \wedge \bar{n} - \frac{x_3}{2}e_2 \wedge e_3 \wedge n \wedge \bar{n}$$

Sphere through a, b, c, and d

$$Sphere : A \wedge B \wedge C \wedge D \wedge X = 0 = \left(-\frac{1}{2}(x_1)^2 - \frac{1}{2}(x_2)^2 - \frac{1}{2}(x_3)^2 + \frac{1}{2}\right)e_1 \wedge e_2 \wedge e_3 \wedge n \wedge \bar{n}$$

Plane through a, b, and d

$$Plane : A \wedge B \wedge n \wedge D \wedge X = 0 = \left(-\frac{x_1}{2} - \frac{x_2}{2} - \frac{x_3}{2} + \frac{1}{2}\right)e_1 \wedge e_2 \wedge e_3 \wedge n \wedge \bar{n}$$

$$\begin{aligned} Hyperbolic\ \ Circle : (A \wedge B \wedge e) \wedge X = 0 = & -x_3e_1 \wedge e_2 \wedge e_3 \wedge n \\ & - x_3e_1 \wedge e_2 \wedge e_3 \wedge \bar{n} \\ & + \left(-\frac{1}{2}(x_1)^2 + x_1 - \frac{1}{2}(x_2)^2 + x_2 - \frac{1}{2}(x_3)^2 - \frac{1}{2}\right)e_1 \wedge e_2 \wedge n \wedge \bar{n} \\ & + x_3e_1 \wedge e_3 \wedge n \wedge \bar{n} \\ & - x_3e_2 \wedge e_3 \wedge n \wedge \bar{n} \end{aligned}$$

```
def properties_of_geometric_objects():
    Print_Function()
    global n, nbar
    g = '# # # 0 0,'+ \
        '# # # 0 0,'+ \
        '# # # 0 0,'+ \
        '0 0 0 0 2,'+ \
        '0 0 0 2 0'
    c3d = Ga('p1 p2 p3 n \\bar{n}',g=g)
    (p1,p2,p3,n,nbar) = c3d.mv()
    print 'g_{ij} =',c3d.g
    P1 = F(p1)
    P2 = F(p2)
    P3 = F(p3)
```

```
    print '\\text{Extracting direction of line from }L = P1\\W P2\\W n'
    L = P1^P2^n
    delta = (L|n)|nbar
    print '(L|n)|\\bar{n} =',delta
    print '\\text{Extracting plane of circle from }C = P1\\W P2\\W P3'
    C = P1^P2^P3
    delta = ((C^n)|n)|nbar
    print '((C^n)|n)|\\bar{n}=',delta
    print '(p2-p1)^(p3-p1)=',(p2-p1)^(p3-p1)
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} (p_1 \cdot p_1) & (p_1 \cdot p_2) & (p_1 \cdot p_3) & 0 & 0 \\ (p_1 \cdot p_2) & (p_2 \cdot p_2) & (p_2 \cdot p_3) & 0 & 0 \\ (p_1 \cdot p_3) & (p_2 \cdot p_3) & (p_3 \cdot p_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Extracting direction of line from $L = P1 \wedge P2 \wedge n$

$(L \cdot n) \cdot \bar{n} = 2p_1 - 2p_2$

Extracting plane of circle from $C = P1 \wedge P2 \wedge P3$

$((C \wedge n) \cdot n) \cdot \bar{n} = 2p_1 \wedge p_2 - 2p_1 \wedge p_3 + 2p_2 \wedge p_3$

$(p2 - p1) \wedge (p3 - p1) = p_1 \wedge p_2 - 1p_1 \wedge p_3 + p_2 \wedge p_3$

```
def extracting_vectors_from_conformal_2_blade():
    Print_Function()
    print r'B = P1\W P2'
    g = '0 -1 #,'+ \
        '-1 0 #,'+ \
        '# # #'
    c2b = Ga('P1 P2 a',g=g)
    (P1,P2,a) = c2b.mv()
    print 'g_{ij} =',c2b.g
    B = P1^P2
    Bsq = B*B
    print '%B^{2} =',Bsq
    ap = a-(a^B)*B
    print "a' = a-(a^B)*B =",ap
    Ap = ap+ap*B
    Am = ap-ap*B
    print "A+ = a'+a'*B =",Ap
    print "A- = a'-a'*B =",Am
    print '%(A+)^{2} =',Ap*Ap
    print '%(A-)^{2} =',Am*Am
    aB = a|B
    print 'a|B =',aB
    return
```

Code Output:

$B = P1 \wedge P2$

$$g_{ij} = \begin{bmatrix} 0 & -1 & (P_1 \cdot a) \\ -1 & 0 & (P_2 \cdot a) \\ (P_1 \cdot a) & (P_2 \cdot a) & (a \cdot a) \end{bmatrix}$$

$B^2 = 1$

$a' = a - (a \wedge B)B = -(P_2 \cdot a)\, P_1 - (P_1 \cdot a)\, P_2$

$$A+ = a' + a'B = -2\left(P_2 \cdot a\right) P_1$$

$$A- = a' - a'B = -2\left(P_1 \cdot a\right) P_2$$

$$(A+)^2 = 0$$

$$(A-)^2 = 0$$

$$a \cdot B = -\left(P_2 \cdot a\right) P_1 + \left(P_1 \cdot a\right) P_2$$

```
def reciprocal_frame_test():
    Print_Function()
    g = '1 # #,'+ \
        '# 1 #,'+ \
        '# # 1'
    ng3d = Ga('e1 e2 e3',g=g)
    (e1,e2,e3) = ng3d.mv()
    print 'g_{ij} =',ng3d.g
    E = e1^e2^e3
    Esq = (E*E).scalar()
    print 'E =',E
    print '%E^{2} =',Esq
    Esq_inv = 1/Esq
    E1 = (e2^e3)*E
    E2 = (-1)*(e1^e3)*E
    E3 = (e1^e2)*E
    print 'E1 = (e2^e3)*E =',E1
    print 'E2 =-(e1^e3)*E =',E2
    print 'E3 = (e1^e2)*E =',E3
    w = (E1|e2)
    w = w.expand()
    print 'E1|e2 =',w
    w = (E1|e3)
    w = w.expand()
    print 'E1|e3 =',w
    w = (E2|e1)
    w = w.expand()
    print 'E2|e1 =',w
    w = (E2|e3)
    w = w.expand()
    print 'E2|e3 =',w
    w = (E3|e1)
    w = w.expand()
    print 'E3|e1 =',w
    w = (E3|e2)
    w = w.expand()
    print 'E3|e2 =',w
    w = (E1|e1)
    w = (w.expand()).scalar()
    Esq = expand(Esq)
    print '%(E1\\cdot e1)/E^{2} =',simplify(w/Esq)
    w = (E2|e2)
    w = (w.expand()).scalar()
    print '%(E2\\cdot e2)/E^{2} =',simplify(w/Esq)
    w = (E3|e3)
    w = (w.expand()).scalar()
    print '%(E3\\cdot e3)/E^{2} =',simplify(w/Esq)
    return
```

Code Output:

$$g_{ij} = \left[\begin{array}{ccc} 1 & (e_1 \cdot e_2) & (e_1 \cdot e_3) \\ (e_1 \cdot e_2) & 1 & (e_2 \cdot e_3) \\ (e_1 \cdot e_3) & (e_2 \cdot e_3) & 1 \end{array}\right]$$

$E = e_1 \wedge e_2 \wedge e_3$

$E^2 = \left(e_1 \cdot e_2\right)^2 - 2\left(e_1 \cdot e_2\right)\left(e_1 \cdot e_3\right)\left(e_2 \cdot e_3\right) + \left(e_1 \cdot e_3\right)^2 + \left(e_2 \cdot e_3\right)^2 - 1$

$E1 = (e2 \wedge e3)E = \left(\left(e_2 \cdot e_3\right)^2 - 1\right) e_1 + \left(\left(e_1 \cdot e_2\right) - \left(e_1 \cdot e_3\right)\left(e_2 \cdot e_3\right)\right) e_2 + \left(-\left(e_1 \cdot e_2\right)\left(e_2 \cdot e_3\right) + \left(e_1 \cdot e_3\right)\right) e_3$

$E2 = -(e1 \wedge e3)E = \left(\left(e_1 \cdot e_2\right) - \left(e_1 \cdot e_3\right)\left(e_2 \cdot e_3\right)\right) e_1 + \left(\left(e_1 \cdot e_3\right)^2 - 1\right) e_2 + \left(-\left(e_1 \cdot e_2\right)\left(e_1 \cdot e_3\right) + \left(e_2 \cdot e_3\right)\right) e_3$

$E3 = (e1 \wedge e2)E = \left(-\left(e_1 \cdot e_2\right)\left(e_2 \cdot e_3\right) + \left(e_1 \cdot e_3\right)\right) e_1 + \left(-\left(e_1 \cdot e_2\right)\left(e_1 \cdot e_3\right) + \left(e_2 \cdot e_3\right)\right) e_2 + \left(\left(e_1 \cdot e_2\right)^2 - 1\right) e_3$

$E1 \cdot e2 = 0$

$E1 \cdot e3 = 0$

$E2 \cdot e1 = 0$

$E2 \cdot e3 = 0$

$E3 \cdot e1 = 0$

$E3 \cdot e2 = 0$

$(E1 \cdot e1)/E^2 = 1$

$(E2 \cdot e2)/E^2 = 1$

$(E3 \cdot e3)/E^2 = 1$