$$(u, v) \to (r, \theta, \phi) = [1, u, v]$$

Unit Sphere Manifold:

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(u) \end{bmatrix}$$

$$a = a^u \boldsymbol{e_u} + a^v \boldsymbol{e_v}$$

$$f = f^u \boldsymbol{e_u} + f^v \boldsymbol{e_v}$$

$$\nabla = \boldsymbol{e_u} \frac{\partial}{\partial u} + \boldsymbol{e_v} \frac{1}{\sin^2(u)} \frac{\partial}{\partial v}$$

$$a \cdot \nabla = a^u \frac{\partial}{\partial u} + a^v \frac{\partial}{\partial v}$$

$$(a \cdot \nabla) \boldsymbol{e}_u = \frac{a^v}{\tan(u)} \boldsymbol{e_v}$$

$$(a \cdot \nabla) \boldsymbol{e}_v = -\frac{a^v}{2} \sin(2u) \boldsymbol{e_u} + \frac{a^u}{\tan(u)} \boldsymbol{e_v}$$

$$(a \cdot \nabla) f = \left(a^u \partial_u f^u - \frac{a^v f^v}{2} \sin(2u) + a^v \partial_v f^u \right) \boldsymbol{e_u} + \left(\frac{a^u f^v}{\tan(u)} + a^u \partial_u f^v + \frac{a^v f^u}{\tan(u)} + a^v \partial_v f^v \right) \boldsymbol{e_v}$$

Tensors on the Unit Sphere

$$V = a_1^u V_u + a_1^v V_v$$

$$T = T_{uu} a_1^u a_2^u + T_{uv} a_1^u a_2^v + T_{vu} a_1^v a_2^u + T_{vv} a_1^v a_2^v$$

Tensor Contraction

$$T[1,2] = (a_1^u)^2 \partial_u^2 T_{uu} + \frac{(a_1^u)^2 \partial_v^2 T_{uu}}{\sin^2(u)} + a_1^u a_1^v \partial_u^2 T_{uv} + a_1^u a_1^v \partial_u^2 T_{vu} + \frac{a_1^u a_1^v \partial_v^2 T_{uv}}{\sin^2(u)} + \frac{a_1^u a_1^v \partial_v^2 T_{vu}}{\sin^2(u)} + (a_1^v)^2 \partial_u^2 T_{vv} + \frac{(a_1^v)^2 \partial_v^2 T_{vv}}{\sin^2(u)}$$

Tensor Evaluation

$$T(a, b) = a^u b^u T_{uu} + a^u b^v T_{uv} + a^v b^u T_{vu} + a^v b^v T_{vv}$$

$$T(a, b + c) = a^u b^u T_{uu} + a^u b^v T_{uv} + a^u c^u T_{uu} + a^u c^v T_{uv} + a^v b^u T_{vu} + a^v b^v T_{vv} + a^v c^u T_{vu} + a^v c^v T_{vv}$$

$$T(a, \alpha b) = \alpha a^u b^u T_{uu} + \alpha a^u b^v T_{uv} + \alpha a^v b^u T_{vu} + \alpha a^v b^v T_{vv}$$

Geometric Derivative With Respect To Slot

$$\nabla_{a_1} T = (a_1^u a_2^u \partial_u T_{uu} + a_1^u a_2^v \partial_u T_{uv} + a_2^u a_1^v \partial_u T_{vu} + a_1^v a_2^v \partial_u T_{vv}) \boldsymbol{e_u} + \frac{1}{\sin^2(u)} (a_1^u a_2^u \partial_v T_{uu} + a_1^u a_2^v \partial_v T_{uv} + a_2^u a_1^v \partial_v T_{vu} + a_1^v a_2^v \partial_v T_{vv}) \boldsymbol{e_v}$$

$$\nabla_{a_2} T = \left(a_1^u a_2^u \partial_u T_{uu} + a_1^u a_2^v \partial_u T_{uv} + a_2^u a_1^v \partial_u T_{vu} + a_1^v a_2^v \partial_u T_{vv}\right) \boldsymbol{e_u} + \frac{1}{\sin^2(u)} \left(a_1^u a_2^u \partial_v T_{uu} + a_1^u a_2^v \partial_v T_{uv} + a_2^u a_1^v \partial_v T_{vu} + a_1^v a_2^v \partial_v T_{vv}\right) \boldsymbol{e_v}$$

Covariant Derivatives

$$\begin{aligned}\mathcal{D}V =& \partial_u V_u a_1^u a_2^u \\ &+ \partial_v V_u a_1^u a_2^v \\ &+ \partial_u V_v a_1^v a_2^u \\ &+ \partial_v V_v a_1^v a_2^v\end{aligned}$$

$$\begin{aligned}\mathcal{D}T =& \partial_u T_{uu} a_1^u a_2^u a_3^u + \partial_v T_{uu} a_1^u a_2^u a_3^v + \partial_u T_{uv} a_1^u a_2^v a_3^u \\ &+ \partial_v T_{uv} a_1^u a_2^v a_3^v + \partial_u T_{vu} a_1^v a_2^u a_3^u + \partial_v T_{vu} a_1^v a_2^u a_3^v \\ &+ \partial_u T_{vv} a_1^v a_2^v a_3^u + \partial_v T_{vv} a_1^v a_2^v a_3^v\end{aligned}$$

$$\mathcal{D}T[1,3](a) = (a^u)^2 a_2^u \partial_u^3 T_{uu} + \frac{(a^u)^2 a_2^u \partial_u \partial_v^2 T_{uu}}{\sin^2(u)} + (a^u)^2 a_2^v \partial_u^2 \partial_v T_{uu} + \frac{(a^u)^2 a_2^v \partial_v^3 T_{uu}}{\sin^2(u)} + a^u a_2^u a^v \partial_u^3 T_{uv} + a^u a_2^u a^v \partial_u^3 T_{vu} + \frac{a^u a_2^u a^v \partial_u \partial_v^2 T_{uv}}{\sin^2(u)} + \frac{a^u a_2^u a^v \partial_u \partial_v^2 T_{vu}}{\sin^2(u)} + a^u a^v a_2^v \partial_u^2 \partial_v T_{uv} + a^u a^v a_2^v \partial_u^2 \partial$$