(This pages uses MathJax to express mathematical expressions requiring internet connection.)

Exercises for Quantum Operators on a Real-Valued Qubit

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Convention: *The default direction of the rotations is counter-clockwise.*

1. The rotation on the unit circle with angle θ is denoted $R(\theta)$. What is the matrix form of $R(\theta)$?

(Hint: Apply each candidate matrix to states $|0\rangle$ and $|1\rangle$ to verify whether the result is the rotated state.)

a) $\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$ **b)** $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ **c)** $\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$ **d)** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ **e)** $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

correct

2. Which one of the following matrices represents the rotation with angle $\frac{\pi}{6}$ on the unit circle?

a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ **b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **c)** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ **d)** $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **e)** $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ correct

3. If $R(\theta) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, what is θ?

a) $\frac{\pi}{4}$ **b)** $\frac{3\pi}{4}$ **c)** $\frac{5\pi}{4}$ **d)** $\frac{7\pi}{4}$ **e)** $\frac{-\pi}{4}$ correct

4. If $R(\theta) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, what is θ?

a) $\frac{\pi}{4}$ **b)** $\frac{3\pi}{4}$ **c)** $\frac{5\pi}{4}$ **d)** $\frac{7\pi}{4}$ **e)** $\frac{-\pi}{4}$ correct

5. If $R(\theta)$ is applied to a qubit initially in state $|0\rangle$ three times, what is the final state?

a) $\begin{pmatrix} \cos(3\theta) \\ \sin(3\theta) \end{pmatrix}$ **b)** $\begin{pmatrix} \cos(3\theta) \\ -\sin(3\theta) \end{pmatrix}$ **c)** $\begin{pmatrix} \sin(3\theta) \\ \cos(3\theta) \end{pmatrix}$ **d)** $\begin{pmatrix} \sin(3\theta) \\ -\cos(3\theta) \end{pmatrix}$ **e)** $\begin{pmatrix} -\sin(3\theta) \\ \cos(3\theta) \end{pmatrix}$ correct

6. If $R(-\theta)$ is applied to a qubit initially in state $|0\rangle$ three times, what is the final state?

a) $\begin{pmatrix} \cos(3\theta) \\ \sin(3\theta) \end{pmatrix}$ **b)** $\begin{pmatrix} \cos(3\theta) \\ -\sin(3\theta) \end{pmatrix}$ **c)** $\begin{pmatrix} \sin(3\theta) \\ \cos(3\theta) \end{pmatrix}$ **d)** $\begin{pmatrix} \sin(3\theta) \\ -\cos(3\theta) \end{pmatrix}$ **e)** $\begin{pmatrix} -\sin(3\theta) \\ \cos(3\theta) \end{pmatrix}$ correct

7. If $R(\theta)$ is applied to a qubit initially in state $|1\rangle$ twice, what is the final state?

a) $\begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$ **b)** $\begin{pmatrix} \cos(2\theta) \\ -\sin(2\theta) \end{pmatrix}$ **c)** $\begin{pmatrix} \sin(2\theta) \\ \cos(2\theta) \end{pmatrix}$ **d)** $\begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$ **e)** $\begin{pmatrix} -\sin(2\theta) \\ \cos(2\theta) \end{pmatrix}$ correct

8. The rotation operator $R(\frac{3\pi}{7})$ is applied to a qubit initially in state $|0\rangle$ n times. If the final state is $|0\rangle$, which one of the followings can be a value of n?

a) 3 **b)** 9 **c)** 10 **d)** 14 **e)** 21 correct

9. We have a qubit in state $|0\rangle$. The rotations $R(\frac{\pi}{3})$ and $R(\frac{\pi}{6})$ are applied m and n times, respectively. If the final state is $-|1\rangle$, what can be the values of (m,n)?

a) $(1,1)$ **b)** $(2,2)$ **c)** $(1,2)$ **d)** $(2,1)$ **e)** $(3,3)$ correct

10. We have a qubit in state $|0\rangle$. The rotations $R(\frac{\pi}{3})$ and $R(-\frac{\pi}{6})$ are applied m and n times, respectively. If the final state is $-|1\rangle$, what can be the values of (m,n)?

a) $(20,11)$ **b)** $(20,9)$ **c)** $(20,7)$ **d)** $(20,5)$ **e)** $(20,3)$ correct

11. The reflection on the unit circle having the line of reflection with angle θ is denoted $Ref(\theta)$. What is the matrix form of $Ref(\theta)$?

(Hint: Apply each candidate matrix to the states $|0\rangle$ and $|1\rangle$ to verify whether the result is the reflected state.)

a) $\begin{pmatrix} \sin(2\theta) & -\cos(2\theta) \\ \cos(2\theta) & \sin(2\theta) \end{pmatrix}$ **b)** $\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$ **c)** $\begin{pmatrix} \sin(2\theta) & \cos(2\theta) \\ -\cos(2\theta) & \sin(2\theta) \end{pmatrix}$
d) $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}$ **e)** $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$ correct

12. If $Ref(\theta) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, what is θ?

(Hint: Apply each candidate matrix to the states $|0\rangle$ and $|1\rangle$ to verify whether the result is the reflected state.)

a) π **b)** $\frac{\pi}{2}$ **c)** $\frac{\pi}{4}$ **d)** $\frac{\pi}{8}$ **e)** 0 correct

13. If $Ref(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, what is θ?

(Hint: Apply each candidate matrix to the states $|0\rangle$ and $|1\rangle$ to verify whether the result is the reflected state.)

a) π **b)** $\frac{\pi}{2}$ **c)** $\frac{\pi}{4}$ **d)** $\frac{\pi}{8}$ **e)** 0 correct

14. If $Ref(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, what is θ?

(Hint: Apply each candidate matrix to the states $|0\rangle$ and $|1\rangle$ to verify whether the result is the reflected state.)

a) $\frac{\pi}{2}$ **b)** $\frac{\pi}{3}$ **c)** $\frac{\pi}{4}$ **d)** $\frac{\pi}{8}$ **e)** 0 correct

15. What is the matrix form of the reflection having the line of reflection $y=-x$?

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **d)** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ correct

16. Which of the followings is identical to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?

(Hint: Test each candidate whether it maps the state $\begin{pmatrix} x \\ y \end{pmatrix}$ to the state $\begin{pmatrix} -x \\ y \end{pmatrix}$.)

a) ZZ **b)** ZX **c)** XZ **d)** XZX **e)** ZXZ correct

17. What is $Ref(\theta) \cdot \begin{pmatrix} \cos\theta' \\ \sin\theta' \end{pmatrix}$?

a) $\begin{pmatrix} \cos(\theta+\theta') \\ \sin(\theta+\theta') \end{pmatrix}$ **b)** $\begin{pmatrix} \cos(\theta-\theta') \\ \sin(\theta-\theta') \end{pmatrix}$ **c)** $\begin{pmatrix} \cos(2\theta-\theta') \\ \sin(2\theta-\theta') \end{pmatrix}$ **d)** $\begin{pmatrix} \cos(-\theta+\theta') \\ \sin(-\theta+\theta') \end{pmatrix}$ **e)** $\begin{pmatrix} \cos(-2\theta+\theta') \\ \sin(-2\theta+\theta') \end{pmatrix}$ correct

18. Let $|u\rangle$ be a quantum state on the unit circle with angle θ'. If we apply the operators $Ref(\theta_1)$ and $Ref(\theta_2)$ in order, what is the angle of the final state?

- **a)** $\theta_1 + \theta_2 - \theta'$
- **b)** $-2\theta_1 + 2\theta_2 + \theta'$
- **c)** $2\theta_1 + 2\theta_2 - \theta'$
- **d)** $-2\theta_1 - 2\theta_2 + \theta'$
- **e)** $2\theta_1 + 2\theta_2 + \theta'$

correct

19. Which one of the following operators maps the state $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ to the state $\begin{pmatrix} \cos(-\theta) \\ \sin(-\theta) \end{pmatrix}$?

(Hint: Determine (i) whether $\sin\theta = \sin(-\theta)$ or not and (ii) whether $\cos\theta = \cos(-\theta)$ or not.)

- **a)** Z
- **b)** H
- **c)** X
- **d)** $-X$
- **e)** $-H$

correct

20. Which one of the following operators is identical to $Ref(\theta)$?

(Hint: Any arbitrary state, say $|u\rangle$, on the unit circle is represented by its angle, say θ'. Find the angle of $Ref(\theta)|u\rangle$ and compare it with the angle of each quantum state obtained by applying the candidate operators.)

- **a)** $R(\theta)X$
- **b)** $R(\theta)R(-\theta)$
- **c)** $R(\theta)R(\theta)Z$
- **d)** $R(\theta)R(\theta)X$
- **e)** $XR(\theta)X$

correct

21. Let $|u_1\rangle = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$ and $|u_2\rangle = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$ be two different quantum states, where $\theta 1, \theta_2 \in (0, \pi)$. If the probabilities of being in states $|0\rangle$ for $|u_1\rangle$ and $|u_2\rangle$ are the same, what is the relation between θ_1 and θ_2?

- ***a)*** $|\theta_1 - \theta_2| = \frac{\pi}{2}$
- ***b)*** $\theta_1 + \theta_2 = \frac{\pi}{2}$
- ***c)*** $\theta_1 + \theta_2 = \pi$
- ***d)*** $|\theta_1 - \theta_2| = \frac{\pi}{4}$
- ***e)*** $\theta_1 + \theta_2 = \frac{3\pi}{2}$

correct

22. Which one of the following pairs of quantum states is perfectly distinguishable?

- **a)** $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
- ***b)*** $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
- ***c)*** $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$
- **d)** $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$
- ***e)*** $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

correct

23. Which one of the following pairs of quantum states is perfectly distinguishable?

- **a)** $\left(\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{2}{7}}|0\rangle - \sqrt{\frac{5}{7}}|1\rangle\right)$
- **b)** $\left(\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$
- **c)** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{2}{7}}|0\rangle - \sqrt{\frac{5}{7}}|1\rangle\right)$
- **d)** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$
- **e)** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, \sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$

correct

24. Which one of the following pairs of quantum states cannot be distinguishable?

- ***a)*** $|0\rangle$ and $|1\rangle$
- ***b)*** $|0\rangle$ and $-|1\rangle$
- ***c)*** $-|1\rangle$ and $|1\rangle$
- ***d)*** $|+\rangle$ and $|-\rangle$
- ***e)*** $-|+\rangle$ and $|-\rangle$

correct

25. Which one of the following pairs of quantum states cannot be distinguishable?

- **a)** $\left(\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{2}{7}}|0\rangle - \sqrt{\frac{5}{7}}|1\rangle\right)$
- **b)** $\left(\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$

○ ***c)*** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{2}{7}}|0\rangle - \sqrt{\frac{5}{7}}|1\rangle\right)$

◉ ***d)*** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, -\sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$

○ ***e)*** $\left(\sqrt{\frac{5}{7}}|0\rangle + \sqrt{\frac{2}{7}}|1\rangle, \sqrt{\frac{5}{7}}|0\rangle - \sqrt{\frac{2}{7}}|1\rangle\right)$

correct

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check all answers ***Score:*** ***25*** *correct answer(s),* ***0*** *incorrect answer(s), and* ***0*** *no answer(s).*