Linear Algebra Notes

krista king
MATH

Linear systems in two unknowns

Throughout Linear Algebra, we'll be really interested in solving systems of linear equations, or linear systems. A **linear system** is a system of equations, defined for a set of unknown variables, where each of the variables is linear (the variables are first degree, or raised to the power of 1).

Later on we'll learn about matrices and how to use them to solve linear systems. So in order to have a foundational understanding of what we're doing when we solve systems, we want to take this lesson to review other basic methods for solving systems.

In an introductory Algebra class, we would have learned three ways to solve systems of linear equations: substitution, elimination, and graphing. Let's review the steps for each of those methods.

Substitution method

1. Get a variable by itself in one of the equations.

2. Take the expression you got for the variable in step 1, and plug it (substitute it using parentheses) into the other equation.

3. Solve the equation in step 2 for the remaining variable.

4. Use the result from step 3 and plug it into the equation from step 1.

Let's look at an example to make things a little more clear.

Example

Find the unique solution to the system of equations.

$$y = x + 3$$

$$2x - 3y = 10$$

Let's solve the system using the substitution method. Since y is already solved for in the first equation, step 1 is completed, and we'll go on to step 2 by substituting $x + 3$ for y in the other equation.

$$2x - 3y = 10$$

$$2x - 3(x + 3) = 10$$

Solve for x. Start by distributing the -3.

$$2x - 3x - 9 = 10$$

Combine like terms.

$$-x - 9 = 10$$

Add 9 to both sides.

$$-x - 9 + 9 = 10 + 9$$

$$-x = 19$$

Multiply both sides by -1.

$$-x(-1) = 19(-1)$$

$$x = -19$$

To find y, we'll plug in -19 for x in the first equation.

$$y = x + 3$$

$$y = -19 + 3$$

$$y = -16$$

The unique solution is $(-19, -16)$.

Elimination method

1. If necessary, rearrange both equations so that the x-terms are first, followed by the y-terms, the equals sign, and the constant term (in that order). If an equation appears to have not constant term, that means that the constant term is 0.

2. Multiply one (or both) equations by a constant that will allow either the x-terms or the y-terms to cancel when the equations are added or subtracted (when their left sides and their right sides are added separately, or when their left sides and their right sides are subtracted separately).

3. Add or subtract the equations.

4. Solve for the remaining variable.

5. Plug the result of step 4 into one of the original equations and solve for the other variable.

Let's look at an example with elimination.

Example

Find the unique solution to the system of equations.

$$y = 3x - 4$$

$$-x + 2y = 12$$

First, we'll rearrange the first equation so that its individual parts are in the correct places for elimination. Subtract $3x$ from both sides.

$$y = 3x - 4$$

$$-3x + y = 3x - 3x - 4$$

$$-3x + y = -4$$

Next, multiply the result above by 2 so that the y-terms will cancel when we subtract the equations.

$$2(-3x + y) = 2(-4)$$

$$-6x + 2y = -8$$

Now we'll subtract the equations.

$$-6x + 2y - (-x + 2y) = -8 - (12)$$

$$-6x + 2y + x - 2y = -8 - 12$$

Combine like terms.

$$-6x + x + 2y - 2y = -20$$

$$-5x + 0 = -20$$

$$-5x = -20$$

Divide both sides by -5.

$$\frac{-5x}{-5} = \frac{-20}{-5}$$

$$x = 4$$

To solve for y, we'll plug in 4 for x in the original first equation.

$$y = 3x - 4$$

$$y = 3(4) - 4$$

$$y = 12 - 4$$

$$y = 8$$

The unique solution is $(4,8)$.

Graphing method

1. Solve for y in each equation.

2. Graph both equations on the same Cartesian coordinate system.

3. Find the point of intersection of the lines (the point where the lines cross).

Let's look at an example using the graphing method.

Example

Graph both equations to find the solution to the system.

$$x + 3y = 12$$

$$2x - y = 5$$

In order to graph these equations, let's put both of them into slope-intercept form. Start with the first equation of the system. We get

$$x + 3y = 12$$

Subtract x from both sides.

$$x - x + 3y = -x + 12$$

$$3y = -x + 12$$

Divide both sides by 3.

$$\frac{3y}{3} = -\frac{x}{3} + \frac{12}{3}$$

$$y = -\frac{1}{3}x + 4$$

Now we take the second equation.

$$2x - y = 5$$

Subtract $2x$ from both sides.

$$2x - 2x - y = -2x + 5$$

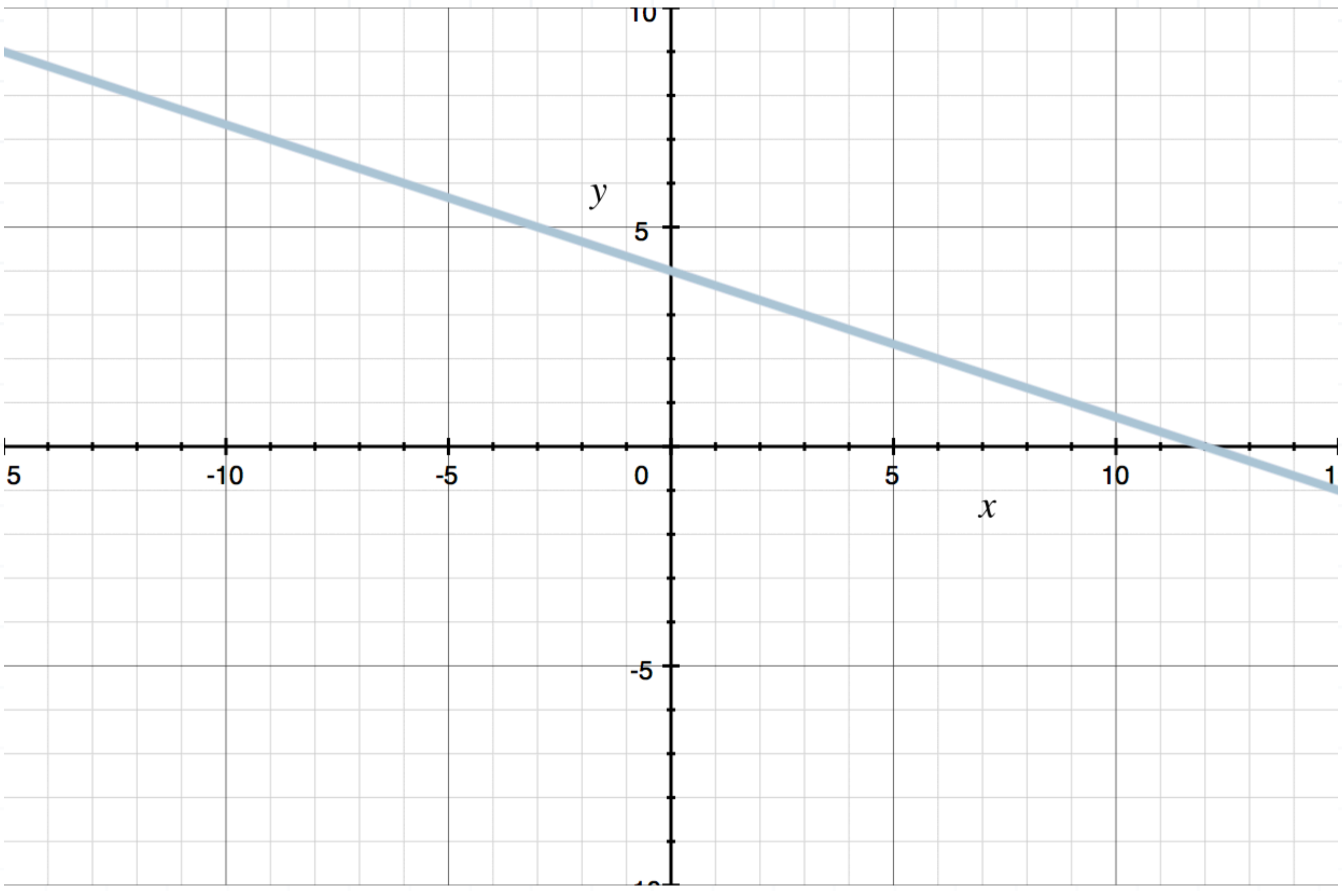
$$-y = -2x + 5$$

Multiply both sides by (-1).

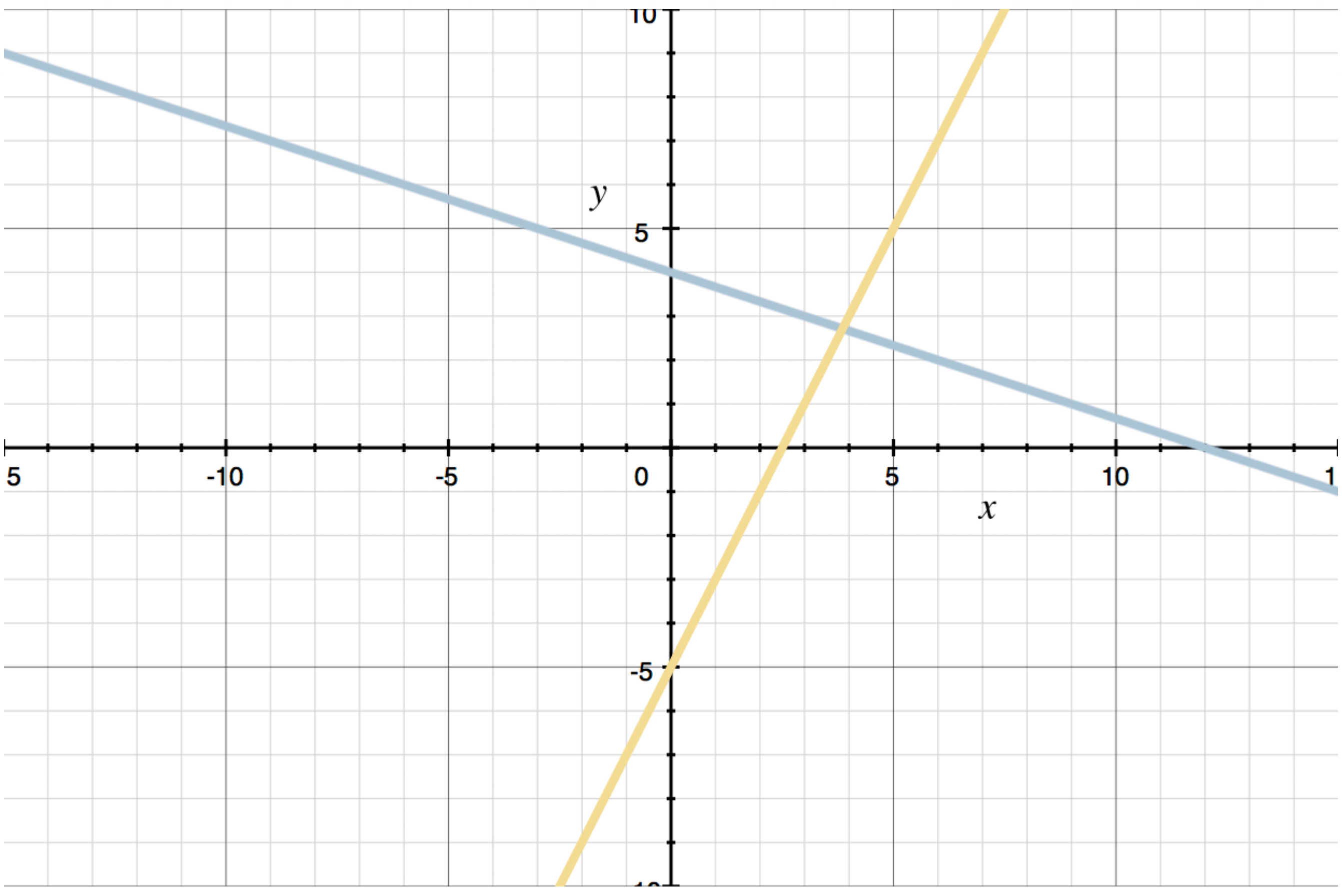
$$(-y)(-1) = (-2x + 5)(-1)$$

$$y = 2x - 5$$

The line $y = -(1/3)x + 4$ intersects the y-axis at 4, and then has a slope of $-1/3$, so its graph is



The line $y = 2x - 5$ intersects the y-axis at -5, and then has a slope of 2, so if you add its graph to the graph of $y = -(1/3)x + 4$, you get



Looking at the intersection point, it appears as though the solution is approximately $(3.75,2.75)$. In actuality, the solution is $(27/7,19/7) \approx (3.86,2.71)$, so our visual estimate of $(3.75,2.75)$ wasn't that far off.

Linear systems in three unknowns

In this lesson we'll look at how to solve systems of three linear equations in three variables.

If a system of three linear equations has solutions, each solution will consist of one value for each variable.

If the three equations in such a linear system are "independent of one another," the system will have either one solution or no solutions. All the systems of three linear equations that you'll encounter in this lesson have at most one solution.

Let's look at an example.

Example

Solve the system of equations.

[1] $-x - 5y + z = 17$

[2] $-5x - 5y + 5z = 5$

[3] $2x + 5y - 3z = -10$

Notice that the coefficients of y in equations [1] and [3] are -5 and 5, respectively. If we add these two equations, the y terms will cancel (we'll eliminate the variable y) and we'll get an equation in only the variables x and z.

$$(-x - 5y + z) + (2x + 5y - 3z) = 17 + (-10)$$

Remove parentheses and combine like terms.

$$-x - 5y + z + 2x + 5y - 3z = 17 - 10$$

$$-x + 2x - 5y + 5y + z - 3z = 17 - 10$$

$$x - 2z = 7$$

You might have also noticed that the coefficients of y in equations [2] and [3] are -5 and 5, respectively, so we can add these two equations to get another equation in only the variables x and z.

$$(-5x - 5y + 5z) + (2x + 5y - 3z) = (5) + (-10)$$

Remove parentheses and combine like terms.

$$-5x - 5y + 5z + 2x + 5y - 3z = 5 - 10$$

$$-5x + 2x - 5y + 5y + 5z - 3z = 5 - 10$$

$$-3x + 2z = -5$$

The coefficients of z in our two new equations are -2 and 2, respectively. If we add these two equations, we can eliminate the variable z, and then solve for x.

$$x - 2z = 7$$

$$-3x + 2z = -5$$

$$(x - 2z) + (-3x + 2z) = 7 + (-5)$$

Remove parentheses and combine like terms.

$$x - 2z - 3x + 2z = 7 - 5$$

$$x - 3x - 2z + 2z = 7 - 5$$

$$-2x = 2$$

$$x = -1$$

Choose one of the new equations, and plug in -1 for x, and then solve for z. We'll choose $x - 2z = 7$.

$$-1 - 2z = 7$$

$$-2z = 8$$

$$z = -4$$

Now choose one of the three original equations, and plug in -1 for x and -4 for z, and then solve for y. We'll choose equation [1].

[1] $-x - 5y + z = 17$

$$-(-1) - 5y + (-4) = 17$$

Simplify and solve for y.

$$1 - 5y - 4 = 17$$

$$-5y + 1 - 4 = 17$$

$$-5y - 3 = 17$$

$$-5y = 20$$

$$y = -4$$

The solution is $(-1, -4, -4)$ or $x = -1$, $y = -4$, and $z = -4$.

Let's do one more.

Example

Use any method to solve the system of equations.

[1] $3a - 3b + 4c = -23$

[2] $a + 2b - 3c = 25$

[3] $4a - b + c = 25$

None of the terms with the same variable have the same coefficient (or coefficients that are equal in absolute value but opposite in sign). So we'll need to multiply one of the equations by some number such that by combining the resulting equation with one of the other two equations, we'll be able to eliminate a variable. Let's multiply equation [2] by 3, so we can eliminate the variable a by subtracting the resulting equation from equation [1].

$$3(a + 2b - 3c) = 3(25)$$

Remove the parentheses.

[4] $3a + 6b - 9c = 75$

Now let's subtract equation [4] from equation [1], which will give us an equation in only the variables b and c.

[1] $3a - 3b + 4c = -23$

[4] $3a + 6b - 9c = 75$

$$(3a - 3b + 4c) - (3a + 6b - 9c) = (-23) - (75)$$

Eliminate the parentheses, and then combine like terms.

$$3a - 3b + 4c - 3a - 6b + 9c = -23 - 75$$

$$3a - 3a - 3b - 6b + 4c + 9c = -23 - 75$$

[5] $-9b + 13c = -98$

We need to get another equation in only the variables b and c. Let's use equations [2] and [3].

This time we need to multiply equation [2] by 4, so we can subtract it from equation [3] and eliminate the variable a.

[2] $a + 2b - 3c = 25$

Remove the parentheses.

$$4(a + 2b - 3c) = 4(25)$$

[6] $4a + 8b - 12c = 100$

Now we'll subtract equation [6] from equation [3].

[3] $4a - b + c = 25$

$$(4a - b + c) - (4a + 8b - 12c) = (25) - (100)$$

Eliminate the parentheses, and then combine like terms.

$$4a - b + c - 4a - 8b + 12c = 25 - 100$$

$$4a - 4a - b - 8b + c + 12c = 25 - 100$$

[7] $-9b + 13c = -75$

With [5] and [7], we now have a system of two equations in the variables b and c.

[5] $-9b + 13c = -98$

[7] $-9b + 13c = -75$

If we subtract [7] from [5], we get

$$(-9b + 13c) - (-9b + 13c) = -98 - (-75)$$

Eliminate the parentheses, and combine like terms.

$$-9b + 13c + 9b - 13c = -98 + 75$$

$$0 = -23$$

Isn't that impossible?

Actually, it isn't impossible, but if something like that happens, it means that the original system of three equations has no solution.

Matrix dimensions and entries

A **matrix** is a rectangular array of values, where each value is an **entry** in both a row and a column.

We know how to solve systems of linear equations using the substitution, elimination, and graphing methods, but what we'll learn throughout this course is that solving linear systems can be made much easier when we use matrices instead of one of these other three methods.

This is especially true as our systems get larger. It's not too bad to use substitution, elimination, or graphing for a system of two equations with two unknowns, but what about a system of 20 equations with 20 unknowns? To tackle a system that large, we need to transition our understanding from these three methods, over to matrix methods.

But first, we need to learn understand the basics about matrices.

Matrix dimensions

A matrix is often described by the number of rows and columns that it has. For instance, a 3×4 matrix is a matrix with 3 rows and 4 columns. In the description "3×4," the number of rows always comes first, and the number of columns always comes second, so remember:

"rows $\times$ columns"

A matrix can be as small as 1×1, with one row and one column, in which case it looks like this:

$$[a]$$

Or it can have infinitely many rows and/or columns. It can have the same number of rows and columns, more rows than columns, or more columns than rows.

A 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A 2×3 matrix: $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

A 3×2 matrix: $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

Matrices are just a way of arranging data, especially large amounts of data. And in this section we'll learn all different ways to work with matrices, like how to add them or multiply them. And these kinds of matrix operations are useful in all kinds of fields, like statistics, economics, data analysis, and computer programming.

Example

Give the dimensions of each matrix.

$$A = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 & 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We always give the dimensions of a matrix as rows $\times$ columns. Matrix A has 3 rows and 3 columns, so A is a 3×3 matrix. Matrix B has 1 row and 5 columns, so B is a 1×5 matrix. Matrix C has 3 rows and 1 column, so C is a 3×1 matrix.

Matrix entries

We call out a particular entry in a matrix using the name of the matrix and the row and column where the entry is sitting. So if the matrix is called K (uppercase letters are used to name matrices), and

$$K = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \end{bmatrix}$$

then if we want the entry in the first row, third column, we write that as $k_{1,3}$, since that's the entry in the first row, third column, of matrix K.

Example

Find $a_{2,3}$, $b_{1,4}$, and $c_{3,1}$.

$$A = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 & 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The value of $a_{2,3}$ is the entry in the second row, third column of matrix A, which is 1, so $a_{2,3} = 1$. The value of $b_{1,4}$ is the entry in the first row, fourth column of matrix B, which is 0, so $b_{1,4} = 0$. The value of $c_{3,1}$ is the entry in the third row, first column of matrix C, so $c_{3,1} = 1$.

Representing systems with matrices

We said that we wanted to start using matrices to solve systems, so we need to learn first how to represent a linear system in a matrix, instead of just as a list of separate equations, like this:

$$3x + 2y = 7$$

$$x - 6y = 0$$

Sometimes for systems like these, you'll use a matrix to represent only the left sides (when all the variable terms are consolidated on the left, and the constant terms are alone on the right sides). A matrix representing the left sides only would be

$$\begin{bmatrix} 3 & 2 \\ 1 & -6 \end{bmatrix}$$

Notice that we've just taken the coefficients on the x and y terms. The first column represents the coefficients on x, 3 and 1, and the second column represents the coefficients on y, 2 and -6. The first row represents the coefficients in the first equation, $3x + 2y = 7$, and the second row represents the coefficients in the second equation, $x - 6y = 0$.

Augmented matrices

But we can also bring the constants from the right side of these equations into the matrix. Whenever we add a column to a matrix that wasn't previously there, we say that we're **augmenting** the matrix, and we call the result an **augmented matrix**. So the augmented matrix for this system could look like this:

$$\begin{bmatrix} 3 & 2 & 7 \\ 1 & -6 & 0 \end{bmatrix}$$

or like this:

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

You can use augmented matrices to represent systems of any size. If we added two more equations to the system, we'd simply add two more rows to the matrix. Or if we added another variable to the system, like z, we'd simply add one more column to the matrix.

Let's do an example with a few more variables.

Example

Represent the system with an augmented matrix called M.

$$-2x + y - t = 7$$

$$x - y + z + 4t = 0$$

You always want to look at all the variables that are included in the system, not just the first equation, since the first equation may not include all the variables.

This particular system includes x, y, z, and t. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. We could set up the matrix like this:

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & | & C_1 \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & | & C_2 \end{bmatrix}$$

Because there's no z-term in the first equation, the value of $m_{1,3}$ will be 0. If we fill in the matrix with that value and all the other coefficients and constants, we get

$$M = \begin{bmatrix} -2 & 1 & 0 & -1 & | & 7 \\ 1 & -1 & 1 & 4 & | & 0 \end{bmatrix}$$

Lining up the variables

Whenever you're building a matrix to represent a system, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation.

That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

Let's do an example where the terms aren't already in order.

Example

Express the system of linear equations as a matrix called B.

$$2x + 3y - z = 11$$

$$7y = 6 - x - 4z$$

$$-8z + 3 = y$$

Before we do anything, we want to put each equation in order, with x, then y, then z on the left side, and the constant on the right side.

$$2x + 3y - z = 11$$

$$x + 7y + 4z = 6$$

$$-y - 8z = -3$$

We could also recognize that there is no x-term in the third equation, but we could add in a 0 "filler" term.

$$2x + 3y - z = 11$$

$$x + 7y + 4z = 6$$

$$0x - y - 8z = -3$$

Pulling all these values into a matrix gives

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & | & C_1 \\ b_{2,1} & b_{2,2} & b_{2,3} & | & C_2 \\ b_{3,1} & b_{3,2} & b_{3,3} & | & C_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

Simple row operations

So now that we know how to transfer a system of equations into a matrix, how do we actually go about "solving" the matrix?

That's where row operations come in. Once we have a linear system represented as a matrix or an augmented matrix, we can use row operations to manipulate and simplify the matrix. Eventually, we'll be able to get the matrix into a form where the solution to the system just reveals itself in the matrix.

Here are the row operations we need to understand in order to be able to simplify matrices:

1. How to switch rows in the matrix

2. How to multiply (or divide) a row by a constant

3. How to add one row to (or subtract one row from) another

Switching two rows

You can switch any two rows in a matrix without changing the value of the matrix. In this matrix, we'll switch rows 1 and 2, which we write as $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & | & 0 \\ 3 & 2 & | & 7 \end{bmatrix}$$

Keep in mind that you can also make multiple row switches. For instance, in this 3×3 matrix, you could first switch the second row with the third row, $R_2 \leftrightarrow R_3$,

$$\begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 7 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$$

and then switch the first row with the second row, $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 7 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 3 \\ 7 & 3 & 4 \\ 1 & 6 & 1 \end{bmatrix}$$

Realize that it's okay to switch rows in a matrix, since a matrix just represents a linear system. It's no different than rewriting the system

$$3x + 2y = 7$$

$$x - 6y = 0$$

as

$$x - 6y = 0$$

$$3x + 2y = 7$$

Switching the order of the equations in a list of equations representing a linear system is all that you're doing when you switch two rows in a matrix.

Example

Write the new matrix after $R_3 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $R_3 \leftrightarrow R_2$ is switching row 2 with row 3. Nothing will happen to row 1. The matrix after $R_3 \leftrightarrow R_2$ is

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 0 & -1 & -8 & | & -3 \\ 1 & 7 & 4 & | & 6 \end{bmatrix}$$

Multiplying a row by a constant

You can multiply any row in a matrix by any non-zero constant without changing the value of the matrix. We often call this value a **scalar** because it "scales" the values in the row. For instance, if we multiply through the first row of this matrix by 2, we don't actually change the value of the matrix.

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 & | & 2 \cdot 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix}$$

How can it be true that multiplying a row by a constant doesn't change the value of the matrix? Aren't the entires in the matrix now different?

Remember that a row in a matrix represents a linear equation. For instance, the matrix

$$\begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix}$$

could represent this linear system:

$$6x + 4y = 14$$

$$x - 6y = 0$$

But given $6x + 4y = 14$, we know we can divide through the equation by 2, and it doesn't change the value of the equation. Dividing through by 2 would just give us $3x + 2y = 7$.

So in the same way, we can divide the 2 back out of the matrix, undoing the operation from before,

$$\begin{bmatrix} 6 & 4 & | & 14 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot 4 & | & \frac{1}{2} \cdot 14 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

and the matrix still has the same value.

Keep in mind that you're not limited to multiplying only one row of a matrix by a non-zero constant. You can multiply as many rows as you like by a constant, and the constants don't even have to be the same.

For example, we can multiply the first row of the matrix by 2 (which we write as $2R_1 \to R_1$), and multiply the second row of the matrix by 3 (which we write as $3R_2 \to R_2$),

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 2\cdot 3 & 2\cdot 2 & | & 2\cdot 7 \\ 3\cdot 1 & 3\cdot -6 & | & 3\cdot 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & | & 14 \\ 3 & -18 & | & 0 \end{bmatrix}$$

and we still won't have changed the value of the matrix, since those constants could be divided right back out again.

Example

Write the new matrix after $3R_1 \leftrightarrow 2R_3$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $3R_1 \leftrightarrow 2R_3$ is multiplying row 1 by a constant of 3, multiplying row 3 by a constant of 2, and then switching those two rows. Nothing will happen to row 2. The matrix after $3R_1$ is

$$\begin{bmatrix} 6 & 9 & -3 & | & 33 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The matrix after $2R_3$ is

$$\begin{bmatrix} 6 & 9 & -3 & | & 33 \\ 1 & 7 & 4 & | & 6 \\ 0 & -2 & -16 & | & -6 \end{bmatrix}$$

The matrix after $3R_1 \leftrightarrow 2R_3$ is

$$\begin{bmatrix} 0 & -2 & -16 & | & -6 \\ 1 & 7 & 4 & | & 6 \\ 6 & 9 & -3 & | & 33 \end{bmatrix}$$

Adding a row to another row

It's also acceptable to add one row to another. Keep in mind though that this doesn't consolidate two rows into one. Instead, we replace a row with the sum of itself and another row. For instance, in this matrix,

$$\begin{bmatrix} 3 & 2 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

we could replace the first row with the sum of the first and second rows, $R_1 + R_2 \to R_1$. When we perform that operation, we're replacing the entries in row 1, but row 2 stays the same.

$$\begin{bmatrix} 3+1 & 2-6 & | & 7+0 \\ 1 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & | & 7 \\ 1 & -6 & | & 0 \end{bmatrix}$$

Of course, you can also replace a row with the difference of itself and another row. But subtracting a row from another is the same as adding the row, multiplied by -1, so because we know we can add rows, it's logical that we can also subtract rows.

Example

Write the new matrix after $R_1 + 4R_3 \to R_1$.

$$\begin{bmatrix} 2 & 3 & -1 & | & 11 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

The operation described by $R_1 + 4R_3 \to R_1$ is multiplying row 3 by a constant of 4, adding that resulting row to row 1, and using that result to replace row 1. The row R_3 is

$$\begin{bmatrix} 0 & -1 & -8 & | & -3 \end{bmatrix}$$

So $4R_3$ would be

$$\begin{bmatrix} 4(0) & 4(-1) & 4(-8) & | & 4(-3) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & -32 & | & -12 \end{bmatrix}$$

Then because $R_1 = \begin{bmatrix} 2 & 3 & -1 & | & 11 \end{bmatrix}$, $R_1 + 4R_3$ is

$$\begin{bmatrix} 2+0 & 3+(-4) & -1+(-32) & | & 11+(-12) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -33 & | & -1 \end{bmatrix}$$

The matrix after $R_1 + 4R_3 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 2 & -1 & -33 & | & -1 \\ 1 & 7 & 4 & | & 6 \\ 0 & -1 & -8 & | & -3 \end{bmatrix}$$

Pivot entries and row-echelon forms

Now that we know how to use row operations to manipulate matrices, we can use them to simplify a matrix in order to solve the system of linear equations the matrix represents.

Our goal will be to use these row operations to change the matrix into either row-echelon form, or reduced row-echelon form.

Let's start by defining pivot entries, since they're part of the definitions of row-echelon and reduced row echelon forms.

Pivot entries

Before we can understand row-echelon and reduced row-echelon forms, we need to be able to identify pivot entries in a matrix.

A **pivot entry**, (or **leading entry**, or **pivot**), is the first non-zero entry in each row. Any column that houses a pivot is called a **pivot column**. So in the matrix

$$\begin{bmatrix} 4 & 1 & 0 & | & 17 \\ 0 & 2 & 5 & | & 10 \\ 0 & 0 & -3 & | & 2 \end{bmatrix}$$

the pivots are 4, 2, and -3. And all three of the columns on the left side are pivot columns, since they each house a pivot entry.

Row-echelon forms

A matrix is in **row-echelon form (ref)** if

1. All the pivot entries are equal to 1.

2. Any row(s) that consist of only 0s are at the bottom of the matrix.

3. The pivot in each row sits in a column to the right of the column that houses the pivot in the row above it. In other words, the pivot entries sit in a staircase pattern, where they stair-step down from the upper left corner to the lower right corner of the matrix.

Row-echelon form might look like this:

$$\begin{bmatrix} 1 & -2 & 0 & | & 6 \\ 0 & 1 & 5 & | & -1 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, the row consisting of only 0s is at the bottom, and the pivots follow a staircase pattern that moves down and to the right, so it's in row-echelon form.

If a matrix is in row-echelon form (the matrix meets the three requirements above for row-echelon form), and if, in each pivot column, the pivot entry is the only non-zero entry, then the matrix is in **reduced row-echelon form (rref)**. Reduced row-echelon form could look like this:

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & 1 & | & -3 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, there are no rows consisting of only 0s, so we don't need to worry about that requirement, the pivots follows a staircase pattern that moves down and to the right, and all three pivot columns include only the pivot entry, and otherwise only 0 entries. The second column includes a non-zero entry, but it's not a pivot column, so that's okay, and this matrix is in reduced row-echelon form.

This is what reduced row-echelon form often looks like for 2×2, 3×3, and 4×4 augmented matrices:

For 2×2:

$$\begin{bmatrix} 1 & 0 & | & a \\ 0 & 1 & | & b \end{bmatrix}$$

For 3×3:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

For 4×4:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & a \\ 0 & 1 & 0 & 0 & | & b \\ 0 & 0 & 1 & 0 & | & c \\ 0 & 0 & 0 & 1 & | & d \end{bmatrix}$$

If you do find a row of zeros in a matrix, either in row-echelon form or reduced row-echelon form, it tells you that the zero row was a combination of some of the other rows. It could be a multiple of another row, the sum or difference of other rows, or some other similar kind of combination.

Sometimes it's fairly simple to put a matrix into row-echelon or reduced row-echelon form.

Example

Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that we can multiply R_2 by $1/5$ (or equivalently, divide R_2 by 5).

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{5}{5} & \frac{0}{5} & -\frac{5}{5} & \frac{0}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we need to put the pivot entries into a staircase pattern. Switch the first and second rows, $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Switch the third and fourth rows, $R_3 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out row is at the bottom, the pivot entries follow a staircase pattern, and all the pivot columns include only the pivot entry, and otherwise all 0 entries. So the matrix is in reduced row-echelon form.

Let's talk for a second about why we would want to put a matrix into rref. Remember that a rref matrix

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

is still representing a system of linear equations. So if we've put the matrix into reduced row-echelon form and then we pull back out the linear equations represented by the matrix, we get

$$1x + 0y + 0z = a$$

$$0x + 1y + 0z = b$$

$$0x + 0y + 1z = c$$

or just

$$x = a$$

$$y = b$$

$$z = c$$

In other words, from reduced row-echelon form, we automatically have the solution to the system! So what we're saying is that, if we put the matrix into its reduced row-echelon form, then we can pull out the value of each variable directly from the matrix. You can almost think about reduced row-echelon form as the simplest, most "cleaned up" version of a matrix.

Gauss-Jordan elimination

Finally, we're at a point where we can start to solve a system using a matrix. To solve a system, our goal will be to use the simple row operations we learned earlier to transform the matrix into row-echelon form, or better yet, reduced row-echelon form.

Gaussian elimination

So we know that it's helpful to put a matrix into reduced row-echelon form, and we've said that we can use matrix row operations to do this, but is there any systematic, orderly way that we go about these row operations?

Yes! **Gauss-Jordan elimination** (or **Gaussian elimination**) is an algorithm (a specific set of steps that can be repeated over and over again) to get the matrix all the way down to reduced row-echelon form. These are the steps:

1. Optional: Pull out any scalars from each row in the matrix.

2. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 3.

3. Multiply through the first row by a scalar to make the leading entry equal to 1.

4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

Let's walk through an example of how to use Gauss-Jordan elimination to change an augmented matrix into reduced row-echelon form and then pull out the values of each variable to get the solution to the system.

Example

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Remember first that this augmented matrix represents the linear system

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

where the entries in the first column are the coefficients on x, the entries in the second column are the coefficients on y, and the entries in the third column are the coefficients on z. The entries in the fourth column are the constants.

Step 1:

Starting with the optional first step from Gauss-Jordan elimination, we could divide through the second row by 5, and that would reduce those values. After $(1/5)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Step 2 (with the first row):

The first entry in the first row is non-zero, so there's no need to swap it with another row.

Step 3 (with the first row):

Multiply row 1 by -1 to get a leading 1 in the first row. After $-R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -(-1) & -(-5) & -(1) & | & -(17) \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ -1 & -1 & 1 & | & 1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Step 4 (with the first row):

Replace row 2 with the sum of rows 1 and 2. After $R_1 + R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 1-1 & 5-1 & -1+1 & | & -17+1 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Replace row 3 with row 3 minus (2 times row 1). After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 2-2(1) & 5-2(5) & -3-2(-1) & | & -10-2(-17) \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 4 & 0 & | & -16 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

We now have $1, 0, 0$ in the first column, which is exactly what we want. It's time to go back to step 2, but this time with the second row.

Step 2 (with the second row):

The second entry in the second row is non-zero, so there's no need to swap it with another row.

Step 3 (with the second row):

Multiply row 2 by $1/4$ to get a leading 1 in the second row. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ \frac{1}{4}(0) & \frac{1}{4}(4) & \frac{1}{4}(0) & | & \frac{1}{4}(-16) \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Step 4 (with the second row):

Replace row 1 with the sum of (-5 times row 2) and row 1. After $-5R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -5(0)+1 & -5(1)+5 & -5(0)-1 & | & -5(-4)-17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Replace row 3 with the sum of (5 times row 2) and row 3. After $5R_2 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 5(0)+0 & 5(1)-5 & 5(0)-1 & | & 5(-4)+24 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & -1 & | & 4 \end{bmatrix}$$

We now have $0, 1, 0$ in the second column, which is exactly what we want. It's time to go back to step 2, but this time with the third row.

Step 2 (with the third row):

The third entry in the third row is non-zero, so there's no need to swap it with another row.

Step 3 (with the third row):

Multiply row 3 by -1 to get a leading 1 in the third row. After $-R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ -(0) & -(0) & -(-1) & | & -(4) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Step 4 (with the third row):

Replace row 1 with the sums of rows 1 and 3. After $R_1 + R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1+0 & 0+0 & -1+1 & | & 3-4 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

We now have 0, 0, 1 in the third column, which is exactly what we want. The matrix is now in reduced row-echelon form since the leading 1 is the first non-zero value in each row, and the leading 1 in each row is to the right of the leading 1 from all the rows above it, and all other values are 0.

From this resulting matrix, we get the solution set

$$1x + 0y + 0z = -1$$

$$0x + 1y + 0z = -4$$

$$0x + 0y + 1z = -4$$

or simplified, we get

$$x = -1$$

$$y = -4$$

$$z = -4$$

That's all it took to find that the solution to the system is $(x, y, z) = (-1, -4, -4)$.

Number of solutions to the linear system

We already know how to solve linear systems using Gaussian elimination to put a matrix into reduced row-echelon form. But up to now, we've only looked at systems with exactly one solution. In fact, systems can have:

- one solution (called the unique solution), or
- no solutions, or
- infinitely many solutions.

The unique solution

We've seen that a unique solution is produced when our reduced row-echelon matrix turns out like this, as an example:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

In this form, we get the unique solution to the system $(x, y, z) = (a, b, c)$. There *is* a solution to the system, and there is only one solution to the system, and it's the point (a, b, c). This is an example with a three-dimensional system, but the same is true for any n-dimensional system.

Infinitely many or no solutions

Sometimes you'll get the matrix in reduced row-echelon form, and you'll end up with something like this:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & c \end{bmatrix}$$

The difference here is that we have a row of zeros at the bottom of the matrix. Here, if c is nonzero, then you'll have a last row that tells you

$$0z = c$$

$$0 = c$$

Remember, we just said c is nonzero. But the equation

$$0 = \text{some non-zero value}$$

can never be true. Because it can never be true, you can conclude that the system has no solutions. In a two-dimensional system, that means you're looking at parallel lines; in a three-dimensional system, that means you're looking at parallel planes.

When there are infinitely many solutions to a system, you might end up with a reduced row-echelon matrix that looks something like this:

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & -3 & | & b \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The zero row at the bottom (zeros across the entire last row) tells you that there will be infinitely many solutions to the system.

Pivot entries and free entries

You already know that the 1's in the first and second row of

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & -3 & | & b \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

are called the **pivot entries** in the matrix. Any other non-zero values on the left side of the matrix, in this case the -3 in the second row, are called **free entries**.

You can almost think about the free entries as independent variables, in the sense that they can be set equal to anything, and you'll still be able to find a solution to the system. The values of the pivot entries are like dependent variables in the sense that their values will depend on the values you set for the free entries.

Example

Say whether the system has one solution, no solutions, or infinitely many solutions.

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & -5 & 1 & | & 17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. To do that, multiply the first row by -1, or $-R_1 \to R_1$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ -5 & -5 & 5 & | & 5 \\ 2 & 5 & -3 & | & -10 \end{bmatrix}$$

Zero out the rest of the first column. First perform $5R_1 + R_2 \to R_2$, then $-2R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 20 & 0 & | & -80 \\ 2 & 5 & -3 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 20 & 0 & | & -80 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Find the pivot entry in the second row with $(1/20)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 5 & -1 & | & -17 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix}$$

Zero out the rest of the second column. First perform $-5R_2 + R_1 \to R_1$, then $5R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & -5 & -1 & | & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & -1 & | & 4 \end{bmatrix}$$

Find the pivot entry in the third row with $-R_3 \rightarrow R_3$.

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -4 \end{bmatrix}$$

Therefore, there is one unique solution to the system, which is $(x, y, z) = (-1, -4, -4)$.

Let's do another example where we get a different result.

Example

Say whether the system has one solution, no solutions, or infinitely many solutions.

$$3a - 3b + 4c = -23$$

$$a + 2b - 3c = 25$$

$$4a - b + c = 25$$

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & -3 & 4 & | & -23 \\ 1 & 2 & -3 & | & 25 \\ 4 & -1 & 1 & | & 25 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with switching the first and second rows.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 3 & -3 & 4 & | & -23 \\ 4 & -1 & 1 & | & 25 \end{bmatrix}$$

Zero out the rest of the first column. First perform $-3R_1 + R_2 \to R_2$, then $-4R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 4 & -1 & 1 & | & 25 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 0 & -9 & 13 & | & -75 \end{bmatrix}$$

Perform $R_3 - R_2 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 25 \\ 0 & -9 & 13 & | & -98 \\ 0 & 0 & 0 & | & 23 \end{bmatrix}$$

We don't have to go any further. The third row now tells us that $0 = 23$, which can't possibly be true. Therefore, the system has no solution.

Matrix addition and subtraction

In the same way that we can add and subtract real numbers, we can also add and subtract matrices. But matrices must have the same dimensions in order for us to be able to add or subtract them.

Dimensions must match

For instance, given a 2×3 matrix, you can only add it to another 2×3 matrix or subtract it from another 2×3 matrix. You couldn't add a 2×3 matrix to a 2×2 matrix, and you couldn't subtract a 3×3 matrix from a 2×4 matrix.

To add matrices, you simply add together entries from corresponding positions in each matrix. For instance, to add 2×2 matrices, you follow this pattern:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Or to add 2×4 matrices, you follow this pattern:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 2+10 & 3+11 & 4+12 \\ 5+13 & 6+14 & 7+15 & 8+16 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 14 & 16 \\ 18 & 20 & 22 & 24 \end{bmatrix}$$

Subtracting matrices works the same way. You simply subtract corresponding entries. For instance, to subtract 2×2 matrices, you follow this pattern:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Matrices can also be part of an equation. For instance, in the same way that $x+3=2$ gets solved as

$$x + 3 = 2$$

$$x = 2 - 3$$

$$x = -1$$

we can also solve an equation that contains matrices, like this:

$$X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$X = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} 6-1 & 8-2 \\ 10-3 & 12-4 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

With the simple linear equation $x + 3 = 2$, we subtracted 3 from both sides to get x by itself, and then simplified $2 - 3$ on the right to find a value for x of $x = -1$. And we really did the same thing with the matrix equation. We subtracted the

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

matrix from both sides to get X by itself, and then simplified the difference of the matrices on the right to find a matrix value for X.

Let's do an example with matrix addition and subtraction.

Example

Solve for B.

$$\begin{bmatrix} 5 & -7 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -1 & -6 \end{bmatrix} = B + \begin{bmatrix} 1 & 0 \\ 17 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 11 & -9 \end{bmatrix}$$

Let's start with the matrix addition on the left side of the equation.

$$\begin{bmatrix} 5+3 & -7+(-4) \\ -1+(-1) & 0+(-6) \end{bmatrix} = B + \begin{bmatrix} 1 & 0 \\ 17 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 11 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -11 \\ -2 & -6 \end{bmatrix} = B + \begin{bmatrix} 1 & 0 \\ 17 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 11 & -9 \end{bmatrix}$$

Subtract matrices on the right.

$$\begin{bmatrix} 8 & -11 \\ -2 & -6 \end{bmatrix} = B + \begin{bmatrix} 1-2 & 0-4 \\ 17-11 & 0-(-9) \end{bmatrix}$$

$$\begin{bmatrix} 8 & -11 \\ -2 & -6 \end{bmatrix} = B + \begin{bmatrix} -1 & -4 \\ 6 & 9 \end{bmatrix}$$

To isolate B, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix} 8 & -11 \\ -2 & -6 \end{bmatrix} - \begin{bmatrix} -1 & -4 \\ 6 & 9 \end{bmatrix} = B$$

$$\begin{bmatrix} 8-(-1) & -11-(-4) \\ -2-6 & -6-9 \end{bmatrix} = B$$

$$\begin{bmatrix} 9 & -7 \\ -8 & -15 \end{bmatrix} = B$$

The conclusion is that the value of B that makes the equation true is this matrix:

$$B = \begin{bmatrix} 9 & -7 \\ -8 & -15 \end{bmatrix}$$

Properties of matrix addition and subtraction

When it comes to addition and subtraction, matrices follow the same rules as real numbers.

Addition

Matrix addition **is commutative and associative**. The fact that it's commutative means that you can add two matrices together in either order, and still get the same answer.

$$A + B = B + A$$

The fact that matrix addition is associative means that you can group the addition in different ways (move the parentheses), and still get the same answer.

$$(A + B) + C = A + (B + C)$$

Subtraction

Matrix subtraction **is not commutative**, and it **is not associative**. The fact that it's not commutative means that you won't get the same result if you subtract matrices in different orders.

$$A - B \neq B - A$$

The fact that matrix subtraction is not associative means that you can't group the subtraction in different ways (move the parentheses) and still get the same answer.

$$(A - B) - C \neq A - (B - C)$$

Scalar multiplication

We've learned about matrix addition and subtraction, and in this lesson we want to start looking at matrix multiplication.

Before we work through matrix multiplication, let's first walk through what it looks like to multiply an entire matrix by a scalar.

Scalar multiplication

We talked earlier about how to multiply a single row by a scalar, but you can also multiply an entire matrix by a scalar.

If we want to multiply a matrix by 3, then 3 is the called the **scalar**, we distribute the scalar across every entry in the matrix, and the result of the scalar multiplication looks like this:

$$3\begin{bmatrix}6 & 2\\ -1 & -4\end{bmatrix}=\begin{bmatrix}3(6) & 3(2)\\ 3(-1) & 3(-4)\end{bmatrix}=\begin{bmatrix}18 & 6\\ -3 & -12\end{bmatrix}$$

Notice that this also translates to dividing through a matrix by a scalar. If you want to divide through a matrix by 6, that's the same as multiplying through the matrix by $1/6$, and we're right back to the same kind of scalar multiplication.

$$\frac{1}{6}\begin{bmatrix}6 & 2\\ -1 & -4\end{bmatrix}=\begin{bmatrix}\frac{1}{6}(6) & \frac{1}{6}(2)\\ \frac{1}{6}(-1) & \frac{1}{6}(-4)\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{3}\\ -\frac{1}{6} & -\frac{2}{3}\end{bmatrix}$$

This translates as well to the kinds of matrix equations we solved in the last lesson.

Example

Solve the equation for K.

$$2\begin{bmatrix}5 & -7\\ -1 & 0\end{bmatrix}+K=-3\begin{bmatrix}2 & 4\\ 11 & -9\end{bmatrix}$$

Apply the scalars to the matrices.

$$\begin{bmatrix}2(5) & 2(-7)\\ 2(-1) & 2(0)\end{bmatrix}+K=\begin{bmatrix}-3(2) & -3(4)\\ -3(11) & -3(-9)\end{bmatrix}$$

$$\begin{bmatrix}10 & -14\\ -2 & 0\end{bmatrix}+K=\begin{bmatrix}-6 & -12\\ -33 & 27\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate K.

$$K=\begin{bmatrix}-6 & -12\\ -33 & 27\end{bmatrix}-\begin{bmatrix}10 & -14\\ -2 & 0\end{bmatrix}$$

$$K=\begin{bmatrix}-6-10 & -12-(-14)\\ -33-(-2) & 27-0\end{bmatrix}$$

$$K=\begin{bmatrix}-16 & 2\\ -31 & 27\end{bmatrix}$$

Multiplying by a zero scalar

What happens when we multiply a matrix by a zero scalar? Well, like any other scalar, we distribute the 0 across every entry in the matrix,

$$(0)\begin{bmatrix}9 & -6 & 2\\ 1 & 0 & -7\end{bmatrix}=\begin{bmatrix}9(0) & -6(0) & 2(0)\\ 1(0) & 0(0) & -7(0)\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and we just end up with a matrix full of zeros. Any matrix that's full of only zeros, regardless of the dimensions of the matrix, we call a "zero matrix."

Zero matrices

A **zero matrix** is a matrix with all zero values, like these:

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \qquad \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

We always name the zero matrix with a capital O. And optionally, you can add a subscript with the dimensions of the zero matrix. Since the values in a zero matrix are all zeros, just having the dimensions of the zero matrix tells you what the entire matrix looks like. As an example, $O_{2\times 3}$ is

$$O_{2\times 3} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Adding and subtracting the zero matrix

Adding the zero matrix to any other matrix doesn't change the matrix's value. And subtracting the zero matrix from any other matrix doesn't change that matrix's value.

Just like with non-zero matrices, matrix dimensions have to be the same in order to be able to add or subtract them.

Adding the zero matrix:

$$\begin{bmatrix}9 & -6 & 2\\1 & 0 & -7\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}9 & -6 & 2\\1 & 0 & -7\end{bmatrix}$$

Subtracting the zero matrix:

$$\begin{bmatrix} 9 & -6 & 2 \\ 1 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & -6 & 2 \\ 1 & 0 & -7 \end{bmatrix}$$

Adding opposite matrices

Adding opposite matrices always results in the zero matrix. Matrices K and $-K$ are **opposite matrices**. So are A and $-A$, and so are X and $-X$. In other words, to get the opposite of a matrix, multiply it by a scalar of -1. So if

$$K = \begin{bmatrix} 9 & -6 & 2 \\ 1 & 0 & -7 \end{bmatrix}$$

then the opposite of K is

$$-K = (-1)\begin{bmatrix} 9 & -6 & 2 \\ 1 & 0 & -7 \end{bmatrix} = \begin{bmatrix} (-1)9 & (-1)(-6) & (-1)2 \\ (-1)1 & (-1)0 & (-1)(-7) \end{bmatrix} = \begin{bmatrix} -9 & 6 & -2 \\ -1 & 0 & 7 \end{bmatrix}$$

If we now add K and $-K$, we'll end up with the zero matrix that has the same dimensions as K and $-K$. Since K and $-K$ are both 2×3, we should get $O_{2\times3}$.

$$K + (-K) = \begin{bmatrix} 9 & -6 & 2 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} -9 & 6 & -2 \\ -1 & 0 & 7 \end{bmatrix}$$

$$K + (-K) = \begin{bmatrix} 9+(-9) & -6+6 & 2+(-2) \\ 1+(-1) & 0+0 & -7+7 \end{bmatrix}$$

$$K + (-K) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{2\times3}$$

Let’s do an example with zero matrices.

Example

Find k and x.

$$\begin{bmatrix} 7 & -1 \\ 4 & 0 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} -7 & 1 \\ k & 0 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 0 & -6 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -9 \\ 0 & 6 \\ -1 & x \end{bmatrix}$$

Adding the zero vector won’t change the value of a matrix, and neither will subtracting out the zero vector, so we can simplify the equation to

$$\begin{bmatrix} 7 & -1 \\ 4 & 0 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} -7 & 1 \\ k & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 0 & -6 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -9 \\ 0 & 6 \\ -1 & x \end{bmatrix}$$

Let’s add the matrices on the left together, and separately add the matrices on the right together.

$$\begin{bmatrix} 7+(-7) & -1+1 \\ 4+k & 0+0 \\ -2+2 & 3+(-3) \end{bmatrix} = \begin{bmatrix} 2+(-2) & 9+(-9) \\ 0+0 & -6+6 \\ 1+(-1) & 1+x \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 4+k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1+x \end{bmatrix}$$

The matrices are equal when entries from corresponding positions in each matrix are equal. So we get

$$4 + k = 0$$

$$k = -4$$

and

$$0 = 1 + x$$

$$x = -1$$

Therefore, we can say that $k = -4$ and $x = -1$ are the values that make the equation true.

Matrix multiplication

We talked before about scalar multiplication, which is when we multiply a matrix by a real-number value. But **matrix multiplication** is what we do when we multiply two matrices together.

Based on what we've already learned about matrix addition and subtraction, you'd think that multiplying matrices is just a matter of multiplying corresponding entries, since matrix addition is just a matter of adding corresponding entries, and matrix subtraction is just a matter of subtracting corresponding entries.

But in fact, we follow an entirely different process to multiply matrices, and we'll walk through exactly what that is in this section.

Dimensions matter

First, when you multiply two matrices A and B together, the order matters. So $A \cdot B$ doesn't have the same result as $B \cdot A$.

This is different than real numbers. We know that $3(4)$ is the same as $4(3)$. That's because real numbers follow the commutative property of multiplication, which means that you can multiply them in any order, and you get the same result in both cases. For matrices, that's not the case; order matters.

The reason the order matters is because of the way we multiply the matrices, which really depends on the dimensions. Here's the thing to remember about dimensions:

The number of columns in the first matrix must be equal to the number of rows in the second matrix.

So for example, you can multiply a 3×2 matrix by any of these:

2×1

2×2

2×3

2×4

...

That's because, when we multiply one matrix by another, we multiply the rows in the first matrix by the columns in the second matrix. Let's say we want to multiply a 2×2 matrix called A by a 2×2 matrix called B.

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

If we call the first and second rows in A rows R_1 and R_2, and call the first and second columns in B columns C_1 and C_2,

$$A = \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix} \end{array}$$

then the product of A and B is

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

Let's look at each entry in the product:

$(AB)_{1,1}$ is the product of the first row and first column

$(AB)_{2,1}$ is the product of the second row and first column

$(AB)_{1,2}$ is the product of the first row and second column

$(AB)_{2,2}$ is the product of the second row and second column

The easy way to tell whether or not you can multiply matrices is to line up their dimensions. For instance, given matrix A is a 2×3 and matrix B is a 3×4, then line up the product AB this way:

$$AB: \ \ 2 \times \boxed{3 \quad 3} \times 4$$

If the middle numbers match like they do here (they're both 3), then you can multiply the matrices to get a valid result, because you have the same number of columns in the first matrix as rows in the second matrix. If you wanted to multiply B by A, you'd line up the product this way:

$$BA: \ 3 \times \boxed{4 \quad 2} \times 3$$

Because those middle numbers aren't equal, you can't multiply B by A (even though A by B was a valid product; that's why order matters!). You don't have the same number of columns in the first matrix as rows in the second matrix, so the product isn't even defined.

Dimensions of the product

Now that you know how to determine whether or not the product of two matrices will be defined, let's talk about the dimensions of the product.

We said before that, because we have the same number of columns in the first matrix as rows in the second matrix, AB will be defined in this case:

$$AB: \ 2 \times \boxed{3 \quad 3} \times 4$$

Once you know that the product AB is defined, you can also quickly know the dimensions of the resulting product. To get those dimensions, just take the number of rows from the first matrix by the number of columns from the second matrix.

$$AB: \ \boxed{2} \times 3 \quad 3 \times \boxed{4}$$

So the dimensions of the product AB will be 2×4. In other words, a 2×3 matrix multiplied by a 3×4 matrix will always result in a 2×4 matrix.

Example

If matrix X is 2×2 and matrix Y is 4×2, say whether XY or YX is defined, and give the dimensions of the product if it is defined.

Line up the dimensions for the products XY and YX.

$$XY: \ 2 \times 2 \quad 4 \times 2$$

$$YX: \ 4 \times 2 \quad 2 \times 2$$

For XY, the middle numbers don't match, so that product isn't defined. For YX, the middle numbers match, so that product is defined.

The dimensions of YX are given by the outside numbers,

$$YX: \ \boxed{4} \times 2 \quad 2 \times \boxed{2}$$

so the dimensions of YX will be 4×2.

Using the dot product to multiply matrices

The **dot product** is the tool we'll use to multiply matrices. When you're calculating a dot product, you want to think about ordered pairs. For instance, we said before that when we take the product AB for

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

the first entry we'll need to find is the dot product of the first row in A and the first column in B. The first row in A is the ordered pair $(2,6)$, and the first column in B is the ordered pair $(-4,1)$.

To take the dot product of these ordered pairs, we take the product of the first values, and then add that result to the product of the second values. In other words, the dot product of $(2,6)$ and $(-4,1)$ is

$$2(-4) + 6(1)$$

$$-8 + 6$$

$$-2$$

So the product of two 2×2 matrices looks like this:

$$\begin{bmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} R_{1,1}C_{1,1} + R_{1,2}C_{2,1} & R_{1,1}C_{1,2} + R_{1,2}C_{2,2} \\ R_{2,1}C_{1,1} + R_{2,2}C_{2,1} & R_{2,1}C_{1,2} + R_{2,2}C_{2,2} \end{bmatrix}$$

Therefore, to find the product of matrices A and B, we get

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-4) + 6(1) & 2(-2) + 6(0) \\ 3(-4) + (-1)(1) & 3(-2) + (-1)(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} -8+6 & -4+0 \\ -12+(-1) & -6+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & -4 \\ -13 & -6 \end{bmatrix}$$

The little dot between the matrices in

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

indicates the dot product. So we could write the dot product of two points x_1 and x_2 as $x_1 \cdot x_2$, or the dot product of two matrices A and B as $A \cdot B$. The little dot tells you "the dot product of these elements."

Because matrix multiplication is always done with the dot product, we don't usually write the dot in between two matrices that we want to multiply, because it's assumed that we're taking the dot product. So you'll usually just see matrix multiplication indicated by two matrices directly next to each other, like this:

$$AB = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

Properties of matrix multiplication

When it comes to multiplication, matrices do not follow the same rules as real numbers.

We've already seen that multiplication **is not commutative**. The fact that it's not commutative means that you can't multiply matrices in a different order and still get the same answer.

$$AB \neq BA$$

Matrix multiplication **is associative**. The fact that it's associative means that you can shift around the parentheses and still get the same answer, as long as you don't change the order of the matrices:

$$(AB)C = A(BC)$$

Matrix multiplication **is distributive**. The fact that it's distributive means that you can distribute multiplication across another value.

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$A(B - C) = AB - AC$$

$$(B - C)A = BA - CA$$

When it comes to the zero matrix, it doesn't matter whether you multiply a matrix by the zero matrix, or multiply the zero matrix by a matrix; you'll get the O matrix either way. But the dimensions of the zero matrix may change, depending on whether it's the first or second matrix in the multiplication.

When $OA = O$, the zero matrix O must have the same number of columns as A has rows.

When $AO = O$, the zero matrix O must have the same number of rows as A has columns.

Let's do an example with these properties of matrix multiplication.

Example

Use matrix multiplication to say whether or not the expression is defined.

$$\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\left(\begin{bmatrix}-4 & -2\\ 1 & 0\\ 8 & -1\end{bmatrix}+\begin{bmatrix}1 & 2 & -1\\ -6 & 0 & 4\\ 2 & -7 & 2\end{bmatrix}\right)$$

Let's first recognize that we have an expression with three matrices that looks like this:

$$A(B+C)$$

We can't add the matrices B and C inside the parentheses, because matrix addition is only defined when the matrix dimensions are the same. B is a 3×2 and C is a 3×3, so we can't do the addition.

We can, however, use the distributive property to distribute matrix A across B and C.

$$\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\begin{bmatrix}-4 & -2\\ 1 & 0\\ 8 & -1\end{bmatrix}+\begin{bmatrix}5 & -1 & 0\\ 4 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ -6 & 0 & 4\\ 2 & -7 & 2\end{bmatrix}$$

Now we have $AB+AC$. The dimensions of the product AB are

$$AB: \boxed{2} \times \boxed{3 \quad 3} \times \boxed{2}$$

The middle numbers match, so the product is defined, and AB will be 2×2.

$$\begin{bmatrix} 5(-4)-1(1)+0(8) & 5(-2)-1(0)+0(-1) \\ 4(-4)-1(1)+2(8) & 4(-2)-1(0)+2(-1) \end{bmatrix} + \begin{bmatrix} 5 & -1 & 0 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -6 & 0 & 4 \\ 2 & -7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -20-1+0 & -10-0+0 \\ -16-1+16 & -8-0-2 \end{bmatrix} + \begin{bmatrix} 5 & -1 & 0 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -6 & 0 & 4 \\ 2 & -7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -10 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} 5 & -1 & 0 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -6 & 0 & 4 \\ 2 & -7 & 2 \end{bmatrix}$$

The dimensions of the product AC are

$$AC: \boxed{2} \times \boxed{3 \quad 3} \times \boxed{3}$$

The middle numbers match, so the product is defined, and AC will be 2×3.

$$\begin{bmatrix} -21 & -10 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} 5(1)-1(-6)+0(2) & 5(2)-1(0)+0(-7) & 5(-1)-1(4)+0(2) \\ 4(1)-1(-6)+2(2) & 4(2)-1(0)+2(-7) & 4(-1)-1(4)+2(2) \end{bmatrix}$$

$$\begin{bmatrix} -21 & -10 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} 5+6+0 & 10-0+0 & -5-4+0 \\ 4+6+4 & 8-0-14 & -4-4+4 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -10 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} 11 & 10 & -9 \\ 14 & -6 & -4 \end{bmatrix}$$

Matrix addition is only defined when the matrices being added have the same dimensions. Here we're trying to add a 2×2 to a 2×3. The dimensions aren't the same, so the sum isn't defined, which means the original expression is not defined, either.

Identity matrices

We already know that multiplying any matrix by a scalar of 1 won't change the matrix.

$$1\begin{bmatrix}6 & 2\\-1 & -4\end{bmatrix}=\begin{bmatrix}1(6) & 1(2)\\1(-1) & 1(-4)\end{bmatrix}=\begin{bmatrix}6 & 2\\-1 & -4\end{bmatrix}$$

But this 1 is just a scalar. Now we want to know if there's an actual *matrix* that we can multiply by another, that, just like the scalar of 1, doesn't change the value of the matrix.

In fact, there *is* a matrix like this, and it's called the **identity matrix**. We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate $I_{2\times 2}$ as just I_2, $I_{3\times 3}$ as just I_3, etc.

We'll talk more later about why the identity matrix is always square. But for now, here's what identity matrices look like:

$$I_2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\qquad I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\qquad I_4=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

When you multiply the identity matrix by another matrix, you don't change the value of the other matrix. Let's see what happens when we multiply the identity matrix by another matrix.

$$I_3A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}7 & 3 & 4\\ 1 & 6 & 1\\ 2 & 2 & 3\end{bmatrix}$$

$$I_3A = \begin{bmatrix}7(1)+1(0)+2(0) & 3(1)+6(0)+2(0) & 4(1)+1(0)+3(0)\\ 7(0)+1(1)+2(0) & 3(0)+6(1)+2(0) & 4(0)+1(1)+3(0)\\ 7(0)+1(0)+2(1) & 3(0)+6(0)+2(1) & 4(0)+1(0)+3(1)\end{bmatrix}$$

$$I_3A = \begin{bmatrix}7+0+0 & 3+0+0 & 4+0+0\\ 0+1+0 & 0+6+0 & 0+1+0\\ 0+0+2 & 0+0+2 & 0+0+3\end{bmatrix}$$

$$I_3A = \begin{bmatrix}7 & 3 & 4\\ 1 & 6 & 1\\ 2 & 2 & 3\end{bmatrix}$$

Notice how multiplying by the identity matrix I_3 didn't change the value of the second matrix.

Dimensions of the identity matrix

Let's prove to ourselves that the identity matrix will always be square. We'll start with some other matrix, like this 3×2:

$$A = \begin{bmatrix}4 & -6\\ 1 & 1\\ -2 & 9\end{bmatrix}$$

Because we know that the identity matrix won't change the value of A, we can set up this equation:

$$I \cdot \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

If we think about the dimensions of A in the context of this equation, we'll see why the identity matrix must be a square. The dimensions of A are 3×2, so let's substitute those into the equation to get a visual picture of the dimensions.

$$I \cdot 3 \times 2 = 3 \times 2$$

Then let's break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

First, we know that in order to be able to multiply matrices at all, we need the same number of columns in the first matrix as we have rows in the second matrix. So we know the identity matrix must have 3 columns.

$$R \times 3 \cdot 3 \times 2 = 3 \times 2$$

We also know that the dimensions of the result matrix on the right, come from the rows of the first matrix and the columns of the second matrix.

$$R \times 3 \cdot 3 \times 2 = 3 \times 2$$

So we know the identity matrix must have 3 rows.

$$3 \times 3 \cdot 3 \times 2 = 3 \times 2$$

Therefore, the identity matrix in this case turns out to be a square 3×3 matrix. And this works for a matrix with any dimensions. Here are some examples:

For a 2×4 matrix, the identity matrix has to be I_2:

$$I \cdot 2 \times 4 = 2 \times 4$$

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

$$R \times 2 \cdot 2 \times 4 = 2 \times 4$$

$$2 \times 2 \cdot 2 \times 4 = 2 \times 4$$

For a 3×1 matrix, the identity matrix has to be I_3:

$$I \cdot 3 \times 1 = 3 \times 1$$

$$R \times C \cdot 3 \times 1 = 3 \times 1$$

$$R \times 3 \cdot 3 \times 1 = 3 \times 1$$

$$3 \times 3 \cdot 3 \times 1 = 3 \times 1$$

Let's do an example problem.

Example

Choose the correct identity matrix for IA, and then find the product of the matrices.

$$A = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

The matrix A is a 2×5, so we'll set up a dimensions equation for IA.

$$I \cdot 2 \times 5 = 2 \times 5$$

$$R \times C \cdot 2 \times 5 = 2 \times 5$$

The number of columns in I must be equal to the number of rows in A.

$$R \times 2 \cdot 2 \times 5 = 2 \times 5$$

The identity matrix must be square, so

$$2 \times 2 \cdot 2 \times 5 = 2 \times 5$$

So we need to multiply I_2 by A. The product of I_2 and matrix A should give us back just the matrix A.

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

$$I_2A = \begin{bmatrix} 1(4)+0(1) & 1(-6)+0(1) & 1(1)+0(-2) & 1(-8)+0(9) & 1(5)+0(0) \\ 0(4)+1(1) & 0(-6)+1(1) & 0(1)+1(-2) & 0(-8)+1(9) & 0(5)+1(0) \end{bmatrix}$$

$$I_2A = \begin{bmatrix} 4+0 & -6+0 & 1+0 & -8+0 & 5+0 \\ 0+1 & 0+1 & 0-2 & 0+9 & 0+0 \end{bmatrix}$$

$$I_2A = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_2.

Properties of the identity matrix

When it comes to the identity matrix, it doesn't matter whether you multiply a matrix by the identity matrix, or multiply the identity matrix by a matrix; you'll get the original matrix either way. But the dimensions of the identity matrix may change, depending on whether it's the first or second matrix in the product.

$IA = A$, but I must have the same number of columns as A has rows

$AI = A$, but I must have the same number of rows as A has columns

The elimination matrix

We've seen how a collection of row operations can change a matrix into reduced row-echelon form, and how, if rref gives us the identity matrix, then we have one unique solution to the linear system represented by the matrix.

What we want to understand now is that each of those individual row operations can be represented by a matrix. Furthermore, the entire collection of all the row operations we perform can be brought together in one matrix, and we call that matrix the **elimination matrix**, E.

One row operation as a matrix

Let's use a matrix from the Gauss-Jordan elimination lesson, and call it A.

$$A = \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

To put this matrix into rref, our first step would be $-R_1 \to R_1$. But here's how we could accomplish this same row operation using a matrix that we multiply by A:

$$E_1A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

Here's what E_1 is telling us:

The second row of E_1, $[0 \quad 1 \quad 0]$, tells us "To get the new second row of A, give me 0 of the first row, plus 1 of the second row, plus 0 of the third row." And that makes sense, since we're not changing the second row of A.

The third row of E_1, $[0 \quad 0 \quad 1]$, tells us "To get the new third row of A, give me 0 of the first row, plus 0 of the second row, plus 1 of the third row." And that makes sense, since we're not changing the third row of A.

But the first row of E_1, $[-1 \quad 0 \quad 0]$, tells us "To get the new first row of A, give me the first row multiplied by a scalar of -1, plus 0 of the second row, plus 0 of the third row."

So E_1 accomplishes the row operation $-R_1 \to R_1$, without changing anything else about the rest of the matrix. And now the resulting matrix after this row operation is the matrix product,

$$E_1A = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}\right)$$

Once $-R_1 \to R_1$ is done, the next step toward reduced row-echelon form would be $5R_1 + R_2 \to R_2$. The only row we want to change is R_2, which means R_1 and R_3 will stay the same. To change the second row, we want a combination (the sum) of 5 of the first row, along with 1 of the second row, but 0 of the third row. So

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}\right)$$

The elimination matrix

Now we've applied the first two row operations to A. And we could simplify these two row operations into one elimination matrix, simply by multiplying E_2 by E_1.

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 5(-1)+1(0)+0(0) & 5(0)+1(1)+0(0) & 5(0)+1(0)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ -5+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} -1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The point here is that we could continue applying more and more row operations to matrix A, and all of them can be combined into one **elimination matrix** called E.

To get A into reduced row-echelon form, we'll actually need the following set of row operations:

1. $-R_1 \to R_1$
4. $R_3 - 2R_1 \to R_3$
7. $5R_2 + R_3 \to R_3$

2. $(1/5)R_2 \to R_2$

3. $R_1 + R_2 \to R_2$

5. $(1/4)R_2 \to R_2$

6. $-5R_2 + R_1 \to R_1$

8. $-R_3 \to R_3$

9. $R_1 + R_3 \to R_1$

There's a simple way to translate these kinds of row operations into the elimination matrix. If the operation is the multiplication of a row by a scalar, like these,

$$-R_1 \to R_1 \qquad (1/4)R_2 \to R_2$$

$$(1/5)R_2 \to R_2 \qquad -R_3 \to R_3$$

then we can put the scalar into the position given by the two subscripts that we see in the operation. For instance, for the row operation $-R_1 \to R_1$, we'd put the scalar -1 into $E_{1,1}$. Or for the row operation $(1/5)R_2 \to R_2$, we'd put the scalar $1/5$ into $E_{2,2}$.

For all other row operations, where we replace a row with the sum or difference of rows, like these,

$$R_1 + R_2 \to R_2 \qquad -5R_2 + R_1 \to R_1 \qquad R_1 + R_3 \to R_1$$

$$R_3 - 2R_1 \to R_3 \qquad 5R_2 + R_3 \to R_3$$

then we put the coefficients from the left side into their corresponding columns, inside the row from the right side. For instance, for the row operation $R_1 + R_2 \to R_2$, we'd put the coefficients 1 and 1 into the first and second columns inside R_2. So we'd put a 1 in $E_{2,1}$ and a 1 in $E_{2,2}$. Or for the row operation $R_3 - 2R_1 \to R_3$, we'd put the coefficients 1 and -2 into the third and first columns inside R_3. So we'd put a 1 in $E_{3,3}$ and a -2 in $E_{3,1}$.

Let's restart the example we've been working with so that we can get more practice with this.

Example

Find the single elimination matrix E that puts A into reduced row-echelon form,

$$A = \begin{bmatrix} -1 & -5 & 1 \\ -5 & -5 & 5 \\ 2 & 5 & -3 \end{bmatrix}$$

where E accounts for the following set of row operations:

1. $-R_1 \to R_1$
2. $(1/5)R_2 \to R_2$
3. $R_1 + R_2 \to R_2$
4. $R_3 - 2R_1 \to R_3$
5. $(1/4)R_2 \to R_2$
6. $-5R_2 + R_1 \to R_1$
7. $5R_2 + R_3 \to R_3$
8. $-R_3 \to R_3$
9. $R_1 + R_3 \to R_1$

The row operation $-R_1 \to R_1$ means we're leaving the second and third rows alone, but multiplying the first row by a scalar of -1, so we'll put a -1 in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $(1/5)R_2 \to R_2$ means we're leaving the first and third rows of the result alone, but multiplying the second row of the result by a scalar of $1/5$, so we'll put a $1/5$ in $E_{2,2}$.

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $R_1 + R_2 \to R_2$ means we're leaving the first and third rows of the result alone, but replacing the second row of the result with 1 of the first row plus 1 of the second row, so we'll put a 1 in $E_{2,1}$ and a 1 in $E_{2,2}$.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have so far for $E_3E_2E_1$.

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(-1)+\frac{1}{5}(0)+0(0) & 0(0)+\frac{1}{5}(1)+0(0) & 0(0)+\frac{1}{5}(0)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+\frac{1}{5}+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{1-3}=\begin{bmatrix}1(-1)+0(0)+0(0) & 1(0)+0(1/5)+0(0) & 1(0)+0(0)+0(1)\\ 1(-1)+1(0)+0(0) & 1(0)+1(1/5)+0(0) & 1(0)+1(0)+0(1)\\ 0(-1)+0(0)+1(0) & 0(0)+0(1/5)+1(0) & 0(0)+0(0)+1(1)\end{bmatrix}$$

$$E_{1-3}=\begin{bmatrix}-1+0+0 & 0+0+0 & 0+0+0\\ -1+0+0 & 0+1/5+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+1\end{bmatrix}$$

$$E_{1-3}=\begin{bmatrix}-1 & 0 & 0\\ -1 & 1/5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Let's do the next set of row operations. The row operation $R_3-2R_1\to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with 1 of the third row and -2 of the first row, so we'll put a 1 in $E_{3,3}$ and a -2 in $E_{3,1}$.

$$E_4E_{1-3}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ -1 & 1/5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The row operation $(1/4)R_2\to R_2$ means we're leaving the first and third rows of the result alone, but replacing the second row of the result with $1/4$ of the second row, so we'll put a $1/4$ in $E_{2,2}$.

$$E_5E_4E_{1-3}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ -1 & 1/5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The row operation $-5R_2+R_1\to R_1$ means we're leaving the second and third rows of the result alone, but replacing the first row of the result with

1 of the first row and -5 of the second row, so we'll put a -5 in $E_{1,2}$ and a 1 in $E_{1,1}$.

$$E_6E_5E_4E_{1-3}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\-1&1/5&0\\0&0&1\end{bmatrix}$$

Let's consolidate what we have so far for $E_6E_5E_4E_{1-3}$.

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1(-1)+0(-1)+0(0)&1(0)+0(1/5)+0(0)&1(0)+0(0)+0(1)\\0(-1)+1(-1)+0(0)&0(0)+1(1/5)+0(0)&0(0)+1(0)+0(1)\\-2(-1)+0(-1)+1(0)&-2(0)+0(1/5)+1(0)&-2(0)+0(0)+1(1)\end{bmatrix}$$

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/4&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1+0+0&0+0+0&0+0+0\\0-1+0&0+1/5+0&0+0+0\\2+0+0&0+0+0&0+0+1\end{bmatrix}$$

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/4&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\-1&1/5&0\\2&0&1\end{bmatrix}$$

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1(-1)+0(-1)+0(2)&1(0)+0(1/5)+0(0)&1(0)+0(0)+0(1)\\0(-1)+\frac{1}{4}(-1)+0(2)&0(0)+\frac{1}{4}(1/5)+0(0)&0(0)+\frac{1}{4}(0)+0(1)\\0(-1)+0(-1)+1(2)&0(0)+0(1/5)+1(0)&0(0)+0(0)+1(1)\end{bmatrix}$$

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1+0+0&0+0+0&0+0+0\\0-1/4+0&0+1/20+0&0+0+0\\0+0+2&0+0+0&0+0+1\end{bmatrix}$$

$$E_{1-6}=\begin{bmatrix}1&-5&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\-1/4&1/20&0\\2&0&1\end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1(-1)-5(-1/4)+0(2) & 1(0)-5(1/20)+0(0) & 1(0)-5(0)+0(1) \\ 0(-1)+1(-1/4)+0(2) & 0(0)+1(1/20)+0(0) & 0(0)+1(0)+0(1) \\ 0(-1)+0(-1/4)+1(2) & 0(0)+0(1/20)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} -1+5/4+0 & 0-1/4+0 & 0-0+0 \\ 0-1/4+0 & 0+1/20+0 & 0+0+0 \\ 0+0+2 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$E_{1-6} = \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Let's do the last set of row operations. The row operation $5R_2 + R_3 \to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with 5 of the second row and 1 of the third row, so we'll put a 5 in $E_{3,2}$ and a 1 in $E_{3,3}$.

$$E_7E_{1-6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $-R_3 \to R_3$ means we're leaving the first and second rows of the result alone, but replacing the third row of the result with -1 of the third row, so we'll put a -1 in $E_{3,3}$.

$$E_8E_7E_{1-6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 1/20 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $R_1 + R_3 \to R_1$ means we're leaving the second and third rows of the result alone, but replacing the first row of the result with 1 of the first row and 1 of the third row, so we'll put a 1 in $E_{1,1}$ and a 1 in $E_{1,3}$.

$$E_9E_8E_7E_{1-6} = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\2&0&1\end{bmatrix}$$

Let's consolidate the rest of the elimination matrices.

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1(1/4)+0(-1/4)+0(2) & 1(-1/4)+0(1/20)+0(0) & 1(0)+0(0)+0(1)\\0(1/4)+1(-1/4)+0(2) & 0(-1/4)+1(1/20)+0(0) & 0(0)+1(0)+0(1)\\0(1/4)+5(-1/4)+1(2) & 0(-1/4)+5(1/20)+1(0) & 0(0)+5(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1/4+0+0 & -1/4+0+0 & 0+0+0\\0-1/4+0 & 0+1/20+0 & 0+0+0\\0-5/4+2 & 0+1/4+0 & 0+0+1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\3/4&1/4&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1(1/4)+0(-1/4)+0(3/4) & 1(-1/4)+0(1/20)+0(1/4) & 1(0)+0(0)+0(1)\\0(1/4)+1(-1/4)+0(3/4) & 0(-1/4)+1(1/20)+0(1/4) & 0(0)+1(0)+0(1)\\0(1/4)+0(-1/4)-1(3/4) & 0(-1/4)+0(1/20)-1(1/4) & 0(0)+0(0)-1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/4+0+0 & -1/4+0+0 & 0+0+0\\0-1/4+0 & 0+1/20+0 & 0+0+0\\0+0-3/4 & 0+0-1/4 & 0+0-1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/4&-1/4&0\\-1/4&1/20&0\\-3/4&-1/4&-1\end{bmatrix}$$

$$E=\begin{bmatrix}1(1/4)+0(-1/4)+1(-3/4) & 1(-1/4)+0(1/20)+1(-1/4) & 1(0)+0(0)+1(-1)\\ 0(1/4)+1(-1/4)+0(-3/4) & 0(-1/4)+1(1/20)+0(-1/4) & 0(0)+1(0)+0(-1)\\ 0(1/4)+0(-1/4)+1(-3/4) & 0(-1/4)+0(1/20)+1(-1/4) & 0(0)+0(0)+1(-1)\end{bmatrix}$$

$$E=\begin{bmatrix}1/4+0-3/4 & -1/4+0-1/4 & 0+0-1\\ 0-1/4+0 & 0+1/20+0 & 0+0+0\\ 0+0-3/4 & 0+0-1/4 & 0+0-1\end{bmatrix}$$

$$E=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -1\\ -\frac{1}{4} & \frac{1}{20} & 0\\ -\frac{3}{4} & -\frac{1}{4} & -1\end{bmatrix}$$

We've found the elimination matrix, and we can check to make sure that it reduces A to the identity matrix.

$$EA=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -1\\ -\frac{1}{4} & \frac{1}{20} & 0\\ -\frac{3}{4} & -\frac{1}{4} & -1\end{bmatrix}\begin{bmatrix}-1 & -5 & 1\\ -5 & -5 & 5\\ 2 & 5 & -3\end{bmatrix}$$

$$EA=\begin{bmatrix}-\frac{1}{2}(-1)-\frac{1}{2}(-5)-1(2) & -\frac{1}{2}(-5)-\frac{1}{2}(-5)-1(5) & -\frac{1}{2}(1)-\frac{1}{2}(5)-1(-3)\\ -\frac{1}{4}(-1)+\frac{1}{20}(-5)+0(2) & -\frac{1}{4}(-5)+\frac{1}{20}(-5)+0(5) & -\frac{1}{4}(1)+\frac{1}{20}(5)+0(-3)\\ -\frac{3}{4}(-1)-\frac{1}{4}(-5)-1(2) & -\frac{3}{4}(-5)-\frac{1}{4}(-5)-1(5) & -\frac{3}{4}(1)-\frac{1}{4}(5)-1(-3)\end{bmatrix}$$

$$EA=\begin{bmatrix}\frac{1}{2}+\frac{5}{2}-2 & \frac{5}{2}+\frac{5}{2}-5 & -\frac{1}{2}-\frac{5}{2}+3\\ \frac{1}{4}-\frac{1}{4}+0 & \frac{5}{4}-\frac{1}{4}+0 & -\frac{1}{4}+\frac{1}{4}+0\\ \frac{3}{4}+\frac{5}{4}-2 & \frac{15}{4}+\frac{5}{4}-5 & -\frac{3}{4}-\frac{5}{4}+3\end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because multiplying the elimination matrix by A gives us the identity matrix, we know that we got the correct elimination matrix.

We won't talk about this until later in the course, but this elimination matrix we've found is called the inverse of A, which we write as A^{-1}. So

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{20} & 0 \\ -\frac{3}{4} & -\frac{1}{4} & -1 \end{bmatrix}$$

As you may suspect, when you multiply an inverse of a matrix by the matrix itself, the result will always be the identity matrix.

$$A^{-1}A = I$$

Vectors

Now that we have a basic understanding of simple matrices and some of their properties, we need to take a little step to the side and introduce vectors. Vectors and matrices are closely related concepts, and throughout the course we'll be using them together to solve problems.

A **vector** has two pieces of information contained within it:

1. the direction in which the vector points, and
2. the **magnitude** of the vector, which is just the length of the vector.

You can express a vector in several ways. Sometimes it's expressed kind of like a coordinate point, $\vec{a} = (3,4)$, or $\vec{b} = (3,4,5)$.

Row and column vectors

But these same vectors $\vec{a} = (3,4)$ and $\vec{b} = (3,4,5)$ can also be expressed as **column matrices** (also called **column vectors**),

$$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

or as a **row matrices** (also called **row vectors**),

$$\vec{a} = \begin{bmatrix} 3 & 4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$$

This is part of the reason why we say that matrices and vectors are closely related. A **column matrix** is a matrix with any number of rows, but exactly one column; a **row matrix** is a matrix with exactly one row, but any number of columns. And it's really common to write vectors as these kinds of matrices.

In fact, for any matrix, each column in the matrix is technically a **column vector**. For example, in matrix A,

$$A = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix}$$

there are five column vectors:

$$a_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, a_4 = \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \text{ and } a_5 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Because each of these column vectors has two components, it means they're vectors in two-dimensional space, $\mathbb{R}^2$. In matrix B,

$$B = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

there are three column vectors,

$$b_1 = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \text{ and } b_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Because each of these column vectors has three components, it means they're vectors in three-dimensional space, $\mathbb{R}^3$.

Similarly, we could also say that A has two **row vectors**,

$$a_1 = [4 \quad -6 \quad 1 \quad -8 \quad 5] \text{ and } a_2 = [1 \quad 1 \quad -2 \quad 9 \quad 0]$$

or that B has three row vectors,

$$b_1 = [7 \quad 3 \quad 4] \text{ and } b_2 = [1 \quad 6 \quad 1] \text{ and } b_3 = [2 \quad 2 \quad 3]$$

When we look at a set of row vectors or column vectors, it's important to understand the space that the vectors occupy. There are two aspects we want to consider: first, the space $\mathbb{R}^n$ in which the vectors lie, and second, the dimension of the "surface" or "space" formed by the vectors specifically.

For instance, given the two vectors $a_1 = [4 \quad -6 \quad 1 \quad -8 \quad 5]$ and $a_2 = [1 \quad 1 \quad -2 \quad 9 \quad 0]$,

- because they each have 5 components, they're vectors in $\mathbb{R}^5$, and
- because there are 2 vectors, they form a two-dimensional plane in $\mathbb{R}^5$.

Or given the three vectors $b_1 = [7 \quad 3 \quad 4]$ and $b_2 = [1 \quad 6 \quad 1]$ and $b_3 = [2 \quad 2 \quad 3]$,

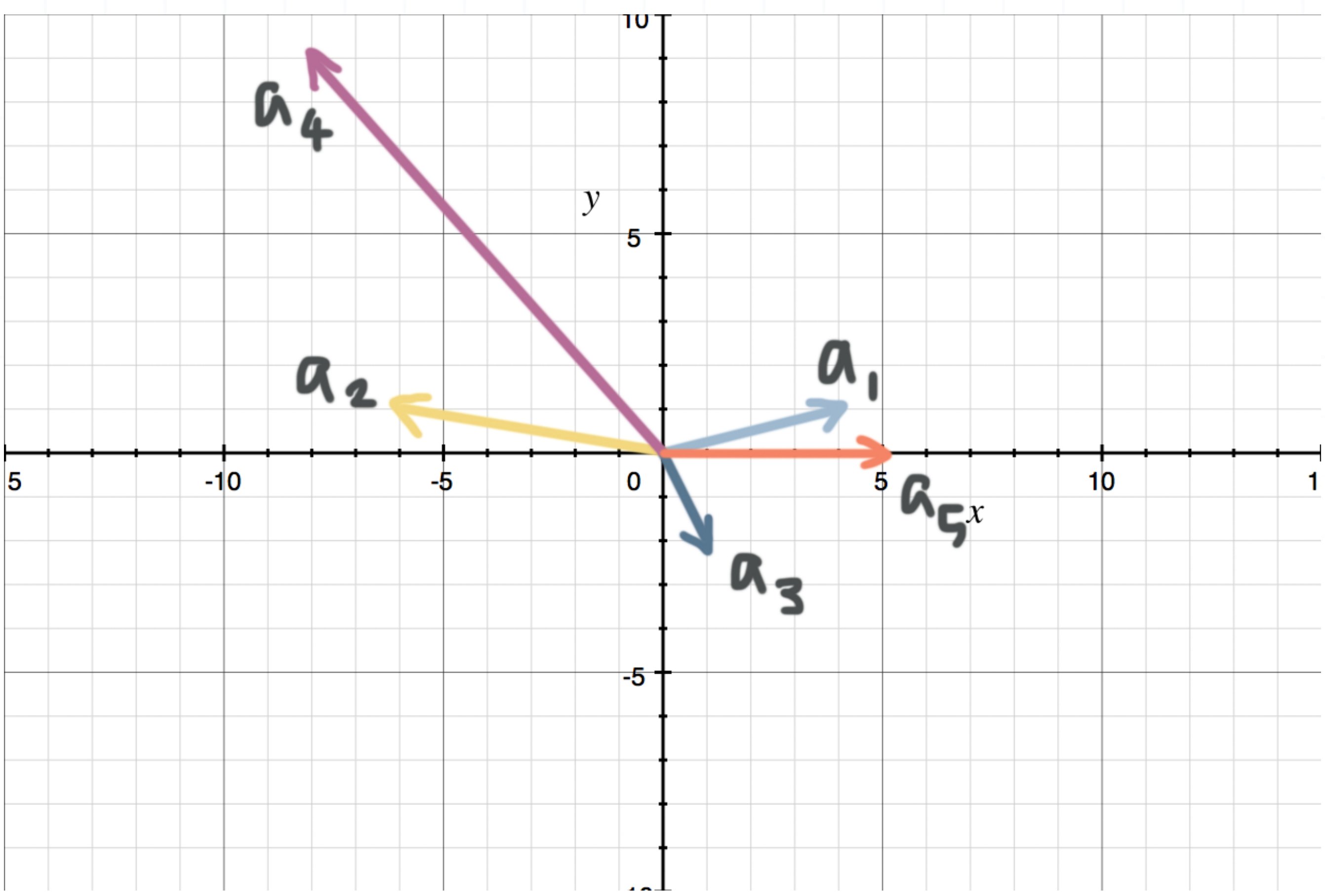
- because they each have 3 components, they're vectors in $\mathbb{R}^3$, and
- because there are 3 vectors, they form a three-dimensional space in $\mathbb{R}^3$ (in other words, they span all of $\mathbb{R}^3$ space).

Vectors can be moved

To sketch a vector, we often start at the origin and move out to the "coordinate point" that's expressed by the vector. Placing the starting point of the vector at the origin means that you're sketching the vector in **standard position**. For instance, the two-dimensional vectors

$$a_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ a_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, \ a_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ a_4 = \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \text{ and } a_5 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

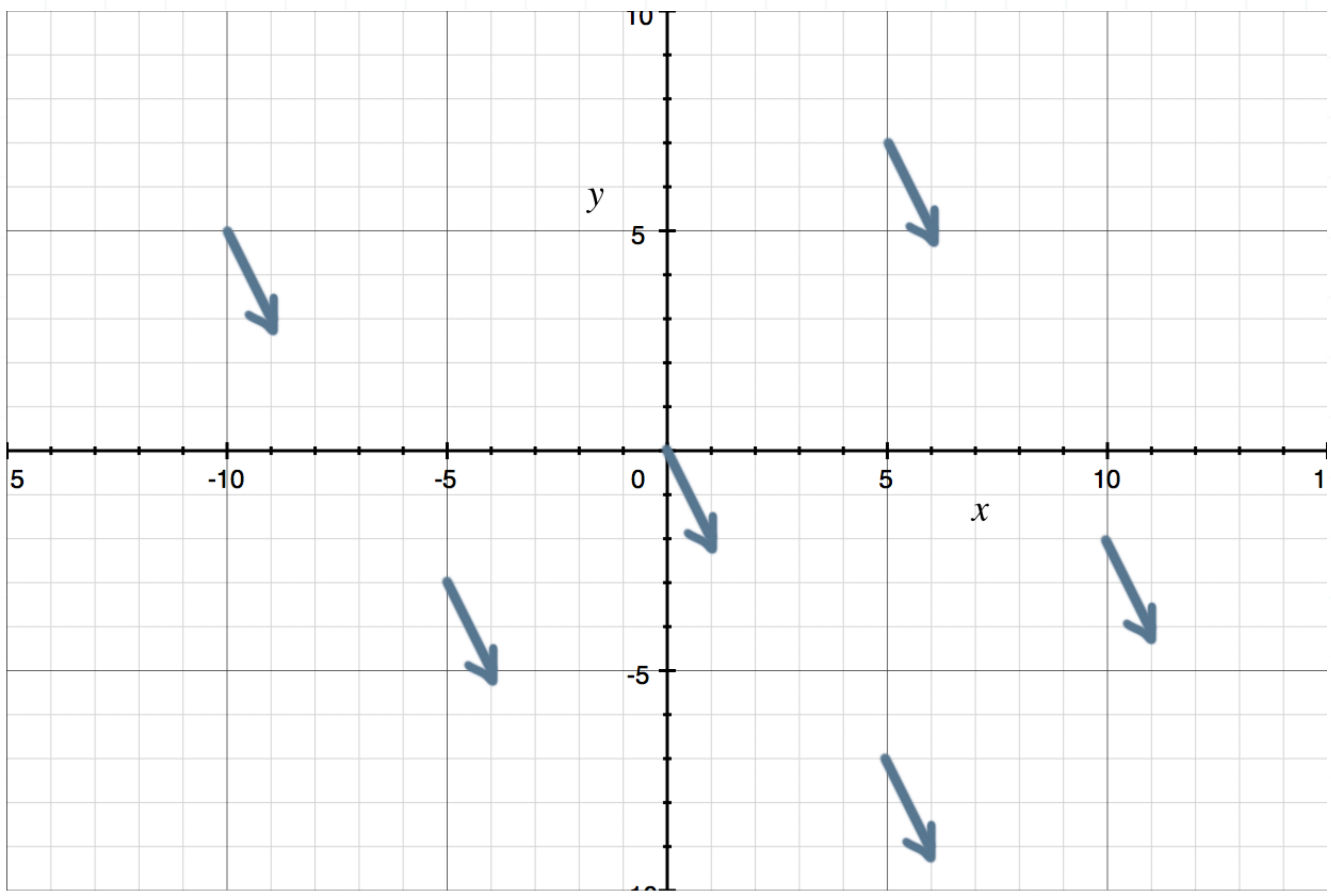
could all be sketched together in $\mathbb{R}^2$:



Each of these vectors has its **initial point** at the origin (each vector "starts" at $(0,0)$), and its **terminal point** at the location described by the vector (each vector "ends" at the coordinate point that you see in each vector).

The little arrow at the end of the vector indicates the terminal point, and then the other end of the vector is the initial point.

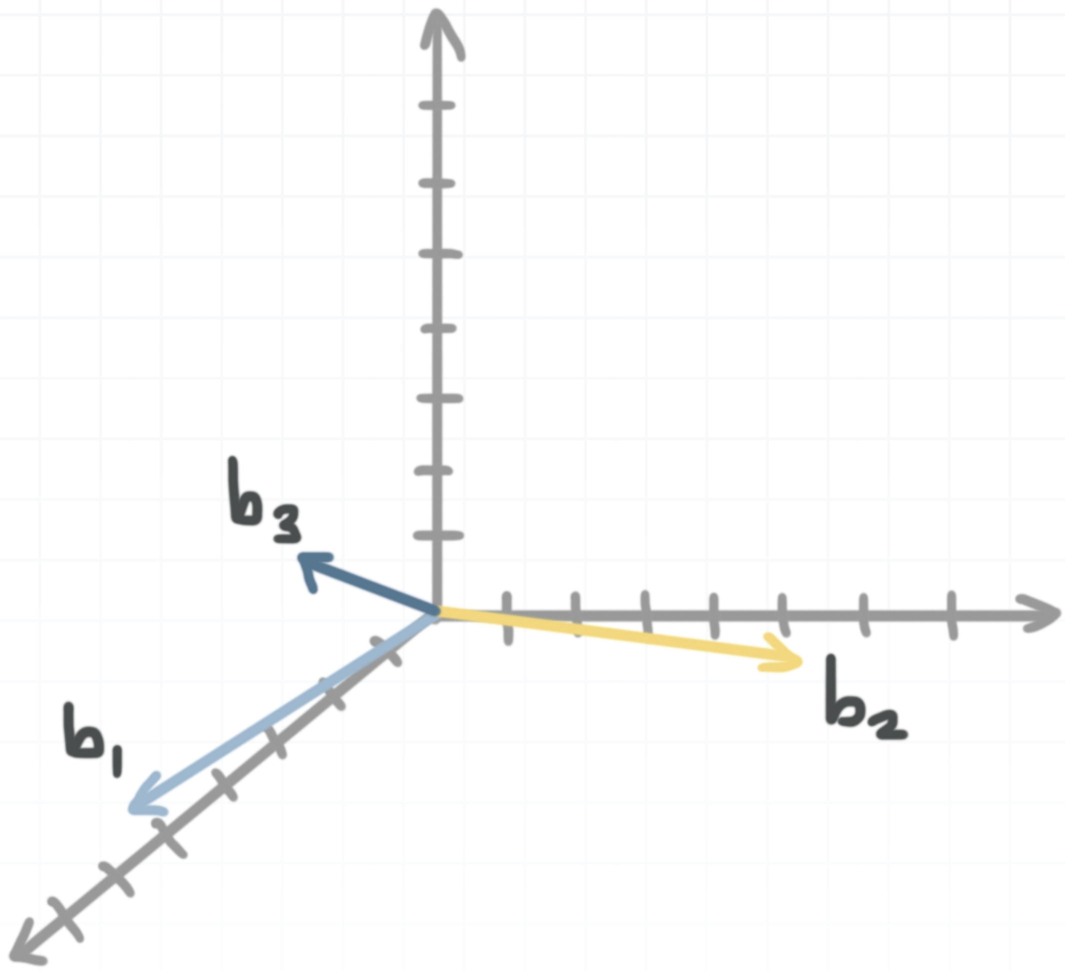
Remember that the information contained in a vector is only its direction and its length, which means vectors don't always have to start at the origin. We can move a vector parallel to itself anywhere in coordinate space, and it'll still be the same vector. Here are many different vectors, every one of which is $a_3 = [1 \quad -2]$.



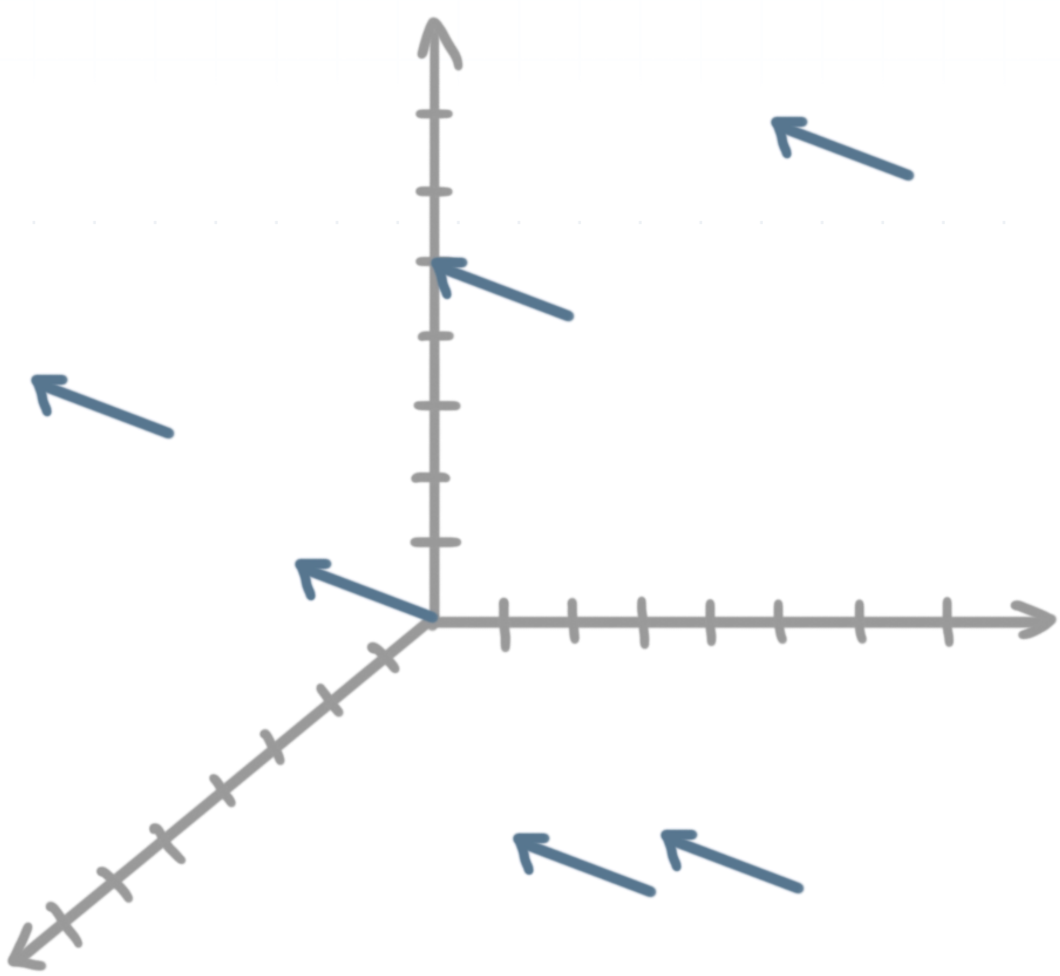
Similarly, the three-dimensional vectors

$$b_1 = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \text{ and } b_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

could all be sketched together in $\mathbb{R}^3$:



And here are many different vectors, every one of which is $b_3 = [4 \quad 1 \quad 3]$.



Let's do an example of sketching vectors.

Example

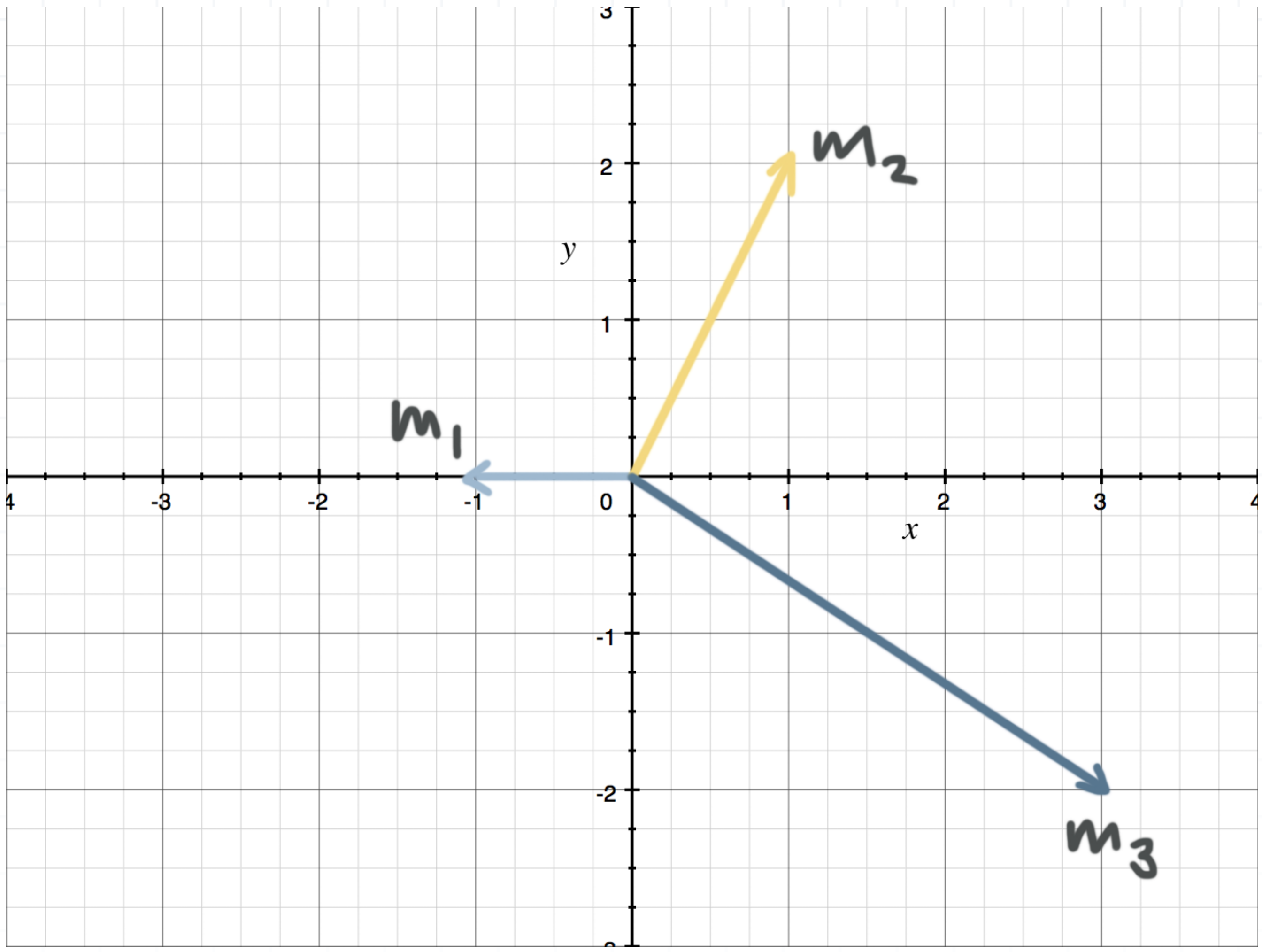
Sketch the column vectors of M in standard position in $\mathbb{R}^2$.

$$M = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix}$$

The column vectors of M are

$$m_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ m_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } m_3 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The vector m_1 in standard position will point from $(0,0)$ to $(-1,0)$; the vector m_2 in standard position will point from $(0,0)$ to $(1,2)$; the vector m_3 in standard position will point from $(0,0)$ to $(3,-2)$. The sketch of the three of them together in $\mathbb{R}^2$ is therefore



m1
m2
m3
x
y

Vector operations

Just like matrices, vectors can be added, subtracted, and multiplied.

Since vectors can be written as column matrices or row matrices, you'll be able to use what you've learned about matrix operations in order to perform the corresponding vector operations.

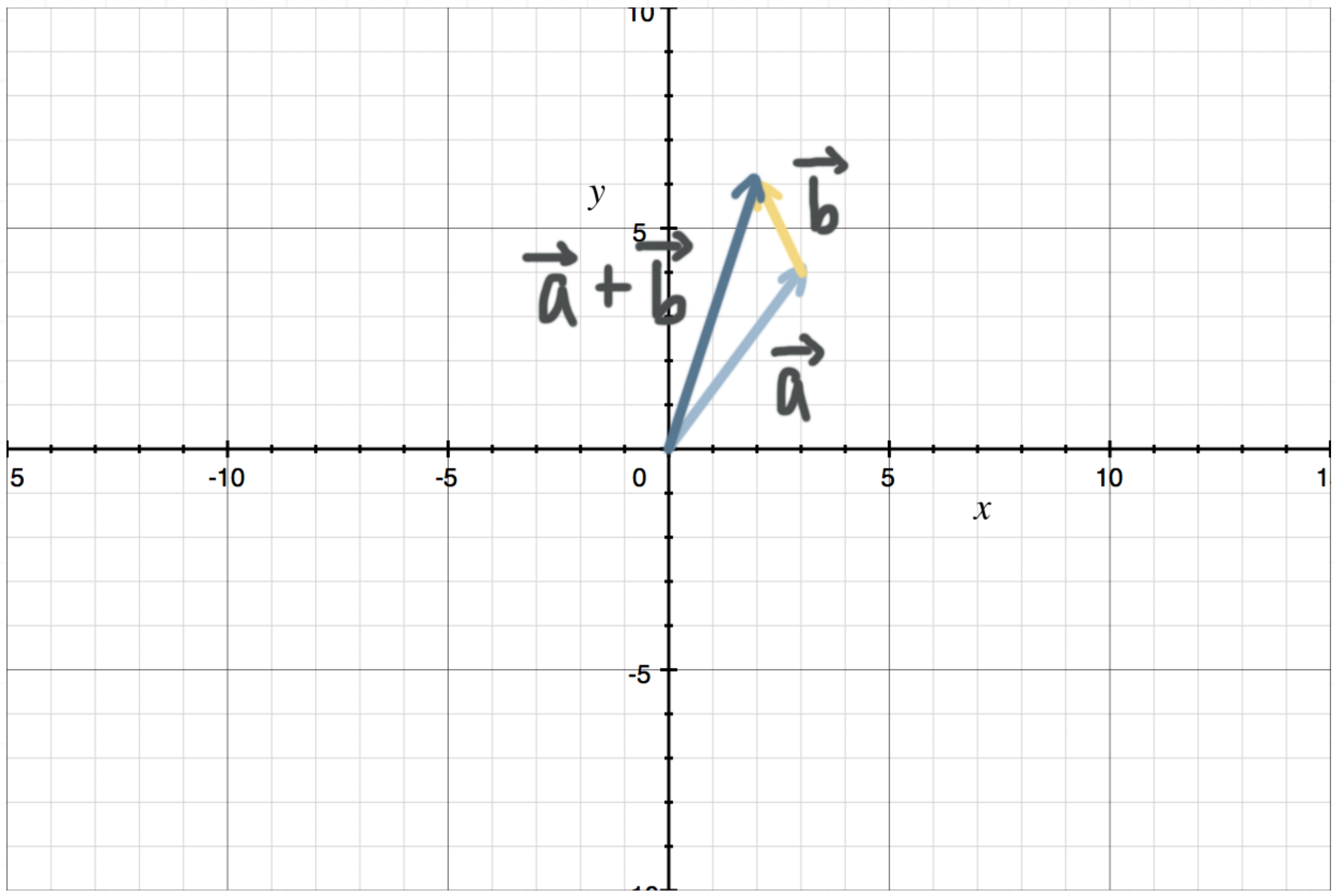
The sum of vectors

To add vectors, just add their corresponding components. Given $\overrightarrow{a}=(3,4)$ and $\overrightarrow{b}=(-1,2)$, the sum of the vectors

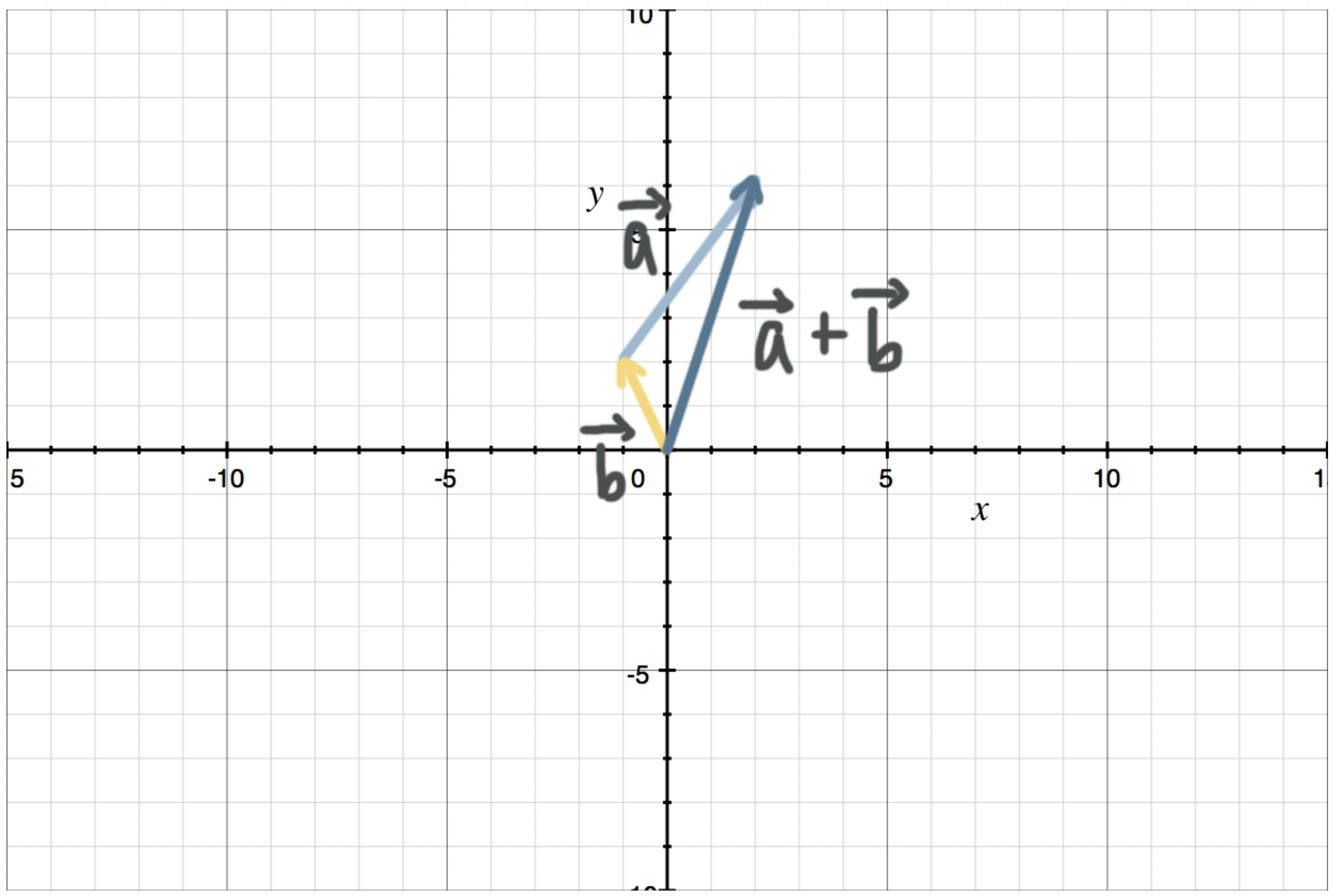
$$\overrightarrow{a}+\overrightarrow{b}=(3+(-1),4+2)$$

$$\overrightarrow{a}+\overrightarrow{b}=(2,6)$$

Graphically, we can see that adding vectors means connecting the terminal point of one to the tail of the other. If we start with $\overrightarrow{a}=(3,4)$ and add $\overrightarrow{b}=(-1,2)$ to it, that looks like this:



If we start with $\overrightarrow{b}=(-1,2)$ and add $\overrightarrow{a}=(3,4)$ to it, that looks like this:



Either way, we end up at $(2,6)$. Because matrix addition is commutative, it makes sense that we end up at the same point, regardless of the order in which we add the matrices. For instance, since $\overrightarrow{a}=(3,4)$ and $\overrightarrow{b}=(-1,2)$ could both be represented as column vectors (or row vectors), we really just have a matrix addition problem, and we know that matrices can be added in either order.

$$A+B=\begin{bmatrix}3\\4\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$$

$$B+A=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$$

And of course, even though we've only added two vectors here, we could find the sum of any number of vectors that we'd like, and the result would always be the same, regardless of the order in which we add them (since matrix addition is commutative). Furthermore, while here we've added two-dimensional vectors, we could change these into n-dimensional vectors, and find the sum of any number of n-dimensional vectors.

The difference of vectors

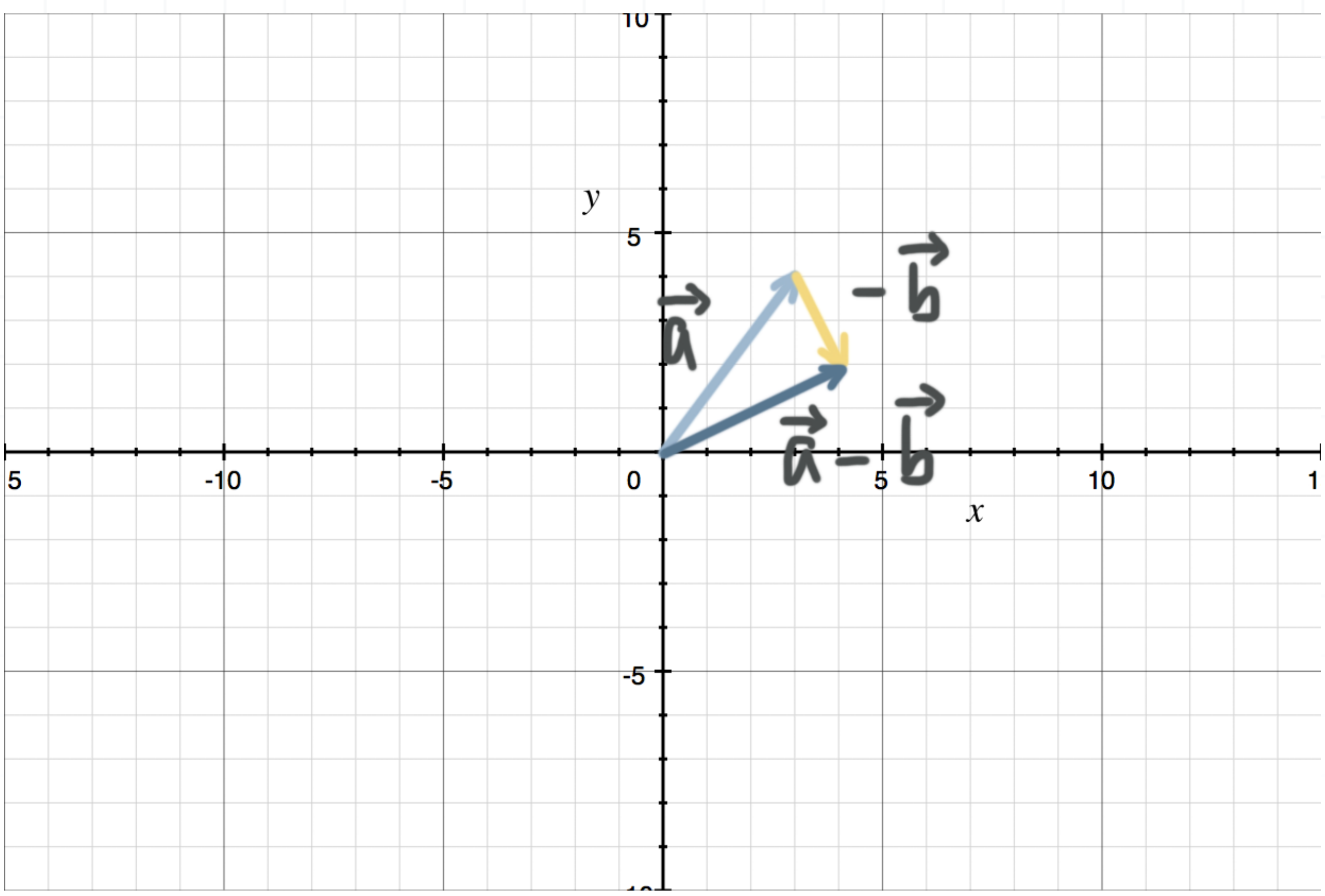
Remember that, unlike matrix addition, matrix subtraction is not commutative. Therefore, it makes a difference whether we subtract $\overrightarrow{b}=(-1,2)$ from $\overrightarrow{a}=(3,4)$ or $\overrightarrow{a}=(3,4)$ from $\overrightarrow{b}=(-1,2)$. We'll get a different result in each case.

But the operation is still simple. We just take the difference of corresponding components from each vector. Here are the differences $\overrightarrow{a}-\overrightarrow{b}$ and $\overrightarrow{b}-\overrightarrow{a}$.

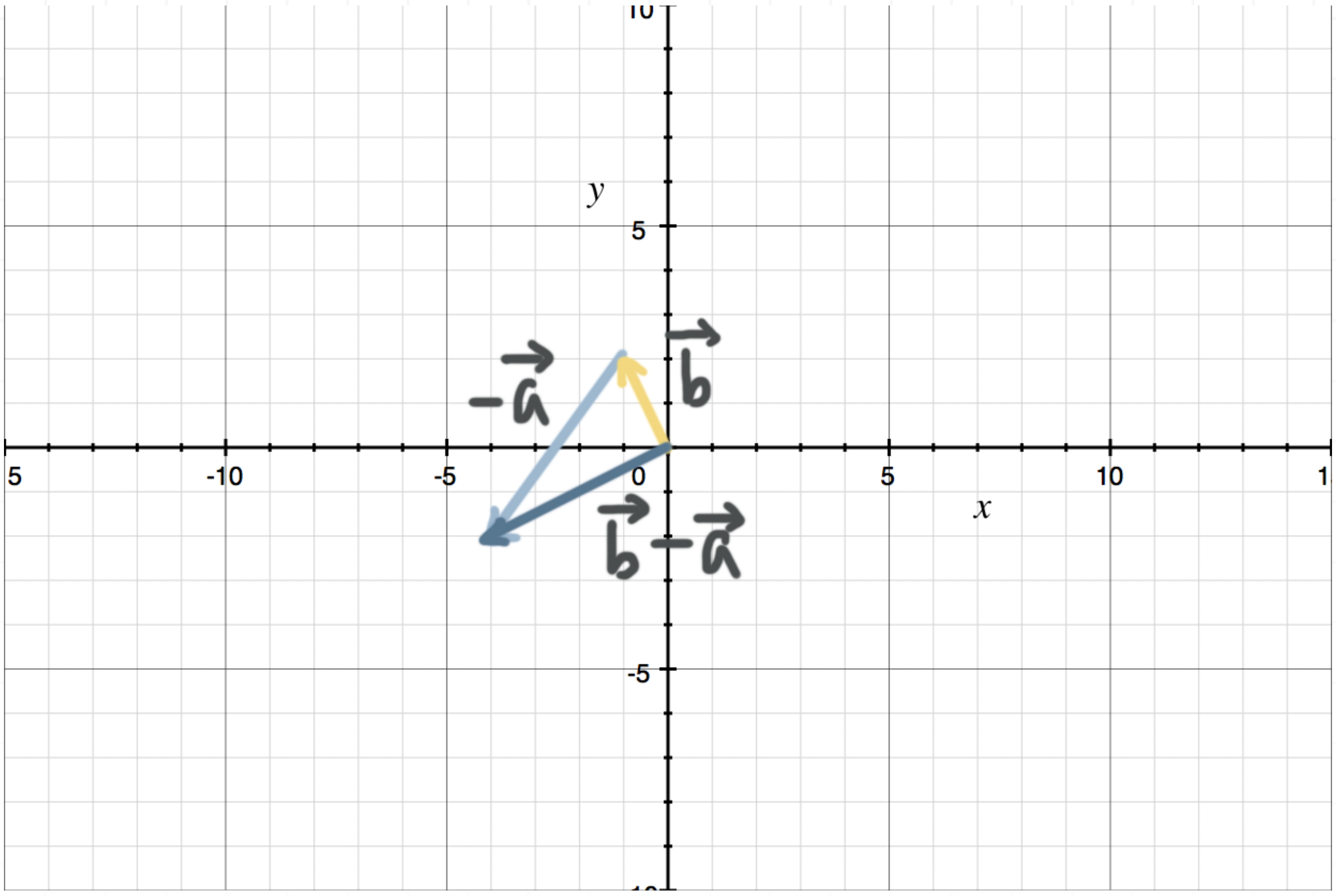
$$\vec{a} - \vec{b} = (3 - (-1), 4 - 2) = (4,2)$$

$$\vec{b} - \vec{a} = (-1 - 3, 2 - 4) = (-4, -2)$$

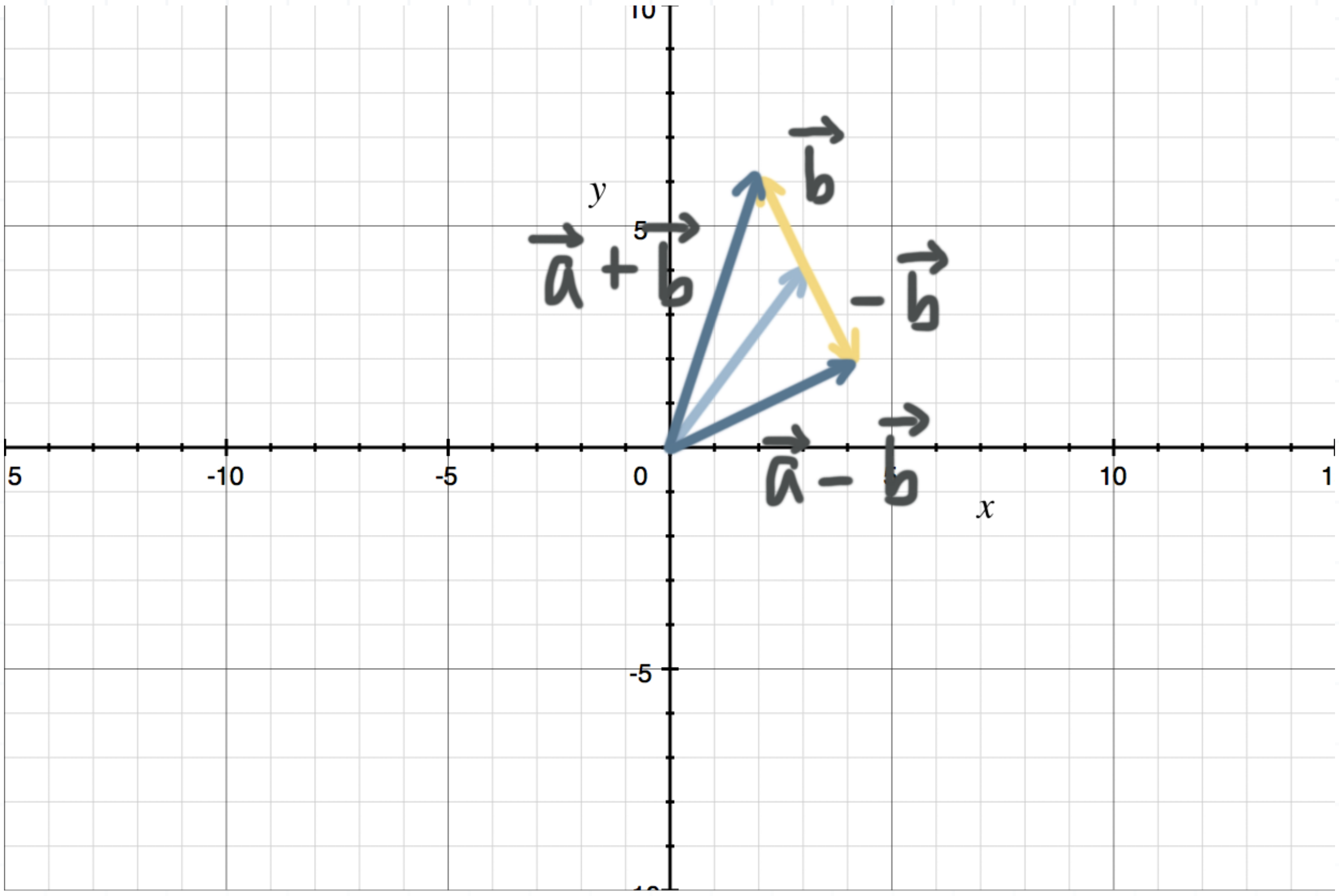
If we sketch the difference of the vectors, we can see that $\vec{a} - \vec{b}$ gets us to $\vec{a} - \vec{b} = (4,2)$,



and that $\vec{b} - \vec{a}$ gets us to $\vec{b} - \vec{a} = (-4, -2)$.



Notice also that $\vec{a}-\vec{b}$ is the same as $\vec{a}+(-\vec{b})$. The vector $-\vec{b}$ has the same length as $\vec{b}$, but moves in exactly the opposite direction. In other words, here are $\vec{a}$, $\vec{b}$, and $-\vec{b}$ sketched together. Notice how $\vec{b}$ and $-\vec{b}$ move in exactly opposite directions. And we can see the results $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.



Let's do an example with vector addition and subtraction.

Example

Find $\overrightarrow{a} + \overrightarrow{b} - \overrightarrow{c} + \overrightarrow{d}$.

$$\overrightarrow{a} = (3,4)$$

$$\overrightarrow{b} = (-1,2)$$

$$\overrightarrow{c} = (3,3)$$

$$\overrightarrow{d} = (-2,0)$$

Just like addition and subtraction with real numbers, we can work left to right. We've already seen the sum $\overrightarrow{a} + \overrightarrow{b}$.

$$\vec{a} + \vec{b} = (3 + (-1), 4 + 2)$$

$$\vec{a} + \vec{b} = (2,6)$$

When we subtract $\vec{c} = (3,3)$ from this, we get

$$\vec{a} + \vec{b} - \vec{c} = (2 - 3, 6 - 3)$$

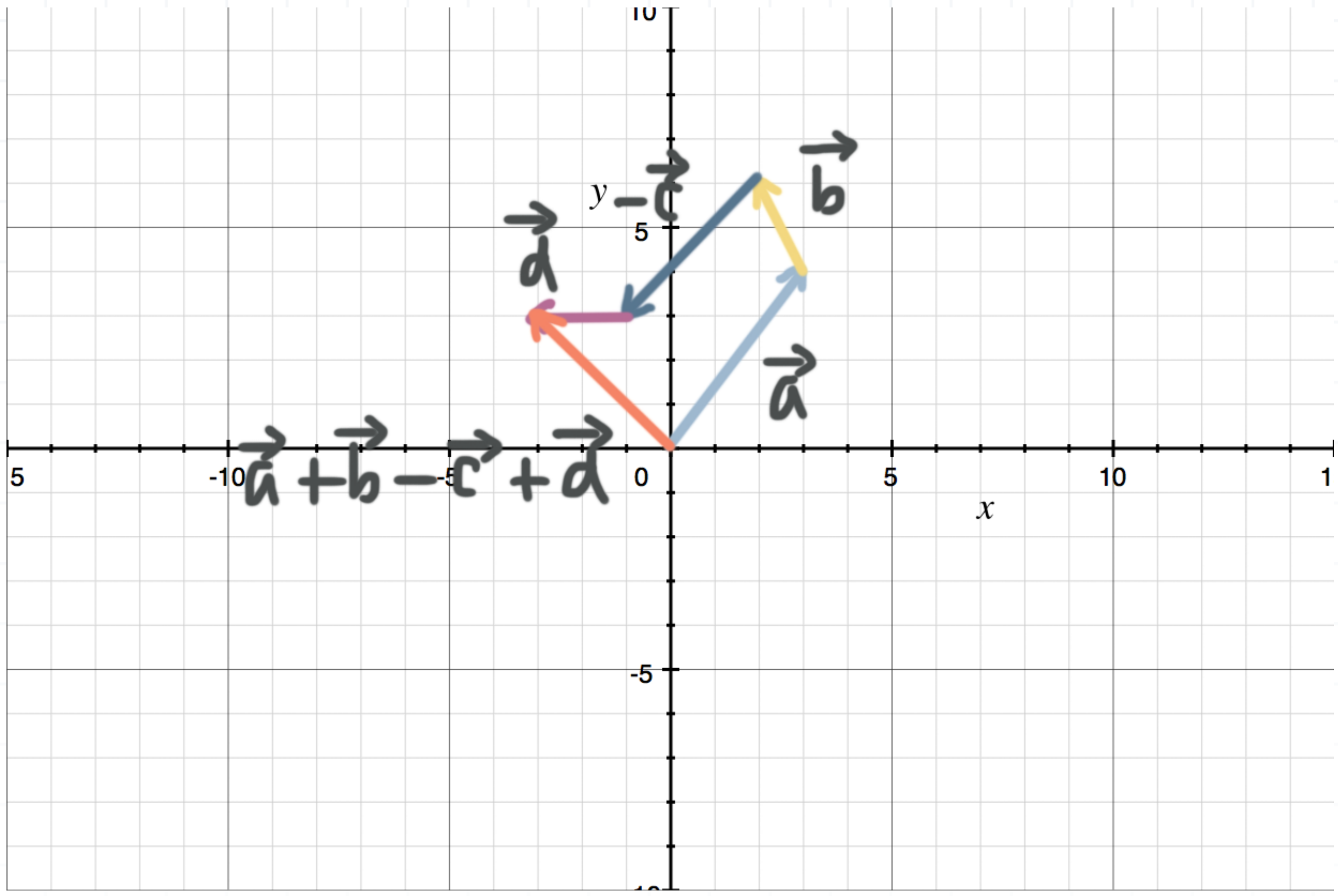
$$\vec{a} + \vec{b} - \vec{c} = (-1,3)$$

When we add $\vec{d} = (-2,0)$ to this, we get

$$\vec{a} + \vec{b} - \vec{c} + \vec{d} = (-1 + (-2), 3 + 0)$$

$$\vec{a} + \vec{b} - \vec{c} + \vec{d} = (-3,3)$$

We can sketch all the individual vectors by connecting the terminal point of each one to the tail of the one before it, plus the vector of the sum $\vec{a} + \vec{b} - \vec{c} + \vec{d}$, together in the same plane, and we see that the sum gets us to the same ending point.

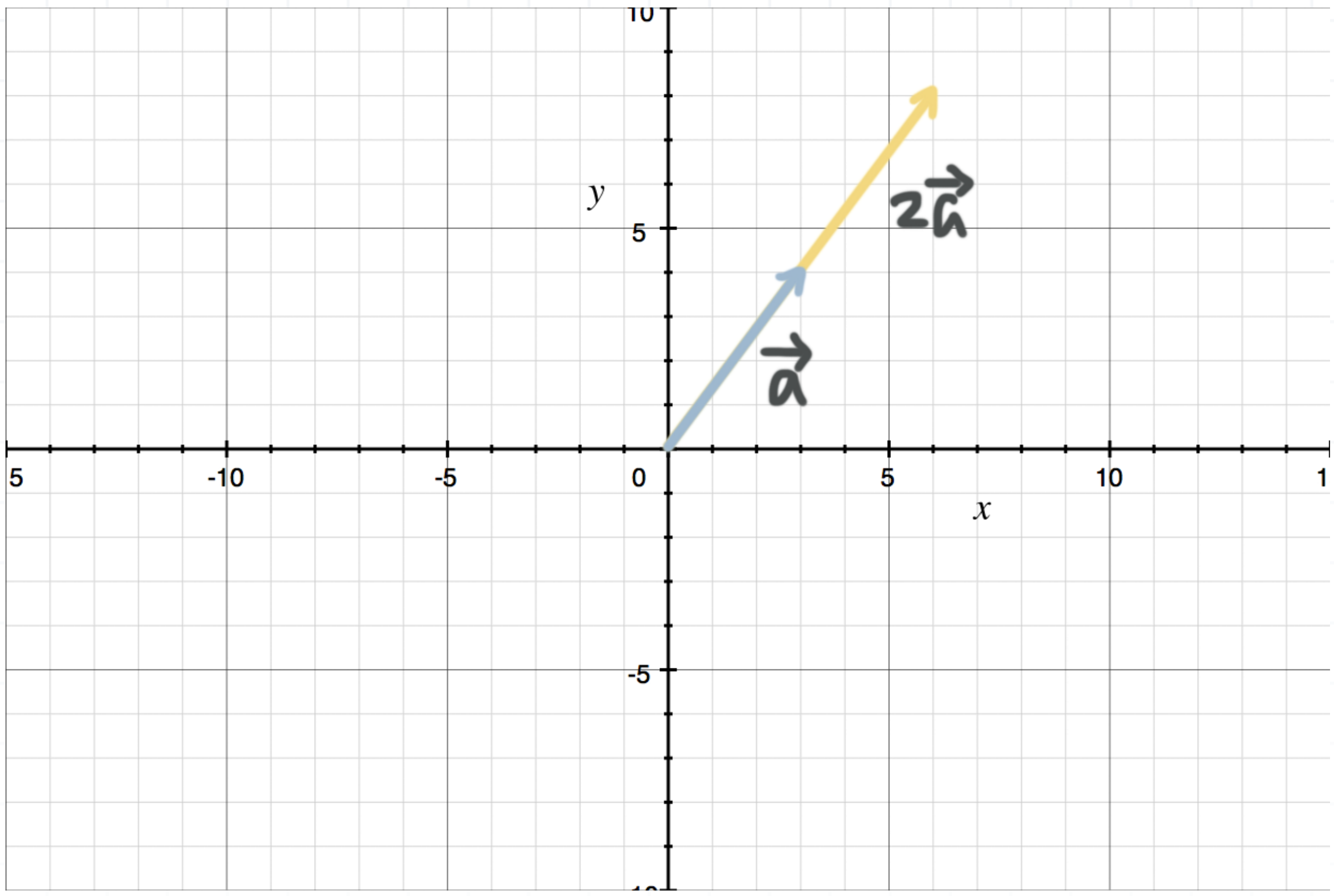


Multiplying a vector by a scalar

Multiplying a vector by a scalar is just like multiplying a column matrix by a scalar. For instance, if we multiply $\vec{a}=(3,4)$ by 2, we get

$$2\vec{a}=2\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2(3)\\2(4)\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$$

Graphically, we can see that the resulting column vector has the same direction, but its magnitude is scaled by the absolute value of the scalar.

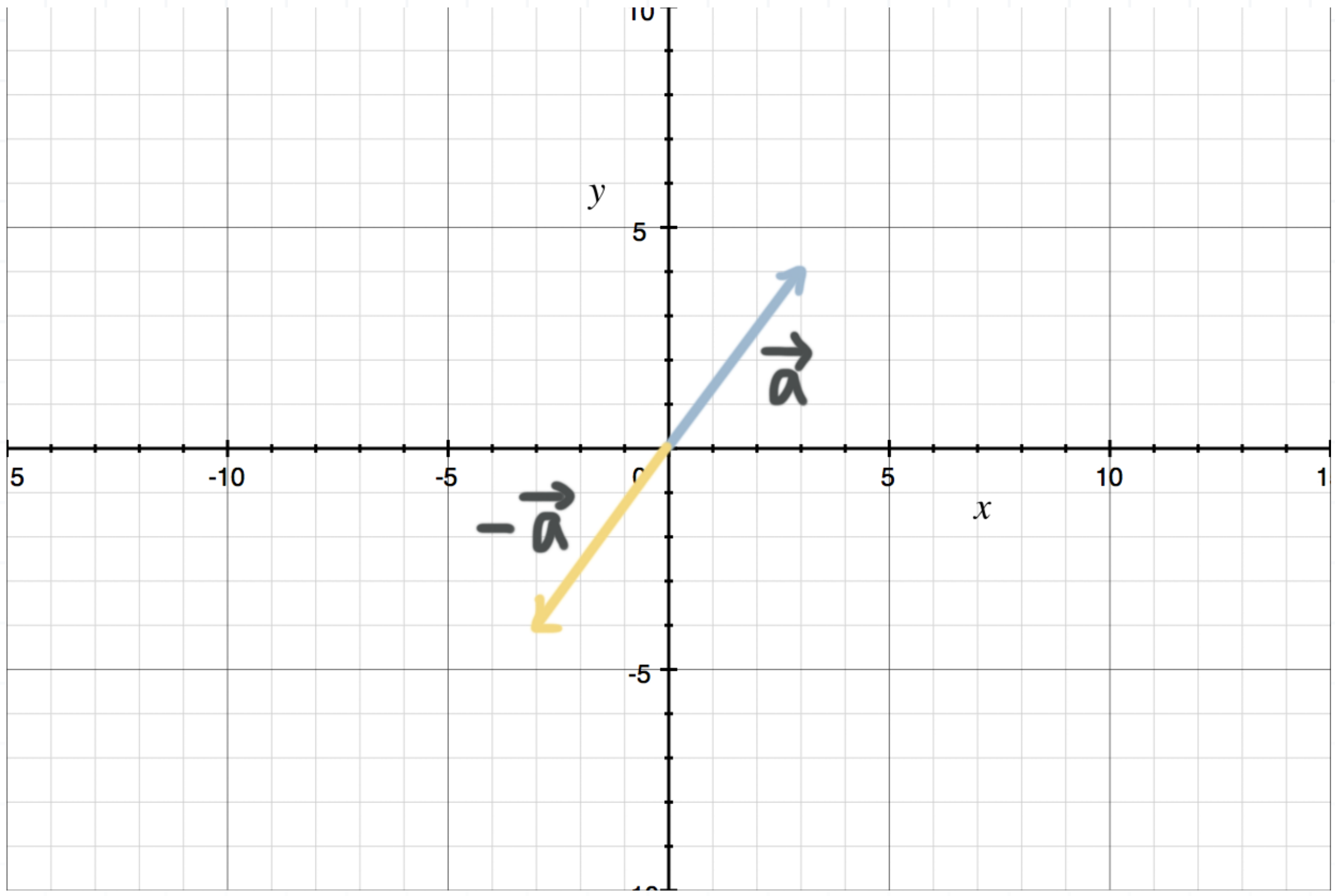


In the graph of these vectors, the yellow $2\vec{a}$ does not begin where $\vec{a}$ ends. The vectors $\vec{a}$ and $2\vec{a}$ both begin at $(0,0)$, and $2\vec{a}$ has double the length of $\vec{a}$.

If you multiply by a negative scalar, the vector will point in exactly the opposite direction, or you could say that its direction rotates 180°. So for $\vec{a}=(3,4)$,

$$-1\vec{a}=-1\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-1(3)\\-1(4)\end{bmatrix}=\begin{bmatrix}-3\\-4\end{bmatrix}$$

and $\vec{a}=(3,4)$ and $-\vec{a}=(-3,-4)$ sketched together looks like this:



Multiplying a vector by a vector

Beyond just multiplying a scalar by a vector, you can also multiply a vector by a vector. We'll talk about this more in a later section, but the product of two vectors is called the **dot product**, and we find it by summing the products of the individual components.

$$\vec{a} \cdot \vec{b} = (a_1, a_2) \cdot (b_1, b_2)$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

For example, the dot product of $\vec{a} = (3,4)$ and $\vec{b} = (-1,2)$ is

$$\vec{a} \cdot \vec{b} = (3,4) \cdot (-1,2)$$

$$\vec{a} \cdot \vec{b} = (3)(-1) + (4)(2)$$

$$\vec{a} \cdot \vec{b} = -3 + 8$$

$$\vec{a} \cdot \vec{b} = 5$$

We can also find the dot product when we write the vectors as matrices. So if we wrote $\vec{a}$ as the matrix A and $\vec{b}$ as the matrix B, then we could have written the dot product as

$$AB = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$AB = [3(-1) + 4(2)]$$

$$AB = [-3 + 8]$$

$$AB = [5]$$

or as

$$BA = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$BA = [-1(3) + 2(4)]$$

$$BA = [-3 + 8]$$

$$BA = [5]$$

When we express the vectors as matrices and then multiply them, it's important to multiply them as a row matrix first, multiplied by a column matrix second. That way, the dimensions are

$$R \times C \quad \times \quad R \times C$$

$$1 \times 2 \quad \times \quad 2 \times 1$$

and the resulting product will be a 1×1 matrix, whose only entry is the value of the dot product. If we instead multiply a column matrix by a row matrix, like

$$AB = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix}$$

then the result isn't the dot product, it's something entirely different. We can tell from the dimensions that the result will be a 2×2 matrix. Because of the way the dimensions can get tricky, we usually just stick with the

$$\overrightarrow{a} \cdot \overrightarrow{b} = (a_1, a_2) \cdot (b_1, b_2)$$

form when we're finding the dot product of two vectors.

Unit vectors and basis vectors

We know that every vector, by its own definition, contains information about its direction and its magnitude (remember that "magnitude" just means "length").

The unit vector

Any vector with a magnitude of 1 is called a **unit vector**, $\vec{u}$. In general, a unit vector doesn't have to point in a particular direction. As long as the vector is one unit long, it's a unit vector.

But oftentimes we're interested in changing a particular vector $\vec{v}$ (with a length other than 1), into an associated unit vector. In that case, that unit vector needs to point in the same direction as $\vec{v}$.

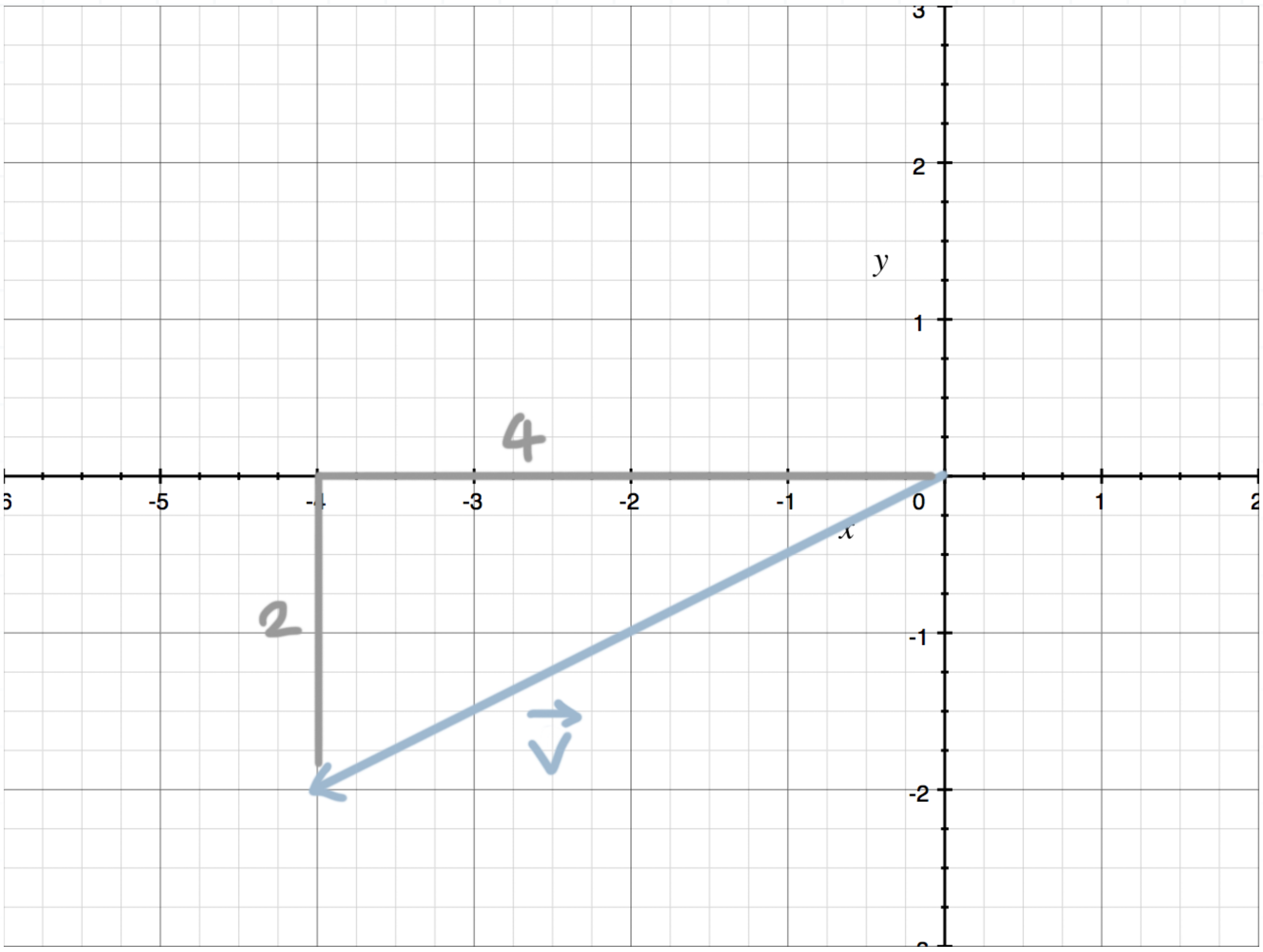
Realize that every vector $\vec{v}$ in space will have a corresponding unit vector. It'll be the vector that points in exactly the same direction as $\vec{v}$, but is only one unit long. You'll be able to find a unit vector for $\vec{v}$, regardless of whether $\vec{v}$ exists in $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$.

Let's look at an example of how to use the Pythagorean theorem find the unit vector that points in the direction of $\vec{v}$, when $\vec{v}$ is in $\mathbb{R}^2$.

Example

Find the unit vector in the direction of $\vec{v}=(-4,-2)$.

Let's start by drawing a picture of the vector $\vec{v}$.



We can then use the Pythagorean theorem to find the length of $\vec{v}$.

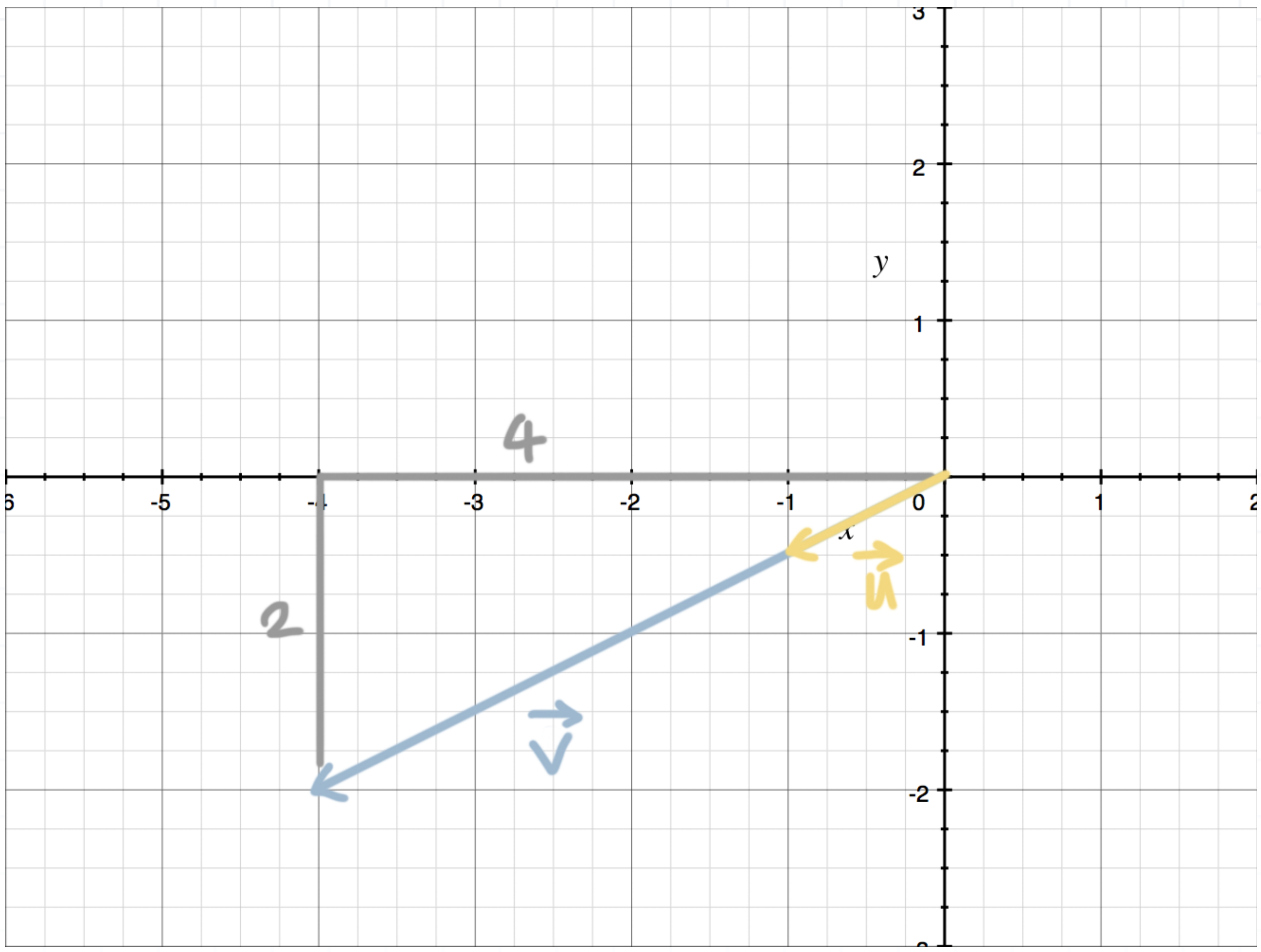
$$||\vec{v}|| = \sqrt{a^2 + b^2}$$

$$||\vec{v}|| = \sqrt{4^2 + 2^2}$$

$$||\vec{v}|| = \sqrt{16 + 4}$$

$$||\vec{v}|| = \sqrt{20}$$

The unit vector $\vec{u}$ is 1 unit long, and sits right on top of $\vec{v}$, pointing in the same direction as $\vec{v}$, so it might look roughly like this:



The smaller triangle formed by the unit vector $\vec{u}$ is similar to the larger triangle formed by $\vec{v}$. So we can set up a proportion to find the horizontal component of $\vec{u}$.

$$\frac{-4}{\sqrt{20}}=\frac{a}{1}$$

$$a=\frac{-4}{\sqrt{20}}=-\frac{2}{\sqrt{5}}$$

Set up a ratio to find the vertical component of the unit vector.

$$\frac{-2}{\sqrt{20}}=\frac{b}{1}$$

$$b = \frac{-2}{\sqrt{20}} = -\frac{1}{\sqrt{5}}$$

Therefore, we can say that the unit vector toward $\vec{v} = (-4, -2)$ has components

$$\vec{u} = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

If we rationalize the denominators here (like we learned to do back in Algebra), we can say that the unit vector that points in the same direction as $\vec{v} = (-4, -2)$ is

$$\vec{u} = \left(-\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}\right)$$

$$\vec{u} = \left(-\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$$

With this last example, we found the unit vector by first using the Pythagorean theorem to find the magnitude of the given vector, and then using a proportion of similar triangles to solve for the components of $\vec{u}$.

But there's a simpler way to find the unit vector that points toward $\vec{v}$. The unit vector that points in the direction of $\vec{v}$ is always given by

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$$

where $||\vec{v}||$ is the magnitude (length) of the vector $\vec{v}$. If $\vec{v}$ is an n-dimensional vector, then its length is the square root of the sum of all of its squared components.

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2}$$

So for instance, to find the unit vector for the three-dimensional vector $\vec{v} = (1,4,-2)$, first find the length of $\vec{v}$.

$$||\vec{v}|| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$$

Then plug $||\vec{v}||$ and $\vec{v}$ into the formula for $\vec{u}$ to find the direction of $\vec{v}$.

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v} = \frac{1}{\sqrt{21}}\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \end{bmatrix}$$

The basis vectors

Oftentimes the unit vector is written as $\hat{u}$, rather than with typical vector notation, $\vec{u}$. The little "hat" above the u is there to tell us that the length of the vector is 1. Anytime you see a vector with the "hat" on it, it means the vector's length is 1, which is why it's typical to use this notation for the unit vector specifically.

There are a few special unit vectors that we'll use a lot in both vector calculus and in linear algebra, which are called the **standard basis vectors**.

In two-dimensional space, we define two specific basis vectors, $\hat{i}=(1,0)$ and $\hat{j}=(0,1)$. As you can see from their components, they both have a length of 1. In three-dimensional space, the basis vectors are $\hat{i}=(1,0,0)$, $\hat{j}=(0,1,0)$, and $\hat{k}=(0,0,1)$.

Sometimes you'll see the basis vectors represented without the "hat," just as the bolded characters $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Linear combinations of the basis vectors

Using these basis vectors for $\mathbb{R}^2$ as a starting point, we can actually build every vector in two-dimensional space, simply by adding scaled combinations of $\hat{i}$ and $\hat{j}$. We'll define this in more detail later on, but these scaled combinations (the sums of scaled vectors) are called **linear combinations**.

For instance, the vector $\overrightarrow{a}=(6,4)$ moves 6 units in the horizontal direction, or 6 times $\hat{i}$. It also moves 4 units in the vertical direction, or 4 times $\hat{j}$. So we could write a linear combination that expresses the vector, where we scale $\hat{i}=(1,0)$ by 6, and scale $\hat{j}=(0,1)$ by 4.

$$\overrightarrow{a}=(6,4)=6\begin{bmatrix}1\\0\end{bmatrix}+4\begin{bmatrix}0\\1\end{bmatrix}$$

$$\overrightarrow{a}=(6,4)=\begin{bmatrix}6\\0\end{bmatrix}+\begin{bmatrix}0\\4\end{bmatrix}$$

$$\vec{a} = (6,4) = \begin{bmatrix} 6+0 \\ 0+4 \end{bmatrix}$$

$$\vec{a} = (6,4) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Which means we can define a new notation to express a vector:

$$\vec{a} = (6,4) = 6\hat{i} + 4\hat{j}$$

We've expressed vectors like a coordinate point, as row and column matrices, and now as a combination of the basis vectors $\hat{i}$ and $\hat{j}$.

Example

Express the vector $\vec{a} = (-3,2,-1)$ using basis vectors.

The vector $\vec{a} = (-3,2,-1)$ is part of $\mathbb{R}^3$, which means we'll need to use the basis vectors for $\mathbb{R}^3$, which are $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, and $\hat{k} = (0,0,1)$.

We're moving -3 units in the direction of the x-axis, 2 units in the direction of the y-axis, and -1 units in the direction of the z-axis.

$$\vec{a} = (-3,2,-1) = -3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{a} = (-3,2,-1) = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{a} = (-3,2,-1) = \begin{bmatrix} -3+0+0 \\ 0+2+0 \\ 0+0-1 \end{bmatrix}$$

$$\vec{a} = (-3,2,-1) = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

So we can express $\vec{a} = (-3,2,-1)$ in terms of basis vectors as

$$\vec{a} = -3\hat{i} + 2\hat{j} - \hat{k}$$

Linear combinations and span

At the end of the last lesson, we took 6 of the basis vector $\hat{i}=(1,0)$ and 4 of the basis vector $\hat{j}=(0,1)$ to express the vector $\vec{a}=(6,4)$ as

$$\vec{a}=6\begin{bmatrix}1\\0\end{bmatrix}+4\begin{bmatrix}0\\1\end{bmatrix}$$

Notice how the vector $\hat{i}=(1,0)$ is multiplied by a scalar of 6, and the vector $\hat{j}=(0,1)$ is multiplied by a scalar of 4. In other words, to express $\vec{a}$, we've only done two operations: 1) we've multiplied vectors by scalars, and 2) we've added these scaled vectors together.

Any expression like this one, which is just the sum of scaled vectors, is called a **linear combination**. Linear combinations can sum any number of vectors, not just two. So $3\vec{a}-2\vec{b}$ is a linear combination, $-\vec{a}+0.5\vec{b}$ is a linear combination, $4.2\vec{a}-7\vec{b}+\pi\vec{c}$ is a linear combination, and so on.

Span of a vector set

The **span** of a set of vectors is the collection of all vectors which can be represented by some linear combination of the set.

That sounds confusing, but let's think back to the basis vectors $\hat{i}=(1,0)$ and $\hat{j}=(0,1)$ in $\mathbb{R}^2$. If you choose absolutely any vector, anywhere in $\mathbb{R}^2$, you can get to that vector using a linear combination of $\hat{i}$ and $\hat{j}$. If I choose $(13,2)$, I can get to it with the linear combination $\vec{a}=13\hat{i}+2\hat{j}$, or if I choose $(-1,-7)$, I can get to it with the linear combination $\vec{a}=-\hat{i}-7\hat{j}$. There's no vector you can find in $\mathbb{R}^2$ that you can't reach with a linear combination of $\hat{i}$ and $\hat{j}$.

And because you can get to any vector in $\mathbb{R}^2$ with a linear combination of $\hat{i}$ and $\hat{j}$, you can say specifically that $\hat{i}$ and $\hat{j}$ **span** $\mathbb{R}^2$. If a set of vectors spans a space, it means you can use a linear combination of those vectors to reach any vector in the space.

In the same way, I can get to any vector, anywhere in $\mathbb{R}^3$, using a linear combination of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, which means $\hat{i}$, $\hat{j}$, and $\hat{k}$ **span** $\mathbb{R}^3$, the entirety of three-dimensional space.

I could also write these facts as

$$\text{Span}(\hat{i}, \hat{j}) = \mathbb{R}^2$$

$$\text{Span}(\hat{i}, \hat{j}, \hat{k}) = \mathbb{R}^3$$

One other point: the span of the zero vector $\vec{O}$ is always just the origin, so $(0,0)$ in $\mathbb{R}^2$, $(0,0,0)$ in $\mathbb{R}^3$, etc.

Span, and linear independence

So our next step is to be able to determine when a vector set spans a space, and when it doesn't. In other words, how can we tell when every point in the space is, or is not, reachable by a linear combination of the vector set?

The answer has to do with whether or not the vectors in the set are linearly independent or linearly dependent. We'll talk about linear (in)dependence in the next lesson, but for now, let's just make three points:

- Any 2 two-dimensional linearly independent vectors will span $\mathbb{R}^2$. The two-dimensional basis vectors $\hat{i}$ and $\hat{j}$ are linearly independent, which is why they span $\mathbb{R}^2$.

- Any 3 three-dimensional linearly independent vectors will span $\mathbb{R}^3$. The three-dimensional basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are linearly independent, which is why they span $\mathbb{R}^3$.

- Any n n-dimensional linearly independent vectors will span $\mathbb{R}^n$. The n-dimensional basis vectors are linearly independent, which is why they span $\mathbb{R}^n$.

So when is a set of vectors linearly dependent, such that they won't span the vector space $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$?

First, we should say that we can never span $\mathbb{R}^n$ with fewer than n vectors. In other words, we can't span $\mathbb{R}^2$ with one or fewer vectors, we can't span $\mathbb{R}^3$ with two or fewer vectors, and we can't span $\mathbb{R}^n$ with $n-1$ or fewer vectors.

Second, assuming we have enough vectors to span the space, generally speaking, those vectors need to be "different enough" from each other that they can cover the whole vector space. It's actually easier to think about when the vectors *won't* be "different enough" to span the vector space:

- When 2 two-dimensional vectors lie along the same line (or along parallel lines), they're called **collinear**, they're linearly dependent, and they won't span $\mathbb{R}^2$.

- When 3 three-dimensional vectors lie in the same plane, they're called **coplanar**, they're linearly dependent, and they won't span $\mathbb{R}^3$.

- When n n-dimensional vectors lie in the same $(n-1)$-dimensional space, they're linearly dependent, and they won't span $\mathbb{R}^n$.

Let's hold off until the next section on more detail about linear dependence and independence, and turn to an example.

Example

Prove that you can use a linear combination of the basis vectors $\mathbf{i}=(1,0)$ and $\mathbf{j}=(0,1)$ to get any vector $\overrightarrow{k}=(k_1,k_2)$ in $\mathbb{R}^2$.

We can set up a vector equation, then write the basis vectors as column vectors.

$$c_1\mathbf{i}+c_2\mathbf{j}=\overrightarrow{k}$$

$$c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}k_1\\k_2\end{bmatrix}$$

This matrix equation can be rewritten as a system of equations:

$$1c_1+0c_2=k_1$$

$$0c_1 + 1c_2 = k_2$$

Simplifying the system leaves us with

$$c_1 = k_1$$

$$c_2 = k_2$$

So what have we shown? We realize that this system means we could pick any vector $\overrightarrow{k} = (k_1, k_2)$ in $\mathbb{R}^2$, and we'd get $k_1 = c_1$ and $k_2 = c_2$, which means our linear combination will simply be a k_1 number of $\mathbf{i}$'s, and a k_2 number of $\mathbf{j}$'s.

So if, for example, the vector we chose in $\mathbb{R}^2$ was $\overrightarrow{k} = (7,4)$, then the linear combination of the basis vectors is

$$k_1\mathbf{i} + k_2\mathbf{j} = \overrightarrow{k}$$

$$7\mathbf{i} + 4\mathbf{j} = \overrightarrow{k}$$

Which means we would need to use 7 of the $\mathbf{i}$ vectors and 4 of the $\mathbf{j}$ vectors in order to reach $\overrightarrow{k} = (7,4)$ from the origin.

Linear independence in two dimensions

A set of vectors are **linearly dependent** when one vector in the set can be represented by a linear combination of the other vectors in the set. Put another way, if one or more of the vectors in the set doesn't add any new information or directionality, then the set is linearly dependent.

As we learned in the last section, only 2 two-dimensional linearly independent vectors are needed to span $\mathbb{R}^2$. Which means, given at set of 3 two-dimensional vectors, the set will always be linearly dependent, since at least one of the vectors could be made from some linear combination of the other two.

Linear dependence with two vectors

As an example, look at the vectors v_1 and v_2.

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

These vectors are linearly dependent, because we can multiply either one by a constant to get the other.

Multiplying v_1 by 4 gives v_2.

Multiplying v_2 by $1/4$ gives v_1.

When the only difference between two vectors is a scalar, then they lie on the same line, they're **collinear**, and we say that they're linearly dependent.

It's also helpful to think about linear dependence as the existence of one or more redundant vectors. In the example with v_1 and v_2, both vectors lie on the same line, which is the line $y = x$. But v_1 can define the entire line, because we can get any point on the line simply by multiplying v_1 by a scalar. But v_2 also defines the entire line for the same reason.

So having v_2 in addition to v_1 really doesn't help us; it doesn't give us any new information outside the line $y = x$. Similarly, having v_1 in addition to v_2 doesn't give us anything new that we didn't already have with v_2. So the two vectors are linearly dependent, because one of them is always redundant. They define the same thing together that they can each already define individually.

On the other hand, the vectors

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

are linearly independent. There's no scalar you can multiply by w_1 that'll give you w_2, and there's no scalar you can multiply by w_2 that'll give you w_1. Which means these vectors aren't collinear, and they therefore span $\mathbb{R}^2$ as a pair of linearly independent vectors.

Linear dependence with three vectors in $\mathbb{R}^2$

As we mentioned earlier, a set of three vectors (in two dimensions) will always be linearly dependent, because even if two of the vectors are linearly independent and span $\mathbb{R}^2$, the third vector will be redundant.

First, let's think about two vectors that aren't linearly dependent (they're linearly independent). Let's just take the basis vectors $\hat{i}$ and $\hat{j}$, but we'll call them v_1 and v_2 for now.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can say that v_1 and v_2 are linearly independent. There's no combination of v_1 vectors that will give us v_2, and similarly no combination of v_2 vectors that will give us v_1, so they're linearly independent.

Spanning $\mathbb{R}^2$

In case you missed it, here's the amazing thing: Given *any* two linearly independent vectors, we can use them to define the entire real plane! It doesn't matter which two vectors we use, as long as they are linearly independent of one another. If they are, then we'll be able to define any point in the plane as a combination of the two linearly independent vectors.

Therefore, given two linearly independent vectors like v_1 and v_2, there's no other vector we can name in the same real plane that won't be a redundant addition to the set. Because every vector in the plane can already be defined by a linear combination of v_1 and v_2, so there's no new information we can get from a third vector in the same plane. For instance, because we know that v_1 and v_2 are linearly independent, the vector set

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

must be linearly dependent. We can create v_3 as a linear combination of v_1 and v_2:

$$v_3 = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 3(1) \\ 3(0) \end{bmatrix} + \begin{bmatrix} 2(0) \\ 2(1) \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 3+0 \\ 0+2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So v_3 hasn't given us any new information. It's redundant when we already have v_1 and v_2, so the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent.

Testing for linear independence

Luckily, there's a reliable way to determine whether a set of vectors is linearly dependent or independent. We set up a system of equations that includes the vectors in the set, and one constant term for each vector. For example, given the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we'd set up the equation

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice how we took the sum of the vectors in the set, put a constant in front of each one, and set the sum equal to the zero vector. To determine linear (in)dependence, we'll always set up the equation this way. Then we can break the equation into a system of equations,

$$c_1(1) + c_2(0) = 0$$

$$c_1(0) + c_2(1) = 0$$

then solve the system. In this case, we find $c_1 = 0$ and $c_2 = 0$. Whether the system is linearly dependent or independent is determined by the values of c_1 and c_2. If the only values that can make the system true are $c_1 = 0$ and $c_2 = 0$, then the vectors are linearly independent. But if either c_1 is nonzero and/or c_2 is nonzero, then the vectors are linearly dependent.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Set up the equation first.

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then break the equation into a system of two linear equations.

$$c_1 + 3c_2 = 0$$

$$4c_1 + 2c_2 = 0$$

Multiply $c_1 + 3c_2 = 0$ by 4 to get

$$4(c_1 + 3c_2) = 4(0)$$

$$4c_1 + 12c_2 = 0$$

and then subtract this equation from $4c_1 + 2c_2 = 0$ to cancel c_1.

$$4c_1 + 2c_2 - (4c_1 + 12c_2) = 0 - 0$$

$$4c_1 + 2c_2 - 4c_1 - 12c_2 = 0$$

$$2c_2 - 12c_2 = 0$$

$$-10c_2 = 0$$

$$c_2 = 0$$

Substitute $c_2 = 0$ back into $c_1 + 3c_2 = 0$ to find the value of c_1.

$$c_1 + 3(0) = 0$$

$$c_1 + 0 = 0$$

$$c_1 = 0$$

The only solution set that makes the system true is $c_1 = 0$ and $c_2 = 0$. Because both values are 0, the given vectors are linearly independent.

Since this is Linear Algebra, we should of course mention that the system of linear equations from the last example,

$$c_1 + 3c_2 = 0$$

$$4c_1 + 2c_2 = 0$$

can always be solved using a matrix instead of using elimination and substitution. Just set up the matrix,

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix}$$

then use Gaussian elimination to put the matrix into echelon form.

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & -10 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

Because we get the identity matrix on the left, and all zero entries on the right, it means we've found $c_1 = 0$ and $c_2 = 0$, and can therefore conclude that the vectors

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

are linearly independent.

Let's do another example where we solve the system using a matrix that we put into row-echelon form.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Set up an equation in which the sum of the linear combination is set equal to the zero vector.

$$c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's use a matrix to solve the system. As we saw in the previous lesson, we really only need the column vectors on the left to go into the matrix.

$$\begin{bmatrix} -2 & 6 & | & 0 \\ 1 & -3 & | & 0 \end{bmatrix}$$

Multiply the first row by $-1/2$.

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 1 & -3 & | & 0 \end{bmatrix}$$

Subtract the first row from the second row, replacing the second row, $R_2 - R_1 \to R_2$.

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The first row of this matrix gives us

$$c_1 - 3c_2 = 0$$

$$c_1 = 3c_2$$

If we substitute back into the linear combination equation,

$$3c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2 \begin{bmatrix} -6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2 \begin{bmatrix} -6 \\ 3 \end{bmatrix} = -c_2 \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$c_2 \begin{bmatrix} -6 \\ 3 \end{bmatrix} = c_2 \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

we see that any $c_1 = 3c_2$ will make the equation true. So we can find any pair (c_1, c_2) that satisfies the equation $c_1 = 3c_2$, like $(c_1, c_2) = (3,1)$, and it proves to us that there's some pair other than $(c_1, c_2) = (0,0)$ that makes the linear combination equation true.

Because $(c_1, c_2) = (0,0)$ is not the only combination that gives the zero vector, it means that the vector set $\{v_1, v_2\}$ is linearly dependent. Which means that v_1 and v_2 can't span $\mathbb{R}^2$.

Linear independence in three dimensions

We can also define linear dependence and independence in three dimensions. If three vectors in three-dimensional space are linearly independent, it means that two of the vectors would be linearly independent of one another in two-dimensional space, and then the third vector would lie outside the plane of the first two vectors.

Only 3 three-dimensional linearly independent vectors are needed to span $\mathbb{R}^3$. Which means, given a set of 4 three-dimensional vectors, the set will always be linearly dependent, since at least one of the vectors could be made from some linear combination of the other three.

Testing for linear independence

To test for linear independence in three dimensions, we can use the same method we used in two dimensions, which was setting the sum of the linear combination of the vectors equal to the zero vector, and then solving the system.

If the only solution is one in which all three constants are 0, then the vectors are linearly independent. But if there's a solution to the system in which one or more of the constants is non-zero, then the vectors are linearly dependent.

Example

Say whether the vectors are linearly dependent or linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

Set up an equation in which the sum of the linear combination is set equal to the zero vector.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let’s use a matrix to solve the system. As we saw in the previous lesson, we really only need the column vectors on the left to go into the matrix.

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -3 \\ 3 & 4 & 0 \end{bmatrix}$$

If we can use Gaussian elimination to put this matrix into reduced row-echelon form, then we’ll know the vector set $\{v_1, v_2, v_3\}$ is linearly independent.

Start by zeroing out the first column below the pivot entry.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 4 & -6 \end{bmatrix}$$

Finish zeroing out the rest of the second column around the pivot.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 22 \end{bmatrix}$$

Find the pivot in the third column.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we were able to get to the identity matrix with Gaussian elimination, we know $(c_1, c_2, c_3) = (0,0,0)$, and therefore that the vector set $\{v_1, v_2, v_3\}$ is linearly independent. Which means we could also say that $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$.

Matrices are great for solving systems of linear equations, but of course we could have also solved the system using the old elimination and substitution method:

[1] $c_1 + 2c_3 = 0$

[2] $2c_1 + c_2 - 3c_3 = 0$

[3] $3c_1 + 4c_2 = 0$

Solve [1] for c_3 in terms of c_1,

$$c_1 + 2c_3 = 0$$

$$2c_3 = -c_1$$

$$c_3 = -\frac{1}{2}c_1$$

and then solve [3] for c_2 in terms of c_1.

$$3c_1 + 4c_2 = 0$$

$$4c_2 = -3c_1$$

$$c_2 = -\frac{3}{4}c_1$$

Now we can substitute $c_2 = -(3/4)c_1$ and $c_3 = -(1/2)c_1$ into [2] to get an equation only in terms of c_1.

$$2c_1 + c_2 - 3c_3 = 0$$

$$2c_1 - \frac{3}{4}c_1 - 3\left(-\frac{1}{2}c_1\right) = 0$$

$$2c_1 - \frac{3}{4}c_1 + \frac{3}{2}c_1 = 0$$

$$\left(2 - \frac{3}{4} + \frac{3}{2}\right)c_1 = 0$$

$$\frac{11}{4}c_1 = 0$$

$$c_1 = 0$$

Then

$$c_2 = -\frac{3}{4}c_1$$

$$c_2 = -\frac{3}{4}(0)$$

$$c_2 = 0$$

and

$$c_3 = -\frac{1}{2}c_1$$

$$c_3 = -\frac{1}{2}(0)$$

$$c_3 = 0$$

This way too, we see that $(c_1, c_2, c_3) = (0,0,0)$, which means that the vector set $\{v_1, v_2, v_3\}$ is linearly independent.

Let's do another example in which the vector set is linearly dependent.

Linear subspaces

We're already familiar with two-dimensional space, $\mathbb{R}^2$, as the xy-coordinate plane. We can also think of $\mathbb{R}^2$ as the vector space containing all possible two-dimensional vectors, $\vec{v} = (x, y)$.

And we know about three-dimensional space, $\mathbb{R}^3$, which is xyz-space. We can think of $\mathbb{R}^3$ as the vector space containing all possible three-dimensional vectors, $\vec{v} = (x, y, z)$.

And even though it's harder (if not impossible) to visualize, we can imagine that there could be higher-dimensional spaces $\mathbb{R}^4$, $\mathbb{R}^5$, etc., up to any dimension $\mathbb{R}^n$. The vector space $\mathbb{R}^4$ contains four-dimensional vectors, $\mathbb{R}^5$ contains five-dimensional vectors, and $\mathbb{R}^n$ contains n-dimensional vectors.

Definition of a subspace

Notice how we've referred to each of these ($\mathbb{R}^2$, $\mathbb{R}^3$, ... $\mathbb{R}^n$) as a "space." Well, within these spaces, we can define subspaces. To give an example, a **subspace** (or **linear subspace**) of $\mathbb{R}^2$ is a set of two-dimensional vectors within $\mathbb{R}^2$, where the set meets three specific conditions:

1. The set includes the zero vector.

2. The set is closed under scalar multiplication.

3. The set is closed under addition.

A vector set is not a subspace unless it meets these three requirements, so let's talk about each one in a little more detail.

1. First, the set has to **include the zero vector**. For example, if we're talking about a vector set V in $\mathbb{R}^2$, $\overrightarrow{v} = (0,0)$ needs to be a member of the set in order for the set to be a subspace. Or if we're talking about a vector set V in $\mathbb{R}^3$, $\overrightarrow{v} = (0,0,0)$ needs to be a member of the set in order for the set to be a subspace.

2. Second, the set has to be **closed under scalar multiplication**. This means that, for any $\overrightarrow{v}$ in the vector set V, $c\overrightarrow{v}$ must also be in V. In other words, we need to be able to take any member $\overrightarrow{v}$ of the set V, multiply it by any real-number scalar c, and end up with a resulting vector $c\overrightarrow{v}$ that's still in V.

   In contrast, if you can choose a member of V, multiply it by a real number scalar, and end up with a vector outside of V, then by definition the set V is not closed under scalar multiplication, and therefore V is not a subspace.

3. Third, the set has to be **closed under addition**. This means that, if $\vec{s}$ and $\vec{t}$ are both vectors in the set V, then the vector $\vec{s} + \vec{t}$ must also be in V. In other words, we need to be able to take any two members $\vec{s}$ and $\vec{t}$ of the set V, add them together, and end up with a resulting vector $\vec{s} + \vec{t}$ that's still in V. (Keep in mind that what we're really saying here is that any linear combination of the members of V will also be in V.)

In contrast, if you can choose any two members of V, add them together, and end up with a vector outside of V, then by definition the set V is not closed under addition.

To summarize, if the vector set V includes the zero vector, is closed under scalar multiplication, and is closed under addition, then V is a **subspace**.

Keep in mind that the first condition, that a subspace must include the zero vector, is logically already included as part of the second condition, that a subspace is closed under multiplication.

That's because we're allowed to choose any scalar c, and $c\vec{v}$ must also still be in V. Which means we're allowed to choose $c = 0$, in which case $c\vec{v}$ will be the zero vector. Therefore, if we can show that the subspace is closed under scalar multiplication, then automatically we know that the subspace includes the zero vector.

Which means we can actually simplify the definition, and say that a vector set V is a **subspace** when

1. the set is closed under scalar multiplication, and

2. the set is closed under addition.

Let's look at an example of a space which is *not* a subspace.

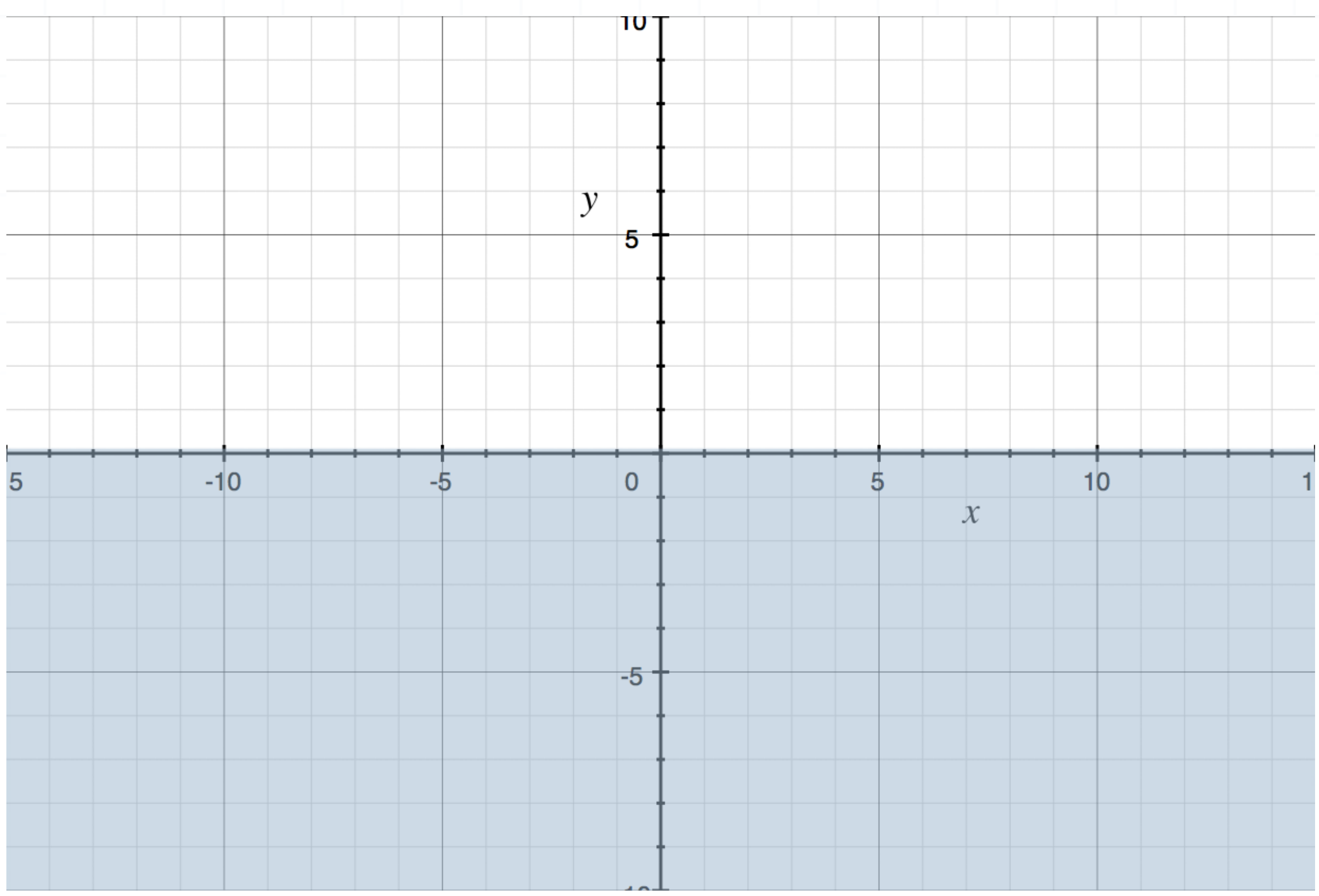
Example

Show that the set is not a subspace of $\mathbb{R}^2$.

$$M = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y \leq 0 \right\}$$

Before we talk about why M is not a subspace, let's talk about how M is defined, since we haven't used this kind of notation very much at this point.

The notation tells us that the set M is all of the two-dimensional vectors (x, y) that are in the plane $\mathbb{R}^2$, where the value of y must be $y \leq 0$. If we show this in the $\mathbb{R}^2$ plane, $y \leq 0$ constrains us to the third and fourth quadrants, so the set M will include all the two-dimensional vectors which are contained in the shaded quadrants:



If we're required to stay in these lower two quadrants, then x can be any value (we can move horizontally along the x-axis in either direction as far as we'd like), but y must be negative to put us in the third or fourth quadrant.

If the set M is going to be a subspace, then we know it includes the zero vector, is closed under scalar multiplication, and is closed under addition. We need to test to see if all three of these are true.

First, we can say M does include the zero vector. That's because there are no restrictions on x, which means it can take any value, including 0, and the restriction on y tells us that y can be equal to 0. Since both x and y can be 0, the vector $\vec{m} = (0,0)$ is a member of M, so M includes the zero vector.

Second, let's check whether M is closed under addition. Let's take two theoretical vectors in M,

$$m_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } m_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

and find their sum.

$$\vec{m}_1 + \vec{m}_2 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\vec{m}_1 + \vec{m}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

Because x_1 and x_2 can both be either positive or negative, the sum $x_1 + x_2$ can be either positive or negative. But because y_1 and y_2 must both be negative, the sum $y_1 + y_2$ can only be negative.

A vector with a negative $x_1 + x_2$ and a negative $y_1 + y_2$ will lie in the third quadrant, and a vector with a positive $x_1 + x_2$ and a negative $y_1 + y_2$ will lie in the fourth quadrant. So the sum $\vec{m}_1 + \vec{m}_2$ still falls within the original set M, which means the set is closed under addition.

Third, and finally, we need to see if M is closed under scalar multiplication. Given a vector in M like

$$\vec{m} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

we need to be able to multiply it by any real number scalar and find a resulting vector that's still inside M. Multiplying $\vec{m} = (2, -3)$ by any positive scalar will result in a vector that's still in M. That's because x will stay positive and y will stay negative, which keeps us in the fourth quadrant.

But multiplying $\vec{m}$ by any negative scalar will result in a vector outside of M! That's because x will become negative (which isn't a problem), but y will become positive, which *is* problem, since a positive y-value will put us outside of the third and fourth quadrants where M is defined. When y becomes positive, the resulting vector lies in either the first or second quadrant, both of which fall outside the set M.

Therefore, while M contains the zero vector and is closed under addition, it is *not* closed under scalar multiplication. And because the set isn't closed under scalar multiplication, the set M is not a subspace of two-dimensional vector space, $\mathbb{R}^2$.

Let's look at another example where the set isn't a subspace.

Example

Show that the set is not a subspace of $\mathbb{R}^2$.

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \ \middle| \ xy = 0 \right\}$$

The vector set V is defined as all the vectors in $\mathbb{R}^2$ for which the product of the vector components x and y is 0. In other words, a vector $v_1 = (1,0)$ is in V, because the product of v_1's components is 0, $(1)(0) = 0$.

Let's try to figure out whether the set is closed under addition. Both v_1 and v_2 are in V.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If we find their sum, we get

$$v_1 + v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_1 + v_2 = \begin{bmatrix} 1 + 0 \\ 0 + 1 \end{bmatrix}$$

$$v_1 + v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The components of $v_1 + v_2 = (1,1)$ do not have a product of 0, because the product of its components are $(1)(1) = 1$. Therefore, v_1 and v_2 are in V, but $v_1 + v_2$ is not in V, which proves that V is not closed under addition, which means that V is not a subspace.

Possible subspaces

In the last example we were able to show that the vector set M is not a subspace. In fact, there are three possible subspaces of $\mathbb{R}^2$.

1. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

2. Any line through the origin $(0,0)$ is a subspace of $\mathbb{R}^2$.

3. The zero vector $\overrightarrow{O} = (0,0)$ is a subspace of $\mathbb{R}^2$.

Similarly, there are four possible subspaces of $\mathbb{R}^3$.

1. $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

2. Any line through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

3. Any plane through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

4. The zero vector $\overrightarrow{O} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

And we could extrapolate this pattern to get the possible subspaces of $\mathbb{R}^n$, as well.

Spans as subspaces

In the last lesson, we looked at the possible subspaces that can exist in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We said that $\mathbb{R}^2$ itself, any line through $(0,0)$, and the zero vector $\vec{O}=(0,0)$ are all subspaces of $\mathbb{R}^2$.

And we said that $\mathbb{R}^3$ itself, any line through $(0,0,0)$, any plane through $(0,0,0)$, and the vector $\vec{O}=(0,0,0)$ are all subspaces of $\mathbb{R}^3$.

With this pattern in mind, we can conclude that every span is a subspace.

Spans are always subspaces

Remember that the span of a vector set is all the linear combinations of that set. The span of any set of vectors is always a valid subspace.

Let's look at an example to see why a span is a subspace.

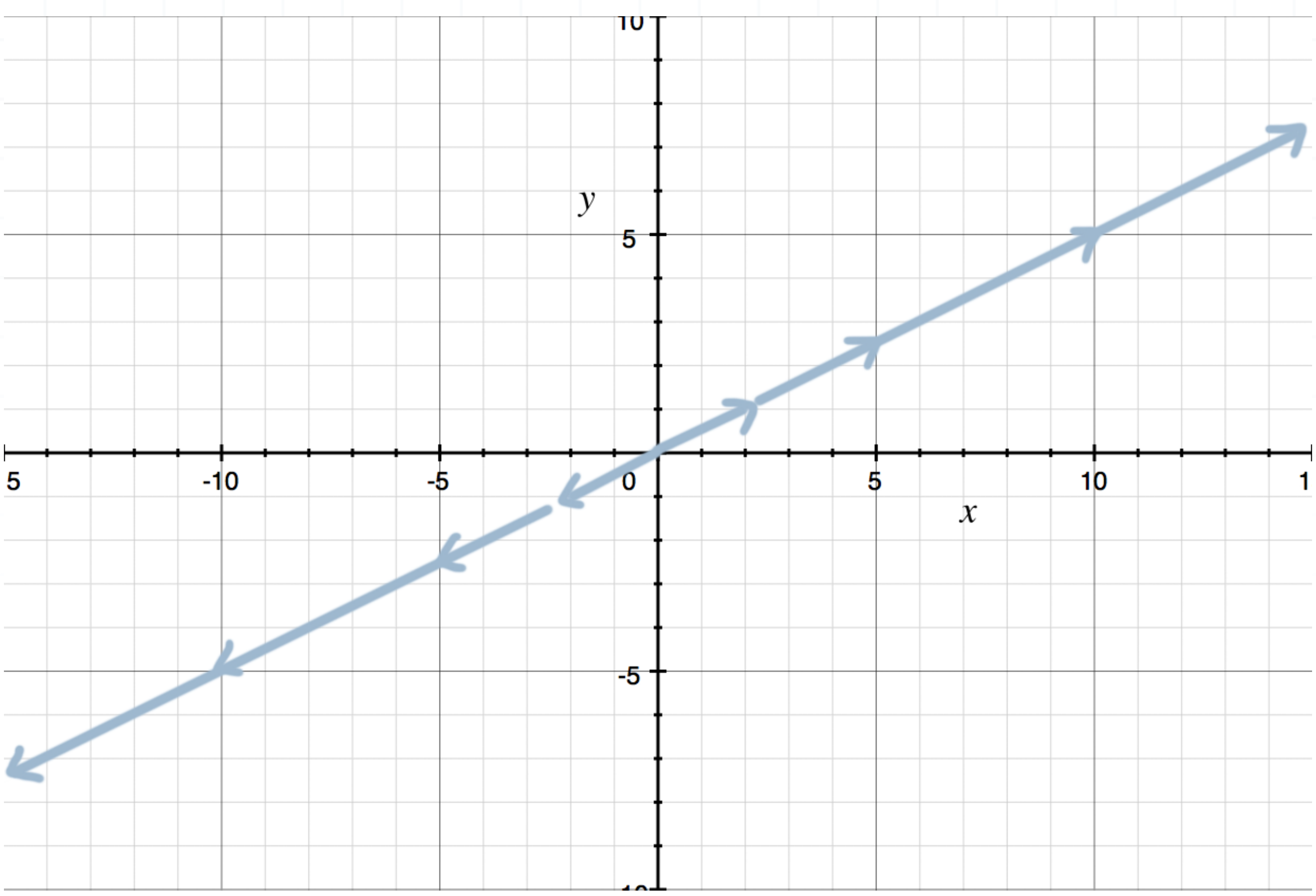
Example

Show that the vector set V is a subspace.

$$V = \text{Span}\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$$

We know that the span of a set of vectors is all of the linear combinations of the vectors in the set. In the set V we only have one vector, so all the linear combinations of the set will only be combinations of the single vector.

You can multiply $\vec{v}=(2,1)$ by any scalar, and/or add and subtract any number of these same vectors, and you'll still get a vector that falls only the same line as the original vector, namely $y=(1/2)x$.



To show that the span represents a subspace, we first need to show that the span contains the zero vector. It does, since multiplying the vector by the scalar 0 gives the zero vector.

Second, we need to show that the span is closed under scalar multiplication. But as we already know, if we multiply the given vector by any scalar, we'll still get a vector that's included in the span, since the resulting vector will still lie along $y=(1/2)x$.

Third, we need to show that the span is closed under addition. Let's imagine that we have a linear combination like this:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can factor out the vector, and rewrite the sum (the linear combination) as

$$(c_1 + c_2 + c_3) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The constant $(c_1 + c_2 + c_3)$ is still just a constant, which means we could rewrite the linear combination as

$$c_4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This result is still just a linear combination of the vectors in the set, which means it's still contained within the span. Therefore, the set is closed under addition.

Because the vector set, which is the span of the single vector, includes the zero vector, is closed under scalar multiplication, and is closed under addition, the span is a subspace.

We can also say that, since the vector set is a line that passes through $(0,0)$, it's a subspace of $\mathbb{R}^2$.

The same logic we used in this example to show that the span was a subspace is a process we can use to show that any span is a subspace.

Basis

The concept of a basis is closely connected to the idea of linear independence. Remember that, in general terms, a set of vectors are linearly independent if none of them are redundant, meaning that none of them can be made from linear combinations of the others. The vectors

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}2\\4\end{bmatrix}$$

are *not* linearly independent (they're linearly dependent), because v_2 is the linear combination $2v_1$. On the other hand,

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix} \text{ and } v_3 = \begin{bmatrix}1\\3\end{bmatrix}$$

are linearly independent, because neither of them can be made with a linear combination of the other.

The difference between span and basis

So we could say, for example, that $\mathbb{R}^2$ is spanned by the vector set V that includes all of these vectors.

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

But this isn't the simplest vector set that we could name to span $\mathbb{R}^2$. We know that the first two vectors are linearly dependent, which means that including $v_2 = (2,4)$ in the set doesn't actually add any new information. We can span $\mathbb{R}^2$ with just v_1 and v_3 alone. So while it's not incorrect to say

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

it's simpler to say

$$\mathbb{R}^2 = \text{Span}(V) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$$

Whenever you have a set of linearly independent vectors that span a vector space like $\mathbb{R}^2$, you can say that the vector set forms a **basis** for that vector space $\mathbb{R}^2$. In other words, if you have a basis for a space, it means you have enough vectors to span the space, but not more than you need. So a vector set is a basis for a space if it

1. spans the space, and

2. is linearly independent.

To take another example, this time in three-dimensional space, both W_1 and W_2 span $\mathbb{R}^3$.

$$\mathbb{R}^3 = \text{Span}(W_1) = \text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

$$\mathbb{R}^3 = \text{Span}(W_2) = \text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}-2\\1\\4\end{bmatrix}\right)$$

But only W_2 forms a basis for $\mathbb{R}^3$. That's because the set of vectors in W_2 is linearly independent, whereas the first two vectors in W_1 are linearly dependent, making one of them redundant and unnecessary. So any set of vectors that is linearly dependent cannot be a basis, since a basis has to consist of only linearly independent vectors.

The basis of the subspace

So what we can say then is that, if any subspace Q is given as the span of a set of vectors (which means the subspace is made up of all of the linear combinations of the set of vectors), and if all the vectors in the set are linearly independent, then the set of vectors is a **basis** for the subspace Q.

Said a different way, if a set of vectors forms the basis of a subspace Q, it means the span of those vectors forms a subspace (which means you can "get to" any vector in the subspace using a linear combination of the vectors in the set), and that the vectors in the set are linearly independent.

So just like before, think about the basis of a subspace as the smallest, or minimum, set of vectors that can span the subspace. There are no "redundant" or "unnecessary" vectors in the set.

Example

Show that the span of the vector set K forms a basis for $\mathbb{R}^2$.

$$K = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

In order for K to form a basis for $\mathbb{R}^2$,

1. the vectors in K need to span $\mathbb{R}^2$, and

2. the vectors in K need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$c_1 = x$$

$$4c_1 + 3c_2 = y$$

We'll substitute $c_1 = x$ into the second equation in the system, and then solve that equation for y.

$$4x + 3c_2 = y$$

$$3c_2 = y - 4x$$

$$c_2 = \frac{1}{3}y - \frac{4}{3}x$$

From this process we can conclude that, given any vector $\vec{v} = (x, y)$ in $\mathbb{R}^2$, we can "get to it" using the values of c_1 and c_2 given by

$$c_1 = x$$

$$c_2 = \frac{1}{3}y - \frac{4}{3}x$$

It doesn't matter which vector we pick in $\mathbb{R}^2$. If we use the values of x and y that we want, and plug them into these equations for c_1 and c_2, we'll get the values of c_1 and c_2 that we need to use in the linear combination in order to arrive at the vector $\vec{v} = (x, y)$. These formulas for c_1 and c_2 won't break, regardless of which (x, y) we pick for the vector, so the vector set K spans $\mathbb{R}^2$.

Then to show that the vectors in K are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$c_1 = 0$$

$$c_2 = \frac{1}{3}(0) - \frac{4}{3}(0), \text{ or } c_2 = 0$$

Because the only values of c_1 and c_2 that give the zero vector are $c_1 = 0$ and $c_2 = 0$, we know that the vectors in K are linearly independent.

Therefore, because the vector set K spans all of $\mathbb{R}^2$, and because the vectors in K are linearly independent, we can say that K forms a basis for $\mathbb{R}^2$.

Standard basis

We've just seen that the specific set K is a basis for $\mathbb{R}^2$. In fact, any set of two linearly independent vectors will form a basis for $\mathbb{R}^2$. In other words, you can pick any two linearly independent vectors in $\mathbb{R}^2$, and as a set, they will form a basis for $\mathbb{R}^2$.

We call the **standard basis** of $\mathbb{R}^2$ the set of unit vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similarly, any three linearly independent vectors in $\mathbb{R}^3$ will form a basis for $\mathbb{R}^3$. The standard basis of $\mathbb{R}^3$ is the set of unit vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If we extrapolate this pattern, what we see is that any set of n linearly independent vectors will form a basis for $\mathbb{R}^n$.

Dot products

We've already learned about the dot product as a method we can use to multiply matrices. The dot product of two two-dimensional vectors is

$$\overrightarrow{u} \cdot \overrightarrow{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2$$

Notice how this dot product formula is really just summing the products of the first components, second components, etc. In other words, we just multiply u_1 by v_1, then add the sum of u_2 and v_2, etc.

And of course, we can expand the dot product concept to matrix multiplication with matrices of any size. (We're not using u here to indicate the unit vector; we're using u as a regular vector name, just like v.)

But the dot product isn't just a matrix multiplication tool. It actually has a definition in and of itself.

The dot product in two dimensions

Remember that we talked earlier about the length of a vector, and we said that the length of a vector in two dimensions is

$$||\overrightarrow{u}|| = \sqrt{u_1^2 + u_2^2}$$

If we dot (take the dot product) a vector with itself, we get

$$\vec{u} \cdot \vec{u} = [u_1 \quad u_2] \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1u_1 + u_2u_2 = u_1^2 + u_2^2$$

The dot product $\vec{u} \cdot \vec{u}$ is $u_1^2 + u_2^2$, which we can see is also the value under the root in the formula for the length of a vector. So if we square both sides of the length formula, we get

$$(||\vec{u}||)^2 = \left(\sqrt{u_1^2 + u_2^2}\right)^2$$

$$||\vec{u}||^2 = u_1^2 + u_2^2$$

And then if we substitute $\vec{u} \cdot \vec{u}$ for $u_1^2 + u_2^2$, we get

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

In words, this formula tells us:

> *"The square of the length of a vector is equal to the vector dotted with itself."*

As we continue working with vectors, this will be a super helpful formula in particular, since it directly relates the length of a vector to its dot product.

Example

Use the dot product to find the length the vector $\vec{v} = (3,4)$.

We know we'll get the square of the length if we dot the vector with itself. Use the formula $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$.

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

$$||\vec{u}||^2 = u_1u_1 + u_2u_2$$

Plugging $\vec{v}$ into this formula gives

$$||\vec{v}||^2 = (3)(3) + (4)(4)$$

$$||\vec{v}||^2 = 9 + 16$$

$$||\vec{v}||^2 = 25$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\vec{v}||^2} = \sqrt{25}$$

$$||\vec{v}|| = 5$$

The length of $\vec{v} = (3,4)$ is 5.

The dot product in n dimensions

We looked at these dot product formulas for two-dimensional vectors, but these formulas are equally valid for three-dimensional vectors, and even n-dimensional vectors. The dot product of two n-dimensional vectors is

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

And if we dot an n-dimensional vector with itself, we get

$$\vec{u} \cdot \vec{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1u_1 + u_2u_2 + \dots + u_nu_n$$

The length of a vector in n-dimensions is

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Just like in two dimensions, the value of a vector dotted with itself is what we see under this square root, so the formula that relates the length of an n-dimensional vector to the dot product of an n-dimensional vector is still the same as the formula in two dimensions:

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

Let's do an example with a higher-dimensional vector.

Example

Use the dot product to find the length the vector $\vec{w} = (2, -3,4,0)$.

We know we'll get the square of the length if we dot the vector with itself. Use the formula $||\overrightarrow{u}||^2 = \overrightarrow{u} \cdot \overrightarrow{u}$.

$$||\overrightarrow{u}||^2 = \overrightarrow{u} \cdot \overrightarrow{u}$$

$$||\overrightarrow{u}||^2 = u_1u_1 + u_2u_2 + u_3u_3 + u_4u_4$$

Plugging $\overrightarrow{w}$ into this formula gives

$$||\overrightarrow{w}||^2 = (2)(2) + (-3)(-3) + (4)(4) + (0)(0)$$

$$||\overrightarrow{w}||^2 = 4 + 9 + 16 + 0$$

$$||\overrightarrow{w}||^2 = 29$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\overrightarrow{w}||^2} = \sqrt{29}$$

$$||\overrightarrow{w}|| = \sqrt{29}$$

The length of $\overrightarrow{w} = (2, -3,4,0)$ is $\sqrt{29}$.

Properties of dot products

Dot products in any dimension are commutative, distributive, and associative, meaning that the following properties all hold:

Commutative

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

Distributive

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(\vec{u} - \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w}$$

Associative

$$(c\vec{u}) \cdot \vec{v} = c(\vec{v} \cdot \vec{u})$$

Let's do a quick example with some of these properties.

Example

Simplify the expression if $\vec{u} = (3,-2)$, $\vec{v} = (-1,4)$, and $\vec{w} = (0,5)$.

$$\vec{w} \cdot (3\vec{u} + 2\vec{v})$$

First, we need to apply the scalars to the vectors to find $3\vec{u}$ and $2\vec{v}$.

$$3\vec{u} = 3(3,-2) = (9,-6)$$

$$2\vec{v} = 2(-1,4) = (-2,8)$$

Then the sum $3\vec{u} + 2\vec{v}$ is

$$3\vec{u} + 2\vec{v} = \begin{bmatrix} 9 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$3\vec{u}+2\vec{v}=\begin{bmatrix}9-2\\-6+8\end{bmatrix}$$

$$3\vec{u}+2\vec{v}=\begin{bmatrix}7\\2\end{bmatrix}$$

Calculate the dot product of $\vec{w}$ and $3\vec{u}+2\vec{v}$.

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=\begin{bmatrix}0 & 5\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix}$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=0(7)+5(2)$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=0+10$$

$$\vec{w}\cdot(3\vec{u}+2\vec{v})=10$$

Cauchy-Schwarz inequality

Now that we understand dot products and some of their properties, and we've learned formulas that are based on the dot product, we can start to explore different applications of the dot product, one of which is a formula called the Cauchy-Schwarz inequality.

The **Cauchy-Schwarz inequality** (also called **Cauchy's inequality**, or **Schwarz's inequality**) tells us that given two nonzero vectors in $\mathbb{R}^n$, the absolute value of the their dot product is always less than or equal to the product of their lengths.

$$|\vec{u}\cdot\vec{v}| \le ||\vec{u}||\ ||\vec{v}||$$

The inequality includes the "less than or equal to" sign, so you might be wondering whether there are specific instances in which the left side of the inequality is less than the right side, and others in which the sides are equivalent. As it turns out, the two sides of the Cauchy-Schwarz inequality are only equal to one another when the two vectors $\vec{u}$ and $\vec{v}$ are collinear, meaning that one vector is a scalar multiple of the other.

$$|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}|| \text{ if and only if } \vec{u} = c\vec{v}$$

If this is not the case (if one vector isn't a scalar multiple of the other), then the left side of the Cauchy-Schwarz inequality will be less than its right side:

$$|\vec{u}\cdot\vec{v}| < ||\vec{u}||\ ||\vec{v}||$$

Testing for linear independence

The Cauchy-Schwarz inequality also gives us another way to test for linear independence.

Remember that we just said that Cauchy-Schwarz will only be an equality, $|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}||$, when the vectors are collinear (when $\vec{u} = c\vec{v}$). We know that collinear vectors are linearly dependent.

So given any $\vec{u}$ and $\vec{v}$,

if $|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}||$, then $\vec{u}$ and $\vec{v}$ are linearly dependent.

if $|\vec{u}\cdot\vec{v}| < ||\vec{u}||\ ||\vec{v}||$, then $\vec{u}$ and $\vec{v}$ are linearly independent.

Of course, knowing whether or not two vectors are linearly independent is important, since only two linearly independent vectors can span a two-dimensional subspace.

Let's work through an example where we use the Cauchy-Schwarz inequality as a test for linear independence.

Example

Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$\vec{u} = (3,4)$ and $\vec{v} = (-6,-8)$

Let's first find the value of the left side of the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|$$

$$|(3)(-6)+(4)(-8)|$$

$$|-18-32|$$

$$|-50|$$

$$50$$

Now find the value of the right side of the Cauchy-Schwarz inequality.

$$||\vec{u}||\ ||\vec{v}||$$

$$\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}$$

$$\sqrt{3^2+4^2}\sqrt{(-6)^2+(-8)^2}$$

$$\sqrt{9+16}\sqrt{36+64}$$

$$\sqrt{25}\sqrt{100}$$

$$5(10)$$

$$50$$

If we plug these two into the Cauchy-Schwarz inequality, we get

$$50=50$$

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\ ||\vec{v}||$$

which tells us that $\vec{u}=(3,4)$ and $\vec{v}=(-6,-8)$ are linearly dependent. We can see the linear dependence also from the fact that each vector is a scalar multiple of the other:

$$\vec{u}=c\vec{v}: \qquad (3,4)=(-1/2)(-6,-8)$$

$$\vec{v}=c\vec{u}: \qquad (-6,-8)=-2(3,4)$$

Therefore, we could conclude that $\vec{u}=(3,4)$ and $\vec{v}=(-6,-8)$ are collinear and do not span $\mathbb{R}^2$.

Vector triangle inequality

The Cauchy-Schwarz inequality comes directly from what we know about the relationship between the dot product of a vector and its length, which we looked at in the lesson about dot products.

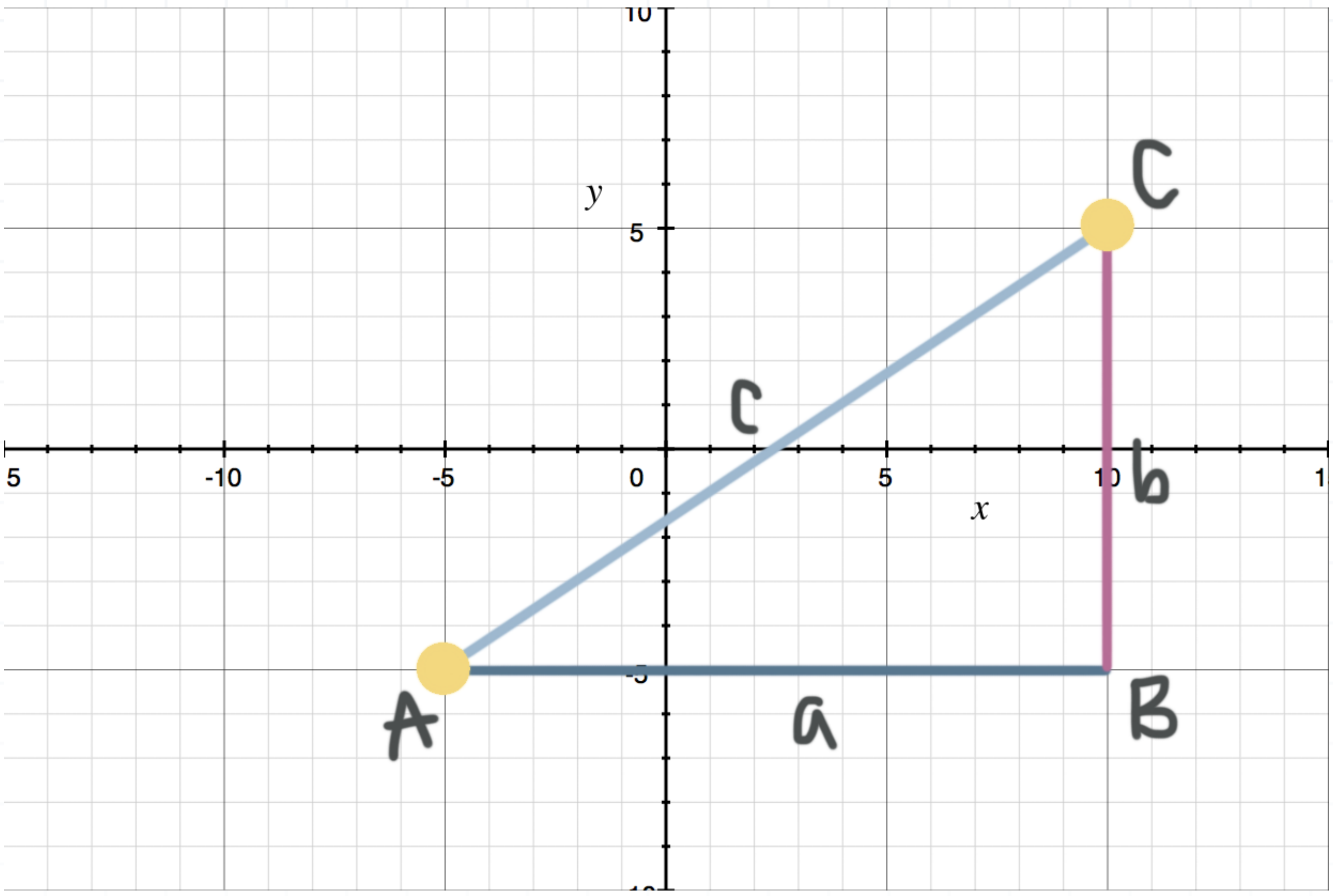
The reason the Cauchy-Schwarz inequality is useful is because, not only can we use it to test for linear independence, but we can build on it to come up with other theorems, including the vector triangle inequality.

The triangle inequality

The vector triangle inequality is identical to the triangle inequality you would have learned in an introductory geometry course, because it says that the sum of the lengths of two sides of a triangle will always be greater than or equal to the length of the third side. When you first learned it in geometry, that inequality looked like this:

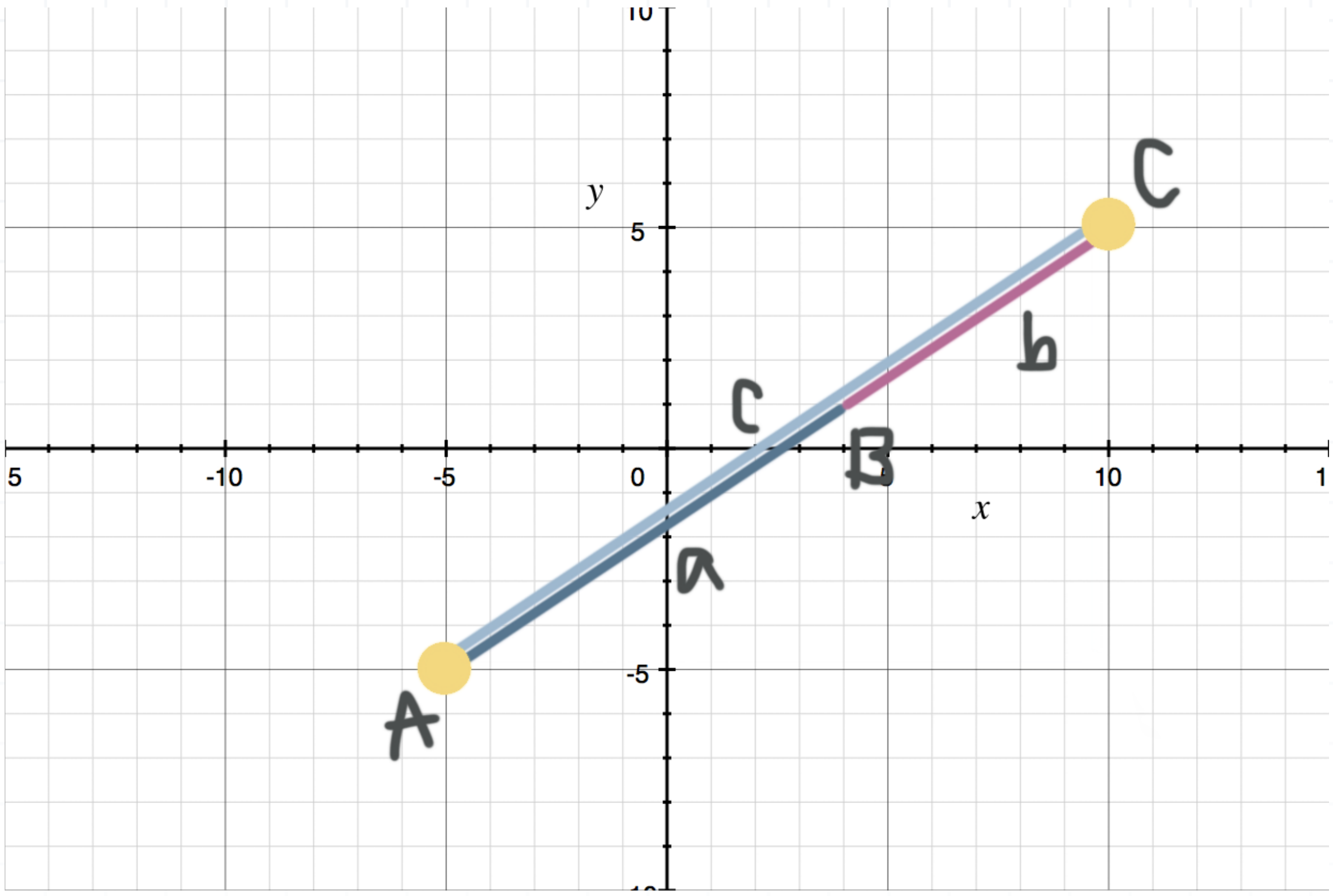
$$c \leq a + b$$

This inequality tells us that the length of c will always be less than or equal to the sum of the lengths of a and b. This makes intuitive sense if we sketch a triangle in the plane.



The length of c is the most direct, efficient route from point A to point C. If we start at A and travel to B first, and then to C, it will take us longer and we'll travel a further distance. So it makes sense that the length of c would naturally be less than the sum of the lengths of a and b.

Only when a and b are collinear is the sum of their lengths equivalent to the length of c.



The vector triangle inequality

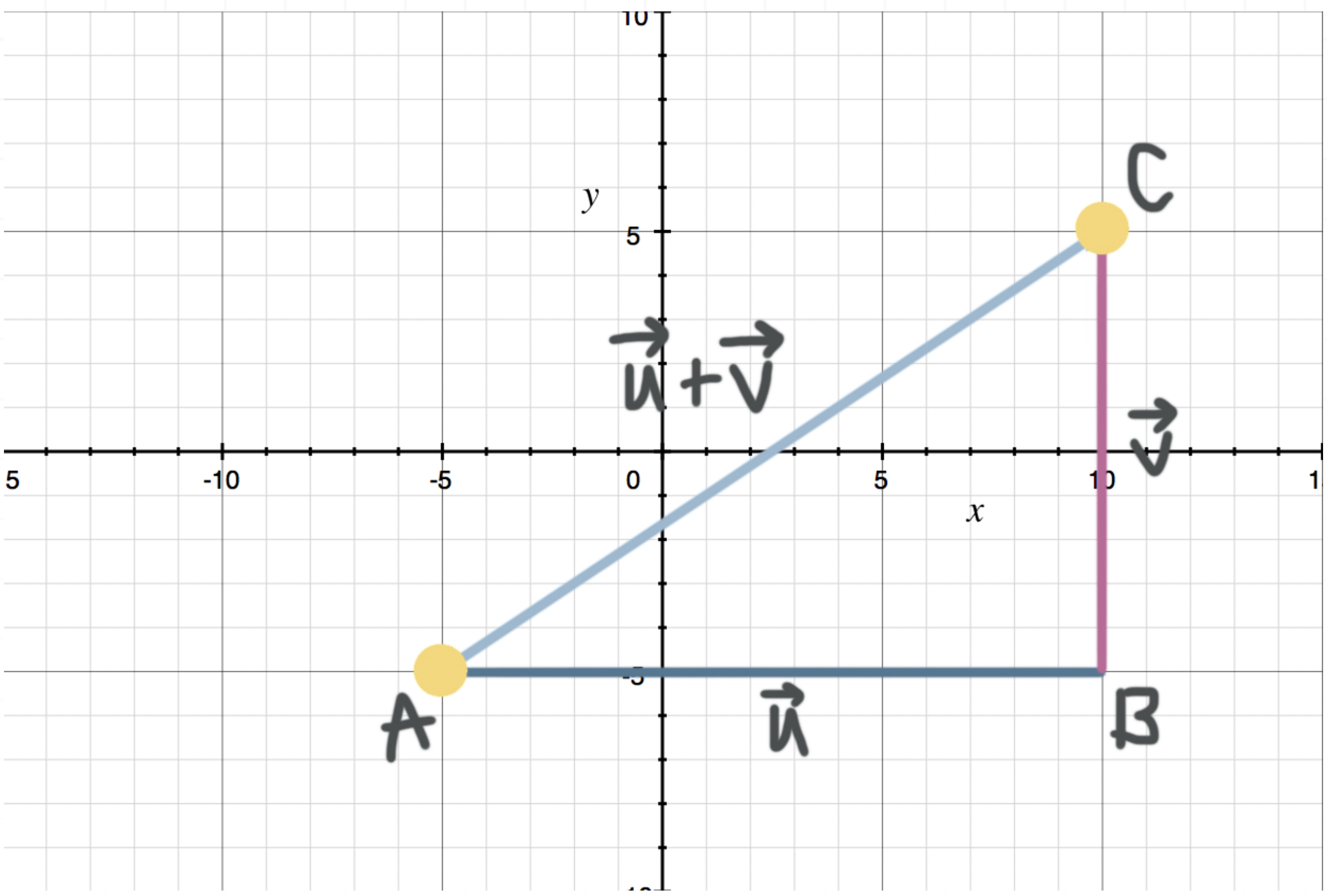
We can translate this triangle inequality so that it's written in terms of vectors, and when we do we call it the **vector triangle inequality**.

As we mentioned before, the reason we're talking about this now is because the vector triangle inequality can be derived directly from the Cauchy-Schwarz inequality.

Just like the basic triangle inequality from geometry, $c \leq a + b$, it tells us that the length of one side of the triangle is always less than or equal to the sum of the lengths of the other two sides.

$$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

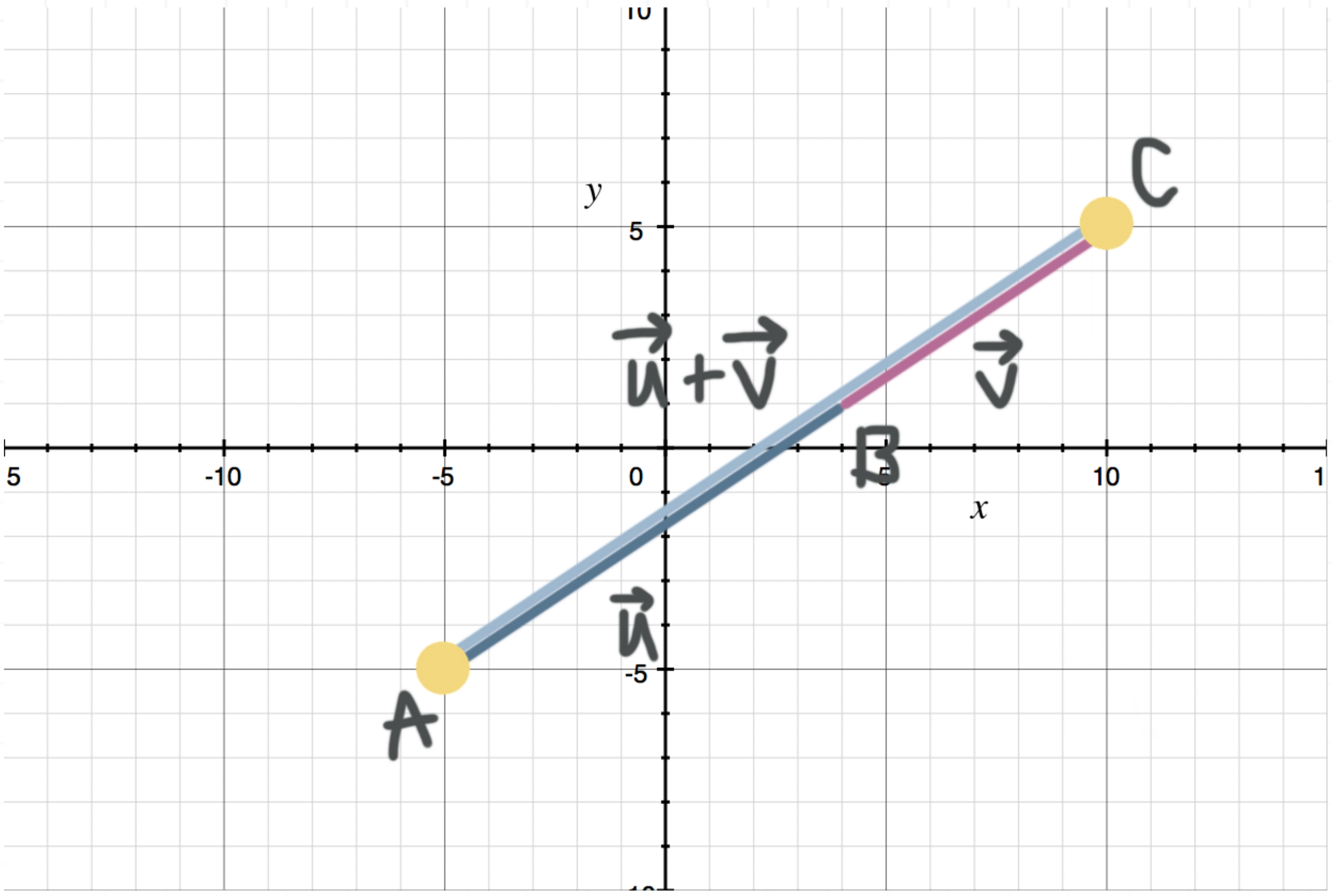
Instead of the triangle $\triangle ABC$ with side lengths a, b, and c, we sketch the triangle whose sides are the vectors $\vec{u}$, $\vec{v}$, and the sum of the vectors $\vec{u}+\vec{v}$.



Just like before, we can see that it's always a shorter, more efficient path to travel along $\vec{u}+\vec{v}$, than it is to travel along $\vec{u}$, and then along $\vec{v}$. Which is why the length of $\vec{u}+\vec{v}$, which we write as $||\vec{u}+\vec{v}||$, will be less than the sum of the lengths of $\vec{u}$ and $\vec{v}$, which we write as $||\vec{u}||+||\vec{v}||$.

And this is true for right triangles (like the ones we're sketching), but also for acute and obtuse triangles. It will always be a shorter path along the single longest side, than the path along the two shorter sides together.

Only when vector $\vec{u}$ and vector $\vec{v}$ are collinear is the sum of their lengths equivalent to the length of the vector $\vec{u}+\vec{v}$.



The triangle inequality for n dimensions

At this point, you might be wondering why we've bothered to redefine the triangle inequality in terms of vectors. After all, $c \leq a+b$ looks much simpler than $||\vec{u}+\vec{v}|| \leq ||\vec{u}||+||\vec{v}||$, and we haven't added any new information to the inequality.

One of the reasons for doing this is because the triangle inequality is only useful in two dimensions. We can't use $c \leq a+b$ in three dimensions or n dimensions.

But the vectors $\vec{u}$ and $\vec{v}$ can be defined in as many dimensions as we choose. Which means that translating the triangle inequality into vector form opens up its usefulness from only two dimensions, to n dimensions, and we can now use the inequality to make conclusions about relationships between vector length in space.

Testing for linear independence

But another reason to know the vector triangle inequality is that, just like the Cauchy-Schwarz inequality, it can be used to test for linear independence.

If two vectors $\vec{u}$ and $\vec{v}$ are collinear and point in the same direction, then the Cauchy-Schwarz inequality will give $||\vec{u}+\vec{v}||=||\vec{u}||+||\vec{v}||$. If the two vectors $\vec{u}$ and $\vec{v}$ are collinear, but they have the same length and point in exactly opposite directions, then $||\vec{u}+\vec{v}||$ will be 0.

Since we know that collinear vectors are linearly dependent, we can use the vector triangle inequality to test the linear (in)dependence of any pair of n-dimensional vectors.

Let's work through an example.

Example

Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$.

$$\vec{u}=(2,-1),\ \vec{v}=(-1,4)$$

First, let's work on finding the value of the right side of the vector triangle inequality.

$$||\vec{u}||+||\vec{v}||$$

$$\sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{2^2+(-1)^2}+\sqrt{(-1)^2+4^2}$$

$$\sqrt{4+1}+\sqrt{1+16}$$

$$\sqrt{5}+\sqrt{17}$$

$$\approx 2.24+4.12$$

$$\approx 6.36$$

Second, let's work on finding the value of the left side of the vector triangle inequality.

$$||\vec{u}+\vec{v}||$$

$$\sqrt{(2-1)^2+(-1+4)^2}$$

$$\sqrt{1^2+3^2}$$

$$\sqrt{1+9}$$

$$\sqrt{10}$$

$$\approx 3.16$$

Since

$$3.16 < 6.36$$

$$||\vec{u} + \vec{v}|| < ||\vec{u}|| + ||\vec{v}||$$

we can conclude that $\vec{u}$ and $\vec{v}$ are not collinear, and that they therefore span $\mathbb{R}^2$.

Angle between vectors

The angle between two vectors, θ, is given by a relationship between the dot product of the vectors and the lengths of the vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

In order to use this formula to find the angle between the vectors, we need both vectors $\vec{u}$ and $\vec{v}$ to be non-zero. That's because, if either vector is the zero vector, we'll be left with the equation $0 = 0\cos\theta$, or just $0 = 0$, and won't be able to solve for the angle between them.

And this makes sense. Because the zero vector has no length and no direction, it wouldn't make sense to look for an angle between the zero vector and another vector. If we're trying to solve for the angle between two vectors, it only makes sense to do so if both of them are non-zero.

Example

Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (2, -1), \ \vec{v} = (-1,4)$$

First, let's find the length of both $\vec{u}$ and $\vec{v}$.

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2} = \sqrt{2^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$$

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2} = \sqrt{(-1)^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}$$

Now we can plug everything into the formula for the angle between vectors.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

$$(2,-1) \cdot (-1,4) = \sqrt{5}\sqrt{17}\cos\theta$$

Calculate the dot product, and simplify.

$$(2)(-1) + (-1)(4) = \sqrt{85}\cos\theta$$

$$-2 - 4 = \sqrt{85}\cos\theta$$

$$-6 = \sqrt{85}\cos\theta$$

$$-\frac{6}{\sqrt{85}} = \cos\theta$$

Take the inverse cosine of each side of the equation to solve for θ.

$$\arccos\left(-\frac{6}{\sqrt{85}}\right) = \arccos(\cos\theta)$$

$$\theta = \arccos\left(-\frac{6}{\sqrt{85}}\right)$$

If we use a calculator to find this arccosine value, we find that the angle between $\vec{u}$ and $\vec{v}$ is $\theta \approx 130.62^\circ$.

Perpendicular vectors

If the vectors are perpendicular, then $\theta = 90^\circ$ (in degrees) or $\theta = \pi/2$ (in radians). If we substitute $\theta = 90^\circ$ into the formula for the angle between vectors, we get an interesting result.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||\cos(90^\circ)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}||(0)$$

$$\vec{u} \cdot \vec{v} = 0$$

We're left with just the dot product of $\vec{u}$ and $\vec{v}$. Which means that, if we calculate a dot product of two non-zero vectors and find that the dot product is 0, we know right away that those vectors are perpendicular.

Notice how we said the two vectors had to be non-zero. Again, if either vector is the zero vector, we'll get 0 for the dot product every time. In other words, if the dot product of two vectors is 0, it means one of two things must be true:

1. at least one of the vectors is the zero vector, or

2. the vectors are perpendicular.

Orthogonality

When we talk about two vectors being perpendicular to one another, it's easy for us to understand this in two dimensions. We picture two vectors that lie at a 90° angle from one another, and we call them perpendicular.

But this concept gets a little murky when we start talking about three dimensions, or even n dimensions. After all, if two vectors are both defined in n dimensions, does it really make much sense to say they're "perpendicular?"

Our intuitive understanding of "perpendicular" really starts to break down in anything higher than two dimensions, so for three or more dimensions, we use a new word to define the idea of "perpendicular," and we'll use the word "orthogonal." If the dot product of two vectors is 0, we say that they're **orthogonal** to one another.

And where before we said that our two vectors both had to be non-zero in order for them to be perpendicular, we no longer exclude the zero vector from the definition once we switch our language from perpendicularity to orthogonality.

So any two vectors whose dot product is 0 are orthogonal (regardless of whether one or both of them is the zero vector). Which means

1. the zero vector is orthogonal to every non-zero vector, and

2. the zero vector is orthogonal to itself.

And of course, just like with perpendicularity, any two non-zero vectors whose dot product is 0 are orthogonal to one another.

Example

Say whether or not the vectors are orthogonal.

$$\vec{u} = (2, -1, 0),\ \vec{v} = (0, 0, 3)$$

If the vectors are orthogonal to one another, then their dot product will be 0. So let's take the dot product of $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = (2, -1, 0) \cdot (0, 0, 3)$$

$$\vec{u} \cdot \vec{v} = (2)(0) + (-1)(0) + (0)(3)$$

$$\vec{u} \cdot \vec{v} = 0 + 0 + 0$$

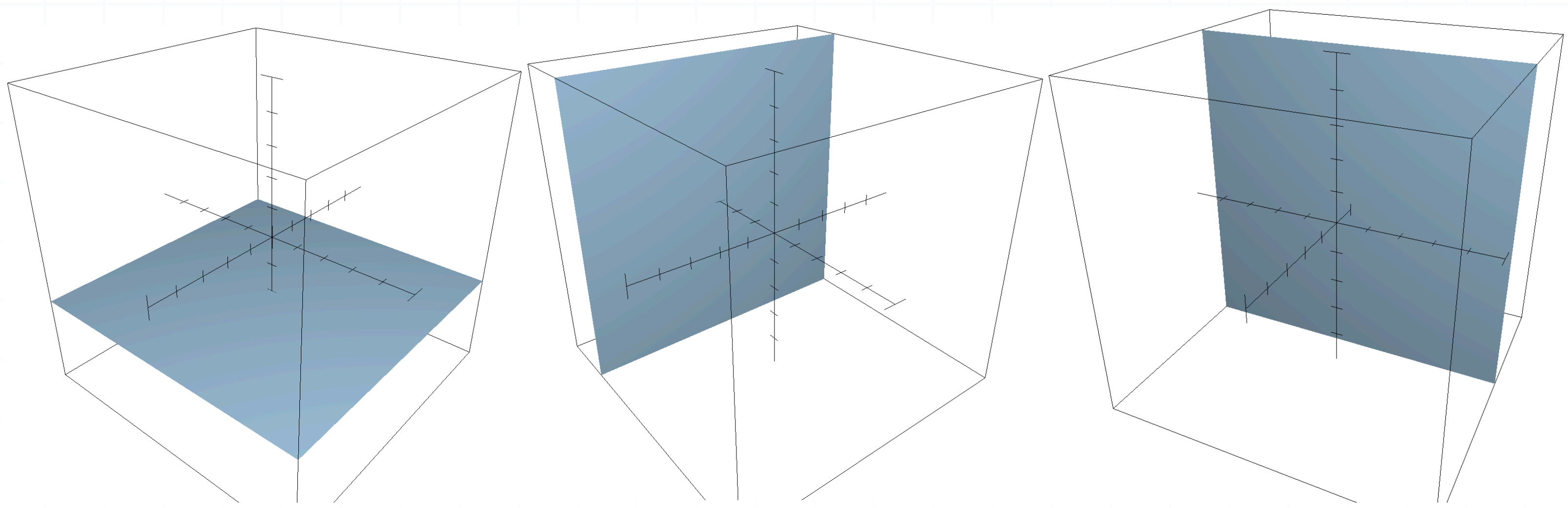
$$\vec{u} \cdot \vec{v} = 0$$

Because the dot product is 0, $\vec{u}$ and $\vec{v}$ are orthogonal to one another.

Equation of a plane, and normal vectors

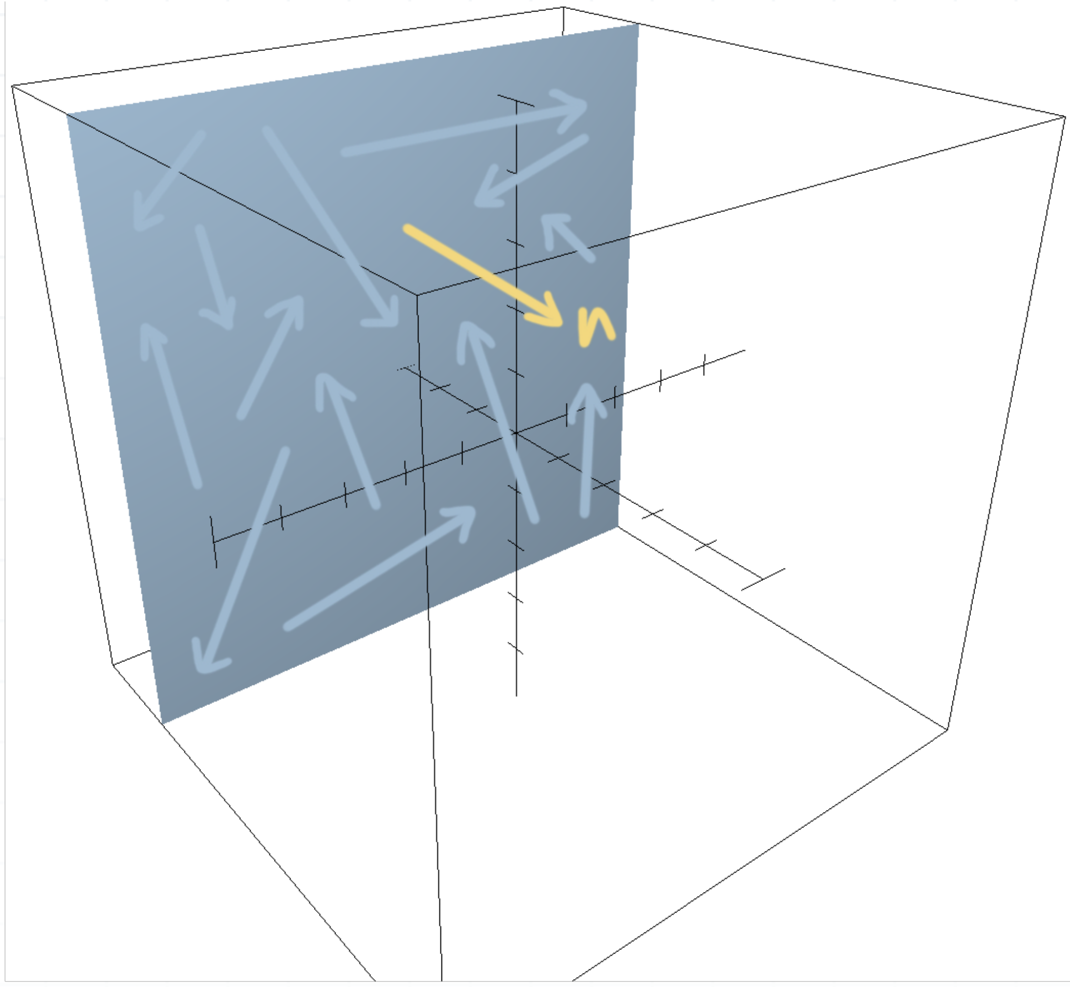
A **plane** is a perfectly flat surface that goes on forever in every direction in three-dimensional space.

Because it's impossible to show a plane going on forever, when we sketch a plane in space, we only sketch a portion of it. These are all examples of planes in space:



Mathematically, we define a **plane** as the set of all vectors that are perpendicular (orthogonal) to one given **normal vector**, which is the vector that's perpendicular (orthogonal) to the plane.

So if you imagine the infinite number of vectors that all lie in the same plane, a normal vector $\vec{n}$ will be perpendicular (orthogonal) to every vector in the plane.



Equation of a plane from the dot product

The standard equation of a plane is given by

$$Ax + By + Cz = D$$

where the normal vector is $\overrightarrow{n} = (A, B, C)$. But you'll also see the equation of a plane written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

The reason this second equation represents a plane is because it's the dot product of the normal vector and a vector in the plane.

Here's why. Remember that two vectors are perpendicular (or orthogonal) when their dot product is 0, which means that we can get the equation of a plane by setting the dot product of the normal vector and a vector in the plane equal to 0.

In other words, let's say that we define two points on the plane (x_0, y_0, z_0) and (x, y, z) with the associated position vectors $\vec{x_0}$ and $\vec{x}$.

$$\vec{x_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Because (x_0, y_0, z_0) and (x, y, z) are both points on the plane, the vector that connects those two points must lie in the plane as well. That vector $(\vec{x} - \vec{x_0})$ is given by

$$(\vec{x} - \vec{x_0}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

If the normal vector to the plane is given by $\vec{n} = (a, b, c)$, then we know that the vector $(\vec{x} - \vec{x_0})$ and the normal vector $\vec{n}$ are orthogonal, which means their dot product is 0.

$$\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is how we get the equation of the plane. Then, given a point on the plane and a normal vector to the plane, the plane's equation will simplify to the standard form $Ax + By + Cz = D$.

For example, given the plane defined by the point $(2,5,-1)$ and the normal vector $\vec{n} = (-1,-2,1)$, the plane's equation would be

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$a(x - 2) + b(y - 5) + c(z - (-1)) = 0$$

$$-1(x - 2) - 2(y - 5) + 1(z - (-1)) = 0$$

$$-x + 2 - 2y + 10 + z + 1 = 0$$

$$-x - 2y + z + 13 = 0$$

$$-x - 2y + z = -13$$

Notice how this plane equation matches the standard equation of the plane, $Ax + By + Cz = D$, with $A = -1$, $B = -2$, $C = 1$, and $D = -13$.

Let's do an example where we build the equation of a plane from its normal vector and a point that lies in the plane.

Example

Find the equation of a plane with normal vector $\vec{n} = (-3,2,5)$ and that passes through $(1,0,-2)$.

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-3(x - 1) + 2(y - 0) + 5(z - (-2)) = 0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-3x + 3 + 2y + 5z + 10 = 0$$

$$-3x + 2y + 5z + 13 = 0$$

$$-3x + 2y + 5z = -13$$

This is the equation of the plane in standard form, with $A = -3$, $B = 2$, $C = 5$, and $D = -13$.

Identifying the normal vector

We've looked at how to build the equation of a plane when we start with a point on the plane and a normal vector to the plane. But we can also work backwards. When we start with the equation of the plane in standard form, we can identify the normal vector from that equation.

Given a plane $Ax + By + Cz = D$, the normal vector to that plane is

$$\overrightarrow{n} = (A, B, C)$$

Because the value of D doesn't change the orientation of the plane, it only shifts it, changing the value of D won't change the normal vector. So the normal vector

$$\vec{n} = (A, B, C)$$

would be normal to all of these planes:

$$Ax + By + Cz = 0$$

$$Ax + By + Cz = \pi$$

$$Ax + By + Cz = -7$$

$$Ax + By + Cz = e$$

This is pretty straightforward, but let's do a quick example just to make sure we know how to grab the normal vector.

Example

Find the normal vector to the plane.

$$-3x + 2y + 5z = -13$$

This is the same plane we found in the last example, so we already know the components of the normal vector. But if we didn't know the normal, we could still simply pick it out of the equation of the plane.

Pulling out the coefficients on x, y, and z, the normal vector is

$$\overrightarrow{n} = (-3,2,5)$$

Notice how the plane equation could be multiplied through by -1, and therefore be rewritten as

$$-1(-3x + 2y + 5z) = -1(-13)$$

$$3x - 2y - 5z = 13$$

This is still the same plane, so another normal vector to the plane could be

$$\overrightarrow{n} = (3, -2, -5)$$

Furthermore, we could multiply both sides of the equation by any other scalar, and we'd get another normal vector. All of these are normal vectors to the same plane, some of which are pointing away from the plane on one side, and others of which are pointing away from the plane on the other side.

Cross products

The **cross product** is a vector that's orthogonal to the two vectors you crossed. So given $\overrightarrow{a}=(a_1,a_2,a_3)$ and $\overrightarrow{b}=(b_1,b_2,b_3)$, their cross product $\overrightarrow{a}\times\overrightarrow{b}$ will be orthogonal to both $\overrightarrow{a}$ and $\overrightarrow{b}$. The formula for the cross product is

$$\overrightarrow{a}\times\overrightarrow{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Notice how the value of the cross product looks like it's given as a matrix, but the matrix is bounded by straight lines instead of bracketed lines. That's because it's not actually a matrix, it's a determinant.

We'll devote an entire lesson section to determinants later on, but for now, we just want to know how to calculate a determinant.

Calculating a determinant

Notice first how the values in the top row of the 3×3 determinant become coefficients on the 2×2 determinants.

Second, notice how $\mathbf{i}$ is positive, $\mathbf{j}$ is negative, and $\mathbf{k}$ is positive. That's because the signs of the coefficients follow a checkerboard pattern. If you're pulling the value in the upper left-hand corner of the determinant, the sign will be positive, but then the signs alternate from there. So $\mathbf{i}$ is positive, $\mathbf{j}$ is negative, and $\mathbf{k}$ is positive because the checkerboard of signs is

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Third, the 2×2 determinants we attach to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are given by the values outside the row and column of the coefficient.

In other words, $\mathbf{i}$ is in the first row and first column, so eliminate the first row and column, and the 2×2 determinant with $\mathbf{i}$ is just everything left over (everything outside the first row and first column).

$$\begin{vmatrix} \mathbf{i} & \cdot & \cdot \\ \cdot & a_2 & a_3 \\ \cdot & b_2 & b_3 \end{vmatrix}$$

And $\mathbf{j}$ is in the first row and second column, so eliminate the first row and second column, and the 2×2 determinant with $\mathbf{j}$ is just everything left over (everything outside the first row and second column).

$$\begin{vmatrix} \cdot & \mathbf{j} & \cdot \\ a_1 & \cdot & a_3 \\ b_1 & \cdot & b_3 \end{vmatrix}$$

Also, $\mathbf{k}$ is in the first row and third column, so eliminate the first row and third column, and the 2×2 determinant with $\mathbf{k}$ is just everything left over (everything outside the first row and third column).

$$\begin{vmatrix} \cdot & \cdot & \mathbf{k} \\ a_1 & a_2 & \cdot \\ b_1 & b_2 & \cdot \end{vmatrix}$$

Let's do an example where we find the cross product of two three-dimensional vectors.

Example

Find the cross product of $\overrightarrow{a} = (1,2,3)$ and $\overrightarrow{b} = (-1,0,3)$.

The cross product would be

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & 3 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}((2)(3) - (3)(0)) - \mathbf{j}((1)(3) - (3)(-1)) + \mathbf{k}((1)(0) - (2)(-1))$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(6 - 0) - \mathbf{j}(3 + 3) + \mathbf{k}(0 + 2)$$

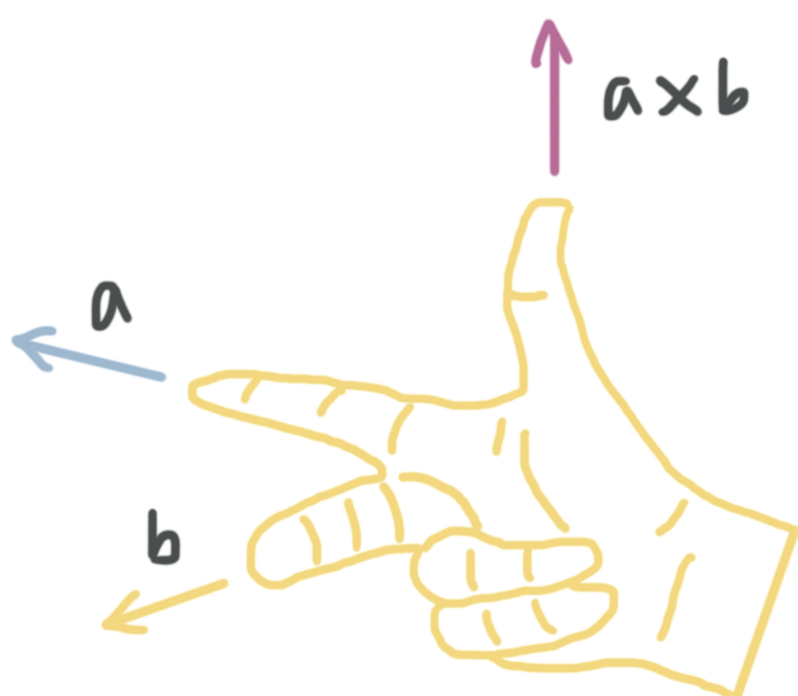
$$\overrightarrow{a} \times \overrightarrow{b} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

So the vector $\vec{a} \times \vec{b} = (6, -6, 2)$ is orthogonal to both $\vec{a} = (1,2,3)$ and $\vec{b} = (-1,0,3)$.

The right-hand rule

Remember that an orthogonal vector can point in multiple directions. For instance, given a vector in two-dimensional space that points toward the positive direction of the x-axis, an orthogonal vector could point toward the positive direction of the y-axis, or toward the negative direction of the y-axis.

In general, the direction of the cross product vector is determined by the **right-hand rule**. If you find the cross product $\vec{a} \times \vec{b}$, it'll point in the direction that your right-hand thumb points, when your right-hand index finger points toward $\vec{a}$ and your right-hand middle finger points toward $\vec{b}$.



Since there are multiple vectors orthogonal to $\vec{a}$ and $\vec{b}$, the right-hand rule is important for determining the correct cross product.

Length of the cross product vector

Earlier we looked at a formula relating the dot product of two vectors to the product of their lengths:

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}|| \cos\theta$$

There's a similar formula for the cross product, which says that the length of the cross product vector is equivalent to the product of the two individual vectors and the sine of the angle between them.

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}|| \sin\theta$$

The length of the cross product is also equivalent to the area of the parallelogram that's formed by the two vectors that were crossed to form the cross product.

Let's do an example of how to find the length of the cross product.

Example

Find the length of the cross product of $\vec{a} = (1,2,3)$ and $\vec{b} = (-1,0,3)$.

First, let's find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{(-1)^2 + 0^2 + 3^2} = \sqrt{1 + 0 + 9} = \sqrt{10}$$

The angle between the vectors will be

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \ ||\vec{b}|| \cos\theta$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \sqrt{14}\sqrt{10} \cos\theta$$

$$(1)(-1) + (2)(0) + (3)(3) = \sqrt{140} \cos\theta$$

$$-1 + 0 + 9 = 2\sqrt{35} \cos\theta$$

$$\frac{8}{2\sqrt{35}} = \cos\theta$$

$$\cos\theta = \frac{4}{\sqrt{35}}$$

$$\theta = \arccos\left(\frac{4}{\sqrt{35}}\right)$$

$$\theta \approx 47.46^\circ$$

Then the length of the cross product is given by

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| \ ||\vec{b}|| \sin\theta$$

$$||\vec{a} \times \vec{b}|| = \sqrt{14}\sqrt{10} \sin(47.46^\circ)$$

$$||\overrightarrow{a}\times\overrightarrow{b}||=\sqrt{140}\sin(47.46^\circ)$$

$$||\overrightarrow{a}\times\overrightarrow{b}||=2\sqrt{35}\sin(47.46^\circ)$$

Using a calculator, we see that the length of the cross product of $\overrightarrow{a}=(1,2,3)$ and $\overrightarrow{b}=(-1,0,3)$ is

$$||\overrightarrow{a}\times\overrightarrow{b}||\approx 8.72$$

We could have also tackled this last example by starting with the cross product vector itself, $\overrightarrow{a}\times\overrightarrow{b}=6\mathbf{i}-6\mathbf{j}+2\mathbf{k}$, which we found in the previous example. If we have the cross product, then all we have to do is plug its components into the formula for the length of a vector.

$$||\overrightarrow{a}\times\overrightarrow{b}||=\sqrt{6^2+(-6)^2+2^2}$$

$$||\overrightarrow{a}\times\overrightarrow{b}||=\sqrt{36+36+4}$$

$$||\overrightarrow{a}\times\overrightarrow{b}||=\sqrt{76}$$

$$||\overrightarrow{a}\times\overrightarrow{b}||\approx 8.72$$

Dot and cross products as opposite ideas

At this point, we know how to calculate dot products and cross products, and we know we can use them to find angles between vectors, test for linear independence, etc.

Now we want to look at what dot products and cross products tell us in general, and how they really represent totally opposite ideas.

What the dot product measures

The **dot product** measures how much two vectors move in the same direction. We can see this somewhat if we look at the formula for the angle between two vectors.

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos\theta$$

When two vectors are perpendicular, the angle between them is 90° (in degrees, or $\pi/2$ in radians). At that angle, $\cos\theta$ is 0, which means we get

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos(90^\circ)$$

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||(0)$$

$$\vec{u}\cdot\vec{v}=0$$

When two vectors point in the same direction, the angle between them is 0° (in degrees, or 0 in radians). At that angle, $\cos\theta$ is 1, which means we get

$$\vec{u}\cdot\vec{v}=||\vec{u}||\ ||\vec{v}||\cos(0^\circ)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||(1)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||$$

When two vectors point in exactly opposite directions, the angle between them is 180° (in degrees, or π in radians). At that angle, $\cos\theta$ is -1, which means we get

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\cos(180^\circ)$$

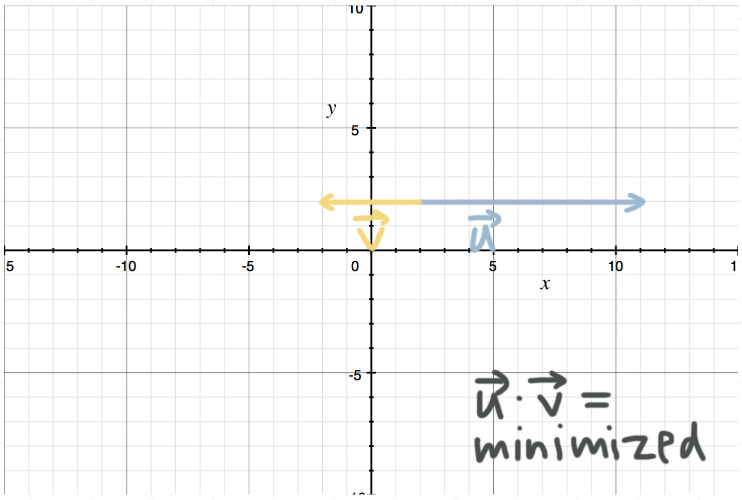
$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||(-1)$$

$$\vec{u} \cdot \vec{v} = -\ ||\vec{u}||\ ||\vec{v}||$$

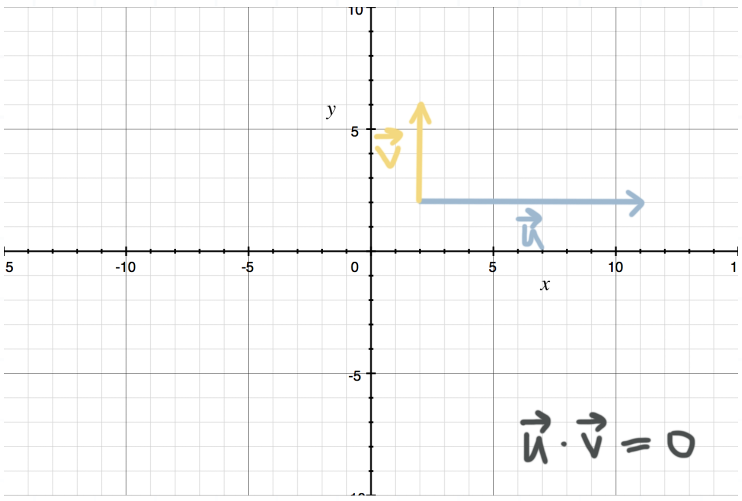
So what we really see here is that the dot product will have its minimum value when vectors point in exactly opposite directions, that the dot product will be 0 when the vectors are orthogonal, and that the dot product will have its maximum value when vectors point in exactly the same direction. When we say that the dot product is maximized, what we really mean is that the dot product, given by $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||$, is just the product of the lengths of the vectors.

Here's a summary of these dot product values:

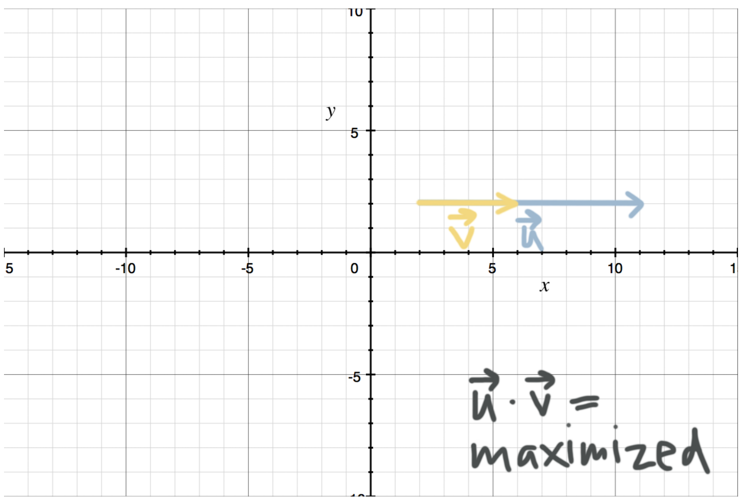
Minimum (and negative) dot product when the vectors point in opposite directions:



Dot product of 0 when the vectors are orthogonal:



Maximum (and positive) dot product when the vectors point in the same direction:



What the cross product measures

The **cross product** measures how much two vectors move in different directions. We can see this somewhat if we look at the formula for the length of the cross product.

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin\theta$$

When two vectors are perpendicular, the angle between them is 90° (in degrees, or $\pi/2$ in radians). At that angle, $\sin\theta$ is 1, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(90^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(1)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||$$

When two vectors point in the same direction, the angle between them is 0° (in degrees, or 0 in radians). At that angle, $\sin\theta$ is 0, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(0^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(0)$$

$$||\vec{u}\times\vec{v}||=0$$

When two vectors point in opposite directions, the angle between them is 180° (in degrees, or π in radians). At that angle, $\sin\theta$ is 0, which means we get

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin(180^\circ)$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(0)$$

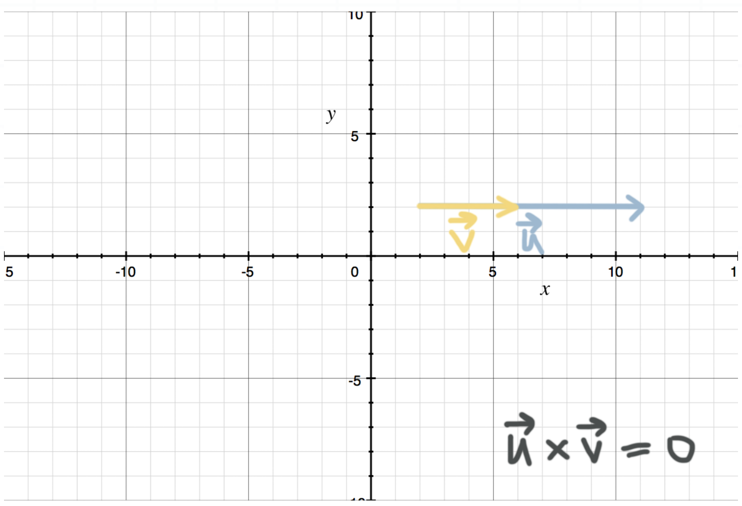
$$||\vec{u}\times\vec{v}||=0$$

So what we really see here is that the length of the cross product is minimized ($||\vec{u} \times \vec{v}|| = 0$) when the vectors are collinear (when they point in the same or opposite directions), and that the length of the cross product is maximized ($||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||$) when the vectors are orthogonal.

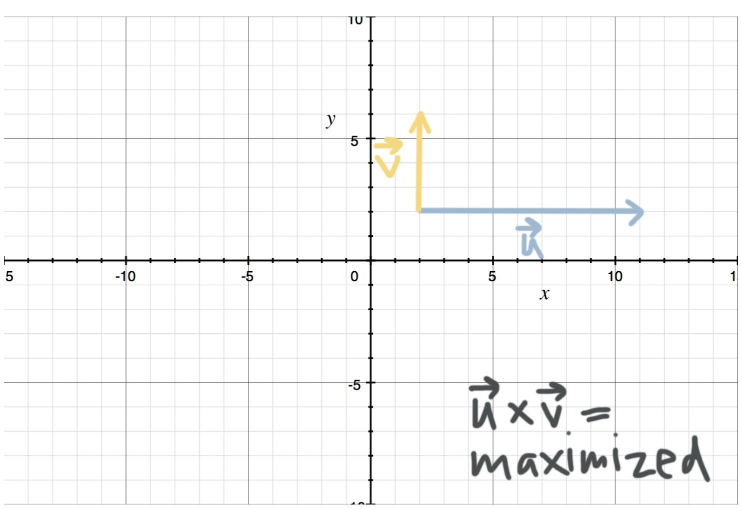
When we say that the length of the cross product is maximized, what we really mean is that the length of the cross product, given by $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \ ||\vec{v}||$, is just the product of the lengths of the vectors.

Here's a summary of these cross product values:

Cross product of 0 when the vectors are collinear:



Maximum cross product when the vectors are orthogonal:



Let’s work through an example of where we find the dot and cross products.

Example

Find the dot product and the length of the cross product of $\vec{u}=(2,0)$ and $\vec{v}=(-4,0)$. Then interpret the results based on what the dot and cross products indicate.

The vector $\vec{u}=(2,0)$ points toward the positive direction of the x-axis. The vector $\vec{v}=(-4,0)$ points toward the negative direction of the x-axis. Which means the angle between them is $\theta=180^\circ$. The length of $\vec{u}=(2,0)$ is 2, and the length of $\vec{v}=(-4,0)$ is 4.

So the dot product is

$$\vec{u}\cdot\vec{v}=\begin{bmatrix}2 & 0\end{bmatrix}\cdot\begin{bmatrix}-4\\0\end{bmatrix}$$

$$\vec{u}\cdot\vec{v}=2(-4)+0(0)$$

$$\vec{u}\cdot\vec{v}=-8$$

And the cross product is

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin\theta$$

$$||\vec{u}\times\vec{v}||=(2)(4)\sin(180^\circ)$$

$$||\vec{u}\times\vec{v}||=(2)(4)(0)$$

$$||\overrightarrow{u} \times \overrightarrow{v}|| = 0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\overrightarrow{u} \cdot \overrightarrow{v} = -8$. The fact that the dot product is negative tells us that the vectors point in exactly opposite directions along the same line.

If the vectors were collinear and the dot product was positive, we'd know that the vectors pointed in exactly the same direction along the same line.

Multiplying matrices by vectors

We already learned how to multiply matrices. As a reminder, matrix multiplication is not commutative, so if A and B are matrices, then $AB \neq BA$. Order matters.

Furthermore, we learned that the number of rows in the second matrix had to equal the number of columns in the first matrix in order for the product to be defined.

Vectors as matrices

And we know that vectors can be written as column matrices (with one column and any finite number of rows) or as row matrices (with one row and any finite number of columns).

That being said, hopefully it won't surprise you too much to say that we can multiply matrices by vectors. After all, we've already been writing vectors as column matrices. So if before we multiplied matrix A by matrix B,

$$AB = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(1) + 0(0) - 2(-2) + 3(0) \\ 0(1) + 4(0) + 3(-2) - 1(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1+0+4+0 \\ 0+0-6-0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

we could instead multiply matrix A by vector $\vec{b}$:

$$A\vec{b} = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$A\vec{b} = \begin{bmatrix} 1(1)+0(0)-2(-2)+3(0) \\ 0(1)+4(0)+3(-2)-1(0) \end{bmatrix}$$

$$A\vec{b} = \begin{bmatrix} 1+0+4+0 \\ 0+0-6-0 \end{bmatrix}$$

$$A\vec{b} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

Both operations (multiplying a matrix by a matrix, and multiplying a matrix by a vector) are valid operations in which we get the same result. And of course, that's because a vector is like a one column matrix or a one row matrix.

We'll need to be really comfortable finding matrix-vector products, so let's do an example.

Example

Find the matrix-vector product, $M\vec{v}$.

$$M = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v} = (-2,1,3)$$

To find $M\vec{v}$, we'll multiply the matrix M by the column vector $\vec{v}$. We know the product is defined, since the matrix has 3 columns and the vector has 3 rows.

$$M\vec{v} = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} 4(-2) - 2(1) + 1(3) \\ 0(-2) + 3(1) + 0(3) \\ -6(-2) + 1(1) - 2(3) \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} -8 - 2 + 3 \\ 0 + 3 + 0 \\ 12 + 1 - 6 \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} -7 \\ 3 \\ 7 \end{bmatrix}$$

Notice how, in this last example, multiplying the matrix by a column vector resulted in another column vector.

This will always be the case. If we multiply an $m \times n$ matrix by an n-row column vector, the result will be an m-row column vector.

On the other hand, if we multiply an m-column row vector by an $m \times n$ matrix, the result will be an n-column row vector. Let's show that by flipping around the last example.

Example

Find the matrix-vector product, $\vec{v}M$.

$$M = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v} = (-2,1,3)$$

To find $\vec{v}M$, we'll multiply the row vector $\vec{v}$ by the matrix M. We know the product is defined, since the vector has 3 columns and the matrix has 3 rows.

$$\vec{v}M = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 0 \\ -6 & 1 & -2 \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -2(4) + 1(0) + 3(-6) & -2(-2) + 1(3) + 3(1) & -2(1) + 1(0) + 3(-2) \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -8 + 0 - 18 & 4 + 3 + 3 & -2 + 0 - 6 \end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix} -26 & 10 & -8 \end{bmatrix}$$

The null space and Ax=O

In the last lesson, we talked about matrix-vector products. In this lesson we want to focus on one really important matrix-vector product, $A\vec{x}=\vec{O}$, where A is a matrix, $\vec{x}$ is a vector, and $\vec{O}$ is the zero vector.

The null space

We talked before about the definition of a subspace, and we said that a set of vectors S was only a subspace if the following three things were all true:

1. S contains the zero vector.

2. S is closed under addition.

3. S is closed under scalar multiplication.

In this lesson, we want to define the **null space**, which is the specific subspace in which the entire universe of vectors $\vec{x}$ in $\mathbb{R}^n$ satisfy the homogeneous equation $A\vec{x}=\vec{O}$ (homogeneous because one side of the equation is 0), where A is any matrix with dimensions $m \times n$.

We indicate the null space with the capital N, and we write

$$N=\left\{\vec{x}\in\mathbb{R}^n \ \mid \ A\vec{x}=\vec{O}\right\}$$

This set notation says that the null space N is all the vectors $\vec{x}$ in $\mathbb{R}^n$ that satisfy $A\vec{x}=\vec{O}$. If we use the matrix A, then we call the null space “the null space of A," or $N(A)$; if we use the matrix B, then we call the null space "the null space of B," or $N(B)$, etc.

The null space as a subspace

The null space N satisfies each of the three conditions of a subspace, so the null space is always a subspace.

First, N contains the zero vector. For instance, an example of the matrix A multiplied by the zero vector could look like this:

$$A\vec{x}=\begin{bmatrix} a_{(1,1)} & a_{(1,2)} & \dots & a_{(1,n)} \\ a_{(2,1)} & \cdot & & \\ \vdots & & \cdot & \cdot \\ a_{(m,1)} & & & a_{(m,n)} \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix} a_{(1,1)}(0)+a_{(1,2)}(0)+\dots+a_{(1,n)}(0) \\ a_{(2,1)}(0)+a_{(2,2)}(0)+\dots+a_{(2,n)}(0) \\ \vdots \\ a_{(m,1)}(0)+a_{(m,2)}(0)+\dots+a_{(m,n)}(0) \end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix} 0+0+\dots+0 \\ 0+0+\dots+0 \\ \vdots \\ 0+0+\dots+0 \end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

$$A\vec{x}=\vec{O}$$

Because we get the zero vector when we multiply any matrix A by the zero vector $\vec{x}$, the null space N contains the zero vector.

Second, the null space is closed under addition. Remember that in order to be closed under addition, the sum of any two vectors in the subspace must also be in the subspace. So for instance, if we pick any two vectors $\vec{x}_1$ and $\vec{x}_2$ that are in the null space, then the sum $\vec{x}_1+\vec{x}_2$ must also be in the null space.

We can show that $\vec{x}_1+\vec{x}_2$ is in the null space if $\vec{x}_1$ and $\vec{x}_2$ are in the null space. Since we chose $\vec{x}_1$ and $\vec{x}_2$ as vectors in the null space, that means they satisfy $A\vec{x}=\vec{O}$, which means both of these equations are true:

$$A\vec{x}_1=\vec{O}$$

$$A\vec{x}_2=\vec{O}$$

If the vector $\vec{x}_1+\vec{x}_2$ is also in the null space, then $A(\vec{x}_1+\vec{x}_2)=\vec{O}$. We know that matrix multiplication (and by extension matrix-vector multiplication, since vectors can be written as matrices) is distributive, which means we can rewrite the equation.

$$A(\vec{x}_1+\vec{x}_2)=\vec{O}$$

$$A\vec{x}_1 + A\vec{x}_2 = \vec{O}$$

We already know that $A\vec{x}_1 = \vec{O}$ and $A\vec{x}_2 = \vec{O}$, so this equation becomes

$$\vec{O} + \vec{O} = \vec{O}$$

$$\vec{O} = \vec{O}$$

So we've shown that the null space N contains $\vec{x}_1 + \vec{x}_2$ if it contains $\vec{x}_1$ and $\vec{x}_2$, which means the null space is closed under addition.

Third, the null space is closed under scalar multiplication. If we again take a vector in N, called $\vec{x}_1$, then $c\vec{x}_1$ must also be in N. Which means $c\vec{x}_1$ has to satisfy $A\vec{x} = \vec{O}$, such that

$$Ac\vec{x}_1 = \vec{O}$$

This equation can always be rewritten as

$$cA\vec{x}_1 = \vec{O}$$

And we already established that $A\vec{x}_1 = \vec{O}$, so we get

$$c\vec{O} = \vec{O}$$

$$\vec{O} = \vec{O}$$

And now we've shown that the null space contains $c\vec{x}_1$ if it contains $\vec{x}_1$, which means the null space is closed under scalar multiplication.

Because the null space always contains the zero vector, is always closed under addition, and is always closed under scalar multiplication, the null space N will always be a valid subspace of $\mathbb{R}^n$.

We'll talk more about this in the next lesson, but let's walk through a simple example of how to show that a vector is in the null space.

Example

Show that $\vec{x}=(2,4)$ is in the null space of A.

$$A=\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}$$

If $\vec{x}=(2,4)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation here, we should get the zero vector.

$$\begin{bmatrix}4(2)-2(4)\\ 2(2)-1(4)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}8-8\\ 4-4\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Because we get a true equation, we know that $\vec{x}=(2,4)$ is in the null space of A.

Linear independence and the null space

The null space of a matrix A tells us about the linear independence of the column vectors of A.

- If the null space $N(A)$ includes only the zero vector, then the columns of A are linearly independent.

- If the null space $N(A)$ includes any other vector in addition to the zero vector, then the columns of A are linearly dependent.

And the reasoning for that makes sense. If there's a nonzero vector in the null space, it tells us that there's a linear combination of the columns of A that gives the zero vector, which means that the columns of A can only be linearly dependent.

The opposite is also true. If all of the column vectors in the matrix are linearly independent of one another, then the zero vector is the only member of the null space. But if the column vectors of the matrix are linearly dependent, then the null space will include at least one nonzero vector.

Null space of a matrix

Now that we understand the general idea of a null space, we want to know how to calculate the null space of a particular matrix. In other words, given a matrix A, we want to be able to find the entire set of vectors $\vec{x}$ that satisfy $A\vec{x}=\vec{O}$.

There may be only one vector in the null space, or there may be many. Either way, we want to be able to find the full set of vectors which are members of the null space.

To find the null space of a particular matrix, we'll plug the matrix into $A\vec{x}=\vec{O}$, rewrite the equation as an augmented matrix, put the augmented matrix into reduced row-echelon form, and then pull the solution from the matrix. The solution we find will be at least part of the null space of A.

Let's do an example.

Example

Find the null space of the matrix A.

$$A=\begin{bmatrix}2 & 1 & -3\\ 4 & 2 & -6\\ 1 & -1 & -6\end{bmatrix}$$

To find the null space of A, we need to find the vector set that satisfies $A\vec{x}=\vec{O}$, so we'll set up a matrix equation.

Because A has three columns, $\vec{x}$ needs to have three rows, so we'll use a 3-row column vector for $\vec{x}$. And multiplying the 3×3 matrix by the 3-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & -6 \\ 1 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this equation, we get a system of equations.

$$2x_1 + x_2 - 3x_3 = 0$$

$$4x_1 + 2x_2 - 6x_3 = 0$$

$$x_1 - x_2 - 6x_3 = 0$$

We can write the system as an augmented matrix,

$$\begin{bmatrix} 2 & 1 & -3 & | & 0 \\ 4 & 2 & -6 & | & 0 \\ 1 & -1 & -6 & | & 0 \end{bmatrix}$$

and then use Gaussian elimination to put it in reduced row-echelon form. Find the pivot entry in the first column by switching the first and third row.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 4 & 2 & -6 & | & 0 \\ 2 & 1 & -3 & | & 0 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 6 & 18 & | & 0 \\ 2 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 6 & 18 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & -1 & -6 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -3 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 3 & 9 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From the reduced row-echelon form of the matrix, we get

$$x_1 - 3x_3 = 0$$

$$x_2 + 3x_3 = 0$$

The pivot entries we found were for x_1 and x_2 (since they were in the first and second columns of our matrix), so we'll solve the system for x_1 and x_2.

$$x_1 = 3x_3$$

$$x_2 = -3x_3$$

This remaining system tells us that x_1 is equivalent to 3 times x_3, and that x_2 is equivalent to -3 times x_3. So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

Because $x_3 = 1x_3$, we were able to fill in the last row of the column matrix on the right with a 1. What this equation tells us is that we can choose any value for x_3, which is acting like a scalar. When we do, we'll get that the set of vectors that satisfy $A\vec{x} = \vec{O}$ is not only $\vec{x} = (3, -3, 1)$, but all linear combinations of $\vec{x} = (3, -3, 1)$. Which means the null space of A is all the linear combinations of $\vec{x} = (3, -3, 1)$.

But remember that when we say "all the linear combinations of a vector set," that's just the span of the vector set! So we can say that the null space of A is the span of $\vec{x} = (3, -3, 1)$.

$$N(A) = \text{Span}\left(\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right)$$

Keep in mind that, because the span of $\vec{x} = (3, -3, 1)$ is the null space of the reduced row-echelon form of A, we can also write the null space equation as

$$N(A) = N(\text{rref}(A))$$

This new equation for the null space tells us that, all we actually have to do to find the null space is put the original matrix A in reduced row-echelon form, and then find the linear combination equation from that rref matrix.

The set of vectors in the null space will be the span of the vector(s) in the linear combination equation.

Let's look at an example where we use this abbreviated process to find the null space of a particular matrix.

Example

Find the null space of K.

$$K = \begin{bmatrix} 1 & -2 & 1 & 3 \\ -3 & 6 & -3 & -9 \\ 4 & -8 & 4 & 12 \end{bmatrix}$$

To find the null space, put the matrix K in reduced row-echelon form.

$$K = \begin{bmatrix} 1 & -2 & 1 & 3 \\ -3 & 6 & -3 & -9 \\ 4 & -8 & 4 & 12 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 4 & -8 & 4 & 12 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(K))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the equation,

$$x_1 - 2x_2 + x_3 + 3x_4 = 0$$

which we can solve for the single pivot variable.

$$x_1 = 2x_2 - x_3 - 3x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of K is the span of the vectors in this linear combination equation.

$$N(K) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The column space and Ax=b

We've already talked about the null space of the matrix A, and $A\vec{x}=\vec{O}$, where $A\vec{x}$ is a matrix-vector product. Now we want to talk about another matrix-vector product, $A\vec{x}=\vec{b}$, and its relationship to the column space of the matrix A.

It turns out that $A\vec{x}=\vec{b}$ has a solution when $\vec{b}$ is a member of the column space of A.

The column space

The **column space** of a matrix A is defined as all possible linear combinations (the span) of the column vectors in A. So if matrix A is an $m\times n$ matrix with column vectors v_1, v_2, ... v_n,

$$A=\begin{bmatrix}v_1 & v_2 & v_3 & \dots & v_n\end{bmatrix}$$

then the column space of A is given as

$$C(A)=\text{Span}(v_1,\ v_2,\ v_3,\ \dots\ v_n)$$

So in $A\vec{x}=\vec{b}$, if $\vec{b}$ is a linear combination of the columns of A (if $\vec{b}$ is in the column space), then we'll be able to find a solution, which means that we'll be able to find some vector $\vec{x}$ that satisfies the equation $A\vec{x}=\vec{b}$.

Notice also that we've just defined the column space as a span, and remember that any span is a valid subspace. So, just like the null space, the column space of a matrix is always a valid subspace.

Linear independence and the null space

Remember before that when we talked about the null space, we said that,

- if the null space $N(A)$ includes only the zero vector, then the columns of A are linearly independent, but

- if the null space $N(A)$ includes any other vector in addition to the zero vector, then the columns of A are linearly dependent.

This is important to the column space because we're essentially saying that we can use the vectors in the null space to make a claim about the linear (in)dependence of the column vectors in the matrix.

If we're able to say that the column vectors of A are linearly independent, then we can also say that the column vectors of A form a basis for the column space. Otherwise, if what we know about the null space tells us that the column vectors of A are linearly dependent, then we know that the column vectors of A cannot form a basis.

In other words, if the column vectors v_1, v_2, v_3, ... v_n in A is a linearly independent set of vectors, then $(v_1, v_2, v_3, \dots v_n)$ would form a basis for the column space of A.

And so to say whether the column vectors of A are linearly independent, we can simply start by finding the null space of A.

Dimensions of the null and column space

For any $m \times n$ matrix A, the null space of A is a subspace in $\mathbb{R}^n$, and the column space of A is a subspace in $\mathbb{R}^m$. More specifically,

- $C(A)$ will be an r-dimensional subspace in $\mathbb{R}^m$, where r is the number of pivot columns in A and m is the total number of rows in A.

- $N(A)$ will be an $(n-r)$-dimensional subspace in $\mathbb{R}^n$, where $(n-r)$ is the number of non-pivot columns (free columns) in A and n is the total number of columns in A.

Let's do an example of how to find the column space of a matrix by first finding the null space.

Example

Find the null space and its dimension, then find the column space of A.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Let's find the null space of A by first putting the matrix into reduced row-echelon form. Find the pivot entry in the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 2 & 2 \\ 5 & 6 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 2 & 2 \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{31}{4} & -\frac{15}{4} \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & 0 & -\frac{8}{31} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix}$$

Then to find the null space of A, set up the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & 0 & -\frac{8}{31} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see from the matrix that x_1, x_2, and x_3 are pivot columns, since those are the columns containing pivot entries, and that x_4 is a free variable. This gives the system of equations

$$x_1 - \frac{3}{31}x_4 = 0$$

$$x_2 - \frac{8}{31}x_4 = 0$$

$$x_3 + \frac{15}{31}x_4 = 0$$

which can be solved for the pivot variables as

$$x_1 = \frac{3}{31}x_4$$

$$x_2 = \frac{8}{31}x_4$$

$$x_3 = -\frac{15}{31}x_4$$

Then the null space of A is all the linear combinations given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix}$$

which means the null space is the span of the single column vector.

$$N(A) = N(\text{rref}(A)) = \text{Span}\left(\begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix} \right)$$

Since the null space is the span of the single column vector, the dimensions of the null space is $\text{Dim}(N(A)) = 1$.

In the case of A, x_4 is "free," which means we can set it equal to any value. So it just becomes a scalar, and we can pick infinitely many values of x_4 to get infinitely many vectors that satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{31} \\ \frac{8}{31} \\ -\frac{15}{31} \\ 1 \end{bmatrix}$$

So the null space of A doesn't just contain the zero vector, which means there's more than one solution to $A\vec{x} = \vec{O}$, which means that the column vectors of A are a linearly dependent set.

Because the vector set

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

is not linearly independent, they don't form a basis for the column space of A. Of course, they span the column space of A, but they can't form a basis for A's column space because they're linearly dependent.

So while the column vectors of A aren't linearly independent, the column space is still just all the linear combinations of the column vectors, which is

$$C(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}\right)$$

Finding the basis for the column space

So if the given vector set isn't a basis for the column space of A, what is? We just need to find the vectors in the set that are redundant, and then remove those, and we'll have the basis. We can do that by setting up an equation with the column vectors from A. Using the same matrix from the last example, we'd get

$$x_1\begin{bmatrix}2\\4\\5\end{bmatrix}+x_2\begin{bmatrix}1\\-2\\6\end{bmatrix}+x_3\begin{bmatrix}3\\8\\-2\end{bmatrix}+x_4\begin{bmatrix}1\\4\\-3\end{bmatrix}=0$$

We said that x_4 was "free," which means we can set it equal to any value that we choose. We'll set $x_4=0$, which will cancel that vector from the equation.

$$x_1\begin{bmatrix}2\\4\\5\end{bmatrix}+x_2\begin{bmatrix}1\\-2\\6\end{bmatrix}+x_3\begin{bmatrix}3\\8\\-2\end{bmatrix}+0\begin{bmatrix}1\\4\\-3\end{bmatrix}=0$$

$$x_1\begin{bmatrix}2\\4\\5\end{bmatrix}+x_2\begin{bmatrix}1\\-2\\6\end{bmatrix}+x_3\begin{bmatrix}3\\8\\-2\end{bmatrix}=0$$

We want to determine if there's any solution to this equation other than $x_1=0$, $x_2=0$, and $x_3=0$.

$$2x_1+x_2+3x_3=0$$

$$4x_1-2x_2+8x_3=0$$

$$5x_1 + 6x_2 - 2x_3 = 0$$

Let’s try to solve the system with an augmented matrix and Gaussian elimination.

$$\begin{bmatrix} 2 & 1 & 3 & | & 0 \\ 4 & -2 & 8 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix}$$

Find the pivot entry in the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 4 & -2 & 8 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & -4 & 2 & | & 0 \\ 5 & 6 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & -4 & 2 & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & \frac{7}{2} & -\frac{19}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & -\frac{31}{4} & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

This reduced row-echelon matrix tells us that the only solution to the system is $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Which means that the first three column vectors of A are linearly independent of one another. None of them are redundant, because we can't make any of the vectors using a linear combination of the others. Therefore, instead of the vector set

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

we can eliminate the last column vector (the one associated with x_4 that's redundant), and then we'll have three linearly independent vectors remaining.

$$\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}$$

Because this set is linearly independent, they can form the basis of the column space of A.

$$\text{Basis of } C(A) \text{ is } \left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}\right)$$

Which means that, instead of writing the column space as

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}, \begin{bmatrix}1\\4\\-3\end{bmatrix}\right)$$

we could write it as

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\6\end{bmatrix}, \begin{bmatrix}3\\8\\-2\end{bmatrix}\right)$$

We've only eliminated a dependent vector, which means both vectors sets will span the same space. But the second set forms a basis for the column space, while the first set doesn't, since the first set isn't linearly independent. With three vectors forming the basis for the column space of A, the dimension of the column space is $\text{Dim}(C(A)) = 3$.

Solving Ax=b

We know that the null space of a matrix is any vector $\vec{x}$ that satisfies $A\vec{x} = \vec{O}$. But now we want the complete solution to the system represented by A. In other words, we don't just want to know which vectors $\vec{x}$ will give the zero vector as a result; now we want to know which vectors $\vec{x}$ will give *any* vector as a result.

In other words, instead of limiting ourselves to only the zero vector, now we want a way to find every $\vec{x}$ that will satisfy $A\vec{x} = \vec{b}$ when we choose any particular $\vec{b}$.

The complementary, particular, and general solutions

We can think of any $\vec{x}$ that satisfies $A\vec{x} = \vec{O}$ as part of the **complementary solution**, and any $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$ as part of the **particular solution**. The **general solution** (or the complete solution) to the system is the sum of the complementary and particular solutions.

That's because all of the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$ and all the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ are part of the complete solution. To distinguish between these solution sets, we call the complementary solution $\vec{x}_n$ (since it's the null space), we call the particular solution $\vec{x}_p$, and we call the general solution just $\vec{x}$.

Then we can say $A\vec{x}_n = \vec{O}$ and $A\vec{x}_p = \vec{b}$. If we add these together, we get

$$A\vec{x}_n + A\vec{x}_p = \vec{O} + \vec{b}$$

$$A(\vec{x}_n + \vec{x}_p) = \vec{b}$$

What this equation shows us is that the full solution set of vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{b}$ will be any vector $\vec{x} = \vec{x}_n + \vec{x}_p$. And that's how we can conclude that the complete solution will be the sum of the complementary and particular solutions.

If you've taken a Differential Equations course (it's okay if you haven't), this should remind you of solving non-homogeneous differential equations. In both cases (here in Linear Algebra with matrices, and in Differential Equations with non-homogeneous equations), we find the set of solutions that satisfy the homogeneous equation where the right side is 0 (or the zero vector $\vec{O}$), and then we find the particular solution that satisfies the non-zero right side. Then the general solution is the sum of the two.

Finding the complete solution set

So to find the full family of solutions to $A\vec{x} = \vec{b}$, we first find the set of all the linear combinations of column vectors that are solutions to the null space equation.

Then we find the particular solution by setting all the free variables equal to 0 (if there *is* any solution, we can find it by setting the free variables equal to 0), plugging in a $\vec{b}$ that satisfies any constraint on $\vec{b}$, and then plugging in to get the single particular vector.

Then the general solution is the sum of the particular solution and the complementary solution.

Let's work through a full example so that we can see how to get all the way to $\vec{x} = \vec{x}_n + \vec{x}_p$.

Example

Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 4 \\ 3 & 4 & 4 & 7 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To find the general solution to $A\vec{x} = \vec{b}$, we need to find the solutions in the null space, and then the particular solution. Let's start with the solutions in the null space, which we'll find by solving $A\vec{x} = \vec{O}$. To get those null space solutions, we'll augment the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & 0 \\ 2 & 2 & 1 & 4 & | & 0 \\ 3 & 4 & 4 & 7 & | & 0 \end{bmatrix}$$

We want to put the augmented matrix into reduced row-echelon form. The first column already contains a pivot of 1, so we'll zero out the rest of the first column.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & 0 \\ 0 & -2 & -5 & -2 & | & 0 \\ 3 & 4 & 4 & 7 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 3 & | & 0 \\ 0 & -2 & -5 & -2 & | & 0 \\ 0 & -2 & -5 & -2 & | & 0 \end{bmatrix}$$

The next step would be to multiply through the second row to find a pivot of 1. But looking ahead, we can see that the second and third rows are identical, so let's use $R_3 - R_2 \to R_3$ to make our calculations easier.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & 0 \\ 0 & -2 & -5 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now find the pivot of 1 in the second row.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & 0 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

There can't be a pivot in the third column, so we'll move to the fourth column. The fourth column can't have a pivot either. In fact, the row of zeros at the bottom of the matrix tells us that the last row is a multiple of some other row or rows in the matrix. In fact, in the original matrix A, we can see that the third row is the sum of the first and second rows.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & 0 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

With the matrix in reduced row-echelon form, we can see that the first and second columns are pivot columns and the third and fourth columns are free columns. Which means x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. Let's parse out a system of equations.

$$1x_1 + 0x_2 - 2x_3 + 1x_4 = 0$$

$$0x_1 + 1x_2 + \frac{5}{2}x_3 + 1x_4 = 0$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

The system simplifies to

$$x_1 - 2x_3 + x_4 = 0$$

$$x_2 + \frac{5}{2}x_3 + x_4 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = 2x_3 - x_4$$

$$x_2 = -\frac{5}{2}x_3 - x_4$$

Then the vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

In other words, any linear combination of these column vectors is a member of the null space; it satisfies $A\vec{x}_n = \vec{O}$. We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. So instead of augmenting the matrix with the zero vector, we augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 2 & 2 & 1 & 4 & | & b_2 \\ 3 & 4 & 4 & 7 & | & b_3 \end{bmatrix}$$

Now we'll again put the matrix into reduced row-echelon form. Zero out the first column below the pivot.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & -2 & -5 & -2 & | & b_2 - 2b_1 \\ 3 & 4 & 4 & 7 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & -2 & -5 & -2 & | & b_2 - 2b_1 \\ 0 & -2 & -5 & -2 & | & b_3 - 3b_1 \end{bmatrix}$$

Find the pivot in the second column.

$$\begin{bmatrix} 1 & 2 & 3 & 3 & | & b_1 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2 + b_1 \\ 0 & -2 & -5 & -2 & | & b_3 - 3b_1 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & -b_1+b_2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2+b_1 \\ 0 & -2 & -5 & -2 & | & b_3-3b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & | & -b_1+b_2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}b_2+b_1 \\ 0 & 0 & 0 & 0 & | & -b_1-b_2+b_3 \end{bmatrix}$$

With the matrix in reduced row-echelon form, we can substitute the values from $\vec{b}=(1,2,3)$, such that the rref matrix becomes

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & -1+2 \\ 0 & 1 & \frac{5}{2} & 1 & | & -\frac{1}{2}(2)+1 \\ 0 & 0 & 0 & 0 & | & -1-2+3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & | & 1 \\ 0 & 1 & \frac{5}{2} & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. To find the vectors that satisfy $A\vec{x}_p=\vec{b}$, we need to set the free variables equal to 0. So let's first rewrite the matrix as a system of equations.

$$1x_1+0x_2-2x_3+1x_4=1$$

$$0x_1+1x_2+\frac{5}{2}x_3+1x_4=0$$

$$0x_1+0x_2+0x_3+0x_4=0$$

The system simplifies to

$$x_1-2x_3+x_4=1$$

$$x_2+\frac{5}{2}x_3+x_4=0$$

Now set the free variables equal to 0, $x_3=0$ and $x_4=0$.

$$x_1 - 2(0) + 0 = 1$$

$$x_2 + \frac{5}{2}(0) + 0 = 0$$

The system becomes

$$x_1 = 1$$

$$x_2 = 0$$

So the particular solution then is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This is the complete solution to the system that's represented by matrix A.

Dimensionality, nullity, and rank

In this lesson we want to talk about the dimensionality of a vector set, which we should start by saying is totally different than the dimensions of a matrix. As we already know, the **dimensions** of a matrix are always given by the number of rows and columns, as

$$\text{dimensions} = \#\text{rows} \times \#\text{columns}$$

But the dimensionality of a vector space refers to something completely different. There are lots of different ways to describe dimensionality, and we can draw lots of conclusions about the space by knowing its dimension, but for now let's just say that the **dimension** of a vector space is given by the number of basis vectors required to span that space.

And speaking of vector spaces, we've just been looking at two really important ones: the null space and the column space. So let's take some time now to talk about the dimension of each of those spaces.

Dimension of the null space (nullity)

The dimension of the null space of a matrix A is also called the **nullity** of A, and can be written as either $\text{Dim}(N(A))$ or $\text{nullity}(A)$.

The nullity of the matrix will always be given by the number of free variables (non-pivot variables) in the system. So if we put the matrix into reduced row-echelon form, we'll be able to quickly identify both the pivot columns (with the pivot variables) and the free columns (with the free variables). The number of free variables is the nullity of the matrix.

The reason that we can get the nullity from the free variables is because every free variable in the matrix is associated with one linearly independent vector in the null space. Which means we'll need one basis vector for each free variable, such that the number of basis vectors required to span the null space is given by the number of free variables in the matrix.

Let's look at an example where we bring back a matrix from the lesson on finding the null space of a matrix.

Example

Find the nullity of K.

$$K = \begin{bmatrix} 1 & -2 & 1 & 3 \\ -3 & 6 & -3 & -9 \\ 4 & -8 & 4 & 12 \end{bmatrix}$$

To find the nullity of the matrix, we need to first find the null space, so we'll set up the augmented matrix for $K\vec{x} = \vec{O}$, then put the matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 1 & 3 & | & 0 \\ -3 & 6 & -3 & -9 & | & 0 \\ 4 & -8 & 4 & 12 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 4 & -8 & 4 & 12 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Normally at this point, we'd rewrite this matrix as a system of equations on our way toward finding the null space. But we can actually find the nullity directly from the rref matrix. We can see that the first column is a pivot column, and the other three columns are free columns, with free variables. Because there are three free variables, the nullity is

$$\text{Dim}(N(K)) = \text{nullity}(K) = 3$$

We can confirm this if we go forward with finding the null space. The rref matrix can be written as just the equation

$$x_1 - 2x_2 + x_3 + 3x_4 = 0$$

which we can solve for the single pivot variable.

$$x_1 = 2x_2 - x_3 - 3x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of K is the span of the vectors in this linear combination equation.

$$N(K) = \text{Span}\left(\begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\0\\1\end{bmatrix}\right)$$

The null space confirms what we found already about the dimension of the null space. We found 3 spanning vectors that form a basis for the null space, which matches the dimension of the null space, $\text{Dim}(N(K)) = \text{nullity}(K) = 3$.

Dimension of the column space (rank)

Similarly, the dimension of the column space of a matrix A is also called the **rank** of A, and can written as either $\text{Dim}(C(A))$ or $\text{rank}(A)$.

The rank of the matrix will always be given by the number of pivot variables in the system. So if we put the matrix into reduced row-echelon form, we'll be able to quickly identify the pivot columns (with the pivot variables). The number of pivot variables is the rank of the matrix.

The reason that we can get the rank from the pivot variables is because every pivot variable in the matrix is associated with one linearly independent vector in the column space. Which means we'll need one basis vector for each pivot variable, such that the number of basis vectors required to span the column space is given by the number of pivot variables in the matrix.

Let's look at an example where we bring back a matrix from the lesson on the column space of a matrix.

Example

Find the rank of A.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form. We already did this in the previous lesson, so we'll abbreviate the steps here.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -4 & 2 & 2 \\ 0 & \frac{7}{2} & -\frac{19}{2} & -\frac{11}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{31}{4} & -\frac{15}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{31} \\ 0 & 1 & 0 & -\frac{8}{31} \\ 0 & 0 & 1 & \frac{15}{31} \end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that the first three columns are pivot columns (with pivot variables), and the last column is a free column. Because there are three pivot variables, the rank is

$$\text{Dim}(C(A)) = \text{rank}(A) = 3$$

Then the column space of A is the span of the first three column vectors of A, since those were the columns that became the pivot columns when we put the matrix into rref.

And this confirms what we found already about the dimension of the column space. We're saying that there are 3 spanning vectors that form a basis for the column space, which matches the dimension of the column space, $\text{Dim}(C(A)) = 3$.

Nullity vs. rank

Notice how, in every matrix, every column is either a pivot column or a free column. What we can say then is that the sum of the nullity and the rank of a matrix will be equal to the total number of columns in the matrix.

$$\#\text{columns} = \text{rank} + \text{nullity}$$

For instance, in a 5-column matrix, if the rank is 3 (because you put the matrix into rref and found 3 pivot columns), then the nullity is $5 - 3 = 2$. Or in a 3-column matrix, if the nullity is 1 (because you put the matrix into rref and found 1 free column), then the rank is $3 - 1 = 2$.

Functions and transformations

If you've taken Algebra or Calculus, you're familiar with the idea of a **function**, which is a rule that maps one value to another.

For instance, the function $f(x) = x + 1$ maps x to $x + 1$. It tells us that, if we put any value x into the function f, the function will give back $x + 1$. In other words, the function will always return an output value that's related to the input value we gave it.

We can also write the function $f(x) = x + 1$ as $f: \ x \mapsto x + 1$, where the arrow with the line on the back literally means "maps to," telling us that f will map every x to $x + 1$.

Functions vs. transformations

We can also use functions to map vectors. For instance, the function

$$f\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}3v_1 - v_2\\-2v_2\end{bmatrix}$$

tells us that, for every vector $\vec{v} = (v_1, v_2)$ that we put into f, the function will give us back a new vector, $\vec{v} = (3v_1 - v_2, -2v_2)$. When a function maps vectors, we call it a **vector-valued function**.

While we usually use functions to map coordinate points, if we're going to map vectors from one space to another, we usually switch over from the language of "functions," to "**transformations**."

In other words, even though functions and transformations perform the same kind of mapping operation, if we want to map vectors, we should really say that the mapping is done by a transformation instead of by a function. So instead of writing

$$f\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3v_1-v_2\\-2v_2\end{bmatrix}$$

to express the transformation of a vector $\overrightarrow{v}=(v_1,v_2)$, it's more appropriate to write

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3v_1-v_2\\-2v_2\end{bmatrix}$$

In the same way that it's most common to use f to indicate a function, it's most common to use T to represent a transformation.

Domain, codomain, and range

Where before we used the notation $f: \ x\mapsto x+1$ to describe the mapping done by the function, we can use a regular arrow like $T: \ A\to B$ to indicate that the transformation T is mapping vectors from the set (or space) A onto vectors in the set (or space) B.

We also want to always consider the space of what we're mapping from and what we're mapping to. For instance, with $T: \ A\to B$, let's say we're mapping from real numbers to real numbers, where both vector sets A and B are defined by real numbers. We could write

$$T: \mathbb{R} \to \mathbb{R}$$

More specifically, if T is mapping from the two-dimensional real plane to the two-dimensional real plane, we could write

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

Keep in mind that, in Linear Algebra, we'll sometimes be mapping "across dimensions," for instance, from two dimensions to three dimensions, or vice versa.

In any transformation, the **domain** is what we're mapping from, and the **codomain** is what we're mapping to. So if $T: \mathbb{R}^2 \to \mathbb{R}^3$, then the domain would be the two-dimensional plane $\mathbb{R}^2$, and the codomain would be three-dimensional space $\mathbb{R}^3$. On the other hand, if $T: \mathbb{R}^3 \to \mathbb{R}^2$, then the domain would be $\mathbb{R}^3$ and the codomain would be $\mathbb{R}^2$.

The **range** is within the codomain. It's the specific set of points that the mapping actually maps to inside the codomain. In other words, T might be mapping us into $\mathbb{R}^3$ in general, but T might not be mapping to every single point in $\mathbb{R}^3$. Whatever set of vectors in $\mathbb{R}^3$ are actually getting mapped to will make up the range of the T.

Example

The transformation T maps every vector in $\mathbb{R}^4$ to the zero vector $\vec{v} = (0,0)$ in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

Because T is mapping vectors in $\mathbb{R}^4$ to vectors in $\mathbb{R}^2$, we can express T as $T: \ \mathbb{R}^4 \to \mathbb{R}^2$, and say that the domain of the transformation is $\mathbb{R}^4$ and its codomain is $\mathbb{R}^2$.

If every vector in $\mathbb{R}^2$ was being mapped to by T, we would say that the range of T is $\mathbb{R}^2$. But the transformation is mapping every vector in $\mathbb{R}^4$ to only the zero vector $\vec{v} = (0,0)$ in $\mathbb{R}^2$. Therefore, the range of T is just the zero vector, $\vec{v} = (0,0)$.

Transformation matrices and the image of the subset

Matrices can be extremely useful when it comes to describing what would otherwise be complex transformations in space.

For instance, let's say you want to know what happens if you take every point in the coordinate plane, shift it up vertically by 2 units, and stretch it out horizontally by 5 units.

Up to now, we don't have a simple way to describe this change mathematically. But that's where transformation matrices come in. They allow us to organize the transformation information into a matrix, which will tell us exactly how every vector in the space should move.

In the last lesson, we learned that a transformation was basically just a function that takes in a vector and transforms it into another vector. In this lesson we want to look at turning a transformation into a matrix, so let's start just by looking at what happens when we use a transformation to move a vector.

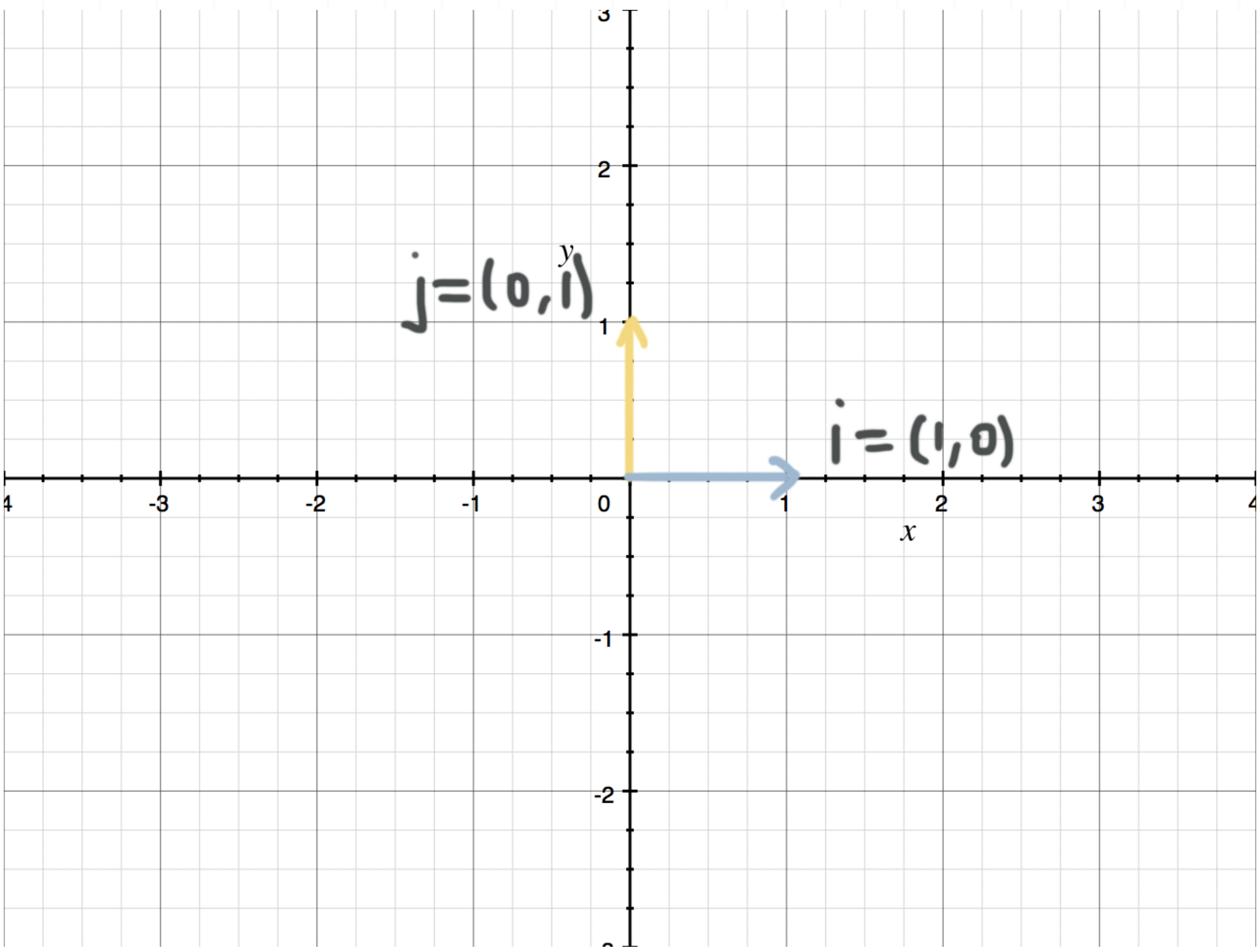
Understanding the transformation matrix

Think about transformations as a series of shifts, stretches, compressions, rotations, etc., that move a vector, or set of vectors, from one position to another. For instance, let's say we have the transformation matrix

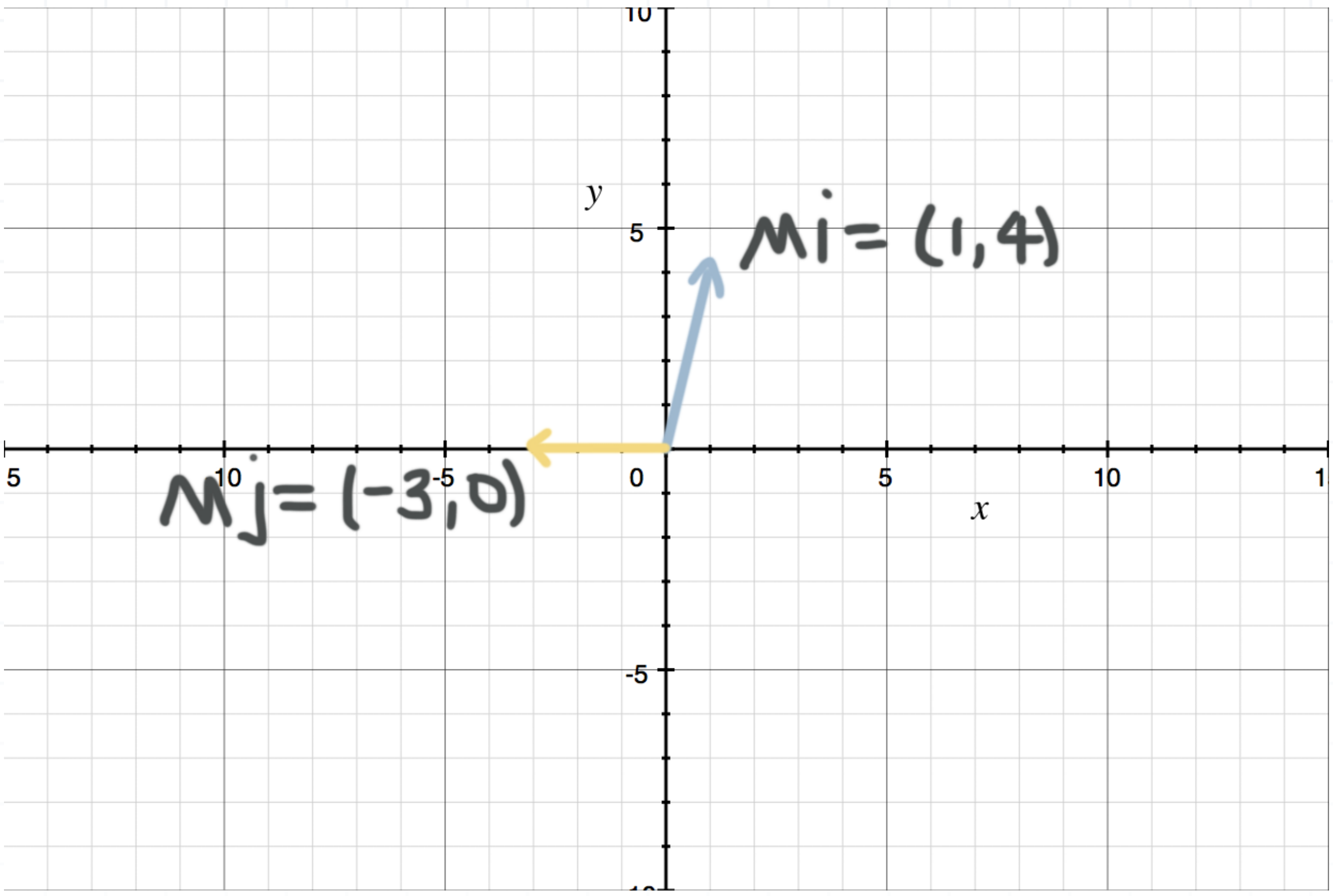
$$M = \begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix}$$

What we want to understand is how the entries in M work together to transform a vector (or vector set).

In a 2×2 transformation matrix, the first column (in this case $(1,4)$) tells us where the standard basis vector $\mathbf{i} = (1,0)$ will land after the transformation. The second column (in this case $(-3,0)$) tells us where the standard basis vector $\mathbf{j} = (0,1)$ will land after the transformation. In other words, given the vectors $\mathbf{i} = (1,0)$ in light blue and $\mathbf{j} = (0,1)$ in yellow,



the transformation matrix M changes $(1,0)$ into $(1,4)$, and changes $(0,1)$ into $(-3,0)$.

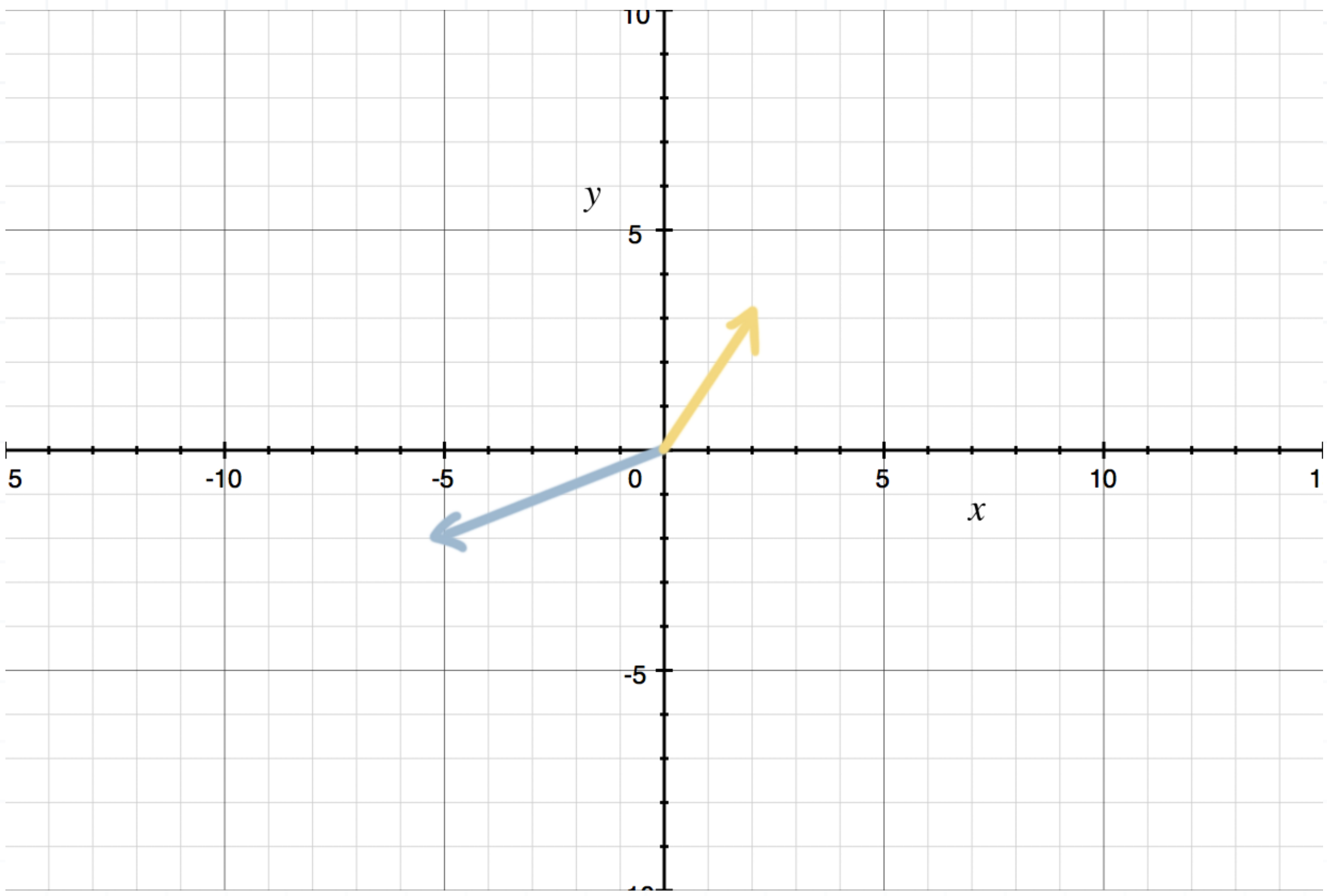


The vector (or vector set) before any transformation has been applied is called the **preimage**, and the vector (or vector set) after the transformation has been applied is called the **image**.

With this in mind, let's do an example where we start with the preimage and image and work backwards to find the transformation matrix.

Example

The graph shows the light blue vector $(1,0)$ and the yellow vector $(0,1)$, after a transformation has been applied. Find the transformation matrix that did the transformation.



The light blue vector now points to $(-5,-2)$. Since the first column of the transformation matrix represents where the unit vector $(1,0)$ lands after a transformation, we can fill in the first column of the transformation matrix.

$$\begin{bmatrix} -5 & \\ -2 & \end{bmatrix}$$

The yellow vector now points to $(2,3)$. Since the second column of a transformation matrix represents where the unit vector $(0,1)$ lands after a transformation, we can fill in the second column of the transformation matrix.

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix}$$

This is the matrix that describes the transformation happening in coordinate space when $(1,0)$ moves to $(-5,-2)$ and when $(0,1)$ moves to $(2,3)$.

We can double-check this by multiplying the transformation matrix we just found by the matrix of column vectors of $\mathbf{i}$ and $\mathbf{j}$.

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5(1) + 2(0) & -5(0) + 2(1) \\ -2(1) + 3(0) & -2(0) + 3(1) \end{bmatrix}$$

$$\begin{bmatrix} -5 + 0 & 0 + 2 \\ -2 + 0 & 0 + 3 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix}$$

There's a really important conclusion that we can draw from this last example problem. The transformation matrix we found doesn't just transform the individual vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$, it models the transformation of every vector in the coordinate plane!

Therefore, armed with this transformation matrix, you can now figure out the transformed location of any other vector in the plane. For instance, let's say you want to know what happens to $\vec{v} = (5,3)$ under the same transformation. Simply multiply the transformation matrix by the vector $\vec{v}$:

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -5(5) + 2(3) \\ -2(5) + 3(3) \end{bmatrix}$$

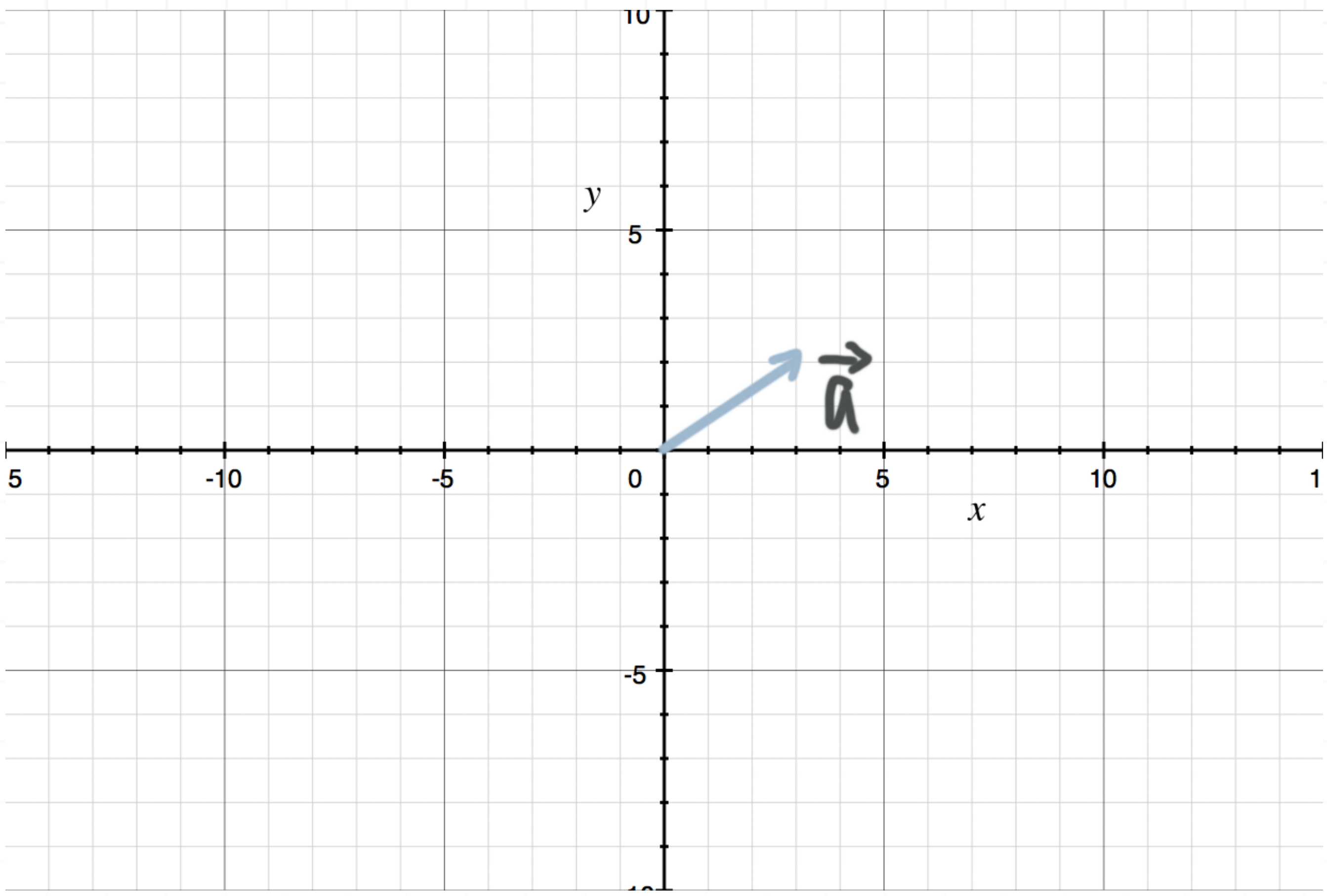
$$\begin{bmatrix} -25 + 6 \\ -10 + 9 \end{bmatrix}$$

$$\begin{bmatrix} -19 \\ -1 \end{bmatrix}$$

Under this transformation, $\vec{v} = (5,3)$ transforms to $T(\vec{v}) = (-19, -1)$. So the transformation matrix transforms the entire coordinate plane, all with one simple matrix!

Transforming vectors

Let's say we have the vector $\vec{a} = (3,2)$.



We can apply a transformation matrix to the vector, and doing so will change it into a transformed vector. Let's say we use the transformation matrix

$$M = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}$$

and apply it to the vector $\vec{a}$. Then the transformation of $\vec{a}$ by M will be the multiplication of M by $\vec{a}$.

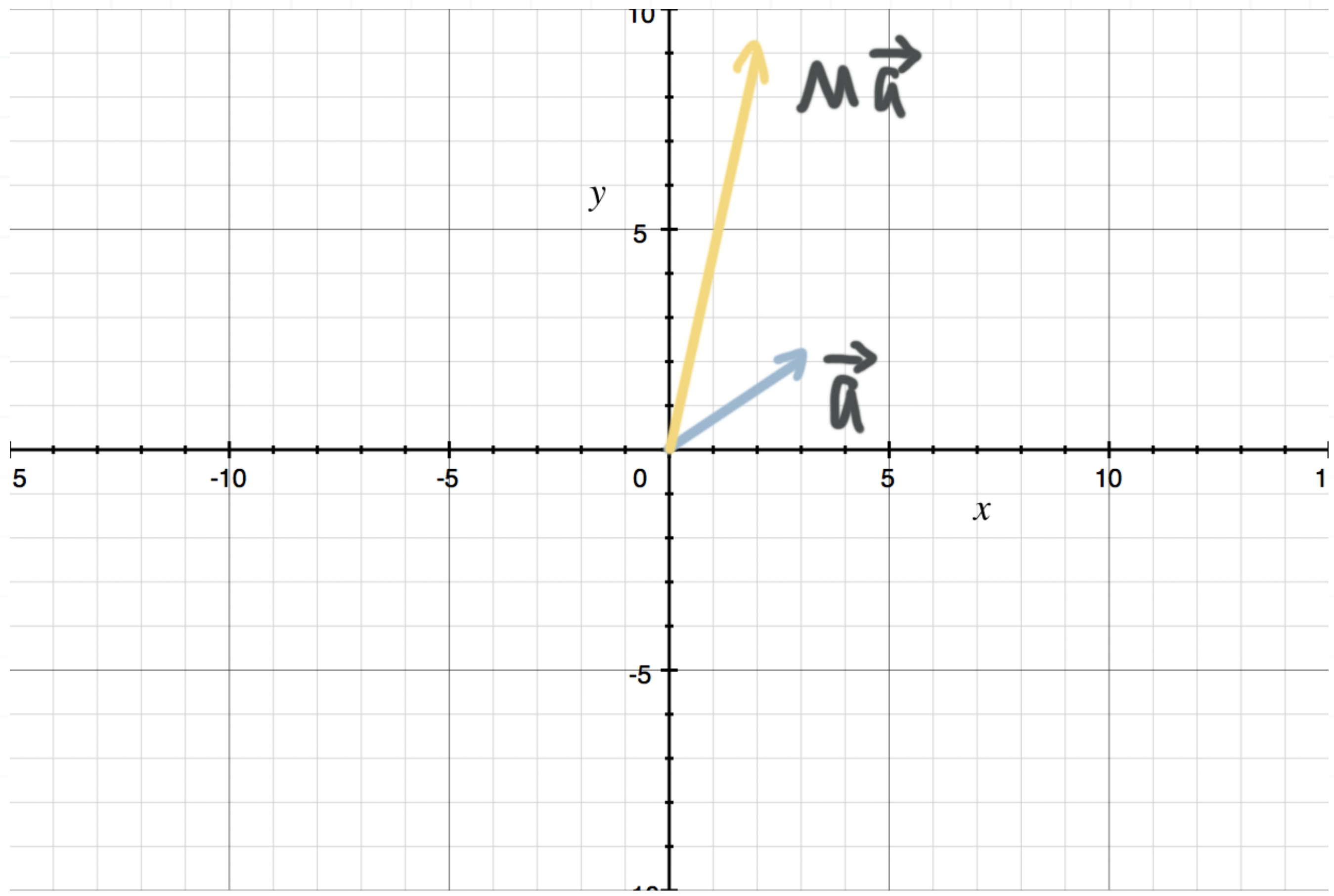
$$M\vec{a} = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} -2(3) + 4(2) \\ 3(3) + 0(2) \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} -6 + 8 \\ 9 + 0 \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

So matrix M transforms $\vec{a} = (3,2)$ into $M\vec{a} = (2,9)$.



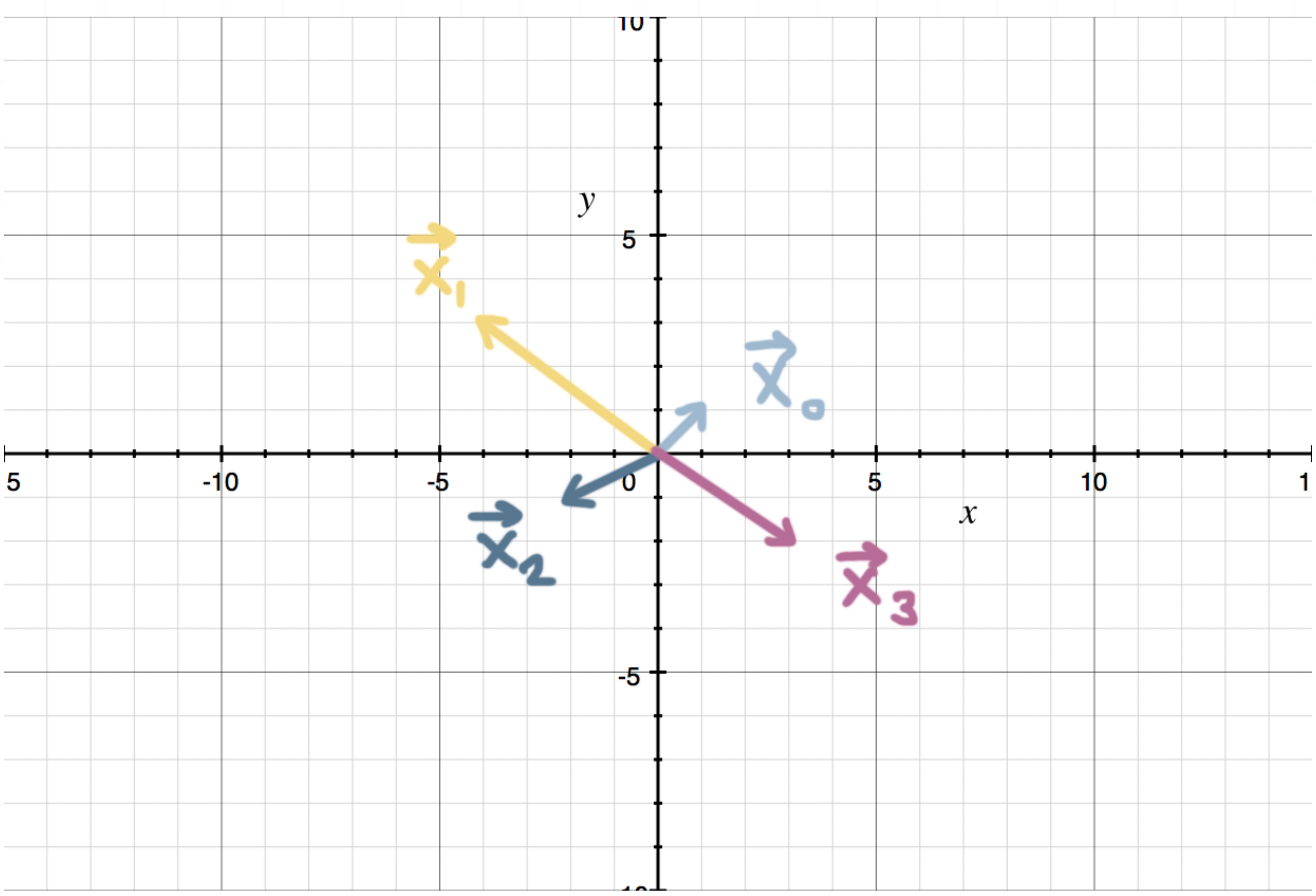
Transforming a set

If instead of being given a single vector $\vec{a}$, we'd been given a set of vectors representing the vertices of some polygon (like a triangle, quadrilateral, pentagon, hexagon, etc.), we can apply a transformation matrix to its vertex vectors, and thereby transform the figure.

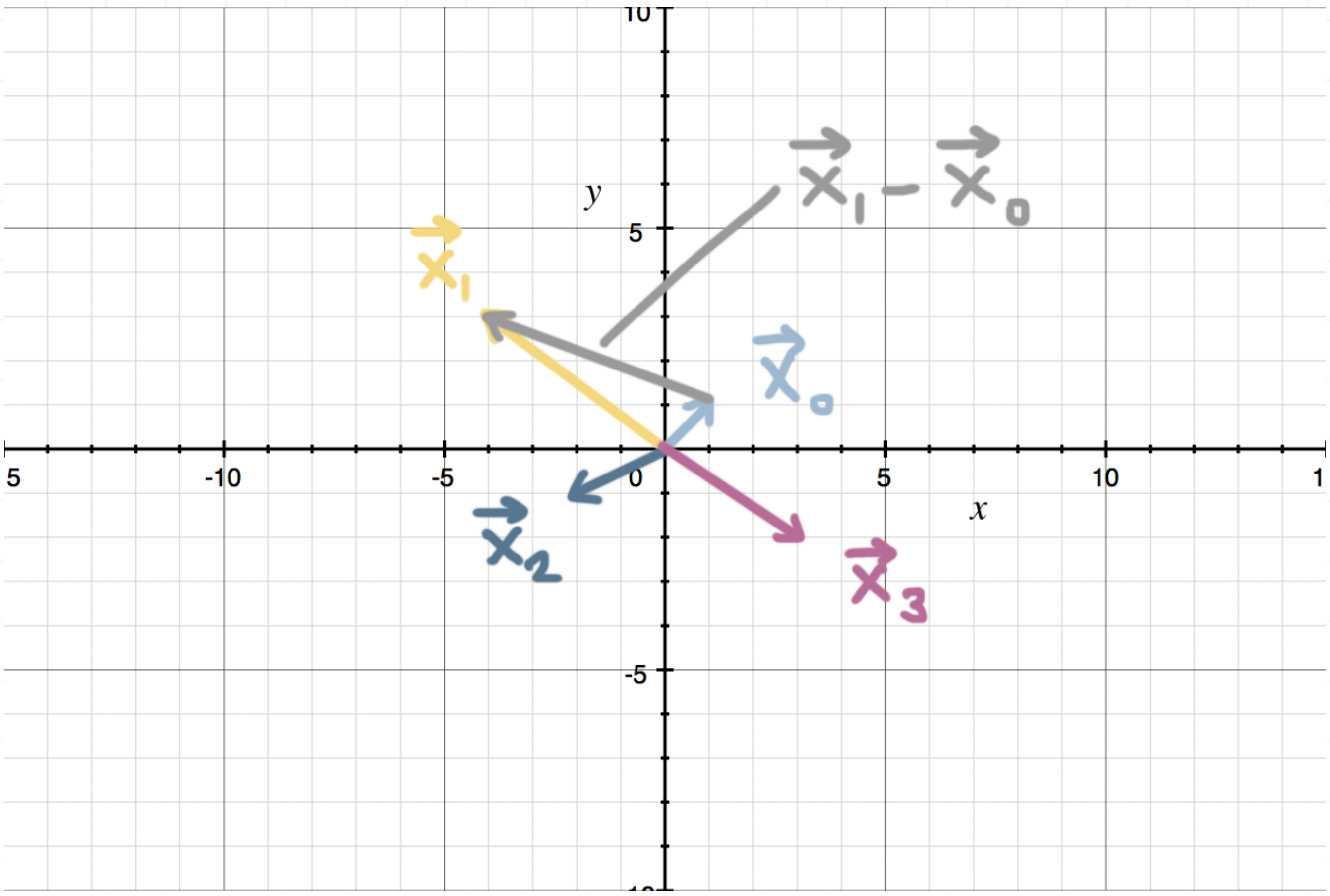
We call the vector set a **subset**, because it's a subset of the space that it's in. For example, you could have a two-dimensional figure, represented by a vector set that's a subset of $\mathbb{R}^2$. Or you could have a three-dimensional figure, represented by a vector set that's a subset of $\mathbb{R}^3$. The original subset is still called the **preimage**, and the transformed subset is still called the **image**. So when we say "the image of the subset under the transformation," we're talking about the set of vectors that define the transformed figure.

For example, let's say we want to transform the quadrilateral Q with vertices $(1,1)$, $(-4,3)$, $(-2,-1)$, and $(3,-2)$. The vertices of Q can be given by position vectors.

$$\vec{x_0}=\begin{bmatrix}1\\1\end{bmatrix}, \vec{x_1}=\begin{bmatrix}-4\\3\end{bmatrix}, \vec{x_2}=\begin{bmatrix}-2\\-1\end{bmatrix}, \vec{x_3}=\begin{bmatrix}3\\-2\end{bmatrix}$$



Now that each vertex of the quadrilateral is defined as a vector, we can describe each side of the quadrilateral in terms of those vectors. For example, we could describe the side connecting $\vec{x}_0$ to $\vec{x}_1$ as the vector $\vec{x_1}-\vec{x_0}$.



So if we call that side Q_0, we can define it as

$$Q_0 = \left\{\vec{x_0} + t(\vec{x_1} - \vec{x_0}) \ \mid \ 0 \leq t \leq 1\right\}$$

This works by limiting t to values between $t = 0$ and $t = 1$. When $t = 0$, the second term will cancel and we'll just be left with the vector $\vec{x_0}$.

$$Q_0 = \vec{x_0} + 0(\vec{x_1} - \vec{x_0})$$

$$Q_0 = \vec{x_0} + 0$$

$$Q_0 = \vec{x_0}$$

And when $t = 1$, the $\vec{x_0}$ will cancel and we'll just be left with the vector $\vec{x_1}$.

$$Q_0 = \vec{x_0} + 1(\vec{x_1} - \vec{x_0})$$

$$Q_0 = \vec{x_0} + \vec{x_1} - \vec{x_0}$$

$$Q_0 = \vec{x}_1$$

So the expression $Q_0 = \left\{\vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \ \mid \ 0 \le t \le 1\right\}$ will define (in terms of vectors) all the points between $\vec{x}_0$ and $\vec{x}_1$, thereby defining the side of the quadrilateral connecting those two vertices. All four sides of the quadrilateral are therefore defined like this:

$$Q_0 = \left\{\vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \ \mid \ 0 \le t \le 1\right\}$$

$$Q_1 = \left\{\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1) \ \mid \ 0 \le t \le 1\right\}$$

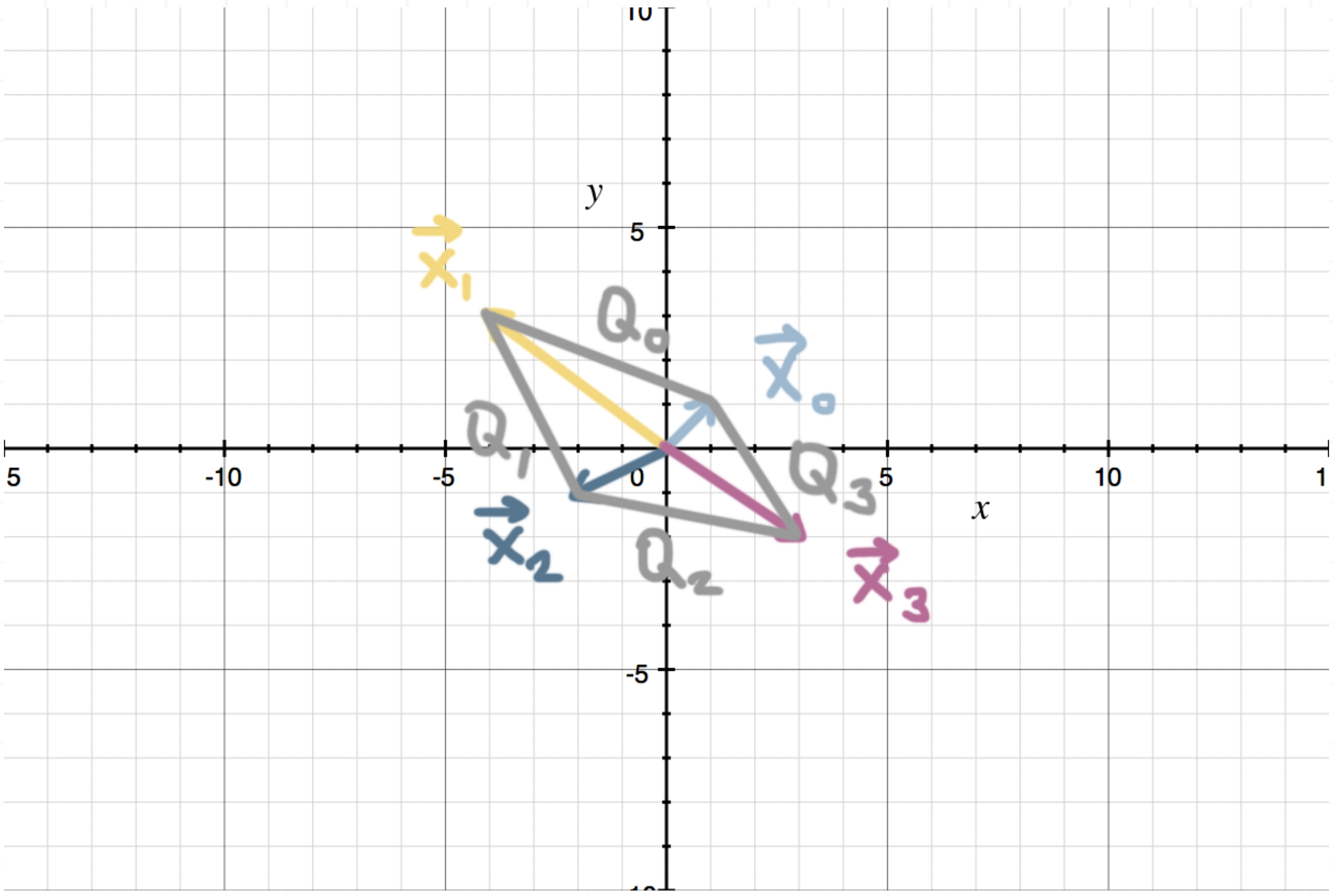
$$Q_2 = \left\{\vec{x}_2 + t(\vec{x}_3 - \vec{x}_2) \ \mid \ 0 \le t \le 1\right\}$$

$$Q_3 = \left\{\vec{x}_3 + t(\vec{x}_0 - \vec{x}_3) \ \mid \ 0 \le t \le 1\right\}$$

With the sides defined, we could say that the set of vectors that defines Q is simply

$$Q = \left\{Q_0, Q_1, Q_2, Q_3, \right\}$$

and that a sketch of Q, including each side of the quadrilateral, and the position vectors that define its vertices, is:



Now we need to pick a transformation. Let's say we want to transform the quadrilateral with the transformation T:

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Apply T to each of the position vectors that defines the vertices of Q.

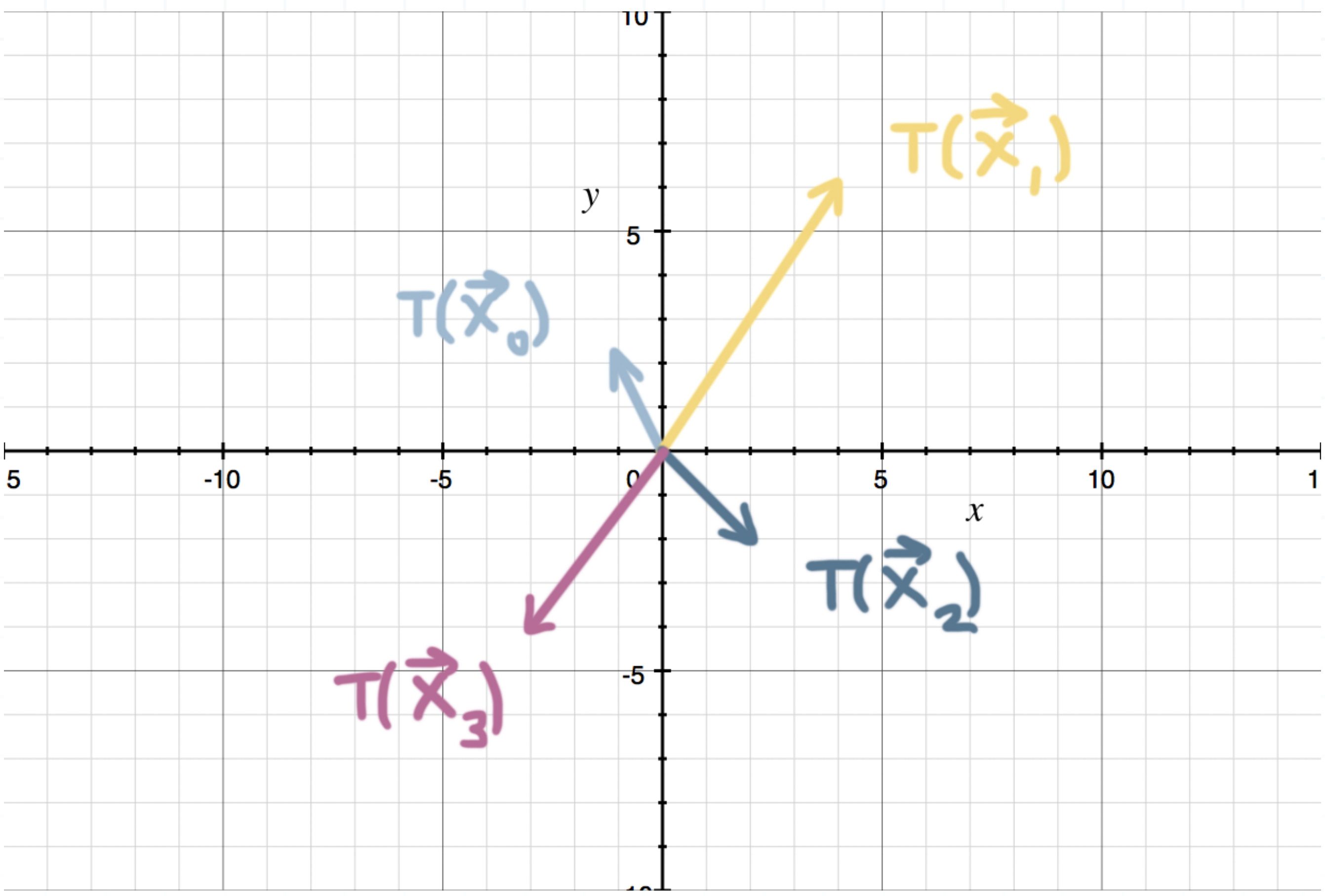
$$T(\vec{x}_0) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1(1) + 0(1) \\ 0(1) + 2(1) \end{bmatrix} = \begin{bmatrix} -1 + 0 \\ 0 + 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T(\vec{x}_1) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1(-4) + 0(3) \\ 0(-4) + 2(3) \end{bmatrix} = \begin{bmatrix} 4 + 0 \\ 0 + 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$T(\vec{x}_2) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1(-2) + 0(-1) \\ 0(-2) + 2(-1) \end{bmatrix} = \begin{bmatrix} 2 + 0 \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$T(\vec{x}_3) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1(3) + 0(-2) \\ 0(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -3 + 0 \\ 0 - 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

These transformed vectors are the position vectors that define the vertices of the transformed quadrilateral. If we sketch the transformed position vectors, we get



The nice thing about transformations of polygons like this one is that we only need to transform the vertices, like we just did. If a straight line connected $\vec{x}_0$ to $\vec{x}_1$ in the original figure (the preimage), then a straight line will also connect $T(\vec{x}_0)$ to $T(\vec{x}_1)$ in the transformed figure (the image). So we simply connect the vertices of the transformation in the same order in which they were connected in the original figure.

We can prove why the straight sides in the transformation follow the same pattern as the straight sides in the original figure if we apply the transformation to a side of the quadrilateral. Let's apply T to Q_0.

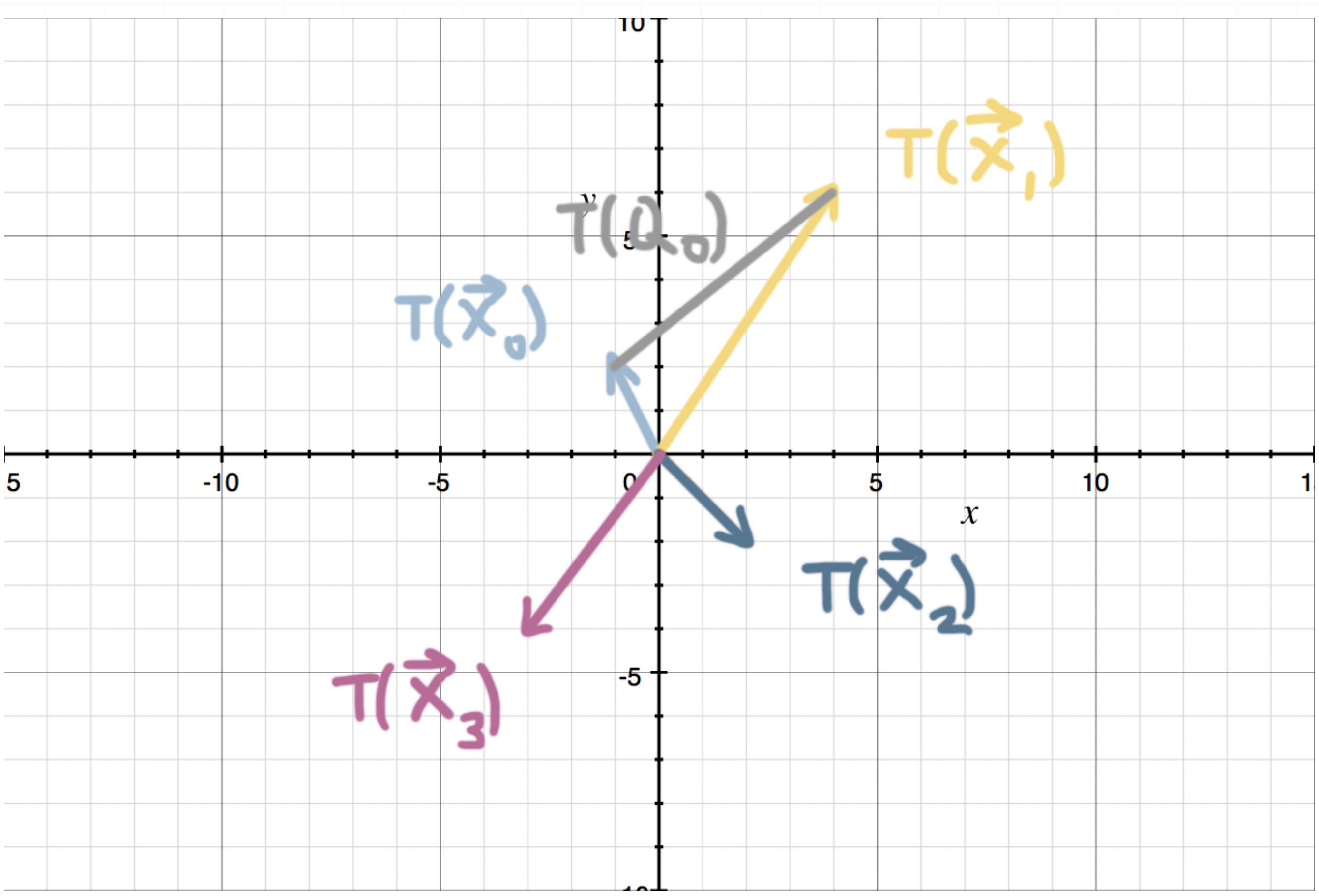
$$T(Q_0) = \left\{T(\vec{x}_0 + t(\vec{x}_1 - \vec{x}_0)) \ \mid \ 0 \le t \le 1\right\}$$

$$T(Q_0) = \left\{ T(\vec{x}_0) + T(t(\vec{x}_1 - \vec{x}_0)) \;\middle|\; 0 \le t \le 1 \right\}$$

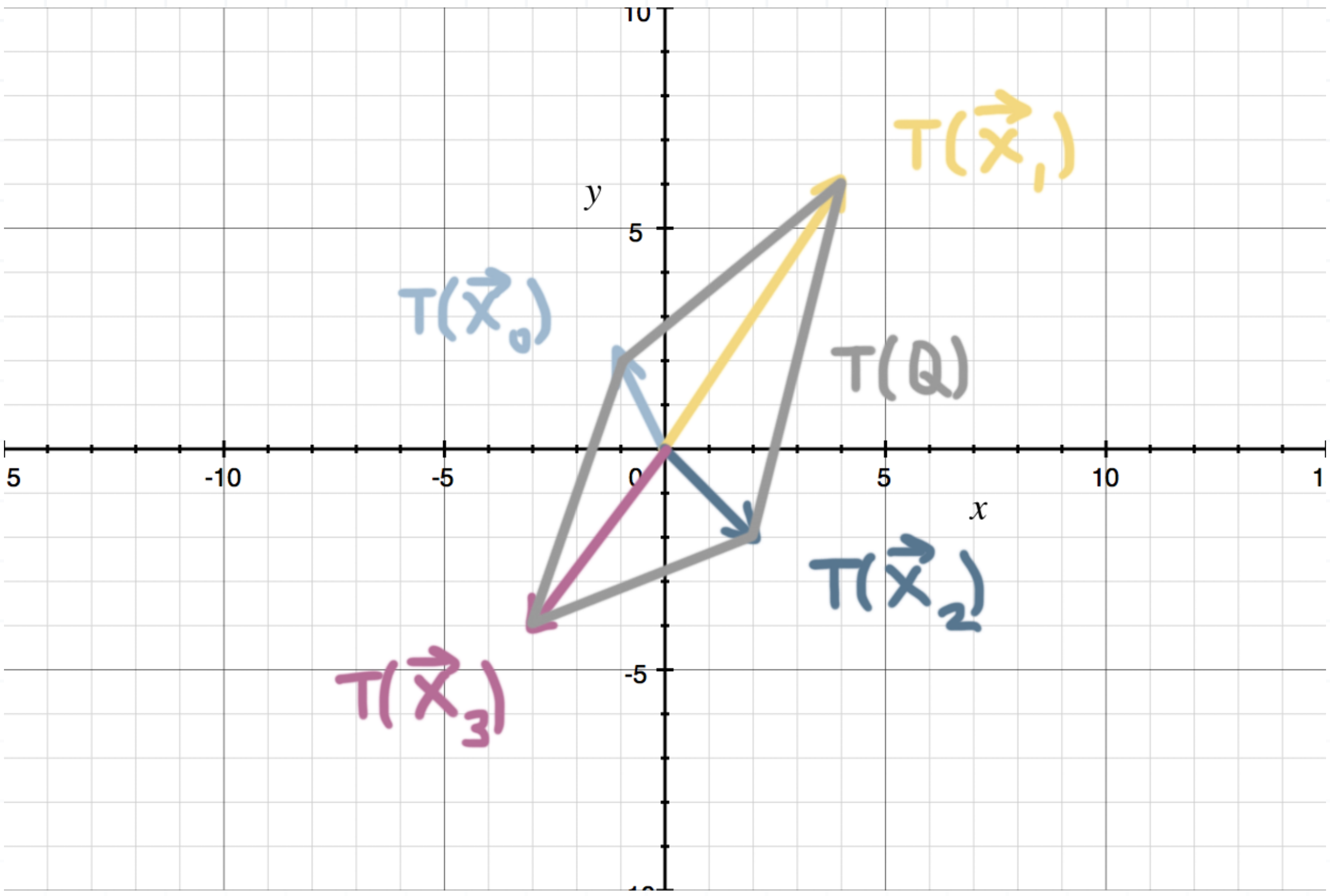
$$T(Q_0) = \left\{ T(\vec{x}_0) + tT(\vec{x}_1 - \vec{x}_0) \;\middle|\; 0 \le t \le 1 \right\}$$

$$T(Q_0) = \left\{ T(\vec{x}_0) + t(T(\vec{x}_1) - T(\vec{x}_0)) \;\middle|\; 0 \le t \le 1 \right\}$$

If we read through $T(\vec{x}_0) + t(T(\vec{x}_1) - T(\vec{x}_0))$, we see that it tells us to start at the transformation of $\vec{x}_0$, and then add t multiples of the vector connecting $T(\vec{x}_1)$ to $T(\vec{x}_0)$. As long as we keep $0 \le t \le 1$, this will give us all the points along a straight line that connects $T(\vec{x}_0)$ to $T(\vec{x}_1)$. And therefore we could sketch the first side of the transformed figure.



This side specifically is called the “image of Q_0 under T.” We could also apply the same transformation to the other three sides of the figure, and we would find similar results, allowing us to connect the rest of the transformed vertices. So the image of the set Q under the transformation T is



$T(\vec{x}_1)$
$T(\vec{x}_0)$
$T(Q)$
$T(\vec{x}_2)$
$T(\vec{x}_3)$
y
x
10
5
-5
-10
0

Preimage, image, and the kernel

In the last lesson we looked at a quadrilateral Q, and we said that Q was a subset of $\mathbb{R}^2$. We used the transformation T to transform Q, and said that $T(Q)$ was the image of Q under T. In other words, applying the transformation T to the preimage Q, we were able to find the image $T(Q)$.

We'll talk more about preimages and images in this lesson, but let's take a quick moment just to say that every transformation is a subspace.

Transformations as subspaces

Now we want to talk about $T(Q)$ as a subspace. Remember that in order for a set to be a subspace, it needs to be closed under addition and closed under scalar multiplication.

As it turns out, by the definition of transformations, a transformation will always be closed under addition and closed under scalar multiplication, which means that the transformation will always be a subspace. So sticking with the example where we transformed Q by the transformation T, we can say that $T(Q)$ is a subspace.

Finding the preimage from the image

Normally, when we talk about the transformation of a set, like $T : A \to B$, we say that the transformation T transforms vectors from the domain A into the codomain B.

But we can also talk about subsets of A and B. For instance, let's say we have a subset A_1 that's inside A. We write that as $A_1 \subseteq A$, which means that A_1 is contained within A as a subset of A. The transformation T will map A_1 to a subset of B, and we can write this transformation of A_1 as $T(A_1)$, and call it the image of A_1 under T.

But sometimes we want to work backwards, starting with a subset of B, we'll call it B_1 where $B_1 \subseteq B$, and trying to find all the points in A that map to B_1. The collection of all the points in A that map to the subset B_1 is the preimage of B_1. Since it's kind of like we're doing a reverse transformation (trying to find all the vectors in A that will map to the subset B_1 under the transformation T), we write the preimage of B_1 under T as $T^{-1}(B_1)$.

In other words,

- if we start with a subset of the domain, under the transformation T it'll map to the image of the subset in the codomain, but

- if we start with a subset of the codomain, under the inverse transformation T^{-1} it'll map to the preimage of the subset in the domain.

Let's actually look at how we'd find the preimage of a set B_1 that's in the codomain, if the set B_1 is made up of two vectors.

Example

Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{ \vec{x} \in \mathbb{R}^2 \ \mid \ T(\vec{x}) \in B_1 \}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle| \ \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & | & 0 \\ 2 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & | & 0 \\ -3 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ -3 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & | & 2 \\ 2 & 0 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & | & 3 \\ -3 & 1 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ -3 & 1 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & \frac{13}{2} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 0$ and $x_2 = 0$. And from the second augmented matrix we get $x_1 = 3/2$ and $x_2 = 13/2$. Therefore,

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} \frac{3}{2} \\ \frac{13}{2} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the subset B_1 under T

Keep in mind that the vector $\vec{x}$ in

$$\begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

is the null space of the transformation matrix. The vector (or set of vectors) $\vec{x}$ that makes this matrix equation true is called the kernel of the transformation T, Ker(T).

In other words, the **kernel** of a transformation T is all of the vectors that result in the zero vector under the transformation T:

$$\text{Ker}(T)=\left\{\vec{x}\in\mathbb{R}^2\ \mid\ T(\vec{x})=\vec{O}\right\}$$

Linear transformations as matrix-vector products

For now, the kinds of transformations we're interested in are linear transformations, and all the transformations we've looked at up to this point are linear transformations.

We've already seen hints of this in the last couple of lessons, but now we'll say explicitly that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if, for any two vectors $\vec{u}$ and $\vec{v}$ that are both in $\mathbb{R}^n$, and for a c that's also in $\mathbb{R}$ (c is any real number), then

- the transformation of their sum is equivalent to the sum of their individual transformations, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, and

- the transformation of a scalar multiple the vector is equivalent to the product of the scalar and the transformation of the original vector, $T(c\vec{u}) = cT(\vec{u})$ and $T(c\vec{v}) = cT(\vec{v})$.

Matrix-vector products

You can always represent a linear transformation as a matrix-vector product. For instance, let's say we're transforming vectors from $\mathbb{R}^2$ to vectors in $\mathbb{R}^3$, so $T : \mathbb{R}^2 \to \mathbb{R}^3$, and that the transformation is described as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 - x_1 \\ -x_1 + 2x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

To turn this transformation into a matrix-vector product, we need to start by thinking about the space we're transforming *from*. In this case, T is transforming vectors from $\mathbb{R}^2$. The identity matrix for $\mathbb{R}^2$ is I_2,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the next step is to apply T to I_2, which we'll do one column at a time.

$$\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$$

For the transformation of the first column of the identity matrix, we plug into T to get

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(0) - 1 \\ -1 + 2(0) \\ 3(1) + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

For the transformation of the second column of the identity matrix, we plug into T to get

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(1) - 0 \\ -0 + 2(1) \\ 3(0) + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Now we get the matrix

$$\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}-1&3\\-1&2\\3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Now, whenever we want to find the transformation T of a vector in $\mathbb{R}^2$, we can simply perform a matrix multiplication problem.

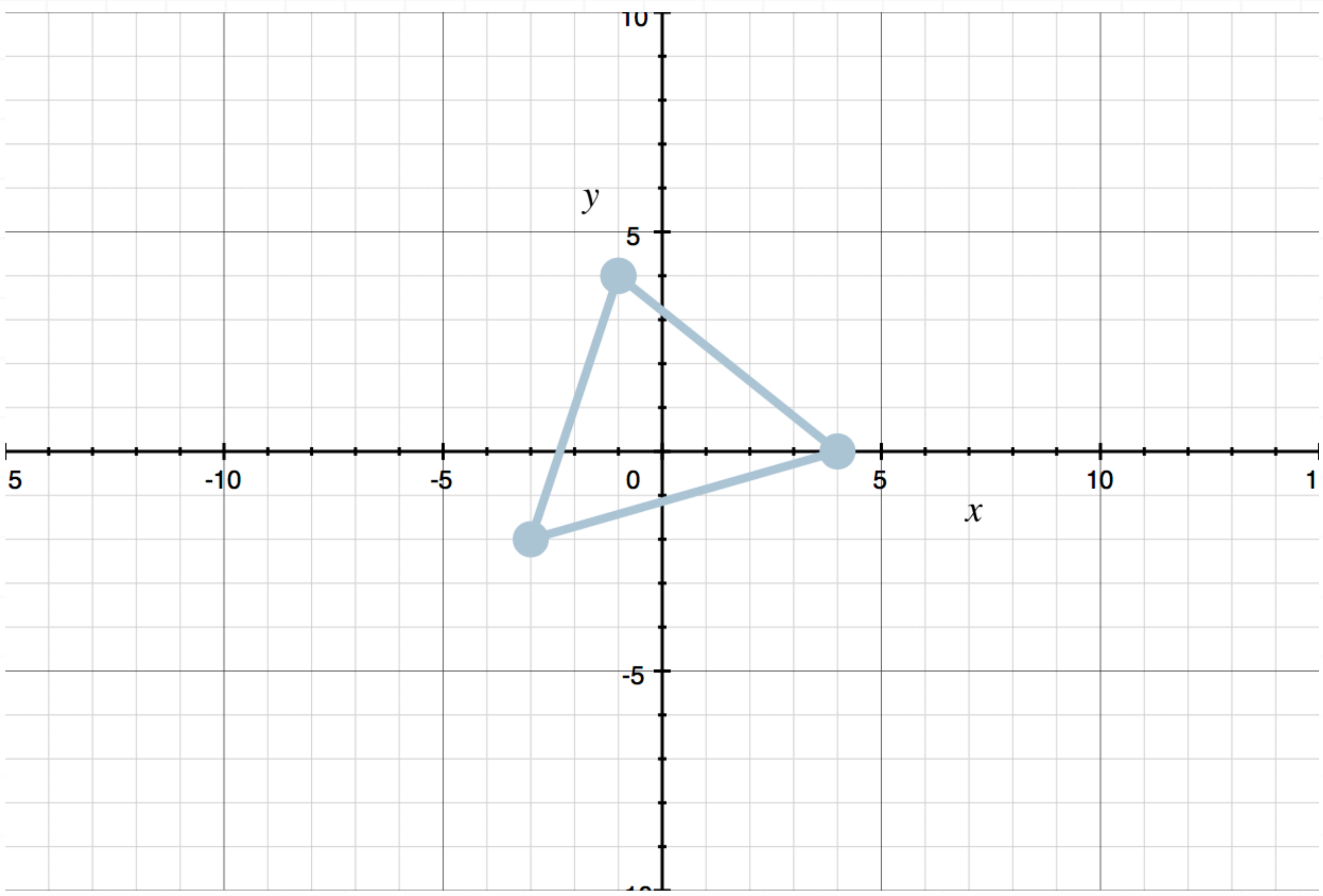
Notice first that we've expressed the transformation as a matrix-vector product, $T(\vec{x})=A\vec{x}$. So now we know that, if we want to transform any vector $\vec{x}$ by T, we can simply multiply the transformation matrix A by the vector $\vec{x}$ that we're trying to transform.

Notice also that if we multiply this 3×2 transformation matrix A by the 2×1 vector $\vec{x}$, we'll get a 3×1 vector as the result. It will always be true that the product of a matrix and a vector will be a vector, and the dimensions of the resulting vector are predictable. We said that this particular transformation T is transforming from $\mathbb{R}^2$ to $\mathbb{R}^3$, so we already know the resulting vector will be in $\mathbb{R}^3$, and we see just by looking at the dimensions of the matrix-vector product that the dimensions of the resulting vector are 3×1, so the resulting vector is, in fact, in $\mathbb{R}^3$.

Example

Use a matrix-vector product to transform the triangle with vertices $(-1,4)$, $(-3,-2)$, and $(4,0)$. The transformation T should include a reflection over the x-axis and a horizontal stretch by a factor of 3.

We don't have to sketch the triangle in order to do the problem, but it'll help us visualize what we're working with.



If each point in the triangle is given by (x, y), a reflection over the x-axis means we'll take the y-coordinate of each point in the triangle and multiply it by -1. So after the reflection, each transformed point will be $(x, -y)$.

To stretch horizontally by a factor of 3, we'll need to multiply every x-value by 3. So after both the reflection and the stretch, each transformed point will be $(3x, -y)$.

Therefore, if a position vector

$$\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

represents a point in the original triangle, then a position vector

$$\vec{v} = \begin{bmatrix} 3v_1 \\ -v_2 \end{bmatrix}$$

represents the corresponding point in the transformed triangle. So a transformation T that expresses the reflection and the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 \\ -v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(1) \\ -0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(0) \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

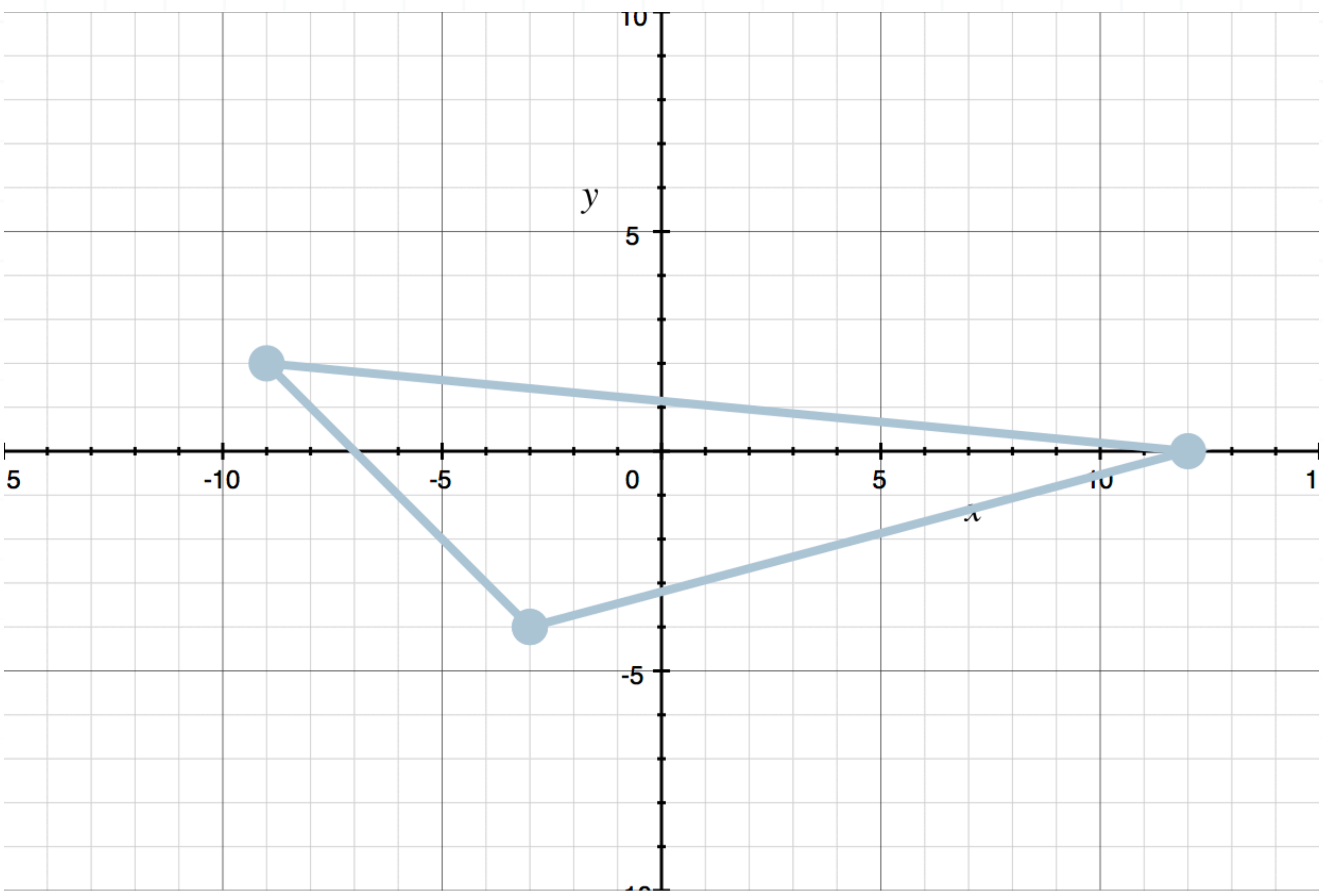
Now that we've built the transformation matrix, we can apply it to each of the vertices of the triangle, $(-1,4)$, $(-3,-2)$, and $(4,0)$.

$$T\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3(-1) + 0(4) \\ 0(-1) - 1(4) \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3(-3) + 0(-2) \\ 0(-3) - 1(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\0\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}3(4)+0(0)\\0(4)-1(0)\end{bmatrix}=\begin{bmatrix}12\\0\end{bmatrix}$$

Therefore, we can sketch the transformed triangle, and see at a glance that it does look roughly like it's been reflected over the x-axis and stretched horizontally to three times its original width.



Linear transformations as rotations

In the previous lesson, we looked at an example of a linear transformation that included a reflection and a stretch. We can apply the same process for other kinds of transformations, like compressions, or for rotations.

But we can also use a linear transformation to rotate a vector by a certain angle, either in degrees or in radians.

The rotation matrix

Instead of using the appropriate identity matrix like we did for reflecting, stretching, and compressing, we'll use a matrix specifically for rotations. But the matrix will still always match the dimension of the space in which we're transforming.

If we're rotating in $\mathbb{R}^2$, we're rotating counterclockwise around the origin through the angle θ, and the transforming rotation matrix will be

$$\text{Rot}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

And the transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ will be

$$\text{Rot}_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If we're rotating in $\mathbb{R}^3$, the transforming rotation matrix will be different depending on which axis we're rotating around.

$$\text{Rot}_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ will be

$$\text{Rot}_{\theta \text{ around } x}(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } y}(\vec{x}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } z}(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So if we know the angle by which we're trying to rotate a vector, we can plug the angle into the rotation matrix, and then multiply the rotation matrix by the vector we want to transform.

We also want to know that rotations follow these properties:

$$\text{Rot}_\theta(\vec{u} + \vec{v}) = \text{Rot}_\theta(\vec{u}) + \text{Rot}_\theta(\vec{v})$$

$$\text{Rot}_\theta(c\vec{u}) = c\text{Rot}_\theta(\vec{u})$$

Let's do an example of a rotation in $\mathbb{R}^2$.

Example

Rotate $\vec{x}$ by an angle of $\theta = 135^\circ$.

$$\vec{x} = (3,2)$$

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 135° will be

$$\text{Rot}_{135^\circ}(\vec{x}) = \begin{bmatrix}\cos(135^\circ) & -\sin(135^\circ)\\ \sin(135^\circ) & \cos(135^\circ)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

First, we'll simplify the rotation matrix. We can get the sine and cosine values at $\theta = 135^\circ$ from the unit circle.

$$\begin{bmatrix}\cos(135^\circ) & -\sin(135^\circ)\\ \sin(135^\circ) & \cos(135^\circ)\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}$$

Then the transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 135° will be rewritten as

$$\text{Rot}_{135^\circ}(\vec{x}) = \begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

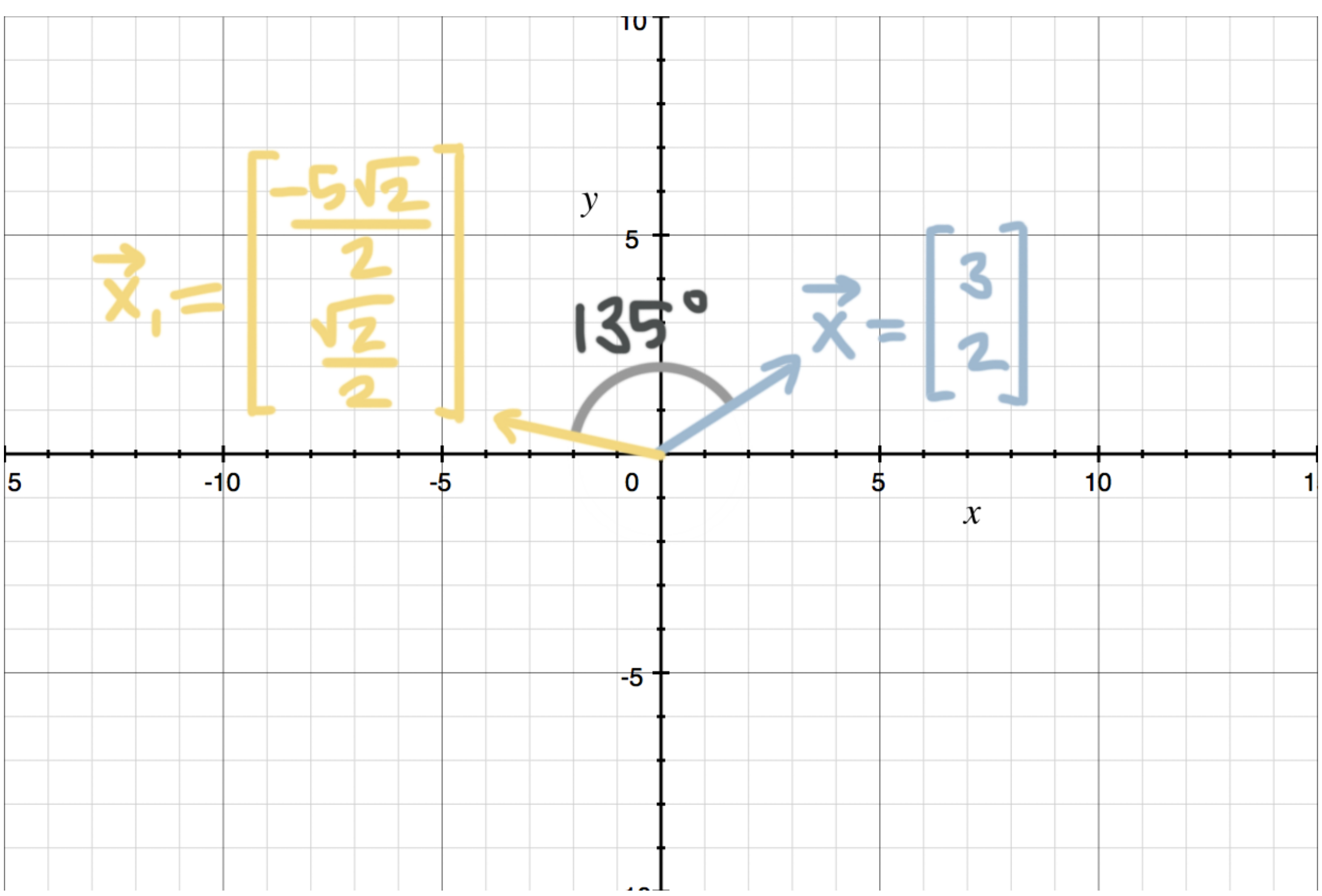
Now we'll apply this specific rotation matrix to $\vec{x}=(3,2)$.

$$\text{Rot}_{135^\circ}\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}$$

$$\text{Rot}_{135^\circ}\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}-\frac{3\sqrt{2}}{2}-\frac{2\sqrt{2}}{2}\\ \frac{3\sqrt{2}}{2}-\frac{2\sqrt{2}}{2}\end{bmatrix}$$

$$\text{Rot}_{135^\circ}\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=\begin{bmatrix}-\frac{5\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}$$

If the original vector is $\vec{x}$, and we call the transformed vector $\vec{x}_1$, then we can sketch $\vec{x}$ and $\vec{x}_1$ and the 135° angle between them.



Adding and scaling linear transformations

Now that we know about linear transformations, we want to think about operations with transformations. So in this lesson we're talking about how to add and scale linear transformations.

Sums of transformations

Given two transformations of $\vec{x}$, $S(\vec{x}):\mathbb{R}^n \to \mathbb{R}^m$ and $T(\vec{x}):\mathbb{R}^n \to \mathbb{R}^m$, remember that you can represent any linear transformation as a matrix-vector product. So rewrite $S(\vec{x})$ as $A\vec{x}$, and rewrite $T(\vec{x})$ as $B\vec{x}$, where A and B are $m \times n$ matrices. Then the sum of the transformations is

$$(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x})$$

$$(S+T)(\vec{x}) = A\vec{x} + B\vec{x}$$

$$(S+T)(\vec{x}) = (A+B)\vec{x}$$

The $(A+B)$ is simply a matrix addition problem, and then $(A+B)\vec{x}$ tells us to multiply the resulting $A+B$ matrix by the vector $\vec{x}$.

Example

Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To add the transformations, we first need to recognize that these transformations are written as matrix-vector products. If we call the matrix in $S(\vec{x})$ the matrix A,

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

and we call the matrix in $T(\vec{x})$ the matrix B,

$$B = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

then the sum of the transformations is simply the sum of these matrices. First, find the sum of the matrices.

$$A + B = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 + (-6) & 1 + 0 \\ -3 + 2 & 0 + (-1) \end{bmatrix}$$

$$A + B = \begin{bmatrix} -4 & 1 \\ -1 & -1 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} -4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Scaled transformations

We can also multiply transformations by scalars. The transformation $T(\vec{x})$, multiplied by a scalar c, would be written as $cT(\vec{x})$. But if we write $T(\vec{x})$ as its matrix-vector product, $B\vec{x}$, then the scaled transformation is $cB\vec{x}$.

$$cT(\vec{x}) = c(B\vec{x}) = (cB)\vec{x}$$

Let's do an example where we apply a scalar to a transformation.

Example

Find the product of a scalar $c = -4$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transformation T is given as a matrix-vector product. If we call the matrix that's in the transformation T the matrix B, then multiplying the transformation by the scalar $c = -4$ gives

$$cT(\vec{x}) = -4 \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First find cB.

$$cB = -4\begin{bmatrix}-6 & 0\\ 2 & -1\end{bmatrix}$$

$$cB = \begin{bmatrix}-4(-6) & -4(0)\\ -4(2) & -4(-1)\end{bmatrix}$$

$$cB = \begin{bmatrix}24 & 0\\ -8 & 4\end{bmatrix}$$

So the scaled transformation would be

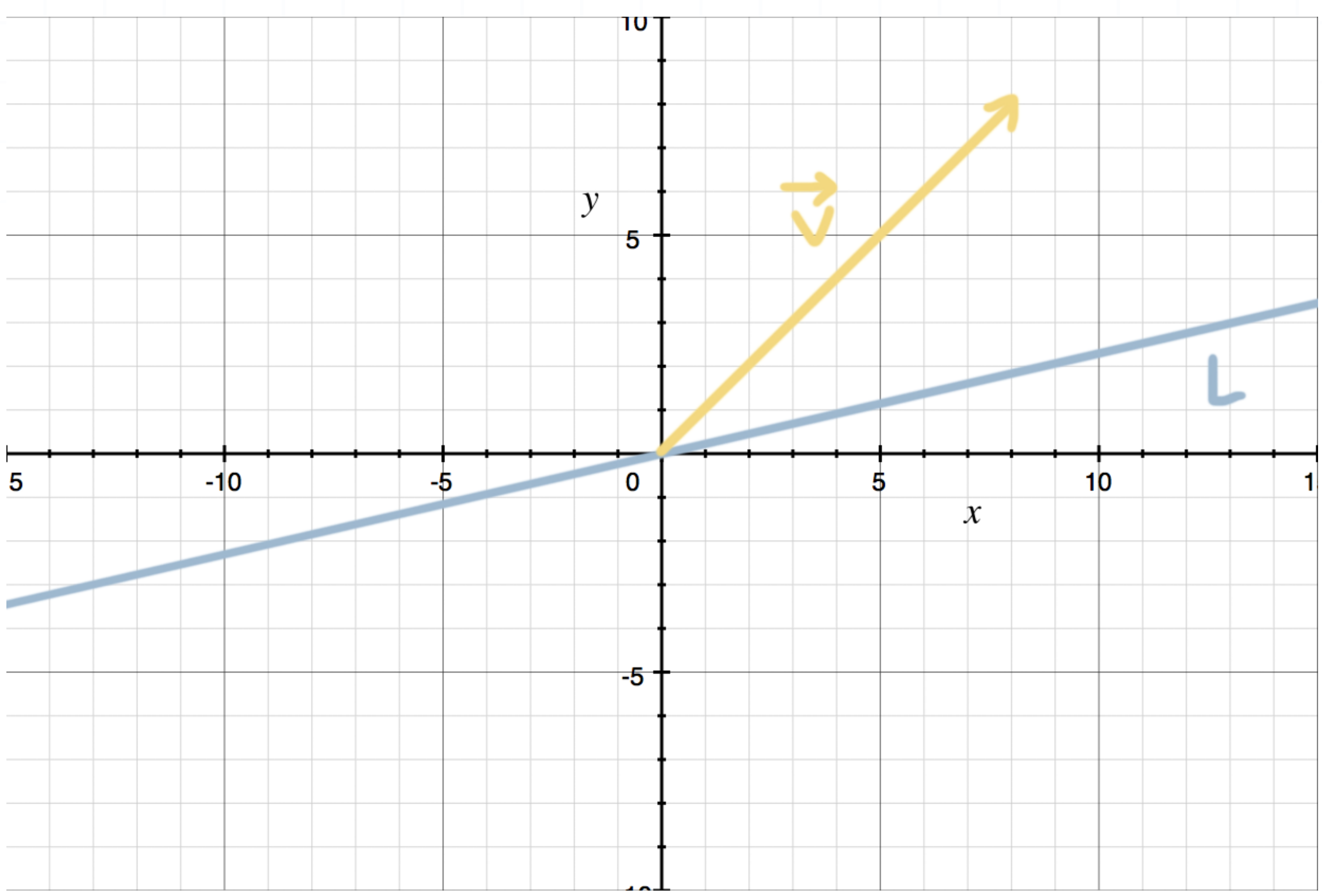
$$cT(\vec{x}) = \begin{bmatrix}24 & 0\\ -8 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Projections as linear transformations

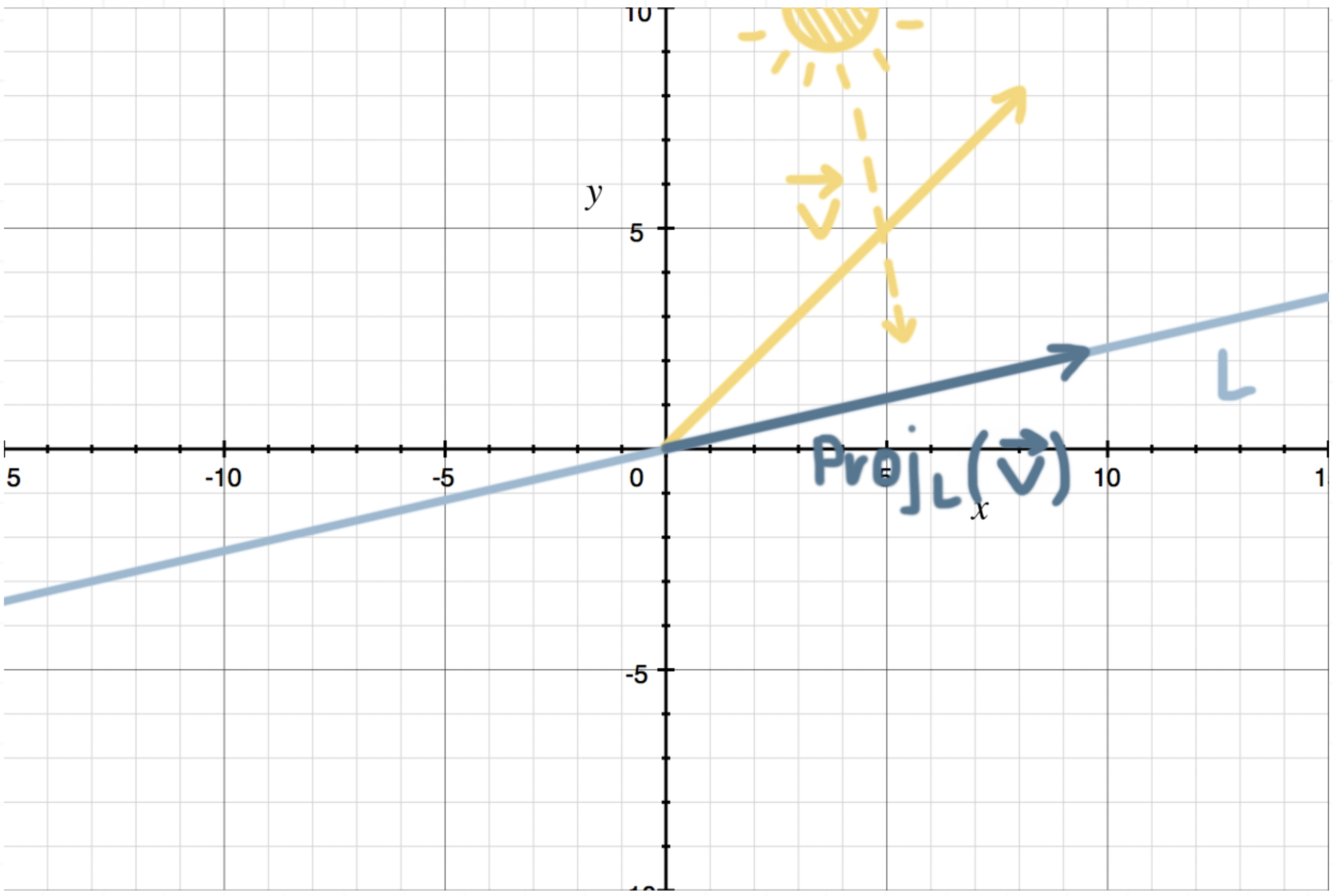
Now we want to start looking at some applications of linear transformations. In this lesson, we want to define a projection (of a vector onto a line), and show how that projection can always be expressed as a linear transformation.

The projection of a vector onto a line

Let's say you're given some line L in two-dimensional space, and a vector $\vec{v}$, like this:



Think of the **projection** of $\vec{v}$ onto L, which we write as $\text{Proj}_L(\vec{v})$, as the shadow that $\vec{v}$ casts onto L. When we imagine this shadow, think about shining a light that's above the vector $\vec{v}$ but also perpendicular to L.



If we name a vector $\vec{x}$ that lies on L, then we could say that the projection of $\vec{v}$ onto L is just some scaled version of $\vec{x}$, because L is equal to the set of all scalar multiples of the vector $\vec{x}$. And we could therefore write the projection as

$$\text{Proj}_L(\vec{v}) = c\vec{x}$$

The vector $\text{Proj}_L(\vec{v})$ is some vector in L, where $\vec{v} - \text{Proj}_L(\vec{v})$ is orthogonal to L. And if we do some work with $\vec{x}$, $\vec{v}$, and c, we can actually determine the value of c as

$$c = \frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}$$

So the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}\right)\vec{x}$$

Let's do an example with another diagram so that we can see what it looks like to actually find the projection vector.

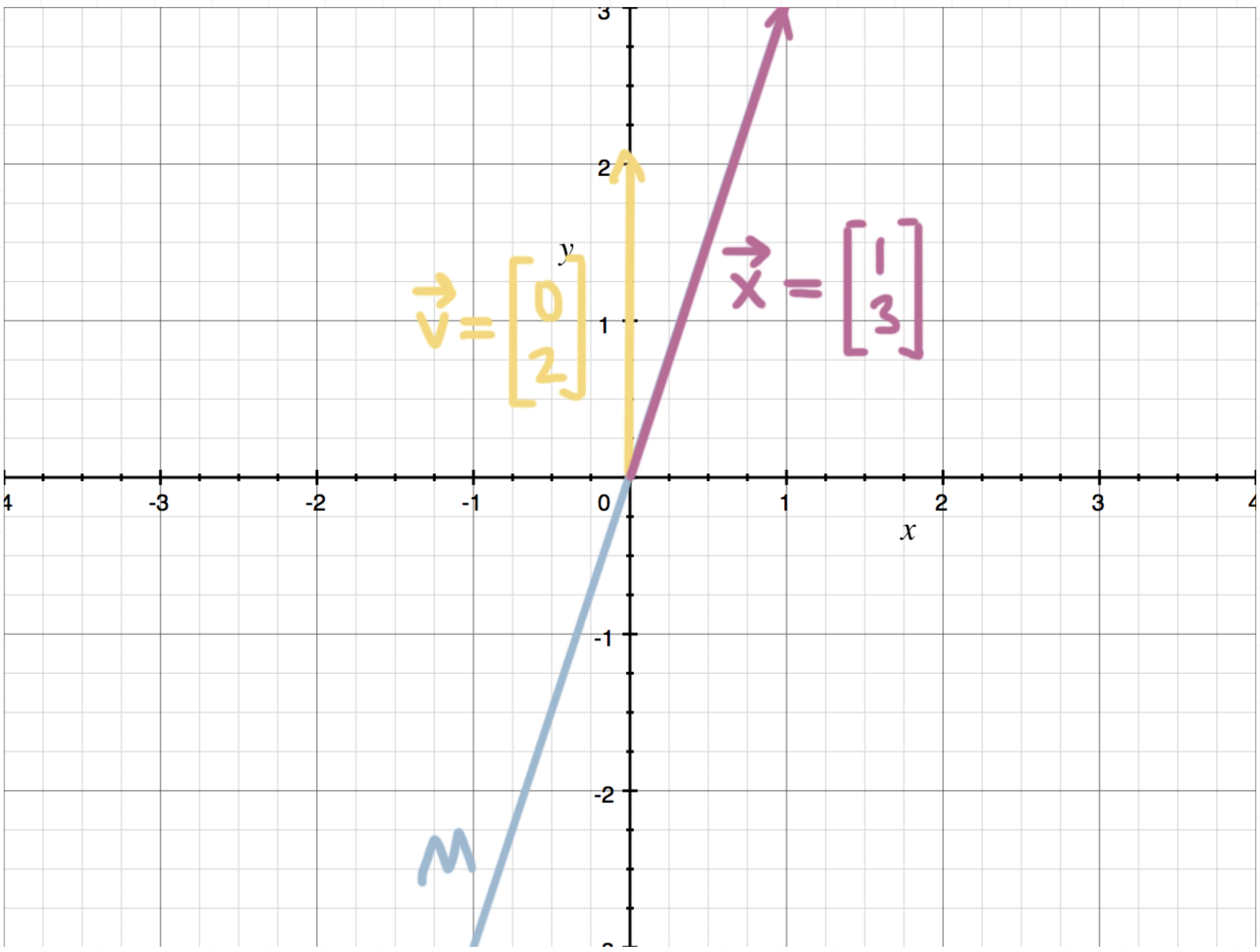
Example

Find the projection of $\vec{v}$ onto M.

$$M = \left\{ c \begin{bmatrix} 1 \\ 3 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The line M is given as all the scaled versions of the vector $\vec{x} = (1,3)$. So we could sketch M, $\vec{v}$, and $\vec{x}$ as



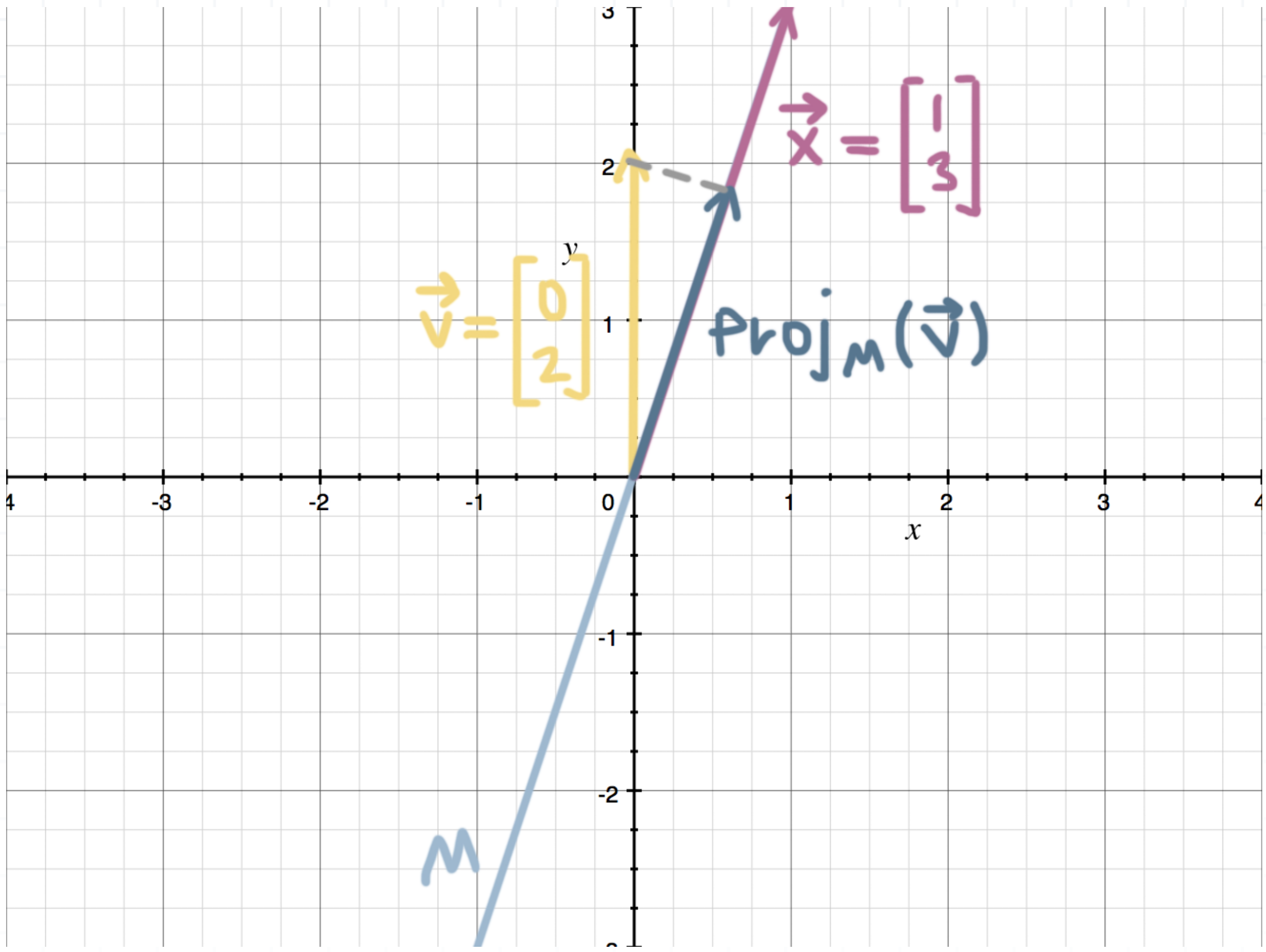
Then the projection of $\vec{v}$ onto M is given by

$$\text{Proj}_M(\vec{v})=\left(\frac{\vec{v}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\right)\vec{x}$$

$$\text{Proj}_M(\vec{v})=\frac{\begin{bmatrix}0 & 2\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1 & 3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\frac{0(1)+2(3)}{1(1)+3(3)}\cdot\begin{bmatrix}1\\3\end{bmatrix}$$

$$\text{Proj}_M(\vec{v})=\frac{6}{10}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\frac{3}{5}\cdot\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix}=\begin{bmatrix}0.6\\1.8\end{bmatrix}$$

We can sketch the projection, $\text{Proj}_M(\vec{v})$,



and we can see how it confirms our idea of the projection as the shadow of $\vec{v}$ onto M.

Normalizing the vector that defines the line

At this point, you might be wondering how we can always get a correct value for the projection, when the vector we use to define the line could change to any vector that's collinear with the line.

After all, in the last example, we defined the line with $\vec{x}=(1,3)$, but we could just have easily defined the line M as $\vec{x}=(2,6)$, $\vec{x}=(-1,-3)$, or any other of the infinitely many vectors that lie along M.

While the projection formula still works, regardless of the vector we pick to lie along M, this brings up the interesting point that we could have picked a unit vector instead. In other words, given a vector $\vec{x}$ that lies along the line, we could normalize it to its corresponding unit vector using

$$\hat{u} = \frac{1}{||\vec{x}||}\vec{x}$$

If we do, then we can actually simplify the projection formula. First, we know that a vector dotted with itself is equivalent to the square of that vector's length. So the projection formula can be rewritten as

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}\right)\vec{x}$$

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{||\vec{x}||^2}\right)\vec{x}$$

If we've normalized $\vec{x}$, then the length of $\vec{x}$ is 1, which means the projection formula becomes

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{1^2}\right)\vec{x}$$

$$\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \vec{x})\vec{x}$$

Furthermore, we've changed $\vec{x}$ into the unit vector $\hat{u}$, so the projection formula is

$$\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u})\hat{u}$$

So let's say we continue on with the previous example. We were told that the line M was defined by the vector $\vec{x}=(1,3)$. We could have normalized that vector as

$$||\vec{x}||=\sqrt{1^2+3^2}=\sqrt{1+9}=\sqrt{10}$$

Then the unit vector associated with $\vec{x}=(1,3)$ would be

$$\hat{u}=\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$$

And the projection of $\vec{v}=(0,2)$ onto M would have been

$$\text{Proj}_M(\vec{v})=(\vec{v}\cdot\hat{u})\hat{u}$$

$$\text{Proj}_M(\vec{v})=\left(\begin{bmatrix}0 & 2\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$$

$$\text{Proj}_M(\vec{v})=\left(0\left(\frac{1}{\sqrt{10}}\right)+2\left(\frac{3}{\sqrt{10}}\right)\right)\begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$$

$$\text{Proj}_M(\vec{v})=\frac{6}{\sqrt{10}}\begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$$

$$\text{Proj}_M(\vec{v}) = \begin{bmatrix} \frac{6}{10} \\ \frac{18}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{9}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1.8 \end{bmatrix}$$

The projection is a linear transformation

As it turns out, the projection of a vector $\vec{v}$ onto the line L is always a linear transformation. Since it's a linear transformation, of course that means that it's closed under addition and closed under scalar multiplication. It's closed under addition because the projection of the sum of two vectors is equivalent to the projection of each vector individually.

$$\text{Proj}_L(\vec{a} + \vec{b}) = ((\vec{a} + \vec{b}) \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = (\vec{a} \cdot \hat{u} + \vec{b} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = (\vec{a} \cdot \hat{u})\hat{u} + (\vec{b} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{a} + \vec{b}) = \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b})$$

And it's closed under scalar multiplication because the projection of the product of a scalar and a vector is equivalent to the product of that scalar and the projection of the vector.

$$\text{Proj}_L(c\vec{a}) = (c\vec{a} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(c\vec{a}) = c(\vec{a} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(c\vec{a}) = c\text{Proj}_L(\vec{a})$$

Because a projection is always a linear transformation, that means we must be able to express it as a matrix-vector product. In other words, $\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u})\hat{u} = A\vec{v}$, and we just need to find the matrix A.

We can do this in n dimensions ($\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, etc.), but let's assume for now that the projection is a transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Then to find A, we start with the I_2 identity matrix, and we'll call the unit vector $\hat{u} = (u_1, u_2)$. The matrix A is then

$$A = \begin{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix}$$

$$A = \begin{bmatrix} \left(1(u_1) + 0(u_2) \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & \left(0(u_1) + 1(u_2) \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix}$$

$$A = \begin{bmatrix} u_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & u_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix}$$

$$A = \begin{bmatrix} u_1^2 & u_2 u_1 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

In other words, given any unit vector $\hat{u} = (u_1, u_2)$ along the line L, the projection of the vector $\vec{v}$ onto L, assuming that we've defined L by a unit vector $\hat{u}$, is

$$\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u})\hat{u} = A\vec{v} = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \vec{v}$$

If we go back to the example we were working with earlier, we were trying to project $\vec{v} = (0,2)$ onto M, where M was defined by the unit vector

$$\hat{u}=\begin{bmatrix}\frac{1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{bmatrix}$$

So using the new formula for the projection as the matrix-vector product, we could say that the projection of $\vec{v}$ onto M is

$$\text{Proj}_M(\vec{v})=\begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}\vec{v}$$

$$\text{Proj}_M(\vec{v})=\begin{bmatrix}\left(\frac{1}{\sqrt{10}}\right)^2 & \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)\\ \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) & \left(\frac{3}{\sqrt{10}}\right)^2\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}$$

First find the matrix A.

$$A=\begin{bmatrix}\left(\frac{1}{\sqrt{10}}\right)^2 & \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)\\ \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) & \left(\frac{3}{\sqrt{10}}\right)^2\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{1}{10} & \frac{3}{10}\\ \frac{3}{10} & \frac{9}{10}\end{bmatrix}$$

Then the projection of $\vec{v}=(0,2)$ onto M is

$$\text{Proj}_M(\vec{v}) = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\text{Proj}_M(\vec{v}) = \begin{bmatrix} \frac{1}{10}(0) + \frac{3}{10}(2) \\ \frac{3}{10}(0) + \frac{9}{10}(2) \end{bmatrix}$$

$$\text{Proj}_M(\vec{v}) = \begin{bmatrix} \frac{6}{10} \\ \frac{18}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{9}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1.8 \end{bmatrix}$$

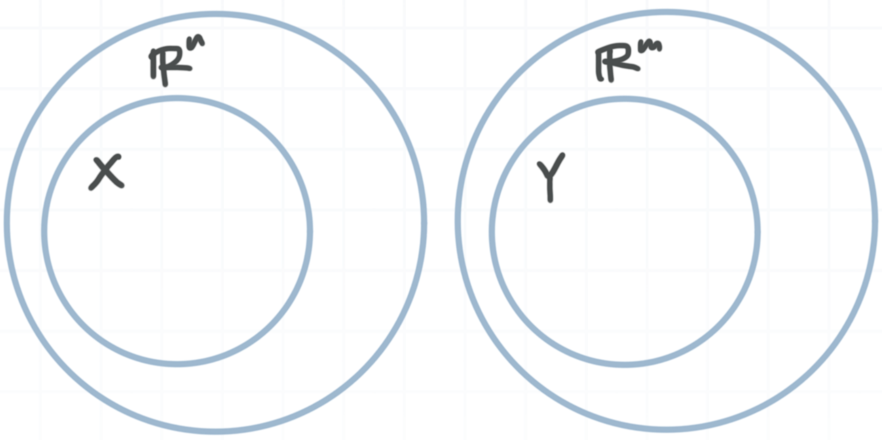
Not only did we get the same result for the projection as what we found earlier, but we also found a matrix for the projection of *any* vector onto M. Now, given any vector $\vec{v}$ in $\mathbb{R}^2$ that we want to project onto M, that projection will always be given by

$$\text{Proj}_M(\vec{v}) = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix} \vec{v}$$

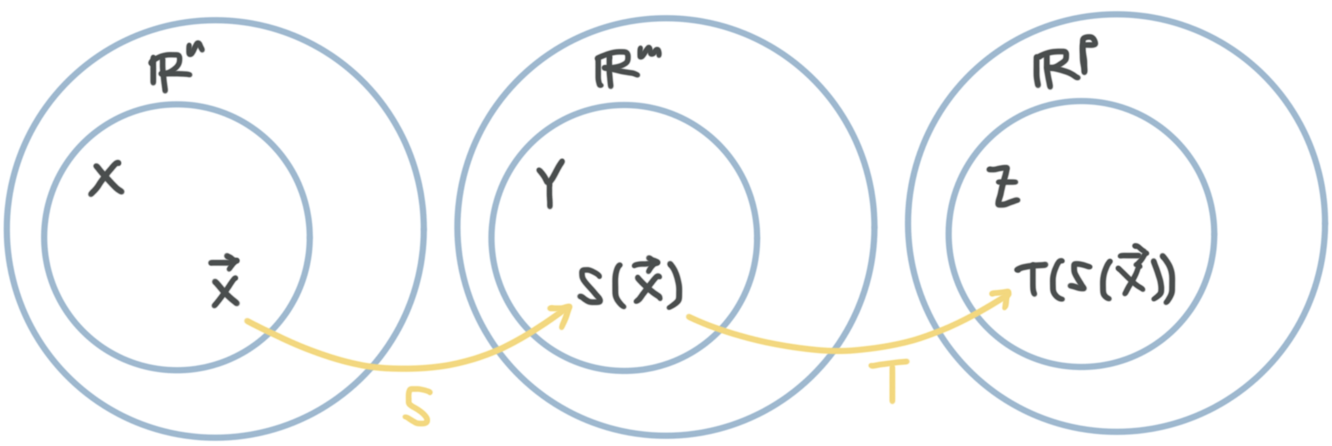
In other words, the reason it's helpful to rewrite the projection as a matrix-vector product is because it generalizes the projection into a transformation matrix, which we can then apply to any vector we want to project onto the line.

Compositions of linear transformations

We've already talked about a linear transformation as transforming vectors in a set X to vectors in a set Y. For instance, we can say that the set X is a subset of $\mathbb{R}^n$ and the set Y is a subset of $\mathbb{R}^m$.



We'll also say that S is a linear transformation that transforms vectors from X to Y, $S: X \to Y$. But let's then say that we want to take vectors from the subset Y and transform them into vectors in a subset Z, which is contained in $\mathbb{R}^p$ space. If T is a linear transformation that transforms vectors from Y to Z, $T: Y \to Z$,



then we can start to think about a composition of transformations. Because at this point, if we want to transform directly from X to Z, we can do so using a composition of the transformations S and T. S will take us from X to Y, and then T will take us from Y to Z, so a composition of T with S will take us all the way from X to Z.

For instance, let's say we have a vector $\vec{x}$ that lies in the subset X, and we want to transform it into Z. We could first transform it into Y using the transformation $S : X \to Y$, and we'd get $S(\vec{x})$. Then we could transform this transformed vector from Y into Z using the transformation $T : Y \to Z$, and we'd get $T(S(\vec{x}))$. We can also write $T(S(\vec{x}))$ as $T \circ S(\vec{x})$, so

$$T \circ S : X \to Z$$

is the composition of T with S.

Compositions as linear transformations

If we know that the transformations S and T are linear transformations, then we can say that the composition of the transformations $T(S(\vec{x}))$ is also a linear transformation. And we know this is true, because we can see that the composition is closed under addition,

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x} + \vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x}) + S(\vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T(S(\vec{x})) + T(S(\vec{y}))$$

$$T \circ S(\vec{x} + \vec{y}) = T \circ S(\vec{x}) + T \circ S(\vec{y})$$

and closed under scalar multiplication.

$$T(S(c\vec{x})) = T(S(c\vec{x}))$$

$$T(S(c\vec{x})) = T(cS(\vec{x}))$$

$$T(S(c\vec{x})) = cT(S(\vec{x}))$$

Additionally, because S and T are linear transformations, they can each be written as a matrix-vector product.

Let's say the transformation S is given by the matrix-vector product $S(\vec{x}) = A\vec{x}$, where A is an $m \times n$ matrix, and the transformation T is given by the matrix-vector product $T(\vec{x}) = B\vec{x}$, where B is a $p \times m$ matrix. Then because the composition is also a linear transformation, it can also be written as a matrix-vector product, and that matrix-vector product is

$$T \circ S(\vec{x}) = T(S(\vec{x})) = T(A\vec{x}) = BA\vec{x} = C\vec{x}$$

where C is a $p \times n$ matrix.

Looking at this equation, we can see that as long as we can find the matrix-matrix product BA, we'll be able to find the transformation of any vector $\vec{x}$ that's in the subset X, transformed all the way through the subset Y and into the subset Z, simply by multiplying the matrix-matrix product BA by the vector $\vec{x}$ that we want to transform.

Example

The transformation $S: X \to Y$ transforms vectors in the subset X into vectors in the subset Y. The transformation $T: Y \to Z$ transforms vectors in the subset Y into vectors in the subset Z. The subsets X, Y, and Z are all in

$\mathbb{R}^2$. Find a matrix that represents the composition of the transformations $T \circ S$, and then use it to transform $\vec{x}$ in X into its associated vector $\vec{z}$ in Z.

$$S(\vec{x}) = \begin{bmatrix} 3x_1 \\ x_1 - x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2x_1 + 4x_2 \\ -x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

We need to start by representing each transformation as a matrix-vector product. Because in both cases we're transforming *from* $\mathbb{R}^2$, we'll use the I_2 identity matrix,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the transformation S to each column of the identity matrix gives

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(1) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(0) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

So the transformation S can be written as the matrix-vector product

$$S(\vec{x}) = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}$$

Applying the transformation T to each column of the I_2 identity matrix gives

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)+4(0)\\-0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)+4(1)\\-1\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix}$$

So the transformation T can be written as the matrix-vector product

$$T(\vec{y})=\begin{bmatrix}2&4\\0&-1\end{bmatrix}\vec{y}$$

If we call the matrix from S

$$A=\begin{bmatrix}3&0\\1&-1\end{bmatrix}$$

and we call the matrix from T

$$B=\begin{bmatrix}2&4\\0&-1\end{bmatrix}$$

then the composition of the transformations can be written as the matrix-vector product

$$T(S(\vec{x}))=BA\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}2&4\\0&-1\end{bmatrix}\begin{bmatrix}3&0\\1&-1\end{bmatrix}\vec{x}$$

Now we can multiply the matrices. Remember that matrix multiplication (which is *not* commutative) is only defined when the number of columns in the first matrix is equivalent to the number of rows in the second matrix. In this case, the first matrix has two columns, and the second matrix has two rows, so the product of the matrices is defined.

$$C = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2(3) + 4(1) & 2(0) + 4(-1) \\ 0(3) - 1(1) & 0(0) - 1(-1) \end{bmatrix}$$

$$C = \begin{bmatrix} 6 + 4 & 0 - 4 \\ 0 - 1 & 0 + 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 10 & -4 \\ -1 & 1 \end{bmatrix}$$

This matrix will allow us to transform any vector $\vec{x}$ from the original subset X through the subset Y and into the subset Z. In other words, it lets us take vectors straight from X all the way to Z.

The composition of the transformations can be written as

$$T(S(\vec{x})) = \begin{bmatrix} 10 & -4 \\ -1 & 1 \end{bmatrix} \vec{x}$$

The problem asks us to transform $\vec{x} = (3,4)$, so we simply find the matrix-vector product.

$$T\left(S\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)\right) = \begin{bmatrix} 10 & -4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}3\\4\end{bmatrix}\right)\right)=\begin{bmatrix}10(3)-4(4)\\-1(3)+1(4)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}3\\4\end{bmatrix}\right)\right)=\begin{bmatrix}30-16\\-3+4\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}3\\4\end{bmatrix}\right)\right)=\begin{bmatrix}14\\1\end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x}=(3,4)$ in the subset X is transformed into the vector $\vec{z}=(14,1)$ in the subset Z.

Inverse of a transformation

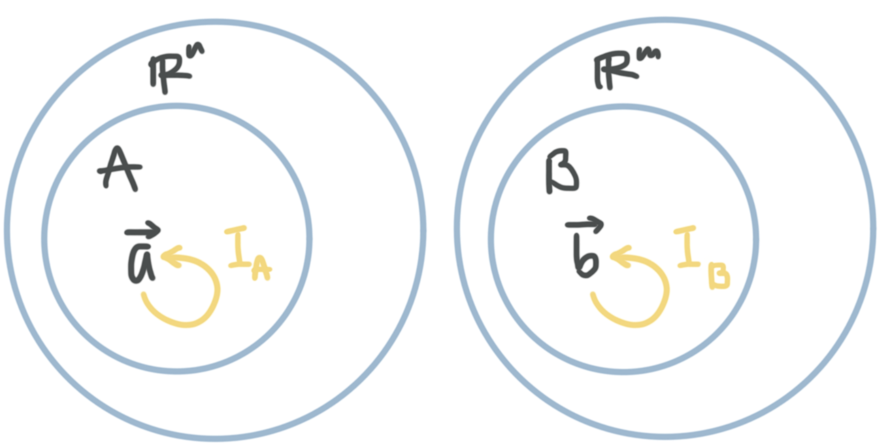
Previously we talked about a transformation as a mapping, something that maps one vector to another.

So if a transformation maps vectors from the subset A to the subset B, such that if $\vec{a}$ is a vector in A, the transformation will map it to a vector $\vec{b}$ in B, then we can write that transformation as $T : A \to B$, or as $T(\vec{a}) = \vec{b}$.

The identity transformation

With that in mind, we want to define the **identity transformation** as the transformation that maps vectors to themselves. There's no trick here. The identity transformation I literally just says, given any vector $\vec{x}$ in X, the transformation will map it to $\vec{x}$, itself. In other words, $I : X \to X$, or as $I_x(\vec{x}) = \vec{x}$.

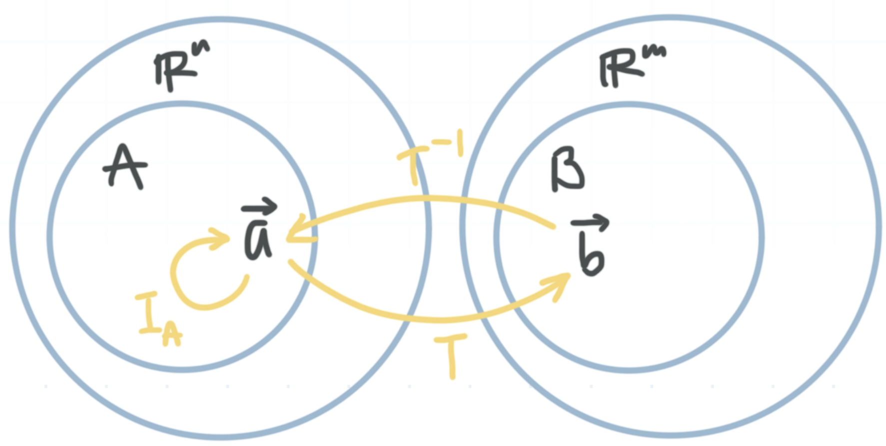
So for a vector $\vec{a}$ in the subset A, the identity transformation would be $I : A \to A$, or $I_A(\vec{a}) = \vec{a}$. And for a vector $\vec{b}$ in the subset B, the identity transformation would be $I : B \to B$, or $I_B(\vec{b}) = \vec{b}$.



Invertibility and inverse transformations

So we know that the identity transformation maps a vector to itself, without, in a way, ever really leaving the subset.

What we want to know now is whether we can use a transformation to map a vector from one subset A to another subset B, and then undo the process, mapping the result we got in B back into A, and end up with the same vector in A that we started with. In a sense, we'd just end up with the identity transformation, but we would have left the subset and come back again.



If we can do this for every vector $\vec{a}$ in A (use a transformation to map $\vec{a}$ in A to $\vec{b}$ in B, and then reverse the process to map $\vec{b}$ back to the $\vec{a}$ we started with), then the transformation is invertible.

In other words, a transformation is **invertible** if it has an **inverse**. If $T: A \to B$, then T is invertible if and only if its inverse $T^{-1}: B \to A$ exists, such that the composition of T with T^{-1} and vice versa are both defined by the identity transformations we talked about earlier:

$$T^{-1} \circ T = I_A$$

$$T \circ T^{-1} = I_B$$

This make sense. If $T : A \to B$ and $T^{-1} : B \to A$, then $T^{-1} \circ T = I_A$ could be written as

$$T^{-1} \circ T = I_A$$

$$T^{-1}(T(\overrightarrow{a})) = I_A(\overrightarrow{a})$$

The transformation $T(\overrightarrow{a})$ maps a vector $\overrightarrow{a}$ in A to some $\overrightarrow{b}$ in B. But then T^{-1} takes that result $\overrightarrow{b}$ in B and maps that vector back to an $\overrightarrow{a}$ in A. In other words, we started at a vector $\overrightarrow{a}$ in A, and ended up right back at the exact same vector $\overrightarrow{a}$ in A, which we know now is just the identity transformation for A, I_A. And we could follow the opposite logical path for $T \circ T^{-1} = I_B$.

And of course, if T and T^{-1} are inverses of one another, it's implied that the domain of T is the codomain of T^{-1}, and that the domain of T^{-1} is the codomain of T.

Invertibility and uniqueness

If a transformation is invertible, there are really three other related conclusions we can make.

1. Its inverse transformation is unique. In other words, an invertible transformation cannot have multiple inverses. It will always have exactly one inverse.

2. When you apply the transformation T to a vector $\vec{a}$ in A, you'll be mapped to one unique vector $\vec{b}$ in B. In other words, the transformation can never map you from $\vec{a}$ in A to multiple values $\vec{b}_1$ and $\vec{b}_2$ in B.

3. When you apply the inverse transformation T^{-1} to a vector $\vec{b}$ in B, you'll be mapped back to one unique vector $\vec{a}$ in A. In other words, the unique inverse transformation can never map you from $\vec{b}$ in B back to multiple vectors $\vec{a}_1$ and $\vec{a}_2$ in A.

Surjective and injective

If T is a transformation that maps vectors in the subset A to vectors in the subset B, $T : A \to B$, and if every vector $\vec{b}$ in B is being mapped to at least once by some vector(s) $\vec{a}$ in A such that $T(\vec{a}) = \vec{b}$, then T is a **surjective** transformation, also called an **onto** transformation. If there's any vector $\vec{b}$ in B which, via T, is not mapped to by any $\vec{a}$ in A, then T is not surjective; it's not onto.

If every vector $\vec{b}$ in $\mathbf{B}$ is being mapped to, then $\mathbf{T}$ is surjective, or onto.

If T is a transformation that maps vectors in the subset A to vectors in the subset B, $T : A \to B$, and if every vector $\vec{b}$ in B is being mapped to by at most one vector $\vec{a}$ in A such that $T(\vec{a}) = \vec{b}$, then T is an **injective** transformation, also called a **one-to-one** transformation. If there is any vector $\vec{b}$ in B which, via T, is mapped to by two or more $\vec{a}$ in A, then T is not injective; it's not one-to-one.

If every $\vec{a}$ maps to a unique $\vec{b}$, then $\mathbf{T}$ is injective, or one-to-one

In other words, if there's a vector $\vec{b}$ in B that's not being mapped to, then the transformation isn't surjective. But if there are multiple vectors $\vec{a}_1$ and $\vec{a}_2$ in A being mapped to the same vector $\vec{b}$ in B, then the transformation isn't injective.

And remember before that we said a transformation was invertible if we could find an inverse transformation, and we know that a transformation will only ever have one inverse.

If we formalize that definition, we're saying that:

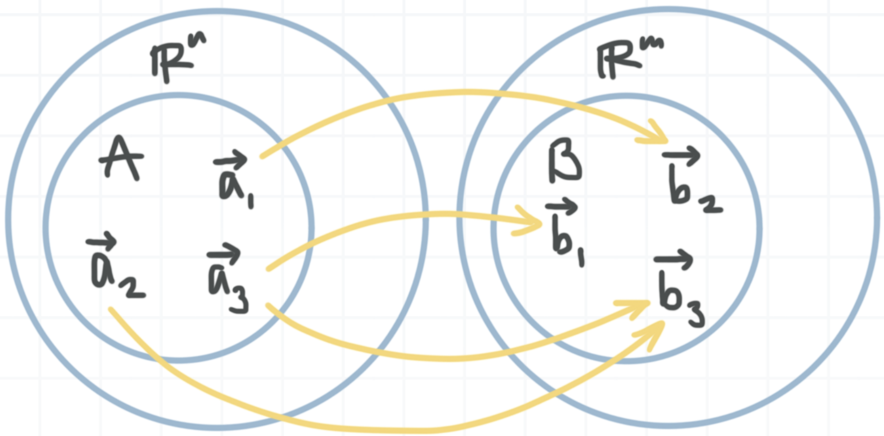
> *A transformation is invertible if, for every $\vec{b}$ in $\mathbf{B}$, there's a unique $\vec{a}$ in A, such that $T(\vec{a}) = \vec{b}$.*

This definition is telling us that *every* $\vec{b}$ is being mapped to, which means the transformation is surjective, or onto. The definition is also telling us that there's a *unique* $\vec{a}$ in A that's mapping to each $\vec{b}$, which means the transformation is injective, or one-to-one.

Therefore, we can say that a transformation T is invertible if and only if T is both surjective and injective.

Example

Say whether the transformation T is invertible.



In order for the transformation to be invertible, it must be both surjective and injective (onto and one-to-one).

If we look at the picture of the transformation, we can see that every $\vec{b}$ in B is being mapped to by an $\vec{a}$ in A, which means the transformation is surjective.

But we can see that both $\vec{a}_2$ and $\vec{a}_3$ are mapping to $\vec{b}_3$, which means we have multiple $\vec{a}$'s in A mapping to the same $\vec{b}$ in B, and therefore that the transformation is *not* injective.

The transformation is invertible if and only if it's both surjective and injective. In this example, it's surjective, but not injective, and therefore, it's not invertible. The transformation does not have an inverse.

Invertibility from the matrix-vector product

Remember we said before that any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix-vector product,

$T(\vec{x}) = M\vec{x}$, where M is an $m \times n$ matrix

In this form, we can say that the transformation T is surjective (onto) if the column space of M is the entire codomain, $\mathbb{R}^m$.

$$C(M) = \mathbb{R}^m$$

The column space of M will be the entire codomain $\mathbb{R}^m$ only when the reduced row-echelon form of M has m pivot entries, one pivot entry in every row. But of course, by the definition of a pivot entry, each pivot entry will be in its own column, which means M also has m pivot columns.

That also means that m basis vectors make up the column space of M, $C(M)$, and that the rank of M is m, $\text{Rank}(M) = \text{Dim}(C(M)) = m$.

These are all just different ways of saying that $T: A \to B$ is surjective, which means that every vector $\vec{b}$ in the codomain B is mapped to at least once by some vector $\vec{a}$ in the domain A via the transformation T.

In other words,

- if M becomes the identity matrix in reduced row-echelon form, or

- if M has m pivot entries, or

- if m basis vectors make up the column space of M, $C(M)$, or

- if the rank of M is m,

then you know that the transformation is surjective.

Let's look at a complete example so that we can see how to figure out whether the transformation is surjective, when we've been given the transformation as a matrix-vector product.

Example

The transformation T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$. Say whether or not T is surjective.

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \vec{x}$$

We can see that T is given as a matrix-vector product. If we call the matrix M, then

$$M = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

To determine whether or not T is surjective, first put M into reduced row-echelon form. Find the pivot entry in the first row.

$$\begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & -2 \\ 0 & 7 \end{bmatrix}$$

Find the pivot entry in the second column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 0 & 7 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now that M is rewritten in reduced row-echelon form,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we can see that it has two pivot entries in two pivot columns. This means there are two columns that make up the basis of the column space of M, $C(M)$, or that the rank of M is 2, $\text{Rank}(M) = \text{Dim}(C(M)) = 2$.

Because the codomain is $\mathbb{R}^4$, and since $2 \neq 4$, we can say that the transformation T is not surjective; it is not onto.

Only square matrices are invertible

We want to go one step further with our understanding of invertibility, by focusing on the matrix M. Let's say for now that M is any $m \times n$ matrix (it has m rows and n columns).

First, there are two things we know:

1. In order for the transformation to be surjective, the rank of M must be m, $\text{Rank}(M) = m$, meaning that we must have a pivot entry in every row.

2. But in order for the transformation to be injective, the rank of M must also be n, $\text{Rank}(M) = n$, meaning that we must have a pivot entry in every column.

It's only possible for $\text{Rank}(M) = m$ and $\text{Rank}(M) = n$ if $m = n$. In other words, a matrix can only be invertible if it's both surjective and injective, but it can only be surjective and injective if $m = n$. So the matrix can only be invertible if it has the same number of rows and columns. And if that's true, then the matrix M has the same number of rows and columns (it's a square matrix), and we can call it an $n \times n$ matrix. So,

Only square matrices can be invertible.

It's important to say that not all square matrices are invertible, but only square matrices have the potential to be invertible. If a square matrix is invertible, then its reduced row-echelon form will be an $n \times n$ matrix where every column is a linearly independent pivot column, which will of course be the $n \times n$ identity matrix I_n.

Because $m = n$, this also implies that the transformation T must map from a domain $\mathbb{R}^n$ to the codomain $\mathbb{R}^n$. In other words, a transformation can only be invertible when it maps $\mathbb{R}^n \to \mathbb{R}^n$.

You could also say it this way. Given a transformation $T(\vec{x}) = M\vec{x}$, if you put the matrix M into reduced row-echelon form and get the identity matrix I_n, then you know that

- the matrix M was a square $n \times n$ matrix, and that
- the transformation T maps $\mathbb{R}^n \to \mathbb{R}^n$, and that
- T is invertible.

Inverse transformations are linear

Inverse transformations are linear transformations

The inverse of an invertible linear transformation T is also itself a linear transformation. Which means that the inverse transformation T^{-1} is closed under addition and closed under scalar multiplication.

$$T^{-1}(\vec{u}+\vec{v})=T^{-1}(\vec{u})+T^{-1}(\vec{v})$$

$$T^{-1}(c\vec{u})=cT^{-1}(\vec{u})$$

In other words, as long as the original transformation T

1. is a linear transformation itself, and

2. is invertible (its inverse is defined, you can find its inverse),

then the inverse of T, T^{-1}, is also a linear transformation.

Inverse transformations as matrix-vector products

Remember that any linear transformation can be represented as a matrix-vector product. Normally, we rewrite the linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ as

$T(\vec{x})=A\vec{x}$, where A is a square $n\times n$ matrix

But because the inverse transformation T^{-1} is a linear transformation as well, we can also write the inverse as a matrix-vector product.

$$T^{-1}(\overrightarrow{x}) = A^{-1}\overrightarrow{x}$$

The matrix A^{-1} is the inverse of the matrix A that we used to define the transformation T.

In other words, given a linear transformation T and its inverse T^{-1}, if we want to express both of them as matrix-vector products, we know that the matrices we use to do so will be inverses of one another.

The reason this is true is because we already know that taking the composition of the inverse transformation T^{-1} with T will always give us the identity matrix I_n, where n is the dimension of the domain and codomain, $\mathbb{R}^n$.

$$(T^{-1} \circ T)(\overrightarrow{x}) = I_n\overrightarrow{x}$$

Going the other way (applying the transformation to the inverse transformation), gives the same result:

$$(T \circ T^{-1})(\overrightarrow{x}) = I_n\overrightarrow{x}$$

The only way these can be true is if the matrices that are part of the matrix-vector products are inverses of one another.

$$(T^{-1} \circ T)(\overrightarrow{x}) = I_n\overrightarrow{x} \qquad \to \qquad (T^{-1} \circ T)(\overrightarrow{x}) = A^{-1}A\overrightarrow{x}$$

$$(T \circ T^{-1})(\overrightarrow{x}) = I_n\overrightarrow{x} \qquad \to \qquad (T^{-1} \circ T)(\overrightarrow{x}) = AA^{-1}\overrightarrow{x}$$

Remember that matrix multiplication is not commutative, which means that, given two matrices A and B, AB is not equal to BA. We can't change the order of the matrix multiplication and still get the same answer. But based on these compositions of T with T^{-1} and vice versa, we're saying

$A^{-1}A$ and AA^{-1} must both equal I_n, which means they must equal each other. The only way that $A^{-1}A$ and AA^{-1} can both equal I_n is if A and A^{-1} are inverses of one another.

This is why, when we represent the inverse transformation T^{-1} as a matrix-vector product, we know that the matrix we use must be the inverse of the matrix we used to represent T. So if we represent T as $T(\vec{x}) = A\vec{x}$ with the matrix A, that means T^{-1} can only be represented as $T^{-1}(\vec{x}) = A^{-1}\vec{x}$ with the inverse of A, A^{-1}.

Finding the matrix inverse

We've said that the inverse transformation T^{-1} can be represented as the matrix-vector product $T^{-1}(\vec{x}) = A^{-1}\vec{x}$. Here's how we can find A^{-1}.

If we start with A, but augment it with the identity matrix, then all we have to do to find A^{-1} is work on the augmented matrix until A is in reduced row-echelon form.

In other words, given the matrix A, we'll start with the augmented matrix

$$[A \mid I]$$

Through the process of putting A in reduced row-echelon form, I will be transformed into A^{-1}, and we'll end up with

$$[I \mid A^{-1}]$$

This seems like a magical process, but there's a very simple reason why it works. Remember earlier in the course that we learned how one row operation could be expressed as an elimination matrix E. And that if we performed lots of row operations, through matrix multiplication of E_1, E_2, E_3, etc., we could find one consolidated elimination matrix.

That's exactly what we're doing here. We're performing row operations on A to change it into I. All those row operations could be expressed as the elimination matrix E. And we're saying that if we multiply E by A, that we'll get the identity matrix, so $EA = I$. But as you know, $A^{-1}A = I$, which means $E = A^{-1}$.

Let's try an example to see how the augmented matrix flips from A into A^{-1}.

Example

Find A^{-1}.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

Because A is a 3×3 matrix, its associated identity matrix is I_3. So we'll augment A with I_3.

$$[A_{3\times3} \mid I_3]$$

$$\begin{bmatrix} -2 & 4 & 0 & | & 1 & 0 & 0 \\ 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Now we need to put A into reduced row-echelon form. We start by switching R_2 and R_1, to get a 1 into the first entry of the first row.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ -2 & 4 & 0 & | & 1 & 0 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Now we'll zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 2 & 8 & | & 1 & 2 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -1 & 4 & | & 0 & 1 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 6 & -4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -28 & | & -3 & -6 & 1 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & 8 & | & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 4 & | & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & | & \frac{1}{14} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 1 & | & \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$

Now that A has been put into reduced row-echelon form on the left side of the augmented matrix, the identity matrix on the right side has been turned into the inverse matrix A^{-1}. So we can say

$$A^{-1} = \begin{bmatrix} -\frac{5}{14} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{14} & \frac{1}{7} & \frac{1}{7} \\ \frac{3}{28} & \frac{3}{14} & -\frac{1}{28} \end{bmatrix}$$

Matrix inverses, and invertible and singular matrices

In the last lesson, we looked at how we could find the inverse of a matrix by augmenting it with its associated identity matrix, and then working the matrix into reduced row-echelon form.

In this lesson, we want to talk more about matrix inverses, and when a matrix inverse is defined at all.

Division as multiplication by the reciprocal

In a way, we can think about a matrix inverse as matrix division. To think about matrix division, we want to remember that dividing by some value is the same as multiplying by the reciprocal of that value. For instance, dividing by 4 is the same as multiplying by $1/4$. So if k is a real number, then we know that

$$k \cdot \frac{1}{k} = 1$$

If we call $1/k$ the inverse of k and instead write it as k^{-1}, then we could rewrite this equation as

$$kk^{-1} = 1$$

and read this as "k multiplied by the inverse of k is 1." What we want to know now is whether this is also true for matrices. If I divide matrix K by matrix K, or multiply matrix K by its inverse, do I get back to 1? In other words, we're trying to prove that

$$K \cdot \frac{1}{K} = I \text{ or } KK^{-1} = I$$

where I is the identity matrix, which of course is the matrix equivalent of 1.

Matrix inverses

Of course, as we already know, matrix division is a valid operation, because multiplying a matrix by its inverse will result in the identity matrix.

We also know already how to find a matrix inverse by augmenting the matrix with the identity matrix, and then using row operations to put the matrix into reduced row-echelon form.

But there's another way to find the matrix inverse, and this method will help us identify when the matrix is invertible. This method requires us to use something called the determinant, which we'll learn about in the next section. For now, we'll just say that the determinant of a matrix M,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\text{Det}(M) = |M| = ad - bc$$

Then the inverse of M is given by

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that the formula for the inverse matrix is a fraction with a numerator of 1 and the determinant as the denominator, multiplied by another matrix.

The other matrix is called the **adjugate** of M, and the adjugate is the matrix in which the values a and d have been swapped, and the values b and c have been multiplied by -1.

Let's do an example where we use this method with the determinant to find the inverse of a matrix.

Example

Find the inverse of matrix K, then find $K \cdot K^{-1}$ and $K^{-1} \cdot K$ to show that you found the correct inverse.

$$K = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}$$

To find the inverse of matrix K, we'll plug into the determinant formula for the inverse of a matrix.

$$K^{-1} = \frac{1}{|K|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$K^{-1} = \frac{1}{\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix}} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \frac{1}{-2(0) - 4(3)} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = -\frac{1}{12} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} -\frac{0}{12} & \frac{4}{12} \\ \frac{3}{12} & \frac{2}{12} \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

This is the inverse of K, but we can prove it to ourselves by multiplying K by its inverse. If we've done our math right, we should get the identity matrix when we multiply them.

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -2(0) + 4\left(\frac{1}{4}\right) & -2\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) \\ 3(0) + 0\left(\frac{1}{4}\right) & 3\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0+1 & -\frac{2}{3}+\frac{4}{6} \\ 0+0 & 1+0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K \cdot K^{-1} = I_2$$

When we multiplied K by its inverse, we get the identity matrix. We also want to make the point that we can multiply in the other direction, and we still get the identity matrix.

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(-2)+\frac{1}{3}(3) & 0(4)+\frac{1}{3}(0) \\ \frac{1}{4}(-2)+\frac{1}{6}(3) & \frac{1}{4}(4)+\frac{1}{6}(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ -\frac{1}{2}+\frac{1}{2} & 1+0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K^{-1} \cdot K = I_2$$

This example shows how to use the determinant formula to find the inverse matrix, and proves that multiplying by the inverse matrix is commutative. Whether we calculate $K \cdot K^{-1}$ or $K^{-1} \cdot K$, we get back to the identity matrix either way.

Invertible and singular matrices

So what does this determinant formula tell us about when the matrix inverse exists? After all, not every matrix has an inverse. Given the formula for the inverse matrix,

$$K^{-1} = \frac{1}{|K|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we can probably spot right away that, because a fraction is undefined when its denominator is 0, we have to say that $|K| \neq 0$. In other words, if the determinant is not equal to 0, then the inverse matrix is defined, and the matrix K is invertible. But if the determinant is equal to 0, then the inverse matrix is undefined, and the matrix K is not invertible. To be specific, if

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse of K is undefined when $ad - bc = 0$, or when $ad = bc$. If we divide both sides of $ad = bc$ by b and d, we get

$$\frac{a}{b} = \frac{c}{d}$$

So if the ratio of a to b (the values in the first row of matrix K) is equal to the ratio of c to d (the values in the second row of matrix K), then you know right away that the matrix K does not have a defined inverse. If the matrix doesn't have an inverse, we call it a **singular matrix**. When the matrix does have an inverse, we say that it's an **invertible matrix**.

Example

Say whether each matrix is invertible or singular.

(a) $M = \begin{bmatrix} 1 & -3 \\ 3 & 5 \end{bmatrix}$

(b) $L = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$

For each matrix, we'll look at whether or not the ratio of a to b is equal to the ratio of c to d. For matrix M, we get

$$\frac{1}{-3} = -\frac{1}{3} \quad \neq \quad \frac{3}{5}$$

Because these aren't equivalent ratios, matrix M is invertible, which means it has a defined inverse. To confirm this, we'll calculate the determinant.

$$\text{Det}(M) = |M| = 1(5) - (-3)(3) = 5 + 6 = 11$$

Since $|M| \neq 0$, M is invertible.

For matrix L, we get

$$\frac{-6}{2} = -3 \quad = \quad \frac{3}{-1} = -3$$

Because these are equivalent ratios, matrix L is not invertible, which means L does not have a defined inverse, and we can therefore say that it's a singular matrix. To confirm this, we'll calculate the determinant.

$$\text{Det}(L) = |L| = (-6)(-1) - 2(3) = 6 - 6 = 0$$

Since $|L| = 0$, L is not invertible, which means L is a singular matrix.

Solving systems with inverse matrices

We know already how to solve systems of linear equations using substitution, elimination, and graphing. This time, we want to talk about how to solve systems using inverse matrices. To walk through this, let's use a simple system.

$$3x - 4y = 6$$

$$-2x + y = -9$$

And for the sake of illustration, let's compare this to the generic system

$$ax + by = f$$

$$cx + dy = g$$

We can always represent a linear system like this as the coefficient matrix, multiplied by the column vector (x, y), set equal to the column vector of the constants (f, g).

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Or, if we call the coefficient matrix M, rename the column vector (x, y) as $\vec{a}$, and rename the column vector (f, g) as $\vec{b}$, then we can say that this equation is equivalent to

$$M\vec{a} = \vec{b}$$

If we multiply both sides by the inverse matrix M^{-1}, we get

$$M^{-1}M\vec{a} = M^{-1}\vec{b}$$

Since $M^{-1}M = I$, we have

$$I\vec{a} = M^{-1}\vec{b}$$

When an identity matrix is multiplied by a vector, $I\vec{a} = \vec{a}$. Therefore,

$$\vec{a} = M^{-1}\vec{b}$$

The reason we went through those steps to multiply by the inverse is because we now have an equation that's solved for $\vec{a}$, which remember is representing the column vector (x, y), which really is just the solution to the system. In other words, this equation is saying that we can find the solutions to the system simply by multiplying the inverse matrix M^{-1} by the column vector (f, g)!

Furthermore, as long as we keep M, and therefore M^{-1} the same, we can substitute any values that we'd like for f and g (the constants on the right side of the system of equations), and $\vec{a} = M^{-1}\vec{b}$ will give us an immediate solution set for (x, y).

Right now that doesn't necessarily feel super useful, but as you use matrices in more advanced ways, it'll become extremely valuable to be able to change the values that make up $\vec{b}$, and immediately get the solution set for (x, y) that comes from $\vec{a}$.

Example

Use an inverse matrix to solve the system.

$$3x - 4y = 6$$

$$-2x + y = -9$$

Start by transferring the system into a matrix equation.

$$M\vec{a} = \vec{b}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{3(1) - (-2)(-4)} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{5} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{4}{5} \\ -\frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\vec{a} = M^{-1}\vec{b}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{5} & -\frac{4}{5} \\ -\frac{2}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{5}(6) - \frac{4}{5}(-9) \\ -\frac{2}{5}(6) - \frac{3}{5}(-9) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{6}{5} + \frac{36}{5} \\ -\frac{12}{5} + \frac{27}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{30}{5} \\ \frac{15}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that the solution to the system is $\vec{a} = (x, y) = (6,3)$.

Representing the system graphically

We can also express the same system from the example,

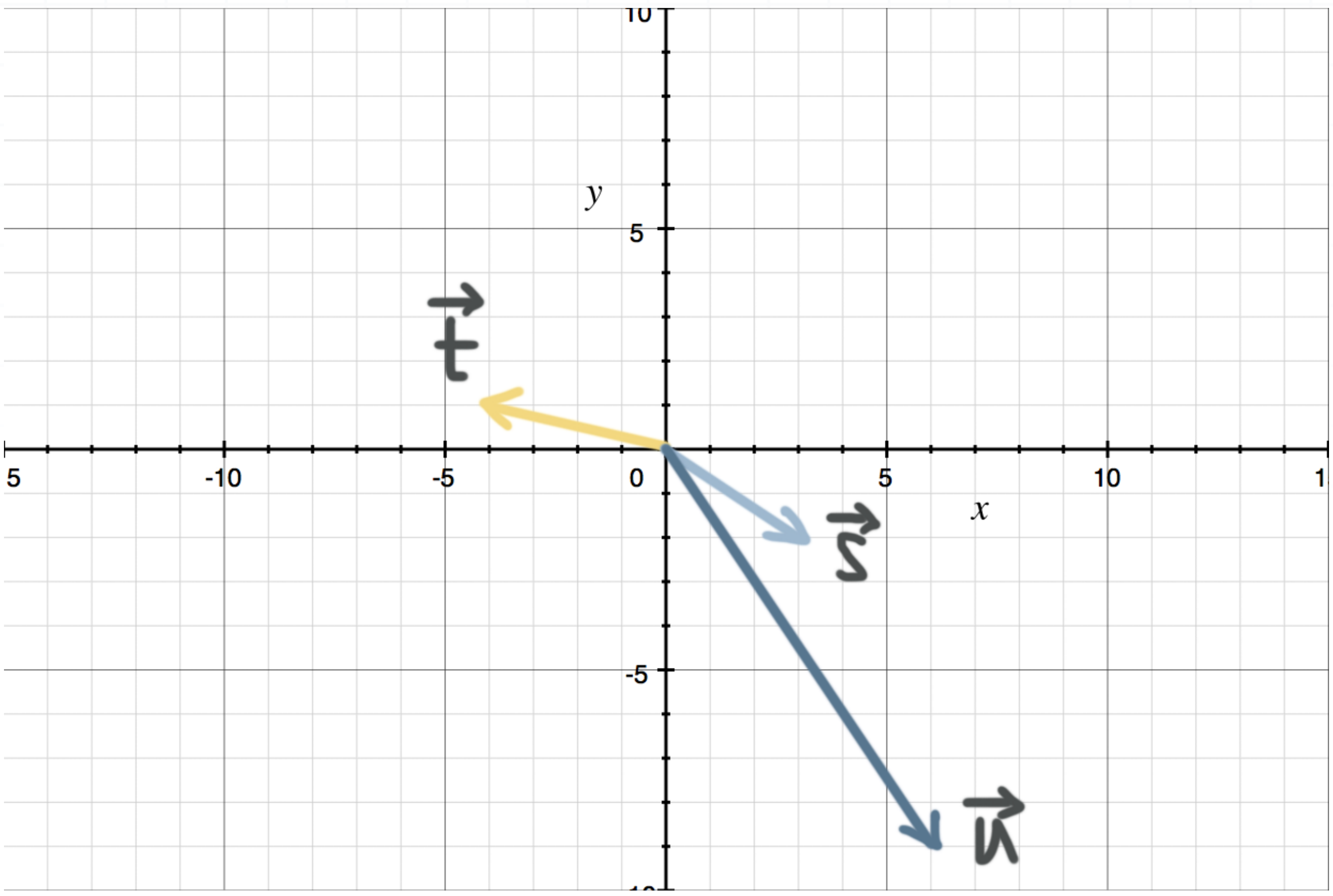
$$3x - 4y = 6$$

$$-2x + y = -9$$

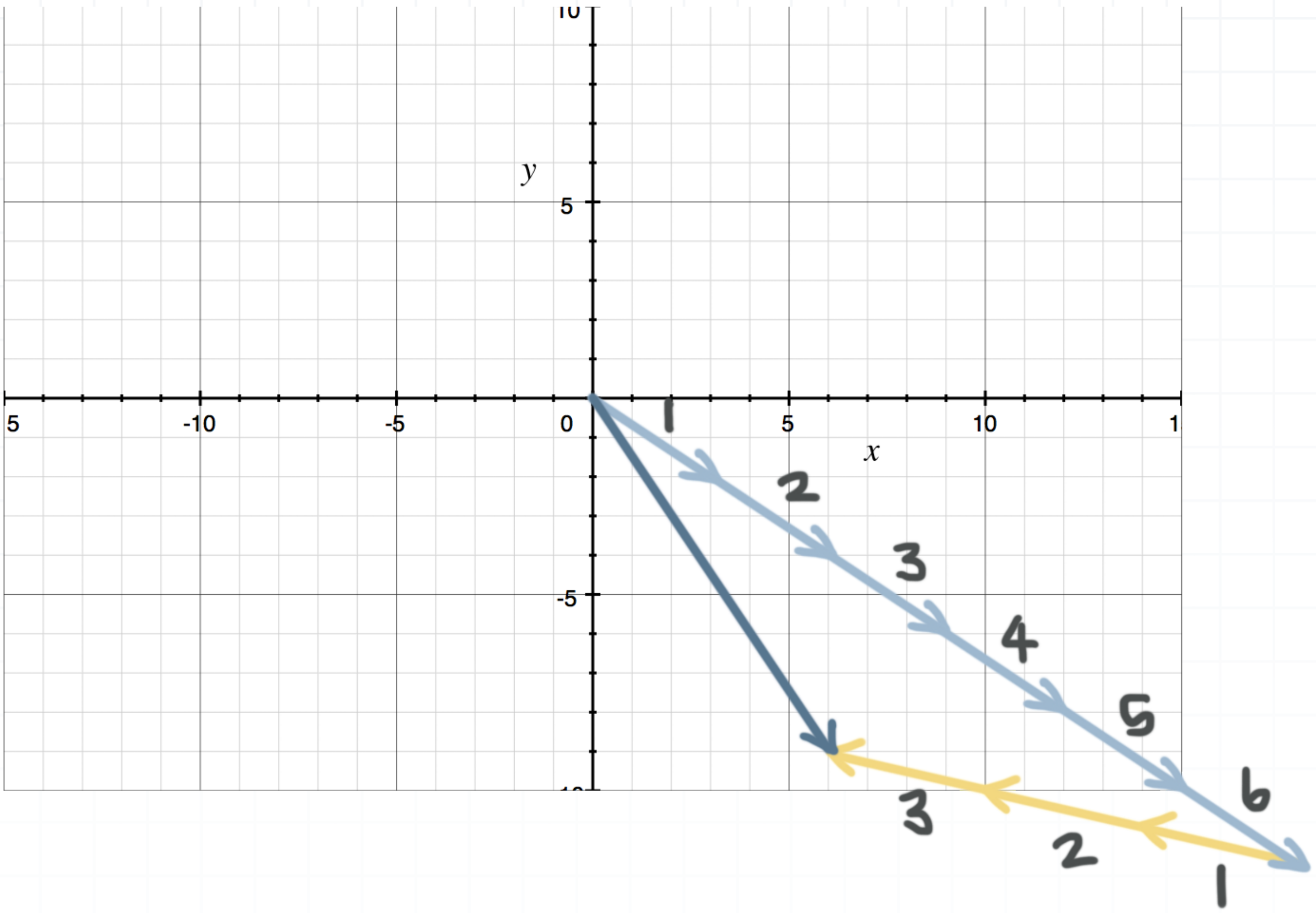
as this matrix equation:

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} x + \begin{bmatrix} -4 \\ 1 \end{bmatrix} y = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

Graphically, we can think about this equation as the vector $\vec{s} = (3, -2)$ for x and the vector $\vec{t} = (-4,1)$ for y, and the resulting vector $\overrightarrow{u} = (6, -9)$. If we put these vectors in the same coordinate plane, we get



We already know from the example that $x = 6$ and $y = 3$. What that solution tells us graphically is that we need to put 6 of the x vectors $\vec{s} = (3, -2)$, and 3 of the y vectors $\vec{t} = (-4,1)$ together, and we'll end up at the same spot as the resulting vector $\overrightarrow{u} = (6, -9)$.


10
y
5
5
-10
-5
0
1
5
10
1
x
2
3
4
5
-5
6
3
2
1

Determinants

In previous lessons, we learned two ways to find the inverse matrix. The first way was to augment the $n \times n$ matrix with the I_n identity matrix, and then put the matrix into reduced row-echelon form. Doing so changes the augmented I_n matrix into the inverse matrix.

The second way was to plug into the inverse matrix formula that uses the determinant,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In this lesson, we want to dig more into the determinant, both for a 2×2 matrix, and for larger matrices as well.

The determinant for 2×2 matrices

First, let's talk more about the formula for the inverse matrix,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We want to notice one really important thing. The inverse matrix is not defined when $ad - bc = 0$. In other words, if $ad - bc = 0$, then we get a 0 in the denominator of the fraction in the formula, which means the fraction is undefined, which means the inverse matrix itself is undefined.

And this fact actually becomes extremely useful for us. Because now we have a test that we can use to determine whether or not a matrix is invertible.

As we've said, the denominator $ad-bc$ is called the **determinant** of the matrix. So we can simply calculate the determinant, and then

- if the determinant is 0, the matrix is not invertible, so you can't find its inverse, but

- if the determinant is nonzero, the matrix is invertible, so you can find its inverse.

We write the determinant of the matrix

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

as any of these:

$$\text{Det}(A)=|A|=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$$

The determinant for 3×3 and $n\times n$ matrices

And the determinant isn't only for 2×2 matrices. We have a formula for the determinant of a 3×3 formula as well.

Given a matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

its determinant is

$$\text{Det}(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Notice how the determinant with a includes all of the entries outside of the row and column that contain a in the 3×3 determinant.

$$\begin{vmatrix} a & \cdot & \cdot \\ \cdot & e & f \\ \cdot & h & i \end{vmatrix}$$

In the same way, the determinant with b includes all of the entries outside of the row and column that contain b,

$$\begin{vmatrix} \cdot & b & \cdot \\ d & \cdot & f \\ g & \cdot & i \end{vmatrix}$$

and the determinant with c includes all of the entries outside of the row and column that contain c.

$$\begin{vmatrix} \cdot & \cdot & c \\ d & e & \cdot \\ g & h & \cdot \end{vmatrix}$$

This will always be how we'll calculate the determinant.

Notice also then that what's left in this 3×3 determinant formula are a few 2×2 determinants. We know how to calculate their determinants already, so we can simplify this formula even further.

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$|A| = aei - afh - bdi + bfg + cdh - ceg$$

$$|A| = aei + bfg + cdh - afh - bdi - ceg$$

We can expand this process beyond 2×2 and 3×3 matrices, to any $n \times n$ matrix. You may have noticed in the formula for the 3×3 determinant that we alternated signs, starting with a positive sign, and we got $+a$, then $-b$, then $+c$. The formula for an $n \times n$ matrix will always follow this same $+, \ -, +, \ -, \ldots$ pattern.

And the formula for an $n \times n$ matrix is recursive. In the same way that the formula the 3×3 determinant reduced the 3×3 determinant to a set of 2×2 determinants, the $n \times n$ determinant formula will reduce the $n \times n$ determinant to a set of $(n-1) \times (n-1)$ determinants, which can then be reduced to a set of $(n-2) \times (n-2)$ determinants, and so on, until eventually you'll be left with only a set of 2×2 determinants, which can be evaluated directly as $ad - bc$.

The following is true for the $n \times n$ matrices as well.

- if the determinant is 0, the matrix is not invertible, so you can't find its inverse, but

- if the determinant is nonzero, the matrix is invertible, so you can find its inverse.

Let's do an example to make sure we know how to use this method for a larger matrix.

Example

Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 0 & 1 & 2 \\ 4 & -2 & 0 & 3 \\ 4 & 1 & 2 & -1 \end{bmatrix}$$

The matrix A is 4×4. Which means we'll find its determinant by reducing to a set of 3×3 determinants with alternating signs,

$$|A| = 1 \begin{vmatrix} 0 & 1 & 2 \\ -2 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} -3 & 1 & 2 \\ 4 & 0 & 3 \\ 4 & 2 & -1 \end{vmatrix} + 3 \begin{vmatrix} -3 & 0 & 2 \\ 4 & -2 & 3 \\ 4 & 1 & -1 \end{vmatrix} - 4 \begin{vmatrix} -3 & 0 & 1 \\ 4 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

and then to a set of 2×2 determinants with alternating signs.

$$|A| = 1 \left[0 \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} + 2 \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} \right]$$

$$-2 \left[-3 \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 4 & 3 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 0 \\ 4 & 2 \end{vmatrix} \right]$$

$$+3 \left[-3 \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} 4 & 3 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 4 & -2 \\ 4 & 1 \end{vmatrix} \right]$$

$$-4\left[-3\begin{vmatrix}-2 & 0\\ 1 & 2\end{vmatrix}-0\begin{vmatrix}4 & 0\\ 4 & 2\end{vmatrix}+1\begin{vmatrix}4 & -2\\ 4 & 1\end{vmatrix}\right]$$

Now that we have only 2×2 determinants remaining, we can evaluate using the $ad-bc$ pattern.

$$|A|=1\left[0((0)(-1)-(3)(2))-1((-2)(-1)-(3)(1))+2((-2)(2)-(0)(1))\right]$$

$$-2\left[-3((0)(-1)-(3)(2))-1((4)(-1)-(3)(4))+2((4)(2)-(0)(4))\right]$$

$$+3\left[-3((-2)(-1)-(3)(1))-0((4)(-1)-(3)(4))+2((4)(1)-(-2)(4))\right]$$

$$-4\left[-3((-2)(2)-(0)(1))-0((4)(2)-(0)(4))+1((4)(1)-(-2)(4))\right]$$

$$|A|=1\left[0(0-6)-1(2-3)+2(-4-0)\right]$$

$$-2\left[-3(0-6)-1(-4-12)+2(8-0)\right]$$

$$+3\left[-3(2-3)-0(-4-12)+2(4+8)\right]$$

$$-4\left[-3(-4-0)-0(8-0)+1(4+8)\right]$$

$$|A|=1\left[0(-6)-1(-1)+2(-4)\right]-2\left[-3(-6)-1(-16)+2(8)\right]$$

$$+3\left[-3(-1)-0(-16)+2(12)\right]-4\left[-3(-4)-0(8)+1(12)\right]$$

$$|A|=1(0+1-8)-2(18+16+16)+3(3-0+24)-4(12-0+12)$$

$$|A|=1(-7)-2(50)+3(27)-4(24)$$

$$|A|=-7-100+81-96$$

$$|A|=-122$$

Because the determinant is nonzero, we know that A is invertible, which means we'll be able to find its inverse.

The determinant along different rows and columns

So far, we've been calculating determinants by focusing on the first row of the matrix. In other words, for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 0 & 1 & 2 \\ 4 & -2 & 0 & 3 \\ 4 & 1 & 2 & -1 \end{bmatrix}$$

from the last example, we used the coefficients with alternating signs from the first row, $+1$, -2, $+3$, and -4, and multiplied them by their associated sub-matrices as determinants.

But we don't have to use the first row. We can actually use any row or any column that we choose, and we'll always get the same value for the determinant. The benefit of this flexibility is that we can try to choose rows that have the most zero values, to make our calculation simpler.

For instance, with the matrix A, let's find the determinant along the second row, since the second row includes a 0. We just need to remember the checkerboard pattern of positive and negative signs. The entry in the first column of the first row is positive, and then everything alternates from there:

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

So the determinant of A along the second row would be

$$|A| = 3\begin{vmatrix} 2 & 3 & 4 \\ -2 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 4 & 0 & 3 \\ 4 & 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 4 \\ 4 & -2 & 3 \\ 4 & 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

Using the second row means that the second term in $|A|$ will be zeroed out.

$$|A| = 3\begin{vmatrix} 2 & 3 & 4 \\ -2 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 4 \\ 4 & -2 & 3 \\ 4 & 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

Now for the first term, we'll find the determinant along the second row, since it includes a 0. We could just as easily use second column, which also includes the same 0, but we'll do the row again.

$$|A| = 3\left[2\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\right] - 1\begin{vmatrix} 1 & 2 & 4 \\ 4 & -2 & 3 \\ 4 & 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

The 0 cancels out the second term completely.

$$|A| = 3\left[2\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\right] - 1\begin{vmatrix} 1 & 2 & 4 \\ 4 & -2 & 3 \\ 4 & 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

We'll find the determinant along the first row for the next determinant, since there are no zeros in the matrix, but we'll find the determinant along the third column for the last determinant (we could just as easily find it along the second row, which includes the same 0). Remember the checkerboard sign pattern.

$$|A| = 3\left[2\begin{vmatrix}3 & 4\\2 & -1\end{vmatrix} - 3\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix}\right]$$

$$-1\left[1\begin{vmatrix}-2 & 3\\1 & -1\end{vmatrix} - 2\begin{vmatrix}4 & 3\\4 & -1\end{vmatrix} + 4\begin{vmatrix}4 & -2\\4 & 1\end{vmatrix}\right]$$

$$+2\left[3\begin{vmatrix}4 & -2\\4 & 1\end{vmatrix} - 0\begin{vmatrix}1 & 2\\4 & 1\end{vmatrix} + 2\begin{vmatrix}1 & 2\\4 & -2\end{vmatrix}\right]$$

The 0 from that third column will zero out the second term.

$$|A| = 3\left[2\begin{vmatrix}3 & 4\\2 & -1\end{vmatrix} - 3\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix}\right]$$

$$-1\left[1\begin{vmatrix}-2 & 3\\1 & -1\end{vmatrix} - 2\begin{vmatrix}4 & 3\\4 & -1\end{vmatrix} + 4\begin{vmatrix}4 & -2\\4 & 1\end{vmatrix}\right]$$

$$+2\left[3\begin{vmatrix}4 & -2\\4 & 1\end{vmatrix} + 2\begin{vmatrix}1 & 2\\4 & -2\end{vmatrix}\right]$$

Now, with a simplified equation, we can more quickly calculate the determinant, and we'll get the same value we did before when we always used the first row.

$$|A| = 3\left[2((3)(-1) - (4)(2)) - 3((2)(2) - (3)(1))\right]$$

$$-1\left[1((-2)(-1)-(3)(1))-2((4)(-1)-(3)(4))+4((4)(1)-(-2)(4))\right]$$

$$+2\left[3((4)(1)-(-2)(4))+2((1)(-2)-(2)(4))\right]$$

$$|A|=3\left[2(-3-8)-3(4-3)\right]$$

$$-1\left[1(2-3)-2(-4-12)+4(4+8)\right]$$

$$+2\left[3(4+8)+2(-2-8)\right]$$

$$|A|=3\left[2(-11)-3(1)\right]-1\left[1(-1)-2(-16)+4(12)\right]+2\left[3(12)+2(-10)\right]$$

$$|A|=3(-22-3)-1(-1+32+48)+2(36-20)$$

$$|A|=3(-25)-1(79)+2(16)$$

$$|A|=-75-79+32$$

$$|A|=-122$$

This is the same result as the one we got when we always found the determinant along the first row, despite the fact that, this time, we chose different rows and columns along which to calculate the determinant.

The determinant by the Rule of Sarrus

Remember before that we found the determinant of the 3×3 matrix

$$A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$$

to be

$$|A| = aei + bfg + cdh - afh - bdi - ceg$$

But instead of remembering this formula, the Rule of Sarrus tells us that each of these terms (aei, bfg, cdh, etc.) is made up of a diagonal in the matrix. If we add the first two columns of the matrix as new columns on the right side of the matrix,

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

then the first three terms, $aei + bfg + cdh$, are given by

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

and the last three terms, $-afh - bdi - ceg$ are given by

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

So this way of finding the determinant just requires us to expand the matrix by adding every column but the last column to the right side of the matrix, adding up the blue set of diagonal products, and then subtracting from that the yellow set of diagonal products.

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

The downside to this simple rule is that it only works this way for 3×3 determinants. Let's use the Rule of Sarrus on a 3×3 matrix.

Example

Use the Rule of Sarrus to find the determinant.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix}$$

We need to add all but the last column to the right side of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -3 & 0 & 1 & -3 & 0 \\ 4 & -2 & 0 & 4 & -2 \end{bmatrix}$$

By the Rule of Sarrus, we'd add the products of the blue diagonals.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -3 & 0 & 1 & -3 & 0 \\ 4 & -2 & 0 & 4 & -2 \end{bmatrix}$$

$$(1)(0)(0) + (2)(1)(4) + (3)(-3)(-2)$$

Then we subtract the products of the yellow diagonals.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -3 & 0 & 1 & -3 & 0 \\ 4 & -2 & 0 & 4 & -2 \end{bmatrix}$$

$$-(3)(0)(4) - (1)(1)(-2) - (2)(-3)(0)$$

Then the determinant is the sum of these two strings of products.

$$|A| = (1)(0)(0) + (2)(1)(4) + (3)(-3)(-2) - (3)(0)(4) - (1)(1)(-2) - (2)(-3)(0)$$

$$|A| = 0 + 8 + 18 - 0 + 2 - 0$$

$$|A| = 8 + 18 + 2$$

$$|A| = 28$$

If we want to double-check that the rule worked, let's also compute the determinant using the previous method.

$$|A| = 1 \begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix} - 2 \begin{vmatrix} -3 & 1 \\ 4 & 0 \end{vmatrix} + 3 \begin{vmatrix} -3 & 0 \\ 4 & -2 \end{vmatrix}$$

$$|A| = 1((0)(0) - (1)(-2)) - 2((-3)(0) - (1)(4)) + 3((-3)(-2) - (0)(4))$$

$$|A| = 1(0 + 2) - 2(0 - 4) + 3(6 - 0)$$

$$|A| = 1(2) - 2(-4) + 3(6)$$

$$|A| = 2 + 8 + 18$$

$$|A| = 28$$

We get the same answer both ways, so we know the Rule of Sarrus correctly calculated the 3×3 determinant.

Cramer's rule for solving systems

Cramer's rule is a simple little rule that lets us use determinants to solve a system of linear equations. The rule says that you can solve for any variable in the system by calculating

$$\frac{D_v}{D}$$

where D_v is the determinant of the coefficient matrix with the answer column values in the variable column you're trying to solve, and where D is the determinant of the coefficient matrix.

Which means that, if we want to find the value of x, we need to find D_x/D, and if we want to find the value of y, we need to find D_y/D.

For example, given the linear system of two equations in two unknowns,

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

we can say that

$$x = \frac{D_x}{D}, \ y = \frac{D_y}{D}, \text{ with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \ D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, \ D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

All that sounds tricky, but let's look at an example to break it down.

Example

Solve for x in the system.

$$2x - 3y = 5$$

$$3x + 12y = -8$$

Because we're looking for the value of x, we want to find D_x/D. We need to start with the coefficient matrix for the system.

$$\begin{bmatrix}2 & -3\\3 & 12\end{bmatrix}$$

The answer column matrix is built from the constants from the right side of the system,

$$\begin{bmatrix}5\\-8\end{bmatrix}$$

D_x is the determinant of the coefficient matrix with the answer column matrix substituted into the x-column, so

$$D_x = \begin{vmatrix}5 & -3\\-8 & 12\end{vmatrix}$$

D is the determinant of the coefficient matrix, so

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 12 \end{vmatrix}$$

Then, putting these values together, Cramer's rule tells us that the value of x in the system is

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 5 & -3 \\ -8 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 12 \end{vmatrix}}$$

$$x = \frac{(5)(12) - (-3)(-8)}{(2)(12) - (3)(-3)}$$

$$x = \frac{36}{33} = \frac{12}{11}$$

Let's do another example where we use Cramer's rule to solve for y.

Example

Use Cramer's rule to solve for the value of y that satisfies the system.

$$3x - 2y = 7$$

$$5x - 8y = 21$$

The coefficient matrix is

$$\begin{bmatrix} 3 & -2 \\ 5 & -8 \end{bmatrix}$$

The answer column matrix is

$$\begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

Then D_y is what we get when we plug the answer column matrix into the second column of the coefficient matrix (because the second column holds the coefficients on y), and then take the determinant of the result.

$$D_y = \begin{vmatrix} 3 & 7 \\ 5 & 21 \end{vmatrix}$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 3 & -2 \\ 5 & -8 \end{vmatrix}$$

Then, putting these values together, Cramer's rule tells us that the value of y in the system is

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 7 \\ 5 & 21 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 5 & -8 \end{vmatrix}}$$

$$y = \frac{(3)(21) - (7)(5)}{(3)(-8) - (-2)(5)}$$

$$y = \frac{28}{-14} = -2$$

Or given a linear system of three equations in three unknowns,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

we can say that

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D},\ \text{with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Let's look at an example of a linear system with three equations.

Example

Use Cramer's rule to solve the system.

$$-7x + 6z = 99$$

$$-4x - 8y - 6z = 40$$

$$8x - y - 9z = -121$$

The coefficient matrix is

$$\begin{bmatrix} -7 & 0 & 6 \\ -4 & -8 & -6 \\ 8 & -1 & -9 \end{bmatrix}$$

The answer column matrix is

$$\begin{bmatrix} 99 \\ 40 \\ -121 \end{bmatrix}$$

Then D_x, D_y, and D_z are what we get when we plug the answer column matrix into the first, second, and third columns, respectively, of the coefficient matrix, and then take the determinant of the result.

$$D_x = \begin{vmatrix} 99 & 0 & 6 \\ 40 & -8 & -6 \\ -121 & -1 & -9 \end{vmatrix}$$

$$D_x = 99 \begin{vmatrix} -8 & -6 \\ -1 & -9 \end{vmatrix} - 0 \begin{vmatrix} 40 & -6 \\ -121 & -9 \end{vmatrix} + 6 \begin{vmatrix} 40 & -8 \\ -121 & -1 \end{vmatrix}$$

$$D_x = 99\left[(-8)(-9) - (-6)(-1)\right] + 6\left[(40)(-1) - (-8)(-121)\right]$$

$$D_x = 99(72 - 6) + 6(-40 - 968)$$

$$D_x = 99(66) + 6(-1{,}008)$$

$$D_x = 6{,}534 - 6{,}048$$

$$D_x = 486$$

and

$$D_y = \begin{vmatrix} -7 & 99 & 6 \\ -4 & 40 & -6 \\ 8 & -121 & -9 \end{vmatrix}$$

$$D_y = -7 \begin{vmatrix} 40 & -6 \\ -121 & -9 \end{vmatrix} - 99 \begin{vmatrix} -4 & -6 \\ 8 & -9 \end{vmatrix} + 6 \begin{vmatrix} -4 & 40 \\ 8 & -121 \end{vmatrix}$$

$$D_y = -7\left[(40)(-9) - (-6)(-121)\right] - 99\left[(-4)(-9) - (-6)(8)\right]$$

$$+6\left[(-4)(-121) - (40)(8)\right]$$

$$D_y = -7(-360 - 726) - 99(36 + 48) + 6(484 - 320)$$

$$D_y = -7(-1{,}086) - 99(84) + 6(164)$$

$$D_y = 7{,}602 - 8{,}316 + 984$$

$$D_y = 270$$

and

$$D_z = \begin{vmatrix} -7 & 0 & 99 \\ -4 & -8 & 40 \\ 8 & -1 & -121 \end{vmatrix}$$

$$D_z = -7 \begin{vmatrix} -8 & 40 \\ -1 & -121 \end{vmatrix} - 0 \begin{vmatrix} -4 & 40 \\ 8 & -121 \end{vmatrix} + 99 \begin{vmatrix} -4 & -8 \\ 8 & -1 \end{vmatrix}$$

$$D_z = -7\left[(-8)(-121) - (40)(-1)\right] + 99\left[(-4)(-1) - (-8)(8)\right]$$

$$D_z = -7(968+40) + 99(4+64)$$

$$D_z = -7(1{,}008) + 99(68)$$

$$D_z = -7{,}056 + 6{,}732$$

$$D_z = -324$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} -7 & 0 & 6 \\ -4 & -8 & -6 \\ 8 & -1 & -9 \end{vmatrix}$$

$$D = -7\begin{vmatrix} -8 & -6 \\ -1 & -9 \end{vmatrix} - 0\begin{vmatrix} -4 & -6 \\ 8 & -9 \end{vmatrix} + 6\begin{vmatrix} -4 & -8 \\ 8 & -1 \end{vmatrix}$$

$$D = -7\left[(-8)(-9) - (-6)(-1)\right] + 6\left[(-4)(-1) - (-8)(8)\right]$$

$$D = -7(72-6) + 6(4+64)$$

$$D = -7(66) + 6(68)$$

$$D = -462 + 408$$

$$D = -54$$

Then, putting these values together, Cramer's rule tells us that the values of x, y, and z in the system are

$$x = \frac{D_x}{D} = \frac{486}{-54} = -9$$

$$y = \frac{D_y}{D} = \frac{270}{-54} = -5$$

$$z = \frac{D_z}{D} = \frac{-324}{-54} = 6$$

The solution to the system is $(x, y, z) = (-9, -5, 6)$.

Modifying determinants

Now that we understand what the determinant is and how to calculate it, we want to look at other properties of determinants so that we can do more with them.

Multiplying a row or a column by a scalar

Given a square matrix A, if you multiply one row (or one column) of A by a scalar k, then the determinant just gets multiplied by k. In other words, for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is $|A| = ad - bc$. But if you multiply any row of A by a scalar k, for instance

$$B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

then the determinant is $\text{Det}(B) = |B| = k|A|$, or $k(ad - bc)$. It doesn't matter which row you multiply by k, the determinant will still be $k(ad - bc)$. This also works for any $n \times n$ matrix.

If you multiply n rows by the constant k, then the determinant of the new matrix will be $k^n|A|$. So if we'd multiplied both rows of A by k,

$$B = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

then the determinant would have been Det$(B) = |B| = k^2|A|$, or $k^2(ad - bc)$.

Sum of two rows

When you have three identical matrices (and this works for any set of $n \times n$ matrices, by the way), except that one row in each matrix is different, and that different row in the third matrix is the sum of the different rows from the first and second matrices, then we know the sum of the determinants of the first and second matrices is equal to the determinant of the third matrix.

For instance, these matrices A, B, and C all have an identical first row, but they have different second rows.

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

Furthermore, the second row in C is the sum of the second rows in A and B.

$$C = \begin{bmatrix} 1 & 0 \\ A_{(2,1)} + B_{(2,1)} = C_{(2,1)} & A_{(2,2)} + B_{(2,2)} = C_{(2,2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 3 + (-2) = 1 & 2 + 1 = 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

When we have this specific case, $|A| + |B| = |C|$.

$$|A| + |B| = |C|$$

$$\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}$$

$$[(1)(2) - (0)(3)] + [(1)(1) - (0)(-2)] = [(1)(3) - (0)(1)]$$

$$(2 - 0) + (1 - 0) = (3 - 0)$$

$$2 + 1 = 3$$

$$3 = 3$$

And like we said, this works the same way for any “different” row in the matrices, and for any set of these kinds of 3×3 or $n \times n$ matrices.

Swapped and duplicate rows

If you switch any row in a matrix A with any other row in the matrix A, the determinant of the new “swapped-row matrix” B is equal to the negative determinant of A. In other words,

$$|B| = -|A|$$

Let’s look at a simple example. If we start with the matrix A,

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

then B is the matrix we get when we switch the rows.

$$B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Then based on this "swapped-row" rule, we should find $|B| = -|A|$. Let's see if we do.

$$|B| = -|A|$$

$$\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$$(3)(0) - (2)(1) = -[(1)(2) - (0)(3)]$$

$$0 - 2 = -(2 - 0)$$

$$-2 = -(2)$$

$$-2 = -2$$

This "swapped-row" rule obviously worked for our 2×2 matrix, but it also works for any other 3×3 or $n \times n$ matrix.

But this rule creates one problem for us. Let's say our matrix has two identical rows, and those are the two that we choose to swap. After we swap them, we end up with the same matrix we started with ($A = B$), since switching two identical rows isn't going to do anything to change the matrix.

Which means that $|A| = |B|$. But we've said that if we switch two rows, then $|B| = -|A|$. From these two determinant equations, we're saying that it must also be true that $|A| = -|A|$. How is that possible? Well, the only way $|A| = -|A|$ can be true is if the determinant is 0, because 0 is the only value that can satisfy an equation $x = -x$.

So what does that tell us? It means that, if any $n \times n$ matrix A has any two rows that are identical, then we know immediately, without doing any calculations, that its determinant is 0, or $|A| = 0$. The same is true if any $n \times n$ matrix A has any two columns that are identical.

And as we know from before, if a matrix determinant is 0, then the matrix isn't invertible, so we can say that any $n \times n$ matrix with any two identical rows is not invertible, so its inverse isn't defined.

Row operations don't change the determinant

When we learned Gaussian elimination for solving systems, we learned how to use row operations to rewrite the matrix. It's important to know that those row operations don't change the value of the determinant, unless, of course, we're multiplying by a scalar, which we talked about at the beginning of this lesson.

Example

Verify that the row operation $R_2 - 3R_1 \to R_2$ doesn't change the value of $|A|$.

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

If we were trying to put this in reduced row-echelon form, we'd start with $R_2 - 3R_1 \to R_2$. The matrix A after the row operations would be

$$A_R = \begin{bmatrix} 1 & 0 \\ 3 - 3(1) & 2 - 3(0) \end{bmatrix}$$

$$A_R = \begin{bmatrix} 1 & 0 \\ 3 - 3 & 2 - 0 \end{bmatrix}$$

$$A_R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Row operations like these don't change the value of $|A|$. The determinant of A before the row operation is

$$|A| = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = (1)(2) - (0)(3) = 2 - 0 = 2$$

And the determinant of A after the row operation is

$$|A_R| = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = (1)(2) - (0)(0) = 2 - 0 = 2$$

We get the same value for the determinant in both cases.

Upper and lower triangular matrices

Upper triangular matrices are matrices in which all entries below the main diagonal are 0. The **main diagonal** is the set of entries that run from the upper left-hand corner of the matrix down to the lower right-hand corner of the matrix. **Lower triangular matrices** are matrices in which all entries above the main diagonal are 0.

Here are the main diagonals in these upper triangular matrices,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and in these lower triangular matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 5 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ 4 & 2 & 3 & -1 \end{bmatrix}$$

We've circled the main diagonal in each matrix, so that we can see that all of the entries in the upper triangular matrix below the main diagonal are 0, and that all of the entries in the lower triangular matrix above the main diagonal are 0.

Determinant of triangular matrices

Because we can find the determinant of a matrix along any row or column that we'd like, for upper triangular matrices, you'd always want to choose the first column (or last row), since it includes the most 0 entries.

For instance, let's say we want to find the determinant of the upper triangular matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Notice that A includes a 0 entry in $a_{(2,3)}$. That's okay. We can have zero values on or above the main diagonal. To be considered an upper triangular matrix, the only thing that matters is that all the entries below the main diagonal are 0.

The determinant of A along the first column is

$$|A| = 1 \begin{vmatrix} -2 & 0 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 5 & 3 \end{vmatrix}$$

The last three terms get zeroed out.

$$|A| = 1 \begin{vmatrix} -2 & 0 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{vmatrix}$$

Let's simplify the remaining 3×3 determinant along the first column again.

$$|A| = 1\left[-2\begin{vmatrix} 5 & 3 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 5 & 3 \end{vmatrix}\right]$$

The last two terms get zeroed out.

$$|A| = 1\left[-2\begin{vmatrix} 5 & 3 \\ 0 & -1 \end{vmatrix}\right]$$

$$|A| = 1\left[-2\begin{vmatrix} 5 & 3 \\ 0 & -1 \end{vmatrix}\right]$$

$$|A| = 1\left[-2((5)(-1) - (3)(0))\right]$$

$$|A| = 1\left[(-2)(5)(-1)\right]$$

$$|A| = (1)(-2)(5)(-1)$$

$$|A| = 10$$

We want to notice two things about this result.

First, the calculation was much easier than a typical 4×4 determinant, so working along the first column is a good strategy when we're calculating an upper triangular matrix determinant.

Second, the value of the determinant was the product $(1)(-2)(5)(-1)$, which is the product of all the entries in the main diagonal of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

In fact, for all upper triangular matrices, this will always be true! Given any upper triangular matrix, you can find the value of the determinant simply by multiplying together all of the entries along the main diagonal of the matrix. This also tells you that, if you have a 0 anywhere along the main diagonal of an upper triangular matrix, that the determinant will be 0. Which means that if the matrix contains a full row of zeros, anywhere in the matrix, that the determinant will be 0.

For $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, the determinant is $|A| = (1)(2)$

For $B = \begin{bmatrix} -2 & 4 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & -4 \end{bmatrix}$, the determinant is $|B| = (-2)(-1)(-4)$

For $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, the determinant is $|C| = (1)(-2)(5)(-1)$

The same is true for lower triangular matrices. If you were calculating the determinant traditionally, you'd want to calculate it along the first row or last column, since those include the most 0 entries.

If you did that, you'd find the determinant of the lower triangular matrix to be the product of the entries along the main diagonal, just like we did for upper triangular matrices.

For $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, the determinant is $|A| = (1)(2)$

For $B = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 4 & -4 \end{bmatrix}$, the determinant is $|B| = (-2)(-1)(-4)$

For $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ 4 & 2 & 3 & -1 \end{bmatrix}$, the determinant is $|C| = (1)(-2)(5)(-1)$

Putting a matrix into upper triangular form or lower triangular form is actually a great way to find the determinant quickly.

Example

Put A into upper or lower triangular form to find the determinant.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

In A, we don't have any more zeros below the main diagonal than above it, or vice versa, so we could really work in either direction. Let's start by rewriting the matrix as the determinant we're trying to find.

$$|A| = \begin{vmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{vmatrix}$$

Now switch the first and second rows so that we have a pivot entry in the first row. Remember that when we switch rows, the determinant gets multiplied by -1.

$$|A| = -\begin{vmatrix} 1 & -1 & 4 \\ -2 & 4 & 0 \\ 0 & 6 & -4 \end{vmatrix}$$

Now let's perform $2R_1 + R_2 \to R_2$.

$$|A| = -\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 8 \\ 0 & 6 & -4 \end{vmatrix}$$

Now perform $-3R_2 + R_3 \to R_3$.

$$|A| = -\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -28 \end{vmatrix}$$

Now that we've got A in upper triangular form, the determinant is just the product of the entries along the main diagonal. Don't forget the negative sign in front of the matrix that we put in for the row switch.

$$|A| = -(1)(2)(-28)$$

$$|A| = 56$$

Using determinants to find area

Now we want to look at an application of matrix determinants. In this lesson, we'll talk about a geometric property of the determinant, which is that the column vectors of a matrix form a parallelogram whose area is the absolute value of the determinant.

Furthermore, we can use this geometric property to find the area of other figures as well, not just parallelograms.

Forming the parallelogram

Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if you separate the matrix into its column vectors, and call them v_1 and v_2,

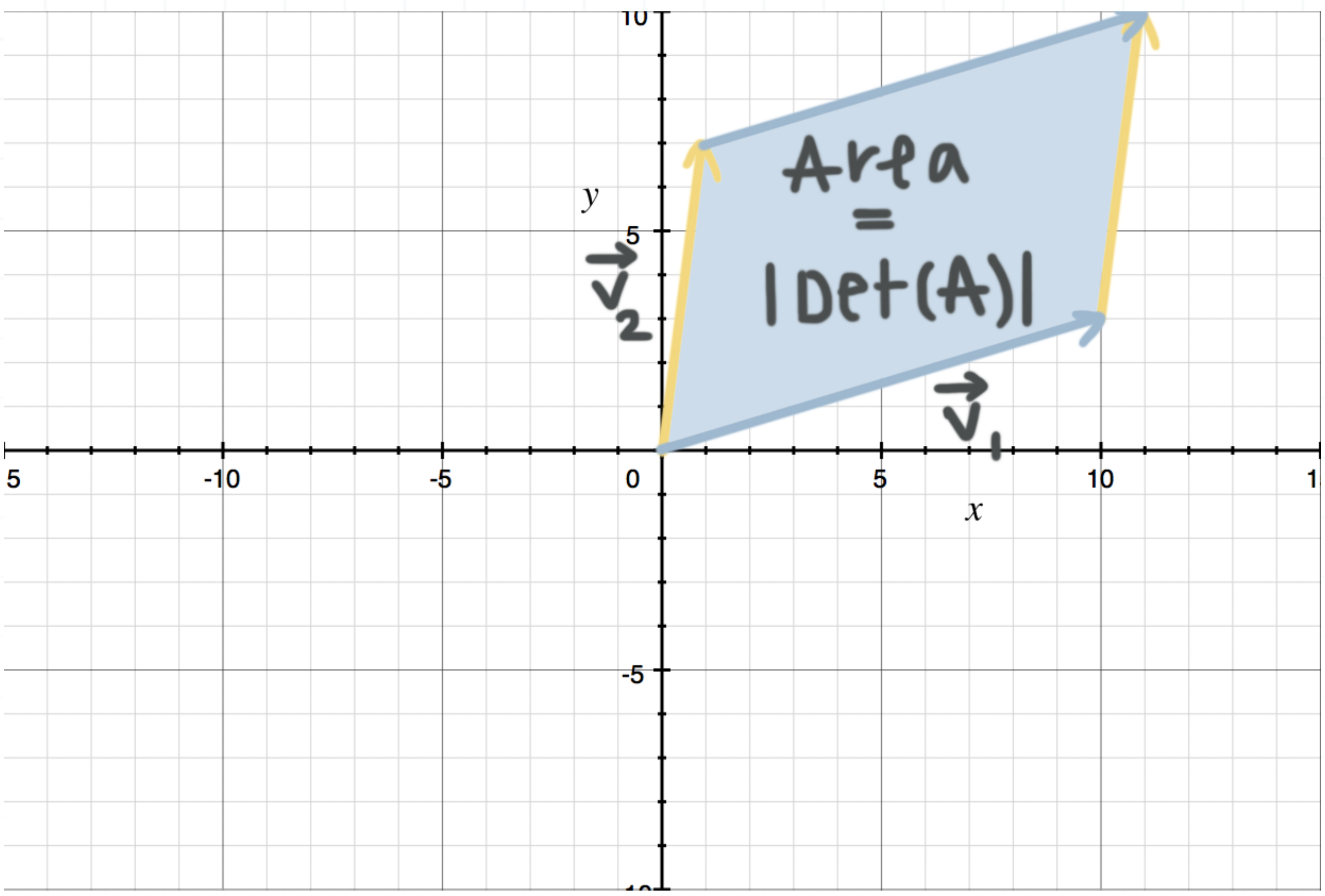
$$v_1 = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

then you can plot v_1 and v_2 in the xy-plane, and create a parallelogram using them as two of its adjacent sides.

The area of that parallelogram is simply the absolute value of the determinant of the matrix A.

$$\text{Area} = |\text{Det}(A)|$$

We're using $\text{Det}(A)$ instead of $|A|$ to avoid confusion between the bars around A that signify the determinant, and the bars in $\text{Area} = |\text{Det}(A)|$ that are there to indicate the absolute value.



Area under a transformation

Furthermore, let's say we're given a figure in the xy-plane, whether it's a parallelogram or any other figure, and let's call that figure f. We could always decide to apply some transformation matrix to f. Let's call the transformation matrix T. Applying T to f will create a new figure, and let's call the new figure g.

We know of course that the area of each figure will be the determinant of a matrix that describes the figure. So if matrix F defines f, then the area of f

is $\text{Area}_f = |\text{Det}(F)|$. And if matrix G defines g, then the area of g would be $\text{Area}_g = |\text{Det}(G)|$.

But we can also describe the area of g in terms of f and the transformation matrix T. As it turns out, the area of g can also be given by

$$\text{Area}_g = |\text{Area}_f \cdot \text{Det}(T)|$$

And this is really helpful, because once we get the figure g after the transformation, we don't have to find its area or calculate its determinant. We can just use the area of f (the figure we started with), multiply that by the determinant of the transformation matrix, and then take the absolute value of that product. In other words, we can find the area of g before we even have the matrix that defines g!

Example

The rectangle R with vertices $(1,1)$, $(1,6)$, $(-3,6)$, and $(-3,1)$ is transformed by T. Find the area of the transformed figure P.

$$T(\overrightarrow{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 0 \end{bmatrix} \overrightarrow{x}$$

All we've been given are the vertices of R and the transformation matrix T, and that's all we need. We don't have to sketch R, we don't have to represent R as a matrix, we don't have to actually apply T to R, nor do we need to know anything else about the transformed figure. All we need to find is the area of R, and the determinant of T.

Horizontally, the rectangle R is defined between -3 and 1, so its width is 4. Vertically, the rectangle R is defined between 1 and 6, so its height is 5. Therefore, the area of R is $\text{Area}_R = 4(5) = 20$.

The determinant of T is

$$\text{Det}(T) = \begin{vmatrix} -3 & 1 \\ 4 & 0 \end{vmatrix} = (-3)(0) - (1)(4) = 0 - 4 = -4$$

Therefore, the area of the transformed figure is

$$\text{Area}_P = |\text{Area}_R \cdot \text{Det}(T)|$$

$$\text{Area}_P = |20(-4)|$$

$$\text{Area}_P = |-80|$$

$$\text{Area}_P = 80$$

The area of the original rectangle R is 20, but after undergoing the transformation T, the area of the new transformed figure P is 80.

So the area of the parallelogram can be given by the determinant of the 2×2 matrix, but it's also true that the determinant of the 3×3 matrix will give the volume of the parallelepiped (three-dimensional parallelogram).

Transposes and their determinants

In this lesson, we want to start talking about matrix transposes.

The **transpose** of a matrix is simply the matrix you get when you swap all the rows and columns. In other words, the first row becomes the first column, the second row becomes the second column, and the nth row becomes the nth column.

You can find the transpose of any matrix with any dimensions, and we indicate the transpose of an $m \times n$ matrix A as an $n \times m$ matrix A^T, and we call it “A transpose.”

Example

Find the transpose of A.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

The first row of A is $[-2 \quad 4 \quad 0]$, so this sequence of entries needs to become the first column of A^T.

$$A^T = \begin{bmatrix} -2 & \cdot & \cdot \\ 4 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix}$$

The second row of A is $[1 \quad -1 \quad 4]$, so this sequence of entries needs to become the second column of A^T.

$$A^T = \begin{bmatrix} -2 & 1 & \cdot \\ 4 & -1 & \cdot \\ 0 & 4 & \cdot \end{bmatrix}$$

The third row of A is $[0 \quad 6 \quad -4]$, so this sequence of entries needs to become the third column of A^T.

$$A^T = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{bmatrix}$$

This works for non-square matrices, too. So the transpose of a matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -3 & 0 & 1 & 2 \\ 4 & -2 & 0 & 3 \end{bmatrix}$$

is

$$B^T = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 0 & -2 \\ 3 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

It's also important to realize that the transpose of a transpose gets you back to the original matrix. So given matrix A and the transpose of matrix A^T, then $(A^T)^T = A$.

Determinant of the transpose

The determinant of a transpose of a square matrix will always be equal to the determinant of the original matrix. In other words, given an $n \times n$ matrix A, the determinant of A and the determinant of A^T are equivalent.

Example

Show that the determinants of A and A^T are equivalent.

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{bmatrix}$$

The determinant of A is

$$|A| = \begin{vmatrix} -2 & 4 & 0 \\ 1 & -1 & 4 \\ 0 & 6 & -4 \end{vmatrix}$$

$$|A| = -2\begin{vmatrix} -1 & 4 \\ 6 & -4 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 0 & -4 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 0 & 6 \end{vmatrix}$$

The last term gets zeroed out.

$$|A| = -2((-1)(-4) - (4)(6)) - 4((1)(-4) - (4)(0)) + 0((1)(6) - (-1)(0))$$

$$|A| = -2(4 - 24) - 4(-4 - 0) + 0(6 - 0)$$

$$|A| = -2(-20) - 4(-4) + 0(6)$$

$$|A| = 40 + 16 + 0$$

$$|A| = 56$$

The matrix A^T is

$$A^T = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{bmatrix}$$

The determinant of A^T is

$$|A^T| = \begin{vmatrix} -2 & 1 & 0 \\ 4 & -1 & 6 \\ 0 & 4 & -4 \end{vmatrix}$$

$$|A^T| = -2\begin{vmatrix} -1 & 6 \\ 4 & -4 \end{vmatrix} - 1\begin{vmatrix} 4 & 6 \\ 0 & -4 \end{vmatrix} + 0\begin{vmatrix} 4 & -1 \\ 0 & 4 \end{vmatrix}$$

The last term gets zeroed out.

$$|A^T| = -2((-1)(-4) - (6)(4)) - 1((4)(-4) - (6)(0)) + 0((4)(4) - (-1)(0))$$

$$|A^T| = -2(4 - 24) - 1(-16 - 0) + 0(16 - 0)$$

$$|A^T| = -2(-20) - 1(-16) + 0(16)$$

$$|A^T| = 40 + 16 + 0$$

$$|A^T| = 56$$

So we've shown that the determinants of A and A^T are equivalent.

$$|A| = |A^T| = 56$$

Transposes of products, sums, and inverses

Now that we understand the idea of a transpose as a matrix with swapped rows and columns, we want to expand our understanding to matrix operations.

More specifically, in this lesson we'll look at how we define the transpose of a matrix product, and then the transpose of the sum of matrices.

Transpose of a matrix product

We have a special rule when it comes to transposing the product of matrices. Let's say that we have two matrices X and Y, $m \times n$ and $n \times m$, respectively, and we multiply X by Y to get the matrix product XY. The transpose of XY will be $(XY)^T$.

This transpose of the product is actually equivalent to the product of the individual transposes, but in the opposite order. In other words, the individual transposes are X^T and Y^T, $n \times m$ and $m \times n$, respectively, and instead of multiplying X by Y, we'd use the opposite order and multiply Y^T by X^T. So

$$(XY)^T = Y^T X^T$$

And this extends to the product of any number of matrices. So for instance, for the product of three matrices X, Y, and Z, the transpose of the matrix XYZ would be

$$(XYZ)^T = Z^T Y^T X^T$$

Let’s do an example to prove to ourselves that the transpose “flips the order” of the product.

Example

Show that $(XY)^T = Y^T X^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

Let’s first calculate the transpose of the product by finding the matrix XY, and then taking its transpose. Multiplying the matrices gives

$$XY = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

$$XY = \begin{bmatrix} 2(-6) + 1(2) & 2(0) + 1(-1) \\ -3(-6) + 0(2) & -3(0) + 0(-1) \end{bmatrix}$$

$$XY = \begin{bmatrix} -12 + 2 & 0 - 1 \\ 18 + 0 & 0 + 0 \end{bmatrix}$$

$$XY = \begin{bmatrix} -10 & -1 \\ 18 & 0 \end{bmatrix}$$

Then the transpose of XY is what we get when we swap the rows and columns:

$$(XY)^T = \begin{bmatrix} -10 & 18 \\ -1 & 0 \end{bmatrix}$$

Now let's try the method where we transpose the matrices individually, and then multiply those transposes in the reverse order. The transpose X^T is

$$X^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

The transpose Y^T is

$$Y^T = \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}$$

So the product Y^TX^T is

$$Y^TX^T = \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

$$Y^TX^T = \begin{bmatrix} -6(2) + 2(1) & -6(-3) + 2(0) \\ 0(2) - 1(1) & 0(-3) - 1(0) \end{bmatrix}$$

$$Y^TX^T = \begin{bmatrix} -12 + 2 & 18 + 0 \\ 0 - 1 & 0 - 0 \end{bmatrix}$$

$$Y^TX^T = \begin{bmatrix} -10 & 18 \\ -1 & 0 \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(XY)^T = Y^T X^T$.

Transpose of a matrix sum

We also know that, given two matrices X and Y, which must be the same size, the transpose of their sum $(X+Y)^T$ is equivalent to the sum of their individual transposes.

$$(X+Y)^T = X^T + Y^T$$

Which means that, just like with the matrix product, we can calculate the transpose two ways.

We can either find the sum of the matrices, and then transpose the sum, or we can take the transpose of each matrix individually, and then sum the individual transposes. And this extends to the sum of any number of matrices.

Example

Show that $(X+Y)^T = X^T + Y^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -6 & 0 \\ 2 & -1 \end{bmatrix}$$

Let's first calculate the transpose of the sum by finding the sum $X+Y$, and then taking its transpose. Adding the matrices gives

$$X+Y=\begin{bmatrix}2 & 1\\ -3 & 0\end{bmatrix}+\begin{bmatrix}-6 & 0\\ 2 & -1\end{bmatrix}$$

$$X+Y=\begin{bmatrix}2+(-6) & 1+0\\ -3+2 & 0+(-1)\end{bmatrix}$$

$$X+Y=\begin{bmatrix}2-6 & 1+0\\ -3+2 & 0-1\end{bmatrix}$$

$$X+Y=\begin{bmatrix}-4 & 1\\ -1 & -1\end{bmatrix}$$

Then the transpose of $X+Y$ is what we get when we swap the rows and columns:

$$(X+Y)^T=\begin{bmatrix}-4 & -1\\ 1 & -1\end{bmatrix}$$

Now let's try the method where we transpose the matrices individually, and then add those transposes. The transpose X^T is

$$X^T=\begin{bmatrix}2 & -3\\ 1 & 0\end{bmatrix}$$

The transpose Y^T is

$$Y^T=\begin{bmatrix}-6 & 2\\ 0 & -1\end{bmatrix}$$

So the sum $X^T + Y^T$ is

$$X^T + Y^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} 2 + (-6) & -3 + 2 \\ 1 + 0 & 0 + (-1) \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} 2 - 6 & -3 + 2 \\ 1 + 0 & 0 - 1 \end{bmatrix}$$

$$X^T + Y^T = \begin{bmatrix} -4 & -1 \\ 1 & -1 \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(X + Y)^T = X^T + Y^T$.

Transpose of a matrix inverse

And when it comes to a square matrix X and its inverse X^{-1}, their transposes are also inverses of one another. In other words, X^T is the inverse of $(X^{-1})^T$. Which means we can also say that the inverse matrix of the transpose is equivalent to the transpose of the inverse matrix.

$$(X^T)^{-1} = (X^{-1})^T$$

Let's do an example to prove that we get the same result, whether we take the inverse and then transpose it, or transpose the matrix and then take the inverse.

Example

Show that $(X^T)^{-1} = (X^{-1})^T$.

$$X = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

Let's see what we get when we transpose X, and then find the inverse of the transpose. The transpose is

$$X^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$$

Then we'll find the inverse of the transpose. Augment the matrix with I_2, find the pivot in the first column, then zero out the rest of the first column.

$$\begin{bmatrix} 2 & -3 & | & 1 & 0 \\ 1 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 1 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & | & -\frac{1}{2} & 1 \end{bmatrix}$$

Find the pivot in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -\frac{3}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 & 1 \\ 0 & 1 & | & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

So the inverse of the transpose is

$$(X^T)^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Now let's see what we get when we find the inverse and then transpose it. To find the inverse, first augment X with I_2, find the pivot in the first column, then zero out the rest of the first column.

$$\begin{bmatrix} 2 & 1 & | & 1 & 0 \\ -3 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ -3 & 0 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & | & \frac{3}{2} & 1 \end{bmatrix}$$

Find the pivot in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & 1 & \frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 & -\frac{1}{3} \\ 0 & 1 & | & 1 & \frac{2}{3} \end{bmatrix}$$

So the inverse is

$$X^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & \frac{2}{3} \end{bmatrix}$$

Then the transpose of the inverse is

$$(X^{-1})^T = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

By calculating both values, we've shown they're equivalent, which means that $(X^T)^{-1} = (X^{-1})^T$.

Null and column spaces of the transpose

Earlier we learned how to find the null and column spaces of a matrix. The null space of a matrix A was the set of vectors $\vec{x}$ that satisfied $A\vec{x} = \vec{O}$.

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{O}\}$$

The column space of A, $C(A)$, was the span of all the column vectors in A.

$$C(A) = \text{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots \vec{a}_n)$$

In this lesson, we want to talk about the null and column spaces of A^T. In other words, if we transpose A to get A^T, what are the null and column spaces of A^T, and how do they relate to the null and column spaces of A?

Spaces of the transpose

If we start with an $m \times n$ matrix A, and if the rank of the matrix is r, then we already know its null and column spaces:

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	$\text{Dim}(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	$\text{Dim}(N(A)) = n - r$

If we then transpose A, that means we're swapping all the rows and columns of A to get A^T. Specifically, we could say that all the rows of A become the columns of A^T. Therefore, we could describe the column space of A^T as both the columns of A^T, but also as the rows of A, since those spaces are equivalent. For that reason, we call the column space of A^T the **row space**.

Like the column space, the **row space** of any $m \times n$ matrix A is all the linear combinations of the rows of A. Of course, the span of the rows of A is equivalent to the span of the columns of A^T, which again, is why we refer to the column space of A^T as the row space of A.

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	$\text{Dim}(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	$\text{Dim}(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	$\text{Dim}(C(A^T)) = r$

The null space of the transpose A^T is the set of vectors $\vec{x}$ that satisfy $A^T\vec{x} = \vec{O}$. If we take the transpose of both sides of this null space equation, we get

$$(A^T\vec{x})^T = (\vec{O})^T$$

We learned earlier that the transpose of a product is the product of the individual transposes, but in the reverse order. So on the left side of this transpose equation, we apply the transpose to both A and $\vec{x}$, but we flip the order of the multiplication, and the equation simplifies to

$$\vec{x}^T(A^T)^T = \vec{O}^T$$

$$\vec{x}^TA = \vec{O}^T$$

So we can say that the null space of the transpose A^T is the set of vectors $\vec{x}^T$ that satisfy $\vec{x}^T A = \vec{O}^T$. We call the null space of the transpose the **left null space**, simply because $\vec{x}^T$ is to the left of A in the equation $\vec{x}^T A = \vec{O}^T$.

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	Dim$(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	Dim$(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	Dim$(C(A^T)) = r$
Left null space of A (null space of A^T)	$N(A^T)$	$\mathbb{R}^m$	Dim$(N(A^T)) = m - r$

Space and dimension

We've been building this table of subspaces, and along the way we've been including the space that contains the subspace, as well as the dimension of each space, where the dimension is always in terms of the number of rows and column in the original matrix, m and n, and the rank of the matrix, r. Now we want to explain those a little further.

Remember first that the null and column spaces are subspaces, which sit inside some vector space $\mathbb{R}^i$, for some positive integer i. If A is an $m \times n$ matrix (it has m rows and n columns), then

- The column space of A and null space of A^T are subspaces of $\mathbb{R}^m$
- The null space of A and column space of A^T are subspaces of $\mathbb{R}^n$

Next remember that the dimension (the number of vectors needed to form the basis) of the column space $C(A)$ is given by the number of pivot columns in the reduced row-echelon form of A. We also call this the rank, so the dimension of $C(A)$ will always be equal to the rank of A. The dimension of the column space of A^T will always be the same value; it'll be equal to the rank of A. So if r is the rank of A, then

$$\text{Dim}(C(A)) = \text{Dim}(C(A^T)) = r$$

Similarly, the number of vectors needed to form the basis of the null space of A (the dimension of $N(A)$), will be the difference of the number of columns in A and the rank of A. And the number of vectors needed to form the basis of the null space of the transpose of A (the dimension of $N(A^T)$), will be the difference of the number of rows in A and the rank of A.

- $\text{Dim}(N(A)) = n - r$

- $\text{Dim}(N(A^T)) = m - r$

Let's work through an example so that we can see how to find each of these values for the transpose matrix.

Example

Find the null and column subspaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces of M^T.

$$M = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 3 & -2 \\ 0 & 7 \end{bmatrix}$$

The transpose of M will be

$$M^T = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 4 & -2 & 7 \end{bmatrix}$$

To find the null space, we'll augment the matrix,

$$\begin{bmatrix} 2 & -1 & 3 & 0 & | & 0 \\ 0 & 4 & -2 & 7 & | & 0 \end{bmatrix}$$

and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 & | & 0 \\ 0 & 4 & -2 & 7 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{4} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{5}{4} & \frac{7}{8} & | & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{4} & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$x_1 + 0x_2 + \frac{5}{4}x_3 + \frac{7}{8}x_4 = 0$$

$$0x_1 + x_2 - \frac{1}{2}x_3 + \frac{7}{4}x_4 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -\frac{5}{4}x_3 - \frac{7}{8}x_4$$

$$x_2 = \frac{1}{2}x_3 - \frac{7}{4}x_4$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{5}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{7}{8} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose, which we call the left null space, is given by

$$N(M^T) = \text{Span}\left(\begin{bmatrix} -\frac{5}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{8} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, M. The original matrix has 4 rows, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$.

The column space of the transpose M^T, which is the same as the row space of M, is simply given by the columns in M^T that contain pivot entries when M^T is in reduced row-echelon form. So the column space of M^T is

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, M. The original matrix has 2 columns, so the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$.

Because there are two vectors that form the basis of $N(M^T)$, the dimension of $N(M^T)$ is 2. This matches the formula in our chart, $\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$. In other words, the dimension of the null space of the transpose will always be the difference of the number of rows in the original matrix and the rank of the original matrix.

Because there are two vectors that form the basis of $C(M^T)$, the dimension of $C(M^T)$ is 2. This matches the rank of the original matrix M. It will always be true that the dimension of the column space of both M and M^T will be equal to the rank of M, and therefore be equivalent to each other.

Let's summarize what we found.

$$N(M^T) = \text{Span}\left(\begin{bmatrix}-\frac{5}{4}\\ \frac{1}{2}\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{7}{8}\\ -\frac{7}{4}\\ 0\\ 1\end{bmatrix}\right) \text{ in } \mathbb{R}^4 \qquad \text{Dim}(N(M^T)) = 2$$

$$C(M^T) = \text{Span}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix}\right) \text{ in } \mathbb{R}^2 \qquad \text{Dim}(C(M^T)) = 2$$

The product of a matrix and its transpose

Now that we know more about a given matrix A and its transpose matrix A^T, we can learn more about the relationship between them.

The product is invertible

For instance, assuming that the columns of A are linearly independent, A^TA is an invertible matrix. That's because we can call A any $m \times n$ matrix, and then A^T is an $n \times m$ matrix. That means A^TA will be a square $n \times n$ matrix.

So if we can then put A^TA into reduced row-echelon form, and we end up with the identity matrix I_n, that means all of the columns in the square $n \times n$ product matrix A^TA are linearly independent, and A^TA is invertible.

Example

Show that A^TA is invertible.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & -1 \end{bmatrix}$$

We've been given A, so first we'll find A^T by swapping all the rows and columns in A.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Then we'll find the product A^TA.

$$A^TA = \begin{bmatrix}1 & 2 & -1\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 0\\ -1 & -1\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1(1)+2(2)-1(-1) & 1(1)+2(0)-1(-1)\\ 1(1)+0(2)-1(-1) & 1(1)+0(0)-1(-1)\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1+4+1 & 1+0+1\\ 1+0+1 & 1+0+1\end{bmatrix}$$

$$A^TA = \begin{bmatrix}6 & 2\\ 2 & 2\end{bmatrix}$$

To see whether or not A^TA is invertible, we'll put the product matrix into reduced row-echelon form. Find the pivot entry in the first column, and then zero out the rest of the first column.

$$A^TA = \begin{bmatrix}1 & \frac{1}{3}\\ 2 & 2\end{bmatrix} \to \begin{bmatrix}1 & \frac{1}{3}\\ 0 & \frac{4}{3}\end{bmatrix}$$

Find the pivot entry in the second column, and then zero out the rest of the second column.

$$A^TA = \begin{bmatrix}1 & \frac{1}{3}\\ 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Because the matrix A^TA in reduced row-echelon form is the identity matrix I_2, that tells us that A^TA is invertible. And that's what we expected to find, since we know that any A^TA is invertible, as long as the columns of A are linearly independent.

Orthogonal complements

Let's remember the relationship between perpendicularity and orthogonality. We usually use the word "perpendicular" when we're talking about two-dimensional space. If two vectors are perpendicular, that means they sit at a 90° angle to one another.

This idea of "perpendicular" gets a little fuzzy when we try to transition it into three-dimensional space or n-dimensional space, but the same idea still does exist in higher dimensions. So to capture the same idea, but for higher dimensions, we use the word "orthogonal" instead of "perpendicular." So two vectors (or planes, etc.) can be orthogonal to one another in three-dimensional or n-dimensional space.

The orthogonal complement

With a refresher on orthogonality out of the way, let's talk about the orthogonal complement. If a set of vectors V is a subspace of $\mathbb{R}^n$, then the **orthogonal complement** of V, called $V^\perp$, is a set of vectors where every vector in $V^\perp$ is orthogonal to every vector in V.

The $\perp$ symbol means "perpendicular," so you read $V^\perp$ as "v perpendicular," or just "v perp."

So if we're saying that V is a set of vectors $\vec{v}$, and $V^\perp$ is a set of vectors $\vec{x}$, then every $\vec{v}$ will be orthogonal to every $\vec{x}$ (or equivalently, every $\vec{x}$ will be orthogonal to every $\vec{v}$), which means that the dot product of any $\vec{v}$ with any $\vec{x}$ will be 0.

So we could express the set of vectors $V^{\perp}$ as

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \quad \text{for every} \quad \vec{v} \in V\}$$

This tells us that $V^{\perp}$ is all of the $\vec{x}$ in $\mathbb{R}^n$ that satisfy $\vec{x} \cdot \vec{v} = 0$, for every vector $\vec{v}$ in V, which is $V^{\perp}$'s orthogonal complement.

And this should make some sense to us. We learned in the past that two vectors were orthogonal to one another when their dot product was 0. For instance, if $\vec{x} \cdot \vec{v} = 0$, that tells us that the vector $\vec{x}$ is orthogonal to the vector $\vec{v}$.

We want to realize that defining the orthogonal complement really just expands this idea of orthogonality from individual vectors to entire subspaces of vectors. So two individual vectors are orthogonal when $\vec{x} \cdot \vec{v} = 0$, but two subspaces are orthogonal complements when every vector in one subspace is orthogonal to every vector in the other subspace.

Example

Describe the orthogonal complement of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$$

The subspace V is a plane in $\mathbb{R}^3$, spanned by the two vectors $\vec{v}_1 = (1,-3,2)$ and $\vec{v}_2 = (0,1,1)$. Therefore, its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to both $\vec{v}_1 = (1,-3,2)$ and $\vec{v}_2 = (0,1,1)$.

$$V^\perp = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$x_1 - 3x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 1 & -3 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 5x_3 = 0$$

$$x_2 + x_3 = 0$$

and we can solve the system for the pivot variables. The pivot entries we found were for x_1 and x_2, so we'll solve the system for x_1 and x_2.

$$x_1 = -5x_3$$

$$x_2 = -x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \right)$$

$V^{\perp}$ is a subspace

We've already assumed that V is a subspace. If we're given any subspace V, then we know that its orthogonal complement $V^{\perp}$ is also a subspace. Of course, that means there must be some way that we know that $V^{\perp}$ is closed under addition and closed under scalar multiplication.

We know $V^{\perp}$ is closed under addition because, if we say that $\vec{v}$ is in V and $\vec{x}_1$ and $\vec{x}_2$ are in $V^{\perp}$, then

$$\vec{x}_1 \cdot \vec{v} = 0$$

$$\vec{x}_2 \cdot \vec{v} = 0$$

because every vector in $V^{\perp}$ is orthogonal to every vector in V. If we add these equations, we get

$$\vec{x}_1 \cdot \vec{v} + \vec{x}_2 \cdot \vec{v} = 0 + 0$$

$$(\vec{x}_1 + \vec{x}_2) \cdot \vec{v} = 0$$

This shows us that the vector $\vec{x}_1 + \vec{x}_2$ will also be orthogonal to $\vec{v}$, which means $\vec{x}_1 + \vec{x}_2$ is also a member of $V^\perp$, which tells us that $V^\perp$ is closed under addition.

And we know that $V^\perp$ is closed under scalar multiplication because, if $\vec{v}$ is in V and $\vec{x}_1$ is in $V^\perp$, then it must also be true that

$$c\vec{x}_1 \cdot \vec{v} = c(\vec{x}_1 \cdot \vec{v}) = c(0) = 0$$

This shows us that the vector $c\vec{x}_1$ will also be orthogonal to $\vec{v}$, which means $c\vec{x}_1$ is also a member of $V^\perp$, which tells us that $V^\perp$ is closed under scalar multiplication.

Complement of the complement

In the same way that transposing a transpose gets you back to the original matrix, $(A^T)^T = A$, the orthogonal complement of the orthogonal complement is the original subspace. So if $V^\perp$ is the orthogonal complement of V, then

$$(V^\perp)^\perp = V$$

Intuitively, this makes sense. If all the vectors in $V^\perp$ are orthogonal to all the vectors in V, then all the vectors in V will be orthogonal to all the vectors in $V^\perp$, so the orthogonal complement of $V^\perp$ will be V.

Orthogonal complements of the fundamental subspaces

Previously, we learned about four fundamental subspaces, which are defined for any matrix A and its transpose A^T:

- the column space $C(A)$, made of the column vectors of A.
- the null space $N(A)$, made of the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$.
- the row space $C(A^T)$, made of the column vectors of A^T, which are also the row vectors of A.
- the left null space $N(A^T)$, made of the vectors $\vec{x}$ that satisfy $\vec{x}A = \vec{O}$.

And we understand the relationship between a subspace V and its orthogonal complement $V^{\perp}$. Every vector $\vec{v}$ in V is orthogonal to every vector $\vec{x}$ in $V^{\perp}$.

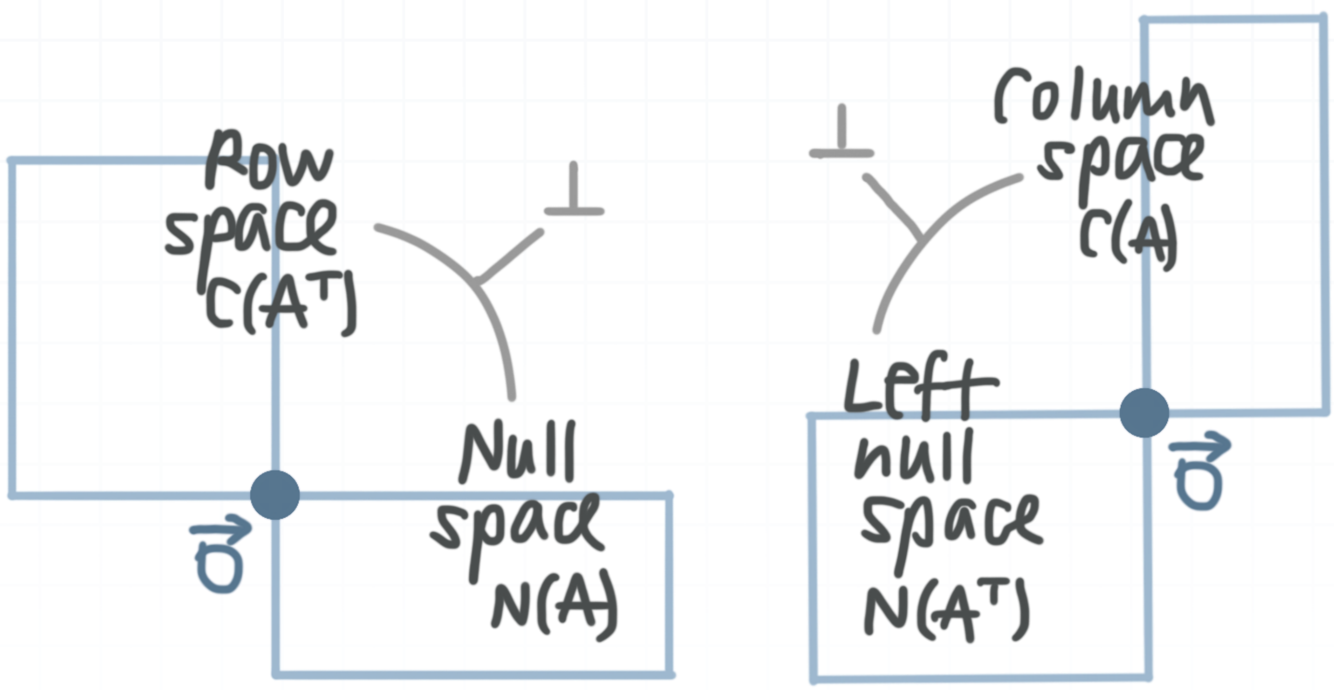
Orthogonality of the fundamental subspaces

With all of this in mind, we can state two important facts:

1. The null space $N(A)$ and row space $C(A^T)$ are orthogonal complements, $N(A) = (C(A^T))^{\perp}$, or $(N(A))^{\perp} = C(A^T)$.

2. The left null space $N(A^T)$ and column space $C(A)$ are orthogonal complements, $N(A^T) = (C(A))^{\perp}$, or $(N(A^T))^{\perp} = C(A)$.

In other words, the row space is orthogonal to the null space, and vice versa, and the column space is orthogonal to the left null space, and vice versa. Because of this fact, we can say that the row space and null space only intersect at the zero vector, and similarly that the column space and left null space only intersect at the zero vector. In fact, there is never any overlap between V and $V^{\perp}$ other than the zero vector.



Dimension of the orthogonal complement

We also want to be able to define the dimension of any orthogonal complement. The good news is that there's actually an easy way to do this. Here's the fact we want to remember:

> The dimension of a subspace V and the dimension of its orthogonal complement $V^{\perp}$ will always sum to the dimension of the space $\mathbb{R}^n$ that contains them both.
>
> $$\text{Dim}(V) + \text{Dim}(V^{\perp}) = n$$

In other words, if we remember that "dimension" really just means "the number of linearly independent vectors needed to form the basis," then we can say that the number of basis vectors of V, plus the number of basis vectors of $V^{\perp}$, will be equal to n, when both V and $V^{\perp}$ are in $\mathbb{R}^n$.

For instance, let's say V is a subspace of $\mathbb{R}^2$, and its dimension is 1. That means V is a line. Because we're in $\mathbb{R}^2$ (two dimensions), the dimension of $V^{\perp}$ must be $2 - \text{Dim}(V) = 2 - 1 = 1$. So $V^{\perp}$ is also a line.

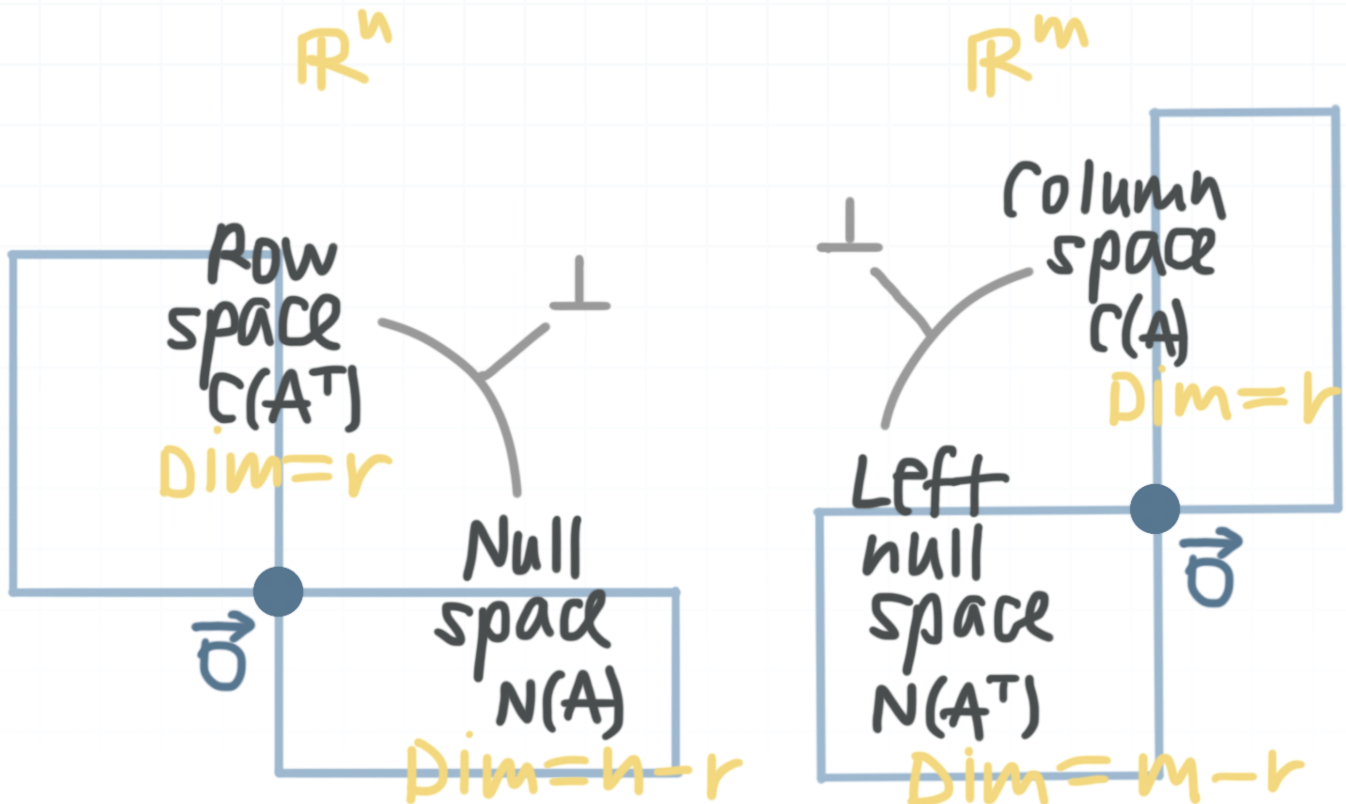
Or to take another example, let's say V is a subspace of $\mathbb{R}^3$, and its dimension is 2. That means V is a plane. Because we're in $\mathbb{R}^3$ (three dimensions), the dimension of $V^{\perp}$ must be $3 - \text{Dim}(V) = 3 - 2 = 1$. So $V^{\perp}$ is a line.

Of course, this same rule applies when we're talking about the four fundamental subspaces as orthogonal complements. If we define a matrix A as an $m \times n$ matrix (a matrix with m rows and n columns), then we know:

- The column space of A is its rank r (the number of linearly independent columns in A). The column space is defined in $\mathbb{R}^m$, because the number of linearly independent columns is given by the number of pivot entries, and the number of pivot entries can't possibly exceed the number of rows m in the matrix A. Because the column space and left null space are orthogonal complements, that means the dimension of the left null space must be $m - r$.

- The row space of A is its rank r (the number of linearly independent rows in A). The row space is defined in $\mathbb{R}^n$, because the number of linearly independent rows is given by the number of pivot entries, and the number of pivot entries can't possibly exceed the number of columns n in the matrix A. Because the row space and null space are orthogonal complements, that means the dimension of the null space must be $n - r$.

So we could add to the picture we sketched earlier, and this time include the dimensions of these fundamental subspaces:



Let's do an example to find the dimensions of orthogonal complements.

Example

For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

First, let's put A into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of A is $r = 3$. And A is a 4×3 matrix, which means there are $m = 4$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamentals subspaces of A are:

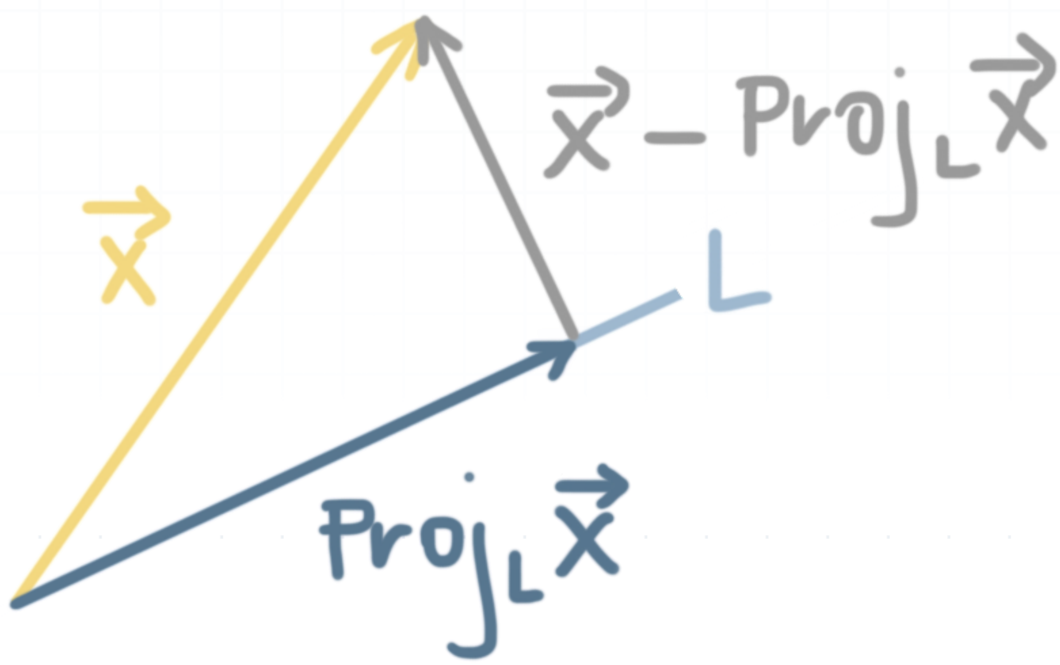
column space, $C(A)$	$r = 3$
null space, $N(A)$	$n - r = 3 - 3 = 0$
row space, $C(A^T)$	$r = 3$
left null space, $N(A^T)$	$m - r = 4 - 3 = 1$

Notice how the dimension of the null space is 0. That means the null space contains only the zero vector.

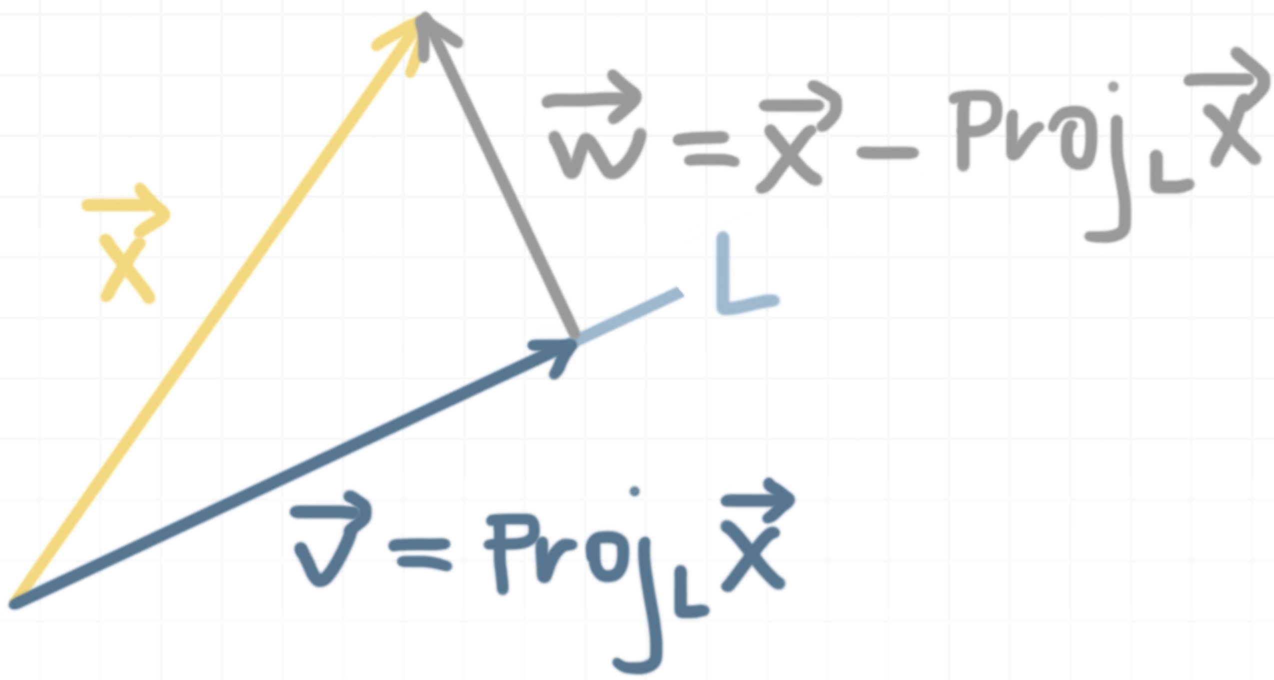
Projection onto the subspace

We learned earlier how to find the projection of a vector onto a line, but we can also project a vector onto a subspace. So let's start by generalizing the definition of a projection onto a line, so that we can see how it might apply to a subspace.

If you project a vector $\vec{x}$ onto a line L, then the projection vector that sits on L is $\text{Proj}_L\vec{x}$, and the vector from L to the terminal point of $\vec{x}$, which is orthogonal to L, is $\vec{x}-\text{Proj}_L\vec{x}$.



But let's say we call $\text{Proj}_L\vec{x}$ the vector $\vec{v}$, and we call $\vec{x}-\text{Proj}_L\vec{x}$ the vector $\vec{w}$. Then we could say that $\vec{x}-\vec{v}=\vec{w}$.



Then we can rewrite $\vec{x} - \vec{v} = \vec{w}$ as

$$\vec{x} = \vec{v} + \vec{w}$$

Because $\vec{w}$ is orthogonal to L, in the same way that $\vec{v}$ is a member of L, $\vec{w}$ is a member of the orthogonal complement of L, or $L^\perp$.

And this doesn't just work for the projection of a vector onto a line. It also works for projecting a vector onto any subspace. And just like with the projection onto a line, if $\vec{x}$ is projected onto a subspace V, and the projection is $\text{Proj}_V\vec{x}$, then the vector $\vec{x}$ is closer to the vector $\text{Proj}_V\vec{x}$ than it is to any other vector in V.

The projection is a linear transformation

If V is a subspace of $\mathbb{R}^n$, then the projection of $\vec{x}$ onto a subspace V is a linear transformation that can be written as the matrix-vector product

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

where A is a matrix of column vectors that form the basis for the subspace V.

There are a couple of important things we want to realize about this formula. First, notice that the product $A(A^TA)^{-1}A^T$ will eventually simplify to one matrix. Second, it's important to say that we cannot distribute the inverse across the matrix product inside the parentheses. In other words, we **cannot** rewrite the formula as

$$\text{Proj}_V\vec{x} = A(A)^{-1}(A^T)^{-1}A^T\vec{x}$$

The only time we'd be allowed to do this is if A is a square, invertible matrix. But if A is a square, invertible matrix, then by definition A defines all of $\mathbb{R}^n$, and the projection of a vector $\vec{x}$ onto V will just be itself, $\vec{x}$. So we only need the projection formula when A is not square and not invertible. But if A is not square and not invertible, such that it makes sense to use the projection formula, then we can't split up the $(A^TA)^{-1}$.

Let's do an example so that we can see how to use it to find the projection of a vector $\vec{x}$ onto the subspace V.

Example

If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}\right)$$

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix}$$

The transpose A^T is then

$$A^T = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

Our next step is to find A^TA.

$$A^TA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) - 2(-2) + 0(0) & 1(2) - 2(1) + 0(-2) \\ 2(1) + 1(-2) - 2(0) & 2(2) + 1(1) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 4 + 0 & 2 - 2 + 0 \\ 2 - 2 - 0 & 4 + 1 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

Then we need to find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} 5 & 0 & | & 1 & 0 \\ 0 & 9 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{1}{5} & 0 \\ 0 & 9 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{1}{5} & 0 \\ 0 & 1 & | & 0 & \frac{1}{9} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$$

Now the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5}(1)+0(2) & \frac{1}{5}(-2)+0(1) & \frac{1}{5}(0)+0(-2) \\ 0(1)+\frac{1}{9}(2) & 0(-2)+\frac{1}{9}(1) & 0(0)+\frac{1}{9}(-2) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5}+0 & -\frac{2}{5}+0 & 0+0 \\ 0+\frac{2}{9} & 0+\frac{1}{9} & 0-\frac{2}{9} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \end{bmatrix} \vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1\left(\frac{1}{5}\right)+2\left(\frac{2}{9}\right) & 1\left(-\frac{2}{5}\right)+2\left(\frac{1}{9}\right) & 1(0)+2\left(-\frac{2}{9}\right) \\ -2\left(\frac{1}{5}\right)+1\left(\frac{2}{9}\right) & -2\left(-\frac{2}{5}\right)+1\left(\frac{1}{9}\right) & -2(0)+1\left(-\frac{2}{9}\right) \\ 0\left(\frac{1}{5}\right)-2\left(\frac{2}{9}\right) & 0\left(-\frac{2}{5}\right)-2\left(\frac{1}{9}\right) & 0(0)-2\left(-\frac{2}{9}\right) \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{1}{5}+\frac{4}{9} & -\frac{2}{5}+\frac{2}{9} & 0-\frac{4}{9} \\ -\frac{2}{5}+\frac{2}{9} & \frac{4}{5}+\frac{1}{9} & 0-\frac{2}{9} \\ 0-\frac{4}{9} & 0-\frac{2}{9} & 0+\frac{4}{9} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{9}{45}+\frac{20}{45} & -\frac{18}{45}+\frac{10}{45} & -\frac{4}{9} \\ -\frac{18}{45}+\frac{10}{45} & \frac{36}{45}+\frac{5}{45} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{29}{45} & -\frac{8}{45} & -\frac{4}{9} \\ -\frac{8}{45} & \frac{41}{45} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix}\vec{x}$$

To simplify the matrix, let's factor out a $1/9$.

$$\text{Proj}_V\vec{x} = \frac{1}{9}\begin{bmatrix} \frac{29}{5} & -\frac{8}{5} & -4 \\ -\frac{8}{5} & \frac{41}{5} & -2 \\ -4 & -2 & 4 \end{bmatrix}\vec{x}$$

This result tells us that we can find the projection of any $\vec{x}$ onto the subspace V by multiplying the transformation matrix

$$\frac{1}{9}\begin{bmatrix}\frac{29}{5} & -\frac{8}{5} & -4\\ -\frac{8}{5} & \frac{41}{5} & -2\\ -4 & -2 & 4\end{bmatrix}$$

by the vector $\vec{x}$.

Least squares solution

So far we've spent a lot of time trying to figure out solutions to $A\vec{x}=\vec{b}$. We're given the matrix A, we're given the vector $\vec{b}$, and we're trying to find any and all $\vec{x}$ that satisfy $A\vec{x}=\vec{b}$.

When $\vec{b}$ is in the column space of A, then there's a solution to $A\vec{x}=\vec{b}$. Because if $\vec{b}$ is in the column space of A, that means $\vec{b}$ can be made from a linear combination of the columns of A. The components of the solution $\vec{x}$ simply tell us how many of each of A's columns to use in the combination in order to get $\vec{b}$.

So, when $\vec{b}$ isn't in the column space of A, that means there's no linear combination of A's columns that can be used to get $\vec{b}$, which means there's no solution $\vec{x}$ that solves $A\vec{x}=\vec{b}$.

The least square solution

When there's no solution to $A\vec{x}=\vec{b}$, we're still interested in the next best thing. In general, we're interested in how close we can get to a solution, even when there isn't one.

The least squares solution of the matrix equation $A\vec{x}=\vec{b}$ is a vector $\vec{x}^*$ in $\mathbb{R}^n$, such that $A\vec{x}^*$ is as "close" to $\vec{b}$ as possible. By "close," we mean that we're minimizing the distance between $A\vec{x}^*$ and $\vec{b}$. In other words, $\vec{x}^*$ will tell us the combination of A's columns that we can use to get as close as possible to $\vec{b}$. The solution $\vec{x}^*$ will satisfy

$$A^T A \vec{x}^* = A^T \vec{b}$$

So to find the least squares solution, we need to find $A^T A$ and $A^T \vec{b}$, and then solve for $\vec{x}^*$.

The reason we're talking about the least squares solution now is because this concept is directly related to what we just learned about the projection of a vector onto a subspace.

Think about the column space $C(A)$ as a subspace, maybe a plane. When there's a solution to $A\vec{x} = \vec{b}$, it means $\vec{b}$ is in the column space, so it's on the plane. But if there's no solution to $A\vec{x} = \vec{b}$, that means $\vec{b}$ sticks up out of the plane, and the closest $\vec{x}^*$ that we can find to $\vec{b}$ is the projection of $\vec{b}$ into the plane, or the projection of $\vec{b}$ onto the subspace.

So when we're solving for $\vec{x}^*$, we're really solving for the projection of $\vec{b}$ onto the column space $C(A)$, $\text{Proj}_{C(A)}\vec{b}$, which is as close as we can get to $\vec{b}$, even when $\vec{b}$ is not in the column space of A.

Let's do an example to see how this works.

Example

Find the least squares solution to the system.

$$x - y = 3$$

$$2x + y = 1$$

$$-x - 4y = 2$$

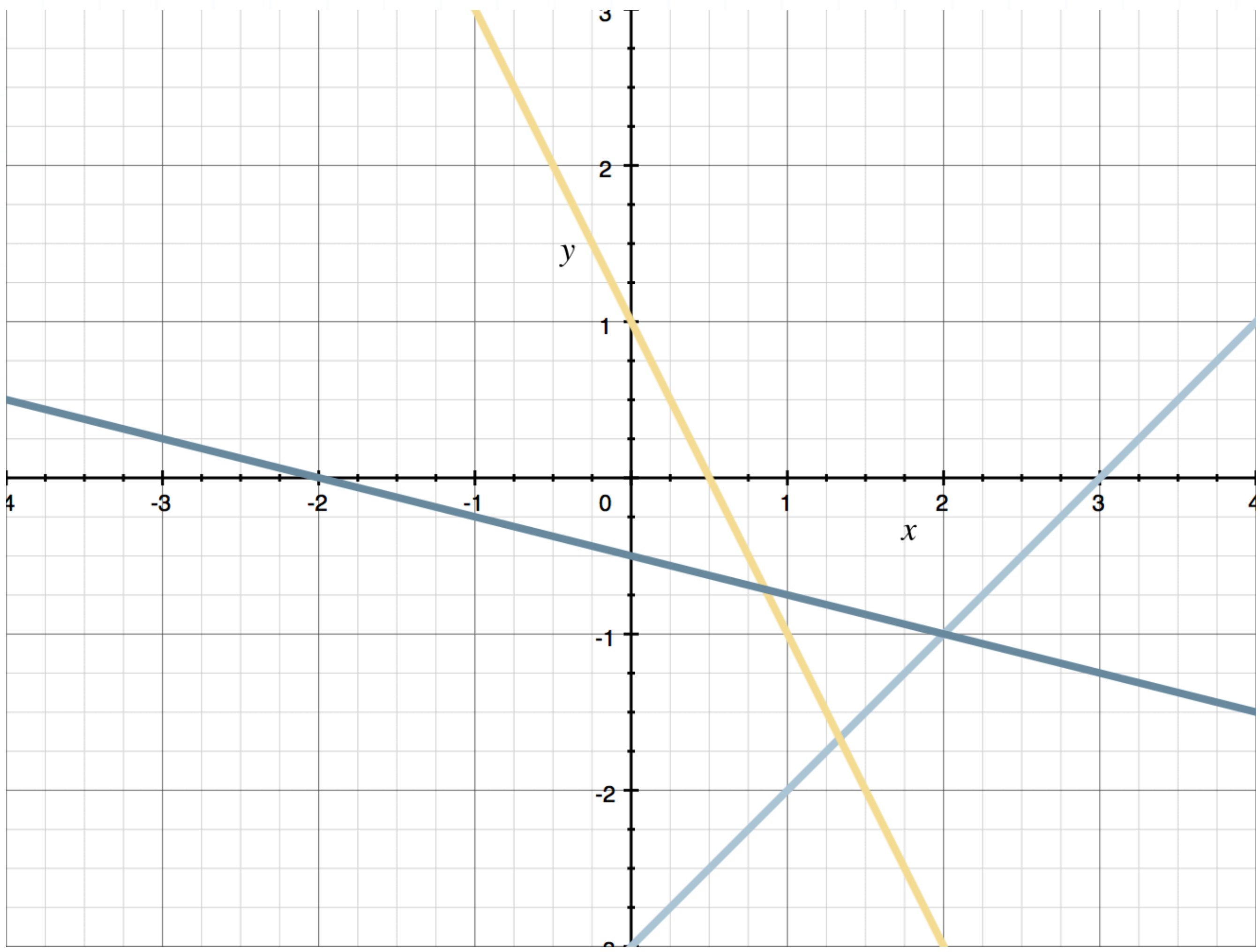
First, let’s graph each line to confirm that the system does not have a solution. If we put each line in slope-intercept form,

$$y = x - 3$$

$$y = -2x + 1$$

$$y = -\frac{1}{4}x - \frac{1}{2}$$

then the graph of all three is



While there are three points at which two of the lines intersect one another, there’s no single point where all three lines intersect, which means there’s no solution to

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

In other words, $\vec{b} = (3,1,2)$ is not in the column space of A, and there's no vector $\vec{x} = (x, y)$ you can find that will make that equation true.

The next best thing we can do is find the point which minimizes the squared distances. By building the matrix equation, we've already found

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & -4 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -4 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & -4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) + 2(2) - 1(-1) & 1(-1) + 2(1) - 1(-4) \\ -1(1) + 1(2) - 4(-1) & -1(-1) + 1(1) - 4(-4) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 4 + 1 & -1 + 2 + 4 \\ -1 + 2 + 4 & 1 + 1 + 16 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & 5 \\ 5 & 18 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b}=\begin{bmatrix}1 & 2 & -1\\ -1 & 1 & -4\end{bmatrix}\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}1(3)+2(1)-1(2)\\ -1(3)+1(1)-4(2)\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}3+2-2\\ -3+1-8\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}3\\ -10\end{bmatrix}$$

Then we get

$$A^TA\vec{x}^*=A^T\vec{b}$$

$$\begin{bmatrix}6 & 5\\ 5 & 18\end{bmatrix}\vec{x}^*=\begin{bmatrix}3\\ -10\end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}6 & 5 & | & 3\\ 5 & 18 & | & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{5}{6} & | & \frac{1}{2}\\ 5 & 18 & | & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{5}{6} & | & \frac{1}{2}\\ 0 & \frac{83}{6} & | & -\frac{25}{2}\end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{5}{6} & | & \frac{1}{2} \\ 0 & 1 & | & -\frac{75}{83} \end{bmatrix}$$

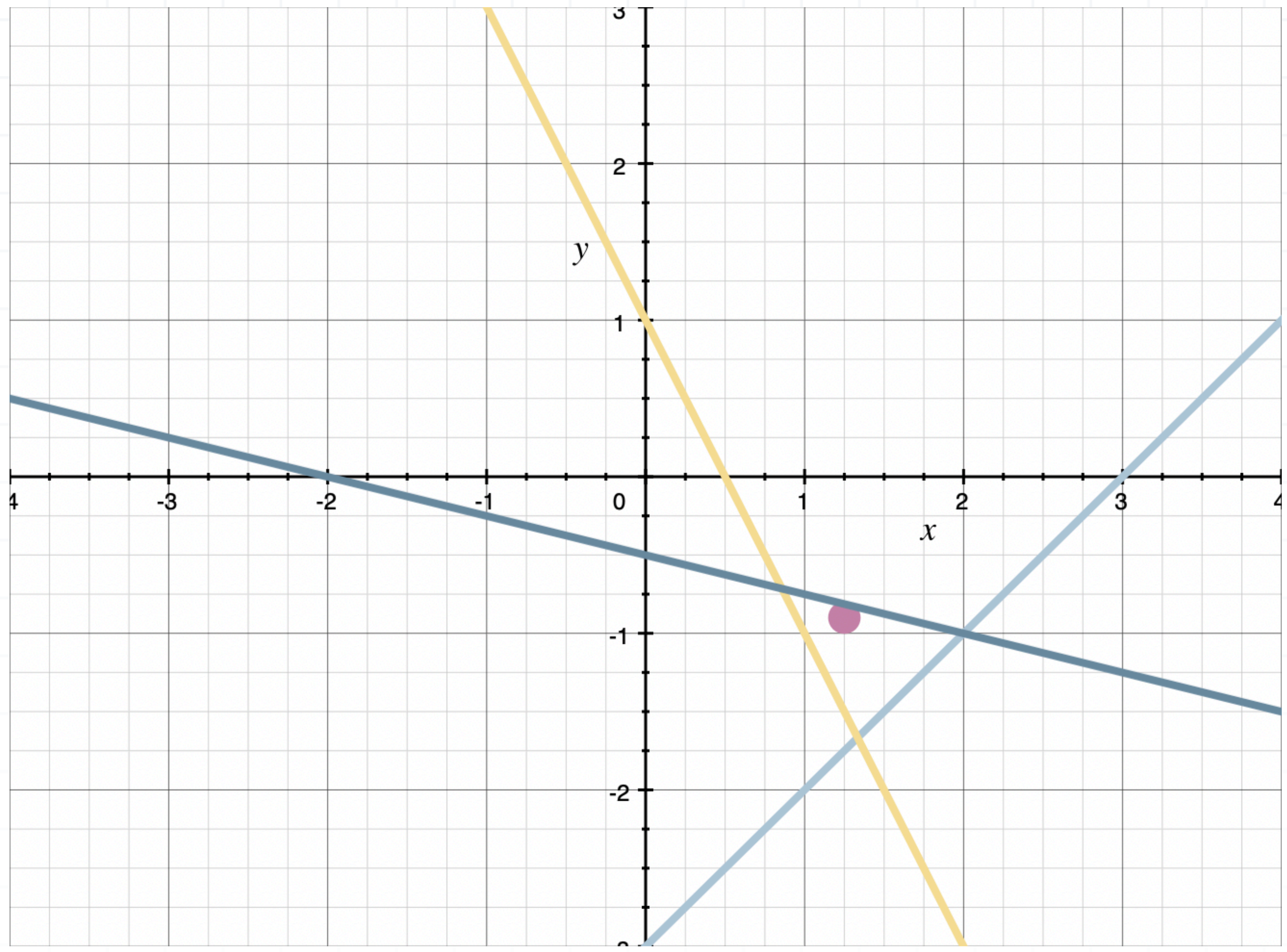
$$\begin{bmatrix} 1 & 0 & | & \frac{104}{83} \\ 0 & 1 & | & -\frac{75}{83} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(\frac{104}{83}, -\frac{75}{83}\right)$$

$$\vec{x}^* \approx (1.25, -0.90)$$

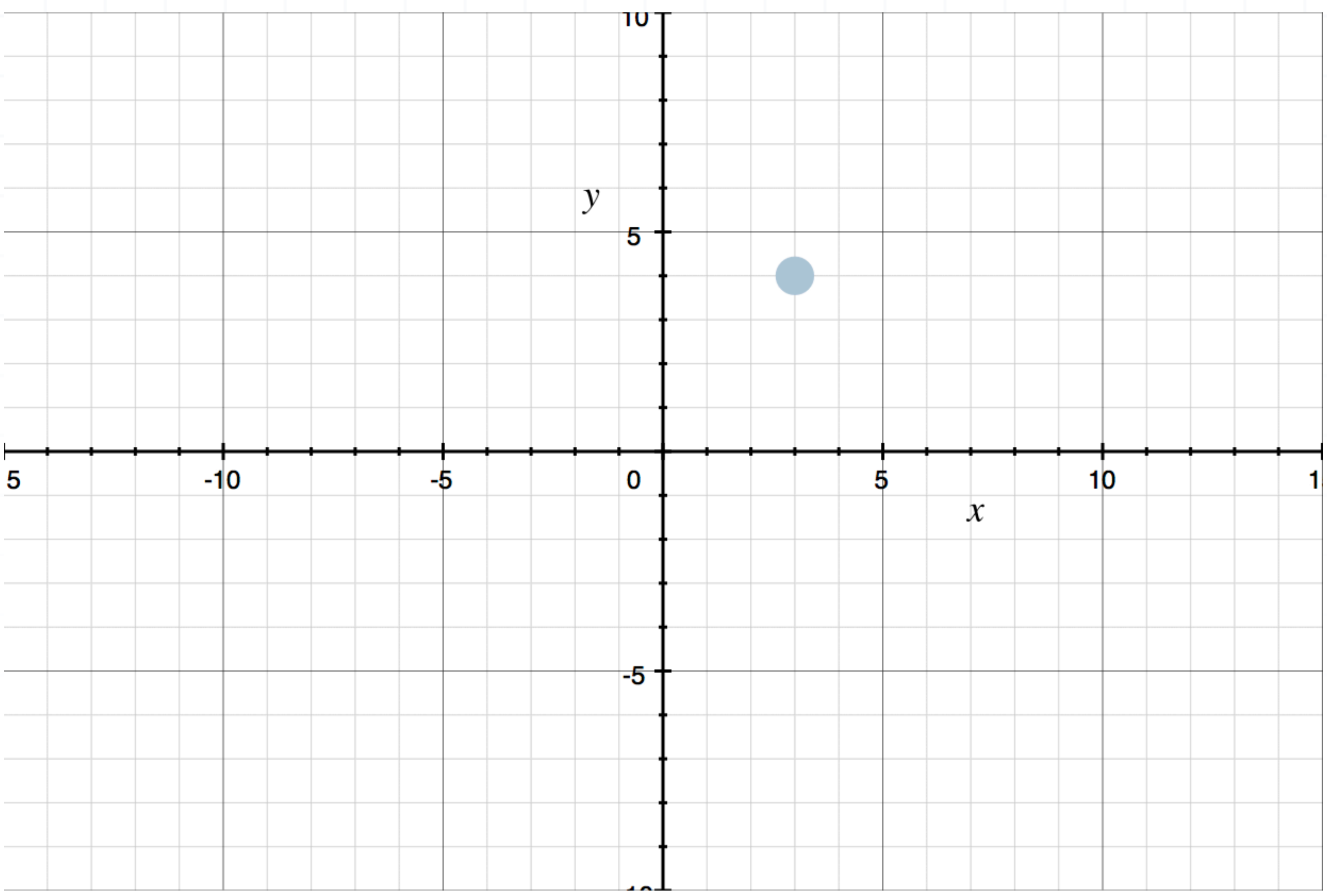
If we graph this solution, alongside the lines we sketched earlier, we can see that the solution is inside the triangle created by the points of intersection of the system.



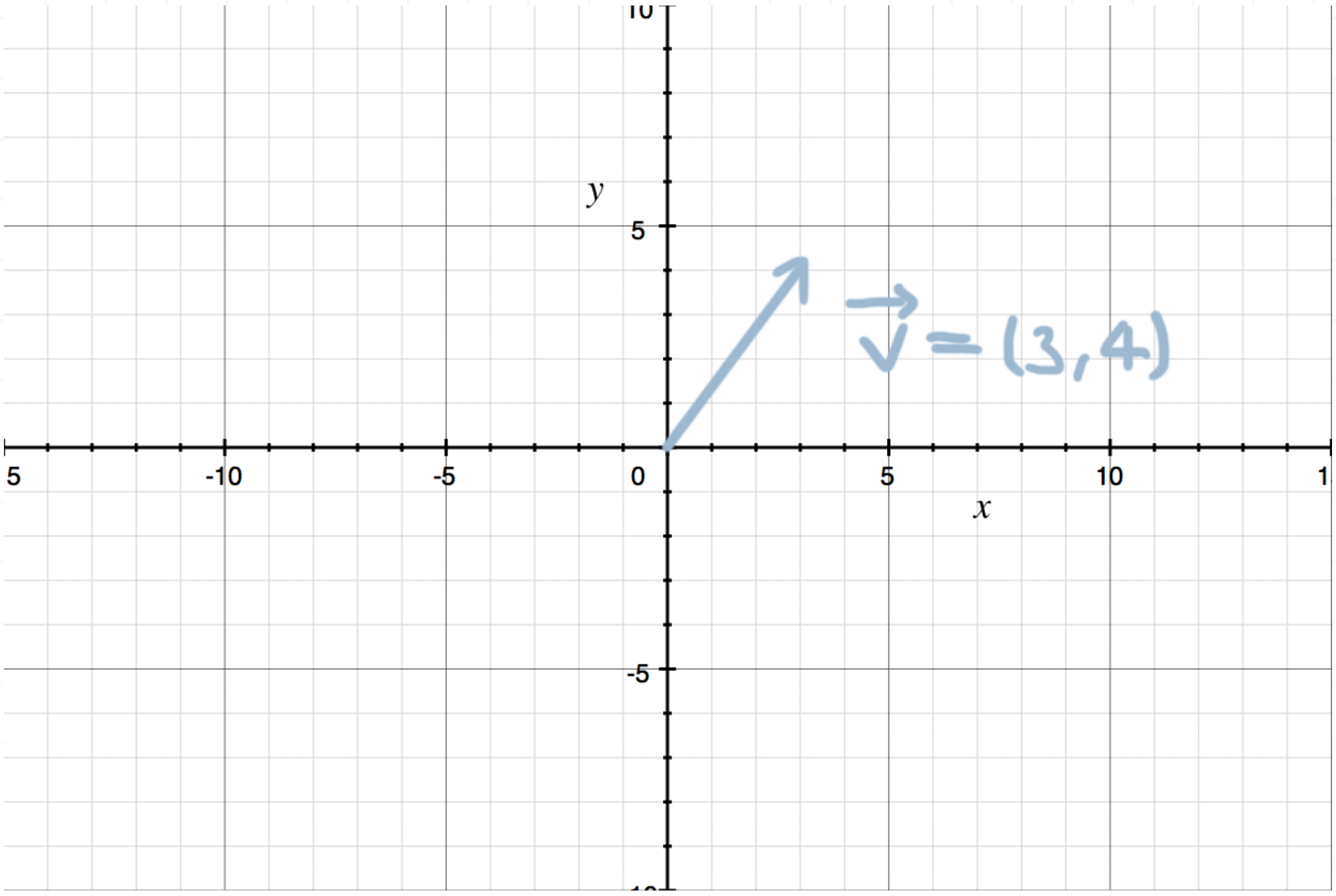
y
x
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
1
2
3
4

Coordinates in a new basis

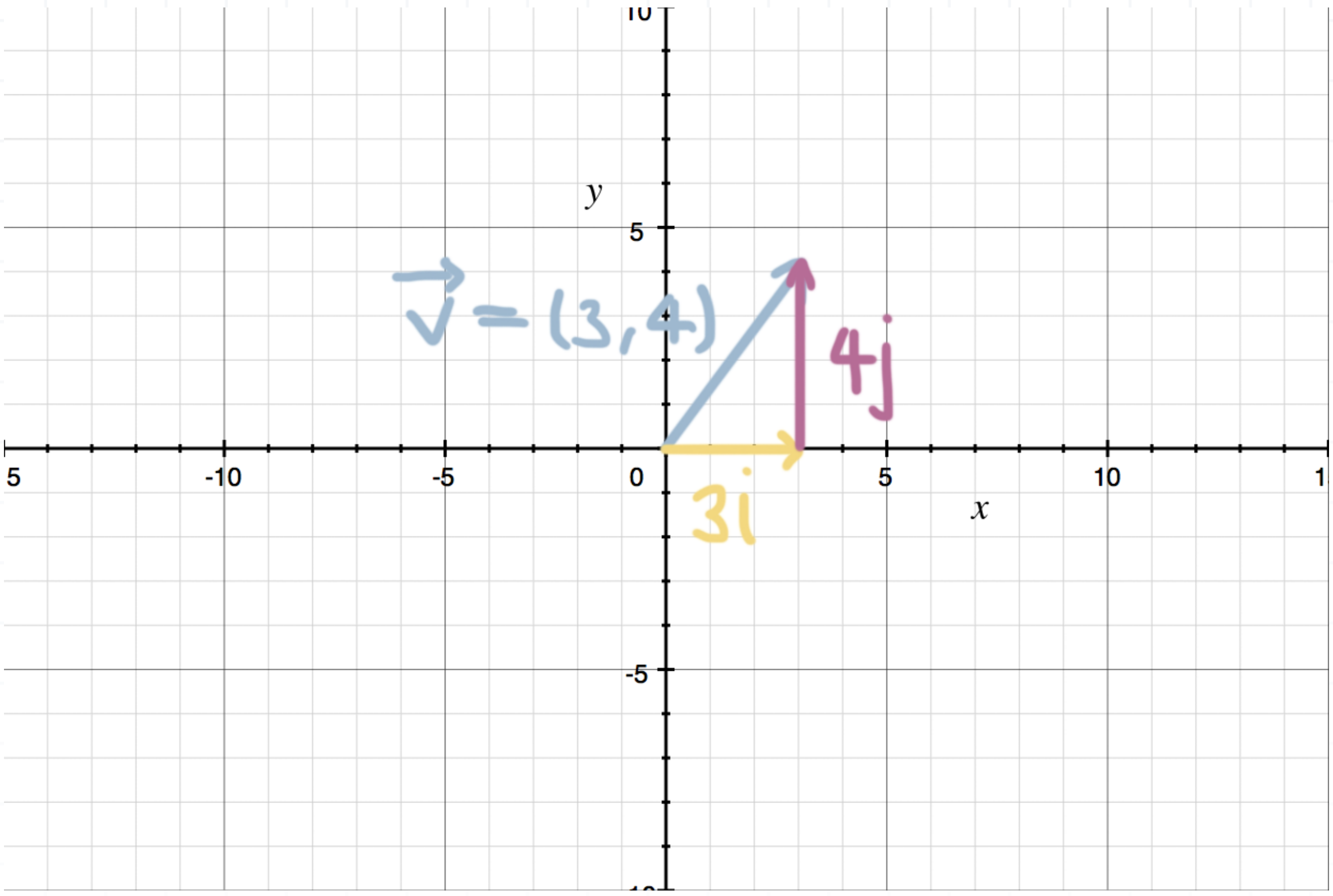
When we first learned to graph, we defined points in space using coordinates. For instance, the point $(3,4)$ told us to move 3 units from the origin horizontally toward the positive direction of the x-axis, and 4 units vertically toward the positive direction of the y-axis.



We've also learned how to define the same point with vectors. The vector $\vec{v}=(3,4)$ will point us to the same point $(3,4)$. So the graph of $\vec{v}=(3,4)$ is



And we know that the same spot can be represented as a combination of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$. The vector $\mathbf{i} = (1,0)$ is the vector that points to $(1,0)$ and the vector $\mathbf{j} = (0,1)$ is the vector that points to $(0,1)$. So we can take 3 of $\mathbf{i}$ and 4 of $\mathbf{j}$ and again get to the same point. So the graph of $3\mathbf{i} + 4\mathbf{j}$ is



In other words, up to now, plotting points has always been done using the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$, or $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in three dimensions. Even when we were originally learning to plot $(3,4)$ back in an introductory Algebra class, and we knew nothing about vectors, we were really learning to plot $3\mathbf{i}+4\mathbf{j}$ in terms of the standard basis vectors, we just didn't know it yet.

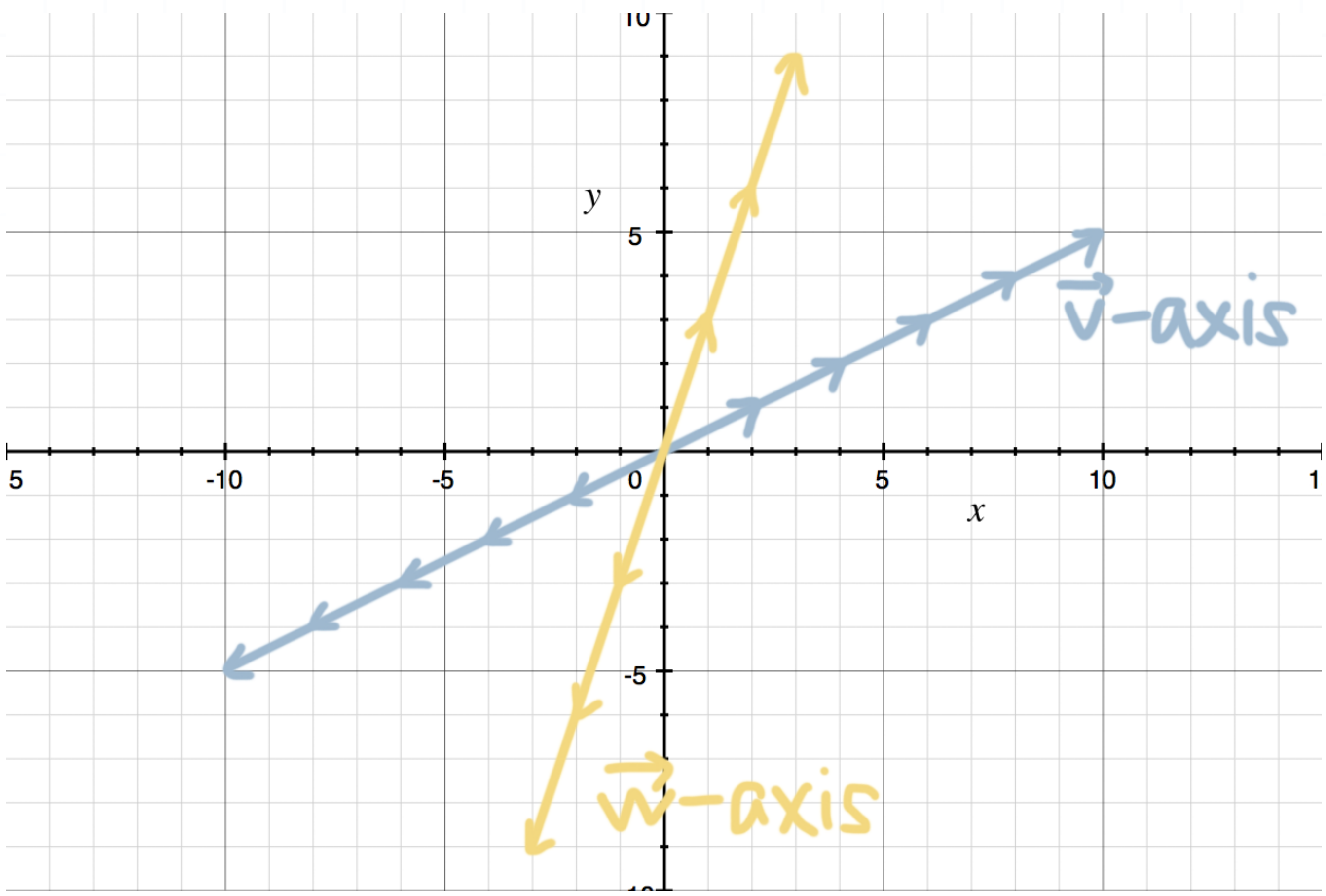
In this lesson, we want to see what it looks like to define points using different basis vectors. In other words, instead of using $\mathbf{i}=(1,0)$ and $\mathbf{j}=(0,1)$, can we use different vectors as the basis instead?

Changing the basis

Well, let's say V is a subspace of $\mathbb{R}^2$. And let's say we wanted our basis vectors to be $\vec{v}=(2,1)$ and $\vec{w}=(1,3)$, such that $B=\{\vec{v},\vec{w}\}$ is basis for $\mathbb{R}^2$, and we want to figure out how to use these vectors to plot the same point we've been working with, $(3,4)$.

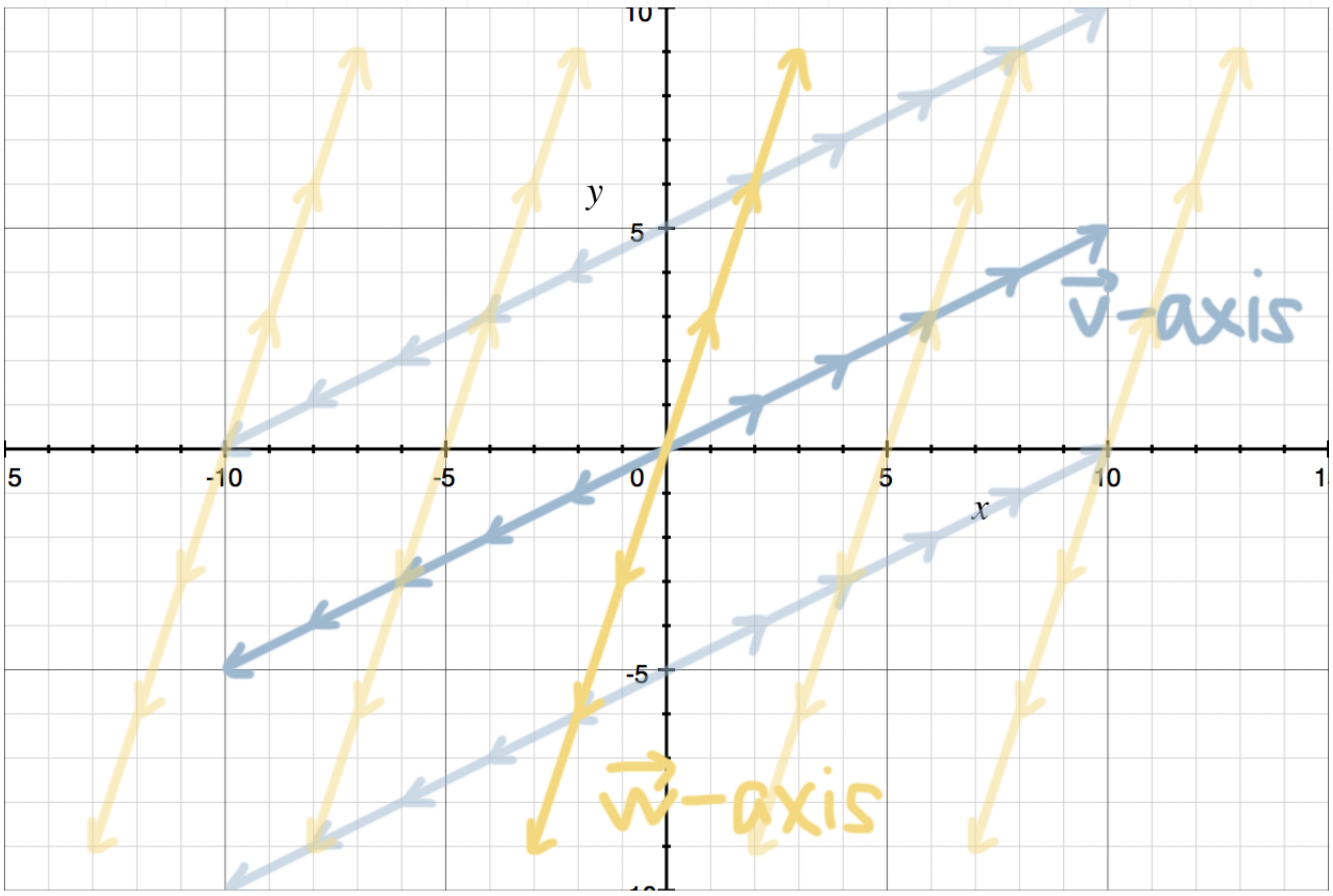
First, we could mark off a new set of axes. Realize that in the standard basis coordinate system, the perfectly horizontal x-axis is marked off in increments of $\mathbf{i}$, and the perfectly vertical y-axis is marked off in increments of $\mathbf{j}$.

Now we want to create a new set of axes, with one axis marked off in increments of $\vec{v}=(2,1)$, and the other axis marked off in increments of $\vec{w}=(1,3)$. Because $\vec{v}$ and $\vec{w}$ are not perfectly horizontal or vertical, the $\vec{v}$- and $\vec{w}$-axes won't be perfectly horizontal or vertical either. Instead, the $\vec{v}$-axis will lie along $\vec{v}=(2,1)$, and the $\vec{w}$-axis will lie along $\vec{w}=(1,3)$.

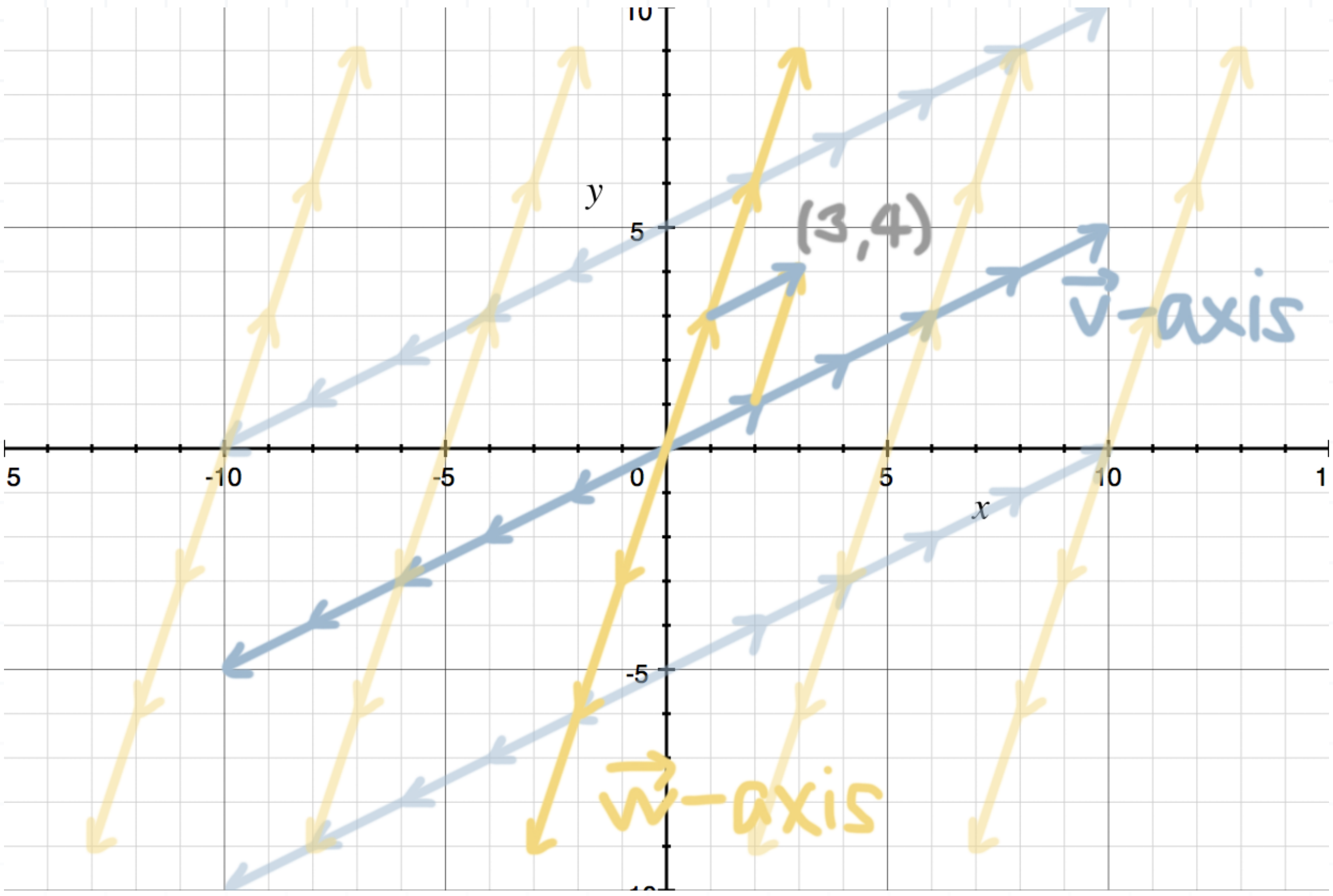


With the new axes sketched in, notice how we've marked them off in increments of $\vec{v}=(2,1)$ and $\vec{w}=(1,3)$, such that we can count how many $\vec{v}$s we’re moving out along the $\vec{v}$-axis, and how many $\vec{w}$s we’re moving out along the $\vec{w}$-axis.

Now we could sketch in a grid pattern that shows the whole “$\vec{v}\,\vec{w}$ coordinate system.”



Now to reach $(3,4)$, we can see that we need 1 of $\vec{v}$ and 1 of $\vec{w}$. We can either go out one unit of $\vec{v}$ toward the positive direction of the $\vec{v}$-axis, and then move one unit of $\vec{w}$ toward the positive direction of the $\vec{w}$-axis, or we can go the opposite way, moving out one unit of $\vec{w}$ toward the positive direction of the $\vec{w}$-axis and then one unit of $\vec{v}$ toward the positive direction of the $\vec{v}$-axis. Either way, we end up at $(3,4)$.



So the vector $\vec{x} \in V$ can be expressed uniquely as $\vec{x} = \vec{v} + \vec{w}$, and the coordinate vector of $\vec{x}$ in the basis B is

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Change of basis matrix

So now that we understand the concept of changing from the standard basis vectors **i** and **j** to another set of basis vectors, like $\vec{v}$ and $\vec{w}$, let's talk about how we'd actually go about doing that.

If we have the vectors that form the basis for the subspace, then we can use them to create a transformation matrix, called the **change of basis matrix**, that will change a vector from one basis to another.

For instance, earlier we changed from $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ to $\vec{v} = (2,1)$ and $\vec{w} = (1,3)$. Let's say that $\vec{v}$ and $\vec{w}$ form the basis for the subspace V, $B = \{\vec{v}, \vec{w}\}$, and some vector $\vec{x}$ can be expressed uniquely as $\vec{x} = c_1\vec{v} + c_2\vec{w}$. Then

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

are the coordinates of $\vec{x}$ relative to the basis B. We can then set up an equation in the form $A[\vec{x}]_B = \vec{x}$, where A is the transformation matrix (the change of basis matrix from the standard basis to the basis B), and $[\vec{x}]_B$ is the vector $\vec{x}$ represented in terms of the basis B.

So if we want to know how the vector $\vec{x} = (3,4)$ will be represented by the new basis given by $\vec{v} = (2,1)$ and $\vec{w} = (1,3)$, then we plug into the equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To figure out how $\vec{x} = (3,4)$ would be represented in V, we'll solve the augmented matrix given by this equation.

$$\begin{bmatrix} 2 & 1 & | & 3 \\ 1 & 3 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 4 \\ 2 & 1 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 4 \\ 0 & -5 & | & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 4 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$$

This tells us that the solution to the equation is $[\vec{x}]_B = (1,1)$, or

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

In other words, if we want to represent $\vec{x} = (3,4)$ in terms of the standard basis vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$, then we need $3\mathbf{i} + 4\mathbf{j}$. But if we want to represent $\vec{x} = (3,4)$ in terms of the alternate basis $\vec{v} = (2,1)$ and $\vec{w} = (1,3)$, then we need $1\vec{v} + 1\vec{w}$, or just $\vec{v} + \vec{w}$. Which matches what we already found earlier when we sketched $(3,4)$ in the alternate basis.

And you can convert the other way, too. If you know you need $1\vec{v} + 1\vec{w}$ to get to $(3,4)$ in the alternate basis, you can do the matrix multiplication to find out the right combination of $\mathbf{i}$ and $\mathbf{j}$ in the standard basis.

$$A[\vec{x}]_V = \vec{x}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2(1) + 1(1) \\ 1(1) + 3(1) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 + 1 \\ 1 + 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let's do another example to make sure we know how to shift back and forth between bases.

Example

Find the combination of basis vectors $\vec{v} = (4, -3)$ and $\vec{w} = (-2,2)$, where $\vec{v}$ and $\vec{w}$ are a basis for $\mathbb{R}^2$, and give $\vec{x} = 8\mathbf{i} - 6\mathbf{j}$.

The vector $\vec{x} = (8, -6)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v} = (4, -3)$ and $\vec{w} = (-2,2)$.

In other words, we're trying to figure out how we would represent $\vec{x} = (8, -6)$, if we were doing it in terms of $\vec{v}$ and $\vec{w}$. So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll solve the augmented matrix.

$$\begin{bmatrix} 4 & -2 & | & 8 \\ -3 & 2 & | & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & 2 \\ -3 & 2 & | & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & 2 \\ 0 & \frac{1}{2} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & 2 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 0 \end{bmatrix}$$

This tells us that

$$[\vec{x}]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

In other words, we need 2 of $\vec{v} = (4, -3)$ and 0 of $\vec{w} = (-2,2)$ in order to get to $\vec{x} = (8, -6)$, in the alternate basis space defined by $\vec{v}$ and $\vec{w}$. So if we want to sketch this result, we first sketch in our $\vec{v}$- and $\vec{w}$-axes.

Then we move out 2 $\vec{v}$-units in the positive direction of the $\vec{v}$-axis, and 0 $\vec{w}$-units along the $\vec{w}$-axis (which means we don't move anywhere at all), and we arrive at $\vec{x} = (8, -6)$.

When A is invertible

If the basis transformation matrix is invertible (if it has an inverse), then we can multiply A^{-1} by both sides of $A[\vec{x}]_B = \vec{x}$.

$$A[\vec{x}]_B = \vec{x}$$

$$A^{-1}A[\vec{x}]_B = A^{-1}\vec{x}$$

$$I[\vec{x}]_B = A^{-1}\vec{x}$$

$$[\vec{x}]_B = A^{-1}\vec{x}$$

Now we have an equation that calculates directly the expression of $\vec{x}$ in the alternate basis.

So if we were to use the values from the last example, we could have first found A^{-1} from A.

$$[A\ |\ I] = \begin{bmatrix} 4 & -2 & | & 1 & 0 \\ -3 & 2 & | & 0 & 1 \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{1}{4} & 0 \\ -3 & 2 & | & 0 & 1 \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & | & \frac{3}{4} & 1 \end{bmatrix}$$

$$[A\ |\ I]=\begin{bmatrix}1 & -\frac{1}{2} & | & \frac{1}{4} & 0\\ 0 & 1 & | & \frac{3}{2} & 2\end{bmatrix}$$

$$[A\ |\ I]=\begin{bmatrix}1 & 0 & | & 1 & 1\\ 0 & 1 & | & \frac{3}{2} & 2\end{bmatrix}$$

Now that the left side of the augmented matrix is in reduced row-echelon form, the right side is the inverse matrix A^{-1}, so

$$A^{-1}=\begin{bmatrix}1 & 1\\ \frac{3}{2} & 2\end{bmatrix}$$

Now to find the representation of $\vec{x}=(8,-6)$ in B, we simply multiply the inverse matrix by the vector.

$$[\vec{x}]_B=A^{-1}\vec{x}$$

$$[\vec{x}]_B=\begin{bmatrix}1 & 1\\ \frac{3}{2} & 2\end{bmatrix}\begin{bmatrix}8\\ -6\end{bmatrix}$$

$$[\vec{x}]_B=\begin{bmatrix}1(8)+1(-6)\\ \frac{3}{2}(8)+2(-6)\end{bmatrix}$$

$$[\vec{x}]_B=\begin{bmatrix}8-6\\ 12-12\end{bmatrix}$$

$$[\vec{x}]_B=\begin{bmatrix}2\\ 0\end{bmatrix}$$

This confirms the result we found before.

Transformation matrix for a basis

Earlier we learned that a linear transformation could always be represented as a matrix-vector product. So we'd say that the linear transformation $T(\vec{x})$ could be written as

$$T(\vec{x}) = A\vec{x}$$

But up to now, we've always been working in the standard basis. Which means that the linear transformation T took vectors $\vec{x}$ that were given in the standard basis, and transformed them using the matrix A into another vector $T(\vec{x})$ in the standard basis.

Transforming from an alternate basis

Now we want to learn how to use the same transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ to transform vectors $[\vec{x}]_B$ given in an alternate basis, into vectors $[T(\vec{x})]_B$ in the alternate basis.

Whether we're transforming vectors in the standard basis or some alternate basis B for the subspace V, the transformation is still linear. Which means that, in the same way we represent a transformation in the standard basis as $T(\vec{x}) = A\vec{x}$, we can represent a transformation in the alternate basis B as

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

If we also define another invertible matrix C as the change of basis matrix that converts vectors between the standard basis and the alternate basis,

$$C[\vec{x}]_B = \vec{x}$$

then we can define a relationship between the matrices A, M, and C. Specifically, we know that $M = C^{-1}AC$.

Let's walk through an example, so that we can see how to use this $M = C^{-1}AC$ relationship.

Example

Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \end{bmatrix}\right)$$

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}=(1,1)$ and $\vec{w}=(-3,-2)$, so

$$C=\begin{bmatrix}1 & -3\\ 1 & -2\end{bmatrix}$$

Now we'll find C^{-1}.

$$[C\mid I]=\left[\begin{array}{cc|cc}1 & -3 & 1 & 0\\ 1 & -2 & 0 & 1\end{array}\right]$$

$$[C\mid I]=\left[\begin{array}{cc|cc}1 & -3 & 1 & 0\\ 0 & 1 & -1 & 1\end{array}\right]$$

$$[C\mid I]=\left[\begin{array}{cc|cc}1 & 0 & -2 & 3\\ 0 & 1 & -1 & 1\end{array}\right]$$

Once the left side of the augmented matrix is I, the right side is the inverse C^{-1}, so

$$C^{-1}=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}$$

With A, C, and C^{-1}, we can find $M=C^{-1}AC$.

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ -3 & 0\end{bmatrix}\begin{bmatrix}1 & -3\\ 1 & -2\end{bmatrix}$$

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2(1)-1(1) & 2(-3)-1(-2)\\ -3(1)+0(1) & -3(-3)+0(-2)\end{bmatrix}$$

$$M=\begin{bmatrix}-2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}2-1 & -6+2\\ -3+0 & 9+0\end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -3 & 9 \end{bmatrix}$$

$$M = \begin{bmatrix} -2(1) + 3(-3) & -2(-4) + 3(9) \\ -1(1) + 1(-3) & -1(-4) + 1(9) \end{bmatrix}$$

$$M = \begin{bmatrix} -2 - 9 & 8 + 27 \\ -1 - 3 & 4 + 9 \end{bmatrix}$$

$$M = \begin{bmatrix} -11 & 35 \\ -4 & 13 \end{bmatrix}$$

The result here is the matrix M from $[T(\vec{x})]_B = M[\vec{x}]_B$ that will transform vectors $[\vec{x}]_B$ in the alternate basis in the domain into vectors $[T(\vec{x})]_B$ in the alternate basis in the codomain.

Now that we have M, given any vector $[\vec{x}]_B$ defined in the alternate basis in the domain, we can simply multiply M by the vector to get another vector, also defined in the alternate basis, in the codomain.

We've been asked to transform $[\vec{x}]_B = (2,1)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$[T(\vec{x})]_B = \begin{bmatrix} -11 & 35 \\ -4 & 13 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -11(2) + 35(1) \\ -4(2) + 13(1) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -22+35 \\ -8+13 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$$

In other words, $[\vec{x}]_B = (2,1)$ is defined in the alternate basis in the domain, and the transformation T maps that vector to $[T(\vec{x})]_B = (13,5)$ in the alternate basis in the codomain.

What if we want to transform a vector in the standard basis in the domain into a vector in the alternate basis in the codomain. We can either,

1. transform the vector in the standard basis in the domain into a vector in the standard basis in the codomain, and then transform the result from the standard basis in the codomain to the alternate basis in the codomain, $\vec{x} \to T(\vec{x}) \to [T(\vec{x})]_B$, where $T(\vec{x}) = A\vec{x}$ and $[T(\vec{x})]_B = C^{-1}T(\vec{x})$, or you can

2. transform the vector in the standard basis in the domain into a vector in the alternate basis in the domain, and then transform the result from the alternate basis in the domain to the alternate basis in the codomain, $\vec{x} \to [\vec{x}]_B \to [T(\vec{x})]_B$, where $[\vec{x}]_B = C^{-1}\vec{x}$ and $[T(\vec{x})]_B = M[\vec{x}]_B$.

We can visualize the pathways between the domain and codomain, and the standard and alternate bases, as

$$
\begin{tikzcd}
 & \text{Domain} & \text{Codomain} \\
\text{Standard Basis} & \vec{x} \arrow[r, "A"] \arrow[d, "C^{-1}"'] & T(\vec{x}) \arrow[d, "C^{-1}"] \\
\text{Alternate Basis} & [\vec{x}]_B \arrow[r, "M"'] & [T(\vec{x})]_B
\end{tikzcd}
$$

Orthonormal bases

We've talked about changing bases from the standard basis to an alternate basis, and vice versa. Now we want to talk about a specific kind of basis, called an **orthonormal basis**, in which every vector in the basis is both 1 unit in length and orthogonal to each of the other basis vectors.

In other words, $B = \{\vec{v_1}, \vec{v_2}, \dots \vec{v_n}\}$ is the orthonormal basis for V if each vector $\vec{v}_i$ in the set V has length 1, such that $\vec{v}_i \cdot \vec{v}_i = 1$ or $||\vec{v}_i||^2 = 1$. And if each vector in the set $\vec{v}_i$ is orthogonal to every other vector in the set $\vec{v}_j$, then $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$.

Realize too that if a set of vectors is orthonormal, that means that all the vectors in the set are also linearly independent.

Example

Verify that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is an orthonormal set, if $\vec{v}_1 = (0,0,1)$ and $\vec{v}_2 = (0,1,0)$.

For the set to be orthonormal, each vector needs to have length 1. So we'll find the length of each vector.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = 0(0) + 0(0) + 1(1) = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = 0(0) + 1(1) + 0(0) = 1$$

Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = 0(0) + 0(1) + 1(0) = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, $\vec{v}_1$ and $\vec{v}_2$ form an orthonormal set, so V is orthonormal.

Converting into an orthonormal basis

Notice in this example that we used two of the three standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\mathbb{R}^3$. In fact, it should make intuitive sense that the standard basis vectors will form an orthonormal set in any $\mathbb{R}^n$.

Considering the fact that the standard basis vectors are extremely easy to use as a basis, it should make sense then that orthonormal bases in general make for good bases.

For instance, to convert a vector from the standard basis to an alternate basis B, normally we would solve a matrix equation like

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } \vec{x} = (x_1, x_2, x_3)$$

Which means we would create an augmented matrix, put it in rref, and from that rref matrix pull out the components of $[\vec{x}]_B$.

But if we're converting to an orthonormal basis specifically, that means the column vectors v_1, v_2, v_3 form an orthonormal set of vectors. An **orthogonal matrix** is a square matrix whose columns form an orthonormal set of vectors. If a matrix is rectangular, but its columns still form an orthonormal set of vectors, then we call it an **orthonormal matrix**.

When a matrix is orthogonal, we know that its transpose is the same as its inverse. So given an orthogonal matrix A,

$$A^T = A^{-1}$$

Orthogonal matrices are always square (an orthonormal matrix can be rectangular, but if we call a matrix orthogonal, we specifically mean that it's a square matrix), so its inverse can be defined, assuming that the square matrix is invertible.

Using an orthonormal basis simplifies many of the operations and formulas that we've learned. Here, if the column vectors v_1, v_2, v_3 in

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

form an orthonormal set, then the matrix problem simplifies to a dot product problem. Specifically, the equation above becomes

$$[\vec{x}]_B = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vec{v}_3 \cdot \vec{x} \end{bmatrix}$$

In other words, instead of putting the augmented matrix into reduced row-echelon form, we just need to take dot products of the vectors that define the orthonormal basis $B = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and the vector $\vec{x}$.

Especially as the dimension gets larger, for instance if we're in $\mathbb{R}^{100}$ instead of just $\mathbb{R}^3$, the matrix problem becomes unmanageable without a computer. But with an orthonormal set, the matrix problem becomes a dot product problem, which is a much easier calculation.

Example

Convert $\vec{x} = (5,6,-1)$ from the standard basis to the alternate basis $B = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}$$

First, let's confirm that the set is orthonormal. Confirm that the length of each vector is 1.

$$||\vec{v}_1||^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 0 + \frac{1}{2} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$

$$||\vec{v}_3||^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$

Confirm that each vector is orthogonal to the others.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} + 0 - \frac{1}{2\sqrt{2}} = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + 0\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2}} + 0 - \frac{1}{2\sqrt{2}} = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{2}{4} + \frac{1}{4} = 0$$

Each of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ has a length 1, and is orthogonal to the other vectors in the set, so the set is orthonormal.

Because the set is orthonormal, the vector $\vec{x} = (5,6,-1)$ can be converted to the alternate basis B with dot products. In other words, instead of solving

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of $[\vec{x}]_B$.

$$[\vec{x}]_B = \begin{bmatrix} \frac{1}{\sqrt{2}}(5) + 0(6) - \frac{1}{\sqrt{2}}(-1) \\ \frac{1}{2}(5) + \frac{\sqrt{2}}{2}(6) + \frac{1}{2}(-1) \\ \frac{1}{2}(5) - \frac{\sqrt{2}}{2}(6) + \frac{1}{2}(-1) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{5}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} \\ \frac{5}{2} + 3\sqrt{2} - \frac{1}{2} \\ \frac{5}{2} - 3\sqrt{2} - \frac{1}{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{6}{\sqrt{2}} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{6\sqrt{2}}{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 3\sqrt{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

This result tells us that $\vec{x} = (5,6,-1)$ can be expressed in the alternate basis B as

$$[\vec{x}]_B = \begin{bmatrix} 3\sqrt{2} \\ 2 + 3\sqrt{2} \\ 2 - 3\sqrt{2} \end{bmatrix}$$

Projection onto an orthonormal basis

Finding the projection of a vector onto a subspace is also much easier when the subspace is defined by an orthonormal basis.

Remember previously that we defined the projection of a vector $\vec{x}$ onto a subspace V as

$$\text{Proj}_V\vec{x}=A(A^TA)^{-1}A^T\vec{x}$$

where A is the matrix made from the column vectors that define V. But if we can define the subspace V with an orthonormal basis, then the projection can be defined as just

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

The reason is because the value $(A^TA)^{-1}$ from the middle of $\text{Proj}_V\vec{x}=A(A^TA)^{-1}A^T\vec{x}$ actually simplifies to the identity matrix when A is a matrix made of orthonormal column vectors. So the projection formula collapses as

$$\text{Proj}_V\vec{x}=A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x}=AIA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

So you can see how defining the subspace with an orthonormal vector set would make calculating the projection of a vector onto the subspace much easier.

Example

Find the projection of $\vec{x}=(5,6,-1)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{2}}{2}\\ \frac{1}{2}\end{bmatrix}\right)$$

We already confirmed in the example in the previous section that $V=\{\vec{v}_1,\vec{v}_2\}$ was an orthonormal vector set. So the projection of $\vec{x}=(5,6,-1)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{2}\\ 0 & \frac{\sqrt{2}}{2}\\ -\frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right) & \frac{1}{\sqrt{2}}(0)+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) & \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\\ 0\left(\frac{1}{\sqrt{2}}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) & 0(0)+\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) & 0\left(-\frac{1}{\sqrt{2}}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ -\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right) & -\frac{1}{\sqrt{2}}(0)+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) & -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\end{bmatrix}\begin{bmatrix}5\\ 6\\ -1\end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{1}{2} + \frac{1}{4} & 0 + \frac{\sqrt{2}}{4} & -\frac{1}{2} + \frac{1}{4} \\ 0 + \frac{\sqrt{2}}{4} & 0 + \frac{1}{2} & 0 + \frac{\sqrt{2}}{4} \\ -\frac{1}{2} + \frac{1}{4} & 0 + \frac{\sqrt{2}}{4} & \frac{1}{2} + \frac{1}{4} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{2}}{4} & -\frac{1}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{2}}{4} \\ -\frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{3}{4}(5) + \frac{\sqrt{2}}{4}(6) - \frac{1}{4}(-1) \\ \frac{\sqrt{2}}{4}(5) + \frac{1}{2}(6) + \frac{\sqrt{2}}{4}(-1) \\ -\frac{1}{4}(5) + \frac{\sqrt{2}}{4}(6) + \frac{3}{4}(-1) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{15}{4} + \frac{3\sqrt{2}}{2} + \frac{1}{4} \\ \frac{5\sqrt{2}}{4} + 3 - \frac{\sqrt{2}}{4} \\ -\frac{5}{4} + \frac{3\sqrt{2}}{2} - \frac{3}{4} \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 4 + \frac{3\sqrt{2}}{2} \\ 3 + \sqrt{2} \\ -2 + \frac{3\sqrt{2}}{2} \end{bmatrix}$$

So the vector in the subspace V which is the shadow of $\vec{x} = (5,6,-1)$ on V is

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 4 + \frac{3\sqrt{2}}{2} \\ 3 + \sqrt{2} \\ -2 + \frac{3\sqrt{2}}{2} \end{bmatrix}$$

In this last example, notice how we took one vector away from the set $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and only used the first two vectors, $V = \{\vec{v}_1, \vec{v}_2\}$. That was intentional.

The orthonormal set $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$. Since $\vec{x} = (5,6,-1)$ is a vector in $\mathbb{R}^3$, finding the projection of $\vec{x} = (5,6,-1)$ onto the subspace $\mathbb{R}^3$ would have simply given us $\vec{x} = (5,6,-1)$ again. In other words, AA^T would give the identity matrix I_3, so

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

would become

$$\text{Proj}_V\vec{x} = I\vec{x}$$

$$\text{Proj}_V\vec{x} = \vec{x}$$

Which shows us that the projection of $\vec{x}$ onto V is just $\vec{x}$ itself. But because we instead changed the subspace to be represented by just $V = \{\vec{v}_1, \vec{v}_2\}$, that means the subspace is now a plane floating in $\mathbb{R}^3$. And that allows us to see the shadow of $\vec{x} = (5,6,-1)$ on the plane, thereby finding the projection of $\vec{x}$ onto V as

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 4 + \frac{3\sqrt{2}}{2} \\ 3 + \sqrt{2} \\ -2 + \frac{3\sqrt{2}}{2} \end{bmatrix}$$

Gram-Schmidt process for change of basis

The **Gram-Schmidt process** is a process that turns the basis of a subspace into an orthonormal basis for the same subspace. Of course, as we've seen, this is extremely useful.

Since working with an orthonormal basis can, for certain applications, be much simpler than working with a non-orthonormal basis, we can save a lot of time by changing the basis of the subspace into an orthonormal basis.

The Gram-Schmidt process changes the basis by dealing with one basis vector at a time, and it works regardless of how many vectors form the basis for the subspace.

How to use Gram-Schmidt

Let's say that a non-orthonormal basis of the subspace V is given by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. In other words,

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

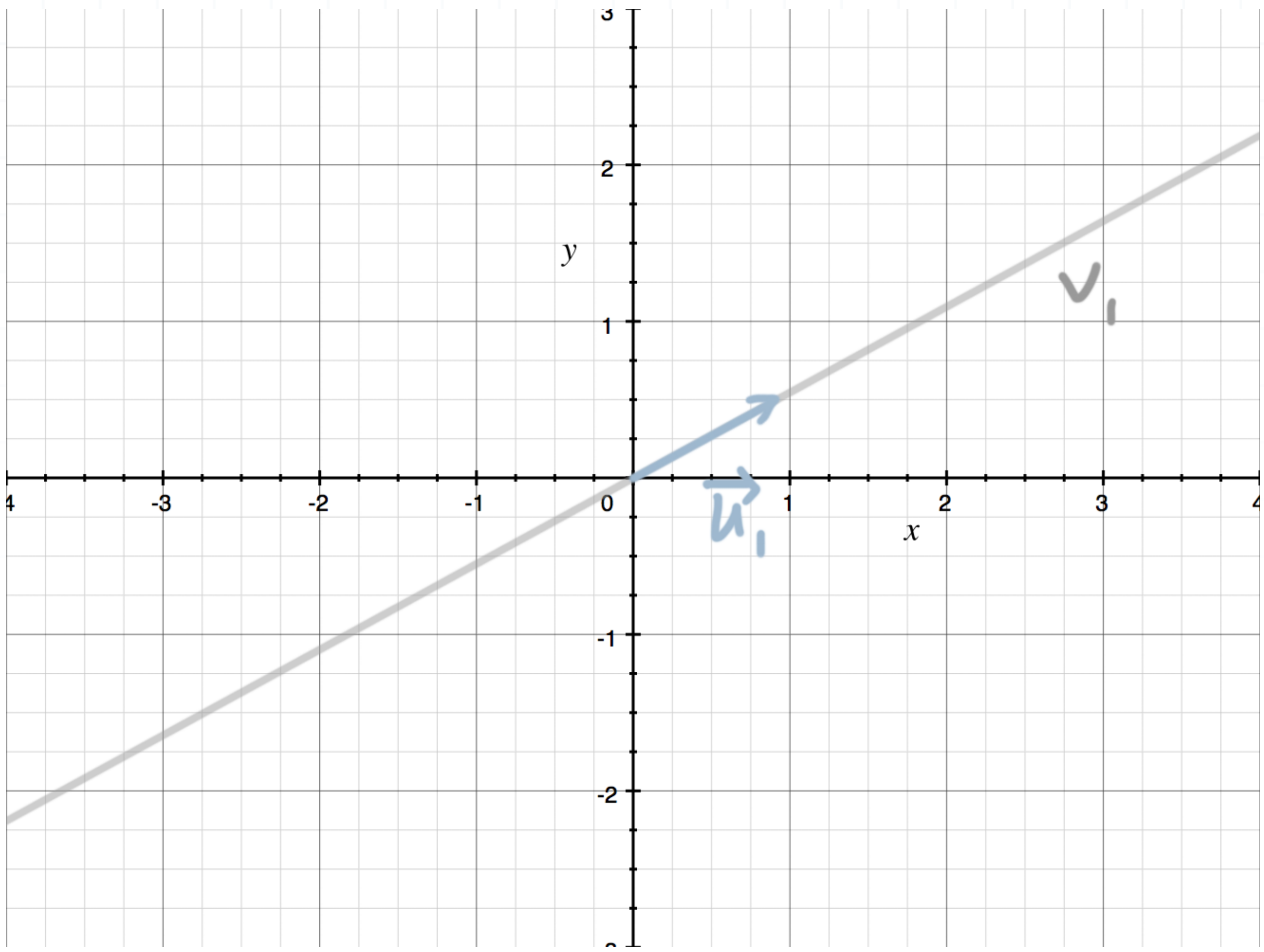
Our first step is to normalize $\vec{v}_1$ (make its length equal to 1). We can make $\vec{v}_1$ normal using

$$\vec{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||}$$

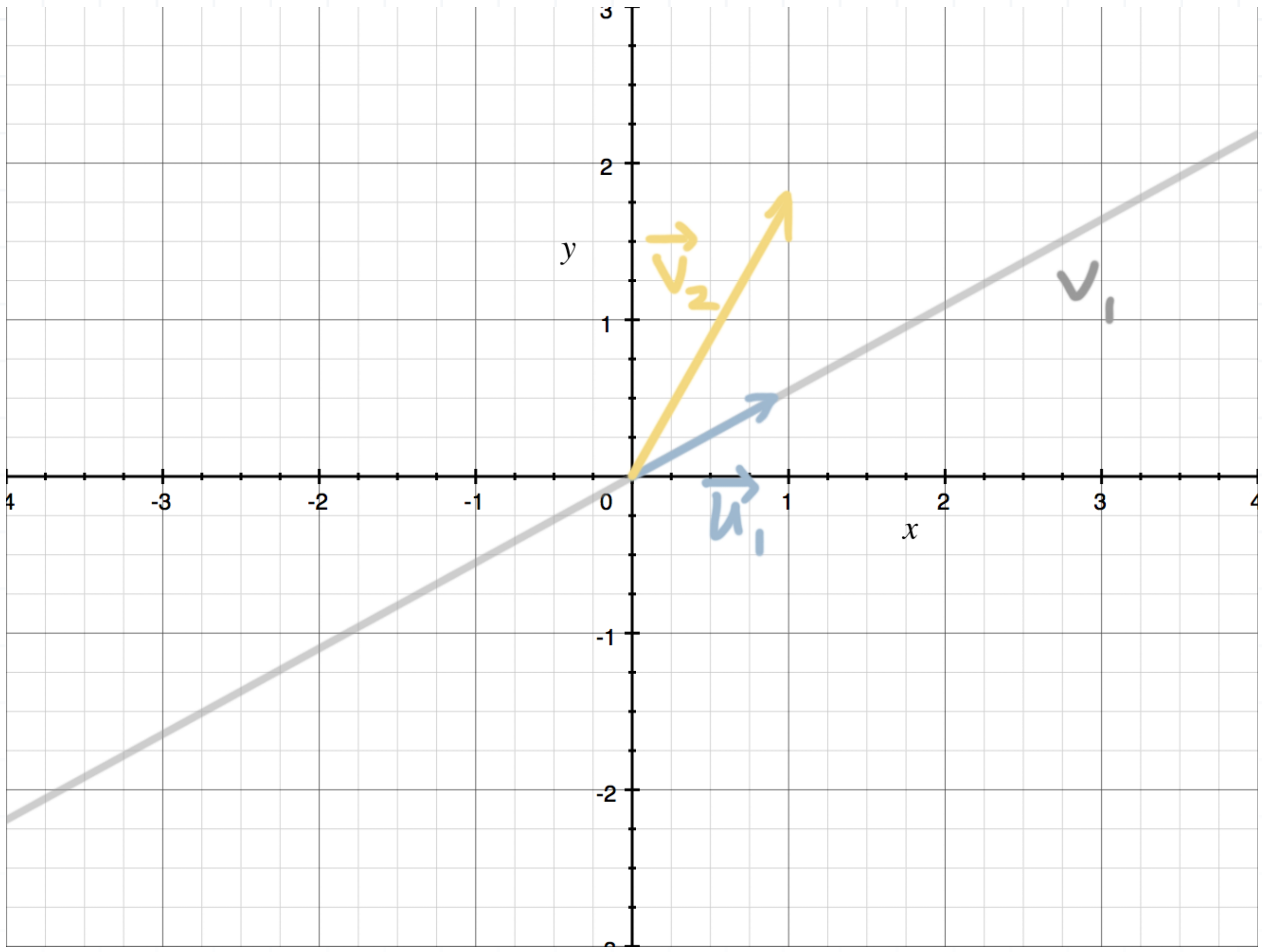
where $||\vec{v}_1||$ is the length of $\vec{v}_1$. Then with $\vec{v}_1$ normalized, the basis of V can be formed by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

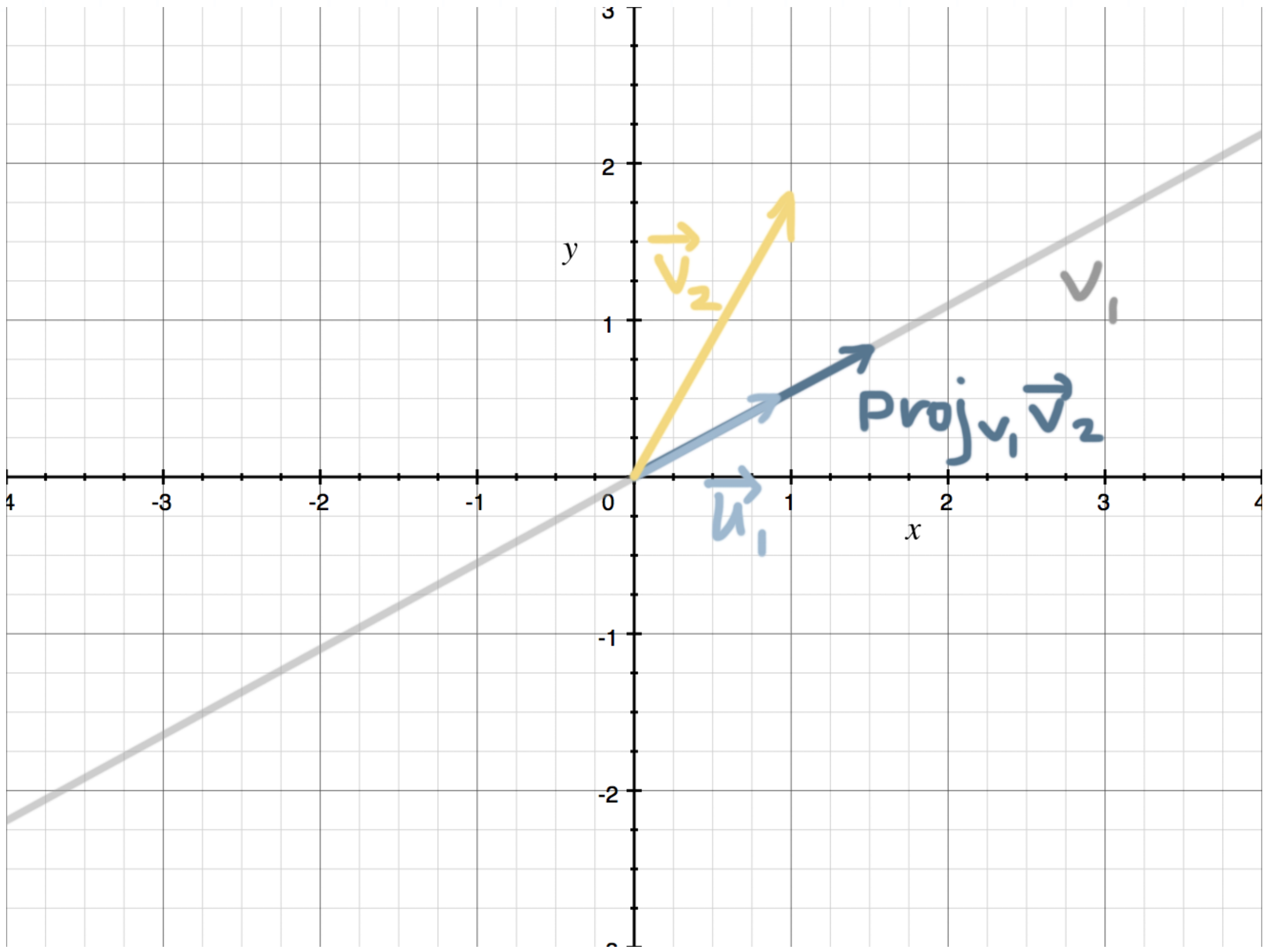
The next step is to replace $\vec{v}_2$ with with a vector that's both orthogonal to $\vec{u}_1$, and normal. To do that, we need to think about the span of just $\vec{u}_1$, which we'll call V_1. Generally, we could imagine that as



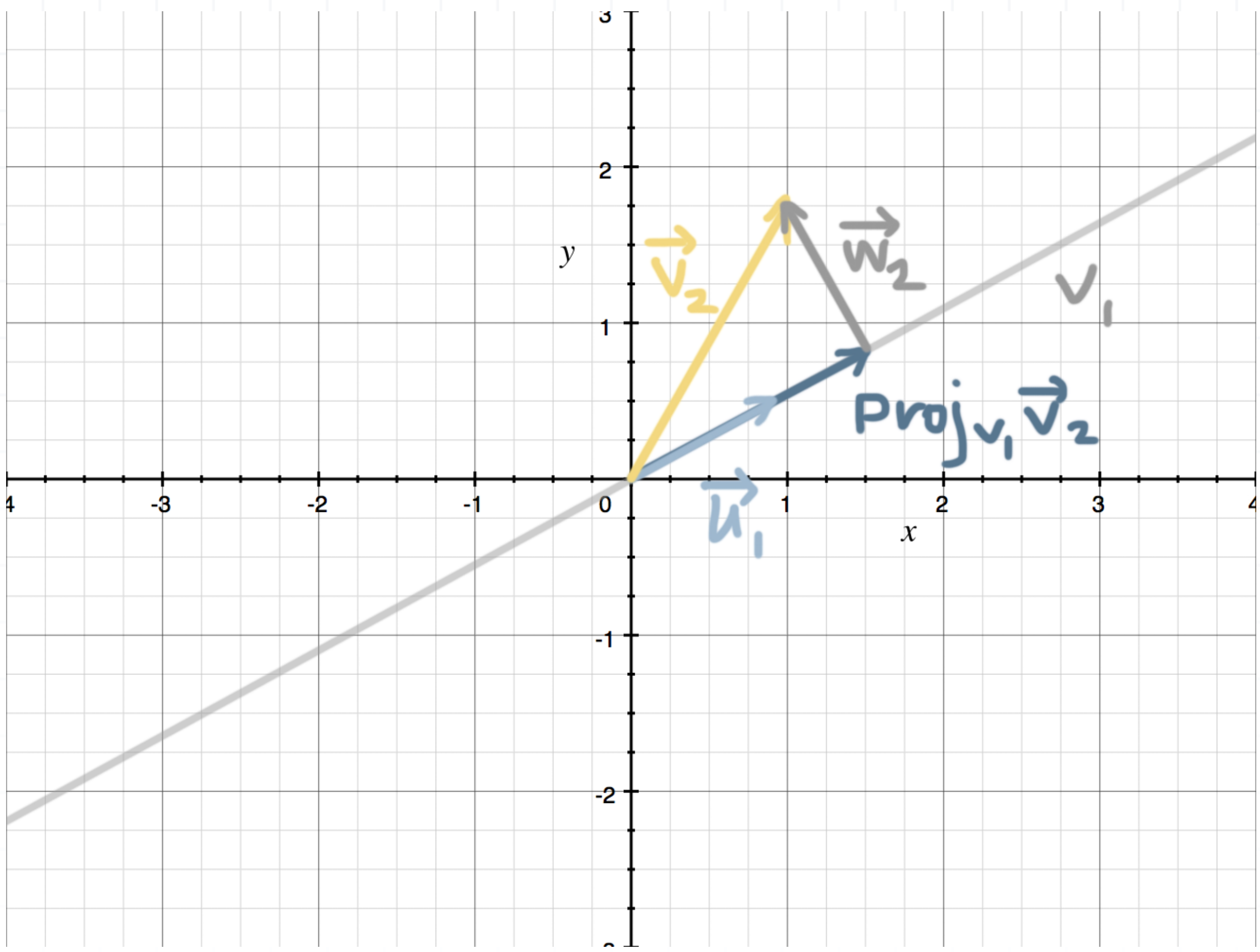
Let's imagine that $\vec{v}_2$ is another vector:



We could sketch the projection of $\vec{v}_2$ onto V_1, $\text{Proj}_{V_1}\vec{v}_2$:



Then $\vec{w}_2$ is the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$, so $\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$,



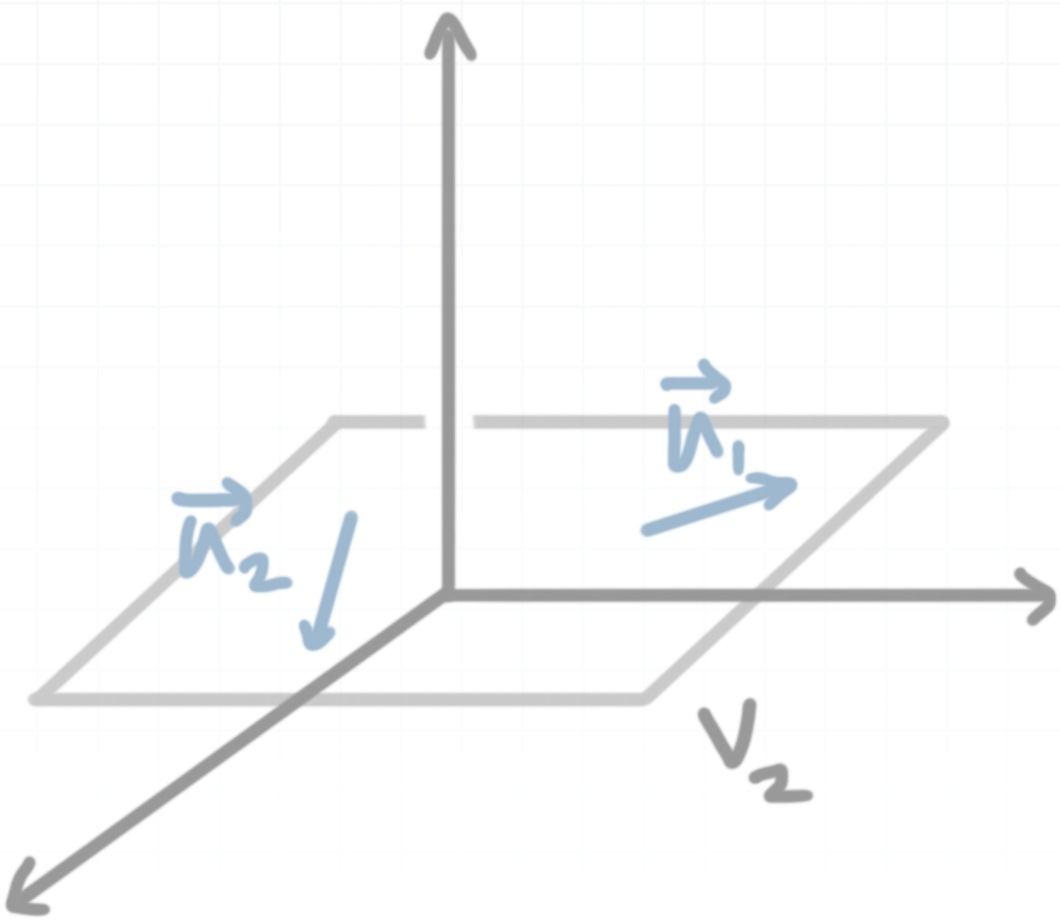
and $\vec{w}_2$ is orthogonal to $\vec{u}_1$. But realize that V_1 is an orthonormal subspace. Since there's only one vector $\vec{u}_1$ that forms the basis for V_1, every vector in the basis is orthogonal to every other vector (because there are no other vectors), and $\vec{u}_1$ is normal, so the basis of V_1 is orthonormal. Which means that we could rewrite the projection $\text{Proj}_{V_1}\vec{v}_2$ as

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

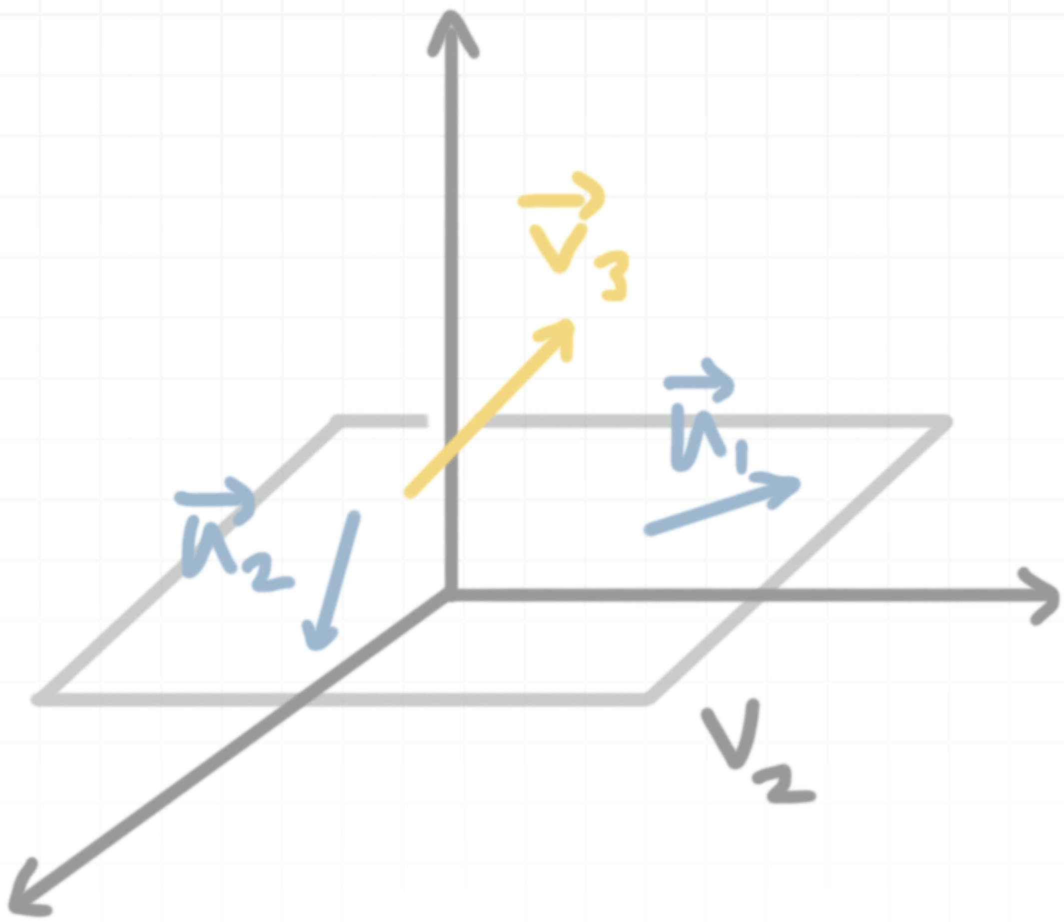
This will give us a vector $\vec{w}_2$ that we can use in place of $\vec{v}_2$. Once we normalize $\vec{w}_2$ the same way we normalized $\vec{v}_1$, we'll call it $\vec{u}_2$, and then we'll be able to say that the basis of V can be formed by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$.

$$V = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{v}_3)$$

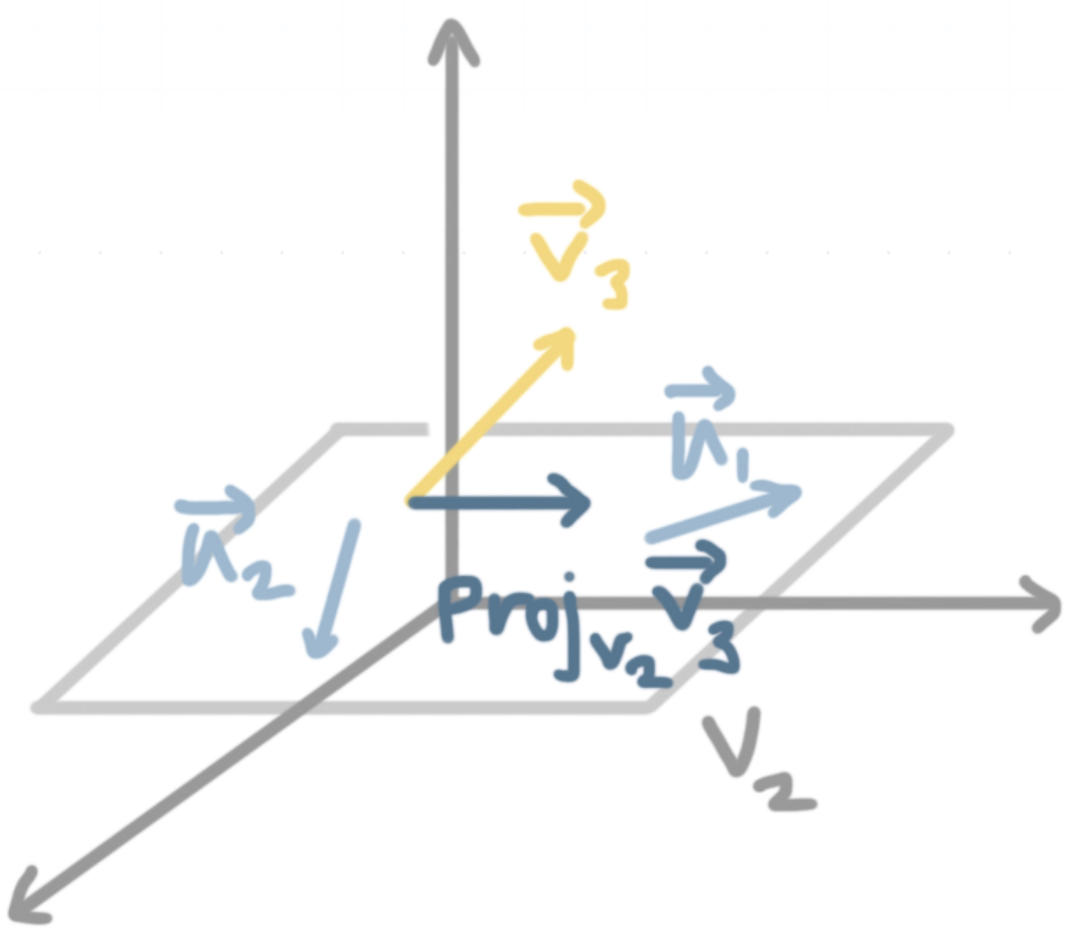
And we would just keep repeating this process with every basis vector. In this case, for the last basis vector $\vec{v}_3$, we'd think about the span of $\vec{u}_1$ and $\vec{u}_2$, which we'll call V_2. The subspace V_2 will be a plane, whereas the subspace V_1 was a line. Generally, we could imagine that as



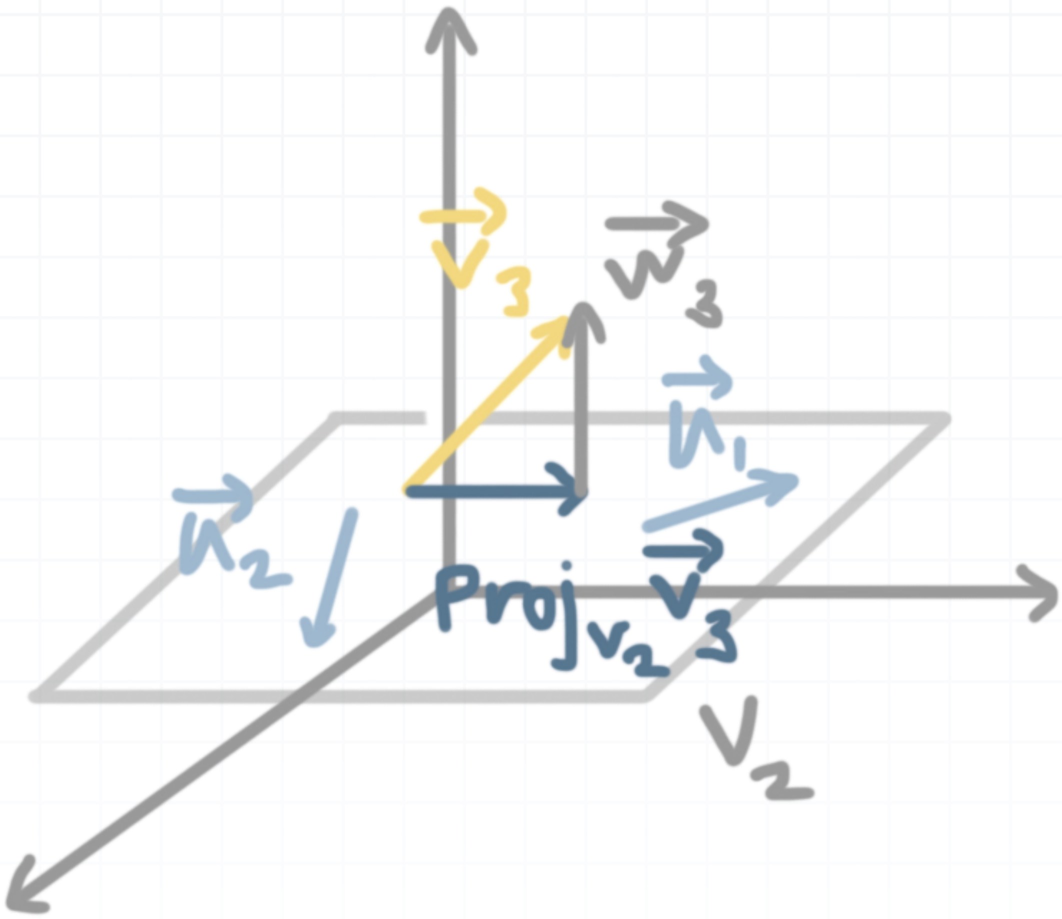
Let's imagine that $\vec{v}_3$ is another vector:



We could sketch the projection of $\vec{v}_3$ onto V_2, $\text{Proj}_{V_2}\vec{v}_3$:



Then $\vec{w}_3$ is the vector that connects $\text{Proj}_{V_2}\vec{v}_3$ to $\vec{v}_3$, so $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$,



and $\vec{w}_3$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. But realize that V_2 is an orthonormal subspace. There are two vectors $\vec{u}_1$ and $\vec{u}_2$ that form the basis for V_2, and those two vectors are orthogonal to one another. And we know that $\vec{u}_1$ and $\vec{u}_2$ are normal, so the basis of V_2 is orthonormal. Which means that we could rewrite the projection $\text{Proj}_{V_2}\vec{v}_3$ as

$$\vec{w}_3 = \vec{v}_3 - \left[(\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}\right]$$

This will give us a vector $\vec{w}_3$ that we can use in place of $\vec{v}_3$. Once we normalize $\vec{w}_3$ the same way we normalized $\vec{v}_1$ and $\vec{w}_2$, we'll call it $\vec{u}_3$, and then we'll be able to say that the basis of V can be formed by $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

$$V = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$$

And as you can see, if we had more basis vectors for V, we could continue this process, finding $\vec{u}_4$, $\vec{u}_5$, etc., until we've converted the entire non-orthonormal basis into an orthonormal basis.

Let's do an example so that we can see how this works with real vectors.

Example

The subspace V is a plane in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\-5\end{bmatrix}\right)$$

Let's define $\vec{v}_1 = (1,2,0)$ and $\vec{v}_2 = (-2,1,-5)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

Then we'll start by normalizing $\vec{v}_1$. The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{1 + 4 + 0} = \sqrt{5}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - \left(\begin{bmatrix}-2\\1\\-5\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - \frac{1}{5}\left(\begin{bmatrix}-2\\1\\-5\end{bmatrix} \cdot \begin{bmatrix}1\\2\\0\end{bmatrix}\right)\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - \frac{1}{5}(-2(1) + 1(2) - 5(0))\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - \frac{1}{5}(-2 + 2 - 0)\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - \frac{1}{5}(0)\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\-5\end{bmatrix} - 0\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$$

This vector $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{(-2)^2 + (1)^2 + (-5)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + 1 + 25}$$

$$||\vec{w}_2|| = \sqrt{30}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V = \text{Span}\left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} \right)$$

$$V = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ \frac{0}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix} \right)$$

$$V = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix}\right)$$

Eigenvalues, eigenvectors, eigenspaces

Any vector $\vec{v}$ that satisfies $T(\vec{v}) = \lambda \vec{v}$ is an **eigenvector** for the transformation T, and λ is the **eigenvalue** that's associated with the eigenvector $\vec{v}$. The transformation T is a linear transformation that can also be represented as $T(\vec{v}) = A\vec{v}$.

The first thing you want to notice about $T(\vec{v}) = \lambda \vec{v}$, is that, because λ is a constant that acts like a scalar on $\vec{v}$, we're saying that the transformation of $\vec{v}$, $T(\vec{v})$, is really just a scaled version of $\vec{v}$.

We could also say that the eigenvectors $\vec{v}$ are the vectors that don't change direction when we apply the transformation matrix T. So if we apply T to a vector $\vec{v}$, and the result $T(\vec{v})$ is parallel to the original $\vec{v}$, then $\vec{v}$ is an eigenvector.

Identifying eigenvectors

In other words, if we define a specific transformation T that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$, then there may be certain vectors in the domain that change direction under the transformation T. For instance, maybe the transformation T rotates vectors by 30°. Vectors that rotate by 30° will never satisfy $T(\vec{v}) = \lambda \vec{v}$.

But there may be other vectors in the domain that stay along the same line under the transformation, and might just get scaled up or scaled down by T. Those are the vectors that will satisfy $T(\vec{v}) = \lambda \vec{v}$, which means that those are the eigenvectors for T. And this makes sense, because

$T(\vec{v}) = \lambda\vec{v}$ literally reads "the transformed version of $\vec{v}$ is the same as the original $\vec{v}$, but just scaled up or down by λ."

The way to really identify an eigenvector is to compare the span of $\vec{v}$ with the span of $T(\vec{v})$. The span of any single vector $\vec{v}$ will always be a line. If, under the transformation T, the span remains the same, such that $T(\vec{v})$ has the same span as $\vec{v}$, then you know $\vec{v}$ is an eigenvector. The vectors $\vec{v}$ and $T(\vec{v})$ might be different lengths, but their spans are the same because they lie along the same line.

The reason we care about identifying eigenvectors is because they often make good basis vectors for the subspace, and we're always interested in finding a simple, easy-to-work-with basis.

Finding eigenvalues

Because we've said that $T(\vec{v}) = \lambda\vec{v}$ and $T(\vec{v}) = A\vec{v}$, it has to be true that $A\vec{v} = \lambda\vec{v}$. Which means eigenvectors are any vectors $\vec{v}$ that satisfy $A\vec{v} = \lambda\vec{v}$.

We also know that there will be 2 eigenvectors when A is 2×2, that there will be 3 eigenvectors when A is 3×3, and that there will be n eigenvectors when A is $n \times n$.

While $\vec{v} = \vec{O}$ would satisfy $A\vec{v} = \lambda\vec{v}$, we don't really include that as an eigenvector. The reason is first, because it doesn't really give us any interesting information, and second, because $\vec{v} = \vec{O}$ doesn't allow us to determine the associated eigenvalue λ.

So we're really only interested in the vectors $\vec{v}$ that are nonzero. If we rework $A\vec{v} = \lambda\vec{v}$, we could write it as

$$\vec{O} = \lambda\vec{v} - A\vec{v}$$

$$\vec{O} = \lambda I_n\vec{v} - A\vec{v}$$

$$(\lambda I_n - A)\vec{v} = \vec{O}$$

Realize that this is just a matrix-vector product, set equal to the zero vector. Because $\lambda I_n - A$ is just a matrix. The eigenvalue λ acts as a scalar on the identity matrix I_n, which means λI_n will be a matrix. If, from λI_n, we subtract the matrix A, we'll still just get another matrix, which is why $\lambda I_n - A$ is a matrix. So let's make a substitution $B = \lambda I_n - A$.

$$B\vec{v} = \vec{O}$$

Written this way, we can see that any vector $\vec{v}$ that satisfies $B\vec{v} = \vec{O}$ will be in the null space of B, $N(B)$. But we already said that $\vec{v}$ was going to be nonzero, which tells us right away that there must be at least one vector in the null space that's not the zero vector. Whenever we know that there's a vector in the null space other than the zero vector, we conclude that the matrix B (the matrix $\lambda I_n - A$) has linearly dependent columns, and that B is not invertible, and that the determinant of B is 0, $|B| = 0$.

Which means we could come up with these rules:

$A\vec{v} = \lambda\vec{v}$ for nonzero vectors $\vec{v}$ if and only if $|\lambda I_n - A| = 0$.

λ is an eigenvalue of A if and only if $|\lambda I_n - A| = 0$.

With these rules in mind, we have everything we need to find the eigenvalues for a particular matrix.

Example

Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We need to find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 2 & 0 - 1 \\ 0 - 1 & \lambda - 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} \right|$$

Then the determinant of this resulting matrix is

$$(\lambda - 2)(\lambda - 2) - (-1)(-1)$$

$$(\lambda - 2)(\lambda - 2) - 1$$

$$\lambda^2 - 4\lambda + 4 - 1$$

$$\lambda^2 - 4\lambda + 3$$

This polynomial is called the **characteristic polynomial**. Remember that we're trying to satisfy $|\lambda I_n - A| = 0$, so we can set this characteristic polynomial equal to 0, and get the **characteristic equation**:

$$\lambda^2 - 4\lambda + 3 = 0$$

To solve for λ, we'll always try factoring, but if the polynomial can't be factored, we can either complete the square or use the quadratic formula. This one can be factored.

$$(\lambda - 3)(\lambda - 1) = 0$$

$\lambda = 1$ or $\lambda = 3$

So assuming non-zero eigenvectors, we're saying that $A\vec{v} = \lambda\vec{v}$ can be solved for $\lambda = 1$ and $\lambda = 3$.

We want to make a couple of important points, which are both illustrated by this last example.

First, the sum of the eigenvalues will always equal the sum of the matrix entries that run down its diagonal. In the matrix A from the example, the values down the diagonal were 2 and 2. Their sum is 4, which means the sum of the eigenvalues will be 4 as well. The sum of the entries along the diagonal is called the **trace** of the matrix, so we can say that the trace will always be equal to the sum of the eigenvalues.

$$\text{Trace}(A) = \text{sum of } A\text{'s eigenvalues}$$

Realize that this also means that, for an $n \times n$ matrix A, once we find $n - 1$ of the eigenvalues, we'll already have the value of the nth eigenvalue.

Second, the determinant of A, $|A|$, will always be equal to the product of the eigenvalues. In the last example, $|A| = (2)(2) - (1)(1) = 4 - 1 = 3$, and the product of the eigenvalues was $\lambda_1\lambda_2 = (1)(3) = 3$.

$$\text{Det}(A) = |A| = \text{product of } A\text{'s eigenvalues}$$

Finding eigenvectors

Once we've found the eigenvalues for the transformation matrix, we need to find their associated eigenvectors. To do that, we'll start by defining an eigenspace for each eigenvalue of the matrix.

The eigenspace E_λ for a specific eigenvalue λ is the set of all the eigenvectors $\vec{v}$ that satisfy $A\vec{v} = \lambda\vec{v}$ for that particular eigenvalue λ.

As we know, we were able to rewrite $A\vec{v} = \lambda\vec{v}$ as $(\lambda I_n - A)\vec{v} = \vec{O}$, and we recognized that $\lambda I_n - A$ is just a matrix. So the **eigenspace** is simply the null space of the matrix $\lambda I_n - A$.

$$E_\lambda = N(\lambda I_n - A)$$

To find the matrix $\lambda I_n - A$, we can simply plug the eigenvalue into the value we found earlier for $\lambda I_n - A$. Let's continue on with the previous example and find the eigenvectors associated with $\lambda = 1$ and $\lambda = 3$.

Example

For the transformation matrix A, we found eigenvalues $\lambda = 1$ and $\lambda = 3$. Find the eigenvectors associated with each eigenvalue.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

With $\lambda = 1$ and $\lambda = 3$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}\right)$$

we get

$$E_1 = N\left(\begin{bmatrix} 1 - 2 & -1 \\ -1 & 1 - 2 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3 - 2 & -1 \\ -1 & 3 - 2 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_1 will satisfy

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & | & 0 \\ -1 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 0 \\ -1 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

So with $v_1 = -v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - v_2 = 0$$

$$v_1 = v_2$$

And with $v_1 = v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

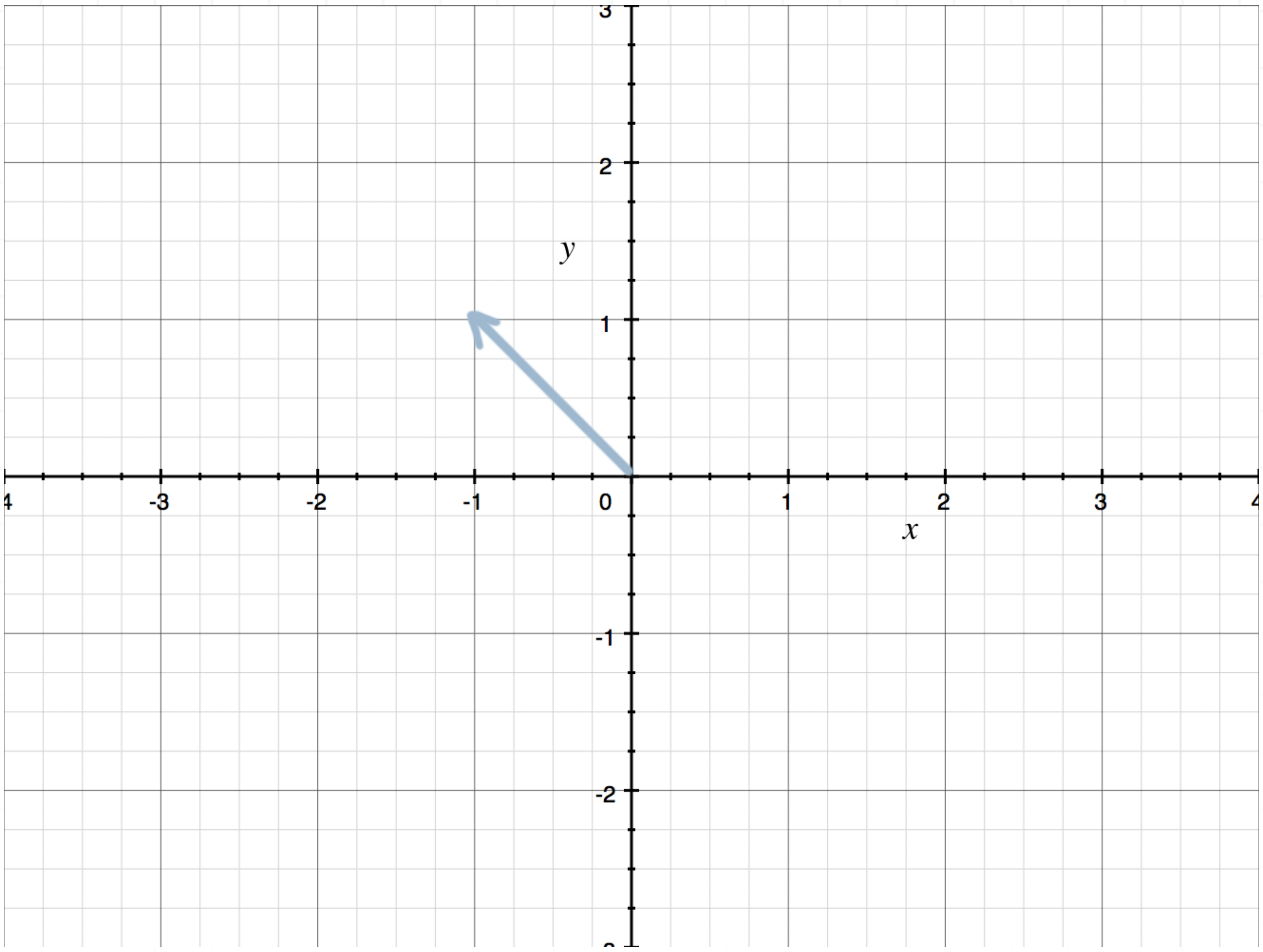
Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

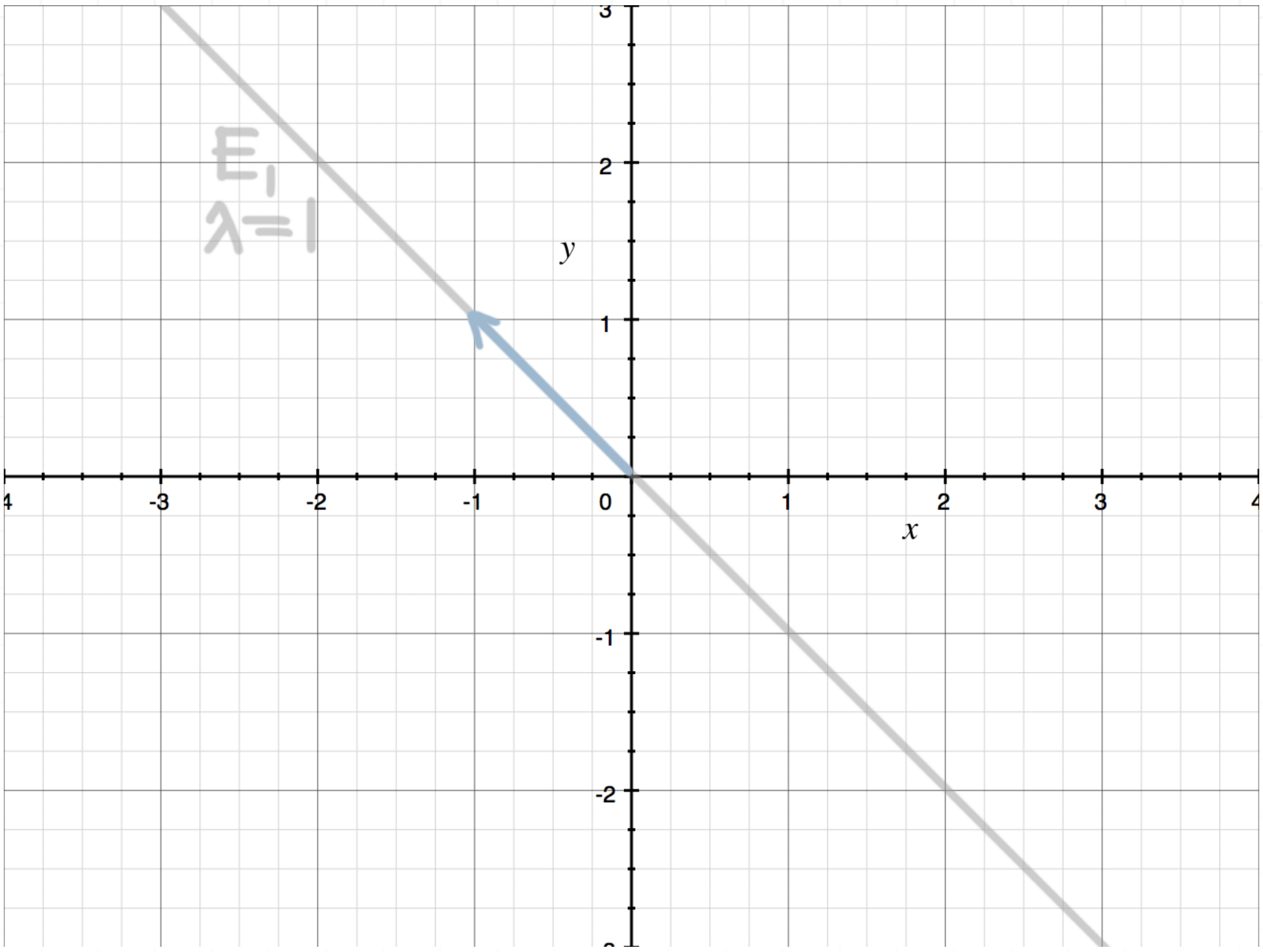
If we put these last two examples together (the first one where we found the eigenvalues, and this second one where we found the associated eigenvectors), we can sketch a picture of the solution. For the eigenvalue $\lambda = 1$, we got

$$E_1 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

We can sketch the spanning eigenvector $\vec{v} = (-1,1)$,



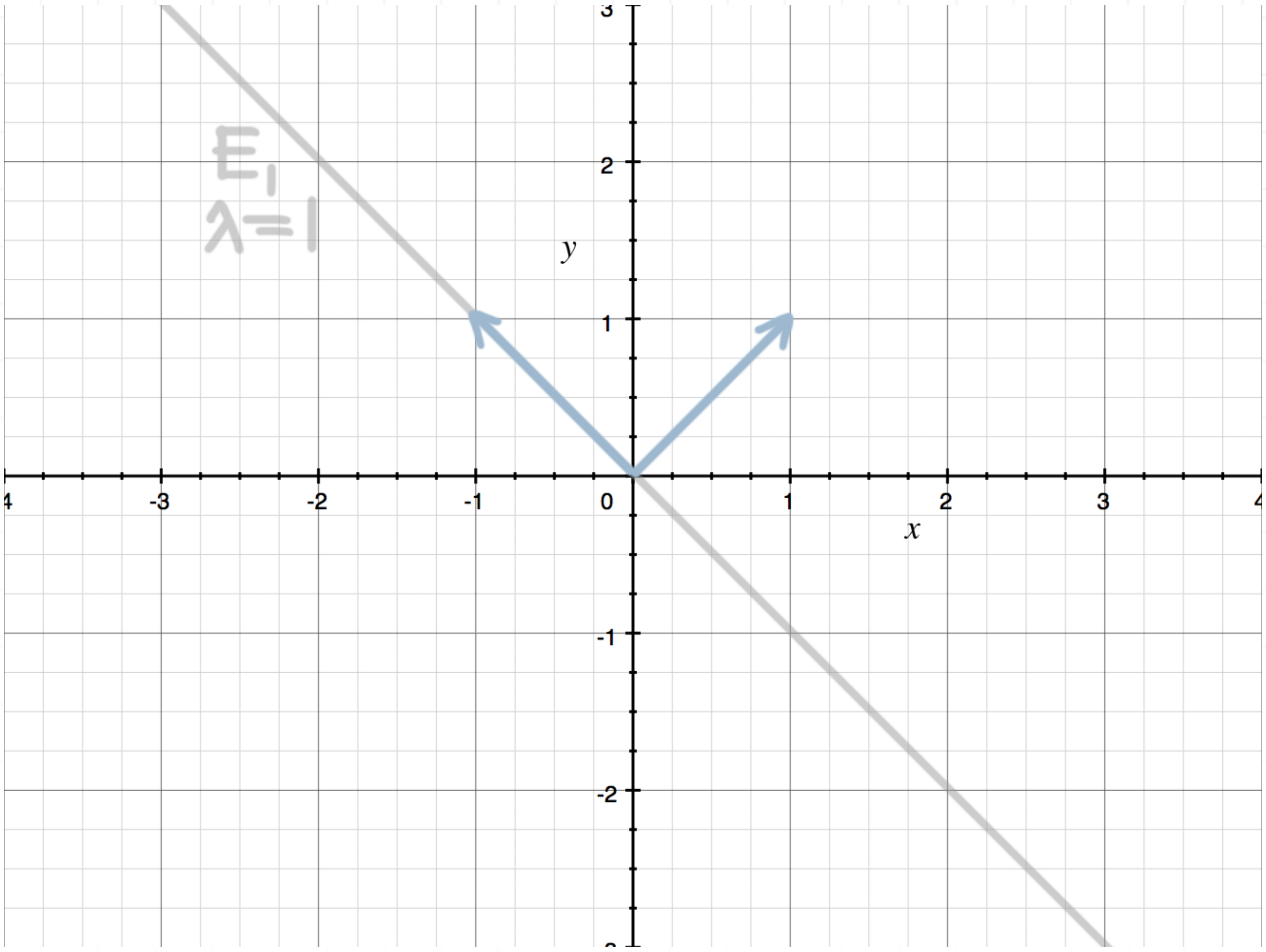
and then say that the eigenspace for $\lambda = 1$ is the set of all the vectors that lie along the line created by $\vec{v} = (-1,1)$.



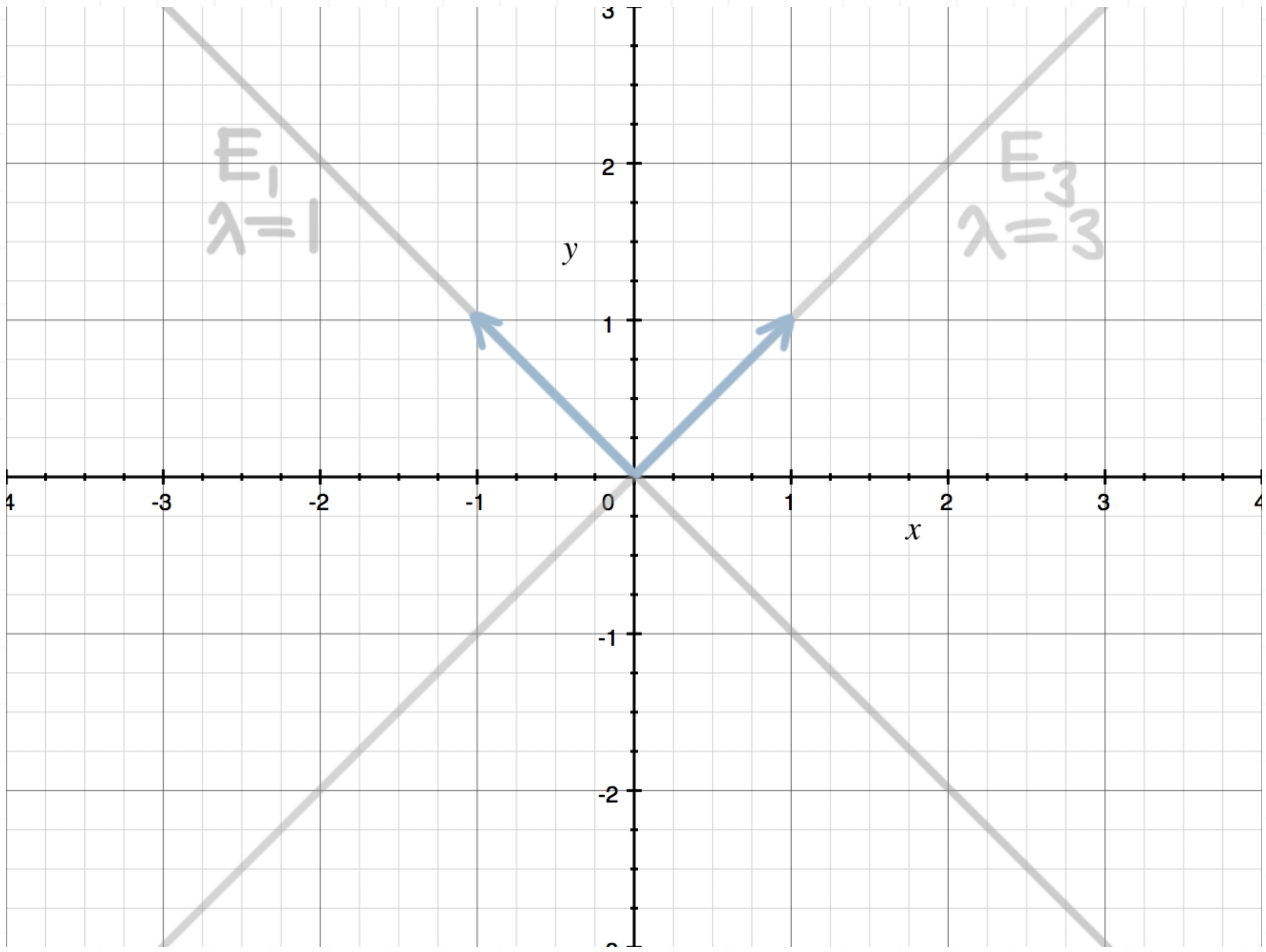
Then for the eigenvalue $\lambda = 3$, we got

$$E_3 = \text{Span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

We can add to our sketch the spanning eigenvector $\vec{v} = (1,1)$,



and then say that the eigenspace for $\lambda = 3$ is the set of all the vectors that lie along the line created by $\vec{v} = (1,1)$.



In other words, we know that, for any vector $\vec{v}$ along either of these lines, when you apply the transformation T to the vector $\vec{v}$, $T(\vec{v})$ will be a vector along the same line, it might just be scaled up or scaled down.

Specifically,

- since $\lambda = 1$ in the eigenspace E_1, any vector $\vec{v}$ in E_1, under the transformation T, will be scaled by 1, meaning that $T(\vec{v}) = \lambda\vec{v} = 1\vec{v} = \vec{v}$, and

- since $\lambda = 3$ in the eigenspace E_3, any vector $\vec{v}$ in E_3, under the transformation T, will be scaled by 3, meaning that $T(\vec{v}) = \lambda\vec{v} = 3\vec{v}$.

Eigen in three dimensions

The process for finding the eigenvalues for a 3×3 matrix is the same as the process we used to do so for the 2×2 matrix.

But calculating the 3×3 determinant and factoring the third-degree characteristic polynomial will be more complex than finding the 2×2 determinant or factoring the second-degree characteristic polynomial.

Let's walk through an example so that we can see the full process.

Example

Find the eigenvalues and eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

We need to find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \right|$$

$$\left|\begin{bmatrix} \lambda-(-1) & 0-1 & 0-0 \\ 0-1 & \lambda-2 & 0-1 \\ 0-0 & 0-3 & \lambda-(-1) \end{bmatrix}\right|$$

$$\left|\begin{bmatrix} \lambda+1 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -3 & \lambda+1 \end{bmatrix}\right|$$

Then the determinant of this resulting matrix is

$$(\lambda+1)\begin{vmatrix} \lambda-2 & -1 \\ -3 & \lambda+1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & \lambda+1 \end{vmatrix} + 0\begin{vmatrix} -1 & \lambda-2 \\ 0 & -3 \end{vmatrix}$$

$$(\lambda+1)[(\lambda-2)(\lambda+1)-(-1)(-3)]-(-1)[(-1)(\lambda+1)-(-1)(0)]$$

$$+0[(-1)(-3)-(\lambda-2)(0)]$$

$$(\lambda+1)[(\lambda-2)(\lambda+1)-3]-(\lambda+1)$$

$$(\lambda+1)(\lambda^2+\lambda-2\lambda-2-3)-(\lambda+1)$$

$$(\lambda+1)(\lambda^2-\lambda-5)-(\lambda+1)$$

Remember that we're trying to satisfy $|\lambda I_n - A| = 0$, so we can set this characteristic polynomial equal to 0 to get the characteristic equation:

$$(\lambda+1)(\lambda^2-\lambda-5)-(\lambda+1)=0$$

To solve for λ, we'll factor.

$$(\lambda+1)[(\lambda^2-\lambda-5)-1]=0$$

$$(\lambda+1)(\lambda^2-\lambda-5-1)=0$$

$$(\lambda+1)(\lambda^2-\lambda-6)=0$$

$$(\lambda+1)(\lambda-3)(\lambda+2)=0$$

$\lambda=-2$ or $\lambda=-1$ or $\lambda=3$

So assuming non-zero eigenvectors, we're saying that $A\vec{v}=\lambda\vec{v}$ can be solved for $\lambda=-2$, $\lambda=-1$, and $\lambda=3$.

With these three eigenvalues, we'll have three eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. Given

$$E_\lambda=N\left(\begin{bmatrix}\lambda+1 & -1 & 0\\ -1 & \lambda-2 & -1\\ 0 & -3 & \lambda+1\end{bmatrix}\right)$$

we get

$$E_{-2}=N\left(\begin{bmatrix}-2+1 & -1 & 0\\ -1 & -2-2 & -1\\ 0 & -3 & -2+1\end{bmatrix}\right)$$

$$E_{-2}=N\left(\begin{bmatrix}-1 & -1 & 0\\ -1 & -4 & -1\\ 0 & -3 & -1\end{bmatrix}\right)$$

and

$$E_{-1}=N\left(\begin{bmatrix}-1+1 & -1 & 0\\ -1 & -1-2 & -1\\ 0 & -3 & -1+1\end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -3 & 0 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3+1 & -1 & 0 \\ -1 & 3-2 & -1 \\ 0 & -3 & 3+1 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -3 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 & | & 0 \\ -1 & -4 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -1 & -4 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -3 & -1 & | & 0 \\ 0 & -3 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -3 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 - \frac{1}{3}v_3 = 0$$

$$v_2 + \frac{1}{3}v_3 = 0$$

or

$$v_1 = \frac{1}{3}v_3$$

$$v_2 = -\frac{1}{3}v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -3 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 & | & 0 \\ -1 & -3 & -1 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -3 & -1 & | & 0 \\ 0 & -1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & | & 0 \\ 0 & -1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + v_3 = 0$$

$$v_2 = 0$$

or

$$v_1 = -v_3$$

$$v_2 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 0 & | & 0 \\ -1 & 1 & -1 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -1 & | & 0 \\ 4 & -1 & 0 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 4 & -1 & 0 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & -3 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 - \frac{1}{3}v_3 = 0$$

$$v_2 - \frac{4}{3}v_3 = 0$$

or

$$v_1 = \frac{1}{3}v_3$$

$$v_2 = \frac{4}{3}v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \right)$$

Let's put these results together. For the eigenvalues $\lambda = -2$, $\lambda = -1$, and $\lambda = 3$, respectively, we got

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right), E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right), \text{ and } E_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \right)$$

Each of these spans represents a line in $\mathbb{R}^3$. So for any vector $\vec{v}$ along any of these lines, when you apply the transformation T to the vector $\vec{v}$, $T(\vec{v})$ will be a vector along the same line, it might just be scaled up or scaled down.

Specifically,

- since $\lambda = -2$ in the eigenspace E_{-2}, any vector $\vec{v}$ in E_{-2}, under the transformation T, will be scaled by -2, meaning that $T(\vec{v}) = \lambda\vec{v} = -2\vec{v}$,

- since $\lambda = -1$ in the eigenspace E_{-1}, any vector $\vec{v}$ in E_{-1}, under the transformation T, will be scaled by -1, meaning that $T(\vec{v}) = \lambda\vec{v} = -1\vec{v} = -\vec{v}$, and

- since $\lambda = 3$ in the eigenspace E_3, any vector $\vec{v}$ in E_3, under the transformation T, will be scaled by 3, meaning that $T(\vec{v}) = \lambda\vec{v} = 3\vec{v}$.

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