Q1

$$X_t = W_t + \beta W_{t-1}$$

With innovation term following,

$$W_t = \eta_t + \theta\eta_{t-1}$$

With $\{\eta\}$ being white noise with $E[\eta^2] = \sigma^2 > 0$

A)

To simplify, we want to express X_t in terms of W_t

$$\to W_{t-1} = \eta_{t-1} + \theta\eta_{t-2}$$

Then,

$$X_t = \eta_t + \theta\eta_{t-1} + \beta(\eta_{t-1} + \theta\eta_{t-2})$$

$$= \eta_t + (\theta + \beta)\eta_{t-1} + \beta\theta\eta_{t-2}$$

By definition, white noise has an expectation of zero for all t, then

$$E[X_t] = E[\eta_t + (\theta + \beta)\eta_{t-1} + \beta\theta\eta_{t-2}]$$

$$= E[\eta_t] + (\theta + \beta)E[\eta_{t-1}] + \beta\theta E[\eta_{t-2}]$$

$$= 0$$

B)

$$ACF = \gamma_X(h)/\gamma_X(0)$$

Where,

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$

For $\gamma_X(0)$,

$$\gamma_X(0) = Cov(X_{t+0}, X_t) = \mathrm{Var(X)} = \mathrm{E[(X - \mu)^2]}$$

$$\mathrm{E[(X - \mu)^2]} = E[(\mathrm{X})^2] = E[(\eta_t + (\theta + \beta)\eta_{t-1} + \beta\theta\eta_{t-2})^2]$$

$$= E[\eta_t{}^2 + (\theta + \beta)^2\eta_{t-1}{}^2 + (\beta\theta)^2\eta_{t-2}{}^2 + 2(\theta + \beta)\eta_t\eta_{t-1} + 2\beta\theta\eta_t\eta_{t-2} + (\theta + \beta)\beta\eta_{t-1}\eta_{t-2}$$

Since White noise, $\eta_t, \eta_{t-1}, \eta_{t-2}$ for any t is uncorrelated, we now have,

$$= E[\eta_t{}^2] + (\theta + \beta)^2 E[\eta_{t-1}{}^2] + (\beta\theta)^2 E[\eta_{t-2}{}^2] + 0 + 0 + 0$$

$$= \sigma^2(1 + (\theta + \beta)^2 + (\beta\theta)^2)$$

$$\gamma_X(0), = \sigma^2(1 + (\theta + \beta)^2 + (\beta\theta)^2)$$

For $\gamma_X(1)$

$$\gamma_X(1) = Cov(X_{t-1}, X_t) =$$

$$E[X_{t-1}, X_t] = E[(\eta_t + (\theta+\beta)\eta_{t-1} + \beta\theta\eta_{t-2})(\eta_{t-1} + (\theta+\beta)\eta_{t-2} + \beta\theta\eta_{t-3})]$$

As before, white noise is uncorrelated, thus only terms where the index is the same will be non-zero, thus the only non-zero indexes are η_{t-1} and η_{t-2}

$$= (\theta+\beta)E[\eta_{t-1}{}^2] + \beta\theta(\theta+\beta)E[\eta_{t-2}{}^2]$$

$$= \sigma^2(\theta+\beta)(1+\beta\theta)$$

$$\gamma_X(1) = \sigma^2(\theta+\beta)(1+\beta\theta)$$

For $\gamma_X(2)$

$$\gamma_X(2) = Cov(X_{t-2}, X_t) =$$

$$E[X_{t-2}, X_t] = E[(\eta_t + (\theta+\beta)\eta_{t-1} + \beta\theta\eta_{t-2})(\eta_{t-2} + (\theta+\beta)\eta_{t-3} + \beta\theta\eta_{t-4})]$$

Again, white noise is uncorrelated, thus only terms where the index is the same will be non-zero, thus the only non-zero index is and η_{t-2}

$$= E[\beta\theta\eta_{t-2}]$$

$$\beta\theta\sigma^2$$

Then finally, h>2

$$\gamma_X(h) = Cov(X_{t-h}, X_t) =$$

$$E[X_{t-h}, X_t] = E[(\eta_t + (\theta+\beta)\eta_{t-1} + \beta\theta\eta_{t-2})(\eta_{t-h} + (\theta+\beta)\eta_{t-h-1} + \beta\theta\eta_{t-h-2})]$$

As h is defined as >2, then there is never any square terms present, and since white noise is uncorrelated, thus only terms where the index is the same will be non-zero,

$$\gamma_X(h) = 0, \qquad h > 2$$

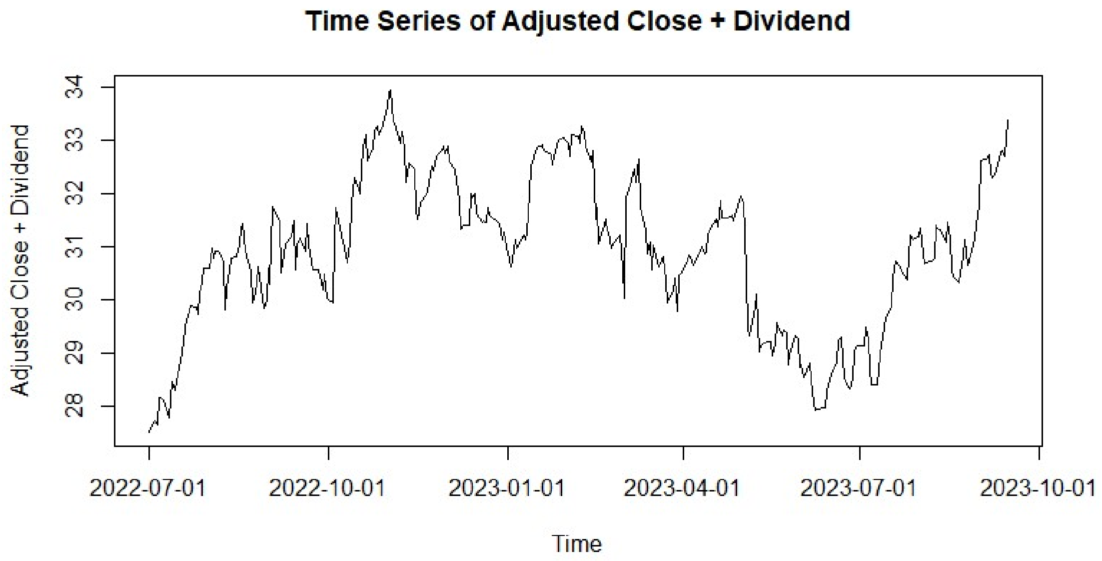
Thus, the ACF function is given by,

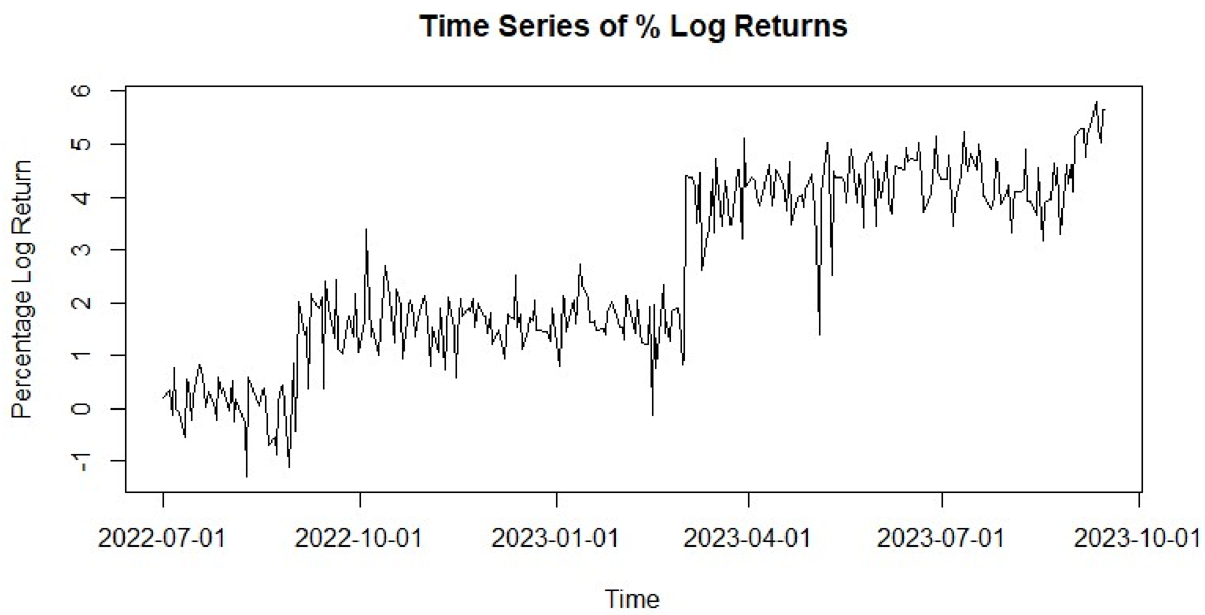
$$ACF := \rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} \dfrac{(\theta+\beta)(1+\beta\theta)}{(1+(\theta+\beta)^2+(\beta\theta)^2)}, & h=1 \\ \dfrac{\beta\theta}{(1+(\theta+\beta)^2+(\beta\theta)^2)}, & h=2 \\ 0, \quad h>2 \end{cases}$$

c) Using the expressions from part b)

Case	β	θ	ρ_1	ρ_2	ρ_3	ρ_4
1	0.5	0	0.4	0	0	0
2	0	0.25	4/17	0	0	0
3	0.5	0.25	54/101	8/101	0	0
4	0.5	-0.5	0	-4/17	0	0

Q2



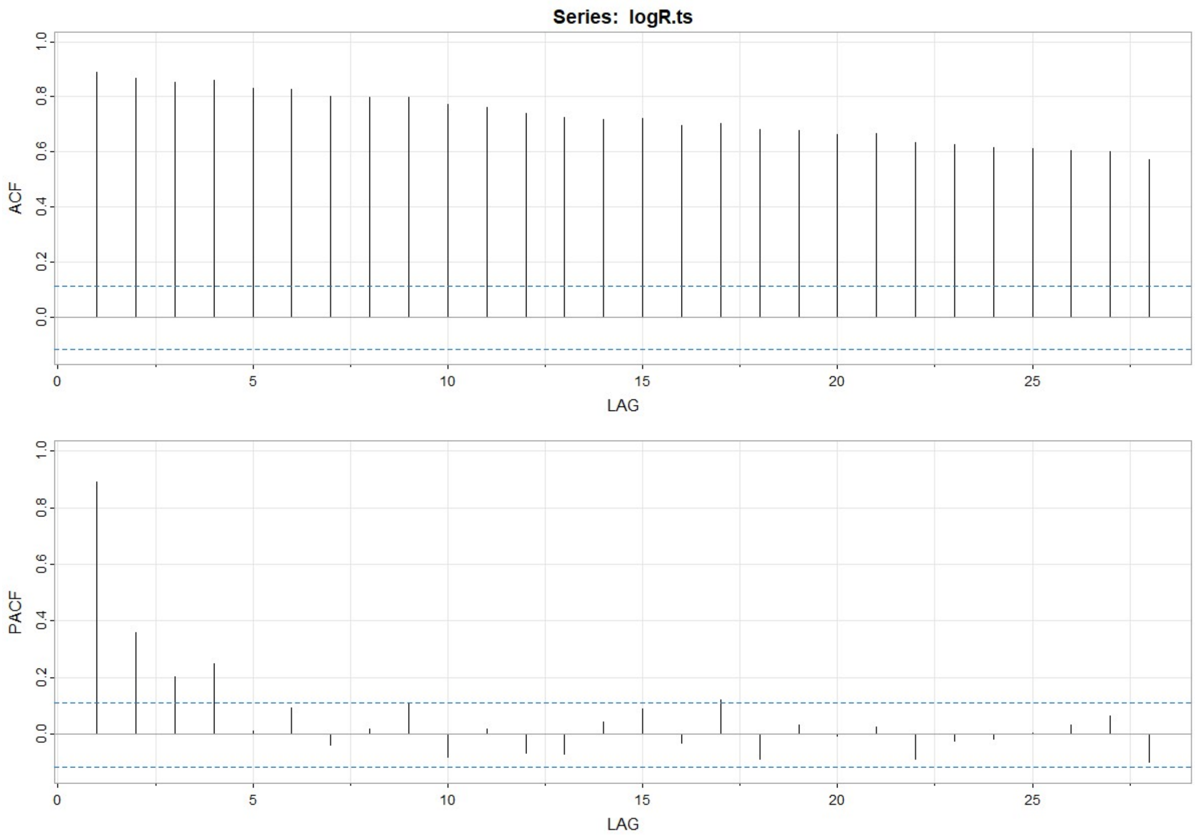



```
close <- AMPOL_Assignment_3$`C lose +Dividend`
logR <- AMPOL_Assignment_3$`percent logreturn`
date <- AMPOL_Assignment_3$Date


close.ts <- timeSeries(data=close, date, units="Adjusted Close +
Dividend")
logR.ts <- timeSeries(data=logR, date, units="Percentage Log Return"
)
```

Q2 PART 1: Percentage Log Returns

Noting that the large jumps to new levels is due to the dividends being accounted for in the adjusted closing price, then we can we assume that we are unable to discern any particular seasonal pattern or trend in the time series.

ACF and PACF: acf2(logR.ts)



The ACF has a clear tail off pattern that linearly decay, meaning there is no cut off. The PACF however shows it cuts off after Lag = 4. This is a signature of an AR model indicating an order of 4 may be applicable. We should also want to test other others in the neighbourhood to find if any extra AR, or possible MA, terms are needed.

AR(3) vs AR(4) vs AR(5)

```
$ttable
      Estimate     SE t.value p.value
ar1     0.4518 0.0552  8.1895  0.0000
ar2     0.2650 0.0591  4.4849  0.0000
ar3     0.2452 0.0556  4.4135  0.0000
xmean   2.6402 0.8682  3.0411  0.0026

$AIC
[1] 2.037743
```

$$\{X_t\}: (1 - \phi_1 B^1 - \phi_2 B^2 - \phi_3 B^3)(X_t - \mu) = W_t$$

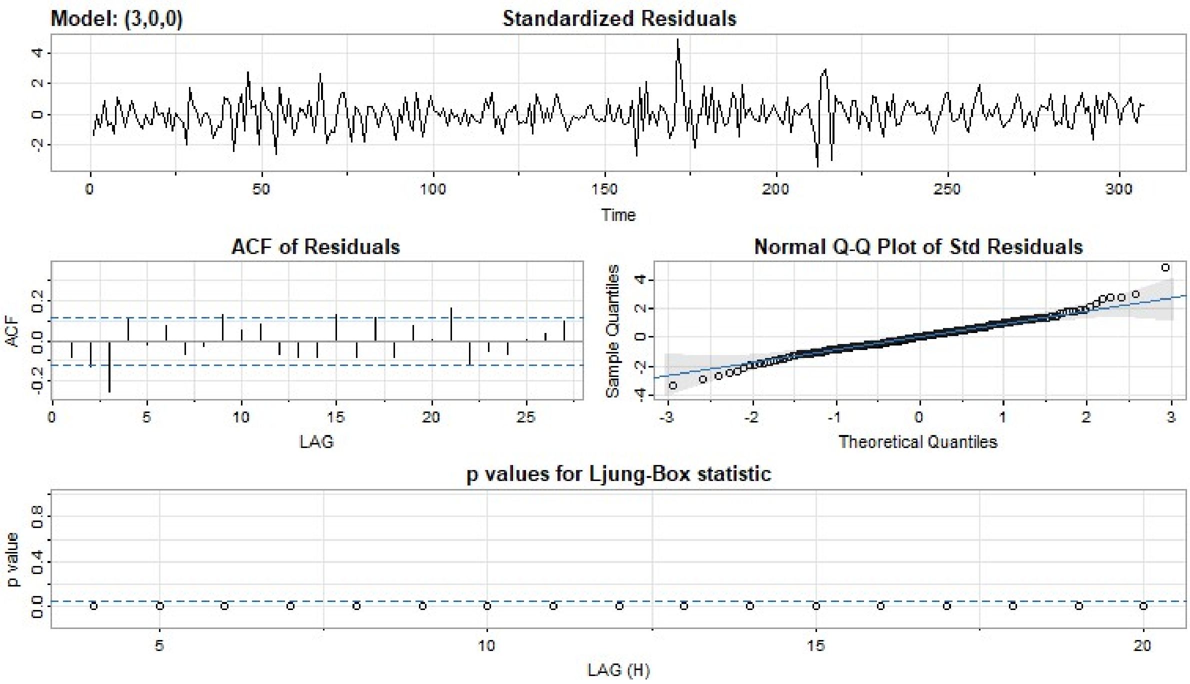
```
$ttable
      Estimate     SE t.value p.value
ar1     0.3776 0.0543  6.9565  0.0000
ar2     0.1869 0.0580  3.2249  0.0014
ar3     0.1120 0.0582  1.9243  0.0553
ar4     0.3003 0.0545  5.5064  0.0000
xmean   2.6779 1.2043  2.2236  0.0269

$AIC
[1] 1.950514
```

$$\{X_t\}: (1 - \phi_1 B^1 - \phi_2 B^2 - \phi_3 B^3 - \phi_4 B^4)(X_t - \mu) = W_t$$

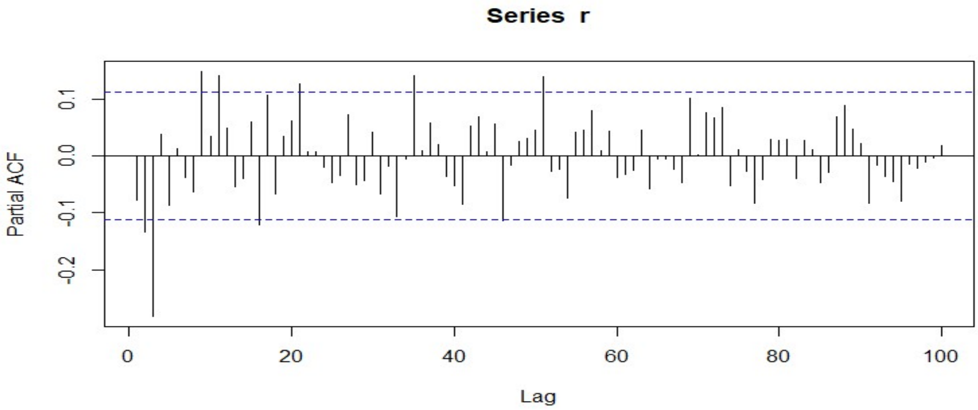
```
$ttable
      Estimate     SE t.value p.value
ar1     0.3525 0.0568  6.2096  0.0000
ar2     0.1777 0.0581  3.0592  0.0024
ar3     0.0970 0.0589  1.6467  0.1007
ar4     0.2693 0.0583  4.6149  0.0000
ar5     0.0833 0.0571  1.4597  0.1454
xmean   2.7412 1.3198  2.0770  0.0387

$AIC
[1] 1.950122
```

$$\{X_t\}: (1 - \phi_1 B^1 - \phi_2 B^2 - \phi_3 B^3 - \phi_4 B^4 - \phi_5 B^5)(X_t - \mu) = W_t$$

While the AR(5) model has the lowest AIC score, we can notice that in both the AR(4) and AR(5) models, there are non-significant coefficients indicated by the p-values > 0.05 significance level. The AR(3) model is the only model with all coefficients deemed significant.



The Standardized Residuals appear randomly distributed and cantered around zero mean. The Norm Q-Q plot also indicates residuals are normally distributed, though there is some tail present. However, some residuals have p-values below the 0.05 significance level, indicating there is still correlation present. This indicates we will still need further model refinement.

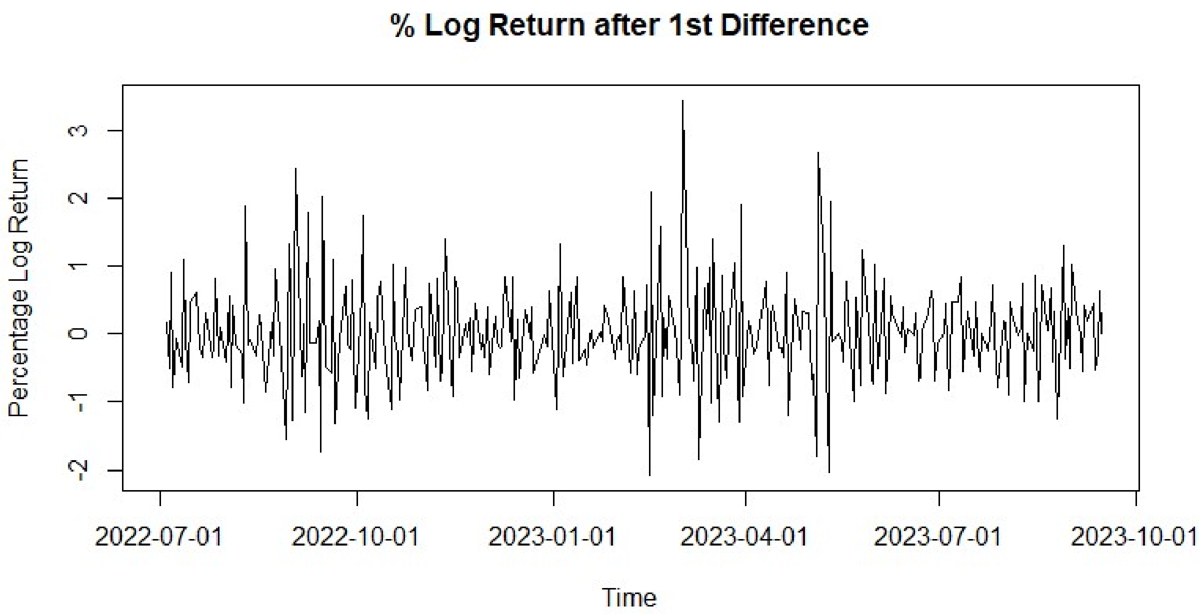


Looking at the PACF of the residuals from the AR(3) model is clearly exhibiting oscillatory behaviour, which indicates we may need to incorporate seasonal components.

At this point, without employing Fourier techniques, we would be essentially employing guess work. Without knowing the natural frequency of the seasonal component, decomposition wouldn't either be applicable. We could use a frequency of 7 days to show weekly cyclical patterns or a frequency of a month to show any intra monthly patterns.

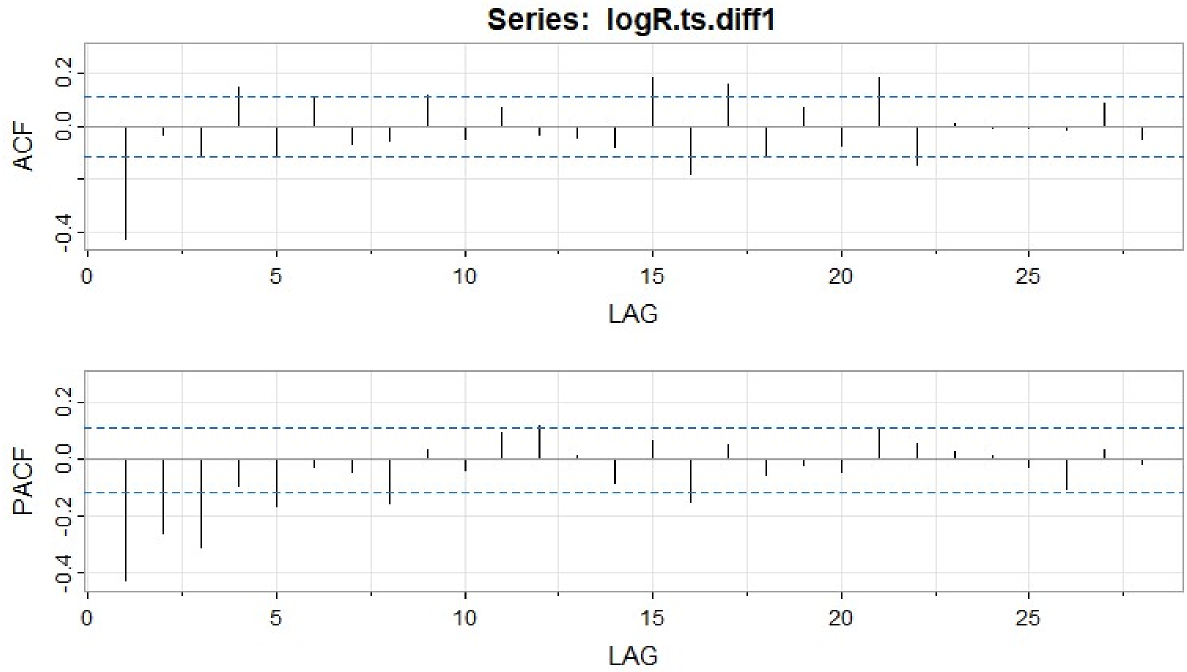
Another method would be to difference the original % Log Returns atleast one or twice to remove any trend.

First Order Differencing: logR.ts.diff1 <-diff(logR.ts, differences = 1)



It seems with a first order differencing we have eliminate the trend entirely. What we are left with now, resembles white noise.

ACF and PACF with a First Order Difference Applied



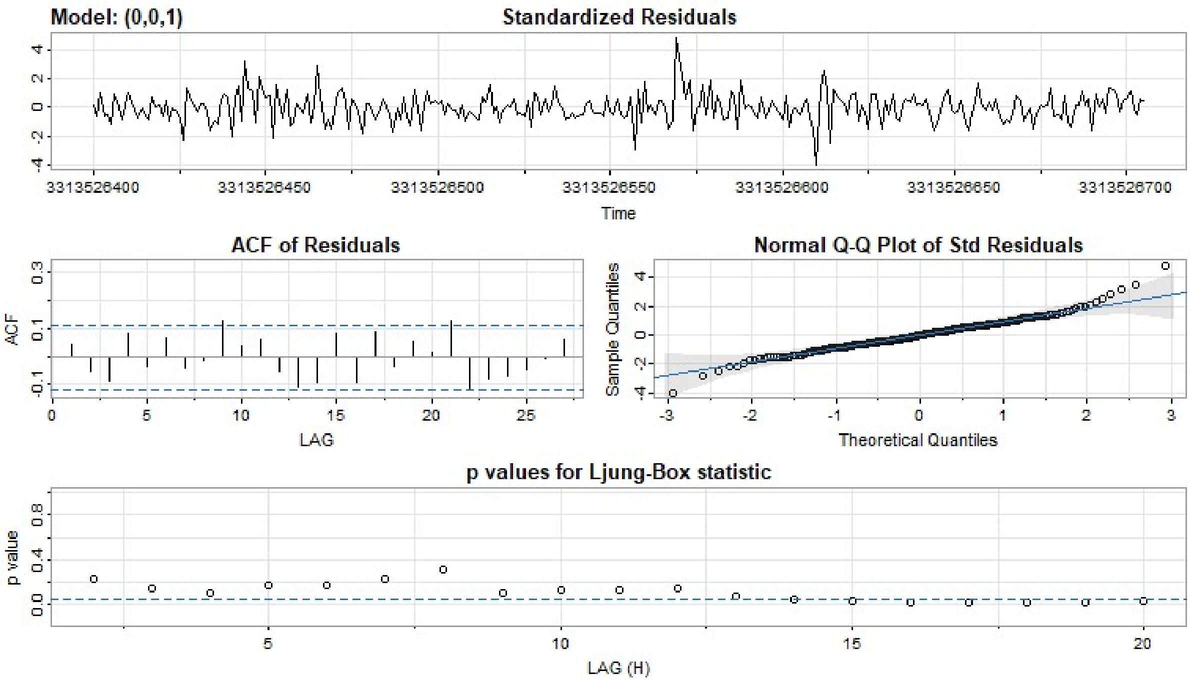
As before, we can see significant correlation at lag = 1 from the ACF, then cuts off. The PACF shows significant correlation up to lag =3, than tails off. This is indicating that we can use an MA model.

```
AR/MA
  0 1 2 3 4 5 6 7 8 9 10 11 12 13
0 x o o x o o o o x o o  o  o  o
1 x o o o o o o o o o o  o  o  o
2 x x x o o o o o o o o  o  o  o
3 x x o x o o o o o o o  o  o  o
4 x x o o x o o o o o o  o  o  o
5 x x x o x x o o o o o  o  o  o
6 x o x o x x o o o o o  o  o  o
7 x o x o x x o o o o o  o  o  o
```

The SACF is also suggesting an MA(1) model.

```
$ttable
      Estimate     SE  t.value p.value
ma1    -0.6893 0.0575 -11.9817  0.0000
ma2    -0.0577 0.0555  -1.0395  0.2994
xmean   0.0173 0.0091   1.9069  0.0575

$AIC
[1] 1.9098
```

The P-value for the 2nd MA term is above the 0.05 significance level, therefore we should reject any further MA terms.



Again, we see that the major problem is the P-values of the residuals still indicating correlations between them, especially at higher lags.

Though note, only a few P-values for the residuals lie below the 0.05 significance level. This is a much better result than the first AR(3) model.

SARIMA(3,0,0) vs SARIMA(0,1,1)

With these two models in consideration, we opt for SARIMA(0,1,1) instead, since for Standardized Residuals, Norm Q-Q are also yielding results indicating normally distributed residuals, while also having a better result for the Ljung-Box Statistic. *(Also worth mentioning, is despite the known occurrence of auto.arima wrongly fitting models, the function agrees with our conclusions, giving the same values)*

```
$ttable
      Estimate     SE  t.value p.value
ma1    -0.7362 0.0382 -19.2530  0.0000
xmean   0.0173 0.0095   1.8326  0.0678
```

$$W_t \sim WN(0,0.3855)$$

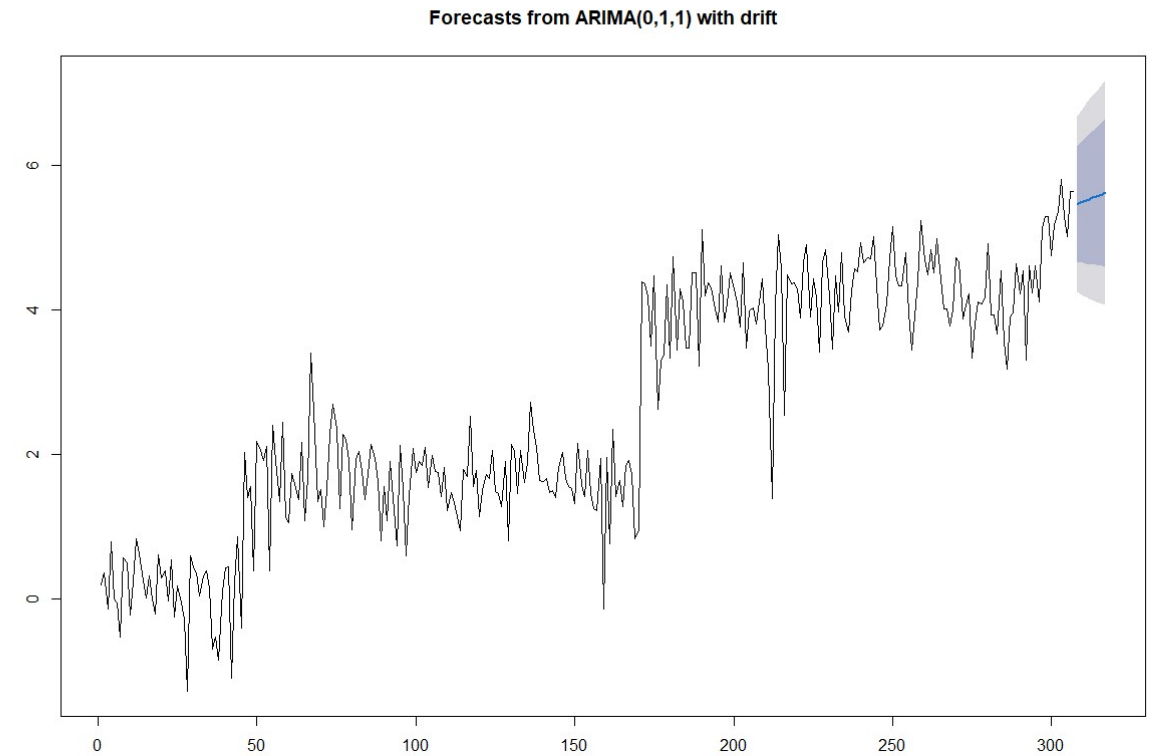
$$\rightarrow x_t = 0.0173 + W_{t-1} + \theta_1 W_{t-1}$$

Forecasting Values

- Before proceeding is also good to note that when wanting to achieve the forecasted data in its original form, we need to revert any transformations. In this case we applied a first order difference. The SARIMA function in this case will take care of this automatically.

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
308	5.467979	4.669700	6.266258	4.247117	6.688841
309	5.485299	4.659705	6.310893	4.222662	6.747936
310	5.502619	4.650585	6.354653	4.199546	6.805693
311	5.519940	4.642262	6.397617	4.177647	6.862232
312	5.537260	4.634667	6.439853	4.156863	6.917657
313	5.554580	4.627741	6.481419	4.137102	6.972058
314	5.571900	4.621433	6.522367	4.118287	7.025514
315	5.589221	4.615699	6.562742	4.100348	7.078093
316	5.606541	4.610499	6.602583	4.083226	7.129856
317	5.623861	4.605796	6.641926	4.066865	7.180857

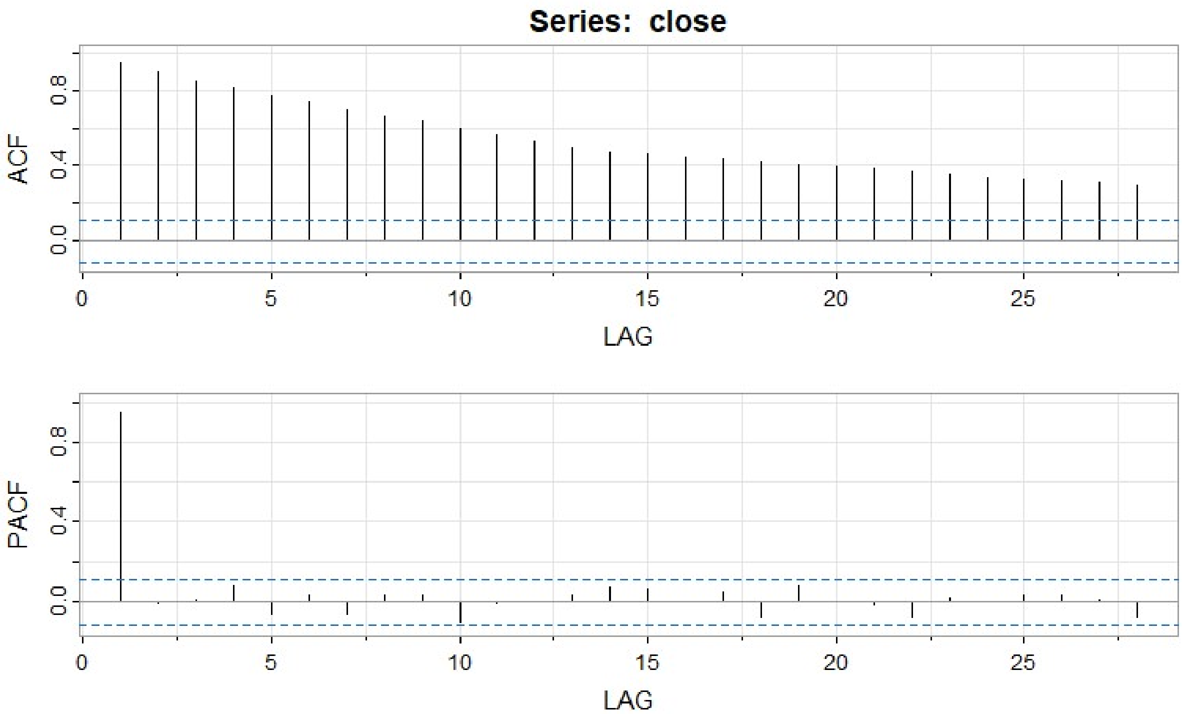
The next 10 forecasted values already in form of the original data



Q2 PART 2: Adjusted Close Series

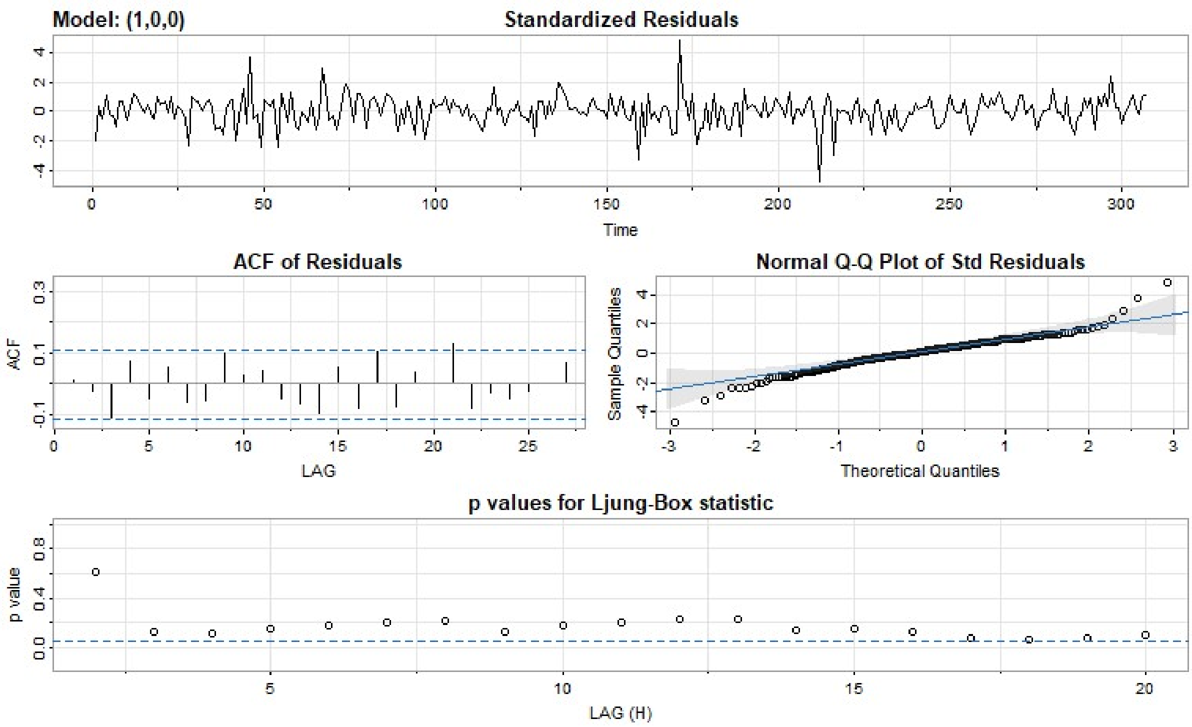
In comparison to the % Log Returns Series, contains clear trend components. We doubt there is any seasonal since the variance along the trends looks to be random.

ACF and PACF of Adj Close Series



As suspected, the ACF has a tail off pattern, not a cut off, and the PACF has high correlation at lag = 1, then cuts. This is clear signature of an AR(1) model.

```
$ttable
      Estimate     SE t.value p.value
ar1     0.9729 0.0144 67.5487       0
xmean  30.8473 0.7245 42.5799       0
```



The standardized residuals appear normally distributed. The Q-Q norm plot supports this conclusion. The ACF of Residuals appears to have no majorly significant correlations, and the

P-values for the Ljung-Box Statistics are all above or on the boundary of the 0.05 significance level. Furthermore, the p-value from a Ljung Box Test on the fitted model is = 0.1026 > 0.05 significance level.

Taken all together, we can conclude with the following model,

$$W_t \sim WN(0,0.146)$$

$$X_t - 30.8473 = 0.9729(X_{t-1} - 30.8473) + W_t$$

Q3

3.8 Verify the calculations for the autocorrelation function of an ARMA(1, 1) process given in Example 3.14. Compare the form with that of the ACF for the ARMA(1, 0) and the ARMA(0, 1) series. Plot the ACFs of the three series on the same graph for $\phi = .6$, $\theta = .9$, and comment on the diagnostic capabilities of the ACF in this case.

The ARMA(1,1) process given in Example 3.14 is the following,

$$X_t = \phi X_{t-1} + \theta W_{t-1} + W_t, \qquad \text{where } |\phi| < 1$$

Example 3.14 The ACF of an ARMA(1, 1)

Consider the ARMA(1, 1) process $x_t = \phi x_{t-1} + \theta w_{t-1} + w_t$, where $|\phi| < 1$. Based on (3.47), the autocovariance function satisfies

$$\gamma(h) - \phi\gamma(h-1) = 0, \quad h = 2, 3, \ldots,$$

and it follows from (3.29)–(3.30) that the general solution is

$$\gamma(h) = c\,\phi^h, \quad h = 1, 2, \ldots. \tag{3.51}$$

To obtain the initial conditions, we use (3.48):

$$\gamma(0) = \phi\gamma(1) + \sigma_w^2[1 + \theta\phi + \theta^2] \quad \text{and} \quad \gamma(1) = \phi\gamma(0) + \sigma_w^2\theta.$$

Solving for $\gamma(0)$ and $\gamma(1)$, we obtain:

$$\gamma(0) = \sigma_w^2 \frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2} \quad \text{and} \quad \gamma(1) = \sigma_w^2 \frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}.$$

To solve for c, note that from (3.51), $\gamma(1) = c\,\phi$ or $c = \gamma(1)/\phi$. Hence, the specific solution for $h \ge 1$ is

$$\gamma(h) = \frac{\gamma(1)}{\phi}\phi^h = \sigma_w^2 \frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}\phi^{h-1}.$$

Finally, dividing through by $\gamma(0)$ yields the ACF

$$\rho(h) = \frac{(1 + \theta\phi)(\phi + \theta)}{1 + 2\theta\phi + \theta^2}\phi^{h-1}, \quad h \ge 1. \tag{3.52}$$

Verifying Derivations

We want derive the following,

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$

For h =0

$$\gamma_X(0) = Cov(X_{t+0}, X_t) = Var(X_t)$$

Thus, we need to find what $Var(X_t) = \sigma^2$ is

$$Var(X_t) = Var(\phi X_{t-1} + \theta W_{t-1} + W_t)$$

Since W_{t-1}, W_t is white noise, they are uncorrelated, and X_{t-1} is not correlated with W_t, since W_t is in the future. The only correlated random variables are X_{t-1} and W_{t-1}

$$= \phi^2 Var(X_{t-1}) + \theta^2 Var(W_{t-1}) + Var(W_t) + 2\phi\theta \mathrm{Cov}(X_{t-1}, W_{t-1})$$

$$= \phi^2\sigma^2 + \theta^2\sigma_W^2 + \sigma_W^2 + 2\phi\theta \mathrm{Cov}(X_{t-1}, W_{t-1})$$

What is left is to derive what $2\phi \mathrm{Cov}(\phi X_{t-1}, W_{t-1})$ is. Since we already have the Expression for X_t, we can find X_{t-1}

$$X_{t-1} = \phi X_{t-2} + \theta W_{t-2} + W_{t-1}$$

Then since we now have the indexes of X_{t-1} and W_{t-1}, we know due to white noise, these are uncorrelated. (In – fact, we can keep iterating this process to find that X_{t-2} is uncorrelated with W_{t-2}.

$$\mathrm{Cov}(X_{t-1}, W_{t-1}) = \sigma_W^2$$

$$\rightarrow \sigma^2 = \phi^2\sigma^2 + \theta^2\sigma_W^2 + \sigma_W^2 + 2\phi\sigma_W^2$$

$$\gamma_X(0) = \sigma^2 = \frac{\sigma_W^2(\theta^2 + 1 + 2\theta\phi)}{1 - \phi^2}$$

For h = 1

$$\gamma_X(1) = Cov(X_{t-1}, X_t) =$$

Given

$$X_{t-1} = \phi X_{t-2} + \theta W_{t-2} + W_{t-1}$$

$$X_t = \phi X_{t-1} + \theta W_{t-1} + W_t,$$

Then, (We want to expand X_t since X_{t-1} has X_{t-2} which we would than have to expand again)

$$Cov(X_{t-1}, X_t) = Cov(X_{t-1}, \phi X_{t-1} + \theta W_{t-1} + W_t)$$

$$= \phi Cov(X_{t-1}, X_{t-1}) + \theta \mathrm{Cov}(W_{t-1}, X_{t-1}) + Cov(W_t, X_{t-1})$$

Noting again that we have white noise and X_{t-1} is not correlated with W_t since it the white noise is in the future, and that $Cov(X_{t-1}, X_{t-1}) = Var(X_t) = \sigma^2$, then,

$$\gamma_X(1) = \phi\sigma^2 + \theta\sigma_W^2$$

Where using the derivation for

$$\gamma_X(0) = \sigma^2 = \frac{\sigma_W^2(\theta^2 + 1 + 2\theta\phi)}{1 - \phi^2}$$

$$\gamma_X(1) = \phi\frac{\sigma_W^2(\theta^2 + 1 + 2\theta\phi)}{1 - \phi^2} + \theta\sigma_W^2$$

$$= \sigma_W^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}$$

Now. For any $h > 1$, we can use the yule walker equation to iterate backwards,

$$\gamma_X(h) = \phi\gamma_X(h-1), \qquad by\ definition$$

Aftering iterating by 1 step backwards,

$$\gamma_X(h) = \phi^{h-1}\gamma_X(h - h + 1) = \phi^{h-1}\gamma_X(1)$$

$$\phi^{h-1}\sigma_W^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}$$

Then to find $\rho_X(h)$,

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \phi^{h-1}\sigma_W^2\frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2} / \frac{\sigma_W^2(\theta^2 + 1 + 2\theta\phi)}{1 - \phi^2}$$

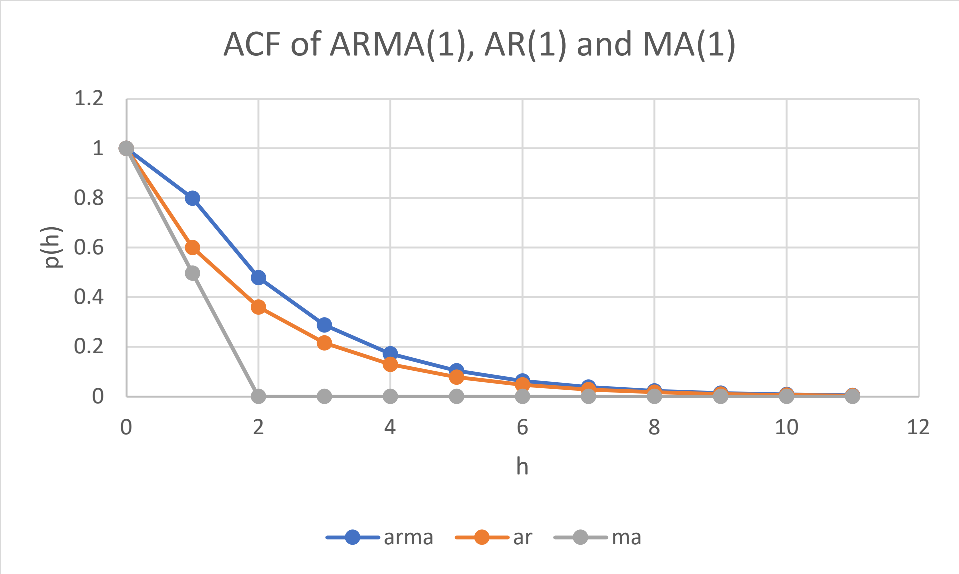
$$= \phi^{h-1}\frac{(1 + \theta\phi)(\phi + \theta)}{\theta^2 + 1 + 2\theta\phi}, \qquad h \geq 1$$

Then standard results of ARMA(1,0) =AR(1), ARMA(0,1)=MA(1), we obtain the following summary

ϕ	θ		ARMA(1,1)	ARMA(1,0)	ARMA(0,1)
0.6	0.9	h=0	1	1	1
0.6	0.9	h=1	$\frac{(1 + \theta\phi)(\phi + \theta)}{\theta^2 + 1 + 2\theta\phi}$	ϕ^1	$\frac{\theta}{1 + \theta^2}$
0.6	0.9	h>1	$\phi^{h-1}\frac{(1 + \theta\phi)(\phi + \theta)}{\theta^2 + 1 + 2\theta\phi}$	ϕ^h	0

Plotting ARMA(1,0) =AR(1), ARMA(0,1)=MA(1), and ARMA(1,1)

ϕ	θ		ARMA(1,1)	ARMA(1,0)	ARMA(0,1)
0.6	0.9	h=0	1	1	1
0.6	0.9	h=1	231/289	0.6	90/181
0.6	0.9	h>1	$0.6^{h-1} * 231/289$	0.6^h	0



Q4

3.1 For an MA(1), $x_t = w_t + \theta w_{t-1}$, show that $|\rho_x(1)| \leq 1/2$ for any number θ. For which values of θ does $\rho_x(1)$ attain its maximum and minimum?

$$\mathrm{E}[\mathrm{X}^2] = E\left[W_t^{\,2} + 2\theta W_t W_{t-1} + \theta^2 W_{t-1}^{\,2}\right] = E\left[W_t^{\,2}\right] + 2E[W_t W_{t-1}] + E[\theta^2 W_{t-1}^{\,2}]$$

$$= \sigma^2 + 0 + \theta^2\sigma^2 = \sigma^2(1+\theta^2)$$

Also note,

$$E[X_{t+1} * X_t] - E[X_{t+1}]E[X_t] = E[X_{t+1} * X_t] - 0$$

$$X_{t+1} * X_t = W_t W_{t+1} + \theta W_t W_{t-1} + \theta W_t^{\,2} + \theta^2 W_t W_{t-1}$$

$$\rightarrow E[X_{t+1} * X_t] = \theta\sigma^2$$

Then,

$$\gamma_X(h) = Cov(X_{t+h}, X_t) = \begin{cases} \sigma^2(1+\theta^2), \ h = 0 \\ \theta\sigma^2, \ h = 1 \\ 0, \quad h > 1 \end{cases}$$

Thus, the ACF is,

$$\boldsymbol{ACF := \rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & h = 0 \\ \dfrac{\theta}{(1+\theta^2)}, & h = 1 \\ 0, & h > 1 \end{cases}}$$

For h =1

$$\rho_X(h) = \frac{\gamma_X(1)}{\gamma_X(0)} = \frac{\theta}{(1+\theta^2)}$$

First derivative w.r.t to Theta

$$\rho_X(h)' = \frac{1-\theta^2}{(1+\theta^2)^2}$$

The second derivative w.r.t theta using the quotient rule,

$$= \frac{-2\theta(-\theta^2+3)}{(\theta^2+1)^3} = \frac{-6\theta+4\theta}{(\theta^2+1)^3}$$

Setting the first derivative equal to zero, and solving for theta,

$$\frac{1-\theta^2}{(1+\theta^2)^2} = 0$$

$$1-\theta^2 = 0$$

$$\theta = +1, -1$$

Substituting these critical values into the 2nd derivative yields,

$$\frac{-2(1)(-(1)^2+3)}{(1^2+1)^3} = -\frac{1}{4} < 0, \qquad \textit{indicating Maximum at } \theta = 1$$

$$\frac{-2(-1)(-(-1)^2+3)}{((-1)^2+1)^3} = \frac{1}{4} > 0, \qquad \textit{indicating Minimum at } \theta = -1$$

Thus, for $\theta = +1, -1$, and there for any Theta

$$\rho_X(1) = \frac{1}{2}, \qquad \rho_X(-1) = -\frac{1}{2},$$

$$\rightarrow -\frac{1}{2} \leq \rho_X(1) \leq \frac{1}{2} \rightarrow |\rho_X(1)| \leq \frac{1}{2}$$

Q5

3.6 For the AR(2) model given by $x_t = -.9x_{t-2} + w_t$, find the roots of the autoregressive polynomial, and then plot the ACF, $\rho(h)$.

Using the Backshift operator,

$$x_t = -0.9B^2x_t + w_t$$

$$(1+0.9B^2)x_t = w_t$$

Characteristic polynomial

$$\varphi(B) = (1+0.9B^2)$$

AR(2) process is casual and stationary if roots of $\varphi(B)$ lie outside the unit circle.

$$(1+0.9Z^2) = 0$$

$$Z = \frac{\sqrt{10}}{3} i = 1.05409i, \qquad z = -\frac{\sqrt{10}}{3} i = -1.05409i$$

Therefore, AR(2) process is casual and stationary.

$$x_t = 0 * x_{t-1} - 0.9B^2 x_t + w_t$$

Using standard results for a general AR(2) model with White noise by Yule Walker Equations,

$$x_t = 0 * x_{t-1} - 0.9x_{t-2} + w_t$$

Represented by

$$x_t = c_1 x_{t-1} + c_2 x_{t-2} + w_t$$

$$\gamma_X(0) = \frac{\sigma_W^2}{1 + c_1 \rho_X(1) - c_2 \rho_X(2)}$$

$$\rho_X(1) = \frac{c_1}{1 - c_2}, \qquad \rho_X(2) = \frac{{c_1}^2 + (1 - c_2)c_2}{1 - c_2}$$

Then in our case,

$$\gamma_X(0) = \frac{\sigma_W^2}{1 + 0.9 \frac{(1 + 0.9) * 0.9}{1 + 0.9}}$$

$$\rho_X(1) = \frac{0}{1 + 0.9} = 0, \qquad \rho_X(2) = \frac{(1 + 0.9) * 0.9}{1 + 0.9} = 0.9$$