Assignment 2 (Week 6) - Spring 2023

Ruben Traicevski 6790021

Q1.

A) To verify that this solution satisfies the PDE, we can use the fundamental solution to the homogenous Heat Equation

The well-known fundamental solution where k $=a^2$ is given by,

$$\phi(x,t) := \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$$

Then we will introduce the convolution operation

$$(f * g(.,s))(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau, s) d\tau$$

Thus, if we convolute our fundamental solution with the initial condition, we will obtain the solution satisfying the initial condition.

Taking that,

$$u(x,0) = \varphi(x)$$

we obtain,

$$\big(\varphi * \phi(x,t)\big)(x) = \int_{-\infty}^{\infty} \varphi(\xi)\phi(x-\xi, \mathrm{t})\mathrm{d}\xi$$

$$\int_{-\infty}^{\infty} \varphi(\xi) \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)\mathrm{d}\xi = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) \exp\left(-\frac{(\xi-\mathrm{x})^2}{4a^2 t}\right)\mathrm{d}\xi$$

Therefore,

$$u(x,t) = \big(\varphi(x) * \phi(x,t)\big)(x)$$

Thus, we can claim that,

$$u_t(x,t) = \big(\varphi * \phi_t(x,t)\big)(x), \qquad u_{xx}(x,t) = \big(\varphi * \phi_{xx}(x,t)\big)(x)$$

This leaves us with to prove,

$$\phi_t(x,t) = a^2 \phi_{xx}(x,t)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)\right) = \frac{\exp\left(\frac{-x^2}{4a^2 t}\right)(x^2 - 2a^2 t)}{8\sqrt{\pi} a^3 t^{5/2}}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)\right) = \frac{\exp\left(\frac{-x^2}{4a^2 t}\right)(x^2 - 2a^2 t)}{8\sqrt{\pi} a^5 t^{\frac{5}{2}}}$$

$$\frac{\exp\left(\frac{-x^2}{4a^2 t}\right)(x^2 - 2a^2 t)}{8\sqrt{\pi} a^3 t^{\frac{5}{2}}} = a^2 \frac{\exp\left(\frac{-x^2}{4a^2 t}\right)(x^2 - 2a^2 t)}{8\sqrt{\pi} a^5 t^{\frac{5}{2}}} = \frac{\exp\left(\frac{-x^2}{4a^2 t}\right)(x^2 - 2a^2 t)}{8\sqrt{\pi} a^3 t^{\frac{5}{2}}}$$

$$LHS = RHS$$

B)

To verify that our solution satisfies the initial condition, i.e,

$$\lim_{t\to 0}\left(\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)\exp\left(-\frac{(\xi-\mathrm{x})^2}{4a^2t}\right)\mathrm{d}\xi\right)$$

we shall take note of the following.

$$\gamma_r(x)=\sqrt{\frac{r}{\pi}}e^{-rx^2}\quad (1)(Guassian\ Delta\ Sequence)$$

Then instead, we will take the fundamental solution / heat kernel,

$$\phi(x,t):=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right)$$

Notice that we can transform the heat kernel into the Gaussian Sequence with the change of variable,

$$Let\ r=\frac{1}{4a^2t},\rightarrow\quad \phi(x,t)=\gamma_r(x)$$

This allows us to see via the convolution used in Q1 a) that,

$$u(x,t)=\int_R^{\cdot}\varphi(\xi)\phi(x-\xi,\mathrm{t})\mathrm{d}\xi=\int_R^{\cdot}\varphi(\xi)\gamma_r(\xi-\mathrm{x},\mathrm{t})\mathrm{d}\xi$$

$$=\int_R^{\cdot}\varphi(\xi)\gamma_r(x-\xi,\mathrm{t})\mathrm{d}\xi\quad as\ \gamma_r(x-\xi)=\gamma_r(\xi-\mathrm{x})$$

To continue, take use of the proceeding,

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}\gamma_r(x)g(x)dx=g(0)\quad (2)$$

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}\gamma_r(x-\alpha)g(x)dx=g(\alpha)\quad (3)$$

Therefore using (1),(2), we can claim that as,

$$t\to 0,\qquad r\to\infty\ \ via\ r=\frac{1}{4a^2t}$$

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}\gamma_r(x-a)\varphi(x)dx=\varphi(\alpha)$$

The result $\varphi(\alpha)$ means that we can achieve any value in $\varphi(\mathrm{x})'\mathrm{s}$ range, by varying α to be any value in $\varphi(\mathrm{x})'\mathrm{s}$ domain, thus verifying that

$$\lim_{t\to 0}(u(x,t)=\varphi(\mathrm{x}))$$

Considering (3), we will make sense of this theorem by linking our Gaussian to the Dirac Delta Function

Our sequence, $\gamma_r(\mathrm{x})$, is a delta sequence because of the following properties

$$1.\ \gamma_r(\mathrm{x})\ \geq 0\ for\ all\ x$$

$$2.\int_{-\infty}^{\infty}\gamma_r(\mathrm{x})dx=1$$

$$3.\ \lim_{r\to\infty}\gamma_r(\mathrm{x})=0, for\ x\neq 0,\qquad \lim_{r\to\infty}\gamma_r(0)=\infty$$

Then we can link the Dirac Delta Function by,

$$Any\ delta\ sequence\ (In\ our\ case, \mathrm{\gamma_r(x)})\ can\ be\ definded\ by\ the\ Dirac\ Delta\ Function\ via$$

$$\mathrm{\delta(x)} = \lim_{r\to\infty} \mathrm{\gamma_r(x)}$$

With this relation to the Dirac Delta Function, we can understand (2),

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \gamma_r(x) g(x) dx = g(0) = \int_{-\infty}^{\infty} \mathrm{\delta(x)} g(x) dx = g(0)$$

and (3) by using the translation property of the Dirac Delta Function

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \gamma_r(x-a) g(x) dx = g(0) = \int_{-\infty}^{\infty} \mathrm{\delta(x-a)} g(x) dx = g(a)$$

Note, for a complied list of the proofs of properties for the Gaussian Delta Sequence, one can be directed to https://proofwiki.org/wiki/Gaussian_Delta_Sequence

Q2

To solve the Laplace Equation in polar coordinate form, we assume the solution has the form,

$$u(r,\theta) = R(r)\mathrm{H}(\theta)$$

It is also important to note that,

$$\mathbf{1.}\boldsymbol{The\ solution\ is\ bounded\ at\ r = 0}$$

$$\mathbf{2.}\boldsymbol{The\ solution\ is\ periodic\ with\ period\ \pi, meaning\ u(r, \theta + 2\pi) = u(r,\theta)}$$

Then our Laplace Equation becomes

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

Then by separating the variables, we obtain the two following ODE's

$$r^2R'' - rR' - k^2R = 0$$

$$H'' + k^2H = 0$$

But because of the periodicity, the ODE's will yield the following solutions,

$$H_k(\theta) = C_k\cos(k\theta) + D_k\sin(\mathrm{k}\theta),\ \ k = 0,1,2,\ldots$$

Notice that 0 is included for k, as it will result in a constant that is also periodic

Then for R,

$$R_0(r) = A_0 + B_0\ln(\mathrm{r}), \qquad k = 0$$

$$R_k(r) = A_kr^k + B_kr^{-k}, \qquad k \neq 0$$

As we need $R(r)$ to be bounded, we must have that,

$$B_0 = B_k = 0$$

$$\longrightarrow R_k(r) = A_kr^k, \qquad R_0(r) = A_0, \qquad k = 0,1,2,..$$

Then using the principle of Superposition using the results above, we have that,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^k (C_k \cos(k\theta) + D_k \sin(\mathrm{k}\theta))$$

Which we can slightly modify to become

$$u(r,\theta) = a_0/2 + \sum_{n=1}^{\infty} (\frac{r}{r_0})^k (a_k \cos(k\theta) + b_k \sin(\mathrm{k}\theta))$$

Then using our Boundary Condition,

$$u(r_0,\theta) = f(\theta) = 1 + \theta, \qquad -\pi \le \theta \le \pi$$

$$u(r_0,\theta) = a_0 + \sum_{k=1}^{\infty} (\frac{r_0}{r_0})^k (a_k \cos(k\theta) + b_k \sin(\mathrm{k}\theta)) = f(\theta)$$

Where the coefficients are found by the following Fourier integrals

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + \theta) d\theta = \frac{1}{\pi} * 2\pi = 2$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + \theta) \cos(k\theta)\, d\theta = \frac{1}{\pi} * \frac{2\sin(k\pi)}{k} = 0, \qquad k = 0,1,2, \ldots$$

$$a_k = 0\; for\; any\; k, \qquad \sin(k\pi) = 0\; for\; any\; k\; (In\; terms\; of\; radians)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + \theta) \sin(\mathrm{k}\theta)\, d\theta = \frac{1}{\pi} * \frac{2\sin(k\pi) - 2k\pi\cos(k\pi)}{k^2} = \frac{1}{\pi} * \frac{-2\pi\cos(k\pi)}{k}, \qquad k = 1,2, \ldots$$

$$= \frac{-2(-1)^k}{k}$$

Thus, our exact solution is found below with the above expressions for the constants,

$$u(r,\theta) = 1 + \textstyle\sum_{k=1}^{\infty} (\frac{r}{r_0})^k (a_k \cos(k\theta) + b_k \sin(\mathrm{k}\theta))$$

$$u(r,\theta) = 1 + \sum_{k=1}^{\infty} \left(\frac{r}{r_0}\right)^k * \frac{-2(-1)^k}{k} * \sin(\mathrm{k}\theta)$$

a)

Letting $r_0 = 2$,

$$u\left(1,\frac{\pi}{4}\right) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k * \frac{-2(-1)^k}{k} * \sin\left(\frac{\pi}{4}\mathrm{k}\right) = 1.5110$$

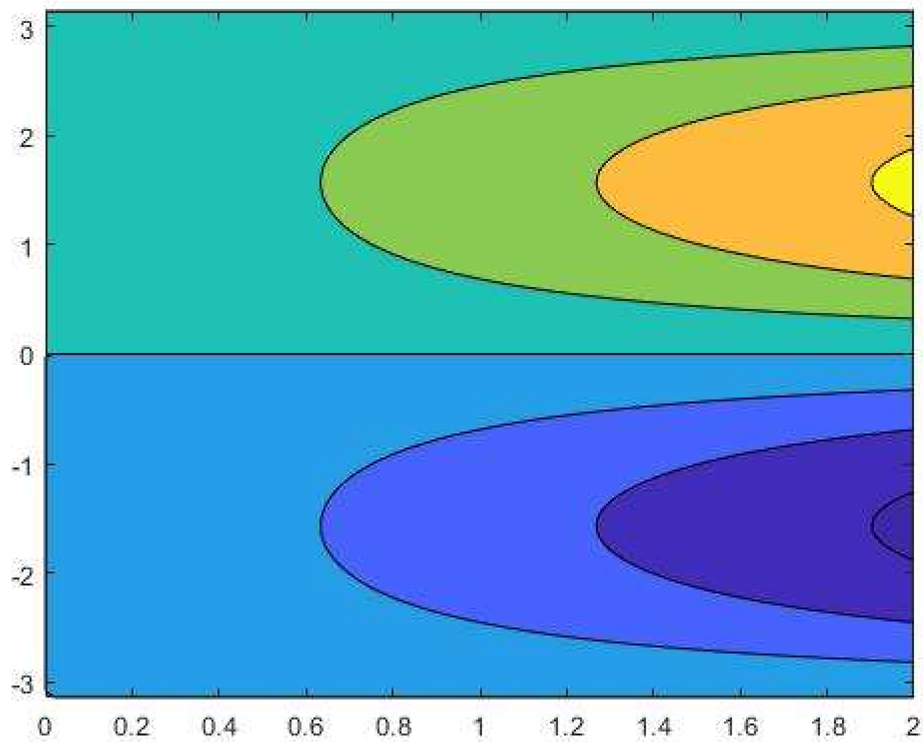
$$u\left(1,\frac{3\pi}{4}\right) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k * \frac{-2(-1)^k}{k} * \sin\left(\frac{3\pi}{4}\mathrm{k}\right) = 2.0009$$

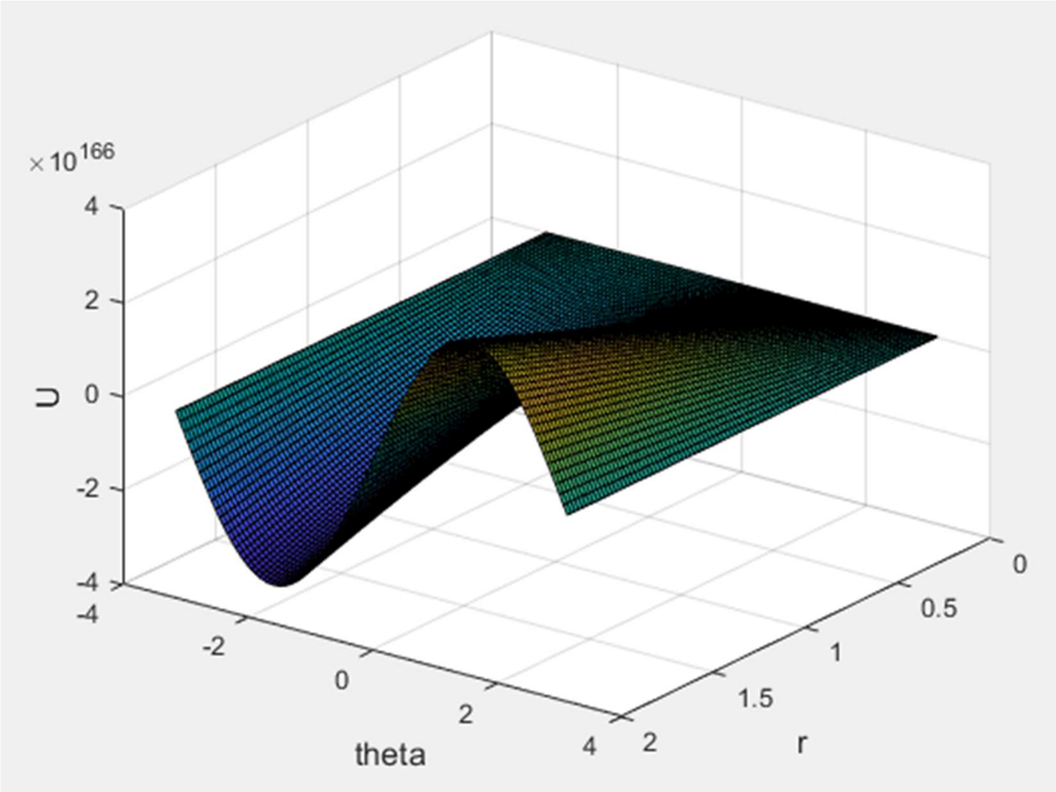
b)

For a contour plot with r = 2, we can use the following code

```
x = linspace(0,ri);
y = linspace (-1*pi,pi)
ri = 2;
[rGrid, thetaGrid] = meshgrid(x,y);
u = zeros(size(rGrid));

for k = 1:100
  u = u + 1+((rGrid/ri)^k).*(-2*((-1)^k)/k).*sin(k*thetaGrid)
end
u = u + 1;
figure
surf(rGrid, thetaGrid, u)
xlabel("r", "FontSize", 12)
ylabel("theta", "FontSize", 12)
zlabel("U", "FontSize", 12)
contourf(rGrid,thetaGrid,u)
```





Q3

To solve the hyperbolic equation, we shall use the result from Chapter 5 that used d'Alembert's approach to the more general system

$$u_{tt} = a^2 u_{xx}$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \varphi(x)$$

The final solution (Chp 5, pg 7/29, (11)) is given as follows,

$$u(x,t) = \frac{\phi(x+at) + \phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \varphi(\alpha)d\alpha$$

In our case, we have that,

$$no\ initial\ velocity, u_t(x,0) = \varphi(x) = 0, \qquad a^2 = 1$$

Thus, our general solution becomes,

$$u(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2}$$

Where our initial condition is a piecewise function

$$u(x,0) = \phi(x) = \begin{cases} 0, & x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

Then inserting our initial condition, we can write it compactly as follows,

$$u(x,t) = \frac{1}{2}\begin{cases} 0, & x < -1-t \\ 1+x+t, & -1-t < x < -t \\ 1-x-t, & -t < x < 1-t \\ 0, & x > 1-t \end{cases} + \frac{1}{2}\begin{cases} 0, & x < -1+t \\ 1+x-t, & -1+t < x < t \\ 1-x+t, & t < x < 1+t \\ 0, & x > 1+t \end{cases}$$

or

$$u(x,t) = \frac{1}{2}\begin{cases} 0, & x+t < -1 \\ 1+x+t, & -1 < x+t < 0 \\ 1-x-t, & 0 < x+t < 1 \\ 0, & x+t > 1 \end{cases} + \frac{1}{2}\begin{cases} 0, & x-t < -1 \\ 1+x-t, & -1 < x-t < 0 \\ 1-x+t, & 0 < x-t < 1 \\ 0, & x-t > 1 \end{cases}$$

b)

To find when the waves have completely separated, we claim that,

$$Let\ A\ be\ the\ furthest\ point\ to\ the\ right\ of\ the\ \phi(x+t)\ region$$

$$\phi(x+t) = 1 - x_A - t = 0$$

$$Let\ B\ be\ the\ furthest\ point\ to\ the\ left\ of\ the\ \phi(x-t)\ region$$

$$\phi(x-t) = 1 - x_B + t = 0$$

Initially, these two points are moving towards each other because they are traveling in the opposite direction. Thus, when these two points meet, is when the waves will first be completely separated.

$$-1 - t_m = 1 + t_m$$

$$t_s = 1$$

This makes sense as our initial condition tell us that both points A and B are displaced from the origin by a displacement of 1, and because they are traveling towards each other at the same speed,

$$t_s = \frac{velocity}{displacement} = \frac{1+1}{1+1} = 1$$

c)

When t=0

$$u(x,t) = \frac{1}{2}\begin{cases} 0, & x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases} + \frac{1}{2}\begin{cases} 0, & x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

When t=1/2 * 1

$$u(x,t) = \frac{1}{2}\begin{cases} 0, & x < -3/2 \\ 1+x+\frac{1}{2}, & -3/2 < x < -1/2 \\ 1-x-\frac{1}{2}, & -1/2 < x < 1/2 \\ 0, & x > 1/2 \end{cases} + \frac{1}{2}\begin{cases} 0, & x < -1/2 \\ 1+x-\frac{1}{2}, & -1/2 < x < 1/2 \\ 1-x+\frac{1}{2}, & 1/2 < x < 3/2 \\ 0, & x > 3/2 \end{cases}$$

When t = $t_s = 1$

$$u(x,t) = \frac{1}{2}\begin{cases} 0, & x < -2 \\ 2 + x, & -2 < x < -1 \\ -x, & -1 < x < 0 \\ 0, & x > 0 \end{cases} + \frac{1}{2}\begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

Sketch of solutions found on next page.