Difference formulas,

$$Forward: \frac{\partial f}{\partial S} = \frac{f_{i+1,j} - f_{i,j}}{\Delta S}, \qquad \frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j}}{\Delta t}$$

$$Backward: \frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i-1,j}}{\Delta S}, \qquad \frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i,j-1}}{\Delta t}$$

$$Central: \frac{f_{i+1,j} - f_{i-1,j}}{\partial S}, \qquad \frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta t}$$

$$Second\ derivative: \frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{2\Delta S^2}$$

discreate time grid given by

$$S = 0, \Delta S, 2\Delta S, 3\Delta S, \dots, (M-1)\Delta S, S_{max}$$

$$i = 1,2,3, \dots, M-2, M-1$$

$$t = 0, \Delta t, 2\Delta t, 3\Delta t, \dots, (N-1)\Delta t, T_{expiration}$$

$$j = N-1, N-2, N-3, \dots, 2, 1, 0$$

BSM PDE,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{dS} = rf$$

Explicit Method

$$rf_{i,j} = \frac{f_{i,j} - f_{i,j-1}}{\partial t} + ri\partial S\frac{f_{i+1,j} - f_{i-1,j}}{2\partial S} + \frac{1}{2}\sigma^2 j^2 \frac{f_{i+1,j} + f_{i-1,j}}{\partial S^2}$$

$$f_{i,j} = a_i f_{i-1,j+1} + b_i f_{i,j+1} + c_i f_{i+1,j+1}$$

With

$$a_i = \frac{1}{2}\Delta t(\sigma^2 i^2 - ri)$$

$$b_i = 1 - \Delta t(\sigma^2 i^2 - ri)$$

$$c_i = \frac{1}{2}\Delta t(\sigma^2 i^2 + ri)$$

Implicit Method

Using a Forward Difference in time and Centred Difference in Price,

$$rf_{i,j} = \frac{f_{i,j+1} - f_{i,j}}{\Delta t} + ri\Delta S\frac{f_{i+1,j} - f_{i-1,j}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta S^2}$$

Rearrange to Obtain,

$$f_{i,j+1} = a_i f_{i-1,j} + b_i f_{i,j} + c_i f_{i+1,j}$$

With the following coefficients,

$$a_i = \frac{1}{2}(ri\Delta t - \sigma^2 i^2)$$

$$b_i = 1 + \Delta t(\sigma^2 i^2 + r)$$

$$c_i = -\frac{1}{2} + \Delta t(\sigma^2 i^2 + ri)$$

Crank Nicolson Method

Combination of both the Implicit and Explicit method, by taking the average of both. This will yield the following,

$$\frac{rf_{i,j-1}}{2} + \frac{rf_{i,j}}{2} = \frac{f_{i,j} - rf_{i,j-1}}{2\Delta t} ri\Delta S\left(\frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta S}\right) + \frac{ri\Delta S}{2}\left(\frac{f_{i+!,j} - rf_{i-1,j}}{2\Delta S}\right) + \frac{\sigma^2 i^2 \Delta S^2}{4}\left(\frac{f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}}{\Delta S^2}\right) + \frac{\sigma^2 i^2 \Delta S^2}{4}\left(\frac{f_{i,j} - 2f_{i,j} + f_{i-1,j}}{\Delta S^2}\right)$$

rearrange equation to obtain,

$$-a_i f_{i-1,j-1} + (1 - b_i) f_{i,j-1} - c_i f_{i+1,j-1} = a_i f_{i-1,j} + (1 - b_i) f_{i,j-1} - c_i f_{i+1,j}$$

Where,

$$a_i = \frac{\Delta t}{4}(\sigma^2 i^2 - ri)$$

$$b_i = \frac{\Delta t}{2}(\sigma^2 i^2 + ri)$$

$$c_i = -\frac{\Delta t}{4} + \Delta t(\sigma^2 i^2 + ri)$$

Discrete Fourier Transform: Stability Analysis

Note: This section assumes prior knowledge of transforming the original BSM PDE into the heat equation for the subsequent analysis...

The Implicit Scheme,

$$-rU_{i-1}^{(n+1)} + (1 + 2r)U_i^{(n+1)} - rU_{i+1}^{(n+1)} = U_i^{(n)}$$

Crank Nicolson Scheme,

$$-rU_{i-1}^{n+1} + 2(1 + r)U_i^{n+1} - rU_{i+1}^{n+1} = f_i^n$$

Discrete Fourier Transform.

From *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach by Daniel J. Duffy*, a method for Stability analysis is to employ the Discrete Fourier Transform on the Finite Difference Scheme's

Let $\mathbf{u} = {}^t(\ldots, u_{-1}, u_0, u_1, \ldots)$ be an infinite sequence of values. Then the DFT (given in Thomas, 1998) is defined as

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-in\xi} u_n \tag{8.30}$$

Then applying this to both schemes, we will obtain, *(Please note d is now the imaginary unit)*

$$Implicit\ DFT: \frac{1}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} e^{di\xi}\left[U_i^{(n+1)}\right] = \widehat{U}_i^{(n)}(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\sum_{i=-\infty}^{\infty} e^{di\xi}\left[-rU_{i-1}^{(n+1)} + (1+2r)U_i^{(n+1)} - rU_{i+1}^{(n+1)}\right]\right)$$

$$= \frac{-r}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} e^{di\xi}\left[U_{i-1}^{(n+1)}\right] + \frac{(1+2r)}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} e^{di\xi}\left[U_i^{(n+1)}\right] + \frac{-r}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} e^{di\xi}\left[U_{i+1}^{(n+1)}\right] \quad (1)$$

Since the sum does not include the index n, we can use the following change of variables to simplify,

$$\frac{1}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} e^{-di\xi}\left[U_{i\pm1}^{(n+1)}\right] = \frac{e^{\pm di\xi}}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-dm}\ \left[U_m^{(n+1)}\right] = e^{\pm d\xi}\widehat{U}_i^{(n+1)}(\xi), \qquad m = i \pm 1$$

Applying this to (1),

$$\widehat{U}_i^{(n)}(\xi) = -re^{-d\xi}\widehat{U}_i^{(n+1)}(\xi) + (1+2r)\widehat{U}_i^{(n+1)}(\xi) - re^{d\xi}\widehat{U}_i^{(n+1)}(\xi)$$

$$= \widehat{U}_i^{(n+1)}(\xi)\left(-re^{-d\xi} + (1+2r) - re^{d\xi}\right)$$

$$= \widehat{U}_i^{(n+1)}(\xi)\left(-2rcos(\xi) + (1+2r)\right)$$

$$\frac{\widehat{U}_i^{(n+1)}(\xi)}{\widehat{U}_i^{(n)}(\xi)} = p(\xi) = \frac{1}{1 + 4r^2sin^2\left(\frac{\xi}{2}\right)}$$

Therefore, we have that, for errors to remain bounded, we apply,

$$|p(\xi)| = \left|\frac{1}{1 + 4r^2sin^2\left(\frac{\xi}{2}\right)}\right| \leq 1$$

$CN\ DFT$:

Following similar arithmetic involving the DFT will yield a similar result,

$$p(\xi) = \frac{1 - 2rsin^2\left(\frac{\xi}{2}\right)}{1 + 2rsin^2\left(\frac{\xi}{2}\right)}$$

$$|p(\xi)| = \left|\frac{1 - 2rsin^2\left(\frac{\xi}{2}\right)}{1 + 2rsin^2\left(\frac{\xi}{2}\right)}\right| \leq 1$$

Thus, we have proved unconditional stability for both *schemes since ANY value of r,*

$$\rightarrow |p(\xi)| \leq 1 \; for \; any \; r$$

$$as \; errors \; are \; bounded \; when |p(\xi)| \leq 1$$

$Explicit\ DFT$:

$$p(\xi) = 1 - 4r^2sin^2\left(\frac{\xi}{2}\right)$$

$$|p(\xi)| = \left|1 - 4r^2sin^2\left(\frac{\xi}{2}\right)\right|$$

$$\left|1 - 4r^2sin^2\left(\frac{\xi}{2}\right)\right| \leq 1$$

This condition is only satisfied when $r \leq \frac{1}{2}$

This implies that the Explicit scheme is conditionally stable, meaning r must be $\leq \frac{1}{2}$ to ensure errors remain bounded.