MATH203 Assignment 6
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Q1 a)

$$(T_n f)(x) = f''(x) - n^2 f(x), x \in (-\pi, \pi), \mathrm{n} \in \mathrm{N}$$

Using definition 6.25, which states that

Let $\mathrm{T} : \mathrm{V} \rightarrow \mathrm{W}$ be a linear map between real vector spaces. The kernel is the set

$\ker(\mathrm{T}) = \{\mathrm{v} \in \mathrm{V} : \mathrm{T}(\mathrm{v}) = 0\mathrm{W}\} \subset \mathrm{V}$. This holds true as, we are given that T_n is a linear map, it is taking functions, f, that have derivatives of all orders from the space F, and T_n is mapping from F to F.

Thus, we can setup our kernel as

$\ker(\mathrm{T}) = \{\mathrm{f} \in F: f''(x) - n^2 f(x) = 0\} \subset \mathrm{F}$

Looking at $f''(x) - n^2 f(x) = 0$, this is a differential equation. We can assume the solution will be of the form $f(x) = e^{\lambda x}$

Substituting this into the DE, we get $\frac{d^2}{dx^2} e^{\lambda x} - n^2 e^{\lambda x} = 0 \; giving \; \lambda^2 e^{\lambda x} - n e^{\lambda x} = 0$

Then by factoring out $e^{\lambda x}$ and noticing that $e^{\lambda x} = 0$ is not a solution for any finite λ, then the zeros come from the polynomial $-n^2 + \lambda^2 = 0$. This gives $\lambda = n \; or \; \lambda = -n$

Using these solutions, from math188, the general solution is the sum of

$f_1(x) = c_1 e^{-nx}$ and $f_2(x) = c_2 e^{nx}$ that equates to $f(x) = c_1 e^{-nx} + c_2 e^{nx}$ where $c_1 \; and \; c_2 \; are \; arbitrary \; real \; constants$

Applying this to our kernel, we get

$\ker(\mathrm{T}) = \{\mathrm{f} \in F: c_1 e^{-nx} + c_2 e^{nx} = f(x), x \in (-\pi, \pi), \mathrm{n} \in \mathrm{N} \text{ and } c_1, c_2 \in \mathrm{R}\} \subset \mathrm{F}$

in which $\ker(\mathrm{T}) = \mathrm{span}\{e^{-nx}, e^{nx}\}$

Q1 b) Using part a, where we concluded that $\ker(\mathrm{T}) = \mathrm{span}\{e^{-nx}, e^{nx}\}$, then to show that this forms a basis, we need to show linear independence.

Using definition 4.20, we shall set $\mathrm{A}e^{-nx} + Be^{nx} = 0$. We can immediately notice that it is linearly independent as $e^{nx} \; or \; e^{-nx}$ can never be = 0,(Also note here that n ∈ N = {1, 2, . . . }) so the only solution is when $\mathrm{A} = \mathrm{B} = 0$. We can also show by using the equations

$\mathrm{A}e^{-nx} + Be^{nx} = 0 \; and \; d/dx(\mathrm{A}e^{-nx} + Be^{nx} = 0)$ which gives

$\mathrm{A}e^{-nx} + Be^{nx} = 0 \; and \; -\mathrm{A}ne^{-nx} + Bne^{nx} = 0$

Using equation 2, we get $\mathrm{A}=\mathrm{B}e^{2nx}$, substitute this into equation 1, and we achieve $2\mathrm{B}e^{nx} = 0$

and as $e^{nx} \; cannot \; equal \; 0, thus \; B = 0 \; and \; this \; also \; shows \; that \; A \; must =$
$0 \; for \; the \; solution.$

This means $\{e^{-nx}, e^{nx}\} \; forms \; a \; basis \; for \; our \; kernel, and \; as \; there \; is \; two \; basis \; vectors, then$

$\mathrm{Dim}(\mathrm{Ker}(T_n)) = 2$

Q2 a)

We should firstly convert our basis for the Kernel into an orthogonal basis, as we should notice that $< e^{-nx}, e^{nx} > = \int_{-\pi}^{\pi} e^{-n} \; e^{nx} dx \; = \; 2\pi$ $(As\ it\ not\ equal\ to\ 0,$
$the\ two\ 'vectors'\ are\ not\ orthogonal)$. To do this, we shall use the Gram-Schmidt process.

Let $u_1 = \; e^{-nx}$ and $v_1 = e^{-nx}$ and $v_2 \; = \; e^{nx}$ (Labelling our basis vectors here)

Let $u_2 = \; v_2$ - $\left[\frac{<,v_2 \, u_1>}{<u_1 \, ,u_1 \, >}\right]$ x u_1

Using our definition of the inner product on F, this evaluates to

$$u_2 = e^{nx} - \frac{\int_{-\pi}^{\pi} e^{-nx} e^{nx} dx}{\int_{-\pi}^{\pi} e^{-2nx} dx} e^{-nx} = e^{nx} \; - \; \left(\frac{2\pi}{\frac{sinh(2\pi n)}{n}}\right) e^{-nx}$$

Note here: $\int_{-\pi}^{\pi} e^{-2nx} dx \; = \frac{-e^{-2\pi n} - e^{2\pi n}}{2n} = \frac{sinh(2\pi n)}{n}$

$using\ the\ hyperbolic\ subsitution\ for\ sinh$

This expression simplifies down to

$$u_2 = e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n)$$

Thus, we have a new basis for the kernel, $\{e^{-n}, e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n)\}$ which indeed is orthogonal as $\int_{-\pi}^{\pi} e^{-nx}(e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n)) \; dx = 0$

Q2 a) Now we can use this orthogonal basis to find the closest point in the span to the constant function 1

Our projection is given by $Proj_{\mathrm{Ker}(T_n)} P_0$

$$= \frac{< P_0 \, , e^{-nx} >}{< e^{-nx}, e^{-nx} >} u_1 + \frac{< P_0 \, , e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n) >}{< e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n), e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n) >} u_2$$

which evaluates to, using the definition $< \mathrm{f, g} > := \int_{-\pi}^{\pi} \mathrm{f(x)g(x)dx}, \mathrm{f, g} \; \in \; \mathrm{F},$

$$\frac{\int_{-\pi}^{\pi} e^{-nx} dx}{\int_{-\pi}^{\pi} e^{-2nx} dx} u_1 \; + \; \frac{\int_{-\pi}^{\pi} e^{nx} - 2\pi n e^{-nx} cs \quad (2\pi n) dx}{\int_{-\pi}^{\pi} (e^{nx} - 2\pi n e^{-nx} csch(2\pi n))^2 dx} u_2$$

$\int_{-\pi}^{\pi} e^{-nx} dx$ using the substitution u =-nx and du =-ndx, equals $\frac{2sinh(\pi n)}{n}$

$\int_{-\pi}^{\pi} e^{-2nx} dx$ using the substitution u =-2nx and du=-2nx dx, equals $\frac{sinh(\pi n)}{n}$

Thus our first term becomes $(2\mathrm{sinh}(\pi n) csch(2\pi n)) u_1$

$\int_{-\pi}^{\pi} e^{nx} \; - \; 2\pi n e^{-nx} csch(2\pi n) dx$ simplifies to $\int_{-\pi}^{\pi} e^{nx} \; dx - \; 2\pi csch(2\pi n) \int_{-\pi}^{\pi} e^{-nx} \; dx$ in which for the integrand e^{nx} we use the substitution u =-nx and du =-ndx, and for the integrand e^{-nx} we use the substitution u =-nx and du =-ndx

This gives $\frac{2si\quad(\pi n)}{n}$ - $2\pi sech(\pi n)$

$$\int_{-\pi}^{\pi} (e^{nx} - 2\pi n e^{-nx} csch(2\pi n))^2 dx \ \ expands\ to$$

$$\int_{-\pi}^{\pi} 4\pi^2 n^2 e^{-2nx} csch^2(2\pi n) + e^{2nx} - 4\pi n csch(2\pi n) \quad dx$$

This evaluates to $\frac{sinh(2\pi n)}{n}$ - $4\pi^2 n csch(2\pi n)$

Thus, our second term becomes $\dfrac{\frac{2sinh(\pi n)}{n} - 2\pi sec\quad(\pi n)}{\frac{sinh(2\pi n)}{n} - 4\pi^2 n csch(2\pi\quad)} u_2$

Our projection then becomes

$$(2\sinh(\pi n) csch(2\pi n))u_1 - \frac{\frac{2sinh(\pi n)}{n} - 2\pi sec\quad(\pi n)}{\frac{sinh(2\pi n)}{n} - 4\pi^2 n csc\quad(2\pi n)} u_2$$

Thus, the closest point in the kernel is

$$(2\sinh(\pi n) csch(2\pi n))\ e^{-nx} - \frac{\frac{2sinh(\pi n)}{n} - 2\pi sech(\pi n)}{\frac{sinh(2\pi\quad)}{n} - 4\pi^2 n csc\quad(2\pi\quad)} (e^{nx} - 2\pi n e^{-nx} csch(2\pi n))$$

Where x ∈ (−π, π) and n ∈ N

Q2 b) Now our projection is

$$Proj_{\mathrm{Ker}(T_n)} P_1 = \frac{< P_1, e^{-nx} >}{< e^{-n}\quad, e^{-nx} >} u_1 + \frac{< P_1, e^{nx} - 2\pi n e^{-nx} csch(2\pi n) >}{< e^{nx} - 2\pi n e^{-nx} csch(2\pi n), e^{nx} - 2\pi n e^{-nx} csch(2\pi n) >} u_2$$

Which gives $\dfrac{\int_{-\pi}^{\pi} x e^{-n}\quad dx}{\int_{-\pi}^{\pi} e^{-2nx} dx} u_1 + \dfrac{\int_{-\pi}^{\pi} x(e^{nx} - 2\pi n e^{-nx} csch(2\pi n))dx}{\int_{-\pi}^{\pi} (e^{nx} - 2\pi n e^{-nx} csch(2\pi n))^2 dx} u_2$

$\int_{-\pi}^{\pi} x e^{-nx} dx$ using integration by parts gives, $\dfrac{2(sinh(\pi n)\text{-}\pi n\cosh(\pi n))}{\mathrm{n}^2}$

$$\int_{-\pi}^{\pi} x(e^{nx} - 2\pi n e^{-n}\quad csch(2\pi n))dx$$

Expands to $\int_{-\pi}^{\pi} x e^{nx} - 2\pi n \mathrm{x} e^{-nx} csch(2\pi n))dx$ which can be solved as

$\int_{-\pi}^{\pi} x e^{nx} \mathrm{dx} + -2\pi n csch(2\pi n) \int_{-\pi}^{\pi} x e^{-nx} dx$ in which we just integration by parts again to achieve

$$\frac{2(\pi n\coth(\pi n)\text{-}1)(\sinh(\pi n) + \pi n\text{sech}(\pi n))}{n^2}$$

Q2b) Then using our integrals previously used before $\int_{-\pi}^{\pi} e^{-2nx}dx$ *and*

$\int_{-\pi}^{\pi}(e^{nx} - 2\pi n e^{-nx} csch(2\pi n))^2 dx$ our projection evaluates to

$$\frac{2cs\quad(\pi n)(\sinh(\pi n)\text{-}\pi n\cosh(\pi n))}{n}(e^{-nx}) +$$

$$\frac{2(\pi n\coth(\pi n)\text{-}1)(\sinh(\pi n)+\pi n se\quad(\pi n))}{n^2(\frac{\sinh(2\pi n)}{n}\text{-}8\pi^2 n\text{csch}(2\pi n))}(e^{nx} - 2\pi n e^{-nx} csch(2\pi n))$$

Which is the closest point in the kernel to P_1 where x ∈ (−π, π) and n ∈ N