BSM PDE Derivation by Ito's Lemma

Ruben Traicevski

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1 The PDE

The Black Scholes PDE is defined as follows

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV, \qquad 0 < t < T,\ \ 0 < S < \infty$$

With boundary conditions needed for whatever suitable derivative. For example, the European Call Option

$$\begin{cases} Call:\ C_{euro}(S,T) = max(S-K,0) & \text{for all S} \\ C_{euro}(0,t) = 0 & \text{for all t} \\ C_{euro}(S,t) \sim S & \text{as S} \to \infty \end{cases}$$

We should also note for later, the assumptions made that underpin the model (This is from the original paper).

1) The short-term interest rate is known and is constant through time

2) The stock pays no dividends or other distributions.

3) The option is "European," that is, it can only be exercised at maturity (Very important for boundary conditions)

4) There are no transaction costs in buying or selling the stock or the option.

5) The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant .

6) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

7) There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

2 Ito's Lemma

Let X_t be a stochastic process that satisfies

$$dX_t = b(t, W_t)dt + \alpha(t, W_t)dt$$

Further suppose we have $F_t = f(t, X_t)$ which is twice continuously differentiable. Then we shall use the following form of Ito's Lemma,

$$dF_t = [\partial_t f(t, X_t) + b(W_t, t)\partial_x f(t, X_t) + \frac{b(W_t, t)^2}{2}\partial_x^2 f(t, X_t)]dt + \alpha(W_t, t)\partial_x f(t, X_t)dW_t$$

3 Derivation

With the above in mind, we will shall show a very similar way of deriving the BSM PDE.

Similarly to the original paper, we shall suppose that the Stock price follows an Ito Process, thus we have an SDE,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where W_t is a Wiener process or Brownian Motion, with σ, the percentage volatility, and μ, percentage drift, are constants.
Now we further suppose that $f(t, S)$ is an at least twice differentiable function for the price of a derivative contingent on S. Then using Ito's lemma with the following, $b(t, W_t) = \mu S_t$ and $\alpha(t, W_t) = \sigma S_t$ gives,

$$dF_t = [\partial_t f(t,S) + \mu S_t \partial_x f(t,S) + \frac{(\sigma S_t)^2}{2} \partial_x^2 f(t,S)]dt + \sigma S_t \partial_x f(t,S) dW_t$$

Now without proving how, we'll claim that Ito Process describing S and our DF_t can be discretized over a time period Δt,

$$\Delta S_t = \mu S_t \Delta t + \sigma S_t \Delta W_t$$

and

$$\Delta F_t = \Delta f = [\partial_t f(t,S) + \mu S_t \partial_x f(t,S) + \frac{(\sigma S_t)^2}{2} \partial_x^2 f(t,S)]\Delta t + \sigma S_t \partial_x f(t,S) \Delta W_t$$

Now we shall consider a delta hedge portfolio, which will consist of one short option $-f$ and long $\frac{\partial f}{\delta S}$ shares at time t. The value of the Portfolio is,

$$P = -f + \frac{\partial f}{\delta S} S$$

or in discrete form,

$$\Delta P = -\Delta f + \frac{\partial f}{\delta S} \Delta S$$

which is the profit or loss over the time period $[t, t + \Delta t]$

Now we will substitute in our two discretized equations into the discretized portfolio,

$$P = -([\partial_t f(t,S) + \mu S_t \partial_x f(t,S) + \frac{(\sigma S_t)^2}{2} \partial_x^2 f(t,S)]\Delta t + \sigma S_t \partial_x f(t,S) \Delta W_t) + \partial_S f(t,S)[\mu S_t \Delta t + \sigma S_t \Delta W_t] S$$

After distributing all the terms and canceling terms, we obtain,

$$\Delta P = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t$$

This is important because the ΔW_t term is now gone, meaning the portfolio is risk less during the time period $[t, t+\Delta t]$.

Then considering the assumptions we defined for the model, the risk less portfolio we have created will earn instantaneous rates of return over the time period $[t, t + \Delta t]$ given by,

$$\Delta P = rP\Delta t$$

Substituting in the two following relations found above,

$$\Delta P = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t \qquad P = -f + \frac{\partial f}{\delta S} S \tag{1}$$

will give

$$(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t = r(-f + \frac{\partial f}{\delta S} S)\Delta t$$

and when rearranged,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf$$

Which is the BSM PDE we gave at the start of the paper