MATH203 Assignment 5
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$$B = \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$$

Q1 a) For the matrix B, notice that B^2 = B x B = B = $\begin{pmatrix} t & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$,

and that for B^3 = B x B x B = B = and that for B^3 = B x B x B= $\begin{pmatrix} t & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$

This indicates that for any n $\geq$ 0, B^n = B for all natural numbers. We can show this by induction.

For n = 1

B^1 = B $\quad \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}^1 = \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$ = RHS.

For n = k

B^k = B^n = B $\quad \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}^k = \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}^n = \begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix}$ = RHS

For n = k + 1

B^{k+1} = B

Using that n = k from the previous assumption, we return to B^k = B^n = B

LHS: $B \ = B^k \ = B^{k+1}$ = B^k x B^1 = B x B = B^2 = B :RHS

And for n = k + 2

LHS: $B \ = B^k \ = B^{k+2}$= B^k x B^1 x B^1 = B x B x B = B^3 = B :RHS

Thus, we see that for any n $\geq$ 0, B^n = B by principal of induction for all natural numbers

b) We can show the matrix of T_B relative to the standard ordered basis given by either using the relation D = $A^{-1}BA$ where A is the matrix formed by the column spaces of the given standard ordered basis indicated by $T_B([v]_B) \ = \ D[v]_B$, or by computing T_B on each basis vector and expressing in terms of the basis

By D = $A^{-1}BA$, A is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since our basis is { e_1 , e_2 } which is the 2x2 identity matrix, and so

$A^{-1} \ = \ A \textit{ which means } D \ = \ B \textit{ showing that the } T_B$ relative to the standard ordered basis given is just B

Likewise, we can show this by

$T_B\binom{1}{0} \ = \ \dfrac{t^2}{t\sqrt{1-t^2}} \ = \ a\binom{1}{0} \ + \ b\binom{0}{1}$ where $\binom{a}{b} = \dfrac{t^2}{t\sqrt{1-t^2}}$. Then by computing $T_B\binom{0}{1}$, *we obtain a similar relation where* $\binom{a}{b} = \dfrac{t\sqrt{1-t^2}}{1-t^2}$, in which the column space gives back B.

b) Then to find the matrix of T_B relative to the ordered basis, we will use the second process given above.

$T_B(\genfrac{}{}{0pt}{}{t}{\sqrt{1-t^2}})$ = $(\genfrac{}{}{0pt}{}{t}{\sqrt{1-t^2}})$ = a$(\genfrac{}{}{0pt}{}{t}{\sqrt{1-t^2}})$ + b$(\genfrac{}{}{0pt}{}{-\sqrt{1-t^2}}{t})$ = $\genfrac{}{}{0pt}{}{at-b\sqrt{1-t^2}}{a\sqrt{1-t^2}\ +bt}$ which by reading off the entries we see that a = 1 and b = 0

Then for $T_B(\genfrac{}{}{0pt}{}{-\sqrt{1-t^2}}{t})$ = $\binom{0}{0}$ = a$(\genfrac{}{}{0pt}{}{t}{\sqrt{1-t^2}})$ + b$(\genfrac{}{}{0pt}{}{-\sqrt{1-t^2}}{t})$ = $\genfrac{}{}{0pt}{}{at-b\sqrt{1-t^2}}{a\sqrt{1-t^2}\ +bt}$ we can see that the solution is a = 0 and b = 0

Thus, we get T_B = $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ when with respect to the ordered basis.

c) The Kernel and Image of T_B $that\ maps\ from\ V\ to\ W(vector\ spaces)$, by definition 6.25, is given by

Ker(T_B) = { $v\ \in R^2 :\ T_B(\boldsymbol{v})\ =\ \boldsymbol{O_w}$ } $\subset$ V and

$\mathrm{Im(T)}\ =\ \{\mathrm{T(v)} : \mathrm{v}\ \in\ \mathrm{V}\}\ \subset\ \mathrm{W}$

For determining the kernel, let….

$T_B(v)\ =\ \begin{pmatrix}t^2 & t\sqrt{1-t^2}\\ t\sqrt{1-t^2} & 1-t^2\end{pmatrix}\binom{a}{b}\ =\ \binom{0}{0}$ which gives $\binom{0}{0} = \begin{pmatrix}at^2+bt\sqrt{1-t^2}\\ at\sqrt{1-t^2}\ +b(1-t^2)\end{pmatrix}$

Solving for a using the first entry, gives $a = \frac{-b\ \sqrt{1-t^2}}{t^2}$ and substituting this into the second equation, gives hat b=0, this also means a=0, meaning the Kernel is the zero vector, and the basis for the kernel does not exist/has dimension zero.

For the Image, Bv = $\begin{pmatrix}t^2 & t\sqrt{1-t^2}\\ t\sqrt{1-t^2} & 1-t^2\end{pmatrix}\binom{a}{b}\ =\ \begin{pmatrix}t^2a\ +bt\sqrt{1-t^2}\\ at\sqrt{1-t^2}\ +b(1-t^2)\end{pmatrix}$

This can be written as $a\begin{pmatrix}t^2\\ t\sqrt{1-t^2}\end{pmatrix} + b\begin{pmatrix}t\sqrt{1-t^2}\\ (1-t^2)\end{pmatrix}$. Notice that $det\begin{pmatrix}t^2 & t\sqrt{1-t^2}\\ t\sqrt{1-t^2} & (1-t^2)\end{pmatrix}\ =\ 0$ meaning one of these vectors is redundant/they are not Linearly Independent, and thus Im(B) = span{$\begin{pmatrix}t^2\\ t\sqrt{1-t^2}\end{pmatrix}$} which is the basis for the Image subspace

d) We can show Eigenvectors do exist for this matrix by obtaining eigenvalues and finding the associated eigenvectors

From (B – λId2)v $=\ 0_{R^2}$ where $\lambda\ \in$ R, firstly solve det(B – λId2) = 0 . The characteristic equation becomes $0\ =\ \lambda^2-\lambda$, giving solutions $\lambda\ =\ 1{,}0$

For the eigenvector associated with 0, sub into (B – λId2)v $=\ 0_{R^2}$, we obtain

$\begin{pmatrix}t^2 & t\sqrt{1-t^2}\\ t\sqrt{1-t^2} & 1-t^2\end{pmatrix}\binom{a}{b} = \binom{0}{0}$ whereby converting the system to augmented matrix and using Gaussian elimination by adding -$\frac{t\sqrt{1-t^2}}{t}$ $x\ row$ 1 to row 2 and then dividing row 1 by t^2 , we achieve

$\begin{pmatrix} 1 & \frac{t\sqrt{1-t^2}}{t} \\ 0 & 0 \end{pmatrix}\binom{a}{b} = \binom{0}{0}$ which gives that $\binom{a}{b} = \begin{pmatrix} \frac{bt\sqrt{1-t^2}}{t} \\ b \end{pmatrix}$, and as this is an eigenvector for any non-zero value of b, where all eigenvectors are scalar multiples. This means we can take b as 1, which gives us the eigenvector of $\begin{pmatrix} \frac{t\sqrt{1-t^2}}{t} \\ 1 \end{pmatrix}$ for the associated eigenvalue of 0.

For the associated eigenvector of eigenvalue 1, we obtain from using our characteristic equation $\begin{pmatrix} t^2-1 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & -t^2 \end{pmatrix}\binom{a}{b} = \binom{0}{0}$ whereby using the same process as before we obtain the matrix equation $\begin{pmatrix} 1 & \frac{-t}{\sqrt{1-t^2}} \\ 0 & 0 \end{pmatrix}\binom{a}{b} = \binom{0}{0}$ from the reduced system. This gives us that $\binom{a}{b} = \begin{pmatrix} \frac{bt}{\sqrt{1-t^2}} \\ b \end{pmatrix}$, and as outlined above, we can again take b as 1 giving $\begin{pmatrix} \frac{t}{\sqrt{1-t^2}} \\ 1 \end{pmatrix}$.

e) To see what is happening geometrically, we can convert the matrix to its trigonometric equivalent by the identity $sin^2(\emptyset) + cos^2(\emptyset) = 1$, the matrix becomes $\begin{pmatrix} sin^2(\emptyset) & sin(\emptyset)cos(\emptyset) \\ sin(\emptyset)cos(\emptyset) & cos^2(\emptyset) \end{pmatrix}$ where vectors in the image space are of the form $\binom{as\quad^2(\emptyset)+bsin(\emptyset)cos(\emptyset)}{asin(\emptyset)cos(\emptyset)+bcos^2(\emptyset)}$ with the corresponding eigenvectors from **d)** are now $\binom{-cot(\emptyset)+csc(\emptyset)sec(\emptyset)}{1}$ for $\lambda = 1$ and $\binom{-cot(\emptyset)}{1}$ for $\lambda = 0$.

We can also use the theorem that holds that a n x n matrix A is diagonalizable if and only if A has n linearly independent eigenvectors, or by seeing if the relation B = PDP^{-1}, where D is the diagonal matrix using our eigenvalues of B, and P is the matrix whose columns are the eigenvectors of B, holds in which we see, does in fact hold. (Note: here I used original form for ease of computation)

$$\begin{pmatrix} t^2 & t\sqrt{1-t^2} \\ t\sqrt{1-t^2} & 1-t^2 \end{pmatrix} = \begin{pmatrix} \frac{t\sqrt{1-t^2}}{t} & \frac{t}{\sqrt{1-t^2}} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -t\sqrt{1-t^2} & t^2 \\ \frac{t-t^3}{\sqrt{1-t^2}} & 1-t^2 \end{pmatrix}$$

Then by applying this relation to our linear transformation, we obtain

$$PDP^{-1}\binom{v_1}{v_2}, = \binom{a}{b}, then\ by\ multiplying\ both\ sides\ by\ P^{-1}, we\ now\ have\ \ DP^{-1}\binom{v_1}{v_2} = P^{-1}\binom{a}{b}$$

And since the same linear transformation P^{-1} is being applied to both 'sides', this new system is equivalent to the old system, where $D\binom{v_1}{v_2}' = \binom{a}{b}' as(\binom{v_1}{v_2}' = P^{-1}\binom{v_1}{v_2}\ and\ \binom{a}{b}' = P^{-1}\binom{a}{b}$

This gives any vector in the transformed space the form of $\binom{0}{at\sqrt{1-t^2}+b(1-t^2)}$ or $\binom{0}{asin(\emptyset)cos(\emptyset)+bcos^2(\emptyset)}$. With this information, we can see the matrix is scaling the space along its eigenvector axis by the factor of the eigenvalues. Passing the basis vector (1,0) for example gives $\binom{0}{sin(\emptyset)cos(\emptyset)}$ while (0,1) gives $\binom{0}{cos^2(\emptyset)}$.