Portfolio Management/Analysis, Notes and Theory

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1 Simple vs Log Returns

1.1 Simple Returns: Time-Weighted Method

The one period Simple Return over any time length (With growth factor, +1)

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

Now suppose we have that for the first period

$$R_1 = \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1$$

Then assuming the gains or losses ,$P_1 - P_0$ are not withdrawn or topped up, then the value of the investment at the start of the second period is P_1
Then for the value of the investment at the end of the second period, with growth factor, is given by

$$R_2 = \frac{P_2 - P_1}{P_1} = \frac{P_2}{P_1} - 1$$

Then to obtain the holding period return over the two successive periods

$$(1 + R_1)(1 + R_2) = \frac{P_2}{P_0}$$

$$(1 + R_1)(1 + R_2) - 1 = \frac{P_2 - P_0}{P_0}$$

Extending this to n periods, assuming returns are reinvested, then the cumulative return or overall return is

$$R = (1 + R_1)(1 + R_2)...(1 + R_n) - 1$$

1.2 Log Returns

Suppose again the one period Simple Return over any time length is (With growth factor, +1)

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

Then we define the One Period Log Return as

$$r_t = ln(1 + R_t)$$

Thus, the n period Log Returns are given by

$$r_t(n) = ln(1 + R_t(n)) = ln[(1 + R_t)(1 + R_{t-1})...(1 + R_{t-n+1})]$$

Where by the log rule $ln(M * N) = ln(M) + ln(N)$

$$r_t(n) = ln(1 + R_t(n)) = r_t + R_{t-1} + ... + r_{t-k+1} = ln(P_t) - ln(P_{t-k})$$

using the relation $R_t = \frac{P_t}{P_{t-1}} - 1$

1.2.1 Logarithmic / Continuously Compounded Return

Define V_f as the final value, including dividends and interest, and V_i as the initial value.
Then the continuously compounded return, also known as force of interest, is given by

$$R_{log} = ln(\frac{V_f}{V_i})$$

and the logarithmic rate of return is,

$$r_{log} = \frac{ln(\frac{V_f}{V_i})}{t}$$

In which can be used to find either the starting or final value,

$$V_f = V_i e^{r_{log} t}$$

1.2.2 r_{log}?

The term $r_{log} = \frac{ln(\frac{V_f}{V_i})}{t}$ is the mean of the log returns, which can be used to compute final value of a continuously compounded return at any time! In comparison, the mean of simple returns is not useful like log returns in find the value of an investment

Continuous Compounding can be regarded as the letting the compounded period become infinitesimally small of Periodic Compounding.

2 Why use Log Returns?

- Logarithmic returns are symmetric, while ordinary returns are not. This means positive and negative percent ordinary returns of equal magnitude do not cancel each other out and result in a net change. Whereas, logarithmic returns of equal magnitude but opposite signs will cancel each other out.
- Also known as continuously compounded return. This means frequency of compounding doesn't matter, which makes the returns of different assets easier to compare
- Logarithmic returns are time additive, meaning the overall return is the sum of individual returns
- It prevents investment prices from becoming negative
- If we suppose returns in periods are independent random variables and are normally distributed, then the sum of these returns is also normally distributed. Thus, we should be taking the Logarithmic return so we can use the time-additive property
- The product of independent random variables that are normally distributed does not result in a new normally distributed variable!

2.1 Independent Random Variables

Let X and Y be independent random variables that are normally distributed , then their sum is also normally distributed.

$$X \sim N(\mu_X, \sigma_X^2)$$

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

then $Z = X + Y$

$$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

The result about the mean holds in all cases, while the result for the variance requires uncorrelatedness, but not independence.

2.1.1 Correlated Random Variables

In the event that the variables X and Y are jointly normally distributed random variables, then X + Y is still normally distributed and the mean is the sum of the means. However, the variances are not additive due to the correlation.
In the case of two variables

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

3 VaR: Value at Risk

"I am X percent certain there will not be a loss of more than V dollars in the next N days."

Mathematical Definition

$$VaR_\alpha(X) = -inf(x \in \mathbb{R} : F_X(x) > \alpha) = F_Y^{-1}(1-\alpha)$$

If X is a random variable for the PnL, $Y := -X$ (loss distribution)
$F_X(x)$ is the CDF of the PnL distribution, while F_Y^{-1} is the inverse CDF of the loss dsitribution

3.1 Historical VaR: No Assumptions on Distribution

Also known as historical simulation or the historical method, makes no assumptions about the distribution of returns.

For example, suppose we want to calculate the 1-day 95 percent VaR for an equity using 100 days of data. The 95th percentile corresponds to the least worst of the worst 5 percent of returns. In this case, because we are using 100 days of data, the VaR simply corresponds to the 5th worst day.

3.2 Parametric Value at Risk

As opposed to Historical Value at Risk, we now make an assumption that returns follow some distribution, commonly a Normal distribution or t-distribution.

Using the definition for VaR found above,

$$VaR_\alpha(X) = F_Y^{-1}(1-\alpha)$$

The $F_X(x)$ is the CDF, while F_Y^{-1} is the inverse of the CDF function

3.2.1 Assuming Normal distribution

When regarding returns are normally distributed

$$VaR_\alpha(X) = \Phi^{-1}(1-\alpha) * \sigma_{portfolio} - \mu_{portfolio}$$

3.2.2 Assuming t-dsitribution

If we define the returns X as being $X = \sigma T$ where T is the t-distribution then we have

$$\sigma = \sigma_{portfolio} * \sqrt{\frac{df-2}{df}}$$

Thus the t distributed VaR is given by

$$VaR_\alpha(X) = \sigma t_{df}^{-1}(1-\alpha) - \mu_{portfolio}$$

3.3 Conditional VaR: Expected Shortfall

The "expected shortfall at q% level" is the expected return on the portfolio in the worst q% of cases

ES is an alternative to value at risk that is more sensitive to the shape of the tail of the loss distribution.
Mathematical Definition

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\lambda(X)d\lambda$$

3.4 Assuming Normal Distribution

$$ES_\alpha(X) = \sigma\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha} - \mu$$

Where φ is the standard normal PDF and Φ is the standard normal CDF.

3.5 Assuming t-Distribution

Again, we have

$$\sigma = \sigma_{portfolio} * \sqrt{\frac{df-2}{df}}$$

Then the expected shortfall is

$$ES_\alpha = \sigma \frac{df + (T^{-1}(\alpha))^2}{df-1} \frac{\tau(T^{-1}(\alpha))}{\alpha} - \mu_{portfolio}$$

Where τ is the standard t-distribution PDF and T is the standard t-distribution CDF

4 Risk-Return Measures

4.1 Sharpe Ratio

The Sharpe ratio seeks to characterize how well the return of an asset compensates the investor for the risk taken.

$$S_a = \frac{E[R_a - R_r]}{\sigma_a} = \frac{E[R_a - R_r]}{\sqrt{Var(R_a - R_r)}}$$

It represents the additional amount of return that an investor receives per unit of increase in risk

4.2 Sortino Ratio

It is a modification of the Sharpe ratio but penalizes only those returns falling below a user-specified target or required rate of return

$$s = \frac{R - T}{DR}$$

Where R is the asset or portfolio average realized return, T is the target or required rate of return, and DR is the target semi-deviation, termed downside deviation. DR was also known as minimum acceptable return MAR.
Under the way DR is defined, we can have different Sortino Ratio's for the same set of returns by adjusting what the minimum acceptable return is.

4.3 Modigliani Ratio

Let D_t be the excess return of the portfolio for some time period t.
Then we can define the sharpe ratio as

$$S = \frac{\bar{\hat{D}}}{\sigma_D}$$

Where $\bar{\hat{D}}$ is the average of all excess returns over some period, and σ_D is the standard deviation of those excess returns.
Then the ratio is defined by

$$M^2 := S * \sigma_B + \bar{\hat{R}}_F$$

Where σ_B is the standard deviation of the excess returns for some benchmark portfolio against which you are comparing the portfolio in question, and $\bar{\hat{R}}_F$ is the average risk free rate for the period.

4.4 Max Draw-down

The draw-down is the measure of the decline from a historical peak. Useful for many reasons such as not relying on assuming underlying returns being normally distributed.
The maximum draw-down (MDD) up to time T is the maximum of the draw-down over the history of the variable.

Draw-down is defined as, for $t \in (0, T)$

$$D(T) = max[max(R_t - R_T), 0]$$

Then the Maximum drawn-down is

$$MDD(T) = maxD(\tau) = max[max(R_t - R_\tau)]$$

where $\tau \in (0, T)$ and the $max(R_t - R_\tau)$ has $t \in (0, \tau)$