Stochastic Calculus

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A stochastic process on the probability space (Ω, F, P) is a family of random variables X_t parameterized by $t \in T$, where $T \subset \mathbb{R}$. If T is an interval, then X_t is a stochastic process in continuous time.

2.1 Filtration

Consider that all the information accumulated until time t is contained by the σ-field $\mathbb{F}_t$. This means $\mathbb{F}_t$ contains the information containing events that have already occurred until time t, and which did not. Since information is growing in time, we have

$$\mathbb{F}_s \subset \mathbb{F}_t \subset \mathbb{F}$$

for any $s, t \in T$ with $s \leq t$. The family $\mathbb{F}_t$ (The family containing all information already occurred until time t) is called a filtration.
This means that the information at time t determines the value of the random variable X_t

2.2 Martingale

A stochastic process X_t, $t \in T$, is called a martingale with respect to the filtration $\mathbb{F}_t$ if

- X_t is integrable for each $t \in T$
- X_t is adapted to the filtration $\mathbb{F}_t$
- $X_s = E[X_t|\mathbb{F}_S$ For all $s < t$

3 Important Stochastic Processes

3.1 Brownian Motion Process

A Brownian Motion process is a stochastic process B_t, $t \geq 0$ which satisfies the following

- 1. The process starts at the origin, $B_0 - 0$
- 2. B_t has independent increments
- 3.The process B_t is continuous in t, thought nowhere differentiable
- 4. The increments $B_t - B_s$ are normally distributed with mean zero and variance $|t - s|$

$$B_t - B_s \sim N(0, |t - s|)$$

Condition 4 gives us that B_t is normally distributed with $E[B_t] = 0$ and $Var[B_t] = t$

$$B_t \sim N(0, t)$$

Consequently, B_t and B_s are not independent, which is different to independent increments

3.2 Wiener Process

The Wiener process is defined similarly to Brownian Motion, and despite not being defined as having normally distributed increments, it can be shown that there is no difference between these two processes by the Levy Theorem. B_t and W_t can be used interchangeably.

A Wiener Process has W_t being defined as normally distributed with mean 0 and variance t, thus its density is

$$\phi_t(x) = \frac{1}{\sqrt{2t\pi}} e^{-\frac{x^2}{2t}}$$

Then its distribution function is

$$F_X(x) = \frac{1}{\sqrt{2t\pi}} \int_a^b exp(-\frac{u^2}{2u})du$$

Then the probability that W_t is between values a and b is given by, where a¡b,

$$P(a \leq W_t \leq b) = \frac{1}{\sqrt{2t\pi}} \int_a^b exp(-\frac{u^2}{2u})du$$

$$Cov(W_s, W_t) = s \qquad Corr(W_s, W_t) = \sqrt{\frac{s}{t}} \tag{1}$$

3.3 Geometric Brownian Motion

With drift μ and volatility σ, GBM is the process defined by, with $t \geq 0$,

$$X_t = e^{\sigma W_t + (\mu - \sigma^2/2)t}$$

This is an extension of exponential Brownian motion, formed by $X_t = e^{W_t}$

3.4 Integrated Brownian Motion

The stochastic process, $t \geq 0$

$$Z_t = \int_0^t W_s ds \qquad Z \sim N(0, \frac{t^3}{3}) \qquad Z_0 = 0 \tag{2}$$

3.5 Exponential Integrated Brownian Motion

Using $Z_t = \int_0^t W_s ds$, then the process is,

$$V_t = e^{Z_t}$$

Since Z_t is normally distributed, then V_t is log normally distributed. The process starts at $V_0 = e^0 = 1$

$$E[V_t] = E[e^{Z_t}] = e^{t^3/6} \qquad Var(V_t) = e^{2t^3/3} - e^{t^3/6} \qquad Cov(V_s, V_t) = e^{(t+3^s)/2} \tag{3}$$

3.6 Brownian Bridge

$$X_t = W_t - tW_1$$

This process is fixed at both 0 and 1 since we can also write

$$X_t = W_t - tW_t - tW_1 + tW_t$$

$$= (1-t)(W_t - W_0) - t(W_1 - W_t)$$

Since the increments $W_t - W_0$ and $W_1 - W_t$ are independent and normally distributed, with

$$W_t - W_0 \sim N(0, t) \qquad W_1 - W_t \sim N(0, 1-t) \tag{4}$$

$$E[X_t] = 0 \qquad Var[X_t] = t(1-t) \tag{5}$$

3.7 Brownian Motion with Drift

$$Y_t = \mu t + W_t$$

Brownian motion with a drift term and

$$E[Y_t] = \mu t \qquad Var[Y_t] \tag{6}$$

3.8 Bessel Process

4 Properties of Stochastic Processes

4.1 Stopping Times

Assume that the decision to stop playing a game before or at time t is determined by the information $\mathbb{F}_{\approx}$ available at time at time t. The the decision can be modeled by a random variable $\tau : \Omega \to [0, \infty]$.
This means that given the information set $\mathbb{F}_{\approx}$, we know whether the event had occurred or not.

4.2 The First Passage of Time

This is a particular type of hitting time which can describe the first time hitting some barrier

4.2.1 First Passage Time Brownian Motion

Let T_a be the first time the Brownian Motion W_t hits a. Then the distribution function of T_a is given by

$$P(T_a \leq t) = P(T_{-a} \leq t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

Brownian Motion will hit a barrier with probability one in a finite amount of time. Likewise, a Brownian process will return to the origin in a finite time with probability 1
This then leads to the relation

$$E[T_a] = \infty$$

However, the fact that Brownian motion returns to the origin or hits a barrier almost surely, is only a characteristic to the first dimension. When extending these concepts to higher dimensions, we must also extend to concept of a barrier and the origin to higher dimensions like a Disk and Ball respectively

4.3 Hitting Times: Brownian Motion with Drift

These results are useful for Mathematical Finance. We assuming the stochastic process of,

$$X_t = \mu t + W_t$$

4.3.1 P(X_t goes up to α before down to $-\beta$)

Assuming a nonzero drift rate, and α,$\beta > 0$ then,

$$= \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}}$$

When the drift rate is zero,

$$\frac{\beta}{\alpha + \beta}$$

This can be used to show that when the drift rate is zero, Brownian motion i s equally likely to go up or down an amount in a given time interval
Assuming nonzero drift rate, and considering $\beta > 0$, the probability that X_t never hits $-\beta$ is given by,

$$= \begin{cases} 1 - e^{-2\mu\beta} & \text{if } \mu > 0 \\ 0 & \text{if } \mu < 0 \end{cases}$$

Recalling that T is the first time when the process hits the boundary,

$$E[X_T] = \frac{\alpha e^{2\mu\beta} + \beta e^{-2\mu\alpha} - \alpha - \beta}{e^{2\mu\beta} - e^{-2\mu\alpha}}$$

Then the expected waiting time is given by,

$$E[T] = \frac{E[X_T]}{\mu}$$

4.3.2 Brownian Motion with Drift and Scaling

4.4 Hitting Time Densities

Define τ as the hitting time when our desired process hits the barrier $x, x > 0$. Then assuming Brownian Motion with drift and scaling, $\mu, \sigma > 0$,

$$X_t = \mu t + \sigma W_t$$

the density function of τ is given by,

$$p(\tau) = \frac{x}{\sigma\sqrt{2\pi\tau^{3/2}}} exp(-\frac{(x - \mu\tau)^2}{2\tau\sigma^2}$$

$$E[\tau] = \frac{x}{\mu} \qquad Var(\tau) = \frac{x\sigma^3}{\mu^3} \tag{7}$$

Now assume, with $x, x > 0$, τ is the first time the process hits $-x$ for the first time, then the density function is,

$$p(\tau) = \frac{x}{\sigma\sqrt{2\pi\tau^{3/2}}} exp(-\frac{(x + \mu\tau)^2}{2\tau\sigma^2}$$

$$E[\tau] = \frac{x}{\mu} exp(-\frac{2\mu x}{\sigma^2})$$

4.4.1 Double Barrier Case

Useful for computing the price for a double barrier derivative

4.5 Limits of Stochastic Processes

4.6 Quadratic Variation

4.7 The Total Variation of Brownian Motion

5 Stochastic Integration (Brownian Motion)

5.1 Mean Square Limit

We say that X_n converges to X in the mean square if

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0$$

Or a more precisely interpreted as,

$$\lim_{n\to\infty} \int_{\Omega} (X_n(\omega) - X(\omega))^2 dP(\omega) = 0$$

5.2 Non anticipating Processes

A process X_t is called non anticipating process if the process is independent of any future increment $W_{t'} - W_t$ for any t and t' with $t < t'$. Consequently the process is independent of Brownian motion in the future

5.3 Increments of Brownian Motions

Useful for Stochastic Integrals

$$E[(W_t - W_s)^2] = t - s$$
$$Var[(W_t - W_s)^2] = 2(t-s)^2$$

$$\frac{W_t - W_s}{\sqrt{t-s}} \sim N(0,1) \qquad \frac{(W_t - W_s)^2}{t-s} \sim \chi^2)1 \tag{8}$$

5.4 The Ito Integral

The Ito integral is a random variable, however it has common properties with the Riemann Integral.
Define $F_t = f(W_t, t)$ to be a non anticipating process which satisfies the non-explosive condition

$$E[\int_a^b F_t^2 dt < \infty$$

Now we will consider partial sums when partitioning the interval from a to b,

$$S_n = \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i})$$

And since the process F_t is defined as being non anticipating, the $W_{t_{i+1}} - W_{t_i}$ are always independent
Then, we define the Ito Integral is the limit of the partial sums S_n,

$$ms \lim_{n\to\infty} S_n = \int_a^b F_t dW_t$$

In which the previous convergence can is taken in the mean square sense,

$$\lim_{n\to\infty} E[(S_n - \int_a^b F_t dW_t)^2] = 0$$

5.5 Examples of Ito Integrals

5.5.1 $F_t = constant$

The partial sums can be computed directly

$$S_n = \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}) = \Sigma_{i=0}^{n-1} c(W_{t_{i+1}} - W_{t_i})$$
$$= c(W_b - W_a)$$

and since thw answer does not depend on n, we obtain,

$$\int_a^b c dW_t = c(W_b - W_a)$$

5.5.2 $F_t = W_t$

$$\int_a^b W_t dW_t = \frac{1}{2}(W_b^2 - W_a^2) - \frac{1}{2}(b-a)$$

5.6 Properties of the Ito Integral

5.7 The Wiener Integral

6 Stochastic Differentiation

Most stochastic processes are not differentiable, thus the derivative operator does not make sense in stochastic calculus
The change in the process X_t between instances t and $t + \Delta t$ is given by $\Delta X_t = X_{t+\Delta t} - X_t$
The Infinitesimal change of a process is when Δt is infinitesimally small, we then obtain,

$$dX_t = X_{t+dt} - X_t$$

6.1 Basic Rules

- Constant Multiple Rule: Given that c is a constant,

$$d(cX_t) = cdX_t$$

- Sum Rule

$$d(X_t + Y_t) = dX_t + dY_t$$

- Difference Rule

$$d(X_t - Y_t) = dX_t - dY_t$$

- Product Rule

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

- Product with a deterministic function f(t)

$$d(f(t)Y_t) = f(t)dY_t + Y_t df(t) + df(t)dY_t$$

- Quotient Rule

$$d\frac{X_t}{Y_t} = \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^3}(dY_t)^2$$

- Quotient Rule with a deterministic function f(t)

$$d\frac{X_t}{f(t)} = \frac{f(t)dX_t - X_t df(t)}{f(t)^2}$$

- Other Important Relations

$$dtdW_t = dt^2 = 0$$
$$d(W_t^2) = 2W_t dW_t + dt$$
$$(dW_t)^2 = dt$$
$$d(tW_t) = W_t dt + tdW_t$$

6.2 Ito's Formula

6.2.1 When $F_t = f(X_t)$

Let X_t be a stochastic process satisfying

$$dX_t = b_t dt + \sigma_t dt$$

with b_t and σ_t being measurable processes (deterministic).
Then define that $F_t = f(X_t)$, with f being twice continuously differentiable, This gives

$$dF_t = [b_t f^{'}(X_t) + \frac{\sigma_t^2}{2} f^{''}(X_t)]dt + \sigma_t f^{'}(X_t)dW_t$$

Ito's formula can also be written in an equivalent integral form

$$F_t = F_0 + \int_0^t (b_s f^{'}(X_s) + \frac{\sigma_s^2}{2} f^{''}(X_s))ds + \int_0^t \sigma_s f^{'}(X_s)dW_s$$

6.3 When $F_t = f(t, X_t)$ and $dX_t = b_t dt + \sigma_t dt$

$$dF_t = [\partial_t f(t, X_t) + b_t \partial_x f(t, X_t) + \frac{\sigma_t^2}{2} \partial_x^2 f(t, X_t)]dt + \sigma_t \partial_x f(t, X_t) dW_t$$

6.3.1 When $F_t = f(t, X_t)$ and $dX_t = b(t, W_t)dt + \sigma(t, W_t)dt$

In the case that we now have a time dependent function, And X_t now satisfies the equation,

$$dX_t = b(t, W_t)dt + \sigma(t, W_t)dt$$

Ito's formula now becomes,

$$dF_t = [\partial_t f(t, X_t) + b(W_t, t)\partial_x f(t, X_t) + \frac{\sigma(W_t, t)^2}{2} \partial_x^2 f(t, X_t)]dt + \sigma(W_t, t)\partial_x f(t, X_t) dW_t$$

6.3.2 Ito Diffusions

If a stochastic process satisfies this SDE $dX_t = b(t, W_t)dt + \sigma(t, W_t)dt$, then it is known as an Ito Diffusion

7 Stochastic Integration Techniques

8 Table of Usual Stochastic Integrals

9 Stochastic Differential Equations

9.1 Finding Mean and Variance of a process