Geometric Brownian Motion

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1 Ito's Lemma

Let X_t be a stochastic process that satisfies

$$dX_t = b(t, W_t)dt + \alpha(t, W_t)dt$$

Further suppose we have $F_t = f(t, X_t)$ which is twice continuously differentiable. Then we shall use the following form of Ito's Lemma,

$$dF_t = [\partial_t f(t, X_t) + b(W_t, t)\partial_x f(t, X_t) + \frac{\sigma(W_t, t)^2}{2}\partial_x^2 f(t, X_t)]dt + \alpha(W_t, t)\partial_x f(t, X_t)dW_t$$

2 Solving The SDE

Consider the following SDE,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where W_t is a Wiener process or Brownian Motion, with σ, the percentage volatility, and μ, percentage drift, are constants. We should also note that $b(t, W_t) = \mu S_t$ and $\alpha(t, W_t) = \sigma S_t$ in our case To begin, we shall claim that $F_t = f(t, X_t) = ln(S_t)$. This comes from observing that the SDE can also be put into the form of

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

in which the right hand side looks like the derivative of a $ln(f(x))$.

Then the partial derivatives are

$$\frac{\partial f(t, X_t)}{\partial X_t} = \frac{1}{S_t} \qquad \frac{\partial^2 f(t, X_t)}{\partial X_t^2} = -\frac{1}{S_t^2} \qquad \frac{\partial f(t, X_t)}{\partial t} = 0 \tag{1}$$

Now we shall input these partial derivatives into Ito's Lemma given above to obtain,

$$dF_t = [0 + \mu S_t \frac{1}{S_t} - \frac{(\sigma S_t)^2}{2}\frac{1}{S_t^2}]dt + \sigma S_t \frac{1}{S_t} dW_t$$

$$= \mu dt + \sigma dW_t - \frac{\sigma^2}{2}dt$$

$$= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

At this point, we can integrate both sides,

$$\int_0^t dF_t = \int_0^t (\mu - \frac{\sigma^2}{2})dt + \int_0^t \sigma dW_t$$

The left hand side is solved by applying the Fundamental Theorem of Calculus, while the integral involving the Wiener process, we can use the identity $\int_a^b dW_t = W_b - W_a$ and noting that $W_0 = 0$ by definition. This equates to,

$$F_t = F_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t$$

Then since at the beginning we used the relation, $F_t = f(t, X_t) = ln(S_t)$, our solution becomes,

$$ln(S_t) = ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t$$

Thus, applying the an exponential to both sides to revert the log transformation, we now have,

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

2.1 Probability Distribution Function

Looking at the above solution, we can define GBM in the short hand way,

$$S_t = S_0 e^{X_t}$$

with $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ which is Brownian Motion with drift and scaling. Taking a logarithm will revert back into Brownian motion, with

$$X_t = ln(S_t/S_0) = ln(S_t) - ln(S_0)$$

$$ln(S_t) = ln(S_0) + X(t)$$

$$ln(S_t) = ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

This is our original form of the solution. At this point we should remind the reader of some key properties of Normal distributions (Note the notation of variables used below is not the same as above).

1. If $X \sim N(\mu, \sigma^2)$, then $Y = aX + b$ is also normally distributed with $Y \sim N(a\mu + b, a^2\sigma^2)$

2. If X_t has normal distribution, than $Y_t = e^{X_t}$ has lognormal distribution. Or another way of saying this, if X_t is log-normally distributed, then $Y_t = ln(X_t)$ has normal distribution, $\rightarrow X_t = e^{Y_t}$

3. A Wiener process is by definition is $\sim N(0,t)$ and has distribution

$$\phi_t(x) = \frac{1}{\sqrt{2t\pi}} e^{-\frac{x^2}{2t}}$$

Thus, going back to our $ln(S_t) = ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, we notice that the right hand side is really just the first property of normal distributions we mentioned above.

Thus, $ln(S_t) = ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ indicates $ln(S_t) \sim N(ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2\sqrt{t})$.

Then knowing that $ln(S_t) \sim N(ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2\sqrt{t})$, if we go back to the 2nd property above, we can then claim that S_t is log normally distributed.

In other words, we can now rewrite the equation just above as $ln(S_t) = F_t$ where $F_t \sim N(ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2\sqrt{t})$, meaning we can revert the log with and exponential and obtain,

$$S_t = e^{F_t} = e^{ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

Summarising,

$$ln(S_t) \sim N(ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2\sqrt{t}) \qquad S_t \sim lognormal(ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2\sqrt{t}) \tag{2}$$

With the PDF of S_t being,

$$p_t(S_t) = \frac{1}{S_t\sigma\sqrt{2\pi t}} exp(-\frac{[ln(S_t/S_0) - (\mu - \frac{1}{2}\sigma^2)t]^2}{2\sigma^2 t})$$

2.1.1 E$[S_t]$ and Var(S_t)

From standard results for log normal distributions we have,

$$E[x] = e^{(\mu + \frac{\sigma^2}{2})} \qquad Var(x) = (e^{\sigma^2} - 1)(e^{2\mu + \sigma^2}) \tag{3}$$

Thus, inputting our mean and variance of from the log normal probability distribution for S_t above gives,

$$E[x] = S_0 e^{\mu t} \qquad Var(x) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1) \tag{4}$$