Lecture 7
Introduction to SAT and SMT

This week

Topics:

- Constraint solvers
- Constraint-based search

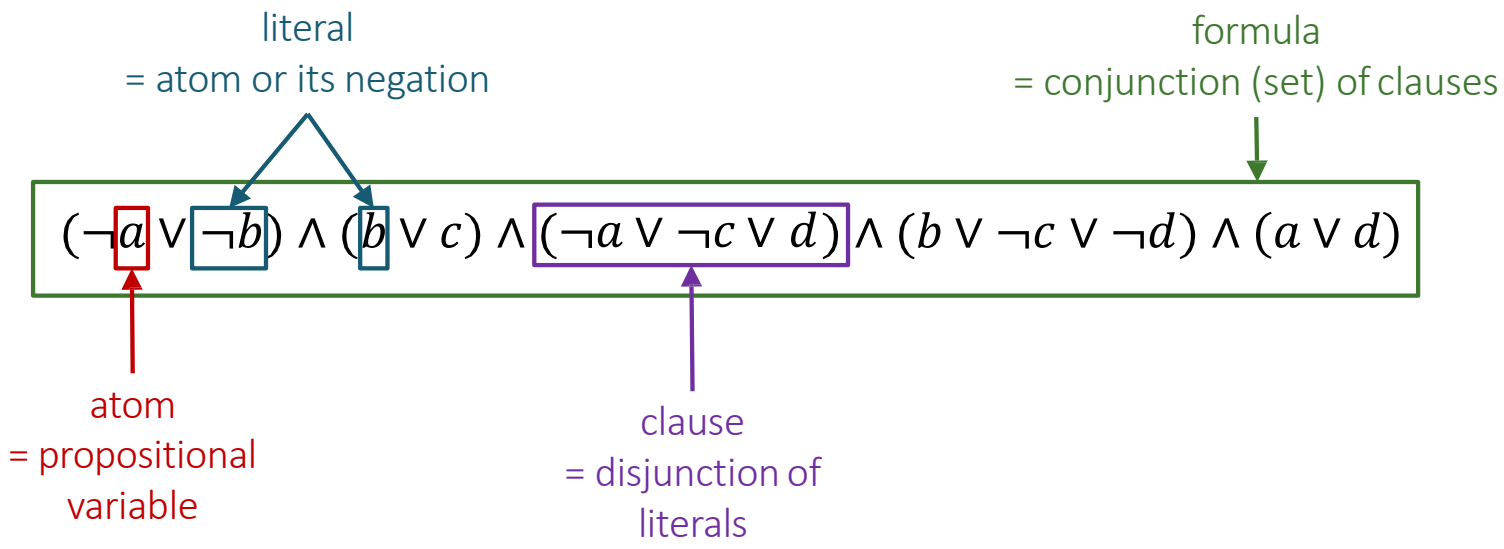
Paper: Sumit Gulwani, Susmit Jha, Ashish Tiwari, Ramarathnam Venkatesan: Synthesis of loop-free programs. PLDI'11

Why do we care?

1. Synthesis is combinatorial search, and so is SAT
2. SAT solvers are really good these days
3. Can use SAT solvers to build synthesizers

The SAT problem

Input: propositional formula in CNF



The SAT problem

Problem: find a *satisfying assignment* (also called a *model*)

- or determine that the formula is *unsatisfiable*

$$(\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg a \vee \neg c \vee d) \wedge (b \vee \neg c \vee \neg d) \wedge (a \vee d)$$

a satisfying assignment:

$\{a \mapsto 0, b \mapsto 1, c \mapsto 0, d \mapsto 1\}$

can be written as a set of literals:

$\{\neg a, b, \neg c, d\}$

or as a formula:

$\neg a \wedge b \wedge \neg c \wedge d$

Naive solution

$$(\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg a \vee \neg c \vee d) \wedge (b \vee \neg c \vee \neg d) \wedge (a \vee d)$$

Build a truth table!

- We can't do fundamentally better: it's an NP-complete problem
- But we can do way better in practice for common instances

$2^{|P|}$

0000	0
0001	0
0010	0
0011	0
0100	0
0101	1
0110	0
0111	1
…	

DPLL: example

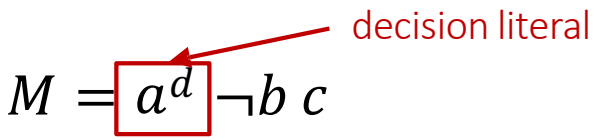
$$\boxed{(\neg a \vee \neg b)} \wedge (b \vee c) \wedge \boxed{(\neg a \vee \neg c \vee d)} \wedge (b \vee \neg c \vee \neg d) \wedge \boxed{(a \vee d)}$$

$M =$		
	$\emptyset$	decide
	a^d	unit-propagate
	$a^d \neg b$	unit-propagate
	$a^d \neg b\, c$	unit-propagate
	$a^d \neg b\, c\, d$	backtrack
	$\neg a$	unit-propagate
	$\neg a\, d$	decide
	$\neg a\, d\, b^d$	unit-propagate
	$\neg a\, d\, b^d \neg c$	SAT!

DPLL algorithm

[Davis, Putnam '60]
[Davis, Logemann, Loveland '62]

State: current model *M* (a sequence of annotated literals)

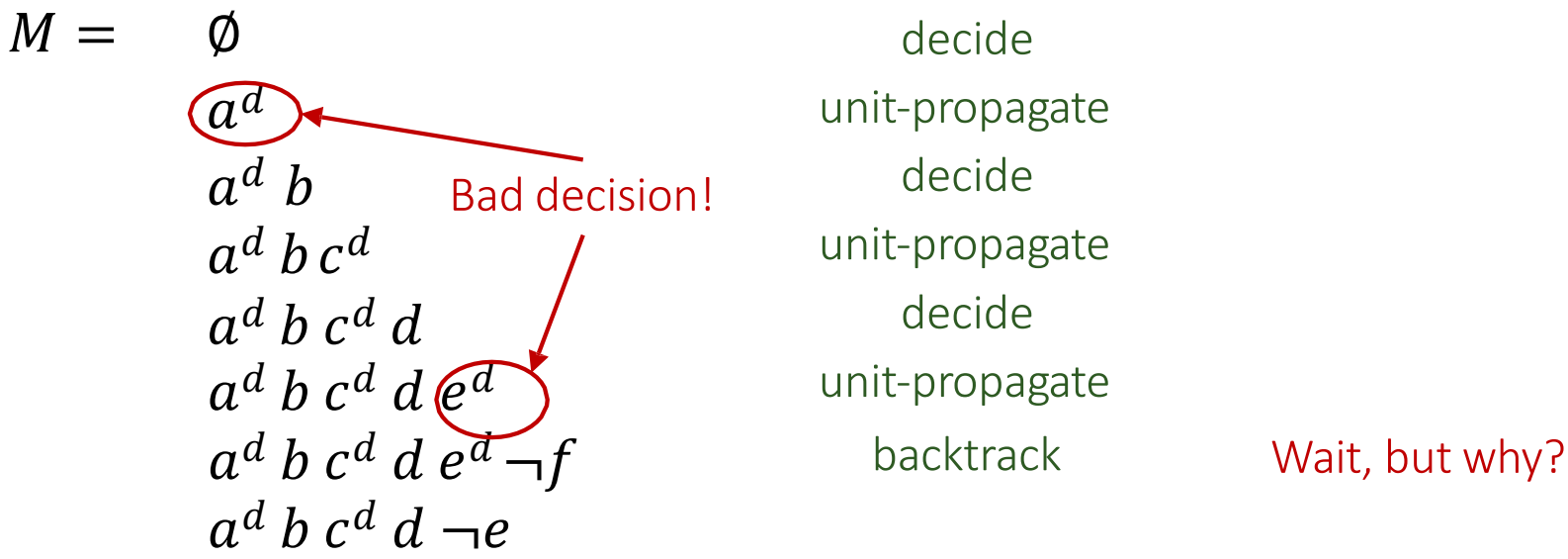


Transitions:

- decide $M \longrightarrow M\ l^d$ if *l* undefined in *M*
- unit-propagate $M \longrightarrow M\ l$ if there is a clause where all literals are false except *l*, which is undefined
- backtrack $M l^d M' \longrightarrow M\ \neg l$ if there is a conflicting clause and *M'* has no decision literals
- fail $M \longrightarrow Unsat$ if there is a conflicting clause and no decision literals

DPLL + clause learning

$$(\neg a \vee b) \wedge (\neg c \vee d) \wedge (\neg e \vee \neg f) \wedge (f \vee \neg b \vee \neg e) \wedge (\neg a \vee \neg e)$$



DPLL + clause learning

$$(\neg a \vee b) \wedge (\neg c \vee d) \wedge (\neg e \vee \neg f) \wedge (f \vee \neg b \vee \neg e) \wedge (\neg a \vee \neg e)$$

$M =$	$\emptyset$	decide
	a^d	unit-propagate
	$a^d\ b$	decide
	$a^d\ b\ c^d$	unit-propagate
	$a^d\ b\ c^d\ d$	decide
	$a^d\ b\ c^d\ d\ e^d$	unit-propagate
	$a^d\ b\ c^d\ d\ e^d\ \neg f$	backjump
	$a^d\ b\ \neg e$	

Beyond propositional logic

What if our formula looks like this?

$$(p \wedge \neg q \vee a = f(b - c)) \wedge (g(g(b) \neq c \vee a - c \leq 7)$$

- talks about integers, functions, sets, lists...

One idea: bit-blast everything and use SAT

- can only find solutions within bounds
- very inefficient, so bounds are small

Better idea: combine SAT with special solvers for theories

- they “natively understand” integers, functions, etc

First-order theories

theory = <function symbols, predicate symbols, axioms>

ground first-order formulas over functions and predicates

Example: theory of Equality and Uninterpreted Functions (EUF)

EUF = <{f, g, h, ...}, {=}, {

$$\forall x.\ x = x$$

$$\forall x\ y.\ x = y \ \Rightarrow\ y = x$$

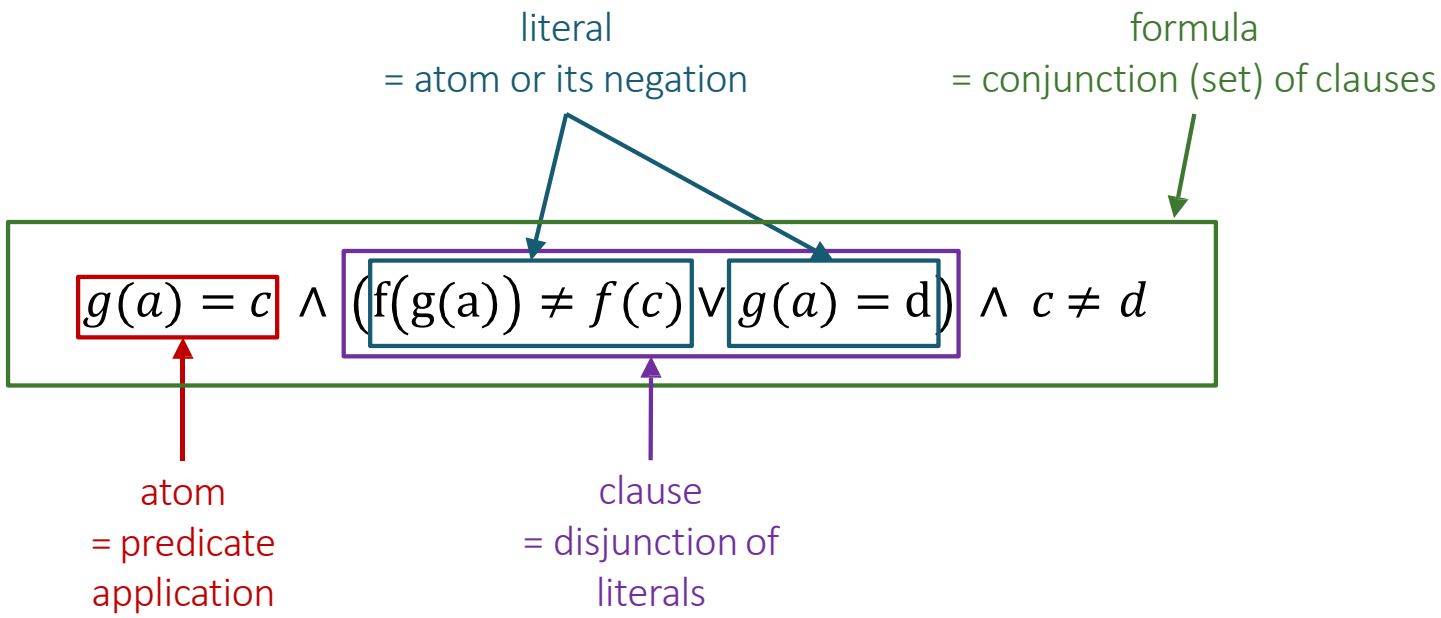
$$\forall x\ y\ z.\ x = y \wedge y = z \ \Rightarrow\ x = z$$

$$\forall x\ y.\ x = y \ \Rightarrow\ f(x) = f(y)$$

}>

The SMT problem

theory +



Theories for our purpose

theory = <function symbols, predicate symbols, ~~axioms~~>

solver

can decide consistency of conjunctions of literals

$f(a) = c$

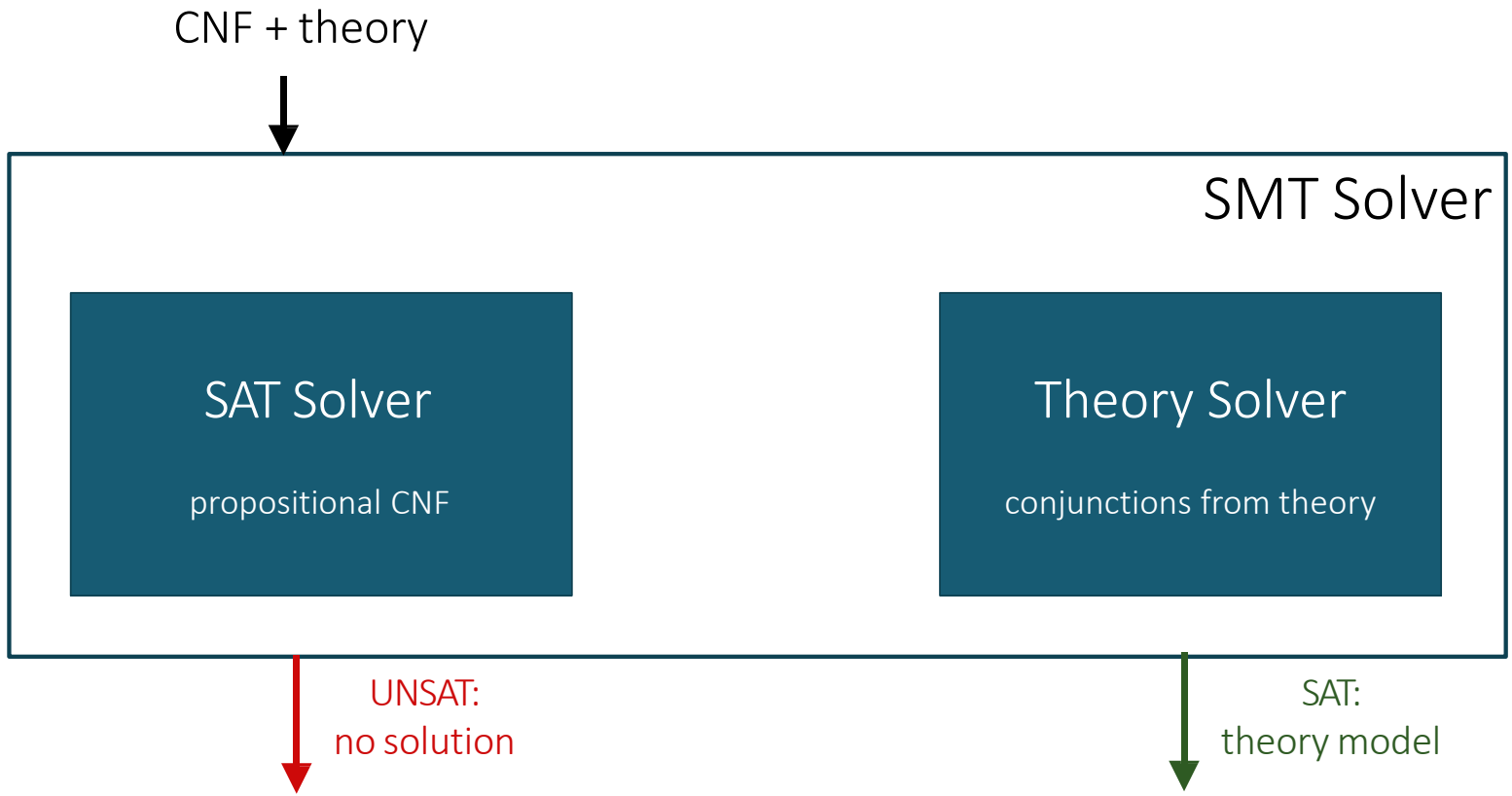
$f(b) \neq d$

$c = d$

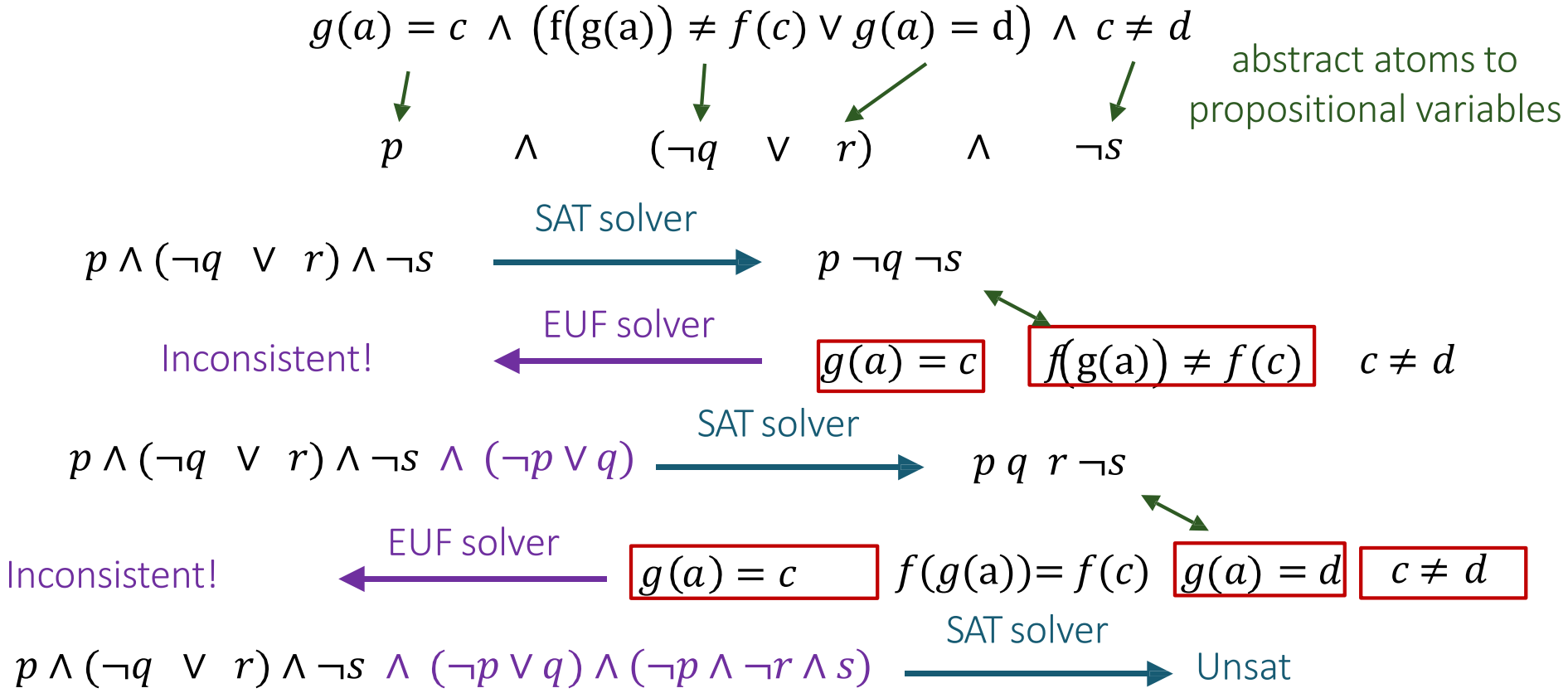
$a = b$

EUF solver → Inconsistent!

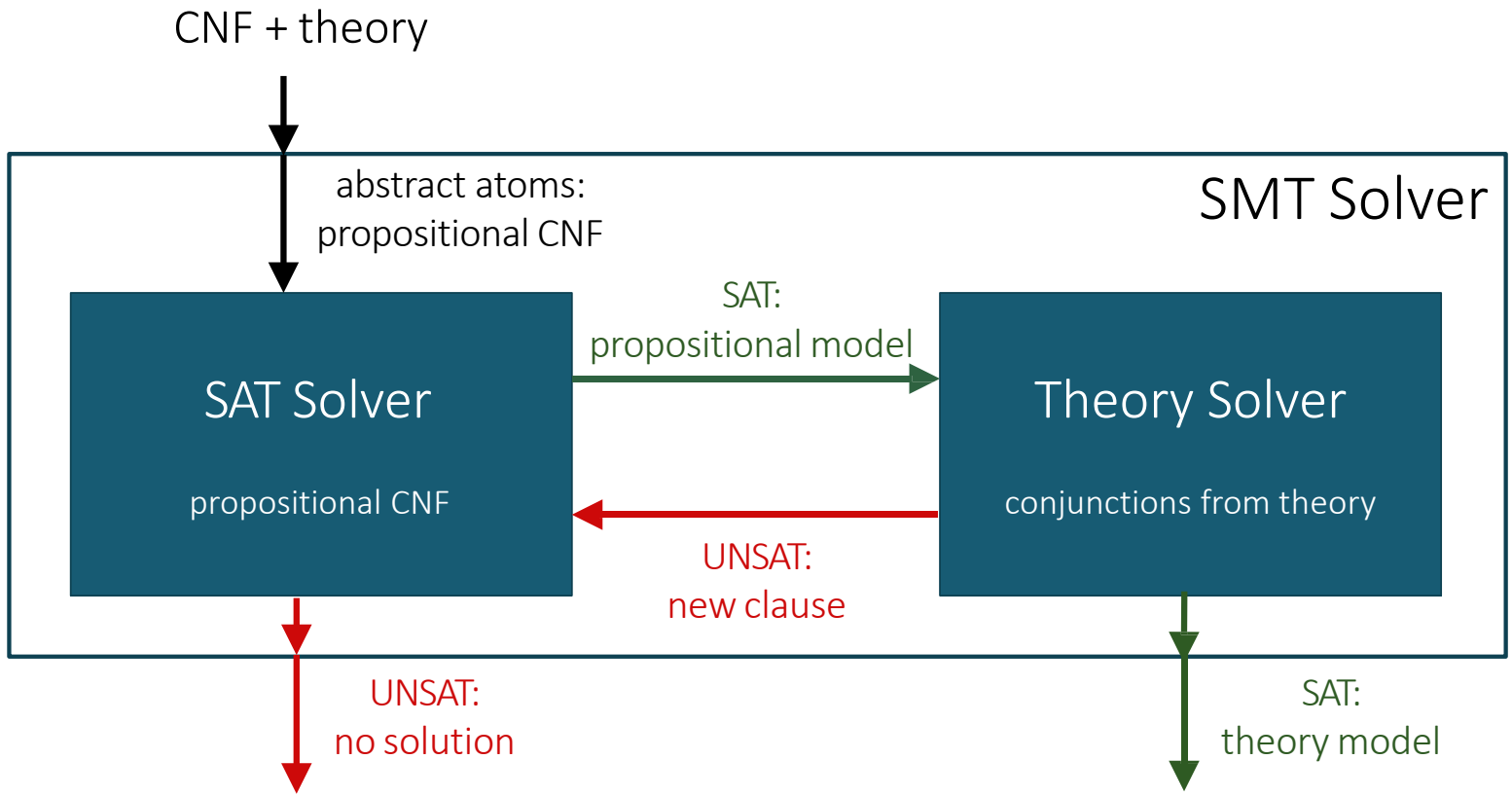
DPLL(T) architecture



Basic DPLL(T)


$g(a) = c \wedge (f(g(a)) \neq f(c) \vee g(a) = d) \wedge c \neq d$
abstract atoms to propositional variables
$p \wedge (\neg q \vee r) \wedge \neg s$
SAT solver
$p\ \neg q\ \neg s$
EUF solver
Inconsistent!
$g(a) = c$
$f(g(a)) \neq f(c)$
$c \neq d$
$p \wedge (\neg q \vee r) \wedge \neg s \wedge (\neg p \vee q)$
SAT solver
$p\ q\ r\ \neg s$
EUF solver
Inconsistent!
$g(a) = c$
$f(g(a)) = f(c)$
$g(a) = d$
$c \neq d$
$p \wedge (\neg q \vee r) \wedge \neg s \wedge (\neg p \vee q) \wedge (\neg p \wedge \neg r \wedge s)$
SAT solver
Unsat

DPLL(T) architecture



DPLL(T) optimizations

Basic

- Check consistency of full propositional models
- Upon inconsistency, add clause and restart
- Check consistency after adding a literal

Advanced

- Check consistency of partial assignment being built
- Upon inconsistency, do conflict analysis and backjump
- Add a theory-propagate rule to DPLL
  - like unit-propagate, but infers all literals that follow from the theory

Popular theories

Equality and Uninterpreted Functions

EUF = <{f, g, h, ...}, {=}, axioms of equality & congruence>

Linear Integer Arithmetic

LIA = <{0, 1, ..., +, -}, {=, ≤}, axioms of arithmetic>

Arrays

Arrays = <{sel, store}, {=}, $\forall a\ i\ v.\ \mathrm{sel}(\mathrm{store}(a, i, v), i) = v$

$\forall a\ i\ j\ v.\ i \neq j \ \Rightarrow\ \mathrm{sel}(\mathrm{store}(a, i, v), j) = \mathrm{sel}(a, j)$ >

Theories can be combined!

Nelson-Oppen combination

Popular SMT solvers

Z3 (Microsoft): https://github.com/Z3Prover/z3/wiki

CVC4 (Stanford): http://cvc4.cs.stanford.edu/web/

Yices (SRI): http://yices.csl.sri.com/

Boolector (JKU Austria): https://boolector.github.io/

SMT-LIB

Uniform format for SMT problems understood by all solvers

```
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (> x 0))
(assert (> y 0))
(assert (> z 0))
(assert (> (* 2 x) (+ y z)))
(check-sat)
(get-model)
```

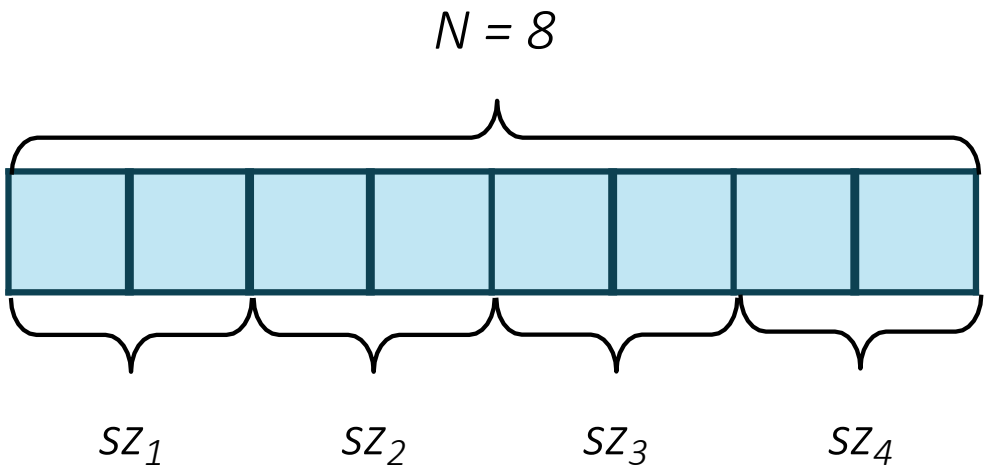
Z3 demo

https://rise4fun.com/Z3/3XCz

https://rise4fun.com/Z3/tutorial

Example: Array Partitioning

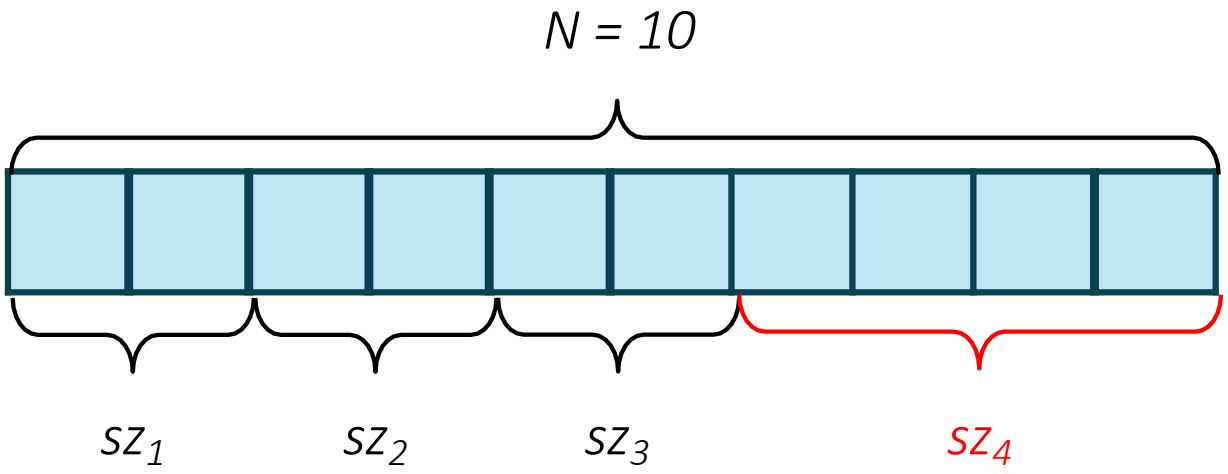
Partition an array of size *N* evenly into *P* sub-ranges



P = 4

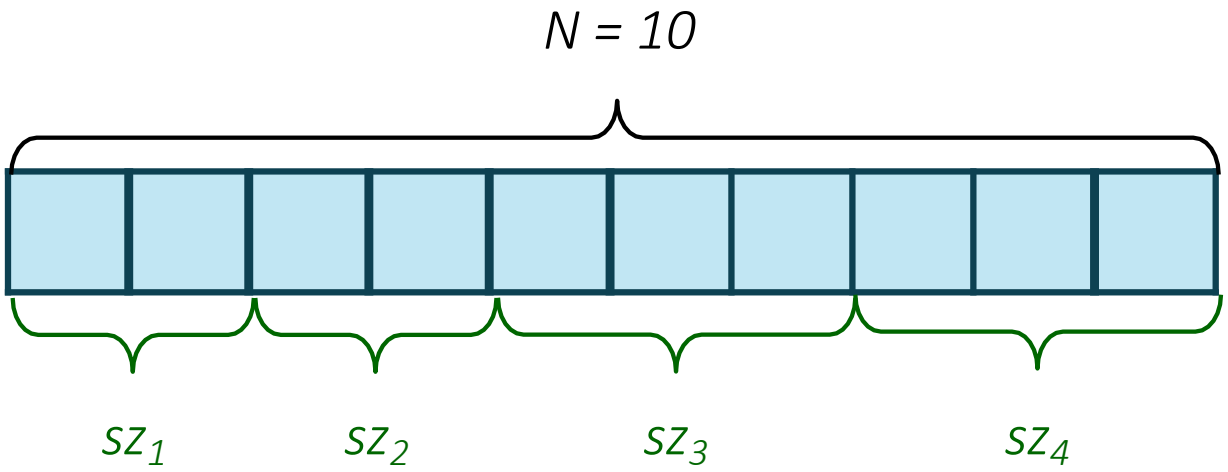
Example: Array Partitioning

Partition an array of size *N* evenly into *P* sub-ranges



Example: Array Partitioning

Partition an array of size *N* evenly into *P* sub-ranges



Can we always make them differ by at most 1?

Why do we care?

If we can encode a synthesis problem as SAT/SMT, we can use solvers to do the search for us

Get some inspiration from how solvers search

- Unit propagation similar to top-down propagation (pruning through inference of consequences of a guess)
- Backjumping / clause learning?
  - Feng, Martins, Bastani, Dillig: Program synthesis using conflict-driven learning. PLDI'18
- Coarse-grained reasoning and gradual refinement like in DPLL(T)?
  - Wang, Dillig, Singh: Program synthesis using abstraction refinement. POPL'18