CSCI 420 Computer Graphics

Lecture 20

Quaternions and Rotations

Rotations

Quaternions

Motion Capture

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Rotations

- Very important in computer animation and robotics
- Joint angles, rigid body orientations, camera parameters
- 2D or 3D

Rotations in Three Dimensions

- Orthogonal matrices:

$$RR^{\mathrm{T}} = R^{\mathrm{T}}R = I$$
$$\det(R) = 1$$

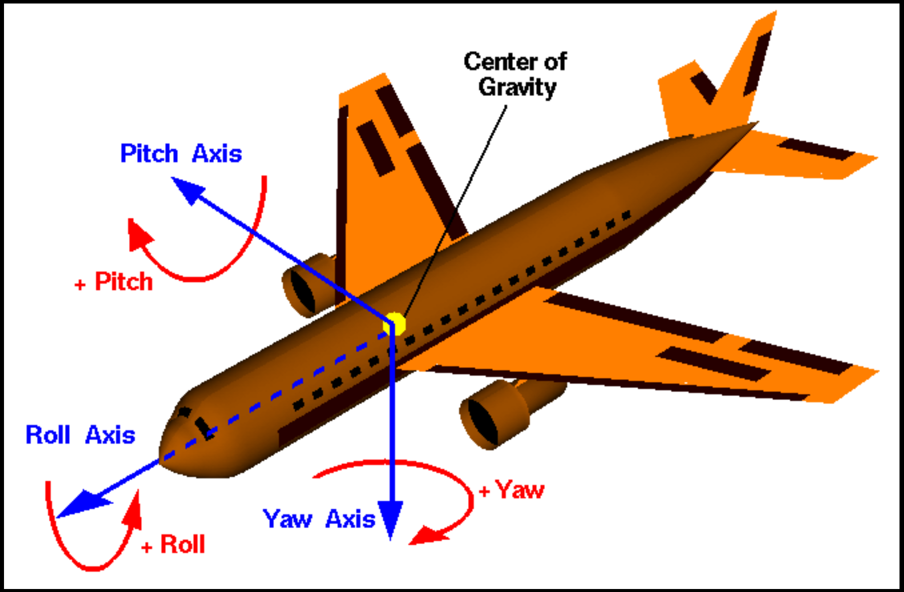
$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Representing Rotations in 3D

- Rotations in 3D have essentially three parameters
- Axis + angle (2 DOFs + 1DOFs)
  – How to represent the axis?
    Longitude / lattitude have singularities
- 3x3 matrix
  – 9 entries (redundant)

Representing Rotations in 3D

- Euler angles
  - roll, pitch, yaw
  - no redundancy (good)
  - gimbal lock singularities



Source: Wikipedia

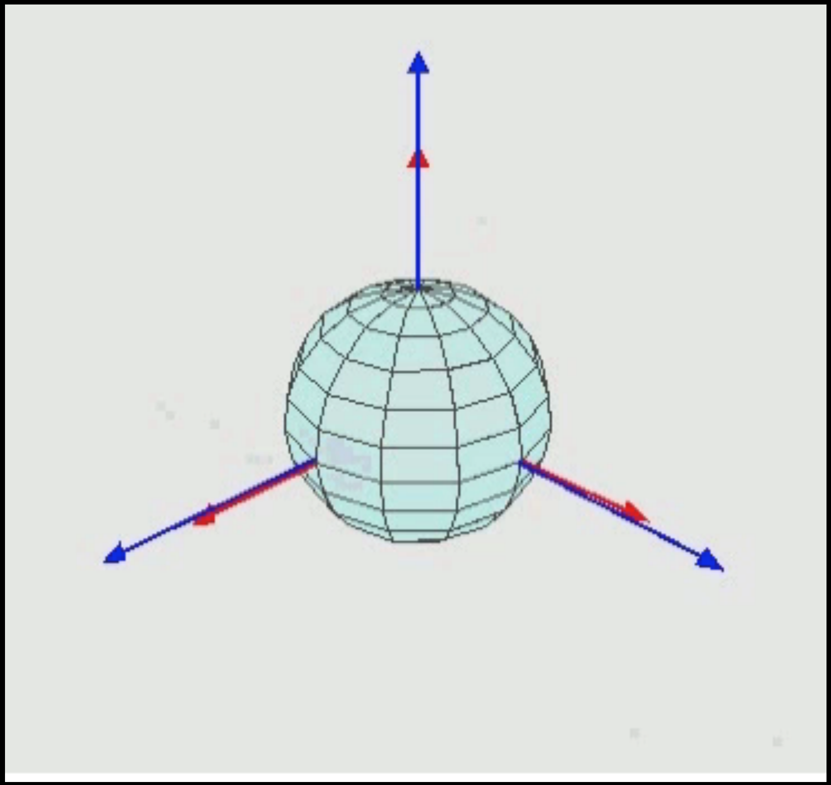
- Quaternions
  - generally considered the “best” representation
  - redundant (4 values), but only by one DOF (not severe)
  - stable interpolations of rotations possible

Euler Angles

1. Yaw
   rotate around y-axis

2. Pitch
   rotate around (rotated) x-axis

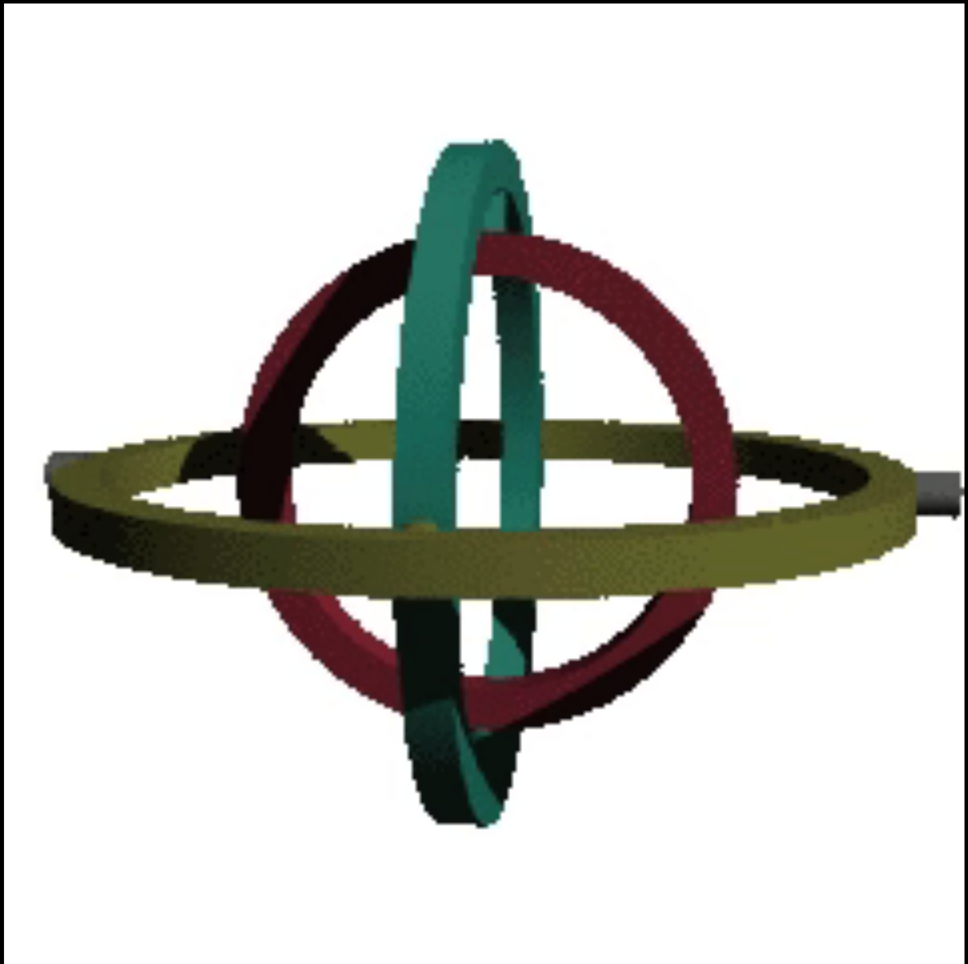
3. Roll
   rotate around (rotated) y-axis



Source: Wikipedia

Gimbal Lock

When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.



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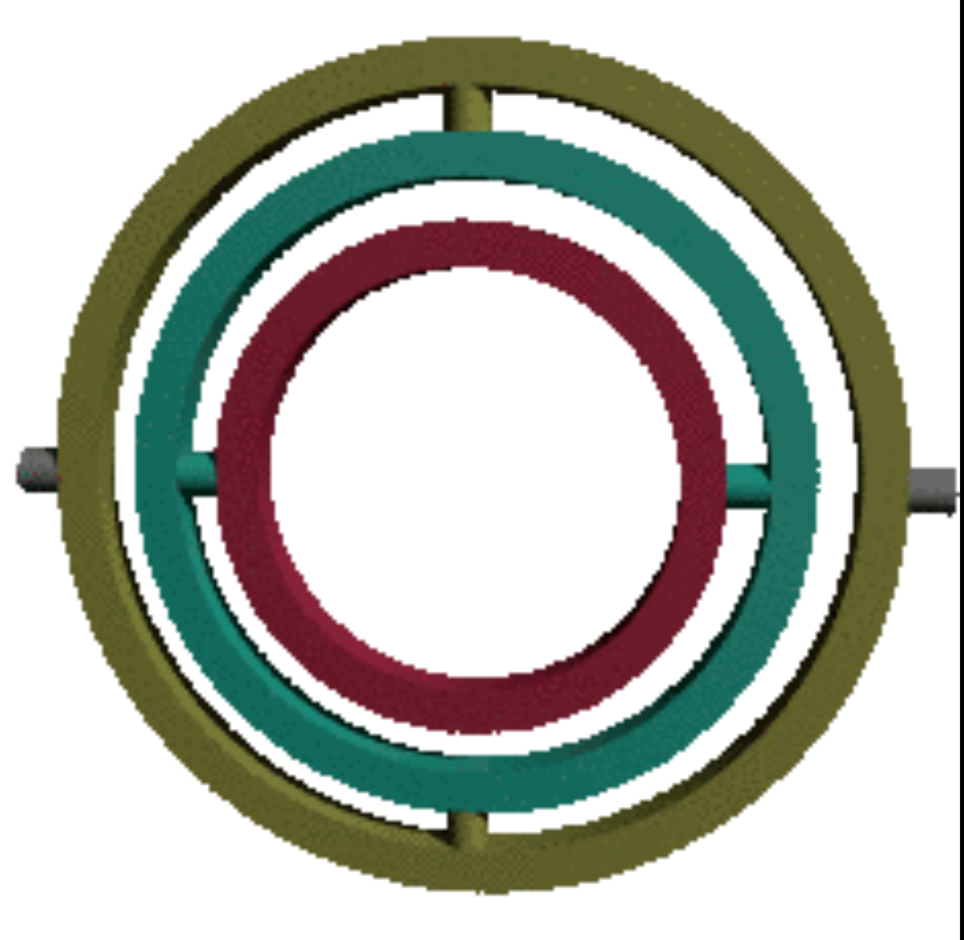

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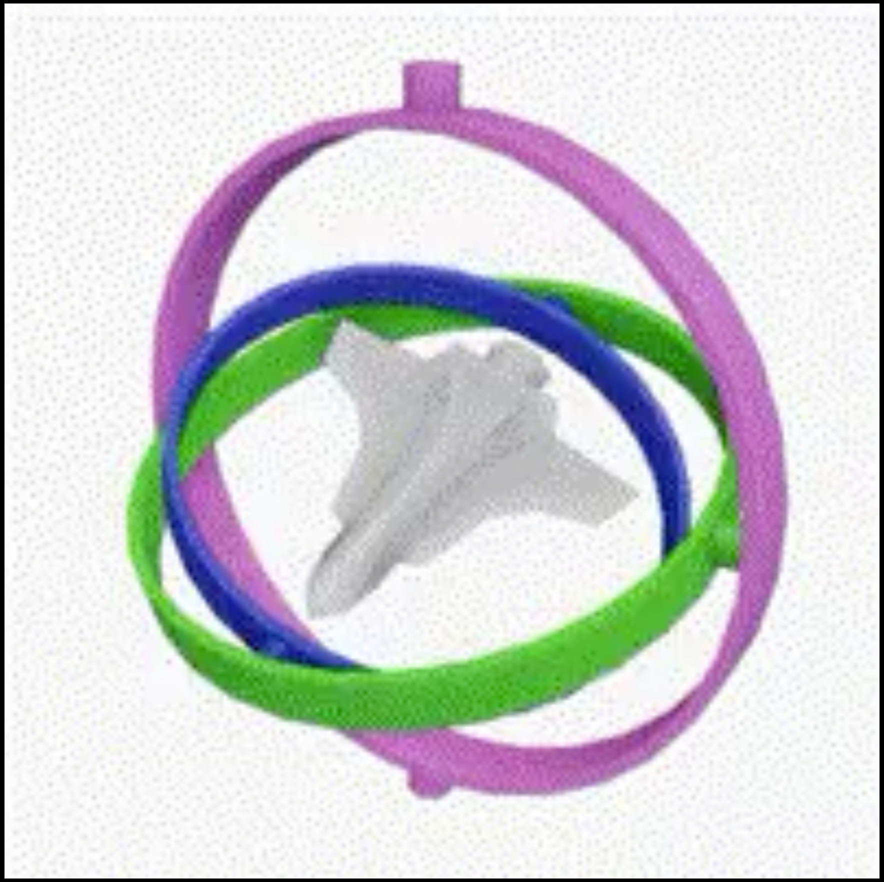


Source: Wikipedia

Outline

- Rotations
- Quaternions
- Motion Capture

Complex numbers

- $i = \sqrt{-1}$
- $z = x + iy$ (complex = real + i*imaginary)

- Solves lots of problems of normal arithmetic and algebra...

Complex numbers

- $i = \sqrt{-1}$
- $z = x + iy$ (complex = real + i*imaginary)
- All complex numbers are also 2D coordinates

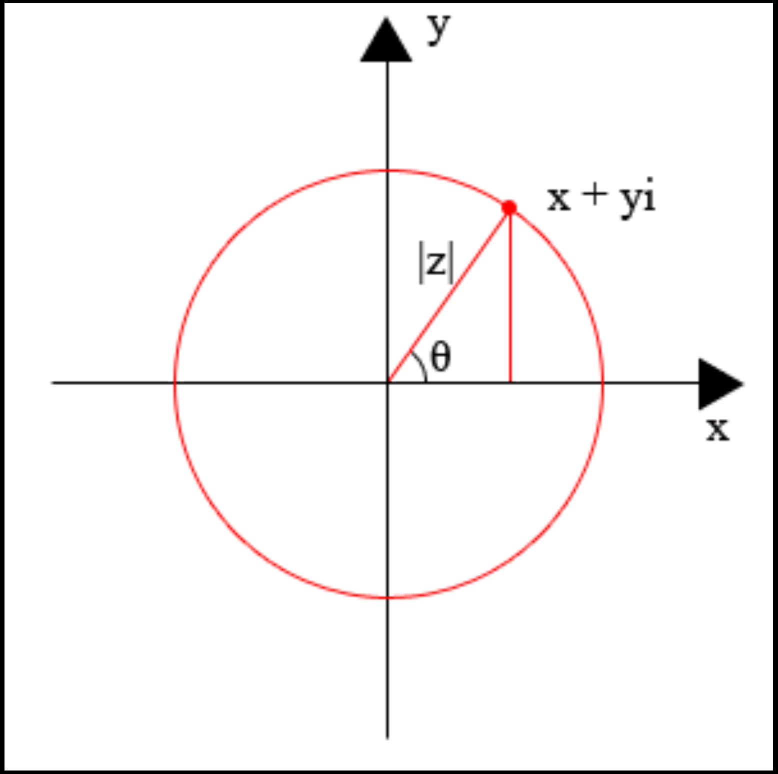


image from Wikipedia

- $|z|$ modulus
- θ argument

Quaternions

- Generalization of complex numbers

- Three imaginary numbers: ***i***, ***j***, ***k***

  $\boldsymbol{i}^2 = -1,\ \boldsymbol{j}^2 = -1,\ \boldsymbol{k}^2 = -1,$
  $\boldsymbol{ij} = \boldsymbol{k},\ \boldsymbol{jk} = \boldsymbol{i},\ \boldsymbol{ki} = \boldsymbol{j},\ \boldsymbol{ji} = -\boldsymbol{k},\ \boldsymbol{kj} = -\boldsymbol{i},\ \boldsymbol{ik} = -\boldsymbol{j}$

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, s,x,y,z are scalars

Quaternions

- Invented by Hamilton in 1843 in Dublin, Ireland

- Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

  $i^2 = j^2 = k^2 = i\ j\ k = -1$

  & cut it on a stone of this bridge.

Source: Wikipedia

Quaternions

- Quaternions are **not** commutative!

  $q_1 \, q_2 \neq q_2 \, q_1$

- However, the following hold:

  $(q_1 \, q_2) \, q_3 = q_1 \, (q_2 \, q_3)$
  $(q_1 + q_2) \, q_3 = q_1 \, q_3 + q_2 \, q_3$
  $q_1 \, (q_2 + q_3) = q_1 \, q_2 + q_1 \, q_3$
  $\alpha \, (q_1 + q_2) = \alpha \, q_1 + \alpha \, q_2$ (α is scalar)
  $(\alpha q_1) \, q_2 = \alpha \, (q_1 q_2) = q_1 \, (\alpha q_2)$ (α is scalar)

- I.e. all usual manipulations are valid, except cannot reverse multiplication order.

Quaternions

- Exercise: multiply two quaternions

  $(2 - \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})\ (-1 + \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) = \ldots$

Quaternion Properties

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$
- Norm: $|q|^2 = s^2 + x^2 + y^2 + z^2$
- Conjugate quaternion: $\bar{q} = s - x\,\boldsymbol{i} - y\,\boldsymbol{j} - z\,\boldsymbol{k}$
- Inverse quaternion: $q^{-1} = \bar{q} / |q|^2$
- Unit quaternion: $|q| = 1$
- Inverse of unit quaternion: $q^{-1} = \bar{q}$

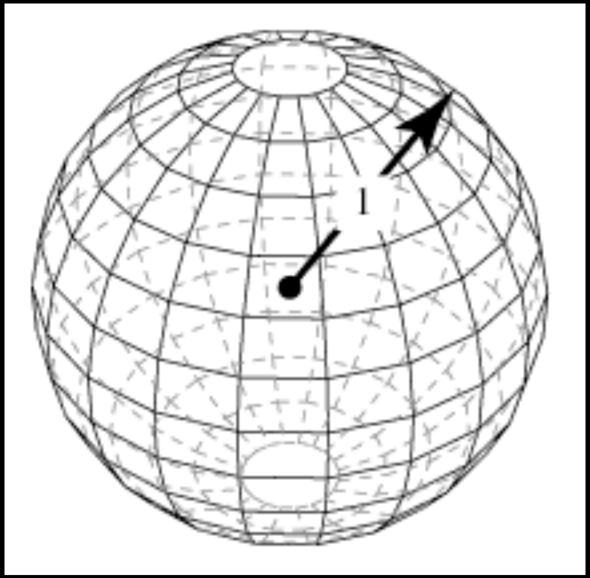
Quaternions and Rotations

- Rotations are represented by ***unit*** quaternions

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

  $s^2 + x^2 + y^2 + z^2 = 1$

- Unit quaternion sphere (unit sphere in 4D)



Source:
Wolfram Research

unit sphere
in 4D

Rotations to Unit Quaternions

- Let (unit) rotation axis be $[u_x, u_y, u_z]$, and angle θ
- Corresponding quaternion is

  $$q = \cos(\theta/2) + \sin(\theta/2)\, u_x\, \boldsymbol{i} + \sin(\theta/2)\, u_y\, \boldsymbol{j} + \sin(\theta/2)\, u_z\, \boldsymbol{k}$$

- Composition of rotations q_1 and q_2 equals $q = q_2\, q_1$
- 3D rotations do not commute!

Unit Quaternions to Rotations

- Let v be a (3-dim) vector and let q be a unit quaternion

- Then, the corresponding rotation transforms vector v to q ***v*** q^{-1}

  (***v*** is a quaternion with scalar part equaling 0, and vector part equaling v)

  For q = a + b ***i*** + c ***j*** + d ***k***

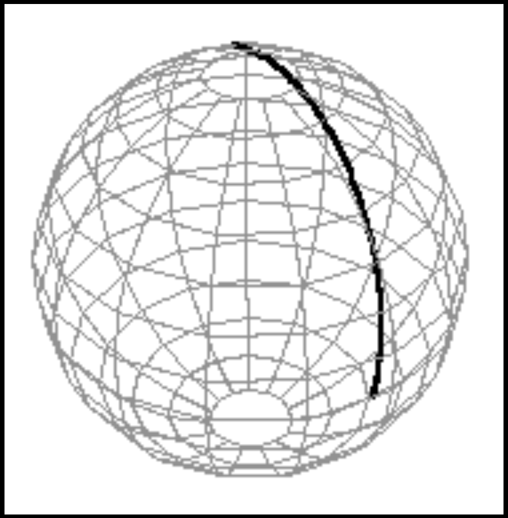
$$R = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Quaternions

- Quaternions q and -q give the same rotation!

- Other than this, the relationship between rotations and quaternions is unique

Quaternion Interpolation



Source:
Wolfram Research

- Better results than Euler angles
- A quaternion is a point on the 4-D unit sphere
  - interpolating rotations requires a unit quaternion at each step -- another point on the 4-D sphere
  - move with constant angular velocity along the great circle between the two points
  - Spherical Linear intERPolation (SLERPing)
- Any rotation is given by 2 quaternions, so pick the shortest SLERP

Quaternion Interpolation

- To interpolate more than two points:
  - solve a non-linear variational constrained optimization (numerically)

- Further information: Ken Shoemake in the SIGGRAPH '85 proceedings (Computer Graphics, V. 19, No. 3, P.245)

Outline

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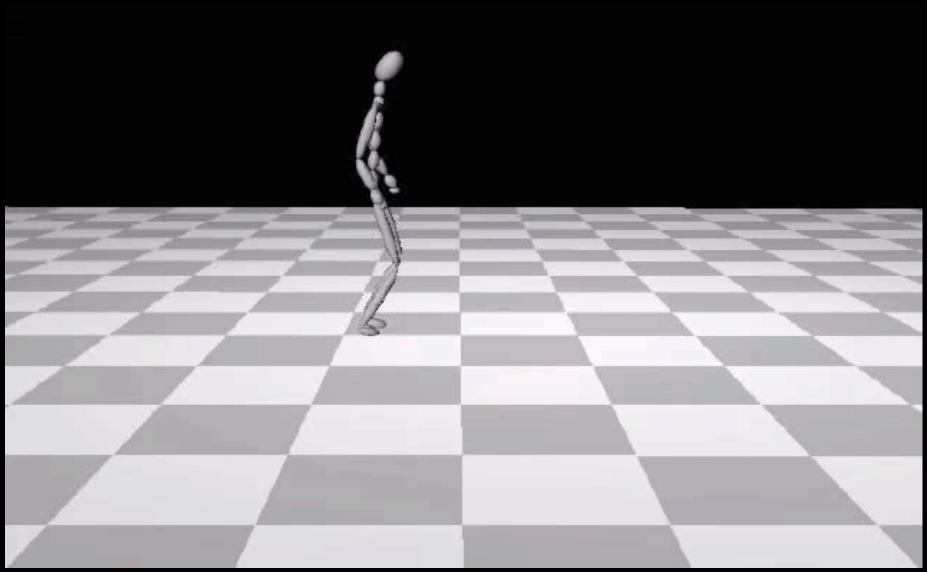
- Motion Capture

What is Motion Capture?

- Motion capture is the process of tracking real-life motion in 3D and recording it for use in any number of applications.

Why Motion Capture?

- Keyframes are generated by instruments measuring a human performer — they do not need to be set manually
- The details of human motion such as style, mood, and shifts of weight are reproduced with little effort



Mocap Technologies: Optical

- Multiple high-resolution, high-speed cameras
- Light bounced from camera off of reflective markers
- High quality data
- Markers placeable anywhere
- Lots of work to extract joint angles
- Occlusion
- Which marker is which? (correspondence problem)
- 120-240 Hz @ 1Megapixel

Facial Motion Capture

Mocap Technologies: Electromagnetic

- Sensors give both position and orientation
- No occlusion or correspondence problem
- Little post-processing
- Limited accuracy

Mocap Technologies: Exoskeleton

- Really Fast (~500Hz)
- No occlusion or correspondence problem
- Little error
- Movement restricted
- Fixed sensors

Motion Capture

- Why not?
  - Difficult for non-human characters
    - Can you move like a hamster / duck / eagle ?
    - Can you capture a hamster’s motion?
  - Actors needed
    - Which is more economical:
      - Paying an animator to place keys
      - Hiring a Martial Arts Expert

When to use Motion Capture?

- Complicated character motion
  - Where “uncomplicated” ends and “complicated” begins is up to question
  - A walk cycle is often more easily done by hand
  - A Flying Monkey Kick might be worth the overhead of mocap
- Can an actor better express character personality than the animator?

Summary

- Rotations
- Quaternions
- Motion Capture