CSCI 420 Computer Graphics
Lecture 16

Geometric Queries for Ray Tracing

Ray-Surface Intersection
Barycentric Coordinates
[Angel Ch. 11]

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Backward Ray Tracing

- Main components of the backward ray tracing algorithm:

1. For each pixel (x,y), fire a ray from COP through (x,y)
2. For each ray & object, calculate closest intersection
3. For closest intersection point **p**
   - Calculate surface normal
   - For each light source, fire shadow ray
   - For each unblocked shadow ray, evaluate local Phong model for that light, and add the result to pixel color

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But how do we calculate the closest intersection?

How do we calculate occlusions for shadow rays?

Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
  - Spheres
  - Planes
  - Polygons
  - Quadrics

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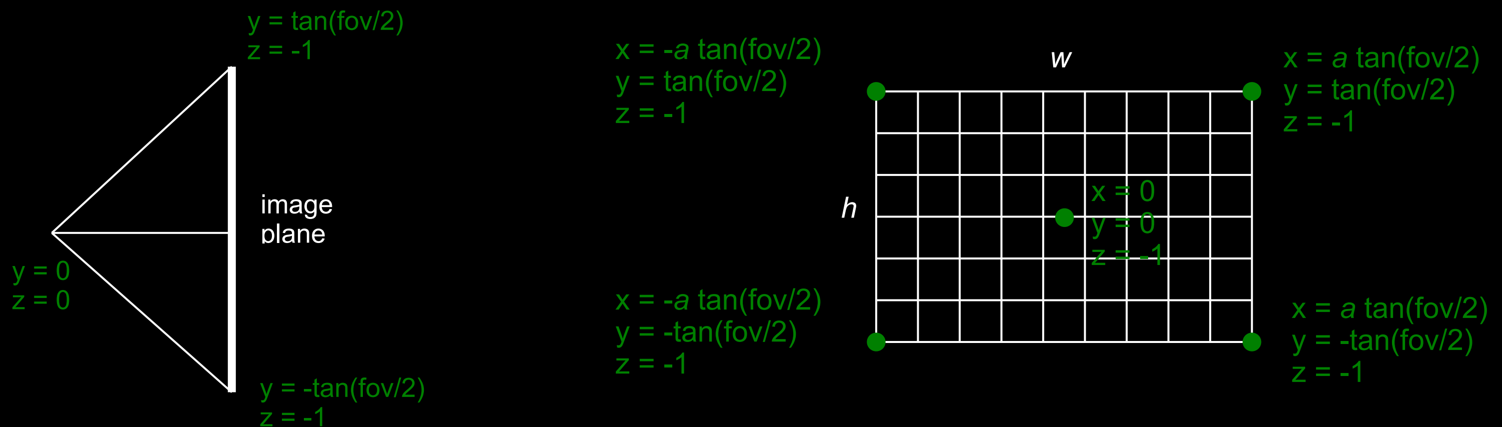
- **Triangles**

Mathematical parametrization of rays

- Ray in parametric form
  - Origin $\mathbf{p}_0 = [x_0 \;\; y_0 \;\; z_0]^T$
  - Direction $\mathbf{d} = [x_d \;\; y_d \;\; z_d]^T$
  - Assume $\mathbf{d}$ is normalized $(x_d^2 + y_d^2 + z_d^2 = 1)$
  - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d}\, t$ for $t > 0$

- Remember from last lecture:



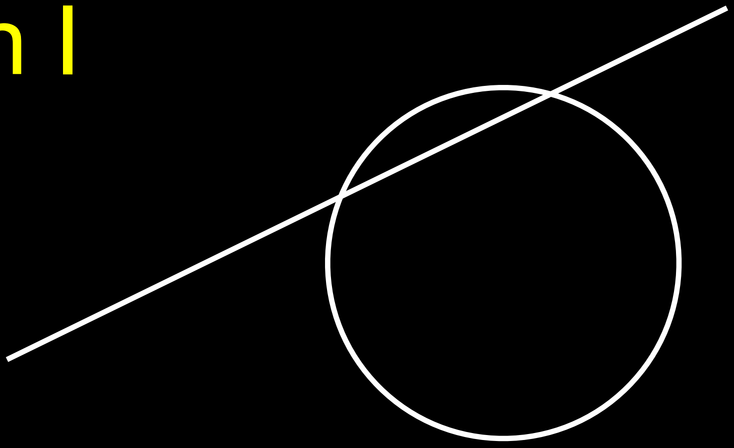
Intersection of Rays and Parametric Surfaces

- Ray in parametric form
  - Origin $\mathbf{p}_0 = [x_0 \ \ y_0 \ \ z_0]^T$
  - Direction $\mathbf{d} = [x_d \ \ y_d \ \ z_d]^T$
  - Assume $\mathbf{d}$ is normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
  - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d}\, t$ for $t > 0$
- Surface in parametric form
  - Point $\mathbf{q} = g(u, v)$, possible bounds on u, v
  - Solve $\mathbf{p}_0 + \mathbf{d}\, t = g(u, v)$
  - Three equations in three unknowns (t, u, v)

Intersection of Rays and Implicit Surfaces

- Ray in parametric form
  - Origin $\mathbf{p}_0 = [x_0 \;\; y_0 \;\; z_0]^T$
  - Direction $\mathbf{d} = [x_d \;\; y_d \;\; z_d]^T$
  - Assume $\mathbf{d}$ normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
  - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d}\, t$ for $t > 0$
- Implicit surface
  - Given by $f(\mathbf{q}) = 0$
  - Consists of all points $\mathbf{q}$ such that $f(\mathbf{q}) = 0$
  - Substitute ray equation for $\mathbf{q}$: $f(\mathbf{p}_0 + \mathbf{d}\, t) = 0$
  - Solve for t (univariate root finding)
  - Closed form (if possible), otherwise numerical approximation

Ray-Sphere Intersection I



- Common and easy case
- Define sphere by
  - Center **c** = $[x_c \ \ y_c \ \ z_c]^T$
  - Radius r
  - Surface f(**q**) = $(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
- Plug in ray equations for x, y, z:

$$x = x_0 + x_d t, \qquad y = y_0 + y_d t, \qquad z = z_0 + z_d t$$

- And we obtain a scalar equation for *t*:

$$(x_0 + x_d t - x_c)^2 + (y_0 + y_d t - y_c)^2 + (z_0 + z_d t - z_c)^2 = r^2$$

Ray-Sphere Intersection II

- Simplify to

$$at^2 + bt + c = 0$$

where

$$a = x_d^2 + y_d^2 + z_d^2 = 1 \qquad \text{since } |d| = 1$$
$$b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c))$$
$$c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2$$

- Solve to obtain t_0 and t_1

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Check if t_0, t_1 > 0 (ray)
Return min(t_0, t_1)

Ray-Sphere Intersection III

- For lighting, calculate unit normal

$$n = \frac{1}{r}\left[(x_i - x_c) \quad (y_i - y_c) \quad (z_i - z_c)\right]^T$$

- Negate if ray originates inside the sphere!
  - (what should we do if the ray originates inside the sphere?)
- Note possible problems with roundoff errors

Simple Optimizations

- Factor common subexpressions
- Compute only what is necessary
  - Calculate $b^2 - 4c$, abort if negative
  - Compute normal only for closest intersection
  - Other similar optimizations

$$at^2 + bt + c = 0$$

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

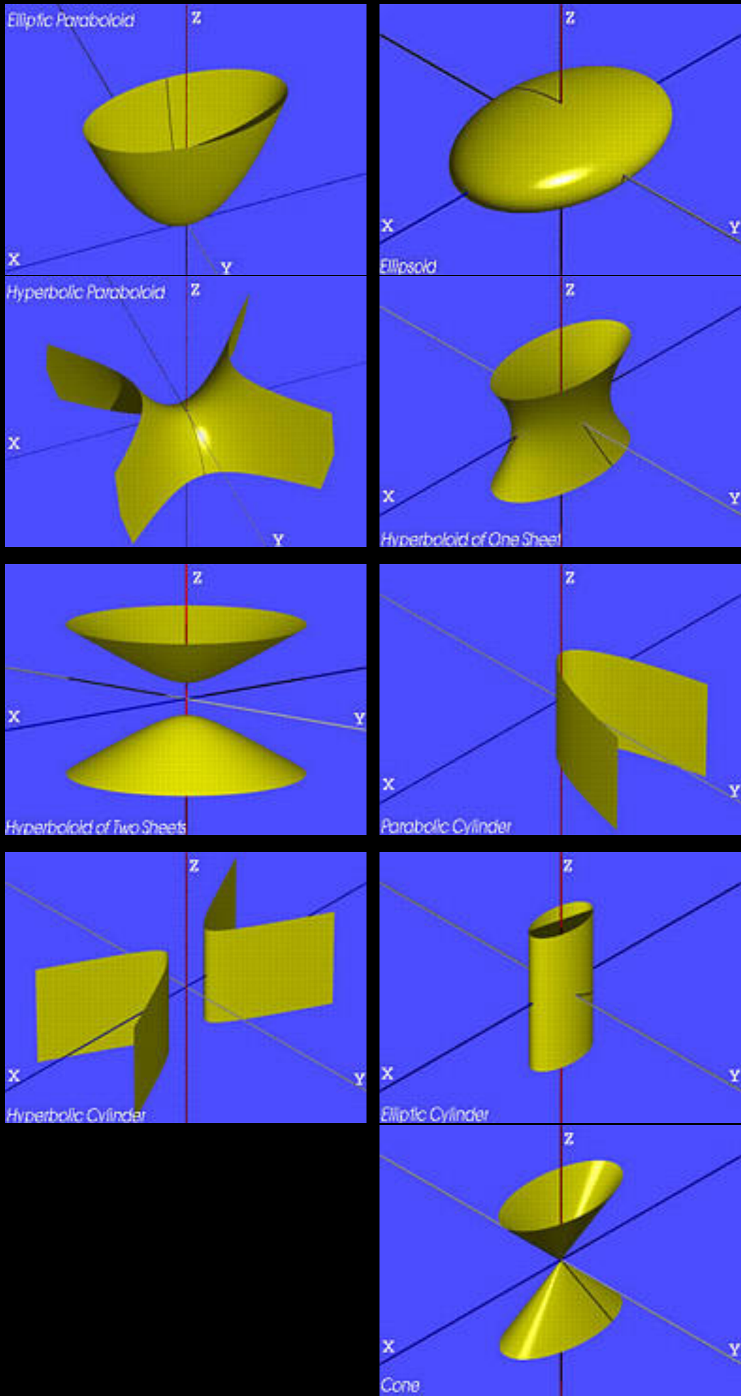
$$a = x_d^2 + y_d^2 + z_d^2 = 1 \qquad \text{since } |d| = 1$$

$$b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c))$$

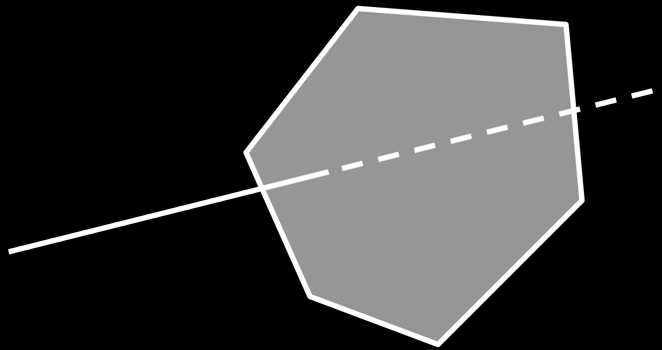
$$c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2$$

Ray-Quadric Intersection

- Quadric f(**p**) = f(x, y, z) = 0,
  where f is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG
- (not on final)



Ray-Polygon Intersection I



- Assume planar polygon in 3D
  1. Intersect ray with plane containing polygon
  2. Check if intersection point is inside polygon
- Plane
  - Implicit form: $ax + by + cz + d = 0$
  - Unit normal: $\mathbf{n} = [a \;\; b \;\; c]^T$ with $a^2 + b^2 + c^2 = 1$
- Substitute:

$$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$

- Solve:

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$

Ray-Polygon Intersection II

- Substitute *t* to obtain intersection point in plane

d coefficient
of plane

- Rewrite using dot product

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}$$

ray
direction

- **n • d** = 0 - danger of division by 0?

Ray-Polygon Intersection II

- Substitute *t* to obtain intersection point in plane
- Rewrite using dot product

d coefficient of plane

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}$$

ray direction

- If **n** • **d** = 0, no intersection (ray parallel to plane)
- If t ≤ 0, the intersection is behind ray origin

Test if point inside polygon

- Could use even-odd rule, or winding rule
- Easier if polygon is in 2D
  (project from 3D to 2D)
- Easier for triangles (tessellate polygons)

Test if point inside polygon

- Could use even-odd rule, or winding rule
- Easier if polygon is in 2D
  (project from 3D to 2D)
- Easier for triangles (tessellate polygons)
- In practice, **everyone tesselates**
  - This also solves problem that polygon might not be completely planar...

Point-in-triangle testing

- Critical for polygonal models

- Project the triangle, and point of plane intersection, onto one of the planes
  $x = 0$, $y = 0$, or $z = 0$
  (pick a plane not perpendicular to triangle)
  (such a choice always exists)

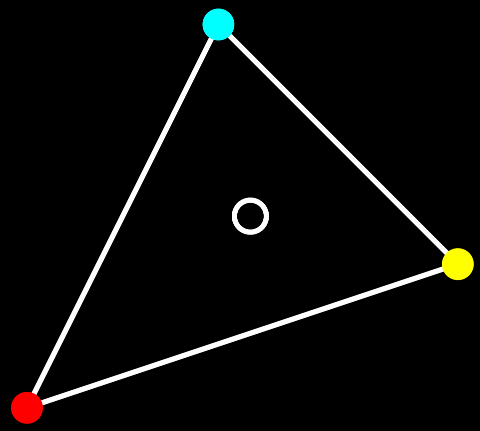
- Then, do the 2D test in the plane, by computing barycentric coordinates

Outline

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates

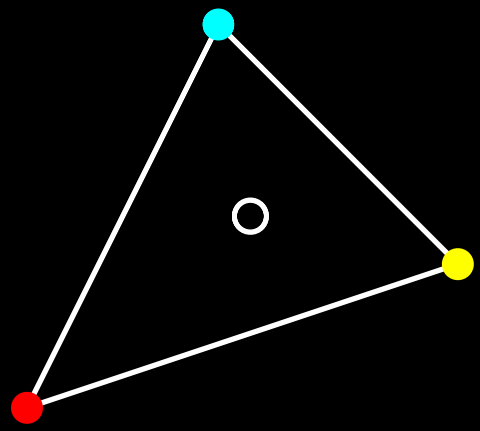
Barycentric coordinates - what are they good for?

- Barycentric coordinates will allow us to express a point in the interior of a triangle using the coordinates of the vertices
- **Useful for many things**
  - inside/outside test
  - pre-fragment shader interpolation



Interpolated Shading for Ray Tracing

- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- Barycentric coordinates



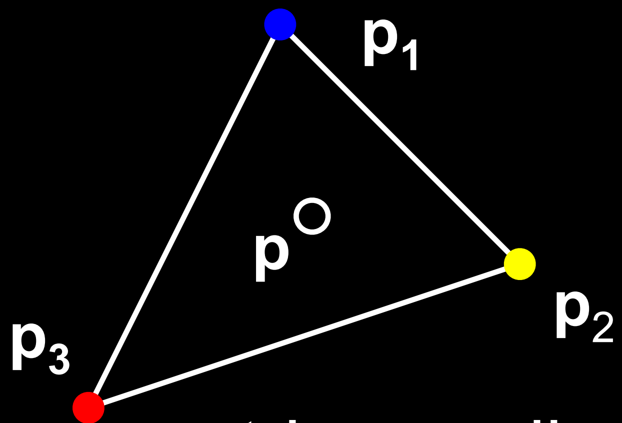
Barycentric Coordinates in 1D

- Linear interpolation
  - $\mathbf{p}(t) = (1 - t)\mathbf{p}_1 + t\,\mathbf{p}_2,\ 0 \le t \le 1$
  - $\mathbf{p}(t) = \alpha\,\mathbf{p}_1 + \beta\,\mathbf{p}_2$ where $\alpha + \beta = 1$
  - $\mathbf{p}$ is between $\mathbf{p}_1$ and $\mathbf{p}_2$ iff $0 \le \alpha, \beta \le 1$
- Geometric intuition
  - Weigh each vertex by ratio of distances from ends

- α, β are called barycentric coordinates

Barycentric Coordinates in 2D

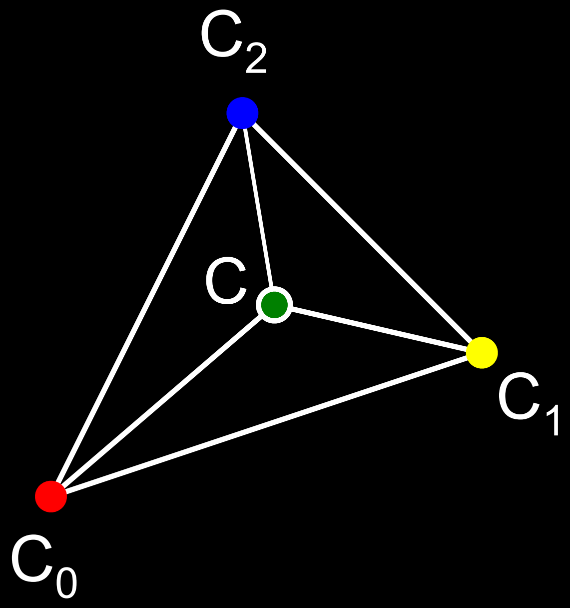
- Now, we have 3 points instead of 2



- Define 3 barycentric coordinates, α, β, γ
- $\mathbf{p} = \alpha\, \mathbf{p}_1 + \beta\, \mathbf{p}_2 + \gamma\, \mathbf{p}_3$
- $\mathbf{p}$ inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1$, $\alpha + \beta + \gamma = 1$
- How do we calculate α, β, γ given $\mathbf{p}$?

Barycentric Coordinates for Triangle

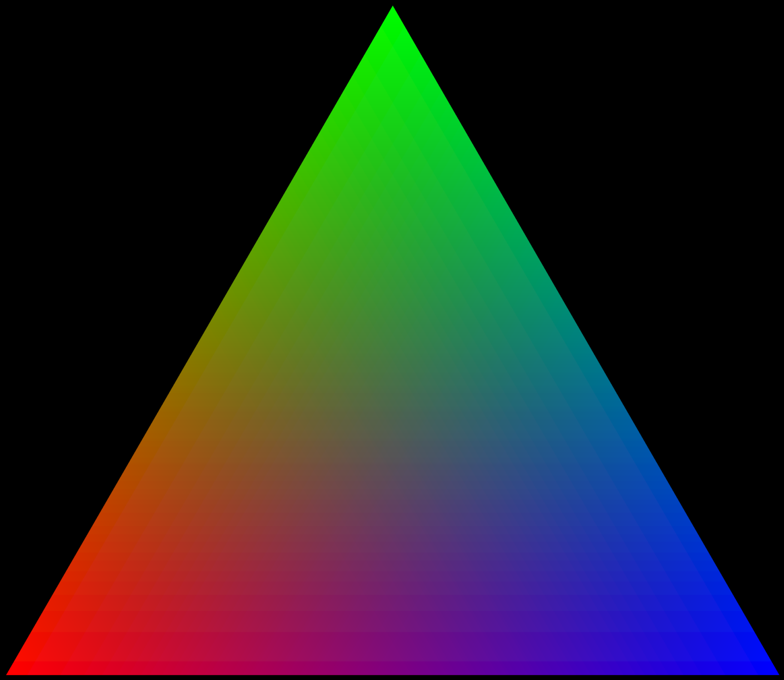
- Coordinates are ratios of triangle areas



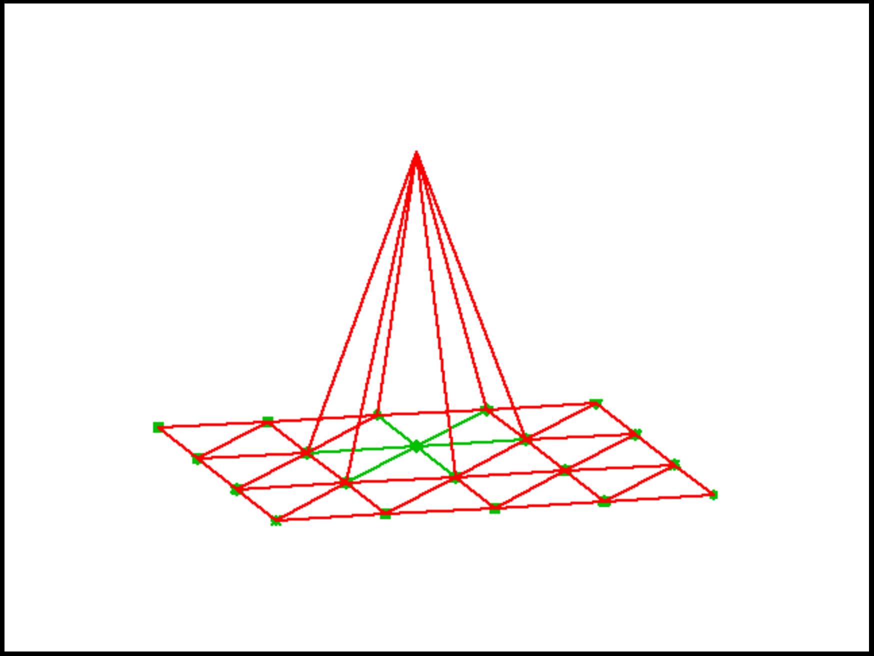
- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.

Barycentric Coordinates for Triangle

- Coordinates are also basis functions
- "Triangle weights"
- Barycentric coordinate functions

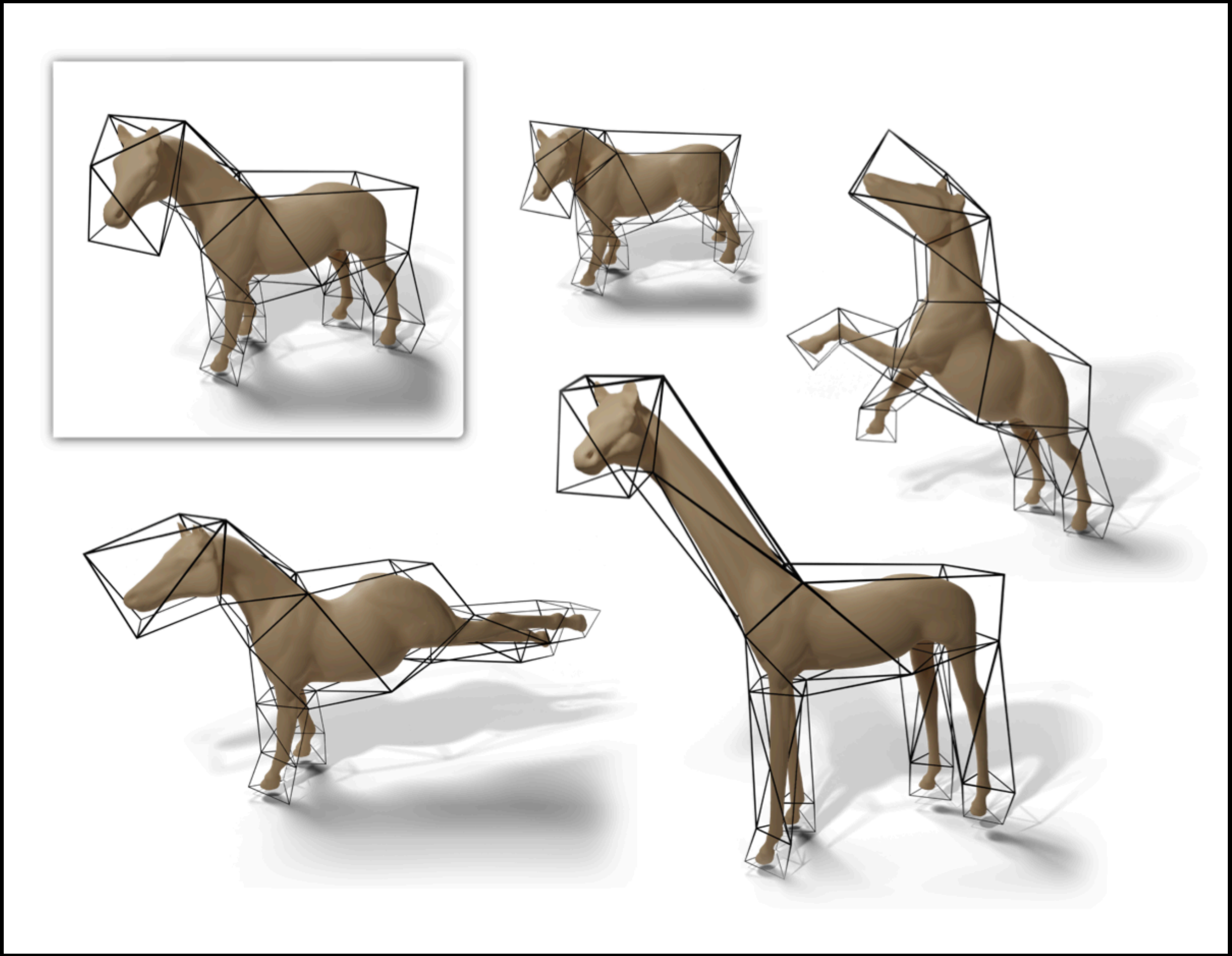


wikipedia.org



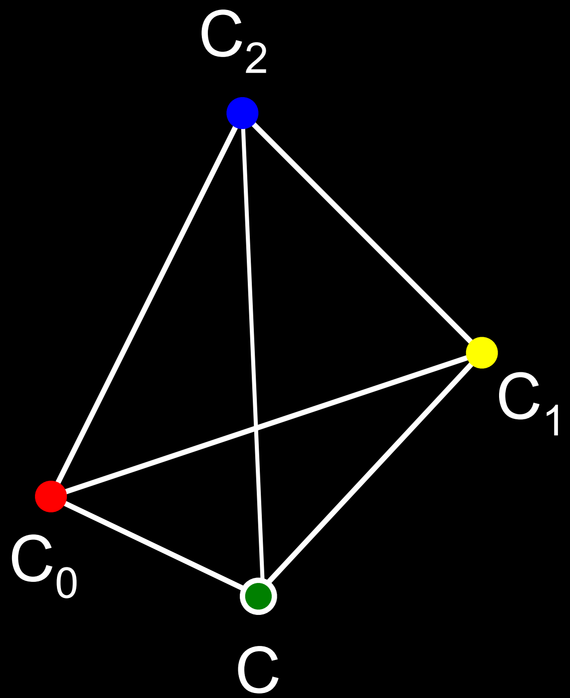
hplgit.github.io

Other uses for Barycentric Coordinates



Dodik et al. 2023. "Variational Barycentric Coordinates"

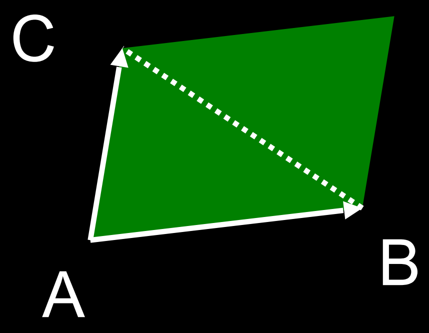
Negative Area



< 0

Point C is outside of the triangle!

Computing Triangle Area in 3D



- Use cross product
- Parallelogram formula
- Area(ABC) = (1/2) |(B – A) x (C – A)|
- How to get correct sign for barycentric coordinates?
  - tricky, but possible:
    compare directions of vectors (B – A) x (C – A), for triangles CC_1C_2 vs $C_0C_1C_2$, etc.
    (either 0 (sign+) or 180 deg (sign-) angle)
  - easier alternative: project to 2D, use 2D formula
  - projection to 2D preserves barycentric coordinates

Computing Triangle Area in 2D

- Suppose we project the triangle to xy plane

- Area(xy-projection(ABC)) =

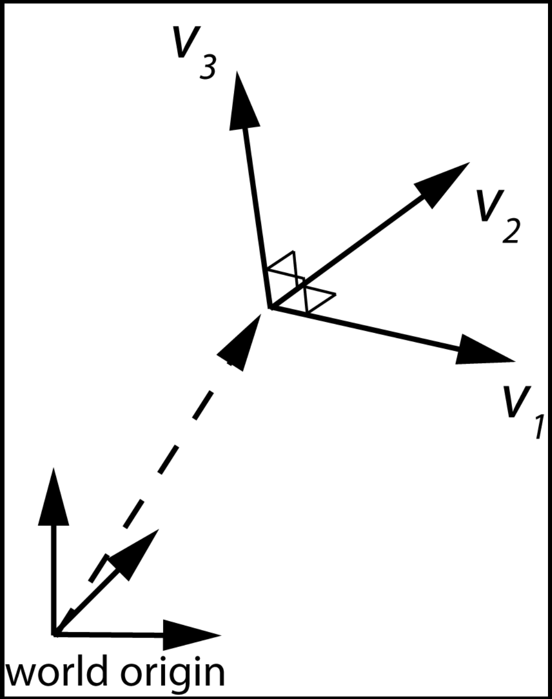
$(1/2)\ ((b_x - a_x)(c_y - a_y) - (c_x - a_x)\ (b_y - a_y))$

- This formula gives correct sign
(important for barycentric coordinates)

- Much easier in 2D, because there is always a constant up direction.

Computing Triangle Area in 2D

- But how to we project to 2D?
- Remember modelview matrices...



$v_1 \cdot v_2 = 0$

$v_2 \cdot v_3 = 0$

$v_1 \cdot v_3 = 0$

$\| v_1 \| = \| v_2 \| = \| v_3 \| = 1$

Orthonormal coordinate system

Change of Coordinate System

- Bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$
- Express basis vectors u_i in terms of v_j

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$
$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$
$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

- Represent in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

Computing Triangle Area in 2D

- But how to we project to 2D?
- There are easier alternatives.
- Just zero the z-coordinate.
  - Barycentric coordinates will be preserved.
  - Must be careful in case the triangle is in the xz or yz planes.

Summary

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates