A Mathematical Theory of Communication

In the more general case with different lengths of symbols and constraints on the allowed sequences, we make the following definition:

Definition: The capacity C of a discrete channel is given by

$$C = \lim_{T \to \infty} \frac{\log N(T)}{T}$$

where $N(T)$ is the number of allowed signals of duration T.

It is easily seen that in the teletype case this reduces to the previous result. It can be shown that the limit in question will exist as a finite number in most cases of interest.

Suppose all sequences of the symbols $S_1, \ldots, S_n$ are allowed and these symbols have durations $t_1, \ldots, t_n$. What is the channel capacity? If $N(t)$ represents the number of sequences of duration t we have:

$$N(t) = N(t - t_1) + N(t - t_2) + \cdots + N(t - t_n)$$

The total number is equal to the sum of the numbers of sequences ending in $S_1, S_2, \ldots, S_n$, and these are $N(t - t_1), N(t - t_2), \ldots, N(t - t_n)$ respectively.

According to a well-known result in finite differences, $N(t)$ is then asymptotic for large t to X_0^t where X_0 is the largest real solution of the characteristic equation:

$$x^{-t_1} + x^{-t_2} + \cdots + x^{-t_n} = 1$$

and therefore

$$C = \log X_0$$

In case there are restrictions on allowed sequences we may still often obtain a difference equation of this type and find C from the characteristic equation.

In the telegraphy case mentioned above:

$$N(t) = N(t-2) + N(t-4) + N(t-5) + N(t-7) + N(t-8) + N(t-10)$$

as we see by counting sequences of symbols according to the last or next to the last symbol occurring. Hence $C = -\log p_0$ where p_0 is the positive root of

$$1 = p^2 + p^4 + p^5 + p^7 + p^8 + p^{10}$$

Solving this we find $C = 0.539$.

A very general type of restriction which may be placed on allowed sequences is the following: We imagine a number of possible states $a_1, a_2, \ldots, a_m$. For each state only certain symbols from the set $S_1, \ldots, S_n$ can be transmitted (different subsets for the different states). When one of these has been transmitted the state changes to a new state depending both on the old state and the particular symbol transmitted. The telegraph case is a simple example of this. There are two states depending on whether or not a space was the last symbol transmitted.