Wireless Autonomous Robot Evacuation from Equilateral Triangles and Squares

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Abstract. Consider an equilateral triangle or square with sides of length 1. A number of robots starting at the same location on the perimeter or in the interior of the triangle or square are required to evacuate from an exit which is located at an unknown location on its perimeter. At any time the robots can move at identical speed equal to 1, and they can cooperate by communicating with each other wirelessly. Thus, if a robot finds the exit it can broadcast "exit found" to the remaining robots which then move in a straight line segment towards the exit to evacuate. Our task is to design robot trajectories that minimize the evacuation time of the robots, i.e., the time the last robot evacuates from the exit. Designing such optimal algorithms turns out to be a very demanding problem and even the case of equilateral triangles turns out to be challenging.

We design optimal evacuation trajectories (algorithms) for two robots in the case of equilateral triangles for any starting position and for squares for starting positions on the perimeter. It is shown that for an equilateral triangle, three or more robots starting on the perimeter cannot achieve better evacuation time than two robots, while there exist interior starting points from which three robots evacuate faster than two robots. For the square, three or more robots starting at one of the corners cannot achieve better evacuation time than two robots, but there exist points on the perimeter of the square such that three robots starting from such a point evacuate faster than two robots starting from this same point. In addition, in either the equilateral triangle or the square it can be shown that a simple algorithm is asymptotically optimal (in the number k of robots, as $k \to \infty$), provided that the robots start at the centre of the corresponding domain.

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1 Introduction

Searching is an important computional task and many different variants have been studied in the mathematical and computer science literature. Numerous fascinating search "excursions" can be found in several books, e.g., [1] (search problems and group testing), [4] (search and game theoretic applications), [6] (cops and robbers problems), [16] (classic pursuit and evasion problems), as well as the seminal treatise [17], to mention a few.

The problem studied here differs from well-known search studies involving multiple robots in that we are not minimizing the time the first robot finds the object of search, but rather the time the last robot reaches the search object. This reminds us of the situation when several people are searching for an exit from a closed area in order to evacuate it and it is important to minimize the time the last person leaves. We call this type of a search an *evacuation problem*, where the object of search is the *exit* and the time the last robot reaches the search object is the *evacuation time*.

In this paper, robot search within a pre-specified geometric domain (in our case, an equilateral triangle or square) is combined with the use of wireless communication between robots in order to minimize the evacuation time. It is clear that finding optimum length evacuation trajectories requires some form of co-operation and co-ordination between the robots (e.g., taking advantage of the possibility of wireless communication between the robots) so that at the time when one of the robots finds the exit, the rest of the robots are as close to it as possible so as to minimize the remaining time to achieve evacuation.

1.1 Model

Three important aspects of the model are required to describe the evacuation problem: (a) the search domain, (b) participating robots and their capabilities, and (c) evacuation algorithms. We elaborate on each of these three aspects in the sequel.

Search Domain. In general, it includes the interior of a non-intersecting, closed curve (e.g., cycle, polygon, or any Jordan curve) in the plane together with the curve itself forming the boundary. The exit is a point on the closed curve at a location unknown to the robots but which they can recognized when they go through it. In a typical setting of interest (which also justifies the term *evacuation*) the robots are enclosed by the boundary of the search domain and can leave it only thorough the exit. The search domains considered in this paper are restricted to equilateral triangles and squares with unit sides.

Robots and their Capabilities. The robots are labeled and may move at maximum speed 1. Although the movement of the robot is unrestricted inside the boundary of the domain, the robot can recognize the boundary when it reaches it and traverse it in its search for the exit. All robots will be assumed to start at the same location. We assume that the robots are aware of their initial position and the search area (for example they can trace the perimeter) and they can change direction when needed at no additional cost. They are also endowed with compass to orient themselves, but they do not know the position of the exit though they know it is located in the boundary. Further, the robots have communication capabilities in that they can communicate with each other wirelessly.

Evacuation Algorithms. An evacuation algorithm is the specification of a trajectory for each robot. A trajectory is a sequence of geometric instructions to be followed by a robot. Such instructions may include move along a specified curve, or move in a particular direction on that curve, or move in a straight line towards the exit, etc. Different robots may follow different trajectories. The closed curve on which the exit is located, as well as the complete evacuation algorithm will be known to each robot. From this knowledge each robot can determine the location of other robots at any time. A robot that finds the exit broadcasts a message "Exit found" and its label to the rest of the robots. Each robot then calculates the location of the exit and moves to it in the shortest possible time (usually along a straight line).

1.2 Related Work

Several previously studied problems can be seen as related to our work, even though they differ in some aspects.

In traditional search problems in an unknown environment, the goal is to construct a map of the environment. In [11], one is seeking for algorithms that achieve a bounded ratio of the worst-case distance traversed in order to see all visible points of the environment, [2] is concerned with exploration where the robot has to construct a complete map of an unknown environment, modelled by a directed, strongly connected graph, and in [3] the robot has to explore either n rectangles or a grid graph with obstacles in a piecemeal fashion. The pursuit-evasion problem [13] is also related to our work; surveys of recent results and autonomous search as related to applications in mobile robotics is given in [8] and additional material with interesting discussions can also be found in [16].

There is extensive literature on robotic exploration in simple polygons. Here we only mention [14] which presents an on-line strategy that enables a mobile robot with vision to explore an unknown simple polygon, and [15] which considers on-line search and exploration problems from the perspective of robot navigation in a simple polygon, with a robot that does not have access to the map of the polygon. Frontier-based exploration which directs mobile robots to regions on the boundary between unexplored space and space that is known to be open is considered in [18]. The problem of creating fast evacuation plans for buildings that are modelled as grid polygons and containing many cells was investigated in [12].

The evacuation problem on equilateral triangles and squares as considered in our paper has, as far as we know, never been studied, but the key motivation of the current study comes from the recent work of [9]. In that paper the authors introduce the evacuation problem as a new search paradigm for autonomous robots, whose initial location is the center of a unit cycle, and an unknown exit on the perimeter of the cycle. The authors obtain tight bounds for evacuation time in the wireless and non-wireless (also known as face to face) communication models depending on the number of robots. Also related is the recent work of [10] which considers evacuation for two robots from a cycle in the face to face communication model and [7], a related evacuation study on a line.

1.3 Outline and Results of the Paper

Here is an outline of our main results. In Sect. 2 we study evacuation in an equilateral triangle with unit side. We design optimal evacuation algorithms for two robots in any initial position on the perimeter or inside the equilateral triangle. We then show that three or more robots cannot improve the evacuation time for any starting position on the perimeter, but three robots can improve the search for the exit for some initial position inside the triangle, in particular, for the centre of the triangle. Section 3 considers a unit side square. Results include: (1) we design optimal evacuation algorithms for two robots starting on the perimeter of the square, (2) we show that three or more robots cannot improve the evacuation time when starting at a corner of the square, but (3) three robots can improve the evacuation time for some points on its perimeter, in particular the midpoint of one of its sides. As a consequence of our analysis, the shortest evacuation time is obtained when the starting position of the robots on the perimeter is (a) at the midpoint of any edge for an equilateral triangle, and (b) at a point at distance $1/2 - 1/\sqrt{12}$ from a vertex for a square. In addition, in either the equilateral triangle or the square it can be shown that a simple algorithm is asymptotically optimal (in the number k of robots, as $k \to \infty$), provided that the robots start at the corresponding centre of the domain. Details of missing proofs can be found in the full paper.

2 Equilateral Triangle

For the equilateral triangle with unit sides we first prove tight upper and lower bounds on the evacuation time for two robots. As stated above, the robots are initially located at the same location. To make the proofs easier to understand, despite some repetition, two cases are considered separately depending on whether or not the robots start on the perimeter or in the interior of the triangle.

2.1 Two Robots Starting on the Perimeter of the Triangle

Theorem 1. *Assume that two robots are initially located on the perimeter of an equilateral triangle at distance x from the closest midpoint of an edge of the triangle. Then $x + \frac{3}{2}$ is a tight bound for evacuating these two robots.*

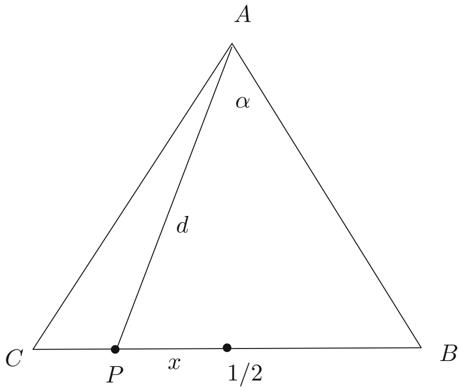

Fig. 1. Evacuating from an equilateral triangle with the two robots starting at a point P at distance x from the midpoint of edge BC.

Proof. Let the robots start at a point P on the perimeter of triangle ABC and at distance x from the midpoint of edge BC (see Fig. 1). To prove the upper bound the robots execute the following algorithm:

1. The robots walk together to the midpoint of edge BC.
2. From the midpoint they move in opposite directions along the perimeter, i.e. Robot 1 towards vertex A via vertex B, and the other Robot 2 towards vertex A via vertex C.
3. When a robot finds the exit it broadcasts “Exit found” to the other robot which immediately goes in a straight line segment to the exit.

The analysis of this algorithm is as follows. The time required for evacuation is $T = T_1 + T_2$, where T_1 is the time from the departure until one of the two robots broadcasts that the exit was found, and T_2 is the time from the broadcast “Exit found” of one robot until the other robot reaches the exit, i.e., the distance of the other robot from the exit. Since the triangle is equilateral, it is easily seen that $T_1 + T_2 \leq x + \frac{3}{2}$, which proves the upper bound.

The value $x + \frac{3}{2}$ is also a lower bound. Without loss of generality assume the starting position of the robots is at point P on edge BC and at distance x from its midpoint, as depicted in Fig. 1. Observe that one of the two vertices A, B is visited first by a robot. We distinguish two cases.

Case 1: A is visited first.

Let $\alpha := \angle PAB$. Since $\alpha < \pi/3$ is the smallest angle of the triangle $\triangle PAB$, it is easy to see that $d := |PA| \geq x + \frac{1}{2}$. Now the adversary puts the exit at B and the robot will need to travel an extra distance 1 to the vertex B; therefore $d + 1$ is a lower bound. However $d + 1 \geq x + \frac{3}{2}$.

Case 2: B is visited first.

Clearly, it takes time at least $x + \frac{1}{2}$ to visit point B from the starting point P. Now, if the adversary puts the exit at A it will take an additional 1 time unit to evacuate. This proves the theorem. ∎

Corollary 1. *The shortest evacuation time is obtained when the two robots start at the midpoint of an edge of the equilateral triangle and the resulting trajectory has length* $\frac{3}{2}$. ■

2.2 Two Robots Starting in the Interior of the Triangle

Let $\mathcal{A}$ be an algorithm for the evacuation of an equilateral triangle by two robots initially placed at the *same* starting position in the interior or in the perimeter of the triangle. When started at position s in the triangle with exit position e, we say $\mathcal{A}$ terminates in time $T(\mathcal{A}, s, e)$ if both robots are at the exit position in time $T(\mathcal{A}, s, e)$. We define $T(\mathcal{A}, s)$ to be the worst-case termination time of algorithm $\mathcal{A}$ for starting position s over all possible exit positions e. Let $d(a, b)$ denote the distance between two points a and b. We prove that $T(\mathcal{A}, s) \geq 3/2 + x$ where x is the minimum distance from s to any of the midpoints of edges.

We start with a series of simple observations and lemmas. Assume without loss of generality that the triangle is $\triangle ABC$, with A, B, C its three vertices, the starting position s is closest to midpoint D and corner A as depicted in Fig. 2. Thus s is located inside $\triangle AFD$.

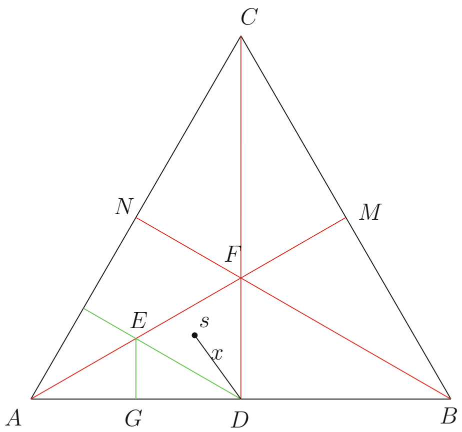

Fig. 2. The triangle to be evacuated. The starting position of the two robots is in $\triangle AFD$. N, M, D are the midpoints of the corresponding sides.

Observation 1 (Distances).

1. $d(s, B) \geq 1/2$, $d(s, C) \geq 1/2$.
2. *If* $s \in \triangle AEG$, *then* $d(s, B) \geq 3/4$.
3. *If* $s \in \triangle AED$, *then* $d(s, C) \geq 3/4$.

Observation 2. *If s is in the quadrilateral $EFGD$, then* $3/2 + x \leq 3/2 + d(D, E) = 3/2 + \frac{\sqrt{3}}{6} < 1.79$.

Observation 3. *If a robot visits a midpoint and then a corner and at this time, there remains an unvisited corner, then* $T(\mathcal{A}, s) \geq 3/2 + x$.

Proof. Place the exit at the as-yet unvisited corner.

Observation 4. *If a robot visits the corner B and then either (a) visits a point P on the edge AB with $d(B, P) \geq 1/2$ before the corner C is visited by either robot, or (b) visits a point Q on the edge BC with $d(B, Q) \geq 1/2$ before the corner A is visited by either robot, then $T(\mathcal{A}, s) \geq 3/2 + x$.*

Proof. If $s \in \triangle AEG$, then the robot takes at least time $3/4$ to visit B, and at least $1/2$ to visit P. At this point since C is not visited by either robot, the adversary can place the exit at C, forcing the robot to take at least $\sqrt{3}/2$ time to reach C. Therefore $T(\mathcal{A}, s) \geq 3/4 + 1/2 + \sqrt{3}/2 > 2$. If instead s is the quadrilateral $EFGD$, the above trajectory takes time at least $1/2+1/2+\sqrt{3}/2 > 1.86 > 1/2 + x$ by Observation 2 for this case.

Observation 5. *Similar to Observation 4 about corner C.*

Lemma 1. *If a robot visits corner B and then visits corner C before the entire edge BA has been explored (by a combination of both robots), then $T(\mathcal{A}, s) > 2$. Also, if it visits corner B and then visits corner A before the entire edge BC has been explored, then $T(\mathcal{A}, s) > 2$.*

We are now ready to show that for two robots the evacuation time when starting inside the triangle follows the same rule as when starting on the perimeter.

Lemma 2. *Similar statement to that of Lemma 1 for a robot that first visits corner C.*

Theorem 2. *Assume that two robots are initially located at point s inside the equilateral triangle, and let $x = \min\{d(s, m) \mid m \text{ is a mid point of an edge}\}$. Then $x + \frac{3}{2}$ is a tight bound for evacuating these two robots.*

Proof. We give a proof of the lower bound by contradiction. Assume $\mathcal{A}$ is a two-robot evacuation algorithm such that $T(\mathcal{A}, s) < 3/2 + x$. Consider the first two distinct corners to be visited by either of the robots. If they were both visited by the same robot, then we place the exit at the remaining corner and thus $T(\mathcal{A}, s) \geq 2$. Therefore, we assume that robot 1 visited corner X and robot 2 visited corner Y where $X, Y \in \{A, B, C\}$ and $X \neq Y$. We consider the three cases separately.

First assume that $X = B$ and $Y = C$. Then Lemmas 1 and 2 imply that the entire edge BC must have been explored before either robot can visit the corner A. However, if either robot visits the midpoint M before visiting the corner A, then by Observation 4, we have $T(\mathcal{A}, s) \geq 3/2 + x$, a contradiction.

Next, suppose $X = A$ and $Y = B$. We claim that if Robot 1 (having visited corner A) visits corner C before exploring the entire segment AD, then $T(\mathcal{A}, s) \geq 2$. Define Q such that $[AQ)$ is the longest contiguous segment visited by Robot 1 before visiting corner C, and suppose for the purpose of contradiction that $y = d(A, Q) \leq 1/2$ (see Fig. 3). By Observation 4, Robot 2 cannot have visited the point Q and neither has Robot 1. The adversary now

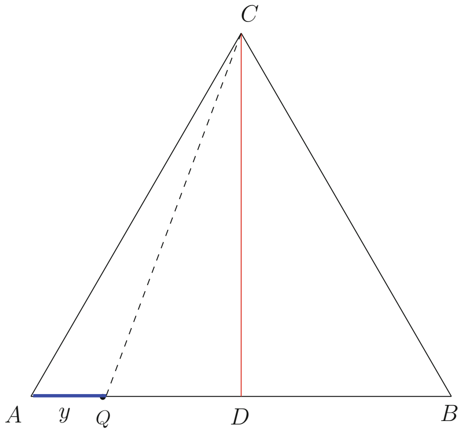

Fig. 3. Q is the leftmost point that is unvisited by A before going to C.

places the exit at point Q. The time taken by Robot 1 to reach Q is at least $d(A,Q)+2d(Q,C) = y+2\sqrt{3/4+(1/2-y)^2} \geq y+2(1-y/2) = 2$. We conclude that Robot 1 must explore the entire segment AD before visiting the corner C. Now, suppose P is a point on the edge AC such that Robot 1 has explored the contiguous segment $[AP)$ before arriving at D, with $d(A,P) = z$. Depending on which of the points P and C is visited first (by either robot), the adversary places the exit at the other point.

First we claim that if $d(A,P) \geq 1/2$, then $T(\mathcal{A}, s) \geq 3/2 + x$. Recall that A is visited before D (by Observation 3). So for robot 1 to explore the segment AP as well as AD takes at least time $d(s,P)+d(P,A)+d(A,D) \geq x+1$. If the exit is placed at C, it takes an additional $\sqrt{3}/2$ time for robot 1 to reach the exit. We conclude that $z < 1/2$, as illustrated in Fig. 4.

Suppose of the two points P and C, the first to be visited is C. If it is robot 1 that arrives first at C, the exit is placed at P, and the time it takes for robot 1 to reach the exit is at least $z + 1/2 + \sqrt{3/2} + 1 - z > 2$. If instead it is robot 2 that arrives first at C, the time it takes to reach the exit at P is at least $1/2 + 1 + 1 - z > 2$ since $z < 1/2$.

Suppose instead that the first of the two points P and C to be visited by either robot is P. The adversary now places the exit at C. If it was robot 1 that arrived first at P, then the time it needs to reach the exit is at least $d(s,P) + d(P,A) + d(A,D) + d(D,P) + d(P,C) = d(s,P) + z + 1/2 + d(D,P) + 1 - z = d(s,P) + d(D,P) + 3/2 > d(s,D) + 3/2 = 3/2 + x$. If instead it was robot 2 that arrived first at P, we consider two cases. Either robot 2 visited P before it visited B or after. If it visited P before it visited B, observe that the closest point to P in the triangle ADF (where s is by assumption) is found by dropping a perpendicular to the line segment AF, say to point L. Now in the right triangle APL, the angle PAL is 30 degrees, and $d(A,P) = z$, therefore, $d(A,L) = z/2$.

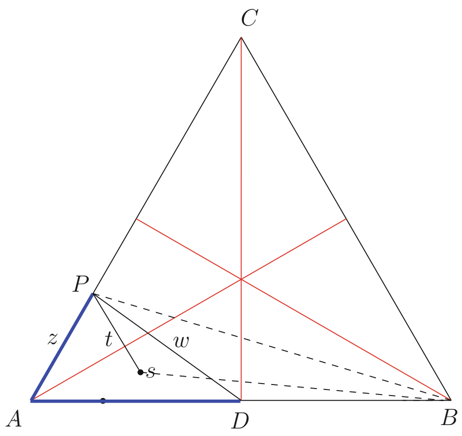

Fig. 4. Robot 1 has visited the segments AP and AD before going to C.

Therefore, the time robot 2 needs to reach C is $d(s,P)+d(P,B)+d(B,C) \geq z/2+\sqrt{3/4+(1/2-z)^2}+1 > z/2+1-z/2+1 = 2$.

Otherwise, it visited B and then P, and then the time it needs is at least $d(s,B)+d(B,P)+d(P,C) \geq d(s,B)+\sqrt{3}/2+1/2$ (see Fig. 4). Now, by Observation 1, if $s \in \triangle AEG$, this time is at least $3/4+\sqrt{3}/2+1/2 > 2$, and if instead $s \in EFGD$, this time is at least $1+\sqrt{3}/2 > 1.86 > 3/2+x$ by Observation 2. Thus if robot 1 visits A and robot 2 visits B, then we are done.

The remaining case $X = A$ and $Y = C$ is very similar to the previous case and details are left to the reader.

Clearly, the evacuation algorithm stated in Theorem 1 gives evacuation time of at most $x+3/2$, thus proving a matching upper bound. ■

2.3 More Than Two Robots

The following rather surprising result, states that more robots don't help for starting points on the perimeter.

Theorem 3. *Assume that $k \geq 3$ robots are initially located at point s on the perimeter of an equilateral triangle and let $x = \min\{d(s,m) : m$ is a mid point of an edge $\}$. Then $x + \frac{3}{2}$ is a tight bound for evacuating these k robots.*

Proof. An examination of the proof of the lower bound of Theorem 1 shows that it does not depend upon the number of robots starting on the perimeter. ■

The above result shows that starting on the boundary of the triangle limits the nature of the search for the exit. But if the robots were to start at an interior point, then they have more options, including travelling through the interior to different points on the perimeter. Thus during an active search, they can not only search more cooperatively but also remain somewhat close to each other to allow faster evacuation when the exit is found. We show how this extra freedom can help:

Theorem 4. *Three robots starting from the centroid of an equilateral triangle can evacuate faster than two robots with the same starting position.*

Above we showed that for any $k \geq 2$ robots there is an optimal algorithm for evacuating an equilateral triangle when the robots start on the perimeter. Here we show there is a straightforward algorithm for evacuation starting at the centroid that achieves performance that is asymptotically optimal as the number of robots, k, goes to infinity.

Theorem 5. *For any $k \geq 2$, there exists an algorithm that achieves evacuation in time $\sqrt{3}/3 + 3/k + 1$ for k robots starting at the centroid of the equilateral triangle. Furthermore, any algorithm for evacuating k robots from the centroid of the equilateral triangle requires at least $\sqrt{3}/3 + 1$ time.*

Remark 1. Consider the case where the robots start at a point along a line from the centroid to one of the corners of the triangle. Let x be the distance from the starting point to the two other corners. It is easy to see, using essentially the same arguments, there is an algorithm that achieves evacuation with k robots in time $1 + x + 3/k$ and that any algorithm will take at least $1 + x$ time. I.e., there is an asymptotically optimal algorithm starting at any of these points as well.

3 Square

In this section we consider the square with unit sides. We first analyze the case of two robots starting on the perimeter searching for the exit.

3.1 Two Robots Starting on the Perimeter of the Square

The surprising fact (to be proved below) is that the starting point for the robots that minimizes the evacuation time is at a point at distance $\frac{1}{\sqrt{12}}$ from the midpoint of an edge. This differs from the equilateral triangle where the shortest evacuation time is obtained when starting from the midpoint of an edge.

Theorem 6. *Assume that two robots are initially located on the perimeter of a square at distance x from the closest vertex of the square. Then $1 + x + \sqrt{1 + (1 - 2x)^2}$ is a tight bound for evacuating these two robots.*

Proof. Consider evacuation of two robots starting from a point on the perimeter of a square $ABCD$ at distance $x \leq 0.5$ from the vertex A (see Fig. 5). To prove the upper bound consider the following algorithm for the robots:

1. Robots move from s along the perimeter in opposite directions. One robot moves on the perimeter towards vertex D via vertex A. The other robot moves on the perimeter towards C via vertex B.

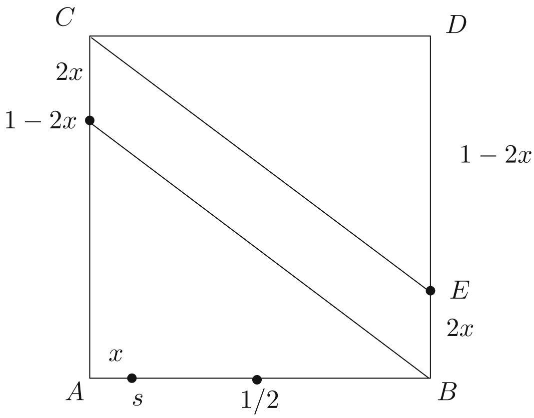

Fig. 5. Square with side 1, $0 \leq x \leq 1/2$. Observe that $d(E,C) = \sqrt{1+(1-2x)^2}$, $d(s,C) = \sqrt{1+x^2}$, $d(s,D) \geq \sqrt{5/4}$, and $d(A,E) = \sqrt{1+4x^2}$

2. If a robot finds the exit it broadcasts "Exit found" to the other robot which moves directly to the exit in a straight line segment, i.e., without necessarily moving on the perimeter.

Next we determine the running time until evacuation. It is easy to see that when the robots are starting at distance x from vertex A, the worst case running time of this algorithm is obtained when the exit is located at C or E. Let $d = d(E,C) = \sqrt{1+(1-2x)^2}$. Thus the worst case running time of this algorithm is $f(x) := 1 + x + d = 1 + x + \sqrt{1+(1-2x)^2}$.

Next we show the lower bound. We consider the visits to the vertices by either of the robots. Suppose at the time C is visited by a robot, the vertex B is as yet unvisited. Then we place the exit at B. The time taken by the robot who visits C to go to the exit is $\geq dist(s,C) + dist(C,B) = \sqrt{1+x^2} + \sqrt{2} \geq f(x)$ for all $0 \leq x \leq 1/2$. Similarly, suppose D is visited before A is visited by either robot. Then we place the exit at A. The time taken by the robot who visits D to get to A is $\geq dist(s,D) + dist(D,A) = \sqrt{1+(1-x)^2} + \sqrt{2} \geq \sqrt{1+x^2} + \sqrt{2} \geq f(x)$.

By the observations above, the following cases are exhaustive for the first two points to be visited:

A and B are the first two vertices to be visited: If the same robot visits both, we place the exit at the vertex diagonally opposite the second visited vertex, causing the robot to take at least $1+x+\sqrt{2} \geq f(x)$. So we assume A was visited by Robot 1 and B by Robot 2. We observe now which is visited next: C or E as shown in Fig. 5, and put the exit at the other one. Suppose C is visited first. If Robot 1 visited C, to get to E, it takes time $d(s,A)+d(A,C)+d(C,E) = f(x)$. If Robot 2 visited C, to get to E it takes time $d(s,B)+d(B,C)+d(C,E) > f(x)$. Suppose instead E was visited first. If Robot 2 visited E first, to get to C, it takes time

$\geq d(s,B)+d(B,E)+d(E,C)=f(x)$. If it was Robot 1 that visited E first, it takes time $d(s,A)+d(A,E)+d(D,C)=x+\sqrt{1+4x^2}+\sqrt{1+(1-2x)^2}\geq f(x)$.

A is visited first, then D: If the same robot visits A and then D, we put the exit at C. The robot takes time $\geq x+\sqrt{2}+1\geq f(x)$. So assume Robot 1 visits A, then Robot 2 visits D. Recall that C cannot be visited before B. Therefore, the next vertex to be visited must be B: we put the exit at C. If it was Robot 1 that visited B, then to get to C it takes time $d(s,A)+d(A,B)+d(B,C)=x+1+\sqrt{2}\geq f(x)$. If it was Robot 2 that visited B, then it takes time $d(s,D)+d(D,B)+d(B,C)>2+\sqrt{2}>f(x)$.

B is visited first, then C: This is similar to the above case and details are left to the reader.

This completes concludes the derivation in all three cases and also the proof of the theorem. ■

Corollary 2. *Among all possible starting positions of the two robots at distance x from a vertex of the unit square the optimal length of a trajectory is obtained when $x=\frac{1}{2}-\frac{1}{\sqrt{12}}$. Moreover, the length of the resulting trajectory for evacuating two robots starting together on the perimeter of a unit square is $\frac{3}{2}+\frac{\sqrt{3}}{2}$.*

Proof. To find the initial position of the robots that minimizes the evacuation time, we differentiate the evacuation time from Theorem 6 with respect to x and we obtain that the maximum is obtained for $x=\frac{1}{2}-\frac{1}{\sqrt{12}}$. Substituting this value of x into the evacuation time function shows that the maximum distance traversed in this case is at most

$$f\left(\frac{1}{2}-\frac{1}{\sqrt{12}}\right)=\frac{3}{2}+\frac{\sqrt{3}}{2},$$

thus proving the corollary. ■

3.2 More Than Two Robots

In this section we show that for a square, using more that two robots to search for the exit can improve the search time for some starting point on the perimeter, but not for all. Thus, it differs from the results for an equilateral triangle. First we show that having more robots doesn't help if the starting point is one of the corners.

Theorem 7. *If $k\geq 2$ robots start evacuation from a corner of the square, then it takes them at least $1+\sqrt{2}$ time before the evacuation is completed.*

As opposed to the case of the triangle, there exist points on the perimeter where even one extra robot helps.

Theorem 8. *Let p be the midpoint of one of the sides of the square. Three robots starting from p can solve the evacuation problem faster than two robots starting from p.*

For the square, our results for $k > 2$ robots are optimal only if the robots start at one of the corners of the square. Again we can show an asymptotically optimal (in k) result holds for the case where they start at the center. We have:

Theorem 9. *For any $k \geq 2$, there exists an algorithm that achieves evacuation in time $3\sqrt{2}/2+4/k$ for k robots starting at the center of the square. Furthermore, any algorithm for evacuating k robots from the center of the square requires at least time $3\sqrt{2}/2$.*

Proof. For the upper bound, starting at one of the corners of the square, mark off points at distance $4/k$ in clockwise order around the perimeter. Send one robot to each marked point and have them search in a clockwise manner the segment between their point and the next marked point. The robot that finds the exit broadcasts "Exit found" (as well as its label) and the other robots move directly to the exit (the position of which they can calculate from the time and the label). The last robot to search its segment finishes at time $\sqrt{2}/2 + 4/k$ and all robots are at most distance $\sqrt{2}$ from the exit at that time so that the all the robots are evacuated by time $3\sqrt{2}/2 + 4/k$.

For the lower bound, we will inform the robots that the exit is either at vertex A or C (opposing corners of the square). Let t be the time when the first robot reaches A or C. If this robot first reaches A the exit will be placed at C and vice versa. Clearly t is greater or equal to $\sqrt{2}/2$ so that at least this robot must take time greater or equal to $\sqrt{2}/2 + \sqrt{2} = 3\sqrt{2}/2$ to exit. ■

4 Conclusion and Open Problems

In this paper, we investigated the problem of evacuating $k \geq 2$ robots from equilateral triangles and squares. Our results assume that the robots start from the same initial point: in the interior or on the perimeter for triangles, and on the perimeter or centre for squares. We design robot trajectories that optimize the evacuation time, and also show that in some specific cases, three robots can evacuate faster than two robots, e.g. when they start at the centroid of the triangle, or start at the midpoint of a side of the square. However, in other cases, having more than two robots makes no difference, e.g. when the robots start anywhere on the perimeter of the triangle or start at a corner of the square.

In addition to problems that remain unsolved from our investigations, there are numerous interesting and surprisingly not easy to solve open problems. Very little is known when the robots do not start at the same point be that on the perimeter or in the interior of the polygon. In this regard, the techniques of parametrized complexity as employed in [5] should be useful. Throughout our investigations we considered only the wireless communication model. However, nothing is known for the face-to-face communication model whereby robots can exchange information only if and when they meet at the same location. Further, as in [9], it would be interesting to investigate the evacuation time for k robots asymptotically in k and compare the wireless and non-wireless models. Evacuation in the more general case of arbitrary simple polygons is entirely open.

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