Evacuating Priority Agents in a Disk with Unknown Exit Location

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1 Preliminaries

Algorithms of search-and-escape involve mobile agents (also called Robots) searching in geometric domains, such as a closed disk, or convex polygon. By working together and communicating with one another, these mobile agents search the domain to find an exit hidden on the perimeter. Many different problems have previously been studied in this topic, such as finding algorithms to evacuate all agents in a domain, or only evacuating a specific subset of these agents.

1.1 Model

All agents in this problem use the same coordinate system and operate in a closed disk, all starting from the center. These agents' algorithms do not all have to be the same, and in fact to most efficiently search the circle, they must all be unique. In our problem, we observe the Priority model of algorithms. In this model, a subset of one or more agents (P) are defined as a Priority (or Queen) and the goal of the algorithm is to evacuate a required subset of these Priority Agents. These algorithms also include a number Helper agents (H), that simply assist in searching the circle for the exit, for a total of (H + P) agents. The Helper agents are not required to evacuate.

Once an exit is found, whether by a Helper or a Priority, the agent may use Wireless communication to immediately broadcast the exit's location and its own identity to all other agents. Upon receiving this broadcasted location, any remaining Priority agents that need to evacuate travel along a chord to the exit and when the required subset has exited, the algorithm terminates. The cost of the algorithm is called the termination time, and is the total worst-case time for the required subset of Priority agents to exit.

1.2 Previous Work

Similar search-and-evacuate problems to this one have previously been studied, including those involving Agents searching on a line, and searching inside of other types of shapes, such as a triangle. In our research thus far, we have mainly looked at the problems regarding the closed unit disk and how to most efficiently search for the exit and evacuate different subsets of agents. We started by studying the algorithms that have been designed for $n = 2, 3$ agents using both the face-to-face and wireless communication models [(Evacuating Robots From a Disk Using Face-to-Face Communication, 2015), (Evacuating Robots Via an Unknown Exit in a Disk, 2015)]. In these algorithms, all agents must evacuate for the algorithm to terminate, and there is no notion of Priority of Helper. Interestingly, in the face-to-face model, agents must be next to each other to communicate.

Afterwards, we looked at problems of a similar type that have been studied, namely those regarding 1 Priority and 1 or more Helper agents searching in a closed disk solely usign wireless communication (God Save the Queen, 2018) (Priority Evacuation From a Disk Using Mobile Robots, 2018)]. In these papers, the results involved getting the only Priority agent to the exit as fast as possible, however, our problem attempts to design an algorithm where only one of multiple Priority agents needs to evacuate.

To facilitate studying these algorithms and seeing results based on test data, we have created an algorithm visualizer. This program uses the different types of movement and communication directives we commonly see in each algorithm to recreate an interactive visualization of the algorithm. To date, all of the algorithms listed in the above papers including our own can be shown, and new ones can be created.

1.3 Our Results

In our initial algorithm using 2 Priority and 1 Helper agent with 1 Priority required to exit, we can show that a termination time upper bound of **3.55 time units** is possible given the specific set of parameters we use to guide the agents. We can achieve this by using a **parameter of angle** $\alpha = 5\pi/9 - 2\sqrt{3}/3$ for the two agents to travel to the perimeter in the third quadrant, i.e, they travel out at an angle of $\pi + \alpha$. This allows us to set the two worst-case time predictions equal. These are the cases where $A)$ $P2$ finds the exit at the very end of its search, or $B)$ H finds the exit in the second quadrant at the angle $\pi - \beta$, where $\beta = (\pi/3 - \alpha)/2$.

As of now, in our second slightly improved algorithm, we predict that we can get a better time than was shown in the initial algorithm, by making $P1$ take a detour at some point during its search, leaving it a constant distance away from H during the last phase of its search. By making it take this detour, we believe that we can improve the upper bound in the case where $P1$ and $P2$ are equidistant from H at the time when H finds the exit. By using a slightly modified α and introducing some new variables, we believe we can achieve an upper bound of less than 3.55 time units.

2 Initial Algorithm

2.1 Evacuation Algorithm $Search_1$

Algorithm $Search_1(\alpha)$			
Robot	**#**	**Trajectory**	**Exit Protocol**
$P1$	0 1	Travel from O to C_0 Travel CCW.	IF: Exit found by $P1$ - Done. ELSE: Travel along chord to exit if closer.
$P2$	0 1	Travel from O to $C_{\pi+\alpha}$ Travel CCW.	IF: Exit found by $P2$ - Done. ELSE: Travel along a chord to exit if closer.
H	0 1	Travel from O to $C_{\pi+\alpha}$ Travel CW.	Broadcast exit location and then wait.

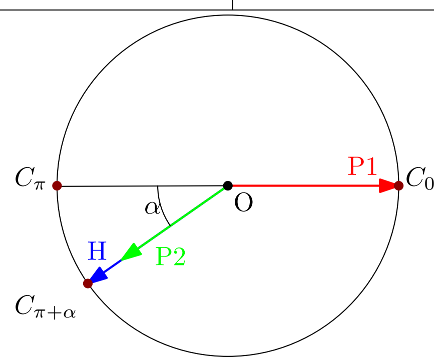
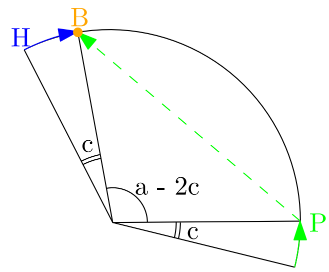


Figure 1: Reference points.

Lemma 1: The upper bound of the distance between two speed-1 agents converging on an unexplored arc of a unit disk with angle $a >= 2\pi/3$ occurs when the agents are $\sqrt{3}$ away from each other at the time of exit discovery.



Proof: Suppose the helper H finds the exit at some point B after the agents have traveled a distance of c along the arc. Then, the total time taken by the priority agent P to get to B, having started at the opposite side of the arc a, would be:

$$t = c + 2\sin(\frac{a - 2c}{2})$$

Then, differentiating with respect to t, we get:

$$\frac{dt}{dc} = 1 - 2\cos(\frac{a - 2c}{2})$$

Finally, setting that equal to zero results in:

$$\frac{2\pi}{3} = a - 2c$$

Thus, the greatest value for t will occur when the agents are $\sqrt{3}$ apart, or at an angle of $\frac{2\pi}{3}$ for $a >= \frac{2\pi}{3}$ at the time of exit discovery.

2.2 Results of $Search_1$

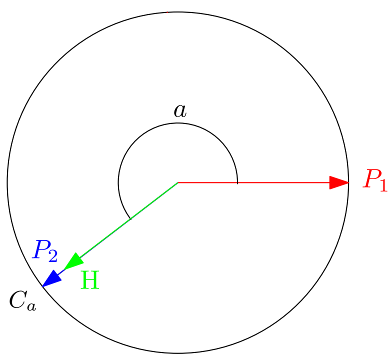
We can show that by sending $P2$ and H out to the perimeter at an angle $a = \frac{5\pi}{9} - \frac{2\sqrt{(3)}}{3}$, we can achieve a lower bound of 3.55 time units for algorithm completion. We achieve this by taking two bounds and equating them. First, we take the time it would take $P2$ to search its entire portion of the circle, as a sort of "goal" time for $P1$ and H to achieve in searching their portions. The time for this lower bound is $t = 1 + \beta$, where $\beta = \pi - \alpha$. Secondly, we can see that for a certain time c after H and $P2$ have started searching the perimeter, finding the exit will be the responsibility of $P2$, as it will be closer if H finds the exit. However, after H passes point $C_{\pi+\alpha-c}$, travelling to the exit found by H

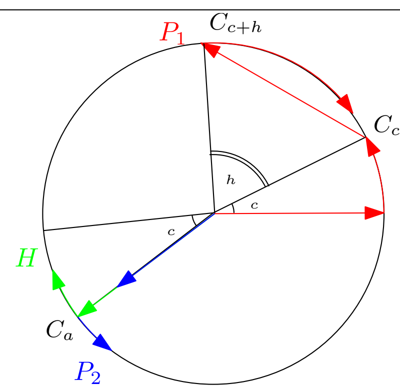
becomes the responsibility of $P1$, and by Lemma 1, we can show that from this point on in the algorithm, the total time it takes $P1$ to evacuate is decreasing. This means that we find our second and third lower bound at the point $C_{\pi+\alpha-c}$, as at this point both $P1$ and $P2$ are equidistant from H when H discovers the exit. Our second bound is $t = 1 + c + 2\sin(c)$, and represents $P2$'s distance from H. Our third bound is $t = 1 + c + 2\sin(\frac{a}{2} - c)$ and represents $P1$'s distance from H.

3 Algorithm Using a Detour

3.1 Evacuation Algorithm $Search_2$

Algorithm $Search_2(a, c, h)$			
Robot	**#**	**Trajectory**	**Exit Protocol**
$P1$	0 1 2 3	Travel from O to C_0. Travel CCW to C_c. Travel along chord to C_{c+h}. Travel CW towards C_c.	IF: Exit found by $P1$ - Done. ELSE: Travel along chord to exit if closer.
$P2$	0 1	Travel from O to C_a Travel CCW towards C_0.	IF: Exit found by $P2$ - Done. ELSE: Travel along a chord to exit if closer.
H	0 1	Travel from O to C_a Travel CW towards C_{c+h}.	Broadcast exit location and then wait.



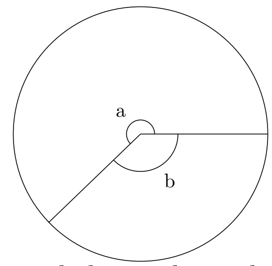


In this algorithm, $P1$ utilizes a detour, abandoning its search along the wall in favor of getting closer to H in anticipation of the exit being found when both priority are equidistant from H at the time of exit discovery. By doing this, we minimize $P2$'s responsibility to H allowing it to search the bottom wall faster. Then, by **Lemma 1** we can prove that for a certain value c and h, the total time until $P1$ evacuates is decreasing until the end of its detour, and constant for the rest of its seach, as it will always be a constant distance h away from the Helper during its last phase.

3.2 Equations

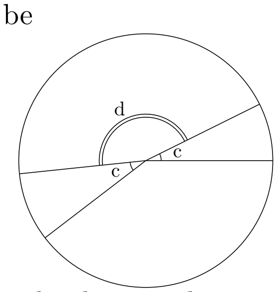
First, we know that we can divide the circle into two sections, where a is the responsibility of the H and $P1$, and the b is the responsibility of $P2$. Therefore:

$$2\pi = a + b$$



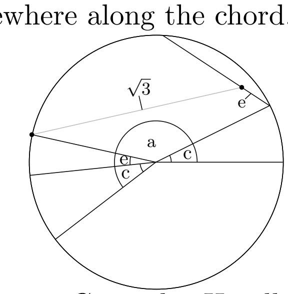
We know that $P1$ and H will have to travel a total of a around the circle, and after a certain time c, the remaining unsearched area will be

$$a = 2c + d$$



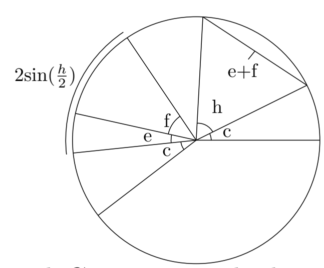
If, after more time e of both agents following the perimeter the distance between the two would be equal to $\sqrt{3}$, which by Lemma 1 will result in the evacuation time increasing, then we instead send $P1$ along a chord of angle h, such that when H reaches point C_{a-c-e}, $P1$ will be travelling somewhere along the chord.

$$c + e = \frac{a - \frac{2\pi}{3}}{2}$$



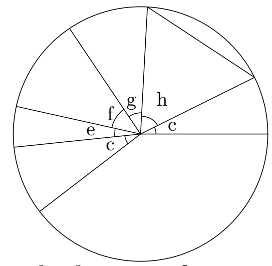
The point at which $P1$ reaches the end of the detour at point C_{c+h}, the H will be at point $C_{a-c-e-f}$, where f is the extra time taken by $P1$ to reach the end of its detour.

$$e + f = 2\sin(\tfrac{h}{2})$$



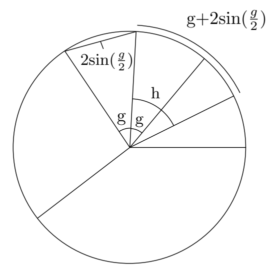
We call g the remaining unsearched arc between C_{c+h} and $C_{a-c-e-f}$ such that:

$d = e + f + g + h$



At this point in the algorithm, we send $P1$ clockwise to search the area from C_{c+h} until C_c and H stays clockwise to search the remaining area from $C_{a-c-e-f}$ until C_{c+h}. At this point the distance between the two agents stays constant for the rest of the algorithm, and this distance is equal to $2\sin(\frac{g}{2})$. Clearly, the optimal value for h is:

$h = g + 2\sin(\frac{g}{2})$



At this point, we have fully partitioned the circle such that:

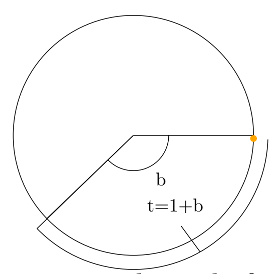
$$2\pi = b + 2c + e + f + g + h$$

which, once simplified, becomes:

$$2\pi = b + 2c + 2\sin(\frac{g + 2\sin(\frac{g}{2})}{2}) + 2g + 2\sin(\frac{g}{2})$$

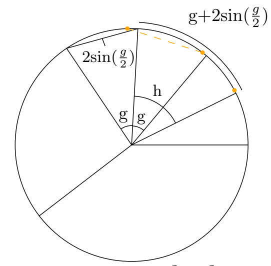
At this point, we identify three upper bounds. One is when $P2$ finds the exit at the end of its search, at

$t = 1 + b$



Another upper bound occurs when either H or $P1$ finds the exit at the end of their search, after time

$t = 1 + c + e + f + h$



Finally, a third bound gets introduced if H finds the exit at a point such that both $P1$ and $P2$ are equidistant from H. For the time before the exit is found at this point of equal distance, evacuation is the responsibility of $P1$, and after this time, it is the responsibility of $P2$. The bound will occur after all the agents have travelled for $c+x$ time, and by this point, $P2$ will have begun its detour. The point of equidistance occurs when $P1$ and H are $2\sin(c+x)$ apart, and as such the distance from $P2$'s point inside the circle to H's point on the perimeter is the same.

$$2\sin(c+x) = \sqrt{2}\sqrt{(x-1)\cos(g+h+x) - x\cos(g+x) - (x-1)x\cos(h) + (x-1)x + 1}$$

4 Stacked Equations Temp