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Optical anisotropy study in coupled quantum wells, a novel source of undisturbed systems

Doctoral Thesis in Applied Sciences
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ABBREVIATIONS

BZ	<i>Brillouin zone</i>
QS	Quantum Structures
QW	Quantum Well
SQW	Single Quantum Well
CQWs	Coupled Quantum Wells
VB	Valence Band
CB	Conduction Band
SCQWs	Symmetric coupled quantum wells
ACQWs	Asymmetric coupled quantum wells
RAS	Reflectance Anisotropy Spectroscopy
PL	Photoluminiscense spectroscopy
PR	Photoreflectance spectroscopy
R	Reflectance spectroscopy
PRD	Photo-Reflectance Differential Spectroscopy
FDM	Finite differnce method
CCD	Charge coupled device
0D	Zero-dimensional
1D	One-dimensional
2D	Two-dimensional
3D	Three-dimensional
fcc	Face-centered cubic
<i>hh</i>	Heavy hole
<i>lh</i>	Light Hole
<i>e</i>	electron
2DEG	Two-dimensional electron gas
BL	Beer-Lambert-Law
TB	Tight-Binding method
PD	Photo-Detector
QM	Quantum Mechanics
$k \cdot p$	Semiempirical theoretical tool to calculate band-structure
TB	Semiempirical Thight-Binding Method
SOC	Spin-Orbit Coupling, also called Spin-Orbit interaction
LFLM	Dr. Luis Felipe Lastras Martinez group.
EFA	Envelope function Approximation
EMA	Effective Mass Approximation
OA	Optical Anisotropy

LIST OF CODES AND PACKAGES

This list denote the *Open-Source* packages, codes, tools, and repositories to develop this work. All inside of this work as images or numerical calculations are subject to the *Open-Source* ideology. Our codes are it housed in own GitHub repository, both personal as the laboratory repository. It's importantly to say that without develop of Open-Source codes like contents in this list, our codes they couldn't been enhanced.

ASE The Atomic Simulation Environment (ASE) is a set of tools and Python modules for setting up, manipulating, running, visualizing and analyzing atomistic simulations. [17]

Spglib Software library for crystal symmetry search [18]

SOLCORE A multi-scale, Python-based library for modelling solar cells and semiconductor materials [19]

Aestimo One-dimensional (1D) self-consistent Schrödinger-Poisson solver for semiconductor heterostructures [20]

VESTA 3D visualization program for structural models, volumetric data such as electron/nuclear densities, and crystal morphologies. [21]

PGF/TikZ PGF is a macro package for creating graphics. It is platform- and format-independent and works together with the most important TeXbackend drivers, including pdfTeX and dvips. It comes with a user-friendly syntax layer called TikZ. [22]

kp-lflm-group $k \cdot p$ julia [23] package developed by Our group research [24]

SYMBOLS

X⁻ Negative Trion

X⁺ Positive Trion

IX Direct Exciton

IX Indirect Exciton

$\text{Al}_x\text{Ga}_{1-x}\text{As}$ AlGaAs semiconductor as a function of Al concentration x

\hbar Planck's constant (eV)

m_0 electron effective mass

(hkl) Family of lattice planes with Miller indices h , k and l

1

PHYSICAL BACKGROUND

In this chapter raise and explains the basis of fundamental physics in semiconductor and quantum structures, implemented to understand the results in this work.

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QUANTUM MECHANICS is basically electron behavior that exhibits many phenomena non explained by classical regime. Quantum structures (QS) are artificially systems conformed by semiconductors where electrons exhibit their quantum nature, this is a great platform to study and create quantum devices. Nowadays, the progress in creation of QS consist in precisely deposition of thin films, in which electrons show fundamentally new electrical and optical properties [25]. Most of these properties consist in quantum behavior as the energy confinement, which is the principal interest to study the electron and their consequent interactions which generates analogous hydrogen atom in semiconductors. Therefore, the interest to studying QS was increasing for many years ago and nowadays, those continue considering an excellent research area.

In this chapter, it presents the fundamental concepts to describe the physical phenomena resultant in this work, without intention to replicate concepts and models already explained in publications with major impact. Therefore, the purpose is to present an own interpretation to highlight the great obtained results.

1.1 Semiconductor Bandstructure

To starting with physical background to understand the QS, it has to start with understand the band structure of semiconductors. The band structure describes the electron behavior in a solid, therefore, we will need to invoke the Schrödinger equation to describe it behavior. But, due inside a solid around 10^{23} valence electrons contribute to the bonding in each cubic centimeter, this results in a many-body complex problem [26], then the general hamiltonian for a solid has the form [27, 28]:

$$H = \frac{1}{2M} \sum_{i=1}^{N_n} \mathbf{P}_j^2 + \frac{1}{2m_0} \sum_{j=1}^{N_e} \mathbf{p}_j^2 + \frac{Z^2}{2} \sum_{i,j=1, i \neq j}^{N_n} V_c(\mathbf{R}_i - \mathbf{R}_j) - Z \sum_{i=1}^{N_n} \sum_{j=1}^{N_e} V_c(\mathbf{r}_j - \mathbf{R}_i) + \frac{1}{2} \sum_{i,j=1, i \neq j}^{N_e} V_c(\mathbf{r}_i - \mathbf{r}_j). \quad (1.1)$$

Where N_n is the number of atomic nuclei, N_e is the number of electrons with mass m_0 , asumming that the nuclei are the same mass it's consider M and charge Z_e . As is obviously, this Hamiltonian is so complicated, the sum of five terms which consists in : kinetic energies to electrons and nuclei, the nucleus-nucleus, nucleus-electron and electron-electron Coulomb interactions, also \mathbf{R}_i are the positions of the nuclei and \mathbf{r}_j are the position of the electrons, the operators \mathbf{P} and \mathbf{p} are momentum operators to nuclei and electrons respectively. Finally, the consider of Coulomb potential V_c [27].

Fortunately, the QS are formed by crystalline materials, the Bloch theorem provide the most important tool to develop required equations. The Bloch theorem establish a periodic potential $U(\mathbf{r})$ for electrons, this due the material is periodic (definition of crystal structure) and the Schrödinger equation it can describe in terms of single electron as:

$$\left[-\frac{\hbar^2}{2m_0} \nabla^2 + U(\mathbf{r}) \right] \psi(r) = \mathbf{E} \psi(\mathbf{r}) \quad (1.2)$$

The principal reason that the periodic potential in a crystal structure is highly important is their translational invariance concept and the consequent symmetry operations that are possible in a crystalline solid. The symmetry concept, as a tool to understand solids, is discussed with major detail in the next chapter. In according to Bloch's theorem we can associate a wave vector \mathbf{k} with each energy state, $E_n(\mathbf{k})$. Thus, it is useful to display the energies $E_n(\mathbf{k})$ as a function of the wave vector \mathbf{k} . This result also knowing as dispersion relation, but in general terms is the electron band structure of the given solid [26].

Even if, the calculation of electron band structure in solids are very complex by the distance between atoms that composes it's, and the Bloch's theorem provide the most important tool to reduce the problem to crystalline structures, exists several methods to calculate the realistic bandstructure for semiconductors that are categorized in two groups: Atomistic methods* (Tight-binding, orthogonalized plane wave methods) and Perturbative methods $(\mathbf{k} \cdot \mathbf{p})^\dagger$. These two main categories with theirs respective methods have special characteristics which becomes in the reasons to choose them. The reasons have to do with to the described bandstructure, this mean, in case of Atomistic methods the entire bands (valence and conduction) can describe, but in case of perturbative methods are reserve to near bandedge bandstructures. So, each of these methods can be chosen and enhanced as the system to study requires. We won't enter in discussion about of which of these methods are the best, the reasons are simple, each method is powerful, and we must be remembered that the complexity of solutions requires that these are solved by numerical techniques therefore this convert it to good approximations. Then, basically the electron behavior inside a semiconductor consist in solutions of the appropriate Schrödinger equation [29].

1.1.1 Valence and Conduction Bands

The most important characteristic of semiconductors even we can call as the fingerprint of these, is their bands structure, this characteristic sort out the solids as insulators, metals, and semiconductors. These are the reason of many mechanisms and phenomena

*In this category can include the ab initio methods, these are the most complex methods due to propose solutions of the many-body problem

[†]In fact, both TB and $\mathbf{k} \cdot \mathbf{p}$ also consider in same kind, because both are *semi-empirical* methods due to they consider experimental parameters.

by which study this structures. In general, the bands of semiconductors composed by valence and conduction bands separated by a region known as bandgap. The bandgap, which is proportional to separation energy of valence and conduction bands also is called as forbidden region, this is because doesn't exist electron states, therefore this gap energy determine the electron conduction in a semiconductor and the difference they have with the insulators and metals as shows Figure 1.1.



Figure 1.1: Band energy diagram for insulators (left), semiconductors (center) and metals (right). The principal difference is the gap energy, for insulators this is longer than semiconductors, although in semiconductors gap energy depends on materials, finally in metals doesn't exist gap energy instead exist an overlap bands characterize these. Dashed line determine Fermi's level.

So, the bandgap determines many characterizes and functionalities in semiconductors. The bandgap energy classifies semiconductors in direct and indirect semiconductors, but doesn't only depend on this energy, in reality the band structure is the liable for this. As before mentioned, it is so difficult to describe electrons behaviors over the solids due to the many body interactions that exists in its, and therefore the Schrödinger equation that's describe electrons behaviors is complicatedly to solve. Fortunately, the semiconductor structures have one of the most important characterize and his has to do with their atomic structure, that's periodically arrangement of atoms. This periodicity is the key to propose solutions and describe the semiconductor band structure. Starting with describe bulk semiconductors, for example GaAs which consists in with the family III-V cube semiconductor so that their lattice structure consist in a two sublattices correspond to each atom which it conform as shows in Figure 1.2. For this case, when the atoms in two sublattice are different, the crystal structure is then called *zinc-blende* [30].

To calculate bandstructures of bulk semiconductors it's important to define specific symmetry direction, this mean that it's not possible to plot dispersion relation. For each three-dimensional wave vector \mathbf{k} , then the plot energy as a function of \mathbf{k} is along

of different high-symmetry directions [26]. In this work the structures to study are composed of semiconductors III-V, being GaAs the bulk in each structure. GaAs is a direct semiconductor so that the [001] direction is the high-symmetry direction then is denoted by Γ point (at $\mathbf{k} = 0$)

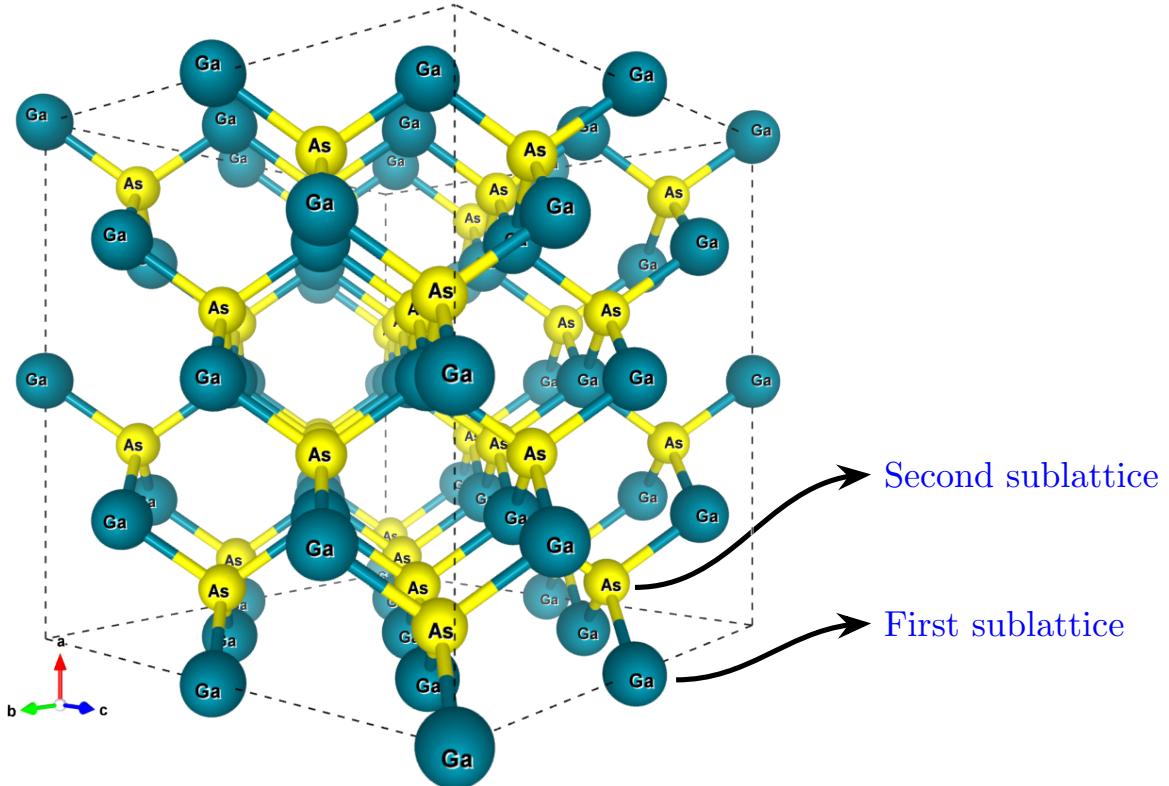


Figure 1.2: GaAs crystal lattice, where the each sublattice correspond of each atom species Ga and As.

As before mentioned several times, it's very complex to compute the Schrödinger equation in the solids. The most “exact” compute is employed by DFT theory, these calculations commonly are called atomistic even some semiempirical models can consider as atomistic, but the semiempirical models are good approximations in comparison with the DFT theory. So, which is the reason to call “exact” solutions to the DFT results? The answer leads us to great discussion and it's not intended to get into controversy, but in general the DFT calculations have the capacity to calculate in terms of electrons interaction and the empirical methods are based in potential choice.

We will, don't into details about band calculations theories and models, but we will make a general reference to the importance in this work. The models more performed in semiconductor heterostructures as GaAs/AlGaAs are semiempirical, this is because DFT theory and their derived models are very limited to carried out in large structures,

their electron interaction nature need high computational perform. So that in comparison with empirical models where the main role is the potential of semiconductor structures, this reduces computational reduce. So, the most models used in semiconductor band calculations are empirical models, these models are distinguished by low computational requires for this reason are considered like approximations. The importance to discuss these concepts will take relevance when we discuss the physics model proposed in this work.

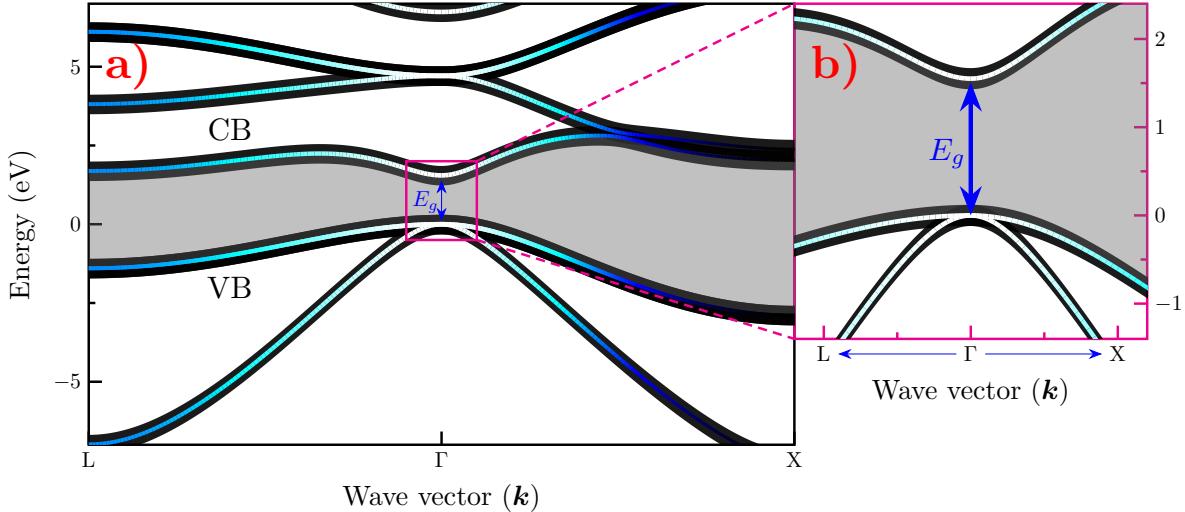


Figure 1.3: Band structure of GaAs, (a) shows the zoom around of Γ to denote the direct band gap and the electrons energy needed to jump from valence to conduction band. (b) denotes the two directions to dispersion of the bands corresponds to Brillouin zone: $\Gamma \rightarrow X$ and $\Gamma \rightarrow L$. [1]

The Section 1.1.1 shows the results of calculations of TB model as discuss it in [31] and the code was implemented by R. Muller [32]. The model purposes by Vogl et al. take into account small number of localized pseudo-orbitals and based the empirical parameters to substitute on TB Hamiltonian. The importance to get bandstructure it's based importance to study optical properties of solid structures, if it doesn't exist band electrons it's like look a place without map, so, the band structure far from being a complex tool it's the key to get the information to investigate the optical properties.

As shown in Figure 1.3(a) the GaAs bandstructure shows that it's a direct semiconductor as a previously mentioned, this gives way to get electron transitions from VB to CB and the energy to success this it. In Figure 1.3(b) it's plotted, the band dispersion around Γ point, it's the most symmetry point. It is well-known that the band dispersion increasing \mathbf{k} along two different directions of the Brillouin zone, from $\mathbf{k} = (0, 0, 0)$ to X point $\mathbf{k} = (2\pi a_L)(1, 0, 0)$ and L point $\mathbf{k} = (2\pi a_L)(1, 1, 1)$. So, these figures, are the typical representation of direct gap to III-V semiconductors around of $\mathbf{k} = 0$, then, is obviously that the shape of dispersion is parabolic. For the GaAs bandstructure calculations shown

in Section 1.1.1 doesn't take into account the contribution of spin*, therefore it's focus on three bands dispersion correspond to a single *s*-like conduction band and two *p*-like valence bands. Is important to say that the characteristic curvature $E-\mathbf{k}$ of dispersion bands corresponds to an electron (*e*) in case of positive curvature while the negative curvature correspond to holes states; heavy (*hh*) and light hole (*lh*) bands, so, this denotes that the transitions, are of dipole nature [1, 28]. All the above disputed is significant to refers of one

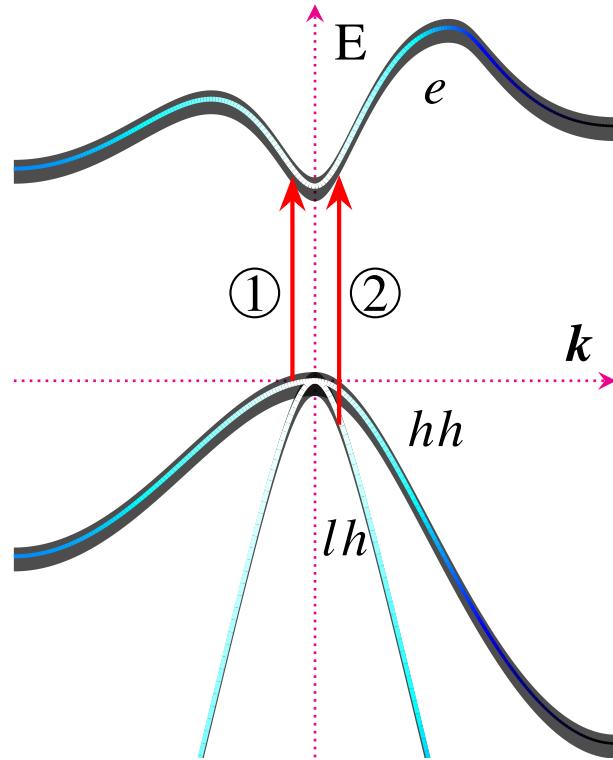


Figure 1.4: Two typical transitions for GaAs near $\mathbf{k} = 0$. The first one correspond to the heavy-hole and second one to the light-hole.

of the most indispensable quantum mechanisms in solids, this is absorption. The electron absorption, specifically interband absorption, give way to a fundamental physical process that involves the principle of many basic studies of semiconductors, applications, and the importance to understand the electron behavior in a semiconductor structures disputed in this work. Then the Figure 1.4 schematizes the two typical transitions in GaAs bulk, this transitions are near to $k=0$, so that is called interband absorption. Is important to remark that the interband transitions are observed in all solids, but the mechanisms are different dependently of their bandstructure, for what, being repetitive when mentioning that the bandstructure is the key to study solids.

In case of GaAs the interband transitions are called as direct transitions, this is because their bandstructure proofs GaAs is a direct Gap semiconductor. This process is determined by quantum mechanical rate $W_{i \rightarrow f}$ for exciting an electron in a initial quantum state

*This spin contribution is called as split-off (so) hole band.

ψ_i to final state ψ_f by absorption of a photon of angular frequency ω [1]. As is very known, this is given by Fermi's golden rule. Later this is disputed according to highlight the model and results obtained. It has been mentioned that the direct transitions are of dipole nature, as before mentioned the CB is type *s*-like while the VB is *p*-like, then it's electric-dipole allowed transitions $p \rightarrow s$. The excitation of electron in CB carries to leave an initial state unoccupied, this is called as a hole creation, then the electron in the final state and the hole is considered as **electron-hole pair**. The next part subject this theme with major focus.

1.1.2 Excitons

The importance to study bandstructure of semiconductors is very clear so far, so that could be said that absorption process is the source of optical properties of solids. It's due to this that the importance to study of semiconductors in the optoelectronics applications. But, this process give rise to formation of one of the most important excitations in the crystal structures. The photon absorption process carries to an electron is excited from CB to VB, this generates an empty location in VB which has positive charge. This positive empty location called as hole, therefore the electron and hole have opposite charge then it's to be expected that they are attracted, so, this creates a bound state called an exciton [33].

From Figure 1.4 and Figure 1.5 is clearly that excitons are commonly presented in direct band gap semiconductors as GaAs, this was denoted by absorption experiments, after mentioned in ?? this is the cause in photoluminescence mechanism.



Figure 1.5: Qualitative scheme of exciton creation in GaAs as direct gap.

1.2

Semiconductor Low-Dimensional Structures

The previous section engaged to explain the principles of semiconductors, this is the bandstructure, the importance of these is practically the fingerprint of all semiconductor, without bandstructure the understanding of these would be improbable. The first approximations were based in GaAs bulk, their cubic symmetry practically defines their nature and consequently the physical effects as excitons existence. But, what happens if joined several semiconductors with same symmetry and structural parameters? The bulk properties and physical properties are the same?. The answers they are well-known, when two materials with relatively same structural parameters, as lattice constant can create a heterojunction, the union of several heterojunction make up a heterostructure.*

The Figure 1.6 is a general scheme of a heterostructure, in this case is presents three

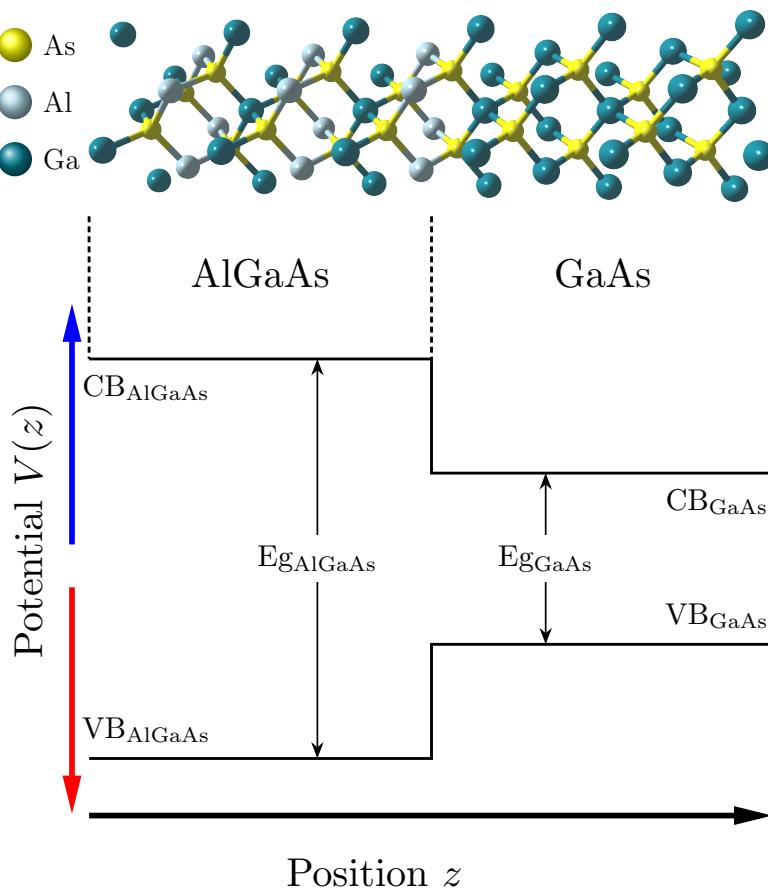


Figure 1.6: General scheme of GaAs/Al_xGa_{1-x}As heterostructure, at top show the scheme of atomic arranged of this heterjunction, the dashed lines are the matched between two dissimilar materials. In bottom shows the band-edge profile.

*The samples studied in this work are heterostructures, for this reason, and by nomenclature it's we refer to that way.

species of atoms Al, As, Ga. These atoms can locate in columns III-V of the periodic table, hence its name of III-V semiconductors. The principal characteristic of these atoms is that it can create matched structures as GaAs, AlAs and ternary alloys as $\text{Al}_x\text{Ga}_{1-x}\text{As}$ with specific Al concentration. The matched semiconductors produce a material with new properties based principally in the difference of bandgap which involves the alloys.

As can see in Figure 1.6 it's consisting a GaAs/ $\text{Al}_x\text{Ga}_{1-x}\text{As}$ heterostructure, these interface is well-matched due to the lattice parameters is relatively equals, therefore and thanks to powerful growth technics as MBE it's possible to get high-quality quantum structures.

Also, the heterostructure composed by two semiconductors with different band gaps generate a discontinuity in either the conduction or the valence band can be represented by a constant potential term [34]. The theory to treatment the electron behavior in these structures, is relatively simple if we consider the above. Although, in this chapter doesn't have intention to explore the theory of electron behavior in that, worth noting that it get one-dimensional potential $V(z)$ to both bands, so the Schrödinger equation can solve simple.

1.2.1 Quantum wells

Doubtless the creation or growth of heterostructures increased the interest in the study of quantum structures, the interactions, and physical behavior of light-matter they would not have been possible without these. The major relevance is due the quantum confinement, the junction of semiconductors results in an interest quantum structures with specific dimensions. From 3D bulk the dimensions reduce to 2D, 1D and 0D dimensional structures. Therefore each of that has interest properties and their correspond applications. The Figure 1.7 schematics the low dimensional heterostructures from 3D bulk, the first low dimensional from 3D to 2D is the Quantum Wells, then from 2D to 1D it have the Quantum Wires finally with 0D have the Quantum Dots. The quantum confinement so is the principal reason to study that structures, the electron behavior which exhibits in it should can to help understand a great variety of quantum mechanical phenomena as electron interaction on a crystal. Suppose a heterostructure composed with a two semiconductor alloys as sandwich, this 2D quantum structure is called a Single Quantum Well (SQW).

This dissimilar semiconductors in terms of their potential energy ($V(z)$) can be schematized as Figure 1.8. The Gap difference of $\text{Al}_x\text{Ga}_{1-x}\text{As}$ and GaAs is due to x Al concentration in $\text{Al}_x\text{Ga}_{1-x}\text{As}$ therefore it obtains a one dimensional potential profile, with that can confinement electrons in a 2D plane along z direction. All of these carries to quantum mechanics formalism, the electron behavior should be obeyed these rules. If we have an electron closed in two potential barriers an L distance, the wave which describe it will be spatially confined. So if we confined many electrons in these potential, we have two important physical aspects: the first one is knowing as Pauli's exclusion principle,



Figure 1.7: Heterostructures from bulk (3D), to Quantum Wells (2D), Quantum Wires (1D) and Quantum Dots (0D).

which as of its Fermion nature prevents carriers with the same spin occupying the same region in of space [34, 35], the second one and one of the most relevant in the birth of the quantum mechanics; the Heisenberg's uncertainty principle. That last, it can say that is the consequence of quantum confinement due to the space reduction of the electrons is expected that momentum increases by an amount of the order \hbar/L . Therefore, the energy of that confined particles increases, and it's referred to as confinement energy [28].

Then the quantum confinement is our started point to understand the optical properties in QWs. As is referred in the figure, the uni-dimensional potential profile can well describe by top conduction- and bottom valence-bands, the band offset in these two of correspond gap energy between that, while Al concentration increases their bandgap and the band offset (Q_c to CB and Q_v to VB) also to.

It's so clearly that the QWs have the potential to presents amazing quantum properties, even if all of these are very important we focus on the optical properties, basically our interest is the light-matter interaction through its result mechanisms.

1.2.2 Preliminary approach of Quantum Confinement effect in QWs

As the title describes, here it will try to explain as the first approach the quantum confinement effect in QWs. If it starts with the scoop, which it can be reduced the



Figure 1.8: GaAs/Al_xGa_{1-x}As Single Quantum Well

electrons space, then this mean that in reciprocal space it has two components k_x and k_y . Then say in crystal symmetry properties to the case of GaAs/Al_xGa_{1-x}As QWs it's Γ the central point, as long as $x < 0.4^*$ the bandstructure depends on confinement energ, so say which the bandstructure depends on confinement energy. In this case, the electronic properties in comparison with a bulk semiconductor properties can solve trough particle-in-a-box as textbook problem as first approach. Nevertheless, even if usually can solver without much mathematical formalism is very essential that it dedicates a chapter with their solution, this is because will employ a physical formalism exclusively to QWs structures. In general way, as it before mentioned the Schrödinger equation solution is the fundamental pillar to understand, where it's taken into account which in a crystal the periodic potential is the key. Here are important remarks before to continue, when it has a heterostructure starting with the bulk model it's clearly that the system doesn't same, the Quantum Mechanics which is behind take into account the symmetry properties, then it can be developed a Hamiltonian to understand that system. In the next chapter will be explained details and the formalism both physical and mathematical to solve and discuss it's. The model which give the tools to get the solutions is called as Effective Mass

*It will be explained in the next section, although it's due to the Gap go from direct to indirect, shortly the symmetry $\Gamma \rightarrow X$.



Figure 1.9: General scheme of typical Schrödinger's equation solutions to one-dimensional potential as (a) where the Eigenenergies of both electron and holes are denoted with same color depending on n value.(b) It's plot, of the subbands in the same case of (a) to both particles.

Approximation, thus their correspond Schrödinger equation is [1, 34, 36–39]:

$$-\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial z^2} \psi(z) + V(z)\psi(z) = E\psi(z), \quad (1.3)$$

where the m^* is the effective mass in each material, and $V(z)$ is the potential profile got by heterostructure materials properties. Therefore, that differential equation can solve as in textbooks explained [1, 34, 36, 38, 40–43]. The idea is thinking as a one particle in a finite potential well, where is well important established the boundary conditions and solve the Schrödinger equation in each part of single QW, this means that need to create a potential function. Then it's can obtain the Eigenfunctions and their correspond Eigenenergies.

The principal idea doesn't is reproducing something which is very well known, the objective of this part is established the scoop of the next chapter. Therefore, before to continue, we will finish with to explain the dispersion in-plane of single QW. As in the QW the one-dimensional potential set up the 1D confinement, is important doesn't confuse which the QW is a 2D structure, but their confinement is along of z direction this mean 1D. Then, the particle can motion in the $x - y$ plane. By this reason, even if consider 3D Schrödinger equation and the above is considered it obtain Equation (1.3) therefore, the solutions in the one-dimensional potential produce discrete states of energy $E_z = E_n$ [34],

where n is the energy level it which produce subbands as shows in Figure 1.9. In contrast, before it called as “energy bands” in the bulk case, now due to the quantum confinement gets subbands to both conduction- and valence bands.

These subbands are the result of the sum of E_z and $E_{x,y}$, which are the 1D confinement energy and the in-plane momentum $k_{x,y}$ then [34]:

$$E = E_n + \frac{\hbar^2 |\mathbf{k}_{x,y}|^2}{2m^*}. \quad (1.4)$$

From equation the effective mass m^* depends on particle, i.e the effective mass to electrons in CB and the holes in VB. So, the most relevant in the solutions is the energy E_n (Figure 1.9(a)) is discrete, this is the quantum confinement in the low-dimensional heterostructures.

1.3 Summary

In this chapter, was exposed the generalities of semiconductor band structure and low-dimensional heterostructures, highlighting or taking in major GaAs/Al_xGa_{1-x}As that's the semiconductors of major importance in this work. The band structure interpretation usually be so hard, and their calculations even more, but the impact and relevance in optical properties of semiconductors starts from that interpretation, from here arises the mathematical arsenal to right physical interpretation. Another significant concept which was treatment as first approach is the effective mass concept, even if, when solved, the bulk Hamiltonian it considers the mass as constant parameter or depending on semiconductor material, contrary in low-dimensional structures have an important role.

In generally, the band structure of semiconductors is the key to understand quantum properties of solids, in this work the relevant is the light-matter. Remember that light-matter interaction in solids can be studied by process resulting in it, as absorption, reflection, transmission, diffraction, scattering, and others [44]. Although, the light-matter interactions are fundamentally quantum electrodynamical, also, can be studied in quantum way through before mentioned process. Firstly, is the photon absorption process can help to understand or calculate the fundamental parameter in semiconductors as is bandgap energy. This parameter is the started point in the study of semiconductors, this is the start point in the map called bandstructure, if it ignores the gap value in the semiconductor to study it couldn't possibly get the principal optical properties of that.

Then, the bandstructure of semiconductors is the map to understand them, without these routes or fundamental parameters couldn't have quantum devices.



PHYSICAL MODEL

In this chapter, it's exposing the fundamental physics to understand the experimental results' through physical model, both numerical and phenomenological. It is emphasis in symmetry properties of semiconductors, which is the base to get our model.

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SEMICONDUCTORS are alloy of materials with pure structural characteristics, it's the alloys which can generate an amazing quantum process that is quantum confinement. But, all of them couldn't possibly be achieved without understand of semiconductor bands, are bands the reason to get semiconductor heterostructures. Now, the objective is proposing the model to study Coupled Quantum Wells. The CQWs are heterostructures grown from a semiconductor substrate as GaAs as previously treated SQW (see Figure 1.8) but coupled by a thin barrier which have a very significant role. But before to mention the objective of study this structures and the physical model which explain their experimental results is important to call about of the symmetry and their relevance to understand the physics of these QS.

Previously, we were very repeating in the importance of semiconductor band structure, also to remark difficult of get them. It was saying the complexity of calculate semiconductor bands is high, by this reason it's developed models to approximate it. But, all of these couldn't be possible if it would not be taken into account the *symmetry* role in physics [45]. Thanks to symmetry, it's possible to understand from Quantum to Universe. It is clear that to talk the symmetry is inevitable to think in geometry or nature patterns inherently way, but in the next sections it's established the symmetry role from the band structure calculation to the importance in the electron behavior in CQWs structures.

2.1

The Symmetry Context

Talk about *symmetry* is talk of shapes or in a romanticism way as the natural harmony that makes something appear beautiful to us [46,47]. But, what is the reason the *symmetry* is very important in physics?. The reason the *symmetry* is very important has to do with a *transformation*, this mean that if a physical system is affected or perturbed by a thing and this appears to be exactly the same before and after that *transformation*, it is said to be *invariant* under that *transformation*. The symmetry of the system is made up of all the transformation operations that leave the system invariant [46].

In this section doesn't have the purpose to be a one more copy or re-interpreted version of a group theory book, yes, the group theory not symmetry theory, it's important to remember that to understand of symmetry physics of solids it's important to understand the group concept. In generally, a group is a set or collection of elements that obey certain criteria and are related to each other through a specific rule of interaction and obey four group axioms [46, 48, 49]. It is important to remark which the heterostructure to will study is composed by $\text{GaAs}/\text{Al}_x\text{Ga}_{1-x}\text{As}$ semiconductors, then we started with the GaAs crystal to propound the symmetry role in the CQWs. Then, already raised the starting point, remember that the crystal solid can be defined as an arrangement of atoms in strictly periodic arrays [50,51], from here arises two concepts: basis and lattice, where that

last is the set of mathematical points to which the basis is attached [50]. These crystal concepts give place of crystal primitive cell in three dimensions also considered as the seed to reproduce a crystal. So, it gets fundamental types of lattices defined by a collection of *symmetry* operations (rotation, translation etc.), then it's compose a lattice point group. In the three-dimensional case, the point symmetry groups require the 14 different lattices types*, where are classified into seven systems. Into these systems it's found the *cubic* system, it which posses three number of lattices. Remember that the GaAs crystal is *cubic*, specifically, is the type FCC lattice. The FCC lattice is easy to imagine, if place an atom in each corner of a cube and in a center of each face of it. Therefore, it is easier to define the planes and crystal directions if we take the cube faces as a reference it gets (hkl) plane and the directions [hkl] which must be perpendicular to a plane (hkl) [50].

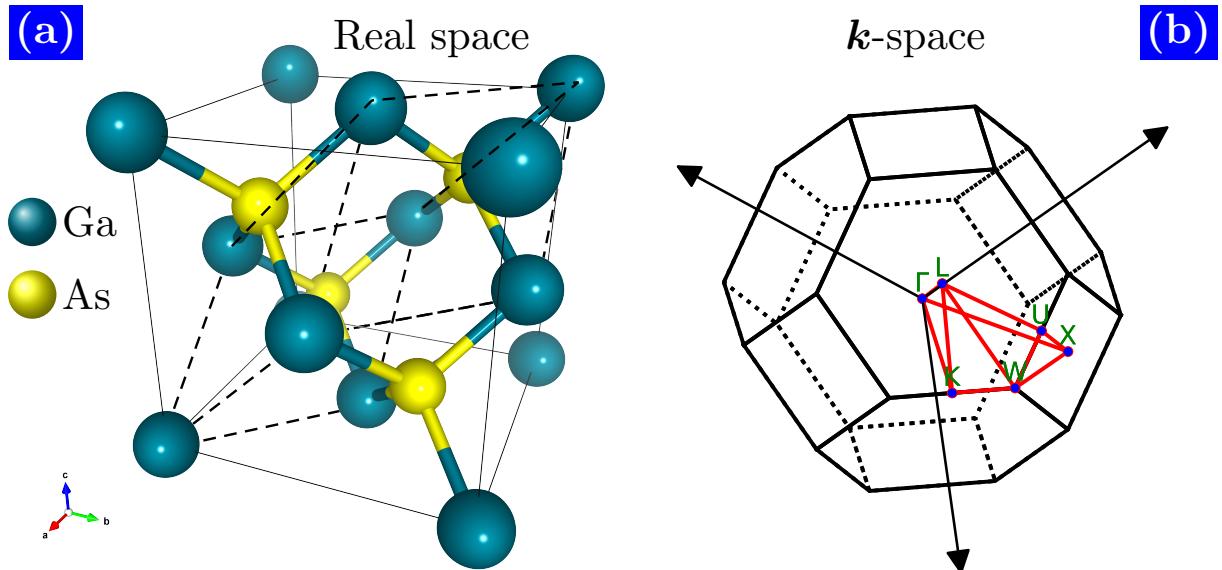


Figure 2.1: (a) GaAs crystal structure in “real space”, this region is known as *unit cell*, into that with dashed line is denoted the primitive cell. This lattice is well-defined by the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} , these vectors are defined as the basis vectors. In (b) is schematized the GaAs crystal structure in k -space, also known as *reciprocal space*.

The lattice is an array of points which make the space lattice of a *crystal* and the repetitions or disposition of these points is controlled by “*symmetry* operations” [52]. The crystal is composed by a space lattice this is a plane lattice[†] which have three *symmetry* operations: *Rotation axes*, *Mirror plane* and *Centre of Symmetry*. If add a one dimension to plane lattice it gets a space lattice, which define the unit cell of a *crystal*, so this adds one more *symmetry* operation which is *Rotation Inversion* or *Roto-Inversion*. Then, can get *symmetry* elements of a *crystal* if apply the four *symmetry* operations and their possible combinations. If collect that *symmetry* elements obtains the *point symmetry* or the *point*

*This lattices are known as Bravais Lattices

[†]2D point pattern array

group of symmetry of a *crystal*. The GaAs crystal as before mentioned is a *cubic* system, but have a defined *cubic* structure called as *cubic zinc sulfide* or simply *zincblende*. This specific *cubic* structure is characterized by arrangement of two type atoms with places coordinates: $000, 0\frac{1}{2}\frac{1}{2}, \frac{1}{2}0\frac{1}{2}, \frac{1}{2}\frac{1}{2}0$ for one type of these as Zn in ZnS or Ga in GaAs structure. In case of the second one atom, it has coordinates : $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}$ for S in ZnS or As in GaAs [50, 53]. The Figure 2.1(a) shows the unit cell of GaAs structure and into them it dashed the primitive cell for the FCC lattice, also denoted the basis vectors \mathbf{a} , \mathbf{b} and \mathbf{c} .

In the case of symmetry of GaAs crystal, is important to remark that this symmetry can also denote in Hermann-Maguin notation $F\bar{4}3m$ which corresponds to three fourfold rotary inversion parallel to the edges of a cube, with four threefold rotation axes parallel to the body diagonal and six mirror planes, each containing a face diagonal [52]. The F label corresponds to cubic system FCC, following by the corresponds operations.

The symmetry context before exposed can view as a macroscopic symmetry about a crystal system, which means, is very interactive to think as a pattern well-ordered can conform a plane lattice and is intuitively work the symmetry operations. But it's not the only symmetry concept in crystal systems, if we enter into crystal it found atoms or molecules which conforms it. So, the internal study of a crystal add two symmetries to the actual worked before. These "microscopic symmetry" [52] the make reference to \mathbf{k} -space or reciprocal space. So, the previous concept of lattice it's also known as direct space lattice. Thanks to X-Ray, Electron or Neutron diffraction techniques, it was possible to study the internal structure o crystal symmetries in the reciprocal space, this trough diffraction phenomena, the propagation of waves into crystal can to form well defined pattern they which are explained by the wave-vector concept [46, 54]. Therefore, is expected that the electron wave function can be denoted with a lattice periodic part $u(\mathbf{r})$ and wavelike part $e^{i\mathbf{k}\cdot\mathbf{r}}$ so, the set of all wave vectors \mathbf{k} corresponds to plane waves due the lattice, this is known as reciprocal lattice [55]. Then taken into account this, and the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} in reciprocal space can describe the total unit cell [46, 55].

Finally, as a result to get the unit cell in reciprocal space and known which this is composed by lattices, these lattices are called as *Brillouin zones*. Practically the *Brillouin zones* are constructed by drawing the vectors \mathbf{K} defining the reciprocal lattice and then bisecting each of these with planes perpendicular to \mathbf{K} [46]*. In Figure 2.1(b) it's schematized the *Brillouin zone* to GaAs crystal structure, specifically this representation is called as *first Brillouin zone*. To GaAs crystal it was defined the symmetry operations which compose the symmetry elements in Hermann-Maguin notation as the point group $F\bar{4}3m$, it's important to consider the Schoenflies notation also, this due people often speak in terms of both, although the Hermann-Maguin notation is consider as the International notation. In Schoenflies notation, the GaAs correspond to T_d point group.

*The wave vector \mathbf{K} is defined in [56] equation (1.5)

2.1.1 The symmetry and the Band Structure

Returning to Figure 2.1(b), the Brillouin zone have labels which they are, importantly, this is because each of these denote a point group symmetry. These points are: Gamma, X, L, W, U, K. In Schoenflies notation these correspond to: $\Gamma \rightarrow O_h$, $X \rightarrow C_{4v}$, $L \rightarrow D_{3d}$, $W \rightarrow C_{4v}$, $U \rightarrow C_{2v}$, being Γ the high symmetry point. Then, why is the importance of the **BZ** role in semiconductor band structure?, the answer is the aim of this subsection. We started with the first section of this work, in it refer the importance of solution of Schrödinger equation, specifically at crystal structures as semiconductors. Here, the most important tool is the Bloch's theorem, it which is developed from periodic property of crystal so, it's possible to approximate.

This context is introductory and general, because this doesn't possible if not consider the symmetry properties in crystals, in fact, *the symmetry of system define the basis function to get the electron band structure* [28, 57–59]. Remember that the concept of *basis function* is a mathematical concept, which in quantum mechanics it's known as *Wave functions*. Also, never to forget which the *symmetry* concept is inherent in physics. Therefore, the Group theory establishes the game rules. So, the **BZ** is the result of Group theory applied in crystal structures, then the **BZ** is the map to understand the electron behavior in crystal structures, this defined the \mathbf{k} points trough high symmetry paths, where this starts at Γ point or $\mathbf{k} = 0$. If observe the Section 1.1.1 the horizontal axis correspond to \mathbf{k} points and labeled the high-symmetry directions from Γ , then this is the \mathbf{k} paths in **BZ** as can see in Figure 2.1(b).

Previously, it's continuously mentioned which band structure calculations are difficult, so, it will start to change the hard word to tricky, this because it's possible to make very good models and approximations taking into account the symmetry of the system. All to begins from symmetry, the well-known models to calculate band structure starts from symmetry arguments of crystal or the semiconductor studied, through succession of symmetry operations it knows until it's invariant, this mean doesn't change under transformation. Here, highlights the invariants concept, which is the connection of symmetry and Quantum Mechanics. The symmetry gets the information of the system, while the QM the information of the state electrons. The Hamiltonian of the crystal has a symmetry which depends on their potential, then the crystal potential posses a point group, which is invariant under any transformation. So, the solution of the Schrödinger equation contains the state of the system. From these tools, can propound the physics of electrons or another quasi-particle inside a semiconductor, for example in perturbation theory, starts from Hamiltonian \mathcal{H}_0 with it specific space group, but under perturbation the Hamiltonian of the system should be the sum of $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$, where this last has the symmetry correspond to a subgroup of the \mathcal{H}_0 group. This is, the principle of this work which after will be discussed with detail. While the solution of Schrödinger equation with the total Hamiltonian \mathcal{H} will result in the energy spectrum $E(\mathbf{k})$ along of the **BZ**.

Being a crystal system and the potential is the periodic, it's to hope which a multiband spectrum. Although here doesn't consider the degeneracy* term, it's evident which the Group Theory has the solution, in general words, are the irreducible representations of the symmetry group which determine the dimension of degeneracy [60].

Thus the band structure as a whole exhibits the symmetry characterized by the crystal [60]. All previous it's about of an ideal crystal, then it's possible to get exact solutions of Schrödinger equation. But, to determine in detail the spectrum $E(\mathbf{k})$ throughout the **BZ**, one needs a numerical solution of the Schrödinger equation. In previous sections, it shows the results of apply **TB** method to GaAs bulk, this method parts from Bond Orbital Model [31, 61, 62], in this case, the basis functions it's forming linear combinations of atomic orbitals (LCAO) to specific symmetry group [57].

In this method, the importance is the arrangement of atoms and their orbitals considered, for Figure 1.2 these are sp^3 . In another way, in the case of $\mathbf{k} \cdot \mathbf{p}$ method, apart from perturbed model, but in both the main idea it's found the \mathbf{k} -points correspond to the symmetry of the system. In another way, in the case of $\mathbf{k} \cdot \mathbf{p}$ method, apart from disturbance model, but in both the main idea it's found the \mathbf{k} -points correspond to the symmetry of the system. The difference apart from their basis is the efficiency in their applied over semiconductor structures, this means that the $\mathbf{k} \cdot \mathbf{p}$ -method is appropriate in a small region of **BZ** to describe $E(\mathbf{k})$, therefor is the preferred option to describe semiconductor bands around of Γ , while if the idea is describing $E(\mathbf{k})$ in an extended region of the **BZ** the **TB**-method is the correct [57, 60]. In any way, the symmetry establishes the basis to get semiconductor band structure, no matter the method this, includes the first principle methods as DFT, which requires the symmetry information of the system to get the pseudopotentials and the geometric optimization to enhance calculations.

Although the symmetry concept is the base, the principal objective in this work you have to see the consequences of symmetry under in a non-direct perturbation, this mean, if it has GaAs bulk-semiconductor this has one defined point group T_d . When applied it a perturbation, as an electromagnetic perturbation the principal symmetry group it's reducing to a subgroup of itself. This is to mean which the subgroup is also invariant. This conception is also knew or called *symmetry breaking*. It this concept, it was employed at first time by Pierre Curie at ended of nineteenth century [63, 64], It this concept, it was employed at first time by Pierre Curie at ended of nineteenth century, he establishes that, if happens thing which doesn't allow system invariance, so the original symmetry it's lowered then, this mean which symmetry is broken. But, what is the importance to focussing on that?. The importance of symmetry breaking is the physical effects that are presents, the properties of semiconductors changes under reduce symmetry and this can observe in band structure. In the next section will be discussed the importance of symmetry breaking in semiconductors and above all over QWs.

*This is due to the linear independent solutions, which corresponds to one energy, this mean m -fold band degeneracy at the point \mathbf{k} [60]

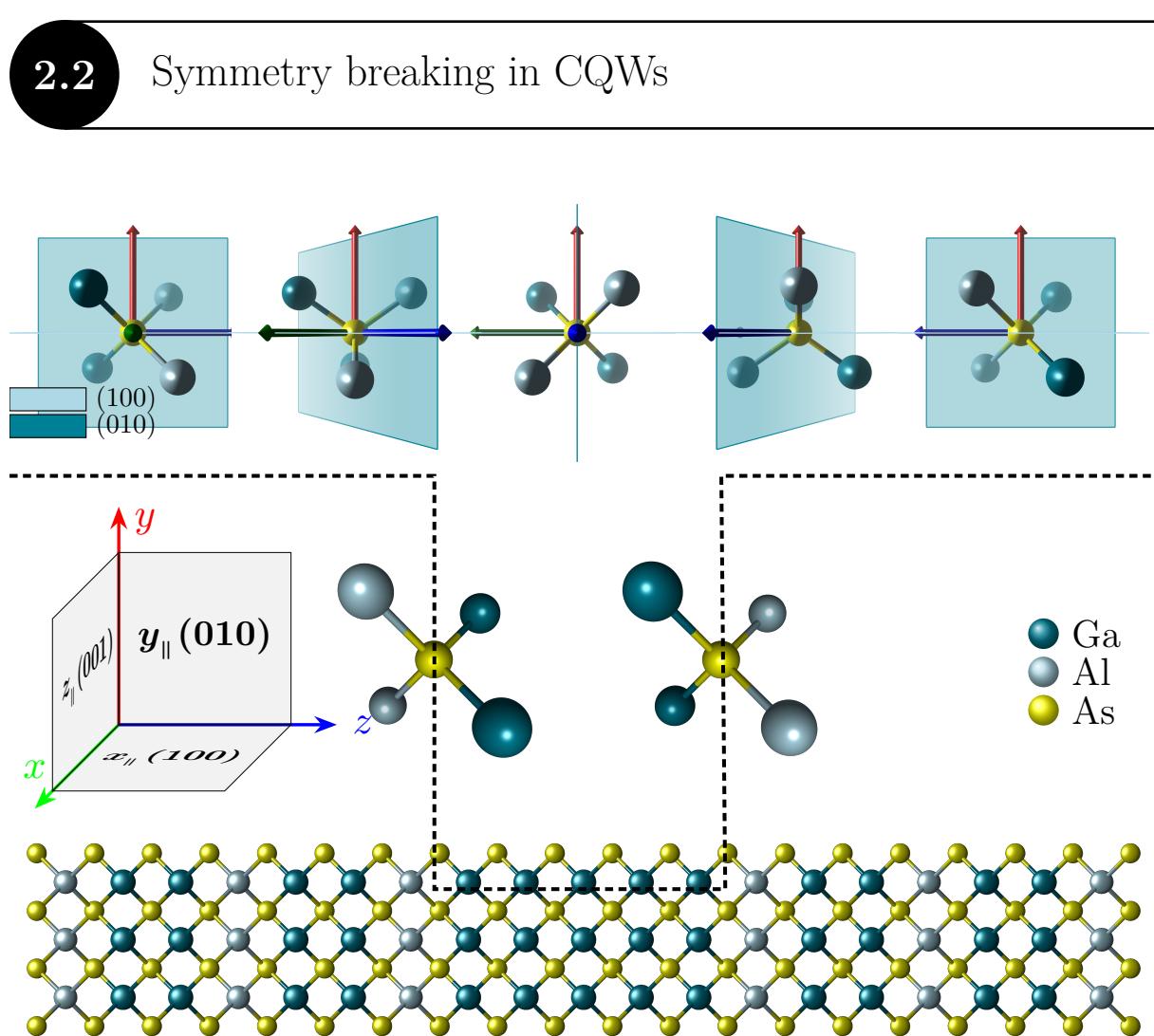


Figure 2.2: In this figure, it's schematized the atom arrangement of $\text{Al}_x\text{Ga}_{1-x}\text{As}/\text{GaAs}/\text{Al}_x\text{Ga}_{1-x}\text{As}$ single QW. At bottom can see the atom structure assuming the Al concentration to barriers, then, in middle of figure draws the conduction band edge as a potential profile, at interfaces between the change from barrier to well and vice versa, are located the matched atoms, here it can observe that at these positions (interface) the symmetry of changed. At top, it's added the atoms' basis where it has taken the Arsenic as central atom, and scheme the continues rotation over the y -axis.

The symmetry breaking, it's the basis for the physical model in this work. Starting from the general and brief concept of the symmetry importance in the Solid State viewed in the past section will arise the symmetry role and the reduced symmetry in CQWs. Before starting the history, it's important to clarify that the SOC isn't considered in this work, although in the future works of the **LFLM** are considered, and they study spin phenomena in CQWs. Then, to start in terms of symmetry the GaAs bulk has T_d point group, without intention to minimize the operations only focussing on their subgroups this due as before mentioned if the symmetry is lowered then the point group reduces to one of subgroup [57].

So, in the superlattice case the symmetry reduces at interfaces due to the change of atom species, that is to say, if it parts from GaAs and added an AlAs or $\text{Al}_x\text{Ga}_{1-x}\text{As}$ lattice the new atom structure reduce symmetry elements that can be done then the point group T_d it's reduce. Let's discuss first the simple QW GaAs/ $\text{Al}_x\text{Ga}_{1-x}\text{As}$ structure grown over [001] direction, if suppose that the $\text{Al}_x\text{Ga}_{1-x}\text{As}$ /GaAs and GaAs/ $\text{Al}_x\text{Ga}_{1-x}\text{As}$ interfaces* are “perfect”, this mean the QWs structurally are perfect, the symmetry of system it's reduced to from $T_d \rightarrow D_{2d}$ [65, 66].

If taken into account a common atom as in this case the As atom as can see in Figure 2.2 and consider that structures growth (001)-oriented lack of translational symmetry [66], then in a single QW, the translational invariance along z axis is lost [67] as can see in Section 2.2. Is so fact which the visualization of symmetry operations ins't trivial, it can support each other with open software library as **Spglib** or **ASE** symmetry functions, these are great tool which has an iterative algoritm under applied recursive operations. If set the arrangement of atoms as the bottom of Figure 2.2, sorting the atoms positions as As-Al-Ga-Ga-As-Al super lattice then at the center of heterostructure the GaAs atoms to finally complement with the heterostructure composed by the first super lattice. Then, it's applies over these the continuous operations under **ASE** package the result is the $P\bar{4}m2$ symmetry in terms of international notation, therefore the D_{2d} point symmetry group. Although that is well-known and described at [65–71], the use of package software will be got a powerful tool to developed future works. So, in order to understand reducing symmetry from $T_d \rightarrow D_{2D}$ of QW grown on the (001), the $z_{\parallel}(001)$ direction becomes inequivalent to both $x_{\parallel}(100)$ and $y_{\parallel}(010)$ directions, hence the symmetry is reducing.

In therm of band structure the symmetry breaking, or symmetry reducing, generate changes, above all over the VB of QWs. This was expected, the reason is the due to the **BZ** zone is reduce to Γ point. From here starts the relevance of Γ and the events which occurs are the next: the first one event which occurs is over VB, due to the **BZ** is reduces to Γ the **VB** it splits for heavy- and light-holes. This is, from bulk as show the band structure in Section 1.1.1 the **VB** is four-fold degenerate, then in QWs the heavy- and light- holes bands splits, so, it gets two-fold degeneracy as shows in Figure 1.9. In fact, when refer to Γ in really it's refer to near band extrema, in case of GaAs and $\text{Al}_x\text{Ga}_{1-x}\text{As}$ being direct band gap semiconductors this really clearly. The consequences of bands split it's the reazon which the effective mass approximation (**EFA**) can works, thanks to that the Schrödinger 's equation can solve as one dimensional equation over both **CB** and **VB** band under structure material parameters. Then, the facility of solve Schrödinger 's equation it's limited to only getting information about transitions, it is important to remember which this solution is over real-space, for this reason, it says that the potential profile is called a band edge profile and can't confuse with the band structure. So, this is the principle of symmetry reduction in QWs structures, in the next section will be addressed the symmetry in CQWs structures and the mechanisms to reduce it.

*Another nomenclature usually used is A/B interfaces, being reference to two dissimilar atoms.

2.2.1 Coupled Quantum Wells

The symmetry breaking doesn't only wight in mathematical aspect, for physics this concept with leads to discover many important and exotic phenomena. When a system reduces its symmetry, their physical properties changes. It's the change in physical properties which reason to study it, in semiconductors as QWs the symmetry breaking starts with modify their band structure (fingerprint) splitting the valence band and reducing the degeneracy, although it can be the first one to appear their consequences get more relevance.

In therms to study changes in physical properties in this work it's focus on optical properties, the light matter interaction gets information about the symmetry breaking through transitions. The GaAs QWs as a direct band gap semiconductor are excellent platform to study light-matter interaction and the effects on symmetry. It's important to remark that from here the excitons played it an important role, in fact, over them falls the physical interpretation of optical properties.

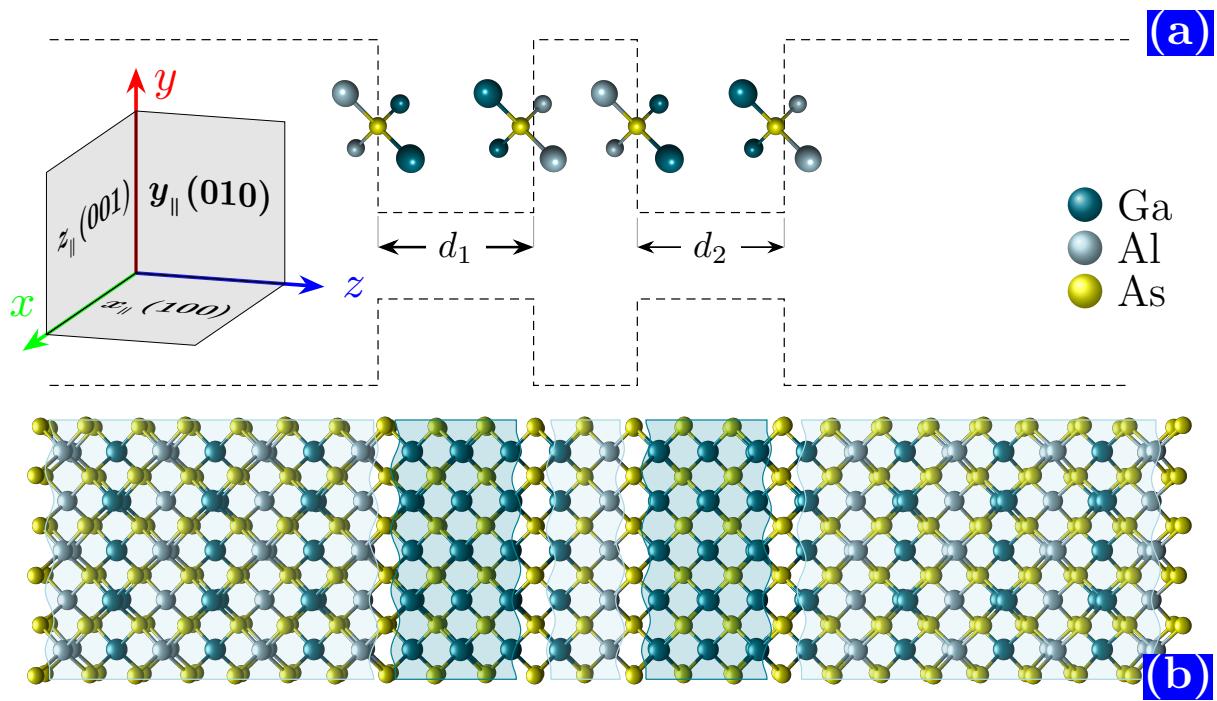


Figure 2.3: General scheme to describe a SCQWs structure. In this case, the barriers are composed by $\text{Al}_x\text{Ga}_{1-x}\text{As}$ and the wells are of GaAs with the same width ($d_1 = d_2$), while the coupling barrier also composed by $\text{Al}_x\text{Ga}_{1-x}\text{As}$. In top (a) denotes both CB and VB edges profiles over z -axis (Real-space) direction. Then in bottom (b), it represents the atoms structure to CQWs.

Then, in the case, of Coupled Quantum Wells (**CQWs**) through excitons get optical properties really it's interesting, although, to get that properties the CQWs should get a double reduce symmetry from T_d , even if that's not obtained it's important to anisotropy

spectroscopy get it as a basis. Then, firstly, it's start with the symmetric coupled quantum wells (**SCQWs**), these are QWs with same width and coupled with a thin barrier, this barrier must be enough thinner so that electron wave function can be overlapping along potential of the two wells. For these structures the symmetry is also D_{2d} as in single QW, it's important to say which, if they aren't consider the interface defects as roughness it's possible to consider ideal interfaces, then the same symmetry operations works in both single QW like a SCQWs.

From ?? it's possible to compare which in contrast with the SQW (Figure 1.8) is the same case with exception to the two QWs, although it's true that the technology of growth it's really accurate, the interfaces aren't be exempt free of them. Even, the rough by possible of Al impurity can cause the possible segregation of this, then it's important the Al concentration x [72, 73]. Even tought, an interface grown over (001) is C_{2v} , if consider structurally perfect as **SQW** or **SCQWs**, the overall symmetry of both interfaces is D_{2d} [65].



Figure 2.4: General scheme corresponds to ACQWs structure. As in Figure 2.3(b), the structure is basically same both in barriers composition and dimensions as well as the coupling barrier, only changes the relative position of this. Then, result that one of QW is wider than other ($d_1 < d_2$), so, this reason causes the asymmetry in structure. Also, here in top (a) draws the potential profile along z , and bottom (b) scheme the atoms structure where is clear that only changes coupling barrier relative position with respect to Figure 2.3(b)

Remember that as Courie mentioned [63, 64, 74]: a system under perturbation reduces their symmetry to a subgroup of original symmetry group, then, if now starts with D_2 symmetry

this subgroup can only reduce to a subgroup of six possibles, into them is C_{2v} subgroup. Previously it mentioned which, exist several mechanisms can reduce the symmetry, these are usually called perturbations. These perturbations can be nature by different sources, *in this work has been it found a novel source which reduces the symmetry, in other words, broken symmetry without needed external source as applied electric or magnetic fields.*

In the next section, it details the reason which it's called a novel source of reduce symmetry, therefore, before continue it's important to mentioned that the simple reason of modify the one well width in CQWs structures makes the system loses fourfold rotations over $z_{\parallel}(001)$ then, the symmetry it reduces.

If compares Figures 2.3 and 2.4 it's clearly that the representative part of the coupling barrier only shift over z , this allows to simulate a **ACQWs** heterostructure, this mean which QW is wider than the other, so, it gets an asymmetric structure which along z losses the rotation symmetry. Also, if it's uses symmetry code packages as **ASE** or **Spplib** which applies consecutively symmetry operations to both CQWs structures, it results in a D_{2d} and C_{2v} for **SCQWs** and **ACQWs** respectively.

2.2.2 Special symmetry reduction from $D_{2d} \rightarrow C_{2v}$

The importance of C_{2v} point group around of QWs system is attractively to study properties of them, over all quantum properties as “spin” [3, 68, 71, 75–77]. Also, it's very important the quantum mixing which exhibits as a result of symmetry breaking, in fact, latter in ?? it presents the result of RAS experiments, which are the result of hole mixing. The anisotropy experimental measured it's caused due to the mixing at **VB**, therefore it's a direct cause of symmetry. Figure 2.5 shows and schematics the roadmap to get a QWs structures with C_{2v} symmetry. This starts at the left with asymmetrical structuresFigure 1.9(a), this asymmetry is related with the potential, exists a variety of them but the objective it's practically the same, the asymmetric potential profile it can be obtained by: asymmetric barrier, this can be due to the change of semiconductor type between adjacent barriers, i.e., AlAs/GaAs/ $\text{Al}_x\text{Ga}_{1-x}\text{As}$ structure, this can interpret as high barrier/well/low barrier [2].Another way of them, is caused of a barrier it's under gradient composition [4, 5]. The end case under asymmetric potential consideration shows in Figure 2.5(c), this case is due in one of the barriers is intentionally “inserted” an atom from other specie [6], this causes an asymmetric potential profile. Finally, it display the our CQWs structure, in comparison with all above firstly it has two wells, therefore reduced symmetry it be

Continued with the map, they have the perturbed structures, they are called like that due to they are under an external perturbation, at center top Figure 2.5(d) it's outlined a QW structure under strain applied, while at bottom center Figure 2.5(e) it's showing a QW structure under electric field applied. So that, in both exists, an external perturbation which causes a losses symmetry. Finally, it displays the ACQWs structureFigure 2.5(f),



Figure 2.5: This roadmap it's developed around of QWs with remarkable potential profile, this mean posses a desire C_{2v} symmetry. It starts with asymmetric QW (a) [2, 3] specifically in potential, this is due to the semiconductor difference in adjacent barriers to well. At center left (b) and bottom left(c) are asymmetric QWs, in the first case the asymmetry [4, 5] it's caused by gradied concentration in a barrier and bottom while the other case the asymmetry it's originated by insert an atom specie different of the barrier [6]. At top center (d), it's show the QW under strain [7–9] which causes the reducing symmetry, while at bottom center(e) it shows the CQWs under applying electric field [10], the relation of both, it's the external perturbation which causes the breaking symmetry. Finally, at right (f) it's presented the ACQWs which has a symmetry breaking due to the relative width of QWs [11].

in comparison with all above. It's notably that this structures has two wells which are coupled by a thin barrier, by this reason as called Coupled Quantum Wells. Then, which is the reason by these structures are novels?.

To discuss this answer, it's important to mention the relevance of CQWs being that these structures are recurrent studied to observe quantum phenomena as exciton (**IX**) condensation [78–80]. It's to be expected that in CQWs can measure indirect transitions, this means that in comparison with a single QWs whereby exists direct transitions (band to band) hardly can measure these. But, the reality of the importance of CQWs being that excitons are very interested to apply in semiconductor devices, the properties of excitons and their interactions really exhibit quantum attractive properties. So, unlike with single QW in CQWs the life of excitons increases [81, 82], in fact, this is one of the principal reasons which that are attractive structures [78, 83, 84]. Also, in terms of spin properties, the CQWs exhibit great potential [85]. Therefore, it being can discuss several interesting quantum properties in comparison that single structures. But, in comparison between CQWs, the symmetric structures need it perturbed to them exhibit these properties, while asymmetric structures are an excellent platform to study quantum properties such as holes mixing, spin, etc. It's then *ACQWs* *an interesting structure which part from being artificial, to get natural perturbation**, even though, as can see doesn't are the unique structures with “natural perturbation” which generates a symmetry breaking, all above mentioned it's reduced to confinement way. This, is the reason to call special symmetry reduction in a structure hardly studied in several sub-areas of solid state.

2.3 Numerical Calculations

All the above cool properties mentioned, could not have been predicted or observed without their knowing their electronic properties. This is many times mentioned, is the fingerprint of semiconductors, so, also mentioned the problem to calculate it. Here was implemented a *simple* model-based in **EFA** method to calculate the confined energies in **CQWs** structures. Before explaining the numerical method to obtain these solutions, it is important to discuss the reason which can apply this method, so, it starts with the definition of band-edge from energy dispersion. In Section 1.1.1 are discussed in general the concept of **VB** and **CB**, also is mentioned the significant methods to calculate them, in case of a heterostructure as **CQWs** the complexity is greater than bulk case. In bulk, the case is well-known several methods to calculate bandstructure, where all of them are developed by the symmetry properties of the system at stake. In the case of the heterostructure, the symmetry is also important, the problem is developed a Hamiltonian capable to describe all of the system, this means, building a Hamiltonian which considers all properties as symmetry,

*Thanks Dr. Raul for magnificent description.

perturbations, and in this case the potential. This work is very hard, even though exists general Hamiltonians to heterostructures, these don't warranty the correct solutions. The history and development of methods and techniques to calculate electronic properties is an area in constant evolution, from the fifties with Kane [12], Luttinger and Kohn [86] in perturbed methods as $\mathbf{k} \cdot \mathbf{p}$ to Slater and Koster [62] which proposed an atomistic method based on a linear combination of atomic orbitals called as **TB**, all of this already discussed and mentioned. So, we developed a model to calculate the confined energies in coupled structures. Even though exist analytical methods to calculate confined energies, in case of CQWs is more difficult to get an exact analytical method, while also exist some analytical methods under approximation description [87–89]. In the Section 1.2.2 it's developed the analytical solution of a simple quantum well taken into account the **EFA** method. It's possible to use this type of methods due to the symmetry reduction from bulk to QW, this due to the split in **VB** which passes from being a fourfold degenerate to be a twofold degenerate. It's precisely in **VB** where is it complicated to solved it.

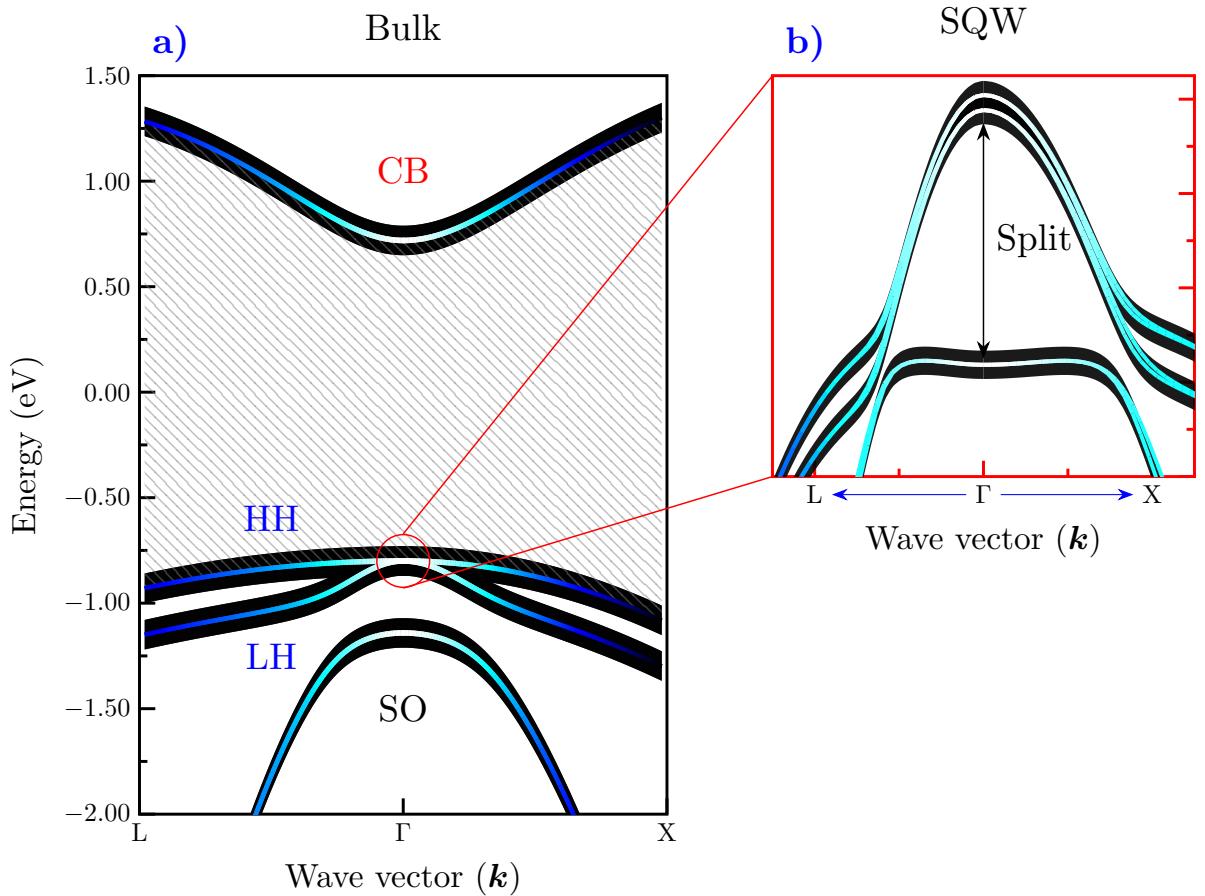


Figure 2.6: (a) The GaAs band structure calculated with 8-band Khane Hamiltonian [12, 13], therefore SOC is consider. At $\Gamma = 0$ in Bulk VB is closed by a circle, here observed the degeneracy, while in (b) it's denoted the split (Δ) between heavy- and light hole bands. Also due SOC consideration, it displays the spin up and down, bands.

The Figure 2.6 it's the result of apply $\mathbf{k} \cdot \mathbf{p}$ method, specifically taken into account 8-band model Hamiltonian [12, 30, 90] which is an extension of four band model [90] and even, this model is raised to bulk semiconductors, but in Figure 2.6(b) shows the results applied in a GaAs SQW heterostructure. To give support to these calculations is important to invoke **EMA**, this method is an efficient method to computational calculations [34, 91] and their basis allows developed electron solutions. In general, exist a variety of Hamiltonian intending to calculate the bandstructure of heterostructures. From conventional Luttinger-Kohn model, [86] to relatively recent models by Burt and Foreman [92–94], which they consider the basis functions depends on symmetry, this is the reason which Hamiltonian is associated to the bulk structure as in this case is Zinc-Blende type.

Although it has been discussed applying this method over heterostructures as the QWs, this work considers it as an “exploratory” tool. The reason for taking him into account in that way is basically their difficult and tedious way to get a correct Hamiltonian that describes our structures, in fact, $\mathbf{k} \cdot \mathbf{p}$ isn't the unique difficult method, also **TB** and others, have laboriousness way to build their Hamiltonians. So, we take the technic and equations from Vurgaftman [30] and solve it numerically to gets the GaAs bulk as shown in Figure 2.6(a) and in the same route, we take the Hamiltonian from Marchewka [13, 95] and solve under **FDM**, without major preamble the idea is to denote the split between heavy- and light-hole bands which is the justification to our numerical calculations proposed. In Equation (2.1) it's presents the 8×8 Hamiltonian solved to get bandstrcuture to GaAs SQW.

$$H_0 = \begin{bmatrix} T & 0 & -\frac{1}{\sqrt{2}}Pk_+ & \sqrt{\frac{2}{3}}Pk_z & \frac{1}{\sqrt{6}}Pk_- & 0 & -\frac{1}{\sqrt{3}}Pk_z & -\frac{1}{\sqrt{3}}Pk_- \\ 0 & T & 0 & -\frac{1}{\sqrt{6}}Pk_+ & \sqrt{\frac{2}{3}}Pk_z & \frac{1}{\sqrt{2}}Pk_- & -\frac{1}{\sqrt{3}}Pk_+ & \frac{1}{\sqrt{3}}Pk_z \\ -\frac{1}{\sqrt{2}}k_-P & 0 & U + V & -\bar{S}_- & R & 0 & \frac{1}{\sqrt{2}}\bar{S}_- & -\sqrt{2}R \\ \sqrt{\frac{2}{3}}k_zP & -\frac{1}{\sqrt{6}}k_-P & -\bar{S}_-^\dagger & U - V & C & R & \sqrt{2}V & -\sqrt{\frac{3}{2}}\bar{S}_- \\ \frac{1}{\sqrt{6}}k_+P & \sqrt{\frac{2}{3}}k_zP & R^\dagger & C^\dagger & U - V & \bar{S}_+^\dagger & -\sqrt{\frac{3}{2}}\bar{S}_+ & -\sqrt{2}V \\ 0 & \frac{1}{\sqrt{2}}k_+P & 0 & R^\dagger & \bar{S}_+ & U + V & \sqrt{2}R^\dagger & \frac{1}{\sqrt{2}}\bar{S}_+ \\ -\frac{1}{\sqrt{3}}k_zP & -\frac{1}{\sqrt{3}}k_-P & \frac{1}{\sqrt{2}}\bar{S}_-^\dagger & \sqrt{2}V & -\sqrt{\frac{3}{2}}\bar{S}_+^\dagger & \sqrt{2}R & U - \Delta & C \\ -\frac{1}{\sqrt{3}}k_+P & \frac{1}{\sqrt{3}}k_zP & -\sqrt{2}R^\dagger & -\sqrt{\frac{3}{2}}\bar{S}_-^\dagger & -\sqrt{2}V & \frac{1}{\sqrt{2}}\bar{S}_+^\dagger & C^\dagger & U - \Delta \end{bmatrix} \quad (2.1)$$

where

$$\begin{aligned} k_{\parallel}^2 &= k_x^2 + k_y^2, k_{\pm} = k_x \pm ik_y, k_z = i\partial/\partial z, \\ T &= E_c(z) + \frac{\hbar^2}{2m_0} [(2F+1)k_{\parallel}^2 + k_z(2F+1)k_z], \end{aligned}$$

$$\begin{aligned}
U &= E_v(z) - \frac{\hbar^2}{2m_0} (\gamma_1 k_{\parallel}^2 + k_z \gamma_1 k_z), \\
V &= -\frac{\hbar^2}{2m_0} (\gamma_2 k_{\parallel}^2 - 2k_z \gamma_2 k_z), \\
R &= -\frac{\hbar^2}{2m_0} \sqrt{3} (\mu k_+^2 - \bar{\gamma} k_-^2), \\
\bar{S}_{\pm} &= -\frac{\hbar^2}{2m_0} \sqrt{3} k_{\pm} (\{\gamma_3, k_z\} + [\kappa, k_z]), \\
\tilde{S}_{\pm} &= -\frac{\hbar^2}{2m_0} \sqrt{3} k_{\pm} \left(\{\gamma_3, k_z\}, \frac{1}{3} [\kappa, k_z] \right), \\
C &= \frac{\hbar^2}{2m_0} k_- [\kappa, k_z]
\end{aligned}$$

and each of these parameters taken from [30]. The Hamiltonian H_0 is defined for [001], and the principal idea is evaluating it in a “discrete” SQW structure, the ”discrete“ term refers to a numerical technique for solving it. The results it shows in fig. 2.6(b), this approximation is enough to evince the **VB** split, even, the results also **CB** information the importance falls on **VB**, this is due that difficult to solve there. Then, the VB split allows applying “simple” numerical methods without needing to define complex Hamiltonian and many parameters. In next section it will discuss the physical and mathematical formulation to calculate the electron energy confined, beginning with the symmetry reduce causes the **VB** split and developed all around of $\Gamma = 0$ this means we work over band-edge potential in real space.

2.3.1 Envelope Function and Effective Mass Aproximation Methods

Furthermore, to get a numerical method and solution robustness without any complex formalism but based on the essence of this work which is the effect of asymmetric width potential in CQWs which causes the symmetry breaking and this physical property involve important optical phenomena, as increase Anisotropy depending on relative width in these. The **EFA** is the mathematical justification to model electron behavior under periodic potential as a crystal, while the **EMA** is the physical model and the considerations to study the electron behavior inside a periodic potential from a crystal [34]. Under that assumptions, it’s possible to review several considerations, that considerations reduce the complexity of solutions and allow simplifying the calculations. Then, in all of this work it develops around of $\Gamma = 0$, this means that will use a one-band model. To resume this last context, it’s importantly to taken into account the next considerations. Firstly, from here we can study the electron and holes behavior by separately, from basic assumption which

*Frequently the **EFA** and **EMA** approximations are considered as the same term, although maybe being correct, given that one depends on the another, is important to remark that the **EMA** depends on the formalism of **EFA**.

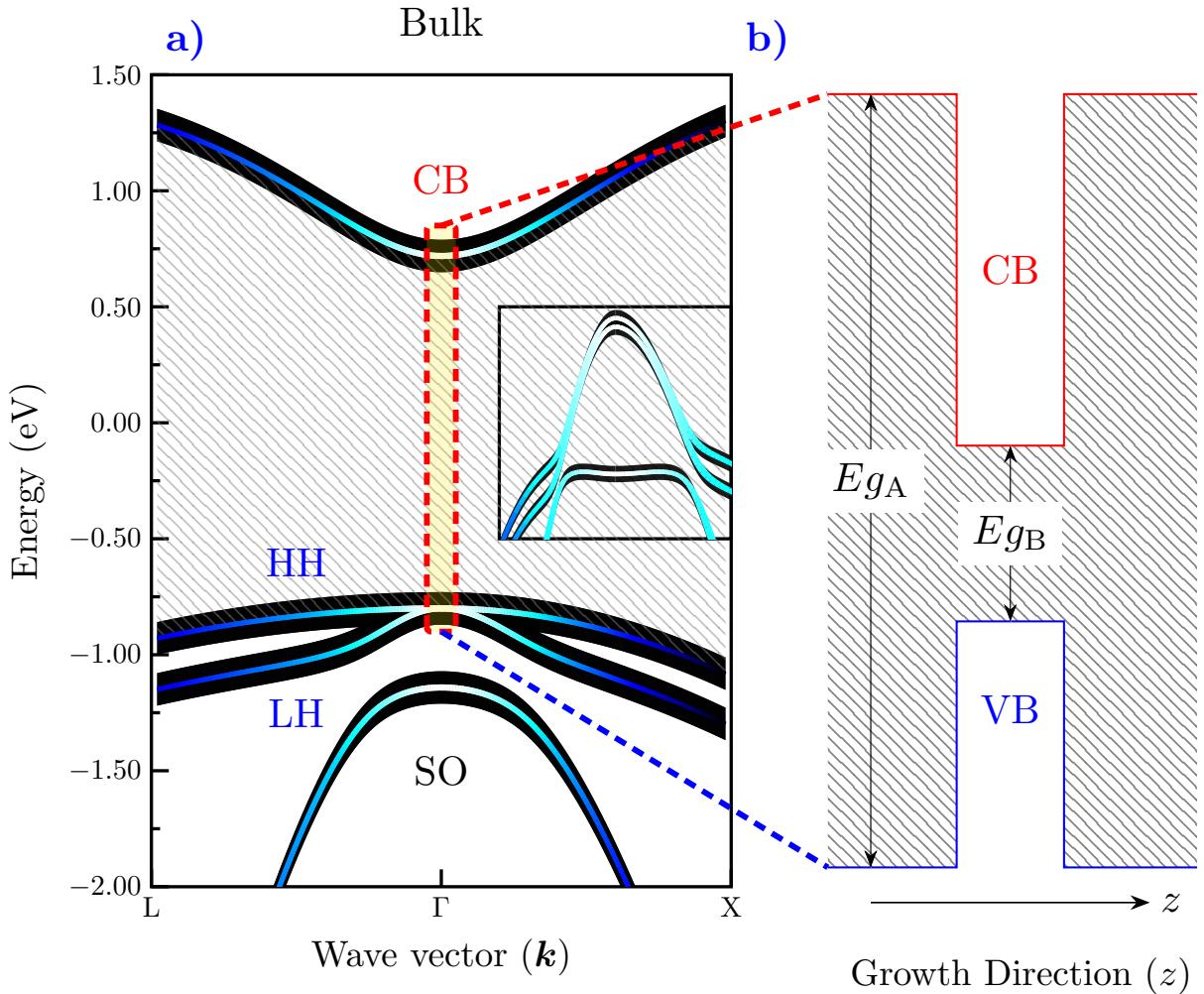


Figure 2.7: (a) it's tyhe focus over $\Gamma = 0$ and inside is the SQW VB $\mathbf{k} \cdot \mathbf{p}$ results, while the

$\Gamma = 0$ this enables to pass from electronic band to band edge (see Figure 2.7). So, it's important to define that tools previosuly mentioned. The **EMA** is a valid approximation in bulk materials, in fact this is an elemental model which due to the good results and simplicity it's can be applied over heterostructures. About of this last, the heterostructures are complex, the dissimilar matched semiconductors which conformed that it, posses doesn't only difference in band-gap energy but also the effective mass.

If join all of that parameters involved it, the model to solve should being proportional to the considers in it, by this reason models as **TB** are complex due to the difficult to define all that parameters, while in $\mathbf{k} \cdot \mathbf{p}$ are well-defined, but contrary, the method frequently is submitted to discussion due to the assumptions as the doesn't "see" the correct point-group symmetries, in fact, frequently assumes which the system already owns it. We can introduce ourselves in a severe discussion about of bandstructure calculations methods, but the reality is can't define a standard method, so that, doing reference to

Harrison et al. [34], we can apply the simplicity method. Then, assuming the simplicity term, in which practically mentioned that, it's important to consider the simple way to get a result as long as this works. So, in this work, will discuss the increase of the **OA** due to the relative width in the wells of the coupled system, then here only considers the potential profile as a really significant parameter, of course, the effective mass and the band-gap are also essential, but these are considered as a part of the model. Being this the basis of the model, this work will employ the formalism of **EFA** where in a general explain, this it gives the envelope functions to calculate the single electron behavior under periodic potential as a heterostructure, therefore, this can allow describing it through position-depend material properties [38, 96, 97], then it's also indispensable to declare the boundary conditions owned by the **EFA**. Therefore with a mathematical formalism, then it's employed the **EMA** which parts of the fundamental parameter which is the effective-mass, then it can work with the 1D-Schrödinger 's equation as a function of effective-mass, as traditional eigen-value equation:

$$H\psi(z) = E\psi(z) \quad (2.2)$$

Then, the Hamiltonian of **EMA** it's developed mainly by the effective-mass, this has the advantage of evaluated along of heterostructure, allowing the change of semiconductors which composed that, so, this makes it a great method. Therefore the Hamiltonian for a particle is given by [38, 96, 98]:

$$H_z = \frac{p_z^2}{2m^*(z)} + V(z). \quad (2.3)$$

In Equation (2.3) it's taken z -axis as a direction perpendicular to interface of QWs, or simply as the growth direction, consequently the potential $V(z)$ it's depend on, and the effective-mass $m^*(z)$ also depends on position z . In Equation (2.3) it's taken z -axis as a direction perpendicular to interface of QWs, or simply as the growth direction, consequently the potential $V(z)$ it's depend on, and the effective-mass $m^*(z)$ also depends on position z . As can see in Figure 2.7(b), it's showing the band-edge profile as a potential profile which depends on semiconductor band-gap along of heterostructure. Also in Equation (2.3), it considers the momentum as p_z^2 as a **1D**, due to the before discussed. Then it's allow to developed a method to solution which depends of position z . Finally, it can denote a **1D** Schrödinger quation under **EMA** to electron also heavy- and light-holes :

$$\left[-\frac{\hbar^2}{2m_{jz}^*} \frac{d^2}{dz^2} + V(z) \right] \psi_{nj}(z) = E_{nj} \psi_{nj}(z), \quad (2.4)$$

$j = e, hh, lh.$

The Equation (2.4) it's then the effective-mass equation implemented in this work. The solution of that, it's detailed in next section where explains the numerical method to solve it. Before to continue, it's important to discuss the **1D** equation. This equation will be applied over each particle ($j = e, hh, lh$), this means, over electron, heavy- and light-holes, where only considers the effective-mass (m_{jz}^*) trough each semiconductor in heterostructure studied and the most important in this work, the potential profile $V(z)$ which depends on relative widths of coupled wells.

2.3.2 Finite Difference Method

In this section it will discuss the numerical method to solve the **1D** Scrödinger 's equation, even tough this is a large discussed theme, here focus on a simple but powerful numerical method. Started from the fact that the Equation (2.4) can it solve along of heterostructure and taken into account each semiconductor with compose it, this means that can discretize that and solved it for electron and holes. Although this, being a **1D** equation, doesn't remove importance in the solution, that is to say, it's needing a robust method. It's important to remark which the spacial discretization depends on Δz . In Figure 2.8 schemes the discrete potential profile considers, this band-edge profile corresponds to a **SCQWs** structure.

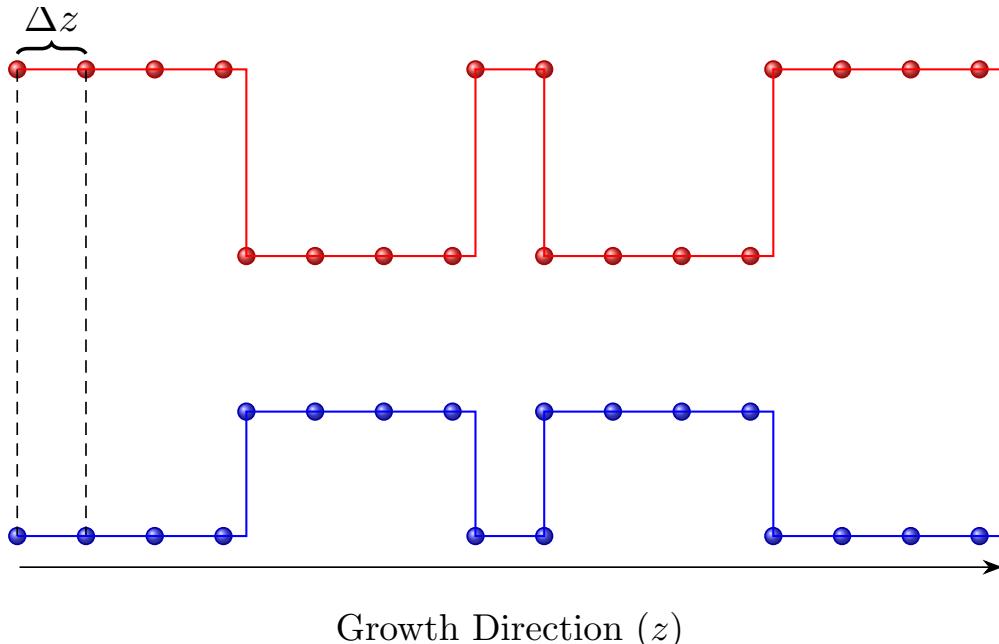


Figure 2.8

2.4Anisotropy model in CQWs

BIBLIOGRAPHY

- [1] M. Fox, “Optical properties of solids,” 2002. (Cited on pages [II](#), [6](#), [7](#), [8](#), and [13](#).)
- [2] B. Koopmans, B. Koopmans, P. Santos, P. Santos, and M. Cardona, “Microscopic reflection difference spectroscopy on semiconductor nanostructures,” *physica status solidi (a)*, vol. 170, no. 2, pp. 307–315, 1998. (Cited on pages [III](#), [25](#), and [26](#).)
- [3] E. A. de Andrada e Silva, G. C. La Rocca, and F. Bassani, “Spin-orbit splitting of electronic states in semiconductor asymmetric quantum wells,” *Phys. Rev. B*, vol. 55, pp. 16 293–16 299, Jun 1997. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.55.16293> (Cited on pages [III](#), [25](#), and [26](#).)
- [4] D. J. English, J. Hübner, P. S. Eldridge, D. Taylor, M. Henini, R. T. Harley, and M. Oestreich, “Effect of symmetry reduction on the spin dynamics of (001)-oriented gaas quantum wells,” *Phys. Rev. B*, vol. 87, p. 075304, Feb 2013. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.87.075304> (Cited on pages [III](#), [25](#), and [26](#).)
- [5] P. S. Eldridge, J. Hübner, S. Oertel, R. T. Harley, M. Henini, and M. Oestreich, “Spin-orbit fields in asymmetric (001)-oriented gaas/ $\text{al}_x\text{ga}_{1-x}\text{as}$ quantum wells,” *Phys. Rev. B*, vol. 83, p. 041301, Jan 2011. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.83.041301> (Cited on pages [III](#), [25](#), and [26](#).)
- [6] J. L. Yu, S. Y. Cheng, Y. F. Lai, Q. Zheng, Y. H. Chen, and C. G. Tang, “Tuning of in-plane optical anisotropy by inserting ultra-thin inas layer at interfaces in (001)-grown gaas/algaas quantum wells,” *Journal of Applied Physics*, vol. 117, no. 1, p. 015302, 2015. (Cited on pages [III](#), [25](#), and [26](#).)
- [7] D. J. English, P. G. Lagoudakis, R. T. Harley, P. S. Eldridge, J. Hübner, and M. Oestreich, “Strain-induced spin relaxation anisotropy in symmetric (001)-oriented gaas quantum wells,” *Phys. Rev. B*, vol. 84, p. 155323, Oct 2011. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.84.155323> (Cited on pages [III](#) and [26](#).)
- [8] C. G. Tang, Y. H. Chen, B. Xu, X. L. Ye, and Z. G. Wang, “Well-width dependence of in-plane optical anisotropy in (001) gaas/algaas quantum wells induced by in-plane uniaxial strain and interface asymmetry,” *Journal of Applied Physics*, vol. 105, no. 10, p. 103108, 2009. (Cited on pages [III](#) and [26](#).)
- [9] Y. Li, F. Liu, X. Ye, Y. Liu, J. Wang, and Y. Chen, “Quantitative investigation of intrinsic shear strain and asymmetric interface conditions in semiconductor superlattices,” *Journal of Applied Physics*, vol. 126, no. 6, p. 065704, 2019. (Cited on pages [III](#) and [26](#).)

- [10] S. H. Kwok, H. T. Grahn, K. Ploog, and R. Merlin, “Giant electropleochroism in gaas-(al,ga)as heterostructures: The quantum-well pockels effect,” *Phys. Rev. Lett.*, vol. 69, pp. 973–976, Aug 1992. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevLett.69.973> (Cited on pages III and 26.)
- [11] O. Ruiz-Cigarrillo, L. F. Lastras-Martínez, E. A. Cerdá-Méndez, G. Flores-Rangel, C. A. Bravo-Velazquez, R. E. Balderas-Navarro, A. Lastras-Martínez, N. A. Ulloa-Castillo, K. Biermann, and P. V. Santos, “Optical anisotropies of asymmetric double gaas (001) quantum wells,” *Phys. Rev. B*, vol. 103, p. 035309, Jan 2021. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.103.035309> (Cited on pages III and 26.)
- [12] E. O. Kane, “Band structure of indium antimonide,” *Journal of Physics and Chemistry of Solids*, vol. 1, no. 4, pp. 249–261, 1957. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/0022369757900136> (Cited on pages III, 28, and 29.)
- [13] E. G. Novik, A. Pfeuffer-Jeschke, T. Jungwirth, V. Latussek, C. R. Becker, G. Landwehr, H. Buhmann, and L. W. Molenkamp, “Band structure of semimagnetic $\text{hg}_{1-y}\text{mn}_y\text{Te}$ quantum wells,” *Phys. Rev. B*, vol. 72, p. 035321, Jul 2005. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.72.035321> (Cited on pages III, 28, and 29.)
- [14] K. Sivalertporn, “Effect of barrier width on the exciton states in coupled quantum wells in an applied electric field,” *Physics Letters A*, vol. 380, no. 22, pp. 1990–1994, 2016. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0375960116300664> (Not cited.)
- [15] N. Debbar, S. Hong, J. Singh, P. Bhattacharya, and R. Sahai, “Coupled gaas/algaas quantum-well electroabsorption modulators for low-electric-field optical modulation,” *Journal of Applied Physics*, vol. 65, no. 1, pp. 383–385, 1989. [Online]. Available: <https://doi.org/10.1063/1.342554> (Not cited.)
- [16] N. V. Tkachenko, *Optical spectroscopy: methods and instrumentations*. Elsevier, 2006. (Not cited.)
- [17] A. H. Larsen, J. J. Mortensen, J. Blomqvist, I. E. Castelli, R. Christensen, M. Dułak, J. Friis, M. N. Groves, B. Hammer, C. Hargus, E. D. Hermes, P. C. Jennings, P. B. Jensen, J. Kermode, J. R. Kitchin, E. L. Kolsbjerg, J. Kubal, K. Kaasbjerg, S. Lysgaard, J. B. Maronsson, T. Maxson, T. Olsen, L. Pastewka, A. Peterson, C. Rostgaard, J. Schiøtz, O. Schütt, M. Strange, K. S. Thygesen, T. Vegge, L. Vilhelmsen, M. Walter, Z. Zeng, and K. W. Jacobsen, “The atomic simulation environment—a python library for working with atoms,” *Journal of Physics: Condensed Matter*, vol. 29, no. 27, p. 273002, 2017. [Online]. Available: <http://stacks.iop.org/0953-8984/29/i=27/a=273002> (Cited on page VI.)

- [18] A. Togo and I. Tanaka, “**Spglib**: a software library for crystal symmetry search,” 2018. (Cited on page VI.)
- [19] D. Alonso-Álvarez, T. Wilson, P. Pearce, M. Führer, D. Farrell, and N. Ekins-Daukes, “Solcore: a multi-scale, python-based library for modelling solar cells and semiconductor materials,” *Journal of Computational Electronics*, vol. 17, no. 3, pp. 1099–1123, 2018. (Cited on page VI.)
- [20] H. Hebal, Z. Koziol, S. Lisesivdin, and R. Steed, “General-purpose open-source 1d self-consistent schrödinger-poisson solver: Aestimo 1d,” *Computational Materials Science*, vol. 186, p. 110015, 2021. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0927025620305061> (Cited on page VI.)
- [21] K. Momma and F. Izumi, “it VESTA3 for three-dimensional visualization of crystal, volumetric and morphology data,” *Journal of Applied Crystallography*, vol. 44, no. 6, pp. 1272–1276, Dec 2011. [Online]. Available: <https://doi.org/10.1107/S0021889811038970> (Cited on page VI.)
- [22] T. Tantau, “Pgf/tikz,” 2007. [Online]. Available: <https://github.com/pgf-tikz/pgf.git> (Cited on page VI.)
- [23] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, “Julia: A fresh approach to numerical computing,” *SIAM Review*, vol. 59, no. 1, pp. 65–98, 2017. [Online]. Available: <https://pubs.siam.org/doi/10.1137/141000671> (Cited on page VI.)
- [24] G. M.-C. O. Ruiz-Cigarrillo, “ $\mathbf{k} \cdot \mathbf{p}$ julia package,” 2017. [Online]. Available: <https://github.com/lflmgroup/kp-lflm-group.git> (Cited on page VI.)
- [25] M. SUNDARAM, S. A. CHALMERS, P. F. HOPKINS, and A. C. GOSSARD, “New quantum structures,” *Science*, vol. 254, no. 5036, pp. 1326–1335, 1991. [Online]. Available: <https://science.sciencemag.org/content/254/5036/1326> (Cited on page 2.)
- [26] J. Piprek, *Handbook of Optoelectronic Device Modeling and Simulation: Fundamentals, Materials, Nanostructures, LEDs, and Amplifiers*, Vol. 1. CRC Press, 2017. (Cited on pages 2, 3, and 5.)
- [27] H. Alloul, *Introduction to the Physics of Electrons in Solids*. Springer Science & Business Media, 2010. [Online]. Available: <https://doi.org/10.1007/978-3-642-13565-1> (Cited on page 2.)
- [28] M. Cardona and Y. Y. Peter, *Fundamentals of semiconductors*. Springer, 2005, vol. 619. (Cited on pages 2, 7, 11, and 19.)
- [29] K. W. Böer and U. W. Pohl, *Semiconductor physics*. Springer, 2018. [Online]. Available: <https://doi.org/10.1007/978-3-319-06540-3> (Cited on page 3.)

- [30] I. Vurgaftman, M. P. Lumb, and J. R. Meyer, *Bands and Photons in III-V Semiconductor Quantum Structures*. Oxford University Press, 2020, vol. 25. (Cited on pages 4, 29, and 30.)
- [31] P. Vogl, H. P. Hjalmarson, and J. D. Dow, “A semi-empirical tight-binding theory of the electronic structure of semiconductors†,” *Journal of Physics and Chemistry of Solids*, vol. 44, no. 5, pp. 365–378, 1983. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/0022369783900641> (Cited on pages 6 and 20.)
- [32] R. Muller, “Tight binding program to compute the band structure of simple semiconductors.” 2017. [Online]. Available: <https://github.com/qftphys/Simple-SPS-tight-binding-code-based-on-parameters-from-Vogl-Hjalmarson-Dow> (Cited on page 6.)
- [33] J. Leonard, *Exciton Transport Phenomena in GaAs Coupled Quantum Wells*. Springer, 2017. (Cited on page 8.)
- [34] P. Harrison and A. Valavanis, *Quantum wells, wires and dots: theoretical and computational physics of semiconductor nanostructures*. John Wiley & Sons, 2016. (Cited on pages 10, 11, 13, 14, 29, 30, and 32.)
- [35] W. Pauli, “Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren,” *Zeitschrift für Physik*, vol. 31, no. 1, pp. 765–783, 1925. (Cited on page 11.)
- [36] S. Chuang, *Physics of Optoelectronic Devices*, ser. Wiley Series in Pure and Applied Optics. Wiley, 1995. [Online]. Available: <https://books.google.com.mx/books?id=ect6QgAACAAJ> (Cited on page 13.)
- [37] J. Singh, *Electronic and Optoelectronic Properties of Semiconductor Structures*. Cambridge University Press, 2003. (Cited on page 13.)
- [38] G. Bastard, *Wave mechanics applied to semiconductor heterostructures*. Wiley, 1990. (Cited on pages 13 and 32.)
- [39] J. H. Davies, *The physics of low-dimensional semiconductors: an introduction*. Cambridge university press, 1998. (Cited on page 13.)
- [40] L. De La Peña, *Introducción a la mecánica cuántica*. Fondo de Cultura económica, 2014. (Cited on page 13.)
- [41] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics*. Cambridge University Press, 2018. (Cited on page 13.)
- [42] J. J. Sakurai and E. D. Commins, “Modern quantum mechanics, revised edition,” 1995. (Cited on page 13.)

- [43] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics, Volume 1: Basic Concepts, Tools, and Applications.* John Wiley & Sons, 2019. (Cited on page 13.)
- [44] N. Rivera and I. Kaminer, “Light–matter interactions with photonic quasiparticles,” *Nature Reviews Physics*, vol. 2, no. 10, pp. 538–561, 2020. (Cited on page 14.)
- [45] B. C. Van Fraassen, *Laws and symmetry.* Clarendon Press, 1989. (Cited on page 16.)
- [46] R. C. Powell, *Symmetry, group theory, and the physical properties of crystals.* Springer, 2010, vol. 824. (Cited on pages 16 and 18.)
- [47] K. Tapp, *Symmetry: A mathematical exploration.* Springer Nature, 2021. (Cited on page 16.)
- [48] J. F. Cornwell, *Group theory in physics: An introduction.* Academic press, 1997. (Cited on page 16.)
- [49] U. Müller, *Symmetry relationships between crystal structures: applications of crystallographic group theory in crystal chemistry.* OUP Oxford, 2013, vol. 18. (Cited on page 16.)
- [50] C. Kittel and P. McEuen, *Kittel’s Introduction to Solid State Physics.* John Wiley & Sons, 2018. (Cited on pages 16, 17, and 18.)
- [51] J. Sólyom, *Fundamentals of the Physics of Solids: Volume 1: Structure and Dynamics.* Springer Science & Business Media, 2007, vol. 1. (Cited on page 16.)
- [52] S. K. Chatterjee, *Crystallography and the World of Symmetry.* Springer Science & Business Media, 2008, vol. 113. (Cited on pages 17 and 18.)
- [53] J. McKelvey, *Solid State and Semiconductor Physics*, ser. A Harper international edition. Harper & Row, 1966. [Online]. Available: <https://books.google.com.mx/books?id=MQNRAAAAMAAJ> (Cited on page 18.)
- [54] C. Malgrange, C. Ricolleau, and M. Schlenker, *Symmetry and Physical Properties of Crystals.* Springer, 2014. (Cited on page 18.)
- [55] N. Ashcroft, A. W. N. Mermin, W. Ashcroft, D. Mermin, N. Mermin, and B. P. Company, *Solid State Physics*, ser. HRW international editions. Holt, Rinehart and Winston, 1976. [Online]. Available: <https://books.google.com.mx/books?id=1C9HAQAAIAAJ> (Cited on page 18.)
- [56] R. C. Powell, *Symmetry in Solids.* New York, NY: Springer New York, 2010, pp. 1–24. [Online]. Available: https://doi.org/10.1007/978-1-4419-7598-0_1 (Cited on page 18.)

- [57] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group theory: application to the physics of condensed matter.* Springer Science & Business Media, 2007. (Cited on pages 19, 20, and 21.)
- [58] R. H. Parmenter, “Symmetry properties of the energy bands of the zinc blende structure,” *Phys. Rev.*, vol. 100, pp. 573–579, Oct 1955. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRev.100.573> (Cited on page 19.)
- [59] P. N. Butcher, N. H. March, and M. P. Tosi, *Crystalline semiconducting materials and devices.* Springer Science & Business Media, 2013. (Cited on page 19.)
- [60] G. Bir and G. Pikus, *Symmetry and Strain-induced Effects in Semiconductors*, ser. A Halsted Press book. Wiley, 1974. [Online]. Available: <https://books.google.com.mx/books?id=38m2QgAACAAJ> (Cited on page 20.)
- [61] W. A. Harrison, “Bond-orbital model and the properties of tetrahedrally coordinated solids,” *Phys. Rev. B*, vol. 8, pp. 4487–4498, Nov 1973. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.8.4487> (Cited on page 20.)
- [62] J. C. Slater and G. F. Koster, “Simplified lcao method for the periodic potential problem,” *Phys. Rev.*, vol. 94, pp. 1498–1524, Jun 1954. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRev.94.1498> (Cited on pages 20 and 28.)
- [63] P. Curie, “Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique,” *Journal de physique théorique et appliquée*, vol. 3, no. 1, pp. 393–415, 1894. (Cited on pages 20 and 24.)
- [64] K. Brading, E. Castellani, and N. Teh, “Symmetry and Symmetry Breaking,” in *The Stanford Encyclopedia of Philosophy*, Fall 2021 ed., E. N. Zalta, Ed. Metaphysics Research Lab, Stanford University, 2021. (Cited on pages 20 and 24.)
- [65] R. Magri and A. Zunger, “Anticrossing and coupling of light-hole and heavy-hole states in (001) GaAs/al_xga_{1-x}As heterostructures,” *Phys. Rev. B*, vol. 62, pp. 10364–10372, Oct 2000. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.62.10364> (Cited on pages 22 and 24.)
- [66] E. L. Ivchenko, A. Y. Kaminski, and U. Rössler, “Heavy-light hole mixing at zinc-blende (001) interfaces under normal incidence,” *Phys. Rev. B*, vol. 54, pp. 5852–5859, Aug 1996. (Cited on page 22.)
- [67] P. Tronc, Y. E. Kitaev, A. G. Panfilov, M. F. Limonov, G. Wang, and V. P. Smirnov, “Bound-state symmetries and optical transitions in gaas/ala_s quantum wells and superlattices with impurities and defects,” *Phys. Rev. B*, vol. 61, pp. 1999–2007, Jan 2000. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.61.1999> (Cited on page 22.)

- [68] M. Glazov, *Electron & Nuclear Spin Dynamics in Semiconductor Nanostructures*, ser. Series on Semiconductor Scienc. OXFORD University Press, 2018. (Cited on pages 22 and 25.)
- [69] O. Krebs and P. Voisin, “Giant optical anisotropy of semiconductor heterostructures with no common atom and the quantum-confined pockels effect,” *Phys. Rev. Lett.*, vol. 77, pp. 1829–1832, Aug 1996. (Cited on page 22.)
- [70] Y. H. Chen, X. L. Ye, J. Z. Wang, Z. G. Wang, and Z. Yang, “Interface-related in-plane optical anisotropy in $\text{gaas}/\text{al}_x\text{ga}_{1-x}\text{As}$ single-quantum-well structures studied by reflectance difference spectroscopy,” *Phys. Rev. B*, vol. 66, p. 195321, Nov 2002. (Cited on page 22.)
- [71] E. L. Ivchenko and S. D. Ganichev, *Spin-Photogalvanics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 245–277. (Cited on pages 22 and 25.)
- [72] N. Chand and S. N. G. Chu, “Origin and improvement of interface roughness in $\text{algaas}/\text{gaas}$ heterostructures grown by molecular beam epitaxy,” *Applied Physics Letters*, vol. 57, no. 17, pp. 1796–1798, 1990. [Online]. Available: <https://doi.org/10.1063/1.104025> (Cited on page 24.)
- [73] K. Tillmann, M. Luysberg, P. Specht, and E. Weber, “Direct compositional analysis of $\text{algaas}/\text{gaas}$ heterostructures by the reciprocal space segmentation of high-resolution micrographs,” *Ultramicroscopy*, vol. 93, no. 2, pp. 123–137, 2002. [Online]. Available: [https://doi.org/10.1016/S0304-3991\(02\)00153-5](https://doi.org/10.1016/S0304-3991(02)00153-5) (Cited on page 24.)
- [74] A. SHUBNIKOV, “On the works of pierre curie on symmetry††originally appeared in russian in uspekhi fizicheskikh nauk 59, 591-602 (1956). translated by l. i. man.” in *Crystal Symmetries*, ser. International Series in Modern Applied Mathematics and Computer Science, I. HARGITTAI and B. VAINSHTEIN, Eds. Amsterdam: Pergamon, 1988, pp. 357–364. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/B9780080370149500078> (Cited on page 24.)
- [75] J.-W. Luo, G. Bester, and A. Zunger, “Supercoupling between heavy-hole and light-hole states in nanostructures,” *Phys. Rev. B*, vol. 92, p. 165301, Oct 2015. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.92.165301> (Cited on page 25.)
- [76] R. Winkler, *Spin-orbit coupling effects in two-dimensional electron and hole systems*. Springer, 2003, vol. 191. (Cited on page 25.)
- [77] S. Döhrmann, D. Hägele, J. Rudolph, M. Bichler, D. Schuh, and M. Oestreich, “Anomalous spin dephasing in (110) gaas quantum wells: Anisotropy and intersubband effects,” *Phys. Rev. Lett.*, vol. 93, p. 147405, Sep 2004. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevLett.93.147405> (Cited on page 25.)

- [78] L. V. Butov, A. Zrenner, G. Abstreiter, G. Böhm, and G. Weimann, “Condensation of indirect excitons in coupled alas/gaas quantum wells,” *Phys. Rev. Lett.*, vol. 73, pp. 304–307, Jul 1994. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevLett.73.304> (Cited on page 27.)
- [79] L. Butov, C. Lai, A. Ivanov, A. Gossard, and D. Chemla, “Towards bose–einstein condensation of excitons in potential traps,” *Nature*, vol. 417, no. 6884, pp. 47–52, 2002. (Cited on page 27.)
- [80] G. Grosso, J. Graves, A. Hammack, A. High, L. Butov, M. Hanson, and A. Gossard, “Excitonic switches operating at around 100 k,” *Nature Photonics*, vol. 3, no. 10, pp. 577–580, 2009. (Cited on page 27.)
- [81] A. T. Hammack, L. V. Butov, J. Wilkes, L. Mouchliadis, E. A. Muljarov, A. L. Ivanov, and A. C. Gossard, “Kinetics of the inner ring in the exciton emission pattern in coupled gaas quantum wells,” *Phys. Rev. B*, vol. 80, p. 155331, Oct 2009. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.80.155331> (Cited on page 27.)
- [82] J. E. Golub, K. Kash, J. P. Harbison, and L. T. Florez, “Long-lived spatially indirect excitons in coupled gaas/al_xga_{1-x}as quantum wells,” *Phys. Rev. B*, vol. 41, pp. 8564–8567, Apr 1990. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.41.8564> (Cited on page 27.)
- [83] K. Sivalertporn, L. Mouchliadis, A. L. Ivanov, R. Philp, and E. A. Muljarov, “Direct and indirect excitons in semiconductor coupled quantum wells in an applied electric field,” *Phys. Rev. B*, vol. 85, p. 045207, Jan 2012. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.85.045207> (Cited on page 27.)
- [84] A. G. Winbow, J. R. Leonard, M. Remeika, Y. Y. Kuznetsova, A. A. High, A. T. Hammack, L. V. Butov, J. Wilkes, A. A. Guenther, A. L. Ivanov, M. Hanson, and A. C. Gossard, “Electrostatic conveyer for excitons,” *Phys. Rev. Lett.*, vol. 106, p. 196806, May 2011. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevLett.106.196806> (Cited on page 27.)
- [85] C. A. Bravo-Velázquez, L. F. Lastras-Martínez, O. Ruiz-Cigarrillo, G. Flores-Rangel, L. E. Tapia-Rodríguez, K. Biermann, and P. V. Santos, “Photoluminescence of double quantum wells: Asymmetry and excitation laser wavelength effects,” *physica status solidi (b)*, vol. n/a, no. n/a, p. 2100612. [Online]. Available: <https://onlinelibrary.wiley.com/doi/abs/10.1002/pssb.202100612> (Cited on page 27.)
- [86] J. M. Luttinger and W. Kohn, “Motion of electrons and holes in perturbed periodic fields,” *Phys. Rev.*, vol. 97, pp. 869–883, Feb 1955. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRev.97.869> (Cited on pages 28 and 29.)

- [87] A. Yariv, C. Lindsey, and U. Sivan, “Approximate analytic solution for electronic wave functions and energies in coupled quantum wells,” *Journal of Applied Physics*, vol. 58, no. 9, pp. 3669–3672, 1985. [Online]. Available: <https://doi.org/10.1063/1.335726> (Cited on page 28.)
- [88] T. Fromherz, “Floquet states and intersubband absorption in strongly driven double quantum wells,” *Phys. Rev. B*, vol. 56, pp. 4772–4777, Aug 1997. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.56.4772> (Cited on page 28.)
- [89] E. Rosencher and B. Vinter, *Optoelectronics*. Cambridge University Press, 2002. (Cited on page 28.)
- [90] C. Galeriu, “ $k \cdot p$ theory of semiconductor nanostructures,” *PhD Dissertations. Worcester Polytechnic Institute, Worcester*, 2005. (Cited on page 29.)
- [91] H. Yeo, J. S. Lee, M. E. Khan, H. S. Kim, D. Y. Jeon, and Y.-H. Kim, “First-principles-derived effective mass approximation for the improved description of quantum nanostructures,” *Journal of Physics: Materials*, vol. 3, no. 3, p. 034012, jul 2020. [Online]. Available: <https://doi.org/10.1088/2515-7639/ab9b61> (Cited on page 29.)
- [92] M. G. Burt, “A new effective-mass equation for microstructures,” *Semiconductor Science and Technology*, vol. 3, no. 12, pp. 1224–1226, dec 1988. [Online]. Available: <https://doi.org/10.1088/0268-1242/3/12/013> (Cited on page 29.)
- [93] B. A. Foreman, “Effective-mass hamiltonian and boundary conditions for the valence bands of semiconductor microstructures,” *Phys. Rev. B*, vol. 48, pp. 4964–4967, Aug 1993. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.48.4964> (Cited on page 29.)
- [94] M. G. Burt, “The justification for applying the effective-mass approximation to microstructures,” *Journal of Physics: Condensed Matter*, vol. 4, no. 32, pp. 6651–6690, aug 1992. [Online]. Available: <https://doi.org/10.1088/0953-8984/4/32/003> (Cited on page 29.)
- [95] M. Marchewka, “Finite-difference method applied for eight-band kp model for $Hg_{1-x}Cd_xTe/HgTe$ quantum well,” *International Journal of Modern Physics B*, vol. 31, no. 20, p. 1750137, 2017. [Online]. Available: <https://doi.org/10.1142/S0217979217501375> (Cited on page 29.)
- [96] B. A. Foreman, “Envelope-function formalism for electrons in abrupt heterostructures with material-dependent basis functions,” *Phys. Rev. B*, vol. 54, pp. 1909–1921, Jul 1996. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.54.1909> (Cited on page 32.)

- [97] D. J. BenDaniel and C. B. Duke, “Space-charge effects on electron tunneling,” *Phys. Rev.*, vol. 152, pp. 683–692, Dec 1966. (Cited on page 32.)
- [98] T. Kamizato and M. Matsuura, “Excitons in double quantum wells,” *Phys. Rev. B*, vol. 40, pp. 8378–8384, Oct 1989. [Online]. Available: <https://link.aps.org/doi/10.1103/PhysRevB.40.8378> (Cited on page 32.)