1. [14 points] Compute the arc length of $y = \ln(\cos x)$ over the interval $[-\pi/4, \pi/4]$. Simplify your answer completely.

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = \ln(\cos x)$$

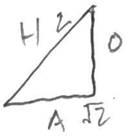
$$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\frac{\sin x}{\cos x} = -\tan x$$

$$S = \int_{-\pi/4}^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx$$

$$S = \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$S = \int_{-\pi/4}^{\pi/4} \sec x\, dx$$

$$S = \int_{-\pi/4}^{\pi/4} \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x}\, dx$$



$$u = \sec x + \tan x$$

$$du = \sec x \tan x + \sec^2 x\, dx$$

$$S = \int \frac{du}{u}$$

$$S = \ln|u|$$

$$S = \ln|\sec x + \tan x| \Big|_{-\pi/4}^{\pi/4}$$

$$S = \ln\left|\frac{2}{\sqrt{2}} + 1\right| - \ln\left|\frac{2}{\sqrt{2}} - 1\right|$$

$$\boxed{S = \ln\left|\frac{2+\sqrt{2}}{\sqrt{2}}\right| - \ln\left|\frac{2-\sqrt{2}}{\sqrt{2}}\right|}$$

$\checkmark$

Divergence
Integral
AST
DCT
LCT
RT/RT

2. [10 points] Determine whether the sequence converges. Explain your reasoning.

a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

sequence converges. As n approaches ∞, the terms of the sequence clearly converge to zero.

b) $\left\{\cos\frac{\pi}{n}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \cos\frac{\pi}{n} = 1$

sequence converges. As n approaches ∞, the terms of the sequence clearly converge to $\cos(0)$, or 1.

3. [5 points] Determine, with justification, whether the series $\sum_{k=1}^{\infty} \pi^k e^{-k}$ converges or diverges.

$\sum_{k=1}^{\infty} \pi^k e^{-k} = \sum_{k=1}^{\infty} \left(\frac{\pi}{e}\right)^k$ is a geometric series with

$r = \frac{\pi}{e} \approx \frac{3.14}{2.7} > 1$

Therefore, $\sum_{k=1}^{\infty} \pi^k e^{-k}$ diverges.

5

telescoping

4. [14 points] Consider the series $\sum_{k=1}^{\infty} \frac{4}{(3k)(3k+2)} = \frac{4}{3\cdot 5} + \frac{4}{6\cdot 8} + \frac{4}{9\cdot 11} + \cdots$.

a) Find the nth partial sum, s_n, of the series. Simplify your answer completely.

$\frac{4}{(3k)(3k+2)} = \frac{a}{3k} + \frac{b}{3k+2} = \frac{2}{3k} - \frac{2}{3k+2}$

$a(3k+2) + b(3k) = 4$

$k=0 \Rightarrow 2a = 4 \Rightarrow a = 2$

$k = -\frac{2}{3} \Rightarrow -2b = 4 \Rightarrow b = -2$

$\boxed{S_n = \sum_{k=1}^{n} \frac{2}{3k} - \frac{2}{3k+2}}$

$= \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{6} - \frac{2}{8}\right) + \cdots + \left(\frac{2}{3n} - \frac{2}{3n+2}\right)$

$= \frac{10-6}{15} + \frac{16-12}{48} \quad \frac{44-36}{99}$

$3\cdot 5 \quad 3\cdot 16 \quad 3\cdot 33$

$\frac{4}{15} + \frac{4}{48}$

$3^1 \quad 3^2$

$= 4\left(\frac{1}{n}\right.$?

$\frac{2}{3} - \frac{2}{5} + \frac{2}{6} - \frac{2}{8}$

+10

$a = \frac{4}{3\cdot 5}$

$3\cdot \quad 3\cdot 3\cdot 11$

5 11 33

b) Compute the sum of the series.

$\sum_{k=1}^{\infty} \frac{4}{(3k)(3k+2)} = \sum_{k=1}^{\infty} \frac{4}{9k^2+6k}$

$4\sum_{k=1}^{\infty} \frac{1}{(3k)(3k+2)} = 4\left(\left(\frac{1}{3}\cdot\frac{1}{5}\right) + \left(\frac{1}{6}\cdot\frac{1}{8}\right) + \left(\frac{1}{9}\cdot\frac{1}{11}\right) + \cdots\right.$

$\frac{4}{15} + \quad \lim_{k\to\infty} \frac{1}{(3k)(3k+2)} = 0$

$48 \quad 99$

$3\cdot 5 \quad 3\cdot 16 \quad 3\cdot 33$

$\frac{2}{3} - \frac{2}{5} + \frac{2}{6} - \frac{2}{8} + \frac{2}{9} - \frac{2}{11}$

$2\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{8}\right.$

$\frac{4}{3}\left(\frac{1}{5}\right. \quad 16 \quad 33$

$5\cdot 1 \quad 5\cdot 3+1 \quad 5\cdot 6+1$

$\frac{1}{n}$

$|a_n|$

5. [12 points] Determine, with justification, whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n}$$

$|a_n| = \left|\frac{1}{\ln(n)}\right|$ let $f(x) = \frac{1}{\ln(x)}$ positive ✓ decreasing ✓ continuous ✓ for all $n \geq 3$

Integral test $\lim_{n\to\infty} \int_3^n \frac{1}{\ln(x)}\, dx$

$a_n = \frac{1}{\ln(n)}$ show that $\frac{1}{\ln x}$ converges

ratio test: $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\ln(n)}{\ln(n)} =$

Limit C.T. $\lim_{n\to\infty} \frac{1}{\ln n} \div \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{n^2}{\ln}$

AST

$a_n = \frac{1}{\ln(n)}$

positive ✓

decreasing ✓

$\lim_{n\to\infty} a_n = 0$ ✓

converges ~~absolutely~~

but not abs

?

?

DCT.

$\frac{1}{\ln(n)} \geq \frac{1}{n}$

$\frac{1}{n}$ diverges ($p = 1$ harmonic)

By DCT $\sum \frac{1}{\ln(n)}$ diverges.

✓

6. [12 points] Consider the series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$.

Use the Integral Test to determine whether the series converges.

let $f(x) = \frac{\arctan(x)}{1+x^2}$ positive ✓ decreasing ✓ continuous ✓ for all $x \geq 1$

$$\lim_{n\to\infty} \int_1^n \frac{\arctan(x)}{1+x^2}\,dx$$

$u = \arctan(x)$ $\quad x: 1 \to n$

$du = \frac{1}{1+x^2}\,dx$ $\quad u: \frac{\pi}{4} \to \arctan(n)$

$$\lim_{n\to\infty} \int_{\frac{\pi}{4}}^{\arctan(n)} u\,du$$

$$\lim_{n\to\infty} \frac{1}{2}u^2 \Big|_{\frac{\pi}{4}}^{\arctan n}$$

$$\lim_{n\to\infty} \frac{1}{2}(\arctan(n))^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

$\tan^{-1}(\infty) = \frac{\sin\theta}{\cos\theta} \; \frac{1}{0} \quad \theta = \frac{\pi}{2}$

$$\frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

The series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$ converges by the integral test ✓

7. [14 points] Consider the power series $\sum_{n=1}^{\infty} \frac{(3x+2)^n}{n^2}$.

a) Calculate the *radius of convergence* for this series.

Ratio test: $\lim_{n\to\infty} \frac{(3x+2)^{n+1} \cdot n^2}{(3x+2)^n \cdot (n+1)^2}$

$|3x+2| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{\infty}{\infty}$

$-\frac{2}{3}$ ✓

$|3x+2| \lim_{n\to\infty} \frac{n^2}{n^2} = |3x+2| < 1$

$3x+2<1$ $\quad$ $3x+2>-1$

$x<-\frac{1}{3}$ $\quad$ $x>-1$

+6

Radius of convergence $= \frac{1}{3}$ ✓

b) Determine the *interval* of convergence for this series.

($\quad$)

-1 $\quad$ $-\frac{1}{3}$

$\boxed{x=-1}$ ✓

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Alternating series Test

$a_n = \frac{1}{n^2}$

positive ✓

decreasing ✓

$\lim_{n\to\infty} a_n = 0$ ✓

converges ✓

$\boxed{x=-\frac{1}{3}}$

$\sum_{n=1}^{\infty} \frac{1^n}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2}$

P-series ✓

$P=2>1$ ✓

converges ✓

interval of convergence : $[-1, -\frac{1}{3}]$ ✓

8

8. [15 points]

a) Write the Maclaurin series expansions of the following functions.

$\sin(x^2)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$$

$$\boxed{\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n+1)!}}$$

$x^2 \cos(x^2)$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{a^n \cdot x^{4n}}{(2n)!}$$

$$\boxed{x^2 \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n)!}}$$

6

b) **Use the Maclaurin series expansions** found above to evaluate the limit $\lim_{x \to 0} \frac{\sin(x^2) - x^2 \cos(x^2)}{x^6}$.

$$\lim_{x \to 0} \left[x^{-6}(\sin(x^2)) - x^{-6}(x^2 \cos(x^2)) \right]$$

$$= \lim_{x \to 0} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n-4}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n-4}}{(2n)!} \right]$$

$$= \lim_{x \to 0} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n-4}}{(2n+1)!} - \frac{(-1)^n x^{4n-4}}{(2n)!} \right]$$

= expand them

$$\frac{1}{(2n+1)!} - \frac{1}{(2n)!}$$

telescoping

$$\left(\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right) + \left(\frac{2}{(2n+1)!} - \frac{2}{}\right.$$

$$\frac{1}{2n! - 2n+1}$$
$$2n+1 \quad 2n$$

$$\frac{1}{1} - \frac{1}{2} + \frac{2}{}$$

$$\frac{2! - 3!}{3!\, 2!}$$

0

$$\left(\frac{x^{-4}}{1} - \frac{x^{-4}}{1}\right) - \left(\frac{1}{3!} - \frac{1}{2!}\right) + \left(\frac{x^4}{5!} - \frac{x^4}{4!}\right) + \ldots$$

+2

$$\frac{\cancel{x} \; \cancel{x^6}}{\cancel{x^2}}$$

$a=0$

9. [14 points]

a) Find the Maclaurin series for x^2e^x. Write the series in sigma notation.

$f(x)=$

$x^2 \cdot e^x = \sum \frac{x^n}{n!}$

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}}$$

3

b) Use the series expansion from part (a) to find a power series expansion of $\int_0^x t^2 e^{-t}\,dt$.

$\sum a_n (x-c)^n$

$e^{-t} = \sum \frac{t^n}{n!}$

$t^2 \cdot e^{-t} = \sum \frac{-t^{n+2}}{n!}$

$\int_0^x \sum \frac{-t^{n+2}}{n!} = \sum_{n=0}^{x} \frac{(-1)^n t^{n+3}}{(n+3)(n!)}\Big|_0^x \cancel{+C}$

$\frac{x^{n+3}}{(n+3)(n!)} \Big|_0^\infty$

$t^2 e^{-t} = \sum_{n=0}^{x} \frac{-t^{n+3}}{(n+3)(n!)} \cancel{+C}$

~~$t=0 \Rightarrow C=0$~~

$$\boxed{\sum_{n=0}^{x} \frac{(-1)^n \cancel{x}^{n+3}}{(n+3)(n!)}}$$

4/7

$\frac{t^3}{n} \cdot -t^n$

c) If we approximate $\int_0^1 t^2e^{-t}\,dt$ using the first three terms from the series found in part (b), what is the error bound for that approximation?

$E_3 \; \cancel{T} \le \cancel{\frac{K|x-a|^{n+1}}{(n+1)!}} \le a_{n+1}$