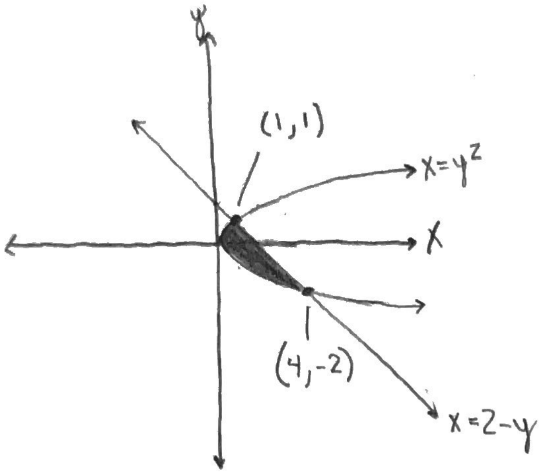
1. [10 points] Find the area of the region bounded by the graphs of $x = y^2$ and $x = 2 - y$.

$\downarrow$

$y = -x + 2$



$y^2 = 2 - y$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2 \rightarrow x = 4$

$y = 1 \rightarrow x = 1$

integrate with respect to y from -2 to 1

$$A = \int_{-2}^{1} (2-y) - (y^2)\, dy$$

$$\int_{-2}^{1} (2 - y - y^2)\, dy$$

$$2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_{-2}^{1}$$

$$\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

? $$\left(\frac{12}{6} - \frac{3}{6} - \frac{2}{6}\right) - \left(-\frac{24}{6} - \frac{12}{6} + \frac{16}{6}\right)$$

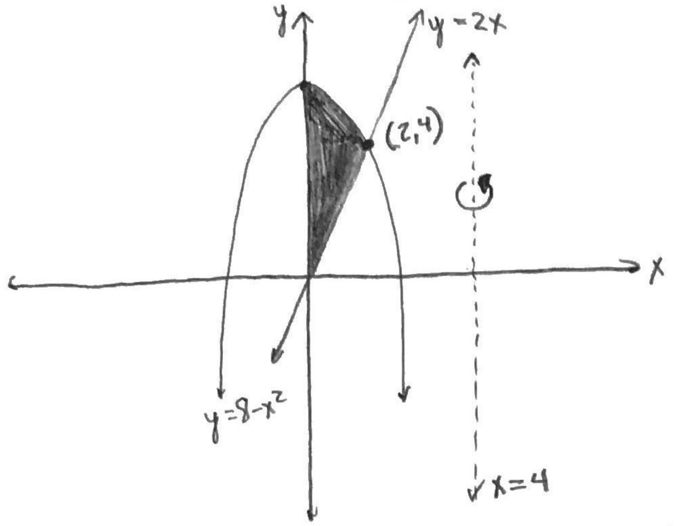
$$\frac{7}{6} - \left(-\frac{20}{6}\right)$$

$$\frac{27}{6} = \boxed{\frac{9}{2}}$$

✓

2. [14 points] We want to find the volume of the solid obtained by rotating the *first quadrant* region bounded by the graphs of $y = 8 - x^2$, $y = 2x$, and the y–axis, about $x = 4$.

a) Graph the region. Label the points of intersection of the two graphs.



$2x = 8 - x^2$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

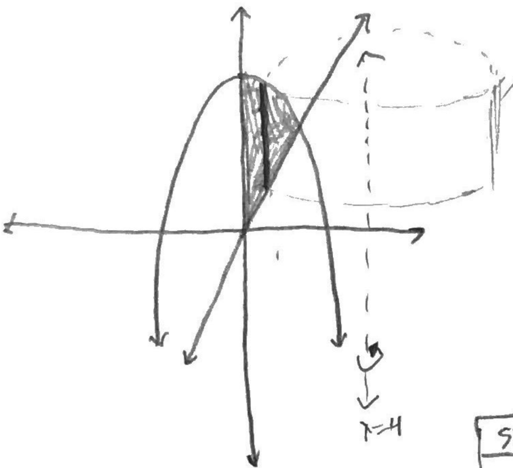
$x = 2 \rightarrow y = 4$

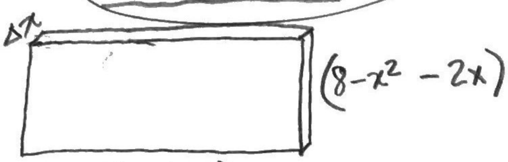
$\cancel{x = -4}$

4

b) Set up an integral, using any method, that gives the volume of the solid. Do **not** evaluate the integral. Indicate which method you are using.

Shell Method





2π (radius)

$2\pi\ (4 - x)$

from $0 \rightarrow 2$

Shell Method 4 limits.

$$V = 2\pi \int (4-x)(8-x^2-2x)\, dx$$

+ 8/10

3. [8 points] $\frac{1}{b-a}\int_a^b f(x)\,dx$

a) Find the average value of the function $f(x) = \cos x$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$f_{ave} = \frac{1}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$$

$$= \frac{1}{\pi}(\sin x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\sin \frac{\pi}{2}\right) - \frac{1}{\pi}\left(\sin -\frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}(1) - \frac{1}{\pi}(-1)$$

$$= \frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}$$

b) Find number c in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $f(c)$ equals its average value.

$$f(c) = \cos(c) = \frac{2}{\pi}$$

$$\boxed{c = \cos^{-1}\left(\frac{2}{\pi}\right)}$$

$1 - \sin^2 = \cos^2$
$\sec^2 - 1 = \tan^2$
$\tan^2 + 1 = \sec^2$

4. [15 points] Use trigonemetric substitution to evaluate the integral.

$$\int \frac{x}{(x^2+9)^3}\,dx$$

$$\int \frac{x}{(x^2+9)^3}\,dx$$

$$\int \frac{x}{\left[9\left(\frac{x^2}{9}+1\right)\right]^3}\,dx$$

$\frac{x}{3} = \tan\theta \rightarrow x = 3\tan\theta$

$\frac{1}{3}dx = \sec^2\theta\,d\theta \rightarrow dx = 3\sec^2\theta\,d\theta$

$$\int \frac{x}{\left[9(\tan^2\theta+1)\right]^3}\,dx$$

$$\int \frac{9\tan\theta\sec^2\theta\,d\theta}{(9\sec^2\theta)^3}$$

$$\int \frac{\tan\theta\,d\theta}{(9\sec^2\theta)^2}$$

$$\frac{1}{81}\int \frac{\tan\theta\,d\theta}{\sec^4\theta}$$

$$\frac{1}{81}\int \frac{\frac{\sin\theta}{\cos\theta}\,d\theta}{\left(\frac{1}{\cos\theta}\right)^4}$$

$$\frac{1}{81}\int \sin\theta\cos^3\theta\,d\theta$$

$$\frac{1}{81}\int \sin\theta\cos\theta\cos^2\theta\,d\theta$$

$$\frac{1}{81}\int \sin\theta\cos\theta(1-\sin^2\theta)\,d\theta$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

$$\frac{1}{81}\int u(1-u^2)\,du$$

$$\frac{1}{81}\int u - u^3\,du$$

$$\frac{1}{81}\left[\frac{u^2}{2} - \frac{u^4}{4}\right] + C$$

$$\boxed{\frac{1}{81}\left[\frac{\sin^2\theta}{2} - \frac{\sin^4\theta}{4}\right] + C}$$

$= ?$

5. [10 points] Evaluate the integral.

$$\int \frac{\sin^3 x}{\cos^4 x}\, dx$$

$\int \tan^3(x) \cdot \frac{1}{\cos(x)}\, dx$

$\int \tan^3(x) \sec(x)\, dx$

$\int \tan^3(x) \sec(x) \cdot \frac{\sec(x)}{\sec(x)}\, dx = \int \frac{\sec x \, \overbrace{\tan^2 x}^{(\sec^2 x - 1)} \sec x \tan x \, dx}{\sec x}$

$u = \sec x$

$du = \sec x \tan x \, dx$

$\int \frac{u(u^2-1)\, du}{u}$

$\int (u^2 - 1)\, du$

$\frac{1}{3}u^3 - u + C$

$\boxed{\frac{\sec^3 x}{3} - \sec x + C}$

✓

☑ change limits
☑ finite answer

6. [10 points] Integrate.

$$\int_4^{25} \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$$

$u = \sqrt{x} + 1$ $\qquad$ $x: 4 \to 25$

$2 \cdot du = \frac{2}{2}\frac{1}{2}x^{-1/2}\,dx$ $\qquad$ $u: 3 \to 6$

$$2\int_3^6 \frac{du}{u^2}$$

$$2\int_3^6 u^{-2}\,du$$

$$2(-u^{-1})\Big|_3^6$$

$$2\left(-\frac{1}{6}\right) - 2\left(-\frac{1}{3}\right)$$

$$-\frac{1}{3} + \frac{2}{3}$$

$$\boxed{\frac{1}{3}}$$

7. [12 points] Integrate.

$$\int \frac{x+1}{(x-1)(x^2+1)}\,dx$$

$$\int \frac{x+1}{(x-1)(x^2+1)}\,dx = \int \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}\,dx = \int \frac{-\frac{1}{2}}{(x-1)} + \frac{\frac{1}{2}x+\frac{1}{2}}{(x^2+1)}\,dx$$

$A(x^2-1) + [Bx+C](x-1) = x+1$

$Ax^2 - A + Bx^2 - Bx + Cx - C = x+1$

$Ax^2 + Bx^2 - Bx + Cx - A - C = x+1$

$\underbrace{(A+B)}_{0}x^2 + \underbrace{(-B+C)}_{1}x + \underbrace{(A-C)}_{1} = x+1$

$A = -B$

$-B + C = 1$

$A - C = 1$

$\cancel{C - B = -A - C}$

$\cancel{2C = -A + B}$

$\cancel{2(1-B) = B + B}$

$\cancel{2 - 2B = 2B}$

$\cancel{\frac{2}{4} = \frac{4B}{4}}$

$\cancel{B = \frac{1}{2}}$

$\cancel{A = -\frac{1}{2}}$

$\cancel{C = \frac{1}{2}}$

$-\frac{1}{2} + C = 1$

$-B + C = -A - C$

$-\frac{1}{2} + C = \frac{1}{2} - C$

$2C = 1$

$$= -\frac{1}{2}\int \frac{1}{(x-1)}\,dx + \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$$

$$= -\frac{1}{2}\ln|x-1| + \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx \quad ?$$

$u = \frac{1}{x^2+1}$ $\quad du = x+1\,dx$

$du = -2x^{-3}$ $\quad v = \frac{1}{2}x^2$

$$-\frac{1}{2}\ln|x-1| + \frac{1}{2}\left[\frac{1}{2}\left(\frac{x^2}{x^2+1}\right) + \frac{1}{4}\int \frac{1}{x}\,dx\right]$$

$$\boxed{-\frac{1}{2}\ln|x-1| + \frac{1}{4}\left(\frac{x^2}{x^2+1}\right) + \frac{1}{8}\ln|x| + C}$$

+6

$\int \frac{1}{1+x^2} = \arctan x$

8. [12 points] Integrate.

$$\int_0^1 \frac{x \arctan x \, dx}{x^2} \; x^2$$

Sub
parts
trig
hyper
partial

$u = \arctan x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2} \qquad v = \frac{1}{2}x^2$

~~$u = x \qquad dv = \arctan x\, dx$~~

~~$du = 1\,dx \qquad v = \frac{1}{\sqrt{1+x^2}}$~~

$\frac{1}{2}x^2 \arctan x - \frac{1}{2}\underbrace{\int \frac{x^2}{1+x^2} dx}$

~~$\frac{x}{\sqrt{1+x^2}} - \int \frac{1}{\sqrt{1+x^2}} dx$~~

$w = 1 + x^2$

$x = \tan\theta$

$dx = \sec\theta^2 d\theta$

$x : 0 \to 1$

$\theta : 0 \to \tan^{-1}(0)$

$\frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{\tan^2\theta \sec^2\theta \, d\theta}{\sec^2\theta}$

$\frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \tan^2\theta \, d\theta$

$\frac{1}{2}x^2 \arctan x \Big|_0 - \frac{1}{6} \tan^3(\tan^{-1}x) \Big|_0^1$

$=$

$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1}$

$= \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}$

$= 1 - \frac{1}{x^2+1}$

$\Rightarrow \int \frac{x^2}{x^2+1} dx = x - \text{Arctan}(x) + C$

9. [12 points] Evaluate the integral to determine whether it converges or diverges.

$$\int_{e^4}^{\infty} \frac{1}{x(\ln x)^{3/2}}\,dx$$

$$\lim_{n\to\infty} \int_{e^4}^{n} \frac{1}{x(\ln x)^{3/2}}\,dx$$

$u = \ln x \qquad x: e^4 \to n$

$du = \frac{1}{x}\,dx \qquad u = 4 \to \ln(n)$

$$\lim_{n\to\infty} \int_{4}^{\ln(n)} \frac{du}{u^{3/2}}$$

$$\lim \int u^{-3/2}\,du$$

$$\lim \frac{-2}{\sqrt{u}} \Big|_{4}^{\ln(n)}$$

$$\lim_{n\to\infty} \left(\frac{-2}{\sqrt{\ln(n)}} - \underbrace{\frac{-2}{\sqrt{4}}}_{\text{finite}} \right) = ?\ \text{finish.}$$

$$\frac{-2}{\sqrt{\ln(\infty)}}$$

$$\underbrace{\frac{-2}{\sqrt{\infty}}}_{\text{approaches } 0}$$

$\boxed{\text{converges}}$