1. (i) (5 points) What is the maximum possible rank of a 4×6 matrix? Explain briefly.

$$\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{bmatrix} \quad \boxed{max = 4}$$

Every row is a non zero row, and there are no zero rows.

Why is that the max? You gave an example. -2

(ii) (5 points) Let $A = \left[\begin{array}{c|c} 2I_2 & B \\ \hline 0_{2\times 2} & 3I_2 \end{array}\right]$, where $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$. Find A^2.

$$\begin{bmatrix} \cancel{2\,0} & \cancel{2\,1} \\ \cancel{0\,2} & \cancel{1\,2} \\ \cancel{0\,0} & \cancel{3\,0} \\ 0\,0 & 0\,3 \end{bmatrix} \left[\begin{array}{cc|cc} 2&0&2&1 \\ 0&2&1&-2 \\ 0&0&3&0 \\ 0&0&0&3 \end{array}\right] = \begin{bmatrix} 2&0&4+6&2+3 \\ 0&4&2+3&-4+-6 \\ 0&0&9&0 \\ 0&0&0&9 \end{bmatrix} = \begin{bmatrix} 2&0&10&6 \\ 0&4&5&-10 \\ 0&0&9&0 \\ 0&0&0&9 \end{bmatrix}$$

-2

(iii) (5 points) What is the nullity of $A_{100\times 4} = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ \vdots&\vdots&\vdots&\vdots \\ 1&1&1&1 \end{bmatrix}$? Explain briefly.

In rref, all rows below first will be zero rows. $\begin{bmatrix} 1&1&1&1 \\ 0&0&0&0 \\ \vdots&\vdots&\vdots&\vdots \\ 0&0&0&0 \end{bmatrix}$

Rank = 1

$\boxed{\text{nullity} = 3}$

✓

(iv) (5 points) Given $A = \begin{bmatrix} -1 & r \\ r & -9 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ determine the value(s) of r, if any, for which $Ax = b$ is not consistent.

There is no "r" that would result in the system being inconsistent because any value of r can still result in the matrix being transformed to Identity matrix.

$$\begin{bmatrix} -1 & r & 2 \\ r & -9 & 6 \end{bmatrix}$$

$-x_1 + rx_2 = 2 \Rightarrow x_1 = rx_2 - 2$

$rx_1 - 9x_2 = 6$

$r(rx_2 - 2) - 9x_2 = 6$

$r^2 x_2 - 2r - 9x_2 = 6$

-5

(v) (5 points) Prove that the inverse of A^T is $(A^{-1})^T$.

$((A\ I)^T)$

If a matrix is invertible, it's rref is I. The transpose of I is itself. The inverse of I is itself also.

$$(A^T)^{-1} = (A^{-1})^T$$
$$(I)^{-1} = (I)^T$$
$$I = I$$

-5

(vi) (5 points) Find the value of r if $\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ r \end{bmatrix}$ are linearly dependent.

$\boxed{r=-6}$

$$3u - 1v = 0$$
$$\begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(vii) (5 points) If a set of vectors contains $\mathbf{0}$, the set is linearly dependent. Why?

All the other vectors can have zero scalars, and the zero vector will have a nonzero scalar and there is still a solution to $Ax = 0$ such that not all the scalars are zero

(viii) (5 points) Show that the vector $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is in the span of $A = \begin{bmatrix} 0 & -1 & -3 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 3 & 9 & 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 0 & -1 & -3 & 1 & -1 \\ 1 & 0 & -2 & 1 & 2 \\ 0 & 3 & 9 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 2 \\ 0 & -1 & -3 & 1 & -1 \\ 0 & 3 & 9 & 0 & 1 \end{array}\right] \xrightarrow[r_3 \to r_3 + 3r_2]{r_2 \to -r_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & 3 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 \end{array}\right]$$

$$\xrightarrow{r_3 \to \frac{1}{3} r_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \xrightarrow[r_2 \to r_2 + r_3]{r_1 \to r_1 - r_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{8}{3} \\ 0 & 1 & 3 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \end{array}$$

rref error

In rref, there exists a solution such that a linear combination of the vectors results in $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. This system is consistent because there are no zero rows with a nonzero on the right.

2. Let

$$E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) (8 points) Write the elementary row operation that transforms the 3×3 identity matrix into E. Then use the inverse of that operation to obtain E^{-1}.

elementary operation : $r_1 - r_2 \to r_1$

inverse operation : $r_1 + r_2 \to r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 + r_2 \to r_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E^{-1}$$

(ii) (8 points) Write the elementary row operation that transforms the 3×3 identity matrix into F. Then use the inverse of that operation to obtain F^{-1}.

elementary operation : $-2r_1 \to r_1$

inverse operation : $-\frac{1}{2} r_1 \to r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2} r_1 \to r_1} \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F^{-1}$$

(iii) (4 points) Find $(EF)^{-1}$. *You can use any method but show your work.*

$$(EF) = \begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 + r_2 \to r_1} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$(EF)^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (10 points) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ using the row operations method.

$$\left[\begin{array}{cccc|cccc} 1&0&0&1&1&0&0&0 \\ 0&1&1&0&0&1&0&0 \\ 1&0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&0&1 \end{array}\right] \xrightarrow{r_3 - r_1 \to r_3} \left[\begin{array}{cccc|cccc} 1&0&0&1&1&0&0&0 \\ 0&1&1&0&0&1&0&0 \\ 0&0&1&-1&-1&0&1&0 \\ 0&1&0&0&0&0&0&1 \end{array}\right] \xrightarrow{r_4 - r_2 \to r_4} \left[\begin{array}{cccc|cccc} 1&0&0&1&1&0&0&0 \\ 0&1&1&0&0&1&0&0 \\ 0&0&1&-1&-1&0&1&0 \\ 0&0&-1&0&0&-1&0&1 \end{array}\right]$$

$$\xrightarrow{r_2 + r_4 \to r_2} \left[\begin{array}{cccc|cccc} 1&0&0&1&1&0&0&0 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&-1&-1&0&1&0 \\ 0&0&-1&0&0&-1&0&1 \end{array}\right] \xrightarrow{r_3 + r_4 \to r_4} \left[\begin{array}{cccc|cccc} 1&0&0&1&1&0&0&0 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&-1&-1&0&1&0 \\ 0&0&0&-1&-1&-1&1&1 \end{array}\right]$$

$$\xrightarrow[-r_4 \to r_4]{r_1 + r_4 \to r_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&0&-1&1&1 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&-1&-1&0&1&0 \\ 0&0&0&1&1&1&-1&-1 \end{array}\right] \xrightarrow{r_3 + r_4 \to r_3} \left[\begin{array}{cccc|cccc} 1&0&0&0&0&-1&1&1 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&1&0&-1 \\ 0&0&0&1&1&1&-1&-1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

4. In this problem A is a ~~4 × 5~~ 4× 6 matrix, and R is the reduced row echelon form of A. The first and the third ~~rows~~ columns of, the matrix R, and another vector $\boldsymbol{b}$ are given below.

R=rref(A)

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 3 \\ -7 \\ -9 \\ 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

(i) (5 points) Can the columns of A generate $\mathcal{R}^4$? Explain your answer.

~~Yes Because the matrix A is 4 x 6 any linear combination of the vectors in A will result in a solution in $\mathbb{R}^4$.~~

-5

(ii) (5 points) Find the fourth column of A

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2v_1 + 3v_3 = v_4$

$$-2\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -7 \\ -9 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -6 \\ -2 \end{bmatrix} + \begin{bmatrix} 9 \\ -21 \\ -27 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ -33 \\ 4 \end{bmatrix} = a_4$$

(iii) (10 points) Find the solution of $R\boldsymbol{x} = \boldsymbol{b}$. Write it in *vector form* (i.e a sum of a fixed vector and a linear combination of vectors with free variables as coefficients).

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & -2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rref ✓

consistent ✓

$x_1 = 2 - 2x_2 + 2x_4 - 2x_6$

free $x_2 = x_2$

$x_3 = -1 - 3x_4 + x_6$

free $x_4 = x_4$

$x_5 = 3 - 3x_6$

free $x_6 = x_6$

$$= \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

5. (10 points) Answer the true/false questions below by circling **True** or **False** in the space provided in front of the statement.
Each question is worth one point.

(i) True False $\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$ is a matrix in reduced row-echelon form.

(ii) True False If the nullity of a matrix A is 0, then the columns of A are linearly independent.

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 3 nullity = 0 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) True False If the second column of the matrix B is all zeros, and if AB is defined, then the second column of AB is all zeros too.

(iv) True False Any elementary matrix is invertible.

(v) True False If B is a 3×4 matrix, then its rows are 4×1 vectors.

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(vi) True False The inverse of an elementary matrix is also an elementary matrix.

(vii) True False The span of two parallel vectors in $\mathcal{R}^2$ is $\mathcal{R}^2$.

(viii) True False If AB is defined, then $(AB)^T = A^T B^T$.

(ix) True False If A is invertible then the system of equations $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution.

(x) True False If some column of matrix A is a pivot column, then the corresponding column in the reduced row echelon form of A is a standard vector.