1. Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. The characteristic polynomial of A is $-(t+2)^2(t-4)$.

(i) (2 points) State the eigenvalues of A and their multiplicities.

$\lambda = -2$ multiplicity $= 2$

$\lambda = 4$ multiplicity $= 1$

✓

(ii) (8 points) For each eigenvalue find a basis for the corresponding eigenspace.

Show work and write your answers very clearly.

$\lambda = -2$

$A + 2I = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{matrix} r_1 \to \frac{1}{3} r_1 \\ r_2 \to \frac{1}{3} r_2 \\ r_3 \to \frac{1}{6} r_3 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{matrix} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_2 - x_3$

$x_2 = x_2$

$x_3 = x_3$

basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ ✓

$\lambda = 4$

$A - 4I = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{matrix} r_1 \to -\frac{1}{3} r_1 \\ r_2 \to r_2 + r_1 \\ r_3 \to r_3 + 2r_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix} \begin{matrix} r_2 \to -\frac{1}{12} r_2 \\ r_3 \to r_3 - r_2 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} r_1 \to r_1 - r_2$

$= \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = \frac{1}{2} x_3$

$x_2 = \frac{1}{2} x_3$

$x_3 = x_3$

basis $\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ ✓

2. (40 points) For each of the eight parts below: write the answer in the space provided, and give a brief explanation.

(i) A is a 5×6 matrix. If nullity$(A) = 4$, what is the nullity of A^T?

Answer: $\boxed{3}$ ✓

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Explanation:
If nullity $= 4$, that means the rank is 2. Thus we must subtract 2 from the number of columns in A^T which is 5.

(ii) Is $W = \left\{ \begin{bmatrix} r \\ -s \\ r+s+1 \end{bmatrix} : r, s \in \mathcal{R} \right\}$ a subspace?

Answer: No

Explanation:
The zero vector is not included. Why? -2

(iii) Suppose $\boldsymbol{v}$ is a nonzero vector and $A = \boldsymbol{v}\boldsymbol{v}^T$. Is $\boldsymbol{v}$ an eigenvector of A?

Answer: Yes.

Explanation:
$A\vec{v} = (vv^T)\vec{v}$. Because $\vec{v}$ is a non zero, this equation is possible. the eigenvalue would be (vv^T) and the eigen vector would be $\vec{v}$. -5

vv^T is not a number.

(iv) Suppose A, B, and C are 3×3 matrices and that $\det(A) = 2$, $\det(B) = 3$, $\det(C) = 2$. What is the value of $\det(-2AB^{-1}C^T)$?

Answer: $\boxed{-\frac{32}{3}}$ ✓

Explanation (calculation):

$$\underbrace{(-1)^3 \cdot (2)^3}_{-2} \cdot \underbrace{2}_{\det A} \cdot \underbrace{\tfrac{1}{3}}_{\det B^{-1}} \cdot \underbrace{2}_{\det C^T} = -8 \cdot 2 \cdot \tfrac{1}{3} \cdot 2 = -\frac{32}{3}$$

(v) If v is an eigenvector of an invertible matrix A, then is it also an eigenvector of A^{-1} ?

Answer: Yes.

Yes

Explanation:

$A\vec{v} = \lambda\vec{v}$

$A^{-1}(A\vec{v}) = A^{-1}(\lambda\vec{v})$

$\vec{v} = \lambda(A^{-1}\vec{v})$

$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$

The new eigenvalue would just be $\frac{1}{\lambda}$

$A\vec{v} = \lambda\vec{v}$

$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$

$\frac{\vec{v}}{\lambda} = \frac{\lambda A^{-1}v}{\lambda}$

$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$

(vi) Suppose A is an $n \times n$ matrix and $\text{rank}(A) < n$. Can Null $(A) = \{0\}$?

Answer: No

Explanation:

Because the rank(A) is less than n, the nullity of A can never be zero. Why? And what does that tell you about Null(A)? -1

(vii) If A is an $m \times n$ matrix then $\dim(\text{Row}(A)) + \dim(\text{Col}(A)) = n$. True or false?

Answer:

False

Explanation:

dim(Row(A)) = rank

dim(Col(A)) = rank

2 · rank isn't always = # of columns

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rank = 3

n = 3

$3+3 \neq 3$

(viii) If $AA^T = I_n$ then what are the possible values for $\det(A)$?

Answer: 1

-5

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Explanation:

Because the determinant of I is one. We must have an A whose determint is also 1. This doesn't change with A^T.

3. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 2 & 4 & 1 & 3 & 2 \\ 3 & 6 & 2 & 5 & 3 \\ 3 & 6 & 0 & 4 & 3 \end{bmatrix}$. Its reduced row echelon form is $R = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(i) (5 points) Write a basis for the row space of A. (You do not have to show any work).

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(ii) (5 points) Write a basis for the column space of A. (You do not have to show any work).

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 4 \end{bmatrix} \right\}$$

$\checkmark$

(iii) (5 points) Find a basis for the null space of A. **You must show your work.**

rref (A). $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -2x_2 - x_5$
$x_2 = x_2$
$x_3 = 0x_2 + 0x_5$
$x_4 = 0x_2 + 0x_5$
$x_5 = x_5$

$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. (10 points) $B = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & -2 & 1 & 2 \\ 3 & -3 & -2 & 1 \end{bmatrix} \end{array}$. Find det($B$) using a cofactor expansion.

Along 1st row

$$\underbrace{1 \cdot \det\begin{vmatrix} 1 & 0 & 4 \\ -2 & 1 & 2 \\ -3 & -2 & 1 \end{vmatrix}}_{1\cdot(1+4)+4(4+3)} - 1\det\underbrace{\begin{vmatrix} 0 & 0 & 4 \\ 1 & 1 & 2 \\ 3 & -2 & 1 \end{vmatrix}}_{4(-2-3)} + 1\det\underbrace{\begin{vmatrix} 0 & 1 & 4 \\ 1 & -2 & 2 \\ 3 & -3 & 1 \end{vmatrix}}_{-1(-5)+4(-3+6)} + 0\det\begin{vmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 3 & -3 & -2 \end{vmatrix}$$

$$33 \quad - (-20) \quad + \quad 17 \quad = \boxed{\cancel{70}}$$

$= 30$

$\neq 70$

-5

5. (10 points) The LU-decomposition of a matrix A is given by $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Use the LU-decomposition method to solve $Ax = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

No credit for using other methods. Must show work.

$A\vec{x} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

$LU\vec{x} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

$L\vec{y} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 13 \\ 2 & 1 & 1 & 4 \end{array}\right]$

$y_1 = 3$

$3(3) + y_2 = 13 \Rightarrow y_2 = 4$

$2(3) + 4 + y_3 = 4 \Rightarrow y_3 = -6$

$U\vec{x} = \vec{y}$

$U\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 3 & -6 \end{array}\right]$

$x_1 + 16 - 24 = 3 \Rightarrow x_1 = 11$

$2x_2 + 2(-6) = 4 \Rightarrow x_2 = 8$

$x_3 = -6$

$\vec{X} = \cancel{\begin{bmatrix} 11 \\ 8 \\ -6 \end{bmatrix}}$

-5

6. Throughout this problem: A is a 3×3 matrix of the form $A = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix}$, i.e. the rows of A are $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$, and $\det(A) = 4$.

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \begin{bmatrix} 2c \\ b \\ a \end{bmatrix} \begin{bmatrix} 2c+2b \\ b \\ a \end{bmatrix} \begin{bmatrix} 2c+2b \\ b \\ a+5b \end{bmatrix} \begin{bmatrix} 2c+2b \\ 3b \\ a+5b \end{bmatrix}$

(i) (5 points) Suppose $B = \begin{bmatrix} 2\boldsymbol{c}+2\boldsymbol{b} \\ 3\boldsymbol{b} \\ \boldsymbol{a}+5\boldsymbol{b} \end{bmatrix}$. Write down the sequence of row operations that would convert the matrix A to the matrix B.

det $=4$

$r_1 \leftrightarrow r_3$ $\quad 4 \cdot -1 = -4$

$r_1 \to 2r_1$ $\quad -4 \cdot 2 = -8$

$r_1 \to r_1 + 2r_2$ $\quad -8 = -8$

$r_3 \to r_3 + 5r_2$ $\quad -8 = -8$

$r_2 \to 3r_2$ $\quad -8 \cdot 3 = -24$

(ii) (5 points) Using the row operations found above, calculate $\det(B)$.

$\det(B) = \boxed{-24}$

(iii) (5 points) Suppose $C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$. What is the determinant of $2A^TC^{-1}$?

$\det(C) = -6 \quad \det(C^{-1}) = -\frac{1}{6}$

$\det 2A^TC^{-1} = 2^3 \cdot (4) \cdot -\frac{1}{6} = \boxed{-\frac{32}{6}}$