1. [15 points] Consider the function $f(x, y, z) = x + 2y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$.

a) Use Lagrange multipliers to find all the critical points of f on the given surface.

$\nabla f = \lambda \nabla g$

$\langle 1,2,2\rangle = \lambda\langle 2x, 2y, 2z\rangle$

$1 = \lambda 2x \rightarrow x = \frac{1}{2\lambda}$

$2 = \lambda 2y \rightarrow y = \frac{2}{2\lambda}$

$2 = \lambda 2z \rightarrow z = \frac{2}{2\lambda}$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 = 3$

$\frac{1}{9\lambda^2} + \frac{4}{4\lambda^2} + \frac{4}{4\lambda^2} = 3$

$\frac{9}{4\lambda^2} = 3$

$4\lambda^2 = 3$

$\lambda^2 = \frac{3}{4}$

$\lambda = \pm\frac{\sqrt{3}}{2}$

$\lambda = \frac{\sqrt{3}}{2}$

$x = \frac{1}{\cancel{2}\frac{\sqrt{3}}{\cancel{2}}} = \frac{1}{\sqrt{3}}$

$y = \frac{2}{\cancel{2}\frac{\sqrt{3}}{\cancel{2}}} = \frac{2}{\sqrt{3}}$

$z = \frac{2}{\cancel{2}\frac{\sqrt{3}}{\cancel{2}}} = \frac{2}{\sqrt{3}}$

$\lambda = -\frac{\sqrt{3}}{2}$

$x = -\frac{1}{\sqrt{3}}$

$y = -\frac{2}{\sqrt{3}}$

$z = -\frac{2}{\sqrt{3}}$

Critical points

$\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

$\left(\frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$

b) Determine the maxima and minima of f on the surface.

$\text{Max} = \frac{1}{\sqrt{3}} + 2\left(\frac{2}{\sqrt{3}}\right) + 2\left(\frac{2}{\sqrt{3}}\right) = \boxed{\frac{9}{\sqrt{3}}}$

$\text{Min} = \frac{-1}{\sqrt{3}} + 2\left(-\frac{2}{\sqrt{3}}\right) + 2\left(-\frac{2}{\sqrt{3}}\right) = \boxed{\frac{-9}{\sqrt{3}}}$

2. [15 points] Consider the integral $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx$.

a) Convert the integral to spherical coordinates and evaluate.

$0 \le x \le 2$

$0 \le y \le \sqrt{4-x^2}$

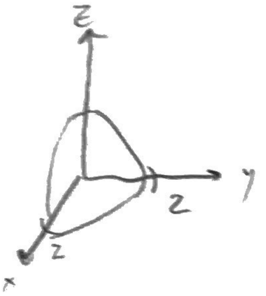
$0 \le z \le \sqrt{4-x^2-y^2}$

$0 \le \rho \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$0 \le \varphi \le \frac{\pi}{2}$

$\frac{1}{8}$ sphere of radius 2



radius = 2

Jacobian = $\rho^2 \sin\varphi$

1st octant

$$\int_0^2\int_0^{\pi/2}\int_0^{\pi/2} \rho^2 \sin\varphi \; d\theta\, d\varphi\, d\rho$$

X

+7/9

$$\frac{\pi}{2}\int_0^2\int_0^{\pi/2} \rho^2 \sin\varphi \; d\varphi\, d\rho$$

$\cancel{\tfrac{1}{8}}\,\tfrac{4}{3}\pi\,\cancel{(8)}$ ✓

$$-\rho^2\cos\varphi \Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$0 - -\rho^2\cos(0)$$

$$\frac{\pi}{2}\int_0^2 \rho^2 \, d\rho$$

$$\frac{\pi}{2}\,\frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=2} = \frac{\pi}{6}\cdot 8 = \boxed{\frac{4\pi}{3}}$$ ✓

b) Convert the integral to cylindrical coordinates. **Do not evaluate.**

$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

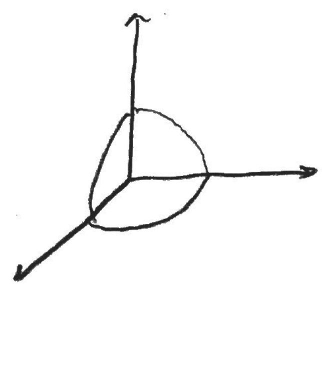
$0 \le z \le \sqrt{4-r^2}$

$0 \le z \le \sqrt{4-r^2}$

Jacobian = r

$$\boxed{\int_0^2\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-r^2}} z\,r\,dz\,d\theta\,dr}$$

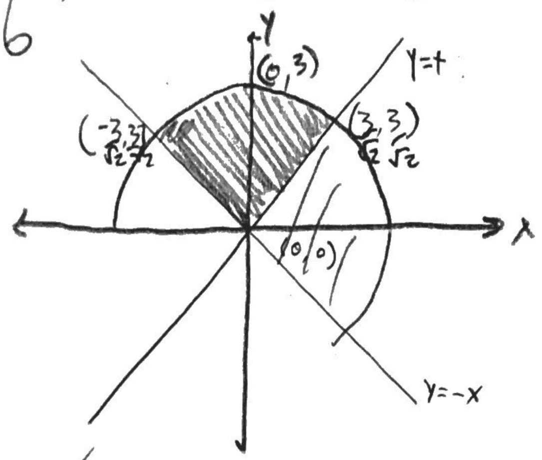
5/6



3. [15 points] Consider the integral $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\, dA$, where $\mathcal{D}$ is the region bounded by $y=-x$, $y=x$, and $x^2+y^2=9$, for $x \geq 0$.

3/6

a) Sketch the region of integration. Label points of intersection.



b) Evaluate the integral by first converting to polar coordinates.

o.k. by (a)

$0 \le r \le 3$

$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

Jacobian = r

$$\int_{\pi/4}^{3\pi/4}\int_0^3 r^2\, dr\, d\theta$$

$$\frac{1}{3}r^3\Big|_0^3$$

$$\int_{\pi/4}^{3\pi/4} 9\, d\theta$$

$$9\theta\Big|_{\pi/4}^{3\pi/4}$$

$$9\cdot\frac{3\pi}{4} - 9\cdot\frac{\pi}{4} = \frac{18\pi}{4} = \boxed{\frac{9\pi}{2}}$$

4. [15 points] *Buzz Lightyear* travels along the helix C given by $\boldsymbol{r}(t) = \langle \sin t, 3t, \cos t\rangle$ and is subject to the force $\boldsymbol{F} = \langle -x, xy, z\rangle$. Find the total work done on Buzz by the force for $0 \le t \le 5\pi$.

P Q R

$\int F \cdot dr$

dx dy dz

$r'(t) = \langle \cos t, 3, -\sin t\rangle$ ✓

$$\int_0^{5\pi} \underbrace{-\sin t}_{P}\underbrace{(\cos t)}_{dx} + \underbrace{3t(\sin t)}_{Q}\underbrace{(3)}_{dy} + \underbrace{\cos t}_{R}\underbrace{(-\sin t)}_{dz}\, dt$$

$9t(\sin t) - 2\sin t\cos t$

$$\int_0^{5\pi} 9t(\sin t)\,dt - 2\int_0^{5\pi} \sin t\cos t\,dt$$

15

$u = \sin t$

$du = \cos t\,dt$

$u = 9t \quad du = 9\,dt$

$dv = \sin t\,dt \quad v = -\cos t$

$$-9t\cos t - \int_0^{5\pi} -9\cos t\,dt \qquad -2\int_0^0 u\,du \qquad = \boxed{45\pi}$$

$\underline{-45\pi}$ $\quad + 9\sin t\Big|_0^{5\pi}$ $\quad \underline{+0}$

$\underline{+0}$

5. [9 points] Is the statement true or false? Give reasons for your answer.

a) For a certain vector field $\boldsymbol{F}$ and oriented path C, we have $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = 2\boldsymbol{i} - 3\boldsymbol{j}$.

False, the result of a vector line integral must be a constant. ✓ +3

b) If C is an oriented curve and $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$, then $\boldsymbol{F} = 0$.

False, this could happen if $r(t)$ is made of all constants. Thus $r'(t)$ would all be zeros. X +2/3

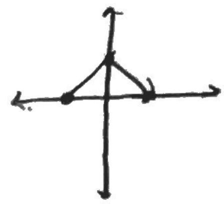
c) $\underbrace{\nabla \times (\nabla f)}_{\text{Curl}(F)} = 0$ for all $f : \mathbb{R}^3 \to \mathbb{R}$.

constant $\Rightarrow f : \mathbb{R}^{\boxed{2}} \to \mathbb{R}$

True, Because f exists this means that the vector field is conservative and thus the curl would be zero.

6. [20 points] Let $\boldsymbol{F}(x,y) = \langle 2xy+1, x^2\rangle$, and let C be the piecewise linear path from $(-1,0)$ to $(0,1)$ to $(1,0)$ (in that order).

a) Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ directly. Do not use potential functions.



$(-1,0) \to (0,1)$

$\langle -1,0\rangle + t\langle 1,1\rangle$

$r(t)\ \langle t-1, t\rangle \quad 0\le t\le 1$

$r'(t) = \langle 1,1\rangle$

$\int_0^1 2t^2-2t+1+(t-1)^2\,dt$

$2t^2-2t+1+t^2-2t+1$

$\int_0^1 3t^2-4t+2\,dt$

$t^3-2t^2+2t\Big|_0^1$

$=1$

$(0,1) \to (1,0)$

$\langle 0,1\rangle + t\langle 1,-1\rangle$

$r(t)\ \langle t, 1-t\rangle \quad 0\le t\le 1$

$r'(t) = \langle 1,-1\rangle$

$\int_0^1 2t-2t^2+1-t^2\,dt$

$\int_0^1 -3t^2+2t+1\,dt$

$-t^3+t^2+t\Big|_0^1$

$=1$

$1+1 = \boxed{2}$

b) Show that $\boldsymbol{F}$ is conservative by finding a potential function for it.

$f = x^2y + x$

3/4

c) Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ using the potential function from part (b).

$f(1,0) - \cancel{f(0,1)} - f(-1,0) =$

$1 - 0 - (-1) = \boxed{2}$

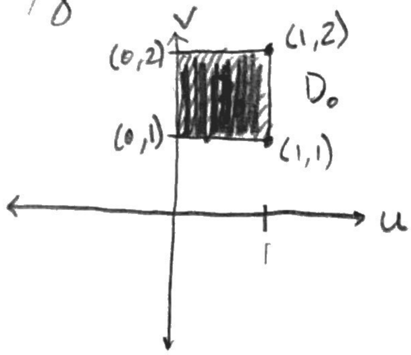
Only use endpoints

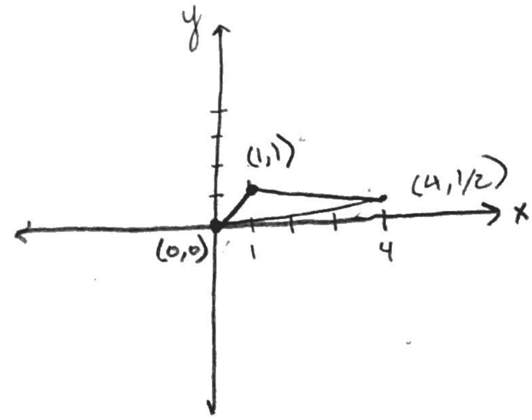
x y

7. [15 points] Let $G(u,v) = \langle uv^2, \frac{u}{v}\rangle$, and let $\mathcal{D}_0$ be the domain in the uv-plane where $0 \le u \le 1$ and $1 \le v \le 2$.

6/8

a) Let $\mathcal{D}$ be the image of $\mathcal{D}_0$ under G. Sketch $\mathcal{D}$, Label its corner points and give equations for its boundary components.





$G(0,1) = (0,0)$

$G(0,2) = (0,0)$

$G(1,1) = (1,1)$

$G(1,2) = (4, \frac{1}{2})$

Along $u=0$

$1 \le v \le 2$

$0 \le x \le 0$

$0 \le y \le 0$

Along $u=1$

$1 \le v \le 2$

$1 \le x \le 4$

$1 \le y \le \frac{1}{2}$

Along $v=1$

$0 \le u \le 1$

$0 \le x \le 1$

$0 \le y \le 1$

Along $v=2$

$0 \le u \le 1$

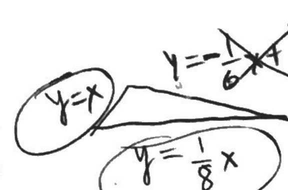
$0 \le x \le 4$

$0 \le y \le \frac{1}{2}$

$\frac{1}{2}$ 3

$\frac{1}{6}$

$\frac{6}{6}$



show work

1/7

b) Let $f(x,y)$ be an arbitrary function. Use the Change of Variables Formula to rewrite $\iint_{\mathcal{D}} f(x,y)\,dA$ as a double integral in u and v. (Your answer should be completely ready to integrate, aside from the unknown function f. In particular, the symbols G, $\mathcal{D}$, and $\mathcal{D}_0$ should appear nowhere in your final answer.)

$$\int_0^1 \int_{\frac{1}{8}x}^{x} f(x,y)\,dy\,dx + \int_1^4 \int_{\frac{1}{8}x}^{-\frac{1}{6}x+\frac{7}{6}} f(x,y)\,dy\,dx$$

$$\int_0^1 \int_1^2 f\left(uv^2, \frac{u}{v}\right) \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| du\,dv$$