The following formulas have consented to their use on this exam. At least one will be relevant.

$$\kappa(t) = \frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3} \qquad \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

$$\kappa(s) = ||\boldsymbol{T}'(s)|| \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

---

**1.** [12 pts] Let $\boldsymbol{r}_1(t) = \langle 2, 1, a\rangle + \langle t, t, 4t\rangle$ and $\boldsymbol{r}_2(s) = \langle 0, -1, 3\rangle + \langle 2s, bs, cs\rangle$ be parametrizations of lines.

$\frac{2}{4}$

a) Find specific values of a, b, and c such that the lines are coincident (ie. they are the same line).

$\langle 1,1,4\rangle \parallel \langle 2,b,c\rangle$ and contain $(2,1,a)$ and $(0,-1,3)$

$\boxed{\begin{array}{l} \cancel{a=8} \\ b=2 \\ c=8 \end{array}}$

$2+t = 2s$
$1+t = -1+2s$
$a+4t = 8s$

$t = 2s-2$
$a + 8s - 8 = 8s$
$a - 8 = 0$
$a = 8$

$3/4$

b) Find specific values of a, b, and c such that the lines are parallel (but not coincident).

$\boxed{\begin{array}{l} a=9 \\ b=2 \\ c=8 \end{array}}$

$a \neq 8$
explain

$\frac{1}{4}$

c) Find specific values of a, b, and c such that the lines intersect (but are not coincident).

$\boxed{\begin{array}{l} a=9 \\ b=6 \\ c=20 \end{array}}$

$2 + t = 0 + 2s$
$1 + t = -1 + bs$
$9 + 4t = 3 + cs$

$3 = 2 + 2s$
$2 = -1 + bs$
$13 = 3 + cs$

$b \neq 2$
$c \neq 8$
if $t=1 \Rightarrow s = \frac{1}{2}$

$2 = -1 + \frac{b}{2}$
$3 = \frac{b}{2}$
$b=6$

$13 = 3 + \frac{c}{2}$
$10 = \frac{c}{2}$
$c = 20$

**2.** [10 pts] Find an equation for the tangent plane to the graph of $f(x,y) = \ln(x^2y + x + y)$ at the point $(x,y) = (-1,2)$.

$$z = f_x(x - x_0) + f_y(y - y_0) + z_0$$

$$z_0 = f(-1,2) = \ln(2 + (-1) + 2) = \ln(3)$$

$$f_x(x,y) = \left.\frac{2xy + 1}{(x^2y + x + y)}\right|_{(-1,2)} = \frac{2(-2) + 1}{2 - 1 + 2} = \frac{-3}{3} = \underline{-1}$$

$$f_y(x,y) = \left.\frac{x^2 + 1}{(x^2y + x + y)}\right|_{(-1,2)} = \frac{1 + 1}{2 - 1 + 2} = \underline{\frac{2}{3}}$$

$$\boxed{z = -1(x + 1) + \frac{2}{3}(y - 2) + \ln(3)}$$

3. [14 pts] Consider the lines $r_1(s) = \langle 1+s, 1+4s, -4-6s \rangle$ and $r_2(t) = \langle t, -3-5t, 2+t \rangle$.

a) Find the point of intersection of $r_1$ and $r_2$.

$1+s = t$

$1+4s = -3-5t$

$-4-6s = 2+t$

$-4-6s = 2+(1+s)$

$-4-6s = 3+s$

$-7 = 7s$

$s=-1 \Rightarrow 1+(-1) = t$

$t=0$

$r_1(-1) = \langle 0, -3, 2 \rangle$

$r_2(0) = \langle 0, -3, 2 \rangle$

Point of intersection: $(0, -3, 2)$

b) Find an equation for the plane that contains both lines.

$v_1 = \langle 1, 4, -6 \rangle$

$v_2 = \langle 1, -5, 1 \rangle$

$v_1 \times v_2 = \langle -26, -7, -9 \rangle$

$-26(0) - 7(-3) - 9(2) = d$

$0 + 21 - 18 = d$

$d = 3$

$\boxed{-26x - 7y - 9z = 3}$

plane:

6/10

**4.** [10 pts] Evaluate the limit, or show that it does not exist.

$$\lim_{(x,y)\to(-1,1)} \frac{x^2+2x+1}{(x+1)^2+(y-1)^2}$$

along $y=-x$: $\lim_{(x,y)\to(-1,1)} \frac{x^2+2x+1}{(x+1)^2+(-x-1)^2} \Rightarrow \lim_{(x,y)\to(-1,1)} \frac{x^2+2x+1}{x^2+\cancel{2x}+1+x^2-\cancel{2x}+1} = \frac{(x+1)^2}{2x^2+2}$

$= \lim_{(x,y)\to(-1,1)} \frac{x+1}{2} = \underline{0}$

along $y=1$: $\lim_{(x,y)\to(-1,1)} \frac{x^2+2x+1}{(x+1)^2} = \lim_{(x,y)\to(-1,1)} \frac{\cancel{(x+1)^2}}{\cancel{(x+1)^2}} = \underline{1}$

limit $\underline{\text{DNE}}$

Along $y=1$

$\lim \frac{(x+1)^2}{(x+1)^2+0} = 1$

Along $x=-1$

$\lim \frac{0}{0+(y-1)^2}$

3/6

**5.** [6 pts] The expression $e^{xy}\cos x = \arcsin(xyz)$ defines $z$ implicitly as a function of $x$ and $y$. Determine $\frac{\partial z}{\partial x}$.

$\frac{d}{dx}\arcsin(xyz) = \cancel{\frac{1}{1+x^2}}$ ?

$e^{xy}\cos x - \arcsin(xyz) = 0$

$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$

define F

$F_x = y e^{xy}\cos x - \sin(x) e^{xy} - \frac{d}{dx}\arcsin(xyz)$

$F_z = 0 - \frac{d}{dz}\arcsin(xyz)$

$$\boxed{-\frac{y e^{xy}\cos(x) - \frac{yz}{1+x^2}}{\frac{xy}{1+z^2}}}$$

6. [20 pts] Consider the helix parametrized by $\boldsymbol{r}(t) = \left\langle 4\sin\left(\frac{t}{3}\right), t, 4\cos\left(\frac{t}{3}\right)\right\rangle$.

a) Which axis does the helix wrap around?

y axis ✓ +2

b) Find a parametrization for the tangent line to the curve at $t = 2\pi$.

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$r'(t) = \left\langle \frac{4}{3}\cos\left(\frac{t}{3}\right), 1, -\frac{4}{3}\sin\left(\frac{t}{3}\right) \right\rangle \Big|_{t=2\pi}$$

$2\pi$ ok

$$= \left\langle -\frac{4}{3}\left(\frac{1}{2}\right), 1, -\frac{\cancel{4}}{\cancel{3}}\left(\frac{\sqrt{3}}{2}\right) \right\rangle \quad +3$$

$$= \boxed{\left\langle -\frac{4}{6}, \cancel{1}, -\frac{2}{\sqrt{3}} \right\rangle}$$

not a line.

$$\vec{u}(t) = \vec{r}(2\pi) + \vec{r}'(2\pi)(t - 2\pi)$$

c) Find an arc-length parametrization for the given curve.

$$s = \int_0^t \sqrt{\left[\frac{4}{3}\cos\left(\frac{t}{3}\right)\right]^2 + 1 + \left[-\frac{4}{3}\sin\left(\frac{t}{3}\right)\right]^2}\, dt$$

$$s = \int_0^t \sqrt{\frac{16}{9}\left(\cos^2\left(\frac{t}{3}\right) + \sin^2\left(\frac{t}{3}\right)\right) + 1}\ \cancel{dt}\ du$$

$$s = \int_0^t \sqrt{\frac{16}{9} + 1}\ \cancel{du}\ dt\ du$$

✓

$$s = \int_0^t \frac{5}{3}\ \cancel{dt}$$

$$s = \frac{5}{3}t \Big|_0^t$$

$$= \frac{5}{3}t - \frac{5}{3}(0)$$

$$s = \frac{5}{3}t$$

$$t = \frac{3}{5}s$$

$$\boxed{r(s) = \left\langle 4\sin\left(\frac{s}{5}\right), \frac{3}{5}s, 4\cos\left(\frac{s}{5}\right) \right\rangle}$$

9/10

7. [6 pts] Consider the curve $C$ parametrized by $\boldsymbol{r}(t) = \langle e^{-t}\sin t, e^{-t}, e^{-t}\cos t\rangle$ for $t \geq 0$. Let $S$ denote the surface $\{(x, y, z) : x^2 + z^2 = y^2\}$.

a) Show that the curve $C$ lies on the surface $S$.

At $t=0$, $r(0) = \langle 0, 1, 1 \rangle$

$x^2 + z^2 = y^2$    x y z

$(0)^2 + (1)^2 = (1)^2$

this does not prove it.

+1

b) What type of surface is $S$?

double cone.

paraboloid going out towards positive direction of y

8. [6 pts] Determine whether the statement is True (T) or False (F). No justification is needed.

a) The unit tangent vector $\boldsymbol{T}(t)$ is always perpendicular to the vector $\boldsymbol{T}'(t)$. T/F

b) The vector projection of a vector $\boldsymbol{v}$ onto a non-zero vector $\boldsymbol{w}$ is always non-zero. T/F

+4/6

c) The linearization of $f(x, y) = 2 - 3x + 4y$ at any point $(a, b)$ is $L(x, y) = 2 - 3x + 4y$. T/F

$f_x = -3$

$f_y = 4$

9. [8 pts] Find a parametrization for the intersection of the surfaces $x^2 - z^2 = y - 1$ and $x^2 + z^2 = 16$.

$x^2 = 16 - z^2$

$16 - z^2 - z^2 = y - 1$

$17 - 2z^2 = y$

$2z^2 = 17 - y$

$z^2 = \frac{17-y}{2}$

$z = \sqrt{\frac{17-y}{2}}$

$z^2 = 16 - x^2$

$x^2 - 16 + x^2 = y - 1$

$2x^2 = y + 15$

$x^2 = \frac{y+15}{2}$

$x = \sqrt{\frac{y+15}{2}}$

let $y = t$

$$\boxed{r(t) = \left\langle \pm\sqrt{\frac{t+15}{2}},\ t,\ \pm\sqrt{\frac{17-t}{2}} \right\rangle}$$

$? \leq t \leq ?$

+4/4

$f(x,y,z) = x^2yz$

$x = t$
$y = 2t$ @ $t=1$
$z = 3t^2$

10. [14 pts] Let $f(x, y, z) = x^2yz$, and let $\boldsymbol{r}(t) = \langle \overset{x}{t}, \overset{y}{2t}, \overset{z}{3t^2} \rangle$.

a) Use the Chain Rule for Paths to evaluate $\frac{d}{dt} f(\boldsymbol{r}(t))$ at $t = 1$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$2xyz \cdot 1 + x^2z \cdot 2 + x^2y \cdot 6t$$

at $t=1$, $x = 1$, $y = 2$, $z = 3$

$$12 + 6 + 12$$

$$= \boxed{30}$$

b) Find the $\overset{\nabla f \cdot \vec{u}}{\text{directional}}$ derivative of $f(x, y, z)$ at $\boldsymbol{r}(1)$ in the direction of $\boldsymbol{r}'(1)$.

$r(1) = \langle 1, 2, 3 \rangle$ (point)

$r'(1) = \langle 1, 2, 6t \rangle = \langle 1, 2, 6 \rangle$ (direction $\Rightarrow \frac{1}{\sqrt{41}}\langle 1, 2, 6 \rangle$

$\|r'(1)\| = \sqrt{1+4+36} = \sqrt{41}$

$\nabla f = \langle 2xyz, x^2z, x^2y \rangle \big|_{(1,2,3)} = \langle 12, 3, 2 \rangle$

$\langle 12, 3, 2 \rangle \cdot \frac{1}{\sqrt{41}} \langle 1, 2, 6 \rangle = \frac{1}{\sqrt{41}} (12 + 6 + 12) = \boxed{\frac{30}{\sqrt{41}}}$

c) Did you get the same answers for parts (a) and (b)? Why or why not?

They are not the same answers because the chain rule does not require the direction vector to have magnitude 1.