Filters

- A circuit that is designed to pass signals with desired frequencies and reject or attenuate others -

Basic Concepts

Frequency Response [1]-[3]

Definition: The variation in the behavior of a circuit with change in signal frequency.

In this report, we mainly discuss the **voltage variation** of a circuit with respect to the **input frequency**.

Voltage gain: (No unit)

$$A_V = \frac{v_{out}}{v_{in}}$$

Phase shift: (Units: rad or °)

$$\varphi = output\ phase - input\ phase$$

Input frequency: (Units: rad/s or Hz)

$$\omega = \frac{d\theta}{dt} = 2\pi f$$

Transfer Function

Definition: The frequency-dependent ratio of a phasor output $\boldsymbol{Y}(\omega)$ to a phasor input $\boldsymbol{X}(\omega)$.

$$\boldsymbol{H}(\omega) = \frac{\boldsymbol{Y}(\omega)}{\boldsymbol{X}(\omega)} = |\boldsymbol{H}(\omega)| \angle\varphi$$

Type:

① **Voltage gain** (No Unit)	$\frac{\mathbf{V}_{out}(\omega)}{\mathbf{V}_{in}(\omega)}$
② **Current gain** (No Unit)	$\frac{\mathbf{I}_{out}(\omega)}{\mathbf{I}_{in}(\omega)}$
③ **Transfer impedance** (Units: Ω)	$\frac{\mathbf{V}_{out}(\omega)}{\mathbf{I}_{in}(\omega)}$
④ **Transfer admittance** (Units: S)	$\frac{\mathbf{I}_{out}(\omega)}{\mathbf{V}_{in}(\omega)}$

Bode Plot [4]

Definition: A graph of the frequency response of a system.

Vertical axis:

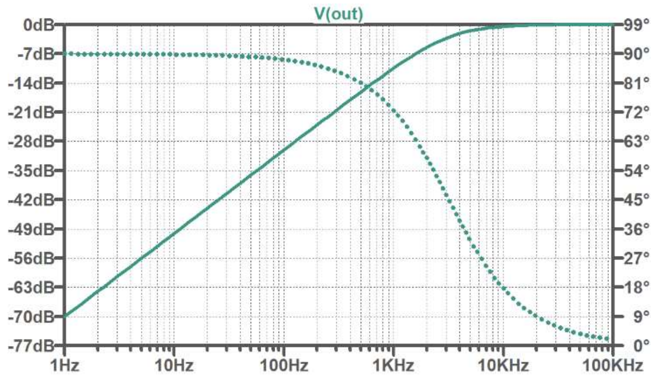
① **Magnitude** (dB)	② **Phase shift** (°)

Horizontal axis: Frequency (logarithmic scale)

Decibel: (Units: dB)

$$H_{dB} = 10\log\left(\frac{P}{P_0}\right) = 10\log\left(\frac{V}{V_0}\right)^2 = 20\log(|\boldsymbol{H}|)$$

Example (— magnitude ··· phase)



Half-Power Point

Definition: The point where the output power is the half of its peak value. It is also known as **half-power bandwidth**.

*Half-power in dB: $10\log(0.5) \approx -3.01\ \text{dB}$

*$10\log(0.5) = 20\log(H) \Rightarrow H = \frac{1}{\sqrt{2}} \approx 0.707$

Bandwidth

Definition: The difference between the two half-power frequencies of a resonant circuit.

$$B = \omega_2 - \omega_1$$

Resonant frequency: $\omega_0 = \sqrt{\omega_1\omega_2}$

Quality factor: $Q = \frac{\omega_0}{B} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$

Classification & Type [5]

Passive

A filter is a passive filter if it consists of **only passive elements** such as **R**, **L**, and **C**.

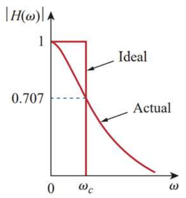
Active

A filter is an active filter if it **consists of active elements** such as **transistors** and **OPAMPs**.

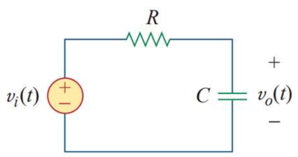
Low Pass

Designed to pass only frequencies up to the cutoff frequency ω_c.

Frequency Response



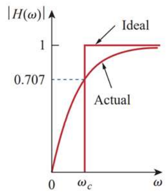
Circuit



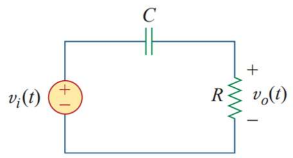
High Pass

Designed to pass only frequencies above the cutoff frequency ω_c.

Frequency Response



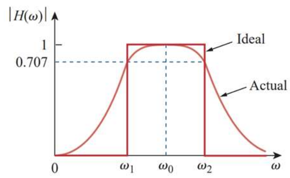
Circuit



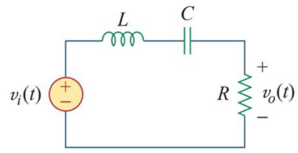
Band Pass

Designed to pass all frequencies between half-power points.

Frequency Response



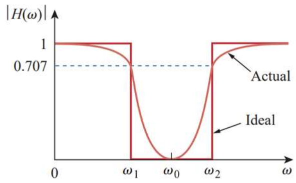
Circuit



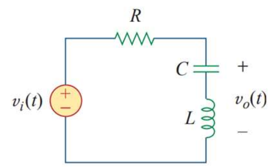
Band Stop

Designed to stop all frequencies between half-power points.

Frequency Response



Circuit

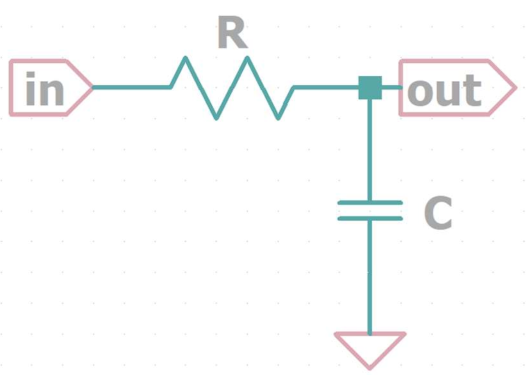


*Images in the above part are from reference [1].

Circuit Analysis

Passive 1st – order low pass filter

Circuit



Cutoff frequency: $\omega_C = \frac{1}{RC}$

*The proof of the cutoff frequency is in page 4.

Analysis

By voltage divider rule,

$$\frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

Transfer function:

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

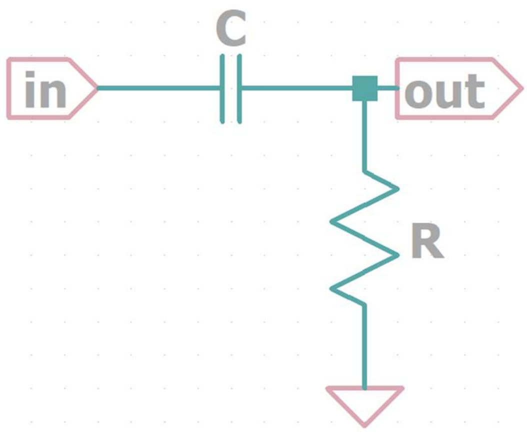
$$= \frac{1}{\sqrt{1^2 + (\omega RC)^2}} \angle -\tan^{-1}(\omega RC)$$

Voltage gain: $A_V = |H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase difference: $\varphi = -\tan^{-1}(\omega RC)$

Passive 1st – order high pass filter

Circuit



Cutoff frequency: $\omega_C = \frac{1}{RC}$

*The proof of the cutoff frequency is in page 4.

Analysis

By voltage divider rule,

$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{1}{1 + \frac{1}{j\omega RC}}$$

Transfer function:

$$\boldsymbol{H}(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{1}{j\omega RC}} = \frac{j\omega RC}{1 + j\omega RC}$$

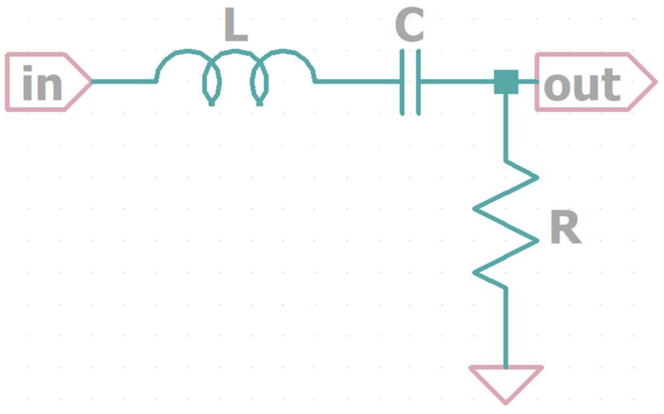
$$= \frac{1}{\sqrt{1^2 + \left(\frac{1}{\omega RC}\right)^2}} \angle \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Voltage Gain: $A_V = |\boldsymbol{H}(\omega)| = \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}$

Phase difference: $\varphi = \tan^{-1}\left(\frac{1}{\omega RC}\right)$

Passive 2nd – order band pass filter

Circuit



*The proofs of following properties are in page 5.

Half-power frequencies:

$$\omega_1, \omega_2 = \mp \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Resonant frequency: $\boldsymbol{\omega_0} = \sqrt{\frac{\boldsymbol{1}}{\boldsymbol{LC}}}$

Bandwidth: $\mathrm{B} = \frac{R}{L}$

Analysis

By voltage divider rule,

$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_L + Z_C} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

Transfer function:

$$\boldsymbol{H}(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j \times \frac{\omega L - \frac{1}{\omega C}}{R}}$$

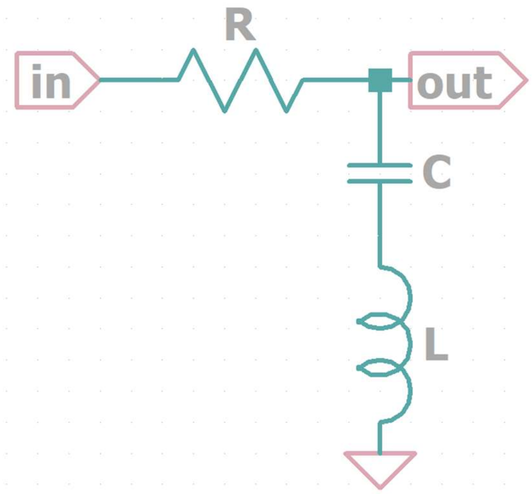
$$= \frac{1}{\sqrt{1^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2}} \angle -\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

Voltage Gain: $A_V = |\boldsymbol{H}(\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2}}$

Phase difference: $\varphi = -\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$

Passive 2nd – order band stop filter

Circuit



*The proofs of following properties are in page 5.

Half-power frequencies:

$$\omega_1, \omega_2 = \mp \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Resonant frequency: $\boldsymbol{\omega_0} = \sqrt{\frac{\mathbf{1}}{\boldsymbol{LC}}}$

Bandwidth: $\mathrm{B} = \frac{R}{L}$

Analysis

By voltage divider rule,

$$\frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{\boldsymbol{Z_L} + \boldsymbol{Z_C}}{\boldsymbol{Z_R} + \boldsymbol{Z_L} + \boldsymbol{Z_C}} = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

Transfer function:

$$\boldsymbol{H}(\omega) = \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{1}{1 - j \times \frac{R}{\omega L - \frac{1}{\omega C}}}$$

$$= \frac{1}{\sqrt{1^2 + \left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)^2}} \angle \tan^{-1}\left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)$$

Voltage Gain: $A_V = |\boldsymbol{H}(\omega)| = \frac{1}{\sqrt{1+\left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)^2}}$

Phase difference: $\varphi = \tan^{-1}\left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)$

Calculate the cutoff frequency:

Low pass filter	High pass filter
$\lvert\boldsymbol{H}(\omega_C)\rvert = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + (\omega_C RC)^2}}$	$\lvert\boldsymbol{H}(\omega_C)\rvert = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + \left(\frac{1}{\omega_C RC}\right)^2}}$
$\Rightarrow 1 + (\omega_C RC)^2 = 2$	$\Rightarrow 1 + \left(\frac{1}{\omega_C RC}\right)^2 = 2$
$\Rightarrow (\omega_C RC)^2 = 1$	$\Rightarrow \left(\frac{1}{\omega_C RC}\right)^2 = 1$

$$\Rightarrow \omega_C RC = \pm 1, \omega_C > 0$$

$$\Rightarrow \omega_C = \frac{1}{RC}$$

Calculate half-power frequencies, resonant frequency and bandwidth:

Band pass filter	Band stop filter
$$\lvert \boldsymbol{H}(\omega)\rvert = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2}}$$ $$\Rightarrow \frac{\omega L - \frac{1}{\omega C}}{R} = \pm 1$$	$$\lvert \boldsymbol{H}(\omega_C)\rvert = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1^2 + \left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)^2}}$$ $$\Rightarrow \frac{R}{\omega L - \frac{1}{\omega C}} = \pm 1$$

$$\Rightarrow \omega L - \frac{1}{\omega C} = \pm R$$

$$\Rightarrow \mathrm{LC}\omega^2 \pm RC\omega - 1 = 0$$

By quadratic formula,

$$\omega = \frac{\mp RC \pm \sqrt{(RC)^2 + 4\mathrm{LC}}}{2LC} = \mp \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Half power frequencies:

Let $\omega_1 < \omega_2$ then find the positive solution,

$$\begin{cases} \omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_2 = \quad \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{cases}$$

Resonant frequency:

By formula of difference of two squares,

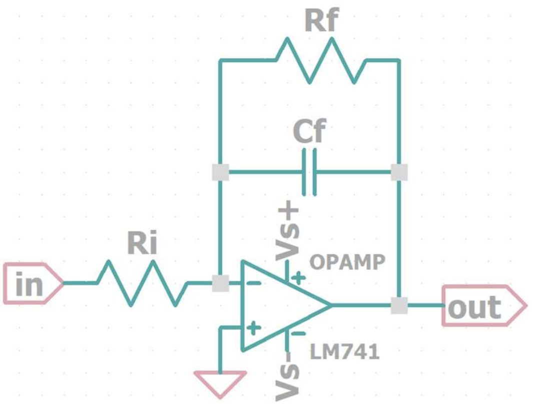
$$\omega_0 := \sqrt{\omega_1\omega_2} = \sqrt{\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}^{\,2} - \left(\frac{R}{2L}\right)^2} = \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \sqrt{\frac{1}{LC}}$$

Bandwidth:

$$B := \omega_2 - \omega_1 = \frac{R}{L}$$

Active 1st – order low pass filter

Circuit



Cutoff frequency: $\omega_C = \frac{1}{R_f C_f}\sqrt{\frac{2{R_i}^2}{{R_f}^2} - 1}$

Proof:

$|\boldsymbol{H}(\omega_C)| = \frac{1}{\sqrt{2}} = \frac{R_f}{R_i}\frac{1}{\sqrt{1+(\omega_C R_f C_f)^2}}$

$\Rightarrow 1 + (\omega_C R_f C_f)^2 = \frac{2{R_i}^2}{{R_f}^2}\ ,\ \omega_C \geq 0$

$\Rightarrow \omega_C = \frac{1}{R_f C_f}\sqrt{\frac{2{R_i}^2}{{R_f}^2} - 1}$

Analysis

**The two inputs of OPAMP are virtual shorted.*

Apply KCL at the inverting input of the OPAMP,

$$\boldsymbol{I} = \frac{\boldsymbol{V}_{in}}{\boldsymbol{Z}_{in}} = \frac{\boldsymbol{V}_{out}}{\boldsymbol{Z}_f}$$

$$\Rightarrow \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{\boldsymbol{Z}_f}{\boldsymbol{Z}_{in}} = \frac{\boldsymbol{Z}_{Rf}||\boldsymbol{Z}_{Cf}}{\boldsymbol{Z}_{Ri}} = \frac{R_f||\frac{1}{j\omega C_f}}{R_i}$$

$$\Rightarrow \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{1}{R_i} \times \frac{1}{\frac{1}{R_f}+j\omega C_f} = \frac{R_f}{R_i}\left(\frac{1}{1+j\omega R_f C_f}\right)$$

Transfer function:

$$\boldsymbol{H}(\omega) = \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{R_f}{R_i}\left(\frac{1}{1+j\omega R_f C_f}\right)$$

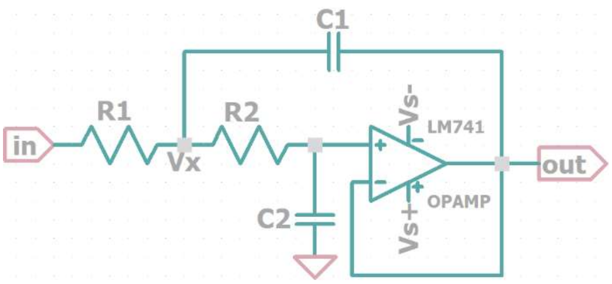
$$= \frac{R_f}{R_i}\frac{1}{\sqrt{1^2 + (\omega R_f C_f)^2}}\ \angle -\tan^{-1}(\omega R_f C_f)$$

Voltage gain: $A_V = |\boldsymbol{H}(\omega)| = \frac{R_f}{R_i}\frac{1}{\sqrt{1+(\omega R_f C_f)^2}}$

Phase difference: $\varphi = -\tan^{-1}(\omega R_f C_f)$

Active 2nd – order low pass filter

Circuit



Resonant frequency: $\omega_0 = \sqrt{\frac{1}{R_1 R_2 C_1 C_2}}$

By the definition of the transfer function of Sallen-Key filter,

$$\boldsymbol{H}(\omega) = \frac{{\omega_0}^2}{s^2 + \frac{{\omega_0}^2}{Q}s + {\omega_0}^2}$$

$$= \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right)s + \frac{1}{R_1 R_2 C_1 C_2}}$$

Analysis

**The two inputs of OPAMP are virtual shorted.*

Let $s = j\omega \Rightarrow s^2 = -\omega^2$.

Apply KCL on node X,

$$\frac{\boldsymbol{V}_{in}-\boldsymbol{V}_x}{\boldsymbol{Z}_{R1}} + \frac{\boldsymbol{V}_{out}-\boldsymbol{V}_x}{\boldsymbol{Z}_{C1}} + \frac{\boldsymbol{V}_{out}-\boldsymbol{V}_x}{\boldsymbol{Z}_{R2}} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{V}_{in} - \boldsymbol{V}_x = R_1(\boldsymbol{V}_x - \boldsymbol{V}_{out})\left(sC_1 + \frac{1}{R_2}\right) \qquad (1)$$

By voltage divider rule,

$$\boldsymbol{V}_x = \frac{\boldsymbol{Z}_{R2}+\boldsymbol{Z}_{C2}}{\boldsymbol{Z}_{C2}}\boldsymbol{V}_{out} = \left(\boldsymbol{1} + \frac{\boldsymbol{Z}_{R2}}{\boldsymbol{Z}_{C2}}\right)\boldsymbol{V}_{out}$$

$$\Rightarrow \boldsymbol{V}_x = (1 + sR_2C_2)\boldsymbol{V}_{out}$$

$$\Rightarrow \boldsymbol{V}_x - \boldsymbol{V}_{out} = sR_2C_2\boldsymbol{V}_{out} \qquad (2)$$

Substitute $\mathbf{V_x}$ and (2) into (1),

$$\boldsymbol{V}_{in} - (1 + sR_2C_2)\boldsymbol{V}_{out} = \left(sC_1 + \frac{1}{R_2}\right)sR_1R_2C_2\boldsymbol{V}_{out}$$

$$\boldsymbol{V}_{in} = (1 + sR_2C_2 + sR_1C_2 + s^2R_1R_2C_1C_2)\boldsymbol{V}_{out}$$

$$\Rightarrow \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{1}{1 + sC_2(R_1 + R_2) + s^2 R_1 R_2 C_1 C_2}$$

Transfer function:

$$\boldsymbol{H}(\omega) = \frac{1}{(1 - \omega^2 R_1 R_2 C_1 C_2) + j\omega C_2(R_1 + R_2)}$$

Voltage gain:

$$A_V = \frac{1}{\sqrt{(1 - \omega^2 R_1 R_2 C_1 C_2)^2 + (\omega C_2(R_1 + R_2))^2}}$$

Phase difference:

$$\varphi = -\tan^{-1}\left(\frac{\omega C_2(R_1 + R_2)}{1 - \omega^2 R_1 R_2 C_1 C_2}\right)$$

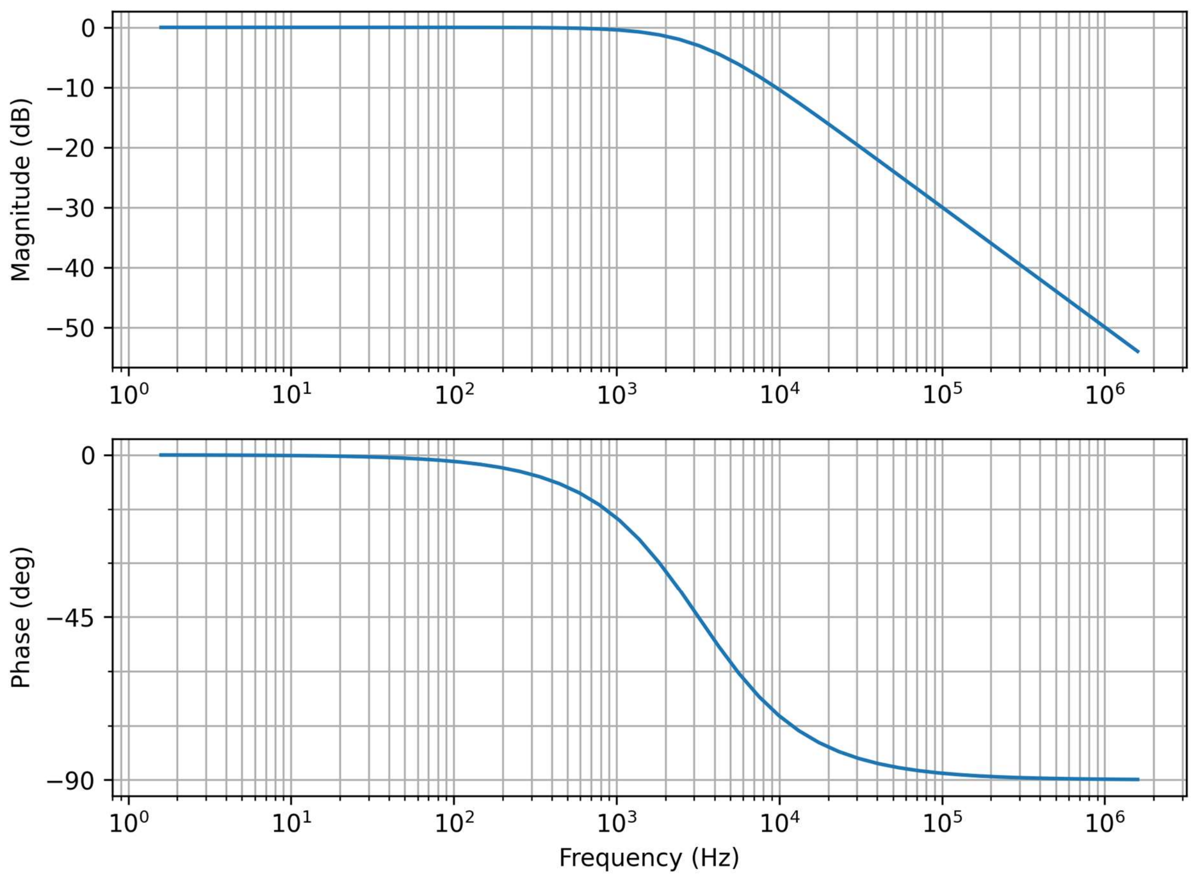
How to calculate the *Measured* phase difference and voltage gain?

Measured Voltage Gain	Measured Phase Difference
$A_{v,Measured} = \frac{V_{out,Measured}}{V_{in,Measured}}$	$\varphi_{Measured} = 2\pi f \Delta t \; rad = 360 f \Delta t \, ^\circ$

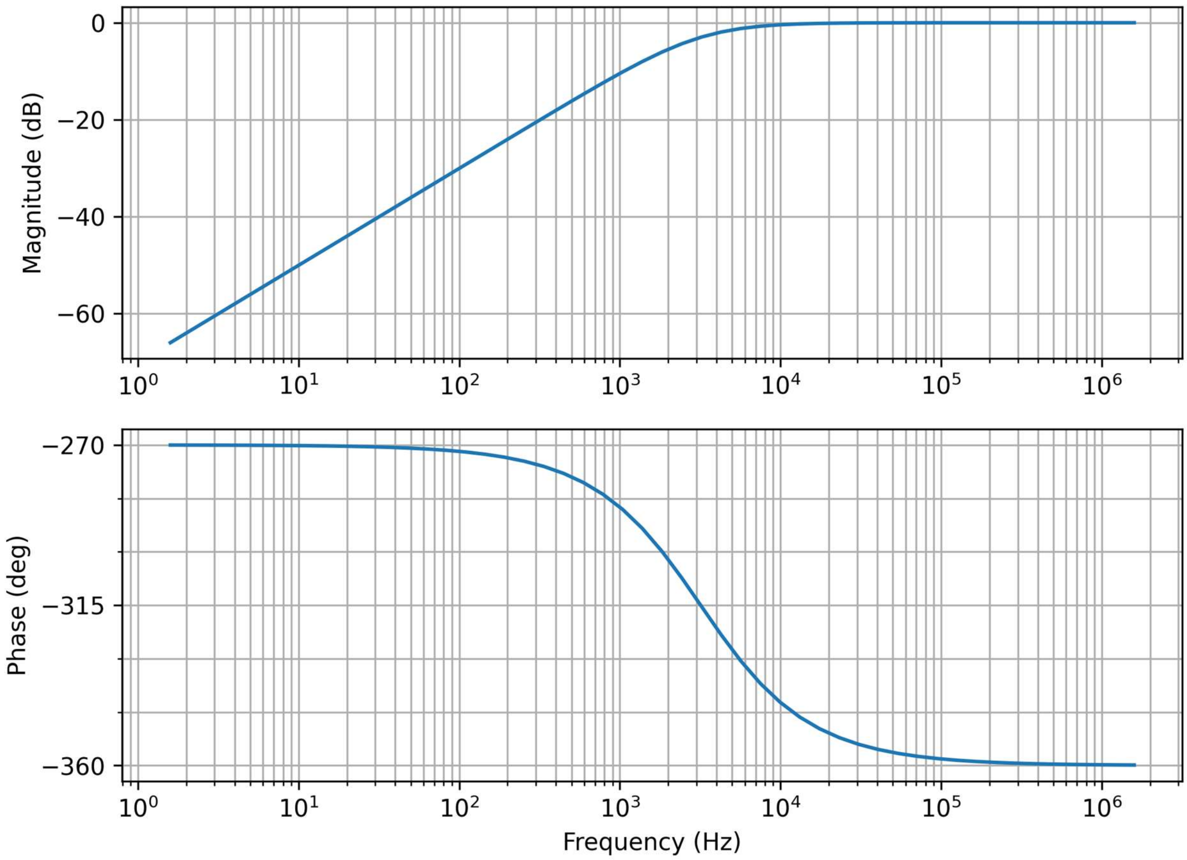
Appendix – Bode plot example of filters in this report

Python code: https://github.com/RUI030/MELab/blob/main/ME_Lab_2_1.ipynb

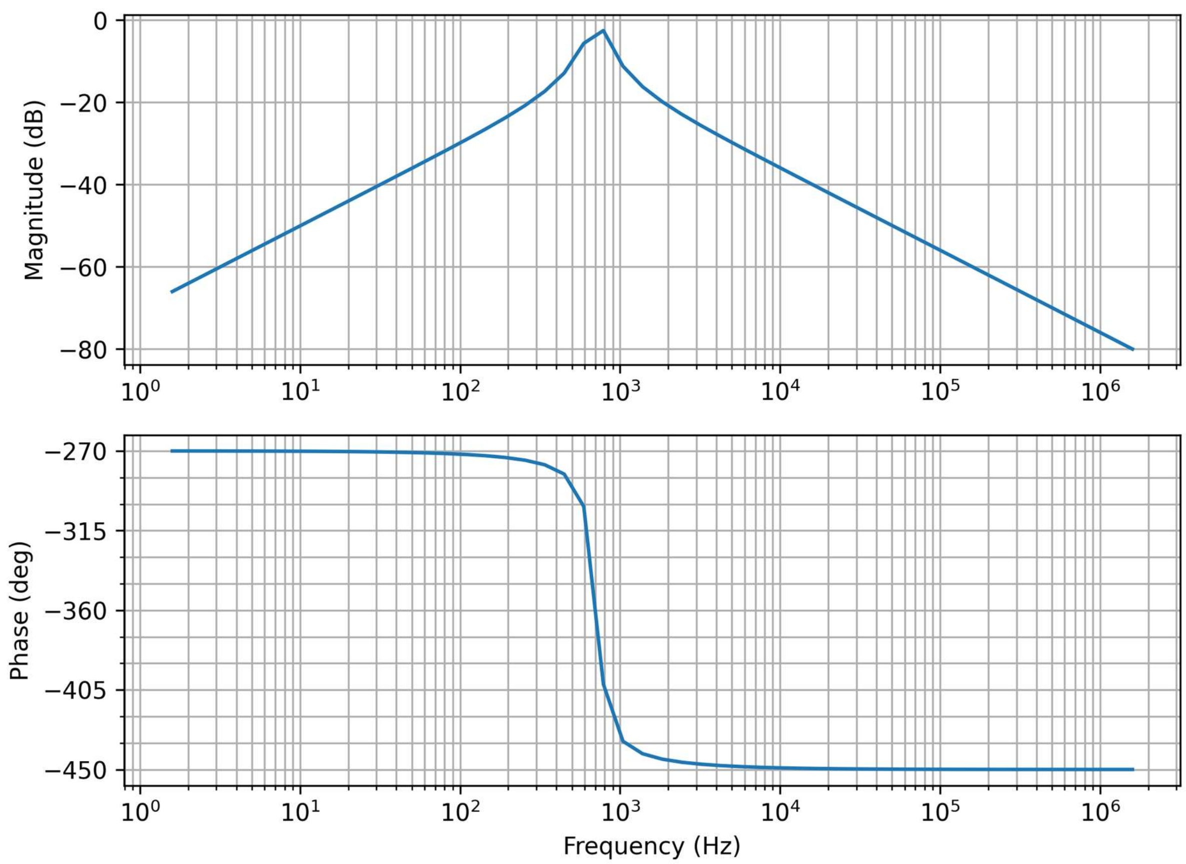
Low Pass



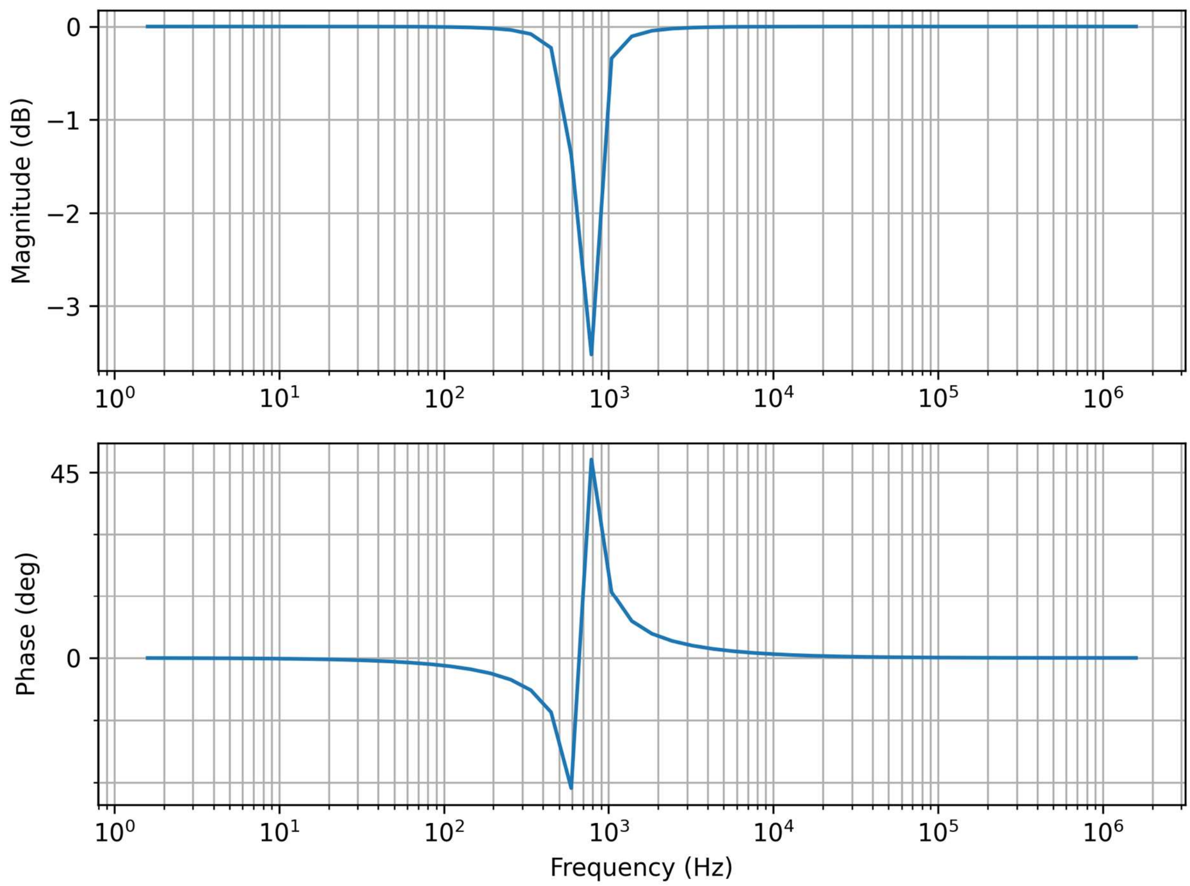
High Pass



Band Pass



Band Stop



Active 1st-Order Low Pass Filter

Similar to 1stpassive low pass filter.

Active 2nd-Order Low Pass Filter (Sallen-Key Filter)