Applied Statistical Analysis I

Multiple linear regression

Trajche Panov, PhD
panovt@tcd.ie

Department of Political Science
Trinity College Dublin

November 5, 2024

Today's Agenda

(1) Lecture recap
(2) Tutorial exercises: What is the relationship between education and Euroscepticism?

Multiple linear regression

Why do we need multiple linear regression? And what is a multiple linear regression model?

Multiple linear regression

Why do we need multiple linear regression?

Working Hours $\xrightarrow{\quad\quad + \quad\quad}$ Political Knowledge

```
## Call:
## lm(formula = polknow ~ work_hours, data = samp)

## Residuals:
##     Min      1Q  Median      3Q     Max
## -7.686  -1.760  -0.061   1.683  10.385

## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 14.59166    1.09142  13.369   <2e-16 ***
## work_hours   0.06791    0.02640   2.572   0.0103 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

## Residual standard error: 2.565 on 998 degrees of freedom
## Multiple R-squared:  0.006585,  Adjusted R-squared:  0.00559
## F-statistic: 6.615 on 1 and 998 DF,  p-value: 0.01025
```

How convincing is this finding?

Multiple linear regression

Why do we need multiple linear regression?

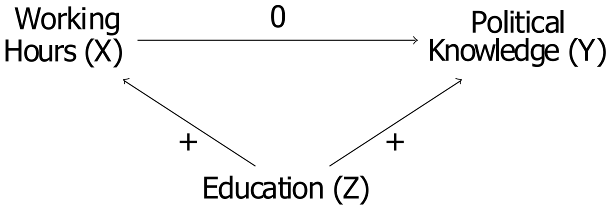


Figure: Education as confounder—Controlling for education is relevant, because it might drive both working hours and political knowledge. Education is causally prior to working hours.

→ Avoid omitted variable bias. Include relevant control variables (Z) which are correlated with both X and Y, and causally prior to X.

Multiple linear regression

Why do we need multiple linear regression?

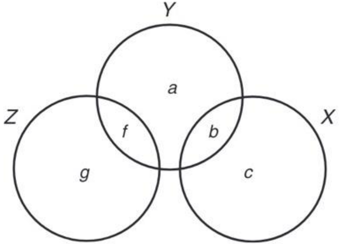


Figure 9.2. Venn diagram in which X and Z are correlated with Y, but not with each other.

"In that case – which, we have noted, is unlikely in applied research – we can safely omit consideration of Z when considering the effects of X on Y. In that figure, the relationship between X and Y – the area b – is unaffected by the presence (or absence) of Z in the model" (Kellstedt and Whitten 2018, 213).

Multiple linear regression

Why do we need multiple linear regression?

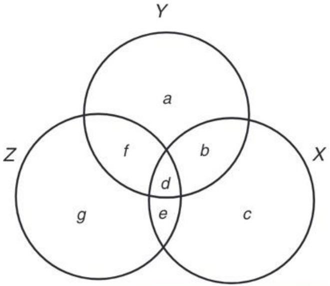


Figure 9.1. Venn diagram in which X, Y, and Z are correlated.

"If, hypothetically, we erased the circle for Z from the figure, we would (incorrectly) attribute all of the area b + d to X, when in fact the d portion of the variation in Y is shared by both X and Z. This is why, when Z is related to both X and Y, if we fail to control for Z, we will end up with a biased estimate of X's effect on Y" (Kellstedt and Whitten 2018, 212).

Multiple linear regression

What is a multiple linear regression model?

$$Y_i = \alpha + \beta_1 X_{i1} + \beta_2 X_{i2} + ... + \beta_k X_{ik} + \epsilon_i$$

- α (intercept): expected value of Y when $X_1 = 0, ..., X_k = 0$.
- β_1 (coefficient): expected change in Y when X_1 increases by one unit, while controlling for the remaining independent variables in the model.
- ...
- β_k (coefficient): expected change in Y when X_k increases by one unit, while controlling for the remaining independent variables in the model.

Multiple linear regression

What is a multiple linear regression model?

```
## Call:
## lm(formula = polknow ~ work_hours + edu, data = samp)

## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.7835 -1.6733  0.0035  1.5941 10.6778

## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 4.854461   1.368601   3.547 0.000408 ***
## work_hours  0.006205   0.025623   0.242 0.808714
## edu         0.767650   0.070797  10.843  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

## Residual standard error: 2.427 on 997 degrees of freedom
## Multiple R-squared:  0.1114,  Adjusted R-squared:  0.1096
## F-statistic: 62.48 on 2 and 997 DF,  p-value: < 2.2e-16
```

The effect of working hours *disappears*.

→ Controlling for working hours, with every additional year of education, the political knowledge increases by 0.76765 scale points.

T-TEST FOR INDIVIDUAL COEFFICIENTS

What is the t-test for individual coefficients?

T-TEST FOR INDIVIDUAL COEFFICIENTS

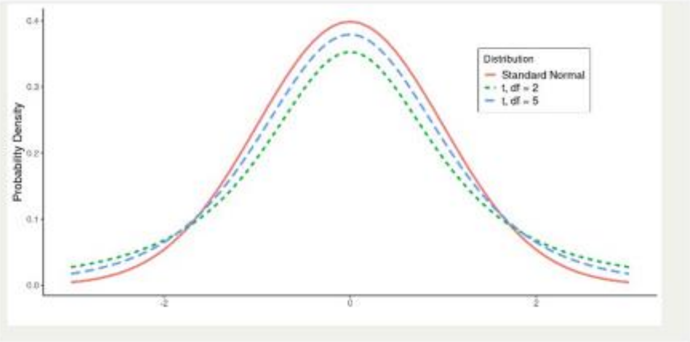
What is the t-test for individual coefficients?

- Null and alternative hypotheses:
  - there is no association between X and Y, $\beta = 0$ (H_0)
  - there is an association between X and Y, $\beta \neq 0$ (H_1)
- Test statistic: "measures the number of standard errors between the estimate and the H_0 value" (Agresti and Finlay 2009, 192).

$$t = \frac{\text{Estimate of parameter} - \text{Null hypothesis value of parameter}}{\text{Standard error of estimate}}$$

$$t = \frac{\hat{\beta} - \beta_{H_0}}{\hat{\sigma}_{\hat{\beta}}} = \frac{\hat{\beta}}{\hat{\sigma}_{\hat{\beta}}}, \quad H_0 \text{ assumes } \beta = 0$$

T-TEST FOR INDIVIDUAL COEFFICIENTS



<u>What is the conclusion?</u> P-value < 0.05, We can reject H_0 with an error probability (p-value) of essentially 0%. → There is an association between house selling price and size

F-TEST

Table 3.5 Regression Output for Supervisor Performance Data

Variable	Coefficient	s.e.	t-Test	p-value
Constant	10.787	11.5890	0.93	0.3616
X_1	0.613	0.1610	3.81	0.0009
X_2	−0.073	0.1357	−0.54	0.5956
X_3	0.320	0.1685	1.90	0.0699
X_4	0.081	0.2215	0.37	0.7155
X_5	0.038	0.1470	0.26	0.7963
X_6	−0.217	0.1782	−1.22	0.2356
$n = 30$	$R^2 = 0.73$	$R_a^2 = 0.66$	$\hat{\sigma} = 7.068$	df = 23

Table 3.2 Description of Variables in Supervisor Performance Data

Variable	Description
Y	Overall rating of job being done by supervisor
X_1	Handles employee complaints
X_2	Does not allow special privileges
X_3	Opportunity to learn new things
X_4	Raises based on performance
X_5	Too critical of poor performance
X_6	Rate of advancing to better jobs

(Chatterjee and Hadi 2015, 59)

F-test

General set-up: Test whether reduced model (RM) is adequate (H_0) or full model (FM) is adequate (H_1).

The reduced model is nested within the full model → compare "the goodness of fit that is obtained when using the full model, to the goodness of fit that results using the reduced model".

$$F = \frac{[SSE(RM) - SSE(FM)]/(p+1-k)}{SSE(FM)/(n-p-1)}$$

(Chatterjee and Hadi 2015, 71–72)

F-TEST

$$F = \frac{[SSE(RM) - SSE(FM)]/(p+1-k)}{SSE(FM)/(n-p-1)}$$

* Sum of squared errors (SSE), denotes *lack of fit* → $SSE(RM) - SSE(FM)$ "represents the increase in the residual sum of squares due to fitting the reduced model".
* We use the ratio, weighted by "respective degrees of freedom to compensate for the different number of parameters involved in the two models".
* p=number of IVs full model, n=number of observations, k=number of paraneters reduced model

(Chatterjee and Hadi 2015, 71–72)

F-TEST

Two versions of the F-test

1. "All the regression coefficients are zero".
2. "Some of the regression coefficients are zero".

(Chatterjee and Hadi 2015, 71)

F-TEST FOR ALL COEFFICIENTS

What is the F-test for all coefficients?

F-test for all coefficients

"All the regression coefficients are zero."

* Reduced model (RM): $Y = \beta_0 + \epsilon$
  all slopes are equal to zero, $\beta_k = 0$ (H_0) → the null model performs better
* Full model (FM): $Y = \beta_0 + \beta_1 X_1 + ... + \beta_p X_p \epsilon$
  at least one slope is different from zero, $\beta_p \neq 0$ (H_1) → the full model performs better

$$F = \frac{[SSE(RM) - SSE(FM)]/(p+1-k)}{SSE(FM)/(n-p-1)} = \frac{[SST - SSE]/p}{SSE/(n-p-1)} = \frac{SSR/p}{SSE/(n-p-1)}$$

"Because the least squares estimate of β_0 in the reduced model is $\bar{y}$, the residual sum of squares from the reduced model is SSE(RM)=SST." "reduced model has one regression parameter and the full model has p+1 regression parameter". "Because SST=SSR+SSE, we can replace SST-SSE by SSR"

(Chatterjee and Hadi 2015, 73)

F-TEST FOR ALL COEFFICIENTS

Table 3.5 Regression Output for Supervisor Performance Data

Variable	Coefficient	s.e.	t-Test	p-value
Constant	10.787	11.5890	0.93	0.3616
X_1	0.613	0.1610	3.81	0.0009
X_2	−0.073	0.1357	−0.54	0.5956
X_3	0.320	0.1685	1.90	0.0699
X_4	0.081	0.2215	0.37	0.7155
X_5	0.038	0.1470	0.26	0.7963
X_6	−0.217	0.1782	−1.22	0.2356
$n = 30$	$R^2 = 0.73$	$R_a^2 = 0.66$	$\hat{\sigma} = 7.068$	df = 23

Table 3.7 Supervisor Performance Data: Analysis of Variance (ANOVA) Table

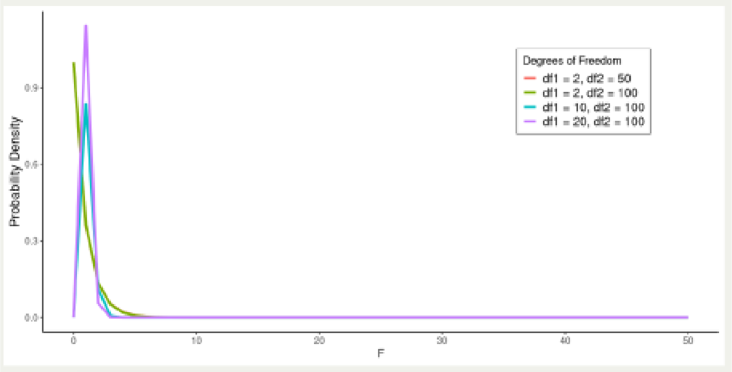
Source	Sum of Squares	df	Mean Square	F-Test
Regression	3147.97	6	524.661	10.5
Residuals	1149.00	23	49.9565	

$$F = \frac{SSR/p}{SSE/(n-p-1)} = \frac{3147.97/6}{1149.00/23} = 10.50$$

How to interpret this value? How likely are we to observe data in sample (this test statistics), under the assumption that H_0 is true?
→ Probability distribution

(Chatterjee and Hadi 2015, 75)

F-test for all coefficients



What is the conclusion? P-value < 0.05, We can reject H_0 with an error probability (p-value) of essentially 0%. → The full model performs better, "not all $\beta's$ can be taken as zero"

(Chatterjee and Hadi 2015, 75).

PARTIAL F-TEST

What is the F-test for some coefficients?

PARTIAL F-TEST

"Some of the regression coefficients are zero".

* Reduced model (RM): $Y = \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \epsilon$
  subset of slopes is equal to zero, $\beta_k = 0$ (H_0) → the reduced model performs better
* Full model (FM): $Y = \beta_0 + \beta_1 X_1 + ... + \beta_p X_p \epsilon$
  at least one slope in the subset is different from zero, $\beta_p \neq 0$ (H_1) → the full model performs better

$$F = \frac{[SSE(RM) - SSE(FM)]/(p+1-k)}{SSE(FM)/(n-p-1)}$$

(Chatterjee and Hadi 2015, 77)

PARTIAL F-TEST

Table 3.5 Regression Output for Supervisor Performance Data

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Constant	10.787	11.5890	0.93	0.3616
X_1	0.613	0.1610	3.81	0.0009
X_2	−0.073	0.1357	−0.54	0.5956
X_3	0.320	0.1685	1.90	0.0699
X_4	0.081	0.2215	0.37	0.7155
X_5	0.038	0.1470	0.26	0.7963
X_6	−0.217	0.1782	−1.22	0.2356
$n = 30$	$R^2 = 0.73$	$R_a^2 = 0.66$	$\hat{\sigma} = 7.068$	df = 23

Table 3.7 Supervisor Performance Data: Analysis of Variance (ANOVA) Table

Source	Sum of Squares	df	Mean Square	F-Test
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Residuals	1149.00	23	49.9565	

(Chatterjee and Hadi 2015, 75)

PARTIAL F-TEST

Table 3.8 Regression Output from the Regression of Y on X_1 and X_3

ANOVA Table

Source	Sum of Squares	df	Mean Square	F-Test
Regression	3042.32	2	1521.1600	32.7
Residuals	1254.65	27	46.4685	

Coefficients Table

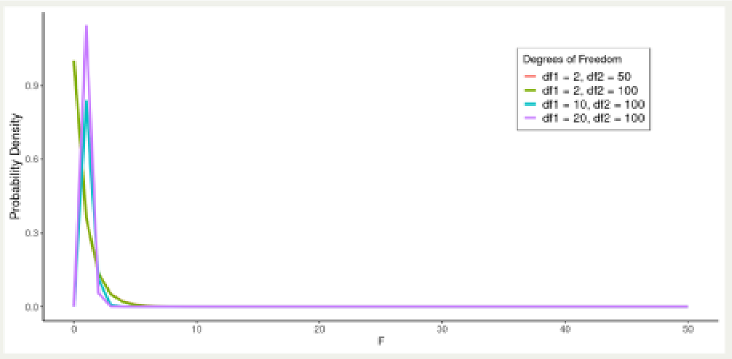
Variable	Coefficient	s.e.	t-Test	p-value
Constant	9.8709	7.0610	1.40	0.1735
X_1	0.6435	0.1185	5.43	< 0.0001
X_3	0.2112	0.1344	1.57	0.1278
$n = 30$	$R^2 = 0.708$	$R_a^2 = 0.686$	$\hat{\sigma} = 6.817$	df = 27

$$F = \frac{[1254.65 - 1149]/4}{1149/23} = 0.0528$$

How to interpret this value? How likely are we to observe data in sample (this test statistics), under the assumption that H_0 is true?
→ Probability distribution

(Chatterjee and Hadi 2015, 76)

Partial F-test

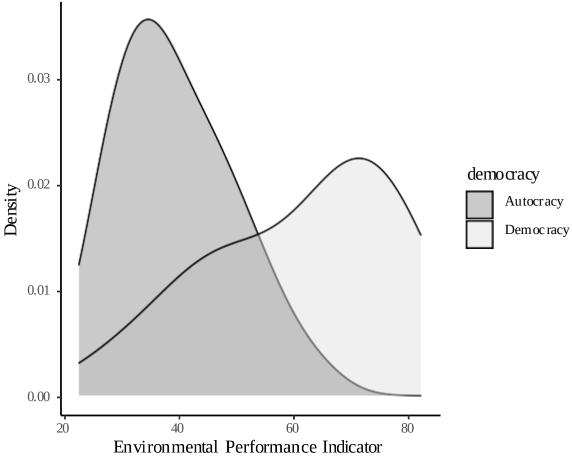


What is the conclusion? P-value > 0.05, We cannot reject H_0. → The reduced model performs better. "The variables X_1 and X_3 together explain the variation in Y as adequately as the full set of six variables" (Chatterjee and Hadi 2015, 77).

Categorical independent variables

What is the reference category?

What is the reference category?



$$\textit{Environmental Performance}_i = \alpha + \beta_1 * \textit{Regime Type}_i$$

What is the reference category?

Dummy variables should take value 0 and 1 for easy interpretation → Re-code existing variables.

```r
# Import data from Quality of Government dataset
qog_data <- read.csv("qog_bas_cs_jan21.csv")

# Generate dummy variable for regime type as factor variable - democracy
# vdem_polyarchy ranges between 0 and 1; cutoff at 0.7
# Countries with score equal or above 0.7 are democracies, those below autocracies
qog_data$democracy <- factor(ifelse(qog_data$vdem_polyarchy >= 0.7, 1, 0))

# Define levels of democracy in factor variable
levels(qog_data$democracy) <- c("Autocracy", "Democracy")

# Summarize generated dummy variable
summary(qog_data$democracy)
```

```
## Autocracy Democracy      NA's
##       119        54        21
```

What is the reference category?

```r
# Generate dummy variable for regime type as factor variable - autocracy
qog_data$autocracy <- factor(ifelse(qog_data$vdem_polyarchy < 0.7, 1, 0))

# Define levels of autocracy in factor variable
levels(qog_data$autocracy) <- c("Democracy", "Autocracy")

# Print first 10 rows in dataset
head(qog_data[c("democracy", "autocracy")], 10)
```

	democracy	autocracy
1	0 Autocracy	1 Autocracy
2	0 Autocracy	1 Autocracy
3	0 Autocracy	1 Autocracy
4	<NA>	<NA>
5	0 Autocracy	1 Autocracy
6	<NA>	<NA>
7	0 Autocracy	1 Autocracy
8	1 Democracy	0 Democracy
9	1 Democracy	0 Democracy
10	1 Democracy	0 Democracy

What happens if we run:

$$\textit{Environmental Performance}_i = \alpha + \beta_1 \textit{Democracy}_i + \beta_2 \textit{Autocracy}_i + \epsilon_i$$

What is the reference category?

$$Environmental\ Performance_i = \alpha + \beta_1 Democracy_i + \beta_2 Auocracy_i + \epsilon_i$$

```
# Fit regression model
m1_trap <- lm(epi_epi ~ democracy + autocracy, data = qog_data)

# Print results
summary(m1_trap)
```

```
lm(formula = epi_epi ~ democracy + autocracy, data = qog_data)

Residuals:
    Min      1Q  Median      3Q     Max
-34.107  -8.860  -0.610   9.293  26.190

Coefficients: (1 not defined because of singularities)
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   39.610      1.138   34.80   <2e-16 ***
democracy1    22.098      2.002   11.04   <2e-16 ***
autocracy1        NA         NA      NA       NA
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Violates assumption of no perfect multicollinearity (essentially a data problem) → One category needs to be excluded = reference category. Interpretation of the model is relative to the reference category.

Binary independent variables

How to include binary independent variables in multiple linear regression?

Binary independent variables

$$\textit{Environmental Performance}_i = \alpha + \beta_1 * \textit{Regime Type}_i + \beta_2 * \textit{Income}_i$$

```
## Call:
## lm(epi_epi ~ democracy + income, data = qog_data)

## Residuals:
##      Min       1Q  Median       3Q      Max
## -53.563   -6.502   0.498    6.773   20.198

## Coefficients:
##                    Estimate Std. Error t value Pr(>|t|)
## (Intercept)         35.3027     1.1269  31.327  < 2e-16 ***
## democracyDemocracy  16.5270     1.8409   8.978 9.08e-16 ***
## income               3.5793     0.4266   8.390 2.92e-14 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

## Residual standard error: 9.982 on 154 degrees of freedom
##   (37 observations deleted due to missingness)
## Multiple R-squared:  0.6175,Adjusted ## R-squared:  0.6126
## F-statistic: 124.3 on 2 and 154 DF,  p-value: < 2.2e-16
```

In comparison to autocracies (= reference category), democracies have a 16.5270 scale point higher score on the Environmental Performance Index, under control of income.

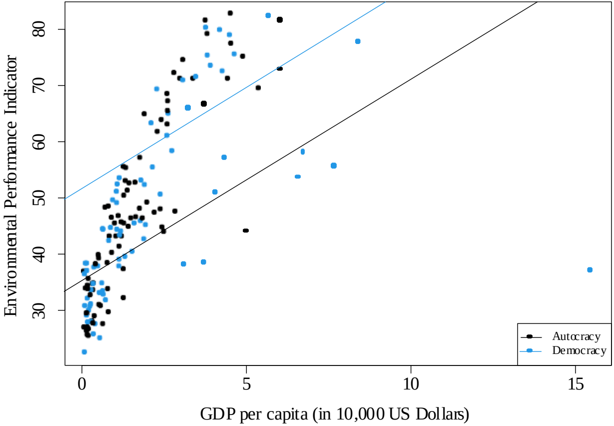
Binary independent variables

$\hat{Y}_i = \alpha + \beta_1 * Regime\ Type_i + \beta_2 * Income_i$

Model for Autocracies:
$\hat{Y}_i = 35.303 + (16.527 * Regime\ Type_i) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (16.527 * 0) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (3.579 * Income_i)$

Model for Democracies:
$\hat{Y}_i = 35.303 + (16.527 * Regime\ Type_i) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (16.527 * 1) + (3.579 * Income_i)$
$\hat{Y}_i = 51.83 + (3.579 * Income_i)$

Binary independent variables



Categorical independent variables

How to select the reference category?

How to select the reference category?

```r
# Run regression model with democracy variable
m1_dem <- lm(epi_epi ~ income + democracy, data = qog_data)

# Run regression model with autocracy variable
m1_aut <- lm(epi_epi ~ income + autocracy, data = qog_data)

# Get regression table with stargazer
stargazer(m1_dem, m1_aut)
```

	Dependent variable:	
	epi_epi	
	(1)	(2)
income	3.579***	3.579***
	(0.427)	(0.427)
democracy1	16.527***	
	(1.841)	
autocracy1		−16.527***
		(1.841)
Constant	35.303***	51.830***
	(1.127)	(1.892)
Observations	157	157
R^2	0.618	0.618
Adjusted R^2	0.613	0.613
F Statistic (df = 2; 154)	124.331***	124.331***
Note:	*p<0.1; **p<0.05; ***p<0.01	

How to select the reference category?

Model 1 for Autocracies:
$\hat{Y}_i = 35.303 + (16.527 * Regime\ Type_i) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (16.527 * 0) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (3.579 * Income_i)$

Model 2 for Autocracies:
$\hat{Y}_i = 51.830 + (-16.527 * Regime\ Type_i) + (3.579 * Income_i)$
$\hat{Y}_i = 51.830 + (-16.527 * 1) + (3.579 * Income_i)$
$\hat{Y}_i = 35.303 + (3.579 * Income_i)$

→ Mathematically identical models.

How do we select the reference category?

How to select the reference category?

```r
# Run regression model with democracy variable
m1 <- lm(epi_epi ~ income + democracy, data = qog_data)

# Get regression table with stargazer
stargazer(m1)
```

	Dependent variable:
	epi_epi
democracy1	16.527***
	(1.841)
income	3.579***
	(0.427)
Constant	35.303***
	(1.127)
Observations	157
R^2	0.618
Adjusted R^2	0.613
F Statistic (df = 2; 154)	124.331***
Note:	*p<0.1; **p<0.05; ***p<0.01

In comparison to autocracies (= reference category), democracies have a 16.5270 scale point higher score on the Environmental Performance Index, under control of income.

Categorical independent variables

How to include categorical independent variables with more than two levels?

Categorical independent variables

Country	X_{region}
Afghanistan	Asia
Albania	EE
Algeria	MENA
Argentina	LA
Australia	Advanced
⋮	⋮

→

Country	X_{region}
Afghanistan	2
Albania	3
Algeria	5
Argentina	4
Australia	1
⋮	⋮

→

Country	X_{Asia}	X_{EE}	X_{LA}	X_{MENA}	$X_{Sub-Saharan}$
Afghanistan	1	0	0	0	0
Albania	0	1	0	0	0
Algeria	0	0	0	1	0
Argentina	0	0	1	0	0
Australia	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

$\textit{School enrollment rate} = \alpha + \beta_1 \textit{Democracy}_i + \beta_2 \textit{Region}_{EE} + \beta_3 \textit{Region}_{LA} + \beta_4 \textit{Region}_{MENA} + \beta_5 \textit{Region}_{Sub-Saharan} + \epsilon_i$

→ Include binary/dummy variables for all levels <u>minus one</u> (=reference category).

- α (intercept): expected value of Y when $X_k = 0$
- β (coefficient): expected change in Y for $X = 1$, in comparison to reference category

Categorical independent variables

→ Convert into factor variable, then R automatically generates dummy variables, with first level as reference category (or change with relevel-function).

```r
# Code dummy variables on the fly
# specify region Sub-Saharan Africa = reference category
lm <- lm(primary_ser ~ democracy + relevel(as.factor(region), ref="Sub-Saharan
    Africa"), data = paglayan2021)

# Print model output
summary(lm)
```

```
Call:
lm(formula = primary_ser ~ democracy + relevel(as.factor(region),
    ref = "Sub-Saharan Africa"), data = paglayan2021)

Coefficients:
                                                    Estimate Std. Error t value Pr(>|t|)
(Intercept)                                           48.060      1.796  26.754  < 2e-16 ***
democracy                                             41.291      1.351  30.557  < 2e-16 ***
ref = "Sub-Saharan Africa")Advanced Economies          3.063      2.143   1.429 0.153007
ref = "Sub-Saharan Africa")Asia and the Pacific       -9.101      2.437  -3.734 0.000192 ***
ref = "Sub-Saharan Africa")Eastern Europe             12.991      2.825   4.599 4.46e-06 ***
ref = "Sub-Saharan Africa")Latin America and the Caribbean -13.090  2.073  -6.315 3.20e-10 ***
ref = "Sub-Saharan Africa")Middle East and North Africa  4.389   2.695   1.629 0.103515
```

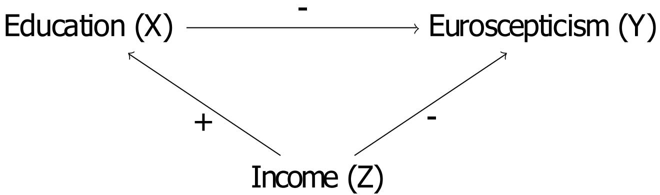
Under control of regime type, Eastern Europe has a student enrollment rate of 12.991 percentage points higher than Sub-Saharan Africa.

What is the relationship between education and Euroscepticism?

$$\text{Education (X)} \xrightarrow{\quad - \quad} \text{Euroscepticism (Y)}$$

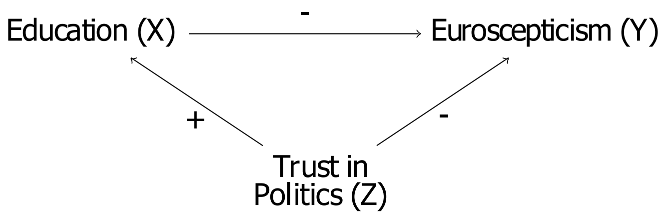
Hypothesis$_1$: The higher the years of education, the lower the level of Euroscepticism.

What is the relationship between education and Euroscepticism?



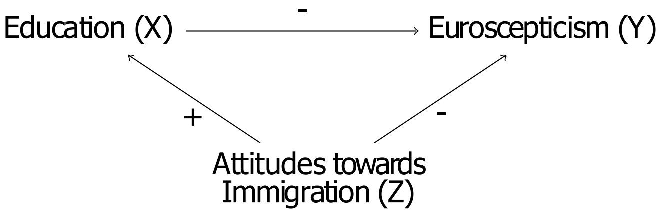
Hypothesis$_2$: The higher the income, the lower the level of Euroscepticism. → Economic dimension

What is the relationship between education and Euroscepticism?



Hypothesis$_3$: The higher the trust in politics, the lower the level of Euroscepticism. → Political dimension

What is the relationship between education and Euroscepticism?



Hypothesis$_3$: The more positive attitudes towards immigration, the lower the level of Euroscepticism. → Cultural dimension

References I

Kellstedt, Paul M., and Guy D. Whitten. 2018. *The fundamentals of political science research.* Cambridge: Cambridge University Press.