EE 368 机器人运动与控制方法（Robotic Motion and Control）
Reference Solutions to Assignment #3

Q-1. For a 3-DOF robot, given the following transformation matrices, find the Jacobian ${}^0J$.

$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & e \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where h, e, f are the lengths of the 1st, 2nd and 3rd link, respectively.

$$
{}^0_4T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & ec_1c_2 + fc_1c_{23} \\ s_1c_{23} & -s_1c_{23} & -c_1 & es_1c_2 + fs_1c_{23} \\ s_{23} & c_{23} & 0 & h + es_2 + fs_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Solution: From the above we can find

$$
{}^0_2R = {}^0_1R\,{}^1_2R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & c_1 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 \\ s_1c_2 & -s_1s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix},
$$

$$
{}^0_3R = {}^0_2R\,{}^2_3R = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 \\ s_1c_2 & -s_1s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}.
$$

So, we have $\displaystyle {}^0J_\omega = [{}^0_1r_3 \quad {}^0_2r_3 \quad {}^0_3r_3] = \begin{bmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$

Since $\displaystyle {}^0P_4 = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} ec_1c_2 + fc_1c_{23} \\ es_1c_2 + fs_1c_{23} \\ h + es_2 + fs_{23} \end{bmatrix}$

$$
\Rightarrow \quad {}^0J_v = \begin{bmatrix} \frac{\partial p_x}{\partial\theta_1} & \frac{\partial p_x}{\partial\theta_2} & \frac{\partial p_x}{\partial\theta_3} \\ \frac{\partial p_y}{\partial\theta_1} & \frac{\partial p_y}{\partial\theta_2} & \frac{\partial p_y}{\partial\theta_2} \\ \frac{\partial p_z}{\partial\theta_1} & \frac{\partial p_z}{\partial\theta_2} & \frac{\partial p_z}{\partial\theta_3} \end{bmatrix} = \begin{bmatrix} -s_1(ec_2 + fc_{23}) & -c_1(es_2 + fs_{23}) & -fc_1s_{23} \\ c_1(ec_2 + fc_{23}) & -s_1(es_2 + fs_{23}) & -fs_1s_{23} \\ 0 & ec_2 + fc_{23} & fc_{23} \end{bmatrix};
$$

$$
\Rightarrow \quad {}^0J = \begin{bmatrix} {}^0J_v \\ {}^0J_\omega \end{bmatrix} = \begin{bmatrix} -s_1(ec_2 + fc_{23}) & -c_1(es_2 + fs_{23}) & -fc_1s_{23} \\ c_1(ec_2 + fc_{23}) & -s_1(es_2 + fs_{23}) & -fs_1s_{23} \\ 0 & ec_2 + fc_{23} & fc_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}
$$

Q-2. Find the Jacobian of the manipulator shown in Figure Q-2. Write it in terms of frame {4} located at the tip of the robot and having the same orientation as frame {3}.

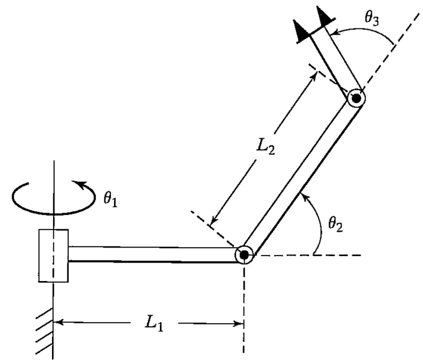


Figure Q-2

Solution: For the 3 DOF robot in Figure Q-2, we can obtain

$$ {}^0_3T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & L_1c_1 + L_2c_1c_2 \\ s_1c_{23} & -s_1s_{23} & -c_1 & L_1s_1 + L_2s_1c_2 \\ s_{23} & c_{23} & 0 & L_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\&\; {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\&\; {}^0_4T = {}^0_3T\,{}^3_4T $$

We could then find ${}^0J(\theta)$ quite easily by differentiating ${}^0P_{YORG}$. Finally, ${}^4J(\theta)$ can be calculated as ${}^4_0R\ {}^0J(\theta)$. This might be tedious, so let's try "standard" velocity propagation:

$$ {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad {}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2\omega_2 = {}^2_1R\ {}^1\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, $$

$$ {}^2v_2 = {}^2_1R(\,{}^1v_1 + \,{}^1\omega_1 \times \,{}^1P_2) = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix} $$

$$ {}^3\omega_3 = {}^3_2R\ {}^2\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}, $$

$$ {}^3v_3 = {}^3_2R(\,{}^2v_2 + \,{}^2\omega_2 \times \,{}^2P_3) = \begin{bmatrix} s_3L_2\dot{\theta}_2 \\ c_3L_2\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2c_2\dot{\theta}_1 \end{bmatrix} $$

$$ {}^4\omega_4 = \,{}^3\omega_3 = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}, $$

$$^{4}v_4 = {}^{4}_{3}R(\,^{3}v_3 + \,^{3}\omega_3 \times \,^{3}P_4) = \,^{3}v_3 + \,^{3}\omega_3 \times \,^{3}P_4 = \begin{bmatrix} s_3 L_2 \dot{\theta}_2 \\ c_3 L_2 \dot{\theta}_2 - L_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -L_1\dot{\theta}_1 - L_2 c_2 \dot{\theta}_1 - L_3 c_{23}\dot{\theta}_1 \end{bmatrix}$$

$$\Longrightarrow \quad {}^{4}J(\theta) = \begin{bmatrix} 0 & s_3 L_2 & 0 \\ 0 & c_3 L_2 + L_3 & L_3 \\ -L_1 - L_2 c_2 - L_3 c_{23} & 0 & 0 \end{bmatrix}$$

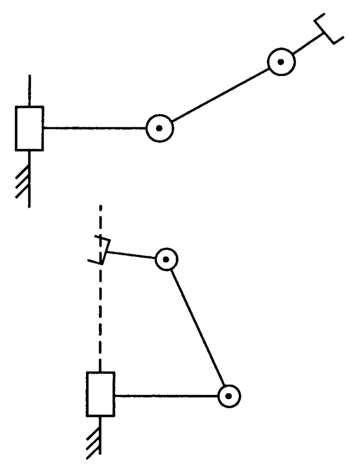
Q-3. For the three-link robot manipulator shown in Figure Q-2, give a set of joint angles for which the manipulator is at a workspace-boundary singularity and another set of angles for which the manipulator is at a workspace-interior singularity.

Solution:

Workspace boundary: Any angle set: $(\theta_1, \theta_2, 0)$

Workspace interior: Any angle set such that

$L_1 + L_2 c_2 + L_3 c_{23} = 0$, θ_1 is arbitrary.



Q-4. Prove that singularities in the force domain exist at the same configurations as the singularities in the position domain.

Solution:

Since $\det[J(\theta)] = \det\,[J^T(\theta)]$, thus when singularities occur in the position domain, $\det[J(\theta)] = 0$, so $\det[J^T(\theta)] = 0$, singularities occur in the force domain as well.