EE 368 机器人运动与控制方法（Robotic Motion and Control）
Reference Solutions to Assignment #4

Q-1. The single-degree-of-freedom "manipulator" in Fig. Q-1 has total mass $m = 1$, with the center of mass at ${}^{1}\boldsymbol{P}_C = [2 \quad 0 \quad 0]^T$, and has inertia tensor ${}^{C}\boldsymbol{I}_1 = \mathrm{diag}\{1 \quad 2 \quad 2\}$. From rest at $t = 0$, the joint angle θ_1 moves in accordance with the time function

$$\theta_1(t) = bt + ct^2$$

in radians. Given the angular acceleration of the link and the linear acceleration of the center of mass in terms of frame {1} as a function of t.

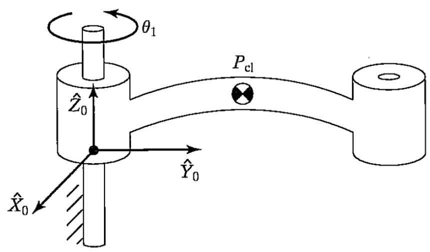

Figure Q-1

Solution: Since $\theta_1(t) = Bt + Ct^2$, so $\dot{\theta}_1(t) = B + 2Ct$, $\ddot{\theta}_1(t) = 2C$, therefore

$${}^{1}\dot{\omega}_1 = \ddot{\theta}_1 \, {}^{1}\hat{Z}_1 = 2C \, {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ 2C \end{bmatrix}$$

$${}^{1}\dot{v}_{C_1} = \begin{bmatrix} 0 \\ 0 \\ 2C \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \otimes \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 4C \\ 0 \end{bmatrix} + \begin{bmatrix} -2\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2(B + 2Ct)^2 \\ 4C \\ 0 \end{bmatrix}.$$

Q-2. Derive the equations of motion for the PR manipulator shown in Fig. Q-2. Neglect friction, but include gravity. (Here, $\hat{X}_0$ is upward.) The inertia tensors of the links are diagonal, with moments $I_{xx1}, I_{yy1}, I_{zz1}$ and $I_{xx2}, I_{yy2}, I_{zz2}$. The centers of mass for the links are given by

$${}^{1}P_{C_1} = \begin{bmatrix} 0 \\ 0 \\ -l_1 \end{bmatrix} \quad \text{and} \quad {}^{1}P_{C_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

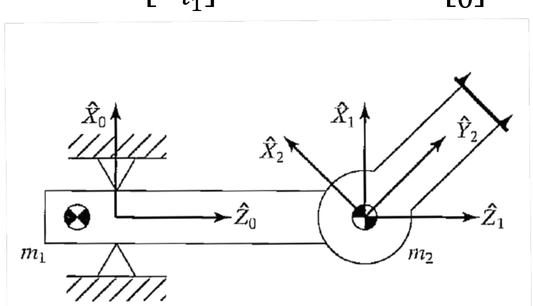

Figure Q-2

Solution: Upon inspection, this problem is quite simple:

$$\boldsymbol{\tau} = \begin{bmatrix} f_1 \\ \tau_2 \end{bmatrix} = \boldsymbol{M}(\Theta)\ddot{\Theta} + \boldsymbol{V}(\Theta, \dot{\Theta}) + \boldsymbol{G}(\Theta)$$

where $\Theta = \begin{bmatrix} d_1 \\ \theta_2 \end{bmatrix}$ and $\boldsymbol{M}(\Theta) = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & I_{zz2} \end{bmatrix}$, $\boldsymbol{V}(\Theta, \dot{\Theta}) = \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boldsymbol{G}(\Theta) = \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Q-3. Derive the Cartesian space form of the dynamics for the two-link planar manipulator of Q-2 in terms of the base frame. *Hint*: Use the Jacobian written in the base frame.

Solution: For the assigned frames, the DH table is as follows:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	90°	0	0	θ_2

and the rotation matrices are:

$${}^0_1R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^1_2R = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ 0 & 0 & -1 \\ s\theta_2 & c\theta_2 & 0 \end{bmatrix},$$

$$\text{thus } {}^0_2R = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ 0 & 0 & -1 \\ s\theta_2 & c\theta_2 & 0 \end{bmatrix}, \; and \; {}^0P_2 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}.$$

Therefore,

$${}^0\dot{P}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix} = {}^0J_v \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^0J_\omega = [{}^0_1r_3 \quad {}^0_2r_3] = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and the Jacobian is

$${}^0J = \begin{bmatrix} {}^0J_v \\ {}^0J_\omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The dynamics in Cartesian space is then

$$\boldsymbol{M}_X(\Theta)\ddot{\Theta} + \boldsymbol{V}_X(\Theta, \dot{\Theta}) + \boldsymbol{G}_X(\Theta) = \boldsymbol{\mathcal{F}}$$

where

$$\boldsymbol{M}_X(\Theta) = \boldsymbol{J}^{+T}(\Theta)\boldsymbol{M}(\Theta)\boldsymbol{J}^{+}(\Theta)$$
$$\boldsymbol{V}_X(\Theta, \dot{\Theta}) = \boldsymbol{J}^{+T}(\Theta)(\boldsymbol{V}(\Theta, \dot{\Theta}) - \boldsymbol{M}(\Theta)\boldsymbol{J}^{+}(\Theta)\dot{\boldsymbol{J}}(\Theta)\dot{\Theta})$$
$$\boldsymbol{G}_X(\Theta) = \boldsymbol{J}^{+T}(\Theta)\boldsymbol{G}(\Theta)$$

where $\boldsymbol{J}^{+}$is the pseudo inverse of $\boldsymbol{J}$ and $\boldsymbol{J}^{+T} = (\boldsymbol{J}^{+})^T$.