① Jacobians

The Jacobian is the mathematical form of the derivatives.

Suppose that we have six functions, each of which is a formulation of six independent variables:

$$y_1 = f_1(x_1, x_2, \cdots x_6)$$
$$y_2 = f_2(x_1, x_2, \cdots, x_6)$$
$$\vdots$$
$$y_6 = f_6(x_1, x_2, \cdots, x_6)$$

In matrix form:

$$\mathbb{Y} = \mathbb{F}(\mathbb{X})$$

where $\mathbb{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{bmatrix}$, $\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix}$

If we wish to calculate the differentials of y_i as a function of differentials of x_j, we simply use the chain-rule of derivatives to obtain the following.

$$\delta y_1 = \frac{\partial f_1}{\partial x_1} \cdot \delta x_1 + \frac{\partial f_1}{\partial x_2} \cdot \delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6} \cdot \delta x_6$$
$$\vdots$$
$$\delta y_6 = \frac{\partial f_6}{\partial x_1} \cdot \delta x_1 + \frac{\partial f_6}{\partial x_2} \cdot \delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6} \cdot \delta x_6$$

In matrix form: $\delta \mathbb{Y} = \dfrac{\partial \mathbb{F}}{\partial \mathbb{X}} \cdot \delta \mathbb{X}$

where $\delta \mathbb{Y} = \begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_6 \end{bmatrix}$, $\delta \mathbb{X} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_6 \end{bmatrix}$

and

$$\boxed{\frac{\partial \mathbb{F}}{\partial \mathbb{X}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \vdots & & & \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}} \triangleq \mathbb{J}_{6\times 6}$$

$m \times n$ in general case

and J is defined as the Jacobian matrix.

The 6x6 matrix of partial derivatives is what we call the Jacobian, $J = \frac{\partial F}{\partial x}$. Note that if the function $f_1(x)$ through $f_6(x)$ are nonlinear, then the partial derivatives are a function of x_i, so we use the notation as follows:

$$\delta \mathbb{Y} = \mathbb{J}(\mathbb{X}) \cdot \delta \mathbb{X}$$

By dividing both sides of the above equation by the differential time element δt, we can think of the Jacobian as mapping from velocities in $\mathbb{X}$ to velocities in $\mathbb{Y}$:

Cartesian velocities

Joint velocities

$$\frac{\delta \mathbb{Y}}{\delta t} = \mathbb{J}(\mathbb{X}) \cdot \frac{\delta \mathbb{X}}{\delta t} \iff \dot{\mathbb{Y}} = \mathbb{J}(\mathbb{X}) \cdot \dot{\mathbb{X}}$$

At any time instant, X has a certain value and $J(X)$ is a linear transformation:

$$\dot{Y} = J(X)\cdot\dot{X}$$

At each new time instant, X has changed and thus so has the linear transformation, which means that Jacobian is time-varying.

$$J(X(t))$$

Consider an n dof manipulator, where joint displacements are denoted by $\theta_1, \theta_2, \cdots \theta_n$ and are put together as an n-dimension vector:

$$\underbrace{\Theta_{n\times1}}_{\text{position}} = \begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_6\end{bmatrix} \Rightarrow \dot{\Theta}_{n\times1} = \begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \vdots\\ \dot\theta_6\end{bmatrix} \quad \text{joint velocities}$$

The Cartesian velocities of the robot e.e. are denoted by ^0v and $^0\omega$ that are put together as a vector of 6-dimension:

$$^0V_{6\times1} = \begin{bmatrix}\overbrace{^0v_{3\times1}}^{\text{linear}}\\ \underbrace{^0\omega_{3\times1}}_{\text{angular}}\end{bmatrix}_{6\times1} \quad \text{Cartesian velocities}$$

where: $^0v \in R^3$ is its linear velocity of the e.e. w.r.t. $\{0\}$

$^0\omega \in R^3$ is its angular velocity of the e.e. w.r.t. $\{0\}$

The Jacobian of a given robot manipulator, ^0J, is defined as a $6\times n$ matrix that relates $\dot{\Theta}$ to ^0V, i.e.

$$\underbrace{^0V_{6\times1}}_{\text{Velocities of the e.e. in Cartesian Space}} = \underbrace{^0J(\Theta)}_{\text{Jacobian}}\cdot\underbrace{\dot{\Theta}}_{\text{Velocities in Joint Space}}$$

Note that the superscript "0" to V and J is to emphasize that both V and J are expressed w.r.t. (the same) frame $\{0\}$ and they are frame dependent!

With $^0V_{6\times1} = {^0J_{6\times n}}(\Theta)\cdot\dot{\Theta}_{n\times1}$

$$\Rightarrow \begin{bmatrix}^0v_{3\times1}\\ ^0\omega_{3\times1}\end{bmatrix} = {^0J}(\Theta)\cdot\dot{\Theta} = \begin{bmatrix}\overbrace{^0J_v(\Theta)_{3\times n}}^{\text{linear velocity}}\\ \underbrace{^0J_\omega(\Theta)_{3\times n}}_{\text{Angular velocity}}\end{bmatrix}\cdot\dot{\Theta}$$

$$\Rightarrow {^0v_{3\times1}} = {^0J_{v_{3\times n}}}(\Theta)\cdot\dot{\Theta}_{n\times1}$$

$$^0\omega_{3\times1} = {^0J_{\omega_{3\times n}}}(\Theta)\cdot\dot{\Theta}_{n\times1}$$

Remarks:

1° In general sense, Jacobian is defined as:

$$\mathcal{J}(\mathcal{X}) = \frac{\partial \mathbb{F}}{\partial \mathcal{X}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \vdots & & & \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}_{6\times 6}$$

2° If one is interested in ${}^0\mathcal{V}$ (linear velocities) only, then

$${}^0\mathcal{V}_{3\times 1} = {}^0\mathcal{J}_{v}(\Theta)_{3\times n} \cdot \dot{\Theta}_{n\times 1}$$

where ${}^0\mathcal{J}_v(\Theta)_{3\times n}$ is the upper 3 rows of ${}^0\mathcal{J}(\Theta)$, corresponding to the linear velocity ${}^0\mathcal{V}$. Indeed,

$${}^0\mathcal{J}(\Theta) = \begin{bmatrix} {}^0\mathcal{J}_v(\Theta) \\ \hdashline {}^0\mathcal{J}_\omega(\Theta) \end{bmatrix} \begin{matrix} \leftarrow \text{Jacobian related to linear velocity} \\ \leftarrow \text{Jacobian related to angular velocity} \end{matrix}$$

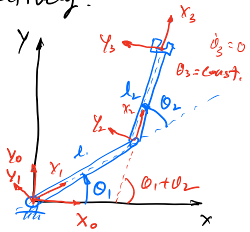
3° If the robot is of planery type, then the size of ${}^0\mathcal{J}(\Theta)$ and also ${}^0\mathcal{J}_v(\Theta)$ and ${}^0\mathcal{J}_\omega(\Theta)$ can be reduced to $3\times n$, $2\times n$ and $1\times n$, respectively.

Example: The 2 dof planery robot

Using the velocity propagation:

$${}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad {}^1\mathcal{V}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2\omega_2 = {}^2_1R \cdot {}^1\omega_1 + \dot{\theta}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$



$${}^2\mathcal{V}_2 = {}^2_1R({}^1\mathcal{V}_1 + {}^1\omega_1 \times {}^1P_2) = \begin{bmatrix} -l_1 S_2 \cdot \dot{\theta}_1 \\ l_1 C_2 \cdot \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$${}^3\omega_3 = {}^3_2R \cdot {}^2\omega_2 + \overset{0}{\overset{\parallel}{\dot{\theta}_3}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$${}^3\mathcal{V}_3 = {}^3_2R({}^2\mathcal{V}_2 + {}^2\omega_2 \times {}^2P_3) = \begin{bmatrix} l_1 S_1 \dot{\theta}_1 \\ l_1 C_1 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$\Rightarrow {}^0\omega_3 = {}^0_3R \cdot {}^3\omega_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$ planery

$${}^0\mathcal{V}_3 = {}^0_3R \cdot {}^3\mathcal{V}_3 = \begin{bmatrix} -l_1 S_1 \dot{\theta}_1 - l_2 S_{12} \cdot (\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 C_1 \dot{\theta}_1 + l_2 C_{12} \cdot (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$\Rightarrow {}^0\mathbb{V} = \begin{bmatrix} {}^0\mathcal{V}_{3\,2\times 1} \\ {}^0\omega_{3\,1\times 1} \end{bmatrix} = \begin{bmatrix} -l_1 S_1 \dot{\theta}_1 - l_2 S_{12} \cdot \dot{\theta}_1 & - l_2 S_{12} \cdot \dot{\theta}_2 \\ l_1 C_1 \dot{\theta}_1 + l_2 C_{12} \cdot \dot{\theta}_1 & + l_2 C_{12} \cdot \dot{\theta}_2 \\ \dot{\theta}_1 & + \dot{\theta}_2 \end{bmatrix}$$

$$\Rightarrow {}^0\mathbb{V}_{3\times 1} = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \\ \hdashline 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \mathcal{J}(\Theta) \cdot \dot{\Theta}$$

linear

angular

$$\Rightarrow {}^0\mathcal{J}(\Theta) = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \\ \hdashline 1 & 1 \end{bmatrix} \begin{matrix} \text{linear} \\ \\ \text{angular} \end{matrix}$$

$$\Rightarrow {}^0\mathcal{J}_v = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \end{bmatrix}_{2\times 2}$$

$${}^0\mathcal{J}_\omega = \begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2}$$

Alternatively, Recall:

$$x(t) = l_1 C_1 + l_2 C_{12}$$
$$y(t) = l_1 S_1 + l_2 S_{12}$$

Differentiate on the eqns: w.r.t. time t:

$$\dot{x}(t) = -l_1 S_1 \dot{\theta}_1 - l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y}(t) = l_1 C_1 \dot{\theta}_1 + l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

linear

$$\Rightarrow \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\Rightarrow \quad {}^0 v = {}^0 J_v(\Theta) \cdot \dot{\Theta}$$

The above approach can be used for any robot manipulators. This is the most convenient way to calculate the LINEAR velocity part of the Jacobian. For angular velocity part of the Jacobian, we note the following alternative way to find ${}^0 J_\omega(\Theta)$.