b. PD control with gravity compensation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

when $\boxed{\dot{\Theta} = \ddot{\Theta} = 0_{n\times1}} \Rightarrow \tau = G(\Theta)$ Gravity term

let robot stand still — Static situation

In this case, $\alpha = I_n$, $\beta = \hat{G}(\Theta)$

where $\hat{G}$ is the model (estimated) of G.

$$\Rightarrow \tau' = \ddot{\Theta}^* = \ddot{\Theta}_d + K_v\dot{E} + K_p E$$

$$\Rightarrow \tau = I\cdot[\ddot{\Theta}_d + K_v\dot{E} + K_p E] + \hat{G}(\Theta)$$

$\Rightarrow$ This PD control with gravity compensation can cause the steady-state error go to zero.

(Same effect as adding the I (integral) term in PID).

III. Lyapunov Stability Theory

The Lyapunov stability analysis or Lyapunov Second (direct) method can be used to conclude on the stability without solving for the solutions of the D.E.s, that govern the system.

While Lyapunov method is useful for examining stability, it generally does not provide any information about the transient response or performance of the system. It is an energy based approach — it is one of the few techniques that can be applied to nonlinear systems to study their stability.

Lyapunov method is concerned with determining the stability of a system described by a D.E.:

$$\dot{X} = f(X) \quad (1^{st}\text{-order D.E.})$$

with $X \in R^{n\times1}$ and $f(\cdot)$ may be nonlinear.

Note that higher-order D.E.s can always be written as a set of first-order D.E.s in a matrix form.

Example: $\ddot{x} + b\dot{x} + 9x^3 = 0$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

$\Rightarrow \quad \ddot{x} + b\dot{x} + 9x^3 = 0 \Leftrightarrow \ddot{x} = -b\dot{x} - 9x^3$

$$\Rightarrow \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -b\dot{x} - 9x^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9x^2 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$= x$, $= \dot{x}$

$$\Rightarrow F(X) = \begin{bmatrix} 0 & 1 \\ -9x^2 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

To prove a system stable by Lyapunov method, one is required to propose a generalized energy function $V(X) \in R^{1\times1}$ that has the properties:

1° $V(X)$ has continuous first-order partial derivatives and $V(X) > 0$ for all X except $V(0) = 0$.

2° $\dot{v}(x) \le 0$ here $\dot{v}(x)$ means the change in $v(x)$ along all system trajectories.

These properties may hold only in a certain region (local stability) or may hold globally (global stability). The intuitive idea is that a positive definite "energy-like" function is stable if it is shown to always decrease or remain constant. Hence, the system is stable in the sense that the size of the state vector is bounded.

stable

when $\dot{v}(x) < 0$ ⇒ asymptotically convergent

⇒ state will go to zero as $t \to \infty$

when $\dot{v}(x) = 0$ ⇒ Deal with this case of $\dot{v}(x) = 0$ by performing a steady-state analysis (via LaSalle's theorem) in order to determine if the stability is asymptotic or if the system under study can "get stuck" (steady-state error exists) somewhere that $v(x) = 0$.

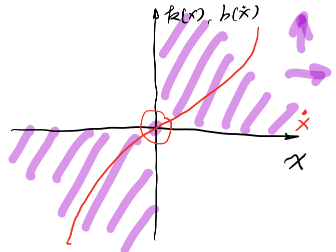
Example #1. $\ddot{x} + b(\dot{x}) + k(x) = 0$

$$\Rightarrow \boxed{\ddot{x} = -b(\dot{x}) - k(x)}$$

The functions $b(\cdot)$ and $k(\cdot)$ are first-and-third quadrant continuous functions such that

$$\dot{x} \cdot b(\dot{x}) \ge 0 \quad \text{for } x \neq 0$$

$$x \cdot k(x) > 0 \quad \text{for } x \neq 0$$



Propose the Lyapunov function (generalized energy function).

$$v(x, \dot{x}) = \underbrace{\frac{1}{2}\dot{x}^2}_{\text{positive}} + \underbrace{\int_0^x k(\lambda)\, d\lambda}_{1^{st}\text{ qudrant} \Rightarrow \text{positive}} > 0$$

Then $\dot{v}(x, \dot{x}) = \frac{1}{2} \cdot 2 \cdot \dot{x}\ddot{x} + k(x) \cdot \dot{x}$

$$= \dot{x}\left(-b(\dot{x}) - k(x)\right) + k(x) \cdot \dot{x}$$

$$= -\dot{x} \cdot b(\dot{x}) - \dot{x} \cdot k(x) + k(x) \cdot \dot{x}$$

$$\Rightarrow \dot{v}(x, \dot{x}) = -\underbrace{\dot{x} \cdot b(\dot{x})}_{>0} < 0 \quad \text{except at } \dot{x} = 0$$

But it is not a function of x ⇒ x could be anything.

In order to conclude on the stability of the system in x, we have to ensure that it is impossible for the system to "get stuck" with nonzero x. To study all the trajectories for which

$$\dot{x} = 0$$

Plug into the original dynamic equation:

$$\boxed{\ddot{x} + b(\dot{x}) + k(x) = 0 \,\Big|\, \dot{x}=0}$$

$\dot{x}=0$, $b(\dot{x})=0$, $\dot{x}=0 \Rightarrow \ddot{x}=0$

$\Rightarrow \quad 0+0+k(x)=0 \quad \Rightarrow \quad x=0$!

$\Rightarrow$ The system will only come to stop if $\dot{x}=0$, $x=0$.

$\Rightarrow$ The system is asymptotically stable. (no. s.s. error)

Example #2. $\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$

PD + $\hat{G}$ control: $\tau = K_p E + K_v \dot{E} + \hat{G}(\Theta)$ $\left(\hat{G}(\Theta) = G(\Theta)\right)$

PD Control with gravity compensation

Task: Θ_d = constant $\Rightarrow$ $\dot{\Theta}_d = 0$, $\ddot{\Theta}_d = 0$

Control: $\tau = K_p E + K_v \dot{E} + G(\Theta)$

$\qquad = K_p E - K_v \dot{\Theta} + G(\Theta)$

Plug control into system model to get the error dynamics equation:

$\Theta_d - \Theta$

$$M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + \cancel{G(\Theta)} = K_p E - K_v\dot{\Theta} + \cancel{G(\Theta)}$$

$$\Rightarrow M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + K_v\dot{\Theta} + K_p\Theta = K_p\Theta_d$$

Consider the Lyapunov function candidate:

$$\mathcal{V} = \underbrace{\tfrac{1}{2}\dot{\Theta}^T \underbrace{M(\Theta)}_{\text{P.D.}}\dot{\Theta}}_{>0} + \underbrace{\tfrac{1}{2}E^T \underbrace{K_p}_{>0} E}_{>0} > 0 \quad \text{except } x=0$$

$$\dot{\mathcal{V}} = \underbrace{\tfrac{1}{2}\ddot{\Theta}^T M(\Theta)\dot{\Theta}}_{1\times1} + \tfrac{1}{2}\dot{\Theta}^T\dot{M}(\Theta)\dot{\Theta} + \underbrace{\tfrac{1}{2}\dot{\Theta}^T M(\Theta)\ddot{\Theta}}_{1\times1}$$

$$+ \tfrac{1}{2}\dot{E}^T K_p E + \tfrac{1}{2}E^T K_p \dot{E} \qquad \dot{E} = -\dot{\Theta} \qquad \dot{\Theta}_d = 0$$

$$\Rightarrow \dot{\mathcal{V}} = \tfrac{1}{2}\dot{\Theta}^T\dot{M}(\Theta)\dot{\Theta} + \dot{\Theta}^T M(\Theta)\ddot{\Theta} - E^T K_p \dot{\Theta}$$

Note that : $\boxed{\tfrac{1}{2}\dot{\Theta}^T \dot{M}(\Theta)\dot{\Theta} = \dot{\Theta}^T V(\Theta,\dot{\Theta})}$

Note that $a^T m b = b^T m a$

$(a^T m b)^T = b^T m^T (a^T)^T = b^T m a$

$$\dot{\mathcal{V}} = \tfrac{1}{2}\dot{\Theta}^T\dot{M}(\Theta)\dot{\Theta} + \dot{\Theta}^T\left[-V(\Theta,\dot{\Theta}) - K_v\dot{\Theta} - K_p\Theta + K_p\Theta_d\right] - E^T K_p\dot{\Theta}$$

$$\dot{\mathcal{V}} = \tfrac{1}{2}\dot{\Theta}^T\dot{M}(\Theta)\dot{\Theta} - \dot{\Theta}^T V(\Theta,\dot{\Theta}) - \dot{\Theta}^T K_v\dot{\Theta}$$

$$\Rightarrow \boxed{\dot{\mathcal{V}} = -\dot{\Theta}^T K_v \dot{\Theta} \le 0} \quad \text{if } K_v > 0$$

$\Rightarrow \dot{\mathcal{V}} = 0 \Rightarrow \dot{\Theta} = 0$, $\ddot{\Theta} = 0$

plug the above into the error dynamics:

$$M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + K_v\dot{\Theta} + K_p\Theta = K_p\Theta_d$$

$$0 + 0 + 0 + K_p\Theta = K_p\Theta_d$$

$\Rightarrow K_p E = 0 \Rightarrow E = 0 \Rightarrow \Theta \xrightarrow{t\to\infty} \Theta_d$

$\Rightarrow$ The system is asymptotically stable !

II Adaptive Control

If dynamic model is not precisely known, what can we do? $\Rightarrow$ use adaptive control !!

a. Regressor Formulation of Robot Dynamics

Robot dynamics: $\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$

can be linearized in terms of a vector of robot dynamic parameters:

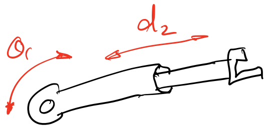
$$\underbrace{\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)}_{\text{non linear}} = \underbrace{Y(\Theta, \dot{\Theta}, \ddot{\Theta})\,\varphi}_{\text{linear}}$$

where $Y(\Theta, \dot{\Theta}, \ddot{\Theta}) \in R^{n \times r}$ is the robot regressor matrix

and $\varphi \in R^{r \times 1}$ is a vector of robot dynamic parameters

Example: 2 DOF RP robot

The dynamics of the robot:



$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + I_{zz_1}\ddot{\theta}_1 + I_{zz_2}\ddot{\theta}_1 + m_2 d_2^2 \ddot{\theta}_1 + 2 m_2 d_2 \dot{\theta}_1 \dot{d}_2 + m_1 l_1 g c_1 + m_2 d_2 g c_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g s_1$$

$$\Rightarrow \underset{2\times1}{\tau} = \begin{bmatrix} \tau_1 \\ \tau_1 \end{bmatrix} = \underbrace{\underset{2\times4}{\begin{bmatrix} l_1^2\ddot{\theta}_1 + l_1 g c_1 & d_2^2\ddot{\theta}_1 + 2d_2\dot{\theta}_1\dot{d}_2 + d_2 g c_2 & \ddot{\theta}_1 & \ddot{\theta}_1 \\ 0 & \ddot{d}_2 - d_2\dot{\theta}_1^2 + g s_1 & 0 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} m_1 \\ m_2 \\ I_{zz_1} \\ I_{zz_2} \end{bmatrix}}}_{2\times1}$$

$$\tau = Y \quad \varphi$$

$$\tau = Y \cdot \varphi$$