A foray into knot theory: the Alexander polynomial

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What is a knot?

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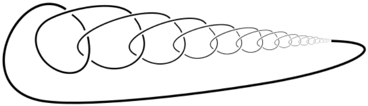

Figure: No wild knots!

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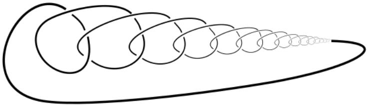

Figure: No wild knots!

Links: Embedding several circles.
Higher dimensional knots: $S^n \hookrightarrow \mathbb{R}^{n+2}$.

Intuitively. Two knots are the same if:

- stretching, bending, moving around
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Definition

Two knots K_1 and K_2 are **equivalent**, $K_1 \sim K_2$, if they are *ambient isotopic*:

- a homotopy of orientation-preserving homeomorphisms $H_t : \mathbb{R}^3 \to \mathbb{R}^3$
- $H_0 = id$, $H_1(K_1) = K_2$.

Note: Isotopy doesn't work

Knot diagrams

Given $K \subset \mathbb{R}^3$, consider a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$.

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Definition

A **knot diagram** of K is a regular projection of K together with height information at each double point.

Knot diagrams

Open question

What are the equivalence classes of knots in $\mathbb{R}^3$?

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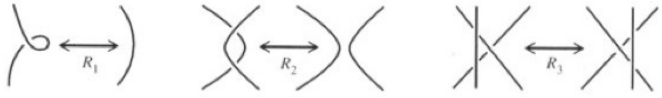
Reidemeister's moves on diagrams:



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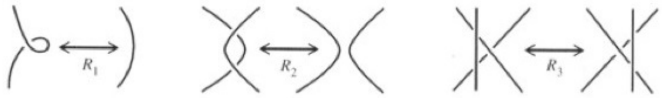

Theorem (Reidemeister)

$\{knots\}/a.i.\ in\ \mathbb{R}^3 = \{knot\ diagrams\}/R_1, R_2, R_3, a.i.\ in\ \mathbb{R}^2$

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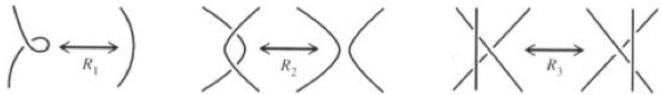

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Definition

A **knot invariant** is a function $F : \{\text{knots}\} \to A$ (some nice space) which has the same value on equivalent knots.

Hence, if $F(K_1) \neq F(K_2)$ we know $K_1 \not\sim K_2$.

Basic Facts

The Alexander polynomial is

1. discovered by James Alexander (1928)
2. a knot invariant of oriented knots
3. $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$

I. The original construction

1. Take an oriented diagram D for a knot K and number the crossings $1, \ldots, n$, the regions $1, \ldots, n+2$.
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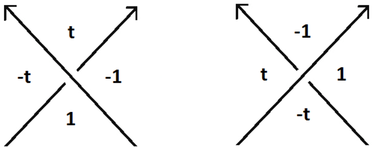

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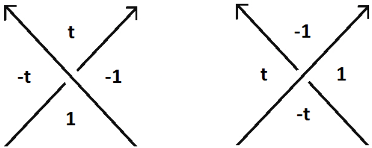


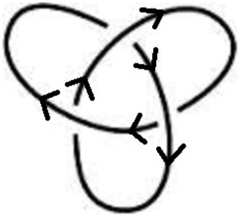
3. New matrix: $\widetilde{M} = M$ with any two columns of adjacent regions deleted.
4. $\Delta_K(t) = \det(\widetilde{M})$

I. Example

Remarks:

- Answer depends on deleted columns (unique up to a factor of $\pm t^k$, $k \in \mathbb{Z}$).
- Comes from the abelianization of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

Let K be the left-hand trefoil knot.



$$\Delta_K(t) = -t^3 + t^2 - t = -t^2(t - 1 + t^{-1})$$

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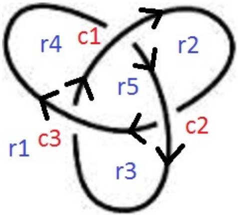

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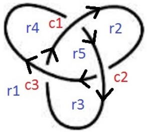

	r1	r2	r3	r4	r5
c1					
c2					
c3					

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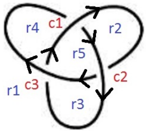

	r1	r2	r3	r4	r5
c1	-t	t	0	1	-1
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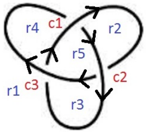

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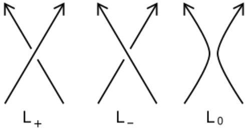
II. Skein relation

The Alexander polynomial for an oriented link L is $\Delta_L(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$ given by:

- $\Delta_{\text{unknot}}(t) = 1$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) - (t^{1/2} - t^{-1/2})\Delta_{L_0}(t) = 0$

where:



Remarks:

- If L is a knot, $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$
- No ambiguity of sign and factors of t^k
- This relation allows us to compute it for all knots since:

Proposition

A knot diagram can always be transformed into the unknot by changing a finite number of crossings.

II. Examples

Example 1. $\Delta_{\bigcirc\bigcirc}(t) = ?$

$$0 \quad = \quad \Delta_{L_+}(t) - \Delta_{L_-}(t) + (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$$

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Example 2. Let H denote the Hopf link. $\Delta_H(t) = ?$

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Hence, $\Delta_H(t) = t^{-1/2} - t^{1/2}$.

II. Examples

Example 3. Let K denote the left-hand trefoil knot. $\Delta_K =?$

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Hence, $\Delta_K(t) = t - 1 + t^{-1}$.

III. Seifert surfaces

Definition

A **Seifert surface** of a knot is an oriented surface whose boundary is the knot.

Theorem

Every knot has a Seifert surface (not unique!).

Definition

If L is an oriented link, its **linking number** $lk(M)$ is obtained by:

$$\frac{1}{2}\sum_{x\in X}(-1)^x$$

where X is the set of crossings between different components of the link in any diagram.

III. The Seifert Matrix

Setup:

- L- oriented link with n components, Σ - a Seifert surface for it with genus g
- $\{[f_i]\}_{i=1}^{2g+n-1}$ - a basis for $H_1(\Sigma, \mathbb{Z})$.
- f_i^+ - the *positive pushoff* of f_i (parallel, just above).

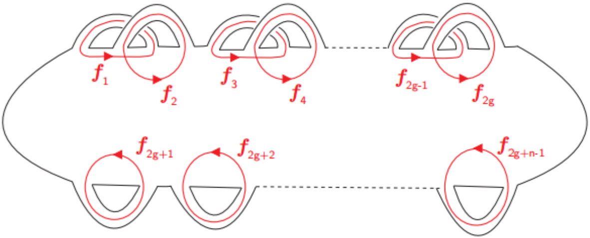


Figure: A standard basis for a genus g surface with n boundary components.

III. The Seifert Matrix

Definition

The **Seifert matrix** M of L is given by $M_{i,j} = lk(f_i, f_j^+)$.

Note: This depends on the surface and the basis.

Definition

The Alexander polynomial of a link L with Seifert matrix M is

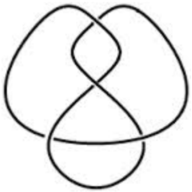
$$\Delta_L(t) = \det(M - tM^T)$$

Remarks:

1. This is defined up to $\pm t^k$.
2. Originates from Deck transformations of an infinite cyclic cover of $\mathbb{R}^3 \setminus L$.

III. Example

The figure eight knot $K_{4,1}$ has the following diagram with a corresponding Seifert surface:



The Seifert matrix is:

$$M = \begin{pmatrix} lk(a^+, a) & lk(a^+, b) \\ lk(b^+, a) & lk(b^+, b) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Then

$$\Delta_{K_{4,1}}(t) = \det(M - tM^T) = \det \begin{pmatrix} 1-t & 1 \\ -t & t-1 \end{pmatrix} = t(-t+3-t^{-1}).$$

Does it behave nicely?

1. Distinguishes all knots with eight or fewer crossings.
2. $\Delta_K(1) = \pm 1$ for any knot K.
3. Palindromic in t and t^{-1}: $\Delta_L(t) = \Delta_L(t^{-1})$ for any link L (up to a $\pm t^k$ factor).
4. One normalization: Require $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$.
5. Given such a polynomial, there is a knot whose Alexander polynomial is the same.
6. Multiplicative under connected sum: $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$.

How useful is it?

Doesn't distinguish:

1. Mirror images and reverses of knots.
2. The unknot.
3. Mutant knots:

Definition

To obtain a **mutant** of a knot, we rotate or reflect a disc intersecting its diagram in four points.

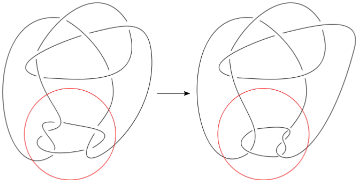
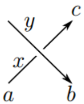


Figure: The Kinoshita-Terasaka and Conway knots.

The multivariable Alexander polynomial

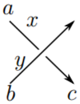
What about links?

1. Label all the arcs of an oriented link L. Label each crossing by the outgoing lower arc. Assign a variable to each link component.
2. Create a matrix with rows indexed by the crossings, columns – by the arcs, using the rule:



$\longmapsto$

	a	b	c
c	-1	$1-x$	y



$\longmapsto$

	a	b	c
c	$-y$	$x-1$	1

3. Compute the ingredients: $\mu(k)$ =the number of times the k-th link component is the over strand in a crossing
rot(k) =the rotation number of the k-th link component

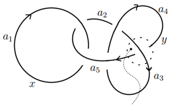
The escape path w_i = a path starting to the right of both outgoing strands at crossing i to the unbounded region. Multiply by x_j^{-1} is arc of j-th component crosses path left to right, x_j otherwise.

Definition (Torres '53)

The multivariable Alexander polynomial of is a link L:

$$\Delta(L) = \frac{(-1)^{i+j}\det(M_i^j)}{w_i(x_j - 1)} \prod_k x_k^{\frac{\mathrm{rot}(k)-\mu(k)}{2}}$$

Example:



For the link L above, $\Delta(L) = y^{-1}(1 - y + y^2)$

Tangles and homology

Definition

A **tangle** is an embedding of n arcs and m circles into $\mathbb{R}^2 \times [0,1]$.

Theorem (Archibald '06)

There is an oriented tangle invariant generalizing the Alexander polynomial, with values in $\Lambda^{top}(X^{out}) \otimes \Lambda^{1/2}(X^{in} \cup X^{out})$ for a tangle with incoming and outgoing strands X^{in} and X^{out} respectively.

Heegaard Floer homology

Theorem (Ozsvath-Szabo '03)

Heegaard Floer Homology is an invariant of closed 3-manifolds which also gives homological invariants of knots in the manifolds. It categorifies the Alexander polynomial which is equal to its Poincare polynomial $\sum_n dim(H_n)p^n$.

The End

Thank you!