# [Proposal] Implementation and Exploration of the Continuous Newton Method

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***Abstract*—This is the project proposal of EECS219A project.**

***Index Terms*—Adaptive step size control, Continuous Newton–Raphson method, Newton–Raphson methods, Nonlinear differential equations**

## I. Introduction

This project will implement continuous Newton-Raphson (NR) method and explore its behavior. The continuous NR method is a method to improve the vanilla NR method by treating the original problem as a differential equations problem and introducing adaptive step size control. The project will implement it in various systems, check its behavior, compare it with the vanilla NR method, explore the effect of several mechanisms and hacks on the continuous NR method. The project may also visualize the continuous NR method.

## II. Paper Review

The paper by Amrein[1] brought out a method to improve the vanilla Newton Raphson method. To solve the problem

$$x \in \Omega : \quad \mathbf{F}(x) = 0 \tag{1}$$

for suffiently small $t_n$, Newton-Raphson method is written in the following way

$$x_{n+1} = x_n - t_n \mathbf{F}'(x_n)^{-1}\mathbf{F}(x_n) \tag{2}$$

Define $\mathbf{N_F} = -\mathbf{F}'(x)^{-1}\mathbf{F}(x_n)$, the above problem can be considered as the discretization of the differential equation problem

$$\begin{cases} \dot{x}(t) = \mathbf{N_F}(x(t)), \quad t \geq 0 \\ x(0) = x_0 \end{cases} \tag{3}$$

Define the set of all points which belong to trajectories leading to $x_\infty$, the desired solution, as $\mathcal{A}(x_\infty)$, which is an attractor. If the iterations stay within the same attractor, the chaotic behavior of Newton method will be tamed. This also explains one reason of why vanilla Newton method fails: step size is too large that the iteration falls outside of the attractor $\mathcal{A}(x_\infty)$.

Consider the linearization at $t = 0, x(0) = x_0$, i.e.

$$\hat{x}(t) = x_0 + t\dot{x}(0) \tag{4}$$

The paper shows that

$$\begin{aligned} \hat{x}(t) - x(t) = \frac{1}{2}t^2 \ \|\mathbf{N_F}(x_0)\|_X \\ +O(t^3) + O(\|x(t) - x_0\|_X^2) \end{aligned} \tag{5}$$

Thence, fixing a tolerance $\tau > 0$ such that

$$\begin{aligned} \tau &= \|\hat{x}(t) - x(t)\|_X \\ &= \frac{t^2}{2} \ \|\mathbf{N_F}(x_0)\|_X + O(t^3) + O(\|x(t) - x_0\|_X^2) \end{aligned} \tag{6}$$

gives the following adaptive step size control for the Newton method:

1) Start the Newton iteration with an initial guess $x_0 \in \mathcal{A}(x_\infty)$.
2) In each iteration step $n = 0, 1, 2, \ldots$, compute

$$t_n = \min\left(\sqrt{\frac{2\tau}{\|\mathbf{N_F}(x_n)\|_X}}, 1\right) \tag{7}$$

3) Compute $x_{n+1}$ based on the Newton iteration, and go to the next step.

It shall be noted that tolerance $\tau$ is set manually a priori, and setting a too large $\tau$ will lead to failure. This also shows that this method lacks a correction strategy for the predicted step size.

## III. Objectives and Timeline

### *A. Paper review*

Time: 1 week or more
Review papers that already apply continuous NR method (in areas other than circuits). Check their implications.

### *B. Basic implementation*

Time: 1 week or less
Content: Implement the continuous NR method. Apply it to systems, including the ones in classes and homeworks.

### *C. Timestep control*

Time: not known. I may do it after other parts are finished and I have learnt more about this topic.
Content: Do as is mentioned on the class wiki: “Timestep control for transient (LTE and NR failure/iteration based), demonstrated on ‘difficult’ problems without using NR convergence aids.”

*D. Hacks check*

Time: more than 2 weeks
Content: Apply several mechanisms and hacks for NR method and check their effect on this new method. Examples: Weighting, RELTOL-ABSTOL criteria, INIT/LIMITING.

*E. Re-examine the paper*

Time: not known. Might be several weeks.
Content: The paper made some assumptions which may produce failures in real practice.

a) *Exponential convergence:* By claiming that $\lim_{t\to\infty} x(t) = x_\infty \in \Omega$ is well-defined with $\mathbf{F}(x_\infty) = 0$, the paper deduced that

$$\mathbf{F}(x(t)) = \mathbf{F}(x_0)e^{-t} \tag{8}$$

However, exponential convergence may not always be the case. For example, non-linear elements in circuits may violate this assumption. Furthermore, I expect that, which is what must happen in engineering practice, this mathematical method will produce some unexpected results. Fixing this will take most of the time.

b) *Correction strategy:* The paper also admitted that this method lacks a correction strategy for the predicted step size. This issue may be fixed by introducing other mechanisms proposed by other papers. I will check this.

*F. Visualization (selective)*

Comment: Not directly related to this course, but seems pretty interesting to me.
Time: (very likely) may not finish before this class ends, but I would like to continue working on it after that.
Content: Devise the visualization of applying different NR method with different plots, similar to the ones in Amrein's paper[1] and 3Blue1Brown's video[2].

Plots like Fig. 1, Fig. 2, Fig. 3 are powerful tools to help understand the behavior of Newton method. But those plots may work for lower-dimensional problems only. A more fascinating plot is the interactive one made by 3Blue1Brown [2], which will help new learners to understand the behavior of Newton method in a more intuitive way.

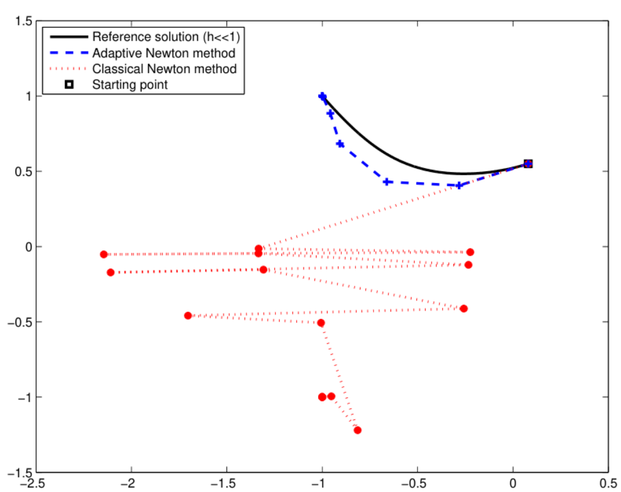


Fig. 1: A plot showing the procedures of Newton methods[1]

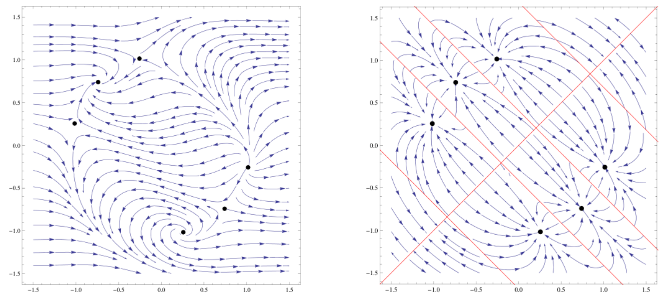

Fig. 2: A plot showing the field of Newton method[1]

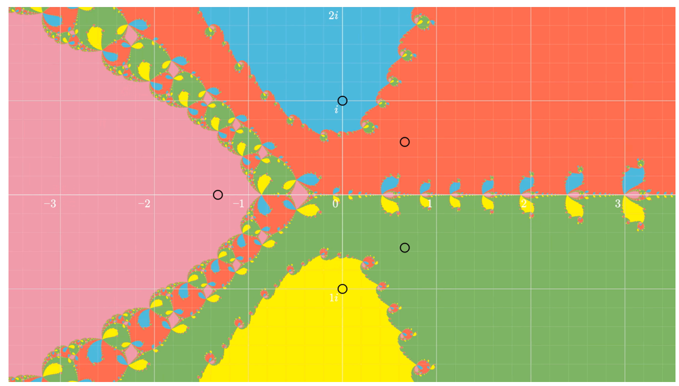

Fig. 3: A plot showing the attraction basins of Newton Method[2]

## IV. Expectations

*A. Deliverables*

1) A report on the implementation and exploration of the continuous Newton method. It will explaning key mechanisms, hacks and their effects, advantages and shortcomings (such as failures).
2) A runnable code implementing the continuous Newton method on several systems.
3) Visualization of the continuous Newton method.

*B. Possible challenges*

For implementations problems, I may refer to existing papers and engineering practices to find inspirations.
For pure theoretical problems which requires too much mathematical knowledge that is beyond my reach, I will try my best. But, if I cannot solve it, I will switch to other tasks in the project, possibly the SELECTIVE ones.

## References

[1] M. Amrein and T. P. Wihler, "An adaptive Newton-method based on a dynamical systems approach," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 9, pp. 2958–2973, Sep. 2014, doi: 10.1016/j.cnsns.2014.02.010.
[2] 3Blue1Brown, "Newton's Fractal (which Newton knew nothing about)." 2021.