Dynamic radioastronomic image reconstruction using Kalman filter under ionospheric perturbations

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Part I
Introduction

A Kalman filter is a linear quadratic estimator, using observations, state-transition model and knowledge about noise and errors to compute a more accurate result than observations alone. Radio astronomical measures being prone to noise and errors, usage of this filtering method seems *a priori* a good way to improve accuracy of reconstruction.
One objective of this paper is to evaluate if measure conditions and results are compatible with the filtering.

Part II
Notations and problem

1 Notations

1.1 General notations

$\mathrm{E}[\bullet]$ denotes the expected value of a random variable/vector/matrix.

Bold symbols represent vectors, normal ones represent scalars

1.2 Hadamard product

Hadamard product, denoted $\odot$, is the element-wise product of same sized matrices :

$$\forall (\mathrm{A},\mathrm{B}) \in \left(\mathbb{C}^{m\times n}\right)^2, \quad \mathrm{A}\odot\mathrm{B} \in \mathbb{C}^{m\times n} \quad \text{and} \quad \forall (i,j) \in [\![1;m]\!] \times [\![1;n]\!], (\mathrm{A}\odot\mathrm{B})_{i,j} = \mathrm{A}_{i,j}\times \mathrm{B}_{i,j}$$

We can further introduce Hadamard inversion :

$$\forall \mathrm{A} \in (\mathbb{C}\setminus\{0\})^{m\times n}, \quad \mathrm{A}^{\circ -1} \in (\mathbb{C}\setminus\{0\})^{m\times n} \quad \text{and} \quad \forall (i,j) \in [\![1;m]\!] \times [\![1;n]\!], \left(\mathrm{A}^{\circ -1}\right)_{i,j} = \left(\mathrm{A}_{i,j}\right)^{-1}$$

And notation $\oslash$ being Hadamard division :

$$\forall (\mathrm{A},\mathrm{B}) \in \mathbb{C}^{m\times n} \times (\mathbb{C}\setminus 0)^{m\times n}, \quad \mathrm{A}\oslash\mathrm{B} \in \mathbb{C}^{m\times n} \quad \text{and} \quad \forall (i,j) \in [\![1;m]\!] \times [\![1;n]\!], (\mathrm{A}\oslash\mathrm{B})_{i,j} = \frac{\mathrm{A}_{i,j}}{\mathrm{B}_{i,j}}$$

1.3 Norms and distances

In order to compute distances between matrices, we note $||\bullet||_{\mathrm{F}}$ Frobenius norm :

$$\mathrm{A} \in \mathscr{M}_{m,n}(\mathbb{K}) \quad ||\mathrm{A}||_{\mathrm{F}} := \sqrt{\mathrm{tr}\left\{\mathrm{A}\mathrm{A}^{\mathrm{H}}\right\}} = \sqrt{\sum_{\substack{1\le i\le m\\ 1\le j\le n}} \left|\mathrm{A}_{ij}\right|^2}$$

Inducing the distance d_1 :

$$\forall (\mathrm{A},\mathrm{B}) \in \left(\mathscr{M}_{m,n}(\mathbb{K})\right)^2 \quad d_1(\mathrm{A},\mathrm{B}) = \frac{||\mathrm{A}-\mathrm{B}||_{\mathrm{F}}}{mn}$$

In order to measure influence of a specific element, the ??? distance d_2 is the following :

$$\forall (\mathrm{A},\mathrm{B}) \in \left(\mathscr{M}_{m,n}(\mathbb{K})\right)^2 \quad d_2(\mathrm{A},\mathrm{B}) = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} \left|\mathrm{A}_{ij} - \mathrm{B}_{i,j}\right|$$

2 Evolution model and Kalman filtering

We consider a set $\{x_k\}$ of n vector as theoretical images taken at a regular time interval. Those images are measured by an antenna mesh which UV plane will be written H, more generally written as observation model. Measurements vectors will be designated by n measures $\{y_k\}$.
We will consider the following model to describe time evolution and observation equation :

$$\begin{cases} \boldsymbol{x}_k = \mathrm{A}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1} \\ \boldsymbol{y}_k = \mathrm{H}\boldsymbol{x}_k + \boldsymbol{v}_k \end{cases}$$

Where :

- A is the linear state-transition model between time $k-1$ and k, assuming evolution is time invariant

- $\{\boldsymbol{w}_k\}$ is the state noise sequence
- $\{\boldsymbol{v}_k\}$ is the measurements noise sequence

In order to build images dynamically with better precision, a Kalman filter has been implemented.

This filter requires more inputs to estimate the state of the system :

- $\{\mathbf{Q}_k\}$, the covariance of the process noise sequence
- $\{\mathbf{R}_k\}$, the covariance of the observation noise sequence

Which can be rewritten as

$$\begin{cases} \mathbf{Q}_k = \mathrm{E}\left[\boldsymbol{w}_{k-1}\boldsymbol{w}_{k-1}^{\mathrm{H}}\right] - \mathrm{E}\left[\boldsymbol{w}_{k-1}\right]\mathrm{E}\left[\boldsymbol{w}_{k-1}\right]^{\mathrm{H}} \\ \mathbf{R}_k = \mathrm{E}\left[\boldsymbol{v}_k\boldsymbol{v}_k^{\mathrm{H}}\right] - \mathrm{E}\left[\boldsymbol{v}_k\right]\mathrm{E}\left[\boldsymbol{v}_k\right]^{\mathrm{H}} \end{cases}$$

Those inputs are not *a priori* known, but it has been shown that using the *autocovariance least-squares* (ALS) method provides a good estimate of those matrices. [CITATION NEEDED]
We will assume that these inputs are known perfectly except when specified otherwise.

Kalman filtering works in 2 steps :

Prediction : This step predicts the state of the system at next time-step through de transition-state model, and the predicted measurement :

$$\begin{cases} \widehat{\boldsymbol{x}}_{k|k-1} = \mathrm{A}\widehat{\boldsymbol{x}}_{k-1|k-1} + \mathbf{m}_{\boldsymbol{w}_k} \\ \widehat{\boldsymbol{y}}_{k|k-1} = \mathrm{H}\widehat{\boldsymbol{x}}_{k|k-1} + \mathbf{m}_{\boldsymbol{v}_k} \end{cases}$$

And the covariance matrix :

$$\mathbf{P}_{k|k-1} = \mathrm{H}\mathbf{P}_{k-1|k-1}\mathrm{H}^{\mathrm{H}} + \mathbf{Q}_k$$

This result can be computed using least-square estimation of the prediction error covariance[1]

Update : Using prior knowledge about noise and observation model, the main goal

Part III
Reliability of Kalman Filter under perturbations

1 Perturbations sources

2 Effects

2.1 Position

2.2 Direction

Part IV
Experimental correction

1 implementation

2 Measures

References

[1] Eric Chaumette. *Introduction to Kalman Filtering and Extend Kalman Filtering*. Nov. 2018.