# Quizlet 4: Semirings

Kevin Liang — Ling 185A — Due: 09/06/2024, 11:59 PM PDT

Your name: Ricardo Varela Tellez

Total: 20 points

## 1 Semirings (14 points)

Remember that a semiring a set $R$ associated with two binary operations: "generalized or" ($\ovee$ operator) and "generalized and" ($\owedge$ operator) with a few additional restrictions for all $x, y, z \in R$.

(i) ($R$, $\ovee$) is a so-called "monoid" in that the following properties are true:

- $\boxed{\text{associative}}$: $x \overset{\text{or}}{\ovee} (y \overset{\text{or}}{\ovee} z) = (x \overset{\text{or}}{\ovee} y) \overset{\text{or}}{\ovee} z$
- $\boxed{\text{neutral element}}$: There is a particular element $\bot \in R$ such that $\bot \overset{\text{or}}{\ovee} x = x \overset{\text{or}}{\ovee} \bot = x$

(ii) ($R$, $\owedge$) is also a monoid in that the following properties are true:

- $\boxed{\text{associative}}$: $x \overset{\text{and}}{\owedge} (y \overset{\text{and}}{\owedge} z) = (x \overset{\text{and}}{\owedge} y) \overset{\text{and}}{\owedge} z$
- $\boxed{\text{neutral element}}$: There is a particular element $\top \in R$ such that $\top \overset{\text{and}}{\owedge} x = x \overset{\text{and}}{\owedge} \top = x$

(iii) The "generalized or" must be commutative: $a \ovee b = b \ovee a$

(iv) The neutral element for "generalized or"($\ovee$) must be the *annihilator* element for "generalized and".
$a \overset{\text{and}}{\owedge} \bot = \bot \overset{\text{and}}{\owedge} a = \bot$

(v) $\owedge$ must distribute $\ovee$

- $a \overset{\text{and}}{\owedge} (b \overset{\text{or}}{\ovee} c) = (a \owedge b) \ovee (a \owedge c)$
- $(b \ovee c) \overset{\text{and}}{\owedge} a = (b \owedge a) \ovee (c \owedge a)$

1. Show that $(\mathbb{R}_{\geq 0}, \times)$ is a monoid (i.e., show that $\times$ is associative and that there is some neutral element $\bot \in \mathbb{R}_{\geq 0}$ such that $\bot \times x = x \times \bot = x$ ). Note $\mathbb{R}_{\geq 0}$ indicates all real numbers that are greater or equal to 0. $\times$ is the multiplication operation. 2 points

2. Show that $(\mathbb{R}_{\geq 0}, +)$ is a monoid (i.e., show that $+$ is associative and that there is some neutral element $\top \in \mathbb{R}_{\geq 0}$ such that $\top + x = x + \top = x$ ). Note $+$ is the addition operation. 2 points

3. Show that $\mathbb{R}_{\geq 0}$ with $\times$ and $+$ is a semiring (i.e., properties (iii), (iv), and (v) are also true). 3 points

4. Choose one of the following and show that it is a semiring (i.e., show that properties (i) through (v) are true for the given set and operations): 7 points

   - $\mathscr{P}(\Sigma^*)$ with $\cdot$ and $\cup$ is a semiring. Specifically, $\mathscr{P}(\Sigma^*)$ is a powerset of $\Sigma^*$ [1], and $X \cdot Y = \{u + + v \mid u \in X, v \in Y\}$ ('++' is string concatenation).
   - $\mathbb{R}_{\geq 0}$ with $\times$ and `max` is a semiring.

[1] powerset is the set of all subsets

1. Show that ($\mathbb{R}_{\geq 0}$, $\times$) is a monoid (i.e., show that $\times$ is associative and that there is some neutral element $\perp \in \mathbb{R}_{\geq 0}$ such that $\perp \times x = x \times \perp = x$ ). Note $\mathbb{R}_{\geq 0}$ indicates all real numbers that are greater or equal to 0. $\times$ is the multiplication operation. 2 points

$\times$ (multiplication) is associative over $\mathbb{R} \geq 0$.
For all $a, b, c \in \mathbb{R} \geq 0$,

$$a \times (b \times c) = (a \times b) \times c$$

This is true by the associative property of multiplication.

Let $\perp$ (neutral number) equal 1.
For all $a \in \mathbb{R} \geq 0$,

$$1 \times a = a \times 1 = a$$

Hence, Neutral element is 1.

Therefore ($\mathbb{R} \geq 0$, $\times$) is a monoid.

2. Show that $(\mathbb{R}_{\geq 0}, +)$ is a monoid (i.e., show that + is associative and that there is some neutral element $\top \in \mathbb{R}_{\geq 0}$ such that $\top + x = x + \top = x$ ). Note + is the addition operation. 2 points

+ ( addition ) is associative over $\mathbb{R} \geq 0$.
for all $a, b, c \in \mathbb{R} \geq 0$,

$$a + (b + c) = (a + b) + c$$

This is true by the associative property of addition.

Let $\top$ (neutral number) equal 0.
for all $a \in \mathbb{R} \geq 0$,

$$0 + a = a + 0 = a$$

Hence, Neutral element is 0.

therefore $(\mathbb{R} \geq 0, +)$ is a monoid.

3. Show that $\mathbb{R}_{\geq 0}$ with $\times$ and $+$ is a semiring (i.e., properties (iii), (iv), and (v) are also true). 3 points

(iii) Since addition is commutative over $\mathbb{R}_{\geq 0}$,
for all $a, b \in \mathbb{R}_{\geq 0}$,

$$a + b = b + a \quad \checkmark$$

(iv) The neutral element for "generalized or" ($\oplus$), which we know is $0$, is the annihilator element for $\times$.
For all $a, b \in \mathbb{R}_{\geq 0}$,

$$a \times 0 = 0 \times a = 0 \quad \checkmark$$

This holds true because 0 times any number is 0.

(v) $\otimes$ ($\times$) must distribute $\oplus$ ($+$).
For all $a, b, c \in \mathbb{R}_{\geq 0}$,

- $a \times (b + c) = (a \times b) + (a \times c) \quad \checkmark$

- $(b + c) \times a = (b \times a) + (c \times a) \quad \checkmark$

The equalities are true because of the distributive property of multiplication over addition in $\mathbb{R}_{\geq 0}$.

Since, (iii), (iv), and (v) are also true, $\mathbb{R}_{\geq 0}$ with $\times$ and $+$ is a semiring.

4. Choose one of the following and show that it is a semiring (i.e., show that properties (i) through (v) are true for the given set and operations): 7 points

- $\mathscr{P}(\Sigma^*)$ with $\cdot$ and $\cup$ is a semiring. Specifically, $\mathscr{P}(\Sigma^*)$ is a powerset of $\Sigma^*$ [1], and $X \cdot Y = \{u++v \mid u \in X, v \in Y\}$ ('++' is string concatenation).
- $\mathbb{R}_{\geq 0}$ with $\times$ and max is a semiring. ←

(i) We know the following from question 1:

× (multiplication) is associative over $\mathbb{R} \geq 0$.
for all $a, b, c \in \mathbb{R} \geq 0$,

$$a \times (b \times c) = (a \times b) \times c$$

This is true by the associative property of multiplication.

Let 1 (neutral number) equal 1.
for all $a \in \mathbb{R} \geq 0$,

$$1 \times a = a \times 1 = a$$

Hence, Neutral element is 1.

therefore $(\mathbb{R} \geq 0, \times)$ is a monoid.

(ii) max is associative over $\mathbb{R} \geq 0$.
for all $a, b, c \in \mathbb{R} \geq 0$,

$$\max(a, \max(b, c)) = \max(\max(a, b), c) \checkmark$$

ex: let $a=1, b=2, c=3$

$$\max(1, \max(2, 3)) = \max(\max(1, 2), 3)$$

$$\max(1, 3) = \max(2, 3)$$

$$3 = 3 \checkmark$$

The order in which we use the max function will not affect the result. So max is associative.

Let T(neutral number) equal 0. (note we can't use negative real numbers)
For all $a \in \mathbb{R} \geq 0$,

$$\max(0, a) = \max(a, 0) = a$$

Hence, Neutral element is 0.

Therefore $(\mathbb{R} \geq 0, \max)$ is a monoid.

(iii) The max function is commutative as the order in which elements are compared won't change the result.
For all $a, b \in \mathbb{R}_{\geq 0}$,

$$\max(a,b) = \max(b,a) \checkmark$$

(iv) The neutral element for "generalized or" (Ⓥ), which we know is 0, is the annihilator element for ×.
For all $a, b \in \mathbb{R}_{\geq 0}$,

$$a \times 0 = 0 \times a = 0 \checkmark$$

This holds true because 0 times any number is 0.

(v) Ⓐ (×) must distribute Ⓥ (max).
For all $a, b, c \in \mathbb{R}_{\geq 0}$,

- $a \times \max(b,c) = \max(a \times b, a \times c)$
- $\max(b,c) \times a = \max(b \times a, c \times a)$

ex: let $a = 1, b = 2, c = 3$

- $1 \times \max(2,3) = \max(1 \times 2, 1 \times 3)$
  $1 \times 3 = \max(2, 3)$
  $3 = 3 \checkmark$
- $\max(2,3) \times 1 = \max(2 \times 1, 3 \times 1)$
  $3 \times 1 = \max(2,3)$
  $3 = 3 \checkmark$

This means multiplying a number by the max of two other numbers is equal to taking the max of the products of the first number and first other number, the first number and second other number.

Since properties (i) through (v) are true, $\mathbb{R}_{\geq 0}$ with $\times$ and max is a semiring.

## 2 Generalized inside values (6 points)

Following our generalized forward values and backward values for generalized FSAs, write out the generalized inside values for CFGs. Try to sketch out the outside values if you are keen.

generalized inside values for CFGs

a. $\text{inside}_G(x)(n) = R(n, x)$

b. $\text{inside}_G(x_1 \ldots x_m)(n) = \bigovee_{1 \le i \le m} \bigovee_{\ell \in N} \bigovee_{r \in N} \big[ R(n, \ell, r) \owedge \text{inside}_G(x_1 \ldots x_i)(\ell) \owedge \text{inside}_G(x_{i+1} \ldots x_m)(r) \big]$

generalized backward values for CFGS

a. $\text{outside}_G(\epsilon, \epsilon)(n) = I(n)$

b. $\text{outside}_G(y_1 \ldots y_m, z_1 \ldots z_q)(n) =$

$\bigovee_{0 \le i \le q} \bigovee_{p \in N} \bigovee_{r \in N} \big[ \text{outside}_G(y_1 \ldots y_m, z_{i+1} \ldots z_q)(p) \owedge R(p, n, r) \owedge \text{inside}_G(z_1 \ldots z_i)(r) \big]$

$\vee \bigovee_{0 \le i \le m} \bigovee_{p \in N} \bigovee_{\ell \in N} \big[ \text{outside}_G(y_1 \ldots y_i, z_1 \ldots z_q)(p) \owedge R(p, \ell, n) \owedge \text{inside}_G(y_{i+1} \ldots y_m)(\ell) \big]$