Unit-V: Numerical Methods for Partial Differential Equations

Course Learning Objectives:

Evaluate the approximate solutions of partial differential equations using numerical methods.

Introduction

Partial differential equations arise in the study of many branches of applied mathematics e.g. in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electro-magnetic theory. Only a few or these equations can be solved by analytical methods which are also complicated and require use of advanced mathematical techniques. In most of these, it is easier to obtain approximate solutions by numerical methods.

CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (1)$$

Such a partial differential equation is said to be

(i) elliptic if $B^2 - 4AC < 0$,

(ii) parabolic if $B^2 - 4AC = 0$,

(iii) hyperbolic if $B^2 - 4AC > 0$.

Solution of Partial Differential equations- Finite difference approximations to partial derivatives

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in fig 1. The points of intersection of the dividing lines are called mesh points or nodal points or grid points.

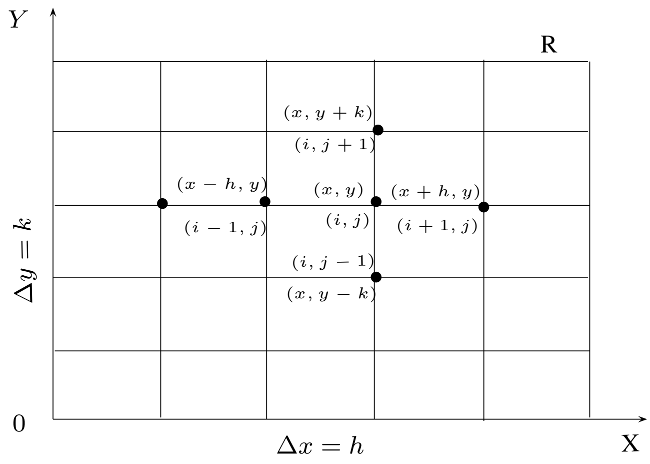

Figure 1

Then the finite difference approximations for the partial derivatives w.r.t x are

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{u(x+h,y)-u(x,y)}{h}+O(h) \quad \text{Forward difference approximation}\\
&= \frac{u(x,y)-u(x-h,y)}{h}+O(h) \quad \text{Backward difference approximation}\\
&= \frac{u(x+h,y)-u(x-h,y)}{2h}+O(h^2) \quad \text{Central difference approximation}\\
\text{and } \frac{\partial^2 u}{\partial x^2} &= \frac{u(x-h,y)-2u(x,y)+u(x+h,y)}{h^2}+O(h^2)
\end{aligned}$$

Writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$\begin{aligned}
u_x &= \frac{u_{i+1,j}-u_{i,j}}{h}+O(h) \quad \text{Forward difference approximation} && (2)\\
&= \frac{u_{i,j}-u_{i-1,j}}{h}+O(h) \quad \text{Backward difference approximation} && (3)\\
&= \frac{u_{i+1,j}-u_{i-1,j}}{2h}+O(h^2) \quad \text{Central difference approximation} && (4)\\
\text{and } u_{xx} &= \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+O(h^2) && (5)
\end{aligned}$$

Similarly we have the approximations for the derivative w.r.t y:

$$\begin{aligned}
u_y &= \frac{u_{i,j+1}-u_{i,j}}{k}+O(k) && (6)\\
&= \frac{u_{i,j}-u_{i,j-1}}{k}+O(k) && (7)\\
&= \frac{u_{i,j+1}-u_{i,j-1}}{2k}+O(k^2) && (8)\\
\text{and } u_{yy} &= \frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}+O(k^2) && (9)
\end{aligned}$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations given in formulae (2) to (9), we obtain the finite-difference analogues of the given equation.

Solution of Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{10}$$

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in figure 2 (assuming that an exact sub-division of R is possible). Replacing the derivatives in the equation (10) by their difference approximations, we have

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

$$or \qquad u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \tag{11}$$

This shows that the value of $u_{i,j}$ at any interior mesh point is the average of its values at

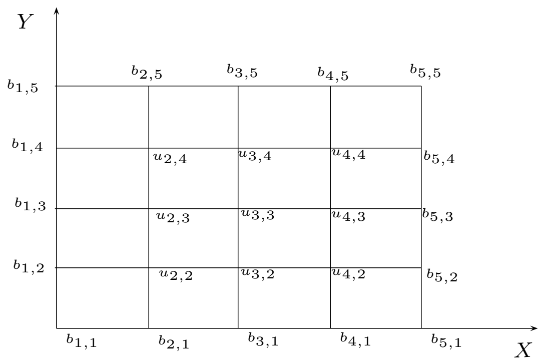

Figure 2

four neighbouring points to the left, right, above and below. The formula (11) is called the **standard** $5-$**point formula** which is exhibited in figure 3. Sometimes a formula similar to

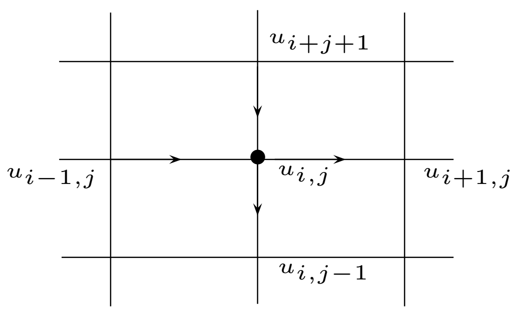

Figure 3

(11) is used which is given by

$$u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}] \tag{12}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points given in formula (12) is called the **diagonal** $5-$**point formula** which is given in figure 4. Although (12) is less accurate than (11), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

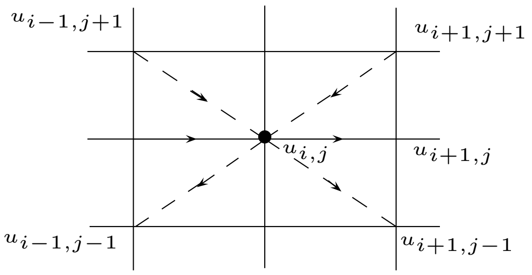

Figure 4

Example 1. *Given the values of $u(x,\, y)$ on the boundary of the sware in fig 5, evaluate the function $u(x,\, y)$ satisfying $u_{xx} + u_{yy} = 0$ at the pivotal points of this figure.*

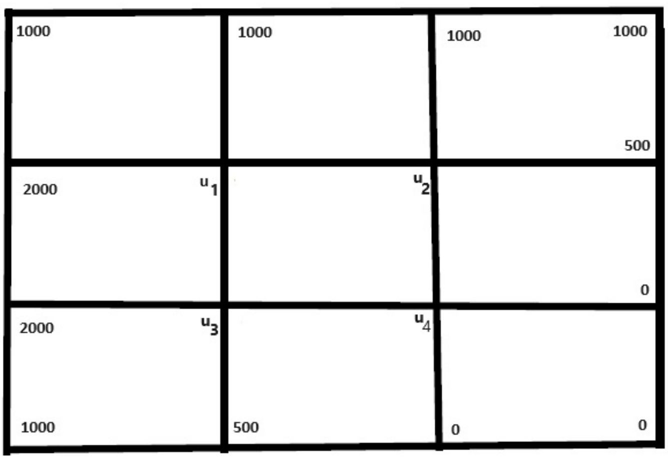

Figure 5

Solution To get the initial values of u_1, u_2, u_3, u_4, we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad (Diag.Formula) \tag{13}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad (Std.Formula) \tag{14}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad (Std.Formula) \tag{15}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad (Std.Formula) \tag{16}$$

We carry out the iteration process using the standard formulae:

$$u_1^{(n+1)} = \frac{1}{4}\left(2000 + u_2^{(n)} + 1000 + u_3^{(n)}\right) \tag{17}$$
$$u_2^{(n+1)} = \frac{1}{4}\left(u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}\right) \tag{18}$$
$$u_3^{(n+1)} = \frac{1}{4}\left(2000 + u_4^{(n)} + u_1^{(n+1)} + 500\right) \tag{19}$$
$$u_4^{(n+1)} = \frac{1}{4}\left(u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0\right) \tag{20}$$

First iteration: (put $n = 0$)

$$\begin{aligned}
u_1^{(1)} &= \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125 \\
u_2^{(1)} &= \frac{1}{4}(1125 + 500 + 1000 + 375) = 750 \\
u_4^{(1)} &= \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000 \\
u_5^{(1)} &= \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438
\end{aligned}$$

Second iteration: (put $n = 1$)

$$\begin{aligned}
u_1^{(2)} &= \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188 \\
u_2^{(2)} &= \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782 \\
u_3^{(2)} &= \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032 \\
u_4^{(2)} &= \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454
\end{aligned}$$

Third iteration: (put $n = 2$)

$$\begin{aligned}
u_1^{(3)} &= \frac{1}{4}(2000 + 782 + 1000 + 1032) \approx 1204 \\
u_2^{(3)} &= \frac{1}{4}(1204 + 500 + 1000 + 454) \approx 789 \\
u_3^{(3)} &= \frac{1}{4}(2000 + 454 + 1204 + 500) \approx 1040 \\
u_4^{(3)} &= \frac{1}{4}(1040 + 0 + 789 + 0) \approx 458
\end{aligned}$$

Similarly, $u_1^{(4)} \approx 1207, u_2^{(4)} \approx 791, u_3^{(4)} \approx 1041, u_4^{(4)} \approx 458$
$u_1^{(5)} \approx 1208, u_2^{(5)} \approx 791.5, u_3^{(5)} \approx 1041.5, u_4^{(5)} \approx 458.25$
Thus there is no significant difference between the fourth and fift iteration values.
Hence $u_1^{(4)} \approx 1208, u_2^{(4)} \approx 792, u_3^{(4)} \approx 1042, u_4^{(4)} \approx 458$

Example 2. *Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for the following square mesh with* *boundary values as shown in figure 6.*

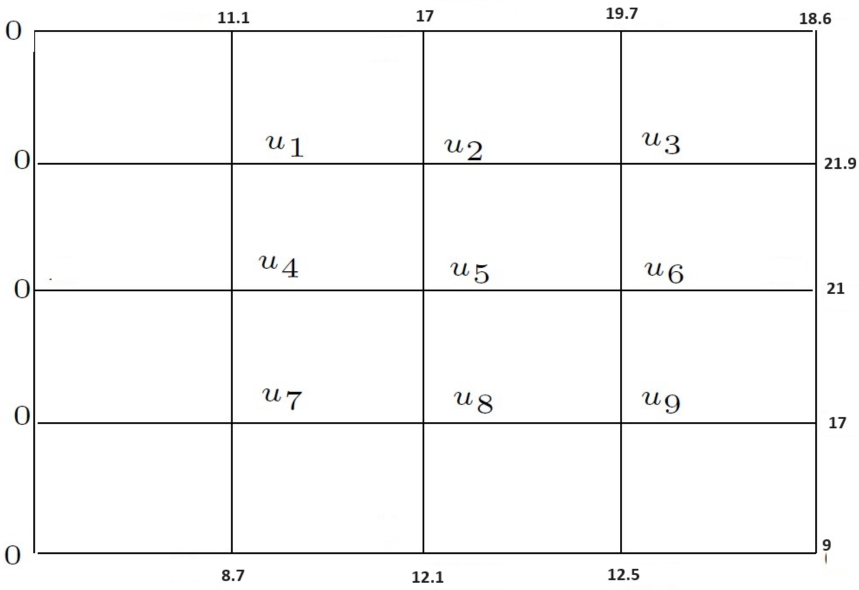

Figure 6

Solution We find the initial values in the follwoing order

$$
\begin{aligned}
u_5 &= \frac{1}{4}(0+17+21+12.1) = 12.5 \qquad (Std.Formula)\\
u_1 &= \frac{1}{4}(0+12.5+0+17) = 7.4 \qquad (Dig.Formula)\\
u_3 &= \frac{1}{4}(12.5+18.6+17+21) = 17.28 \qquad (Dig.Formula)\\
u_7 &= \frac{1}{4}(12.5+0+0+12.1) = 6.15 \qquad (Dig.Formula)\\
u_9 &= \frac{1}{4}(12.5+9+21+12.1) = 13.65 \qquad (Dig.Formula)\\
u_2 &= \frac{1}{4}(17+12.5+7.4+17.3) = 13.55 \qquad (Std.Formula)\\
u_4 &= \frac{1}{4}(7.4+6.2+0+12.5) = 6.52 \qquad (Std.Formula)\\
u_6 &= \frac{1}{4}(17.3+13.7+12.5+21) = 16.12 \qquad (Std.Formula)\\
u_8 &= \frac{1}{4}(12.5+12.1+6.2+13.7) = 11.12 \qquad (Std.Formula)
\end{aligned}
$$

We carry out the iteration process using the standard formulae:

$$\begin{aligned}
u_1^{(n+1)} &= \frac{1}{4}\left(0 + 11.1 + u_4^{(n)} + u_2^{(n)}\right)\\
u_2^{(n+1)} &= \frac{1}{4}\left(u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)}\right)\\
u_3^{(n+1)} &= \frac{1}{4}\left(u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9\right)\\
u_4^{(n+1)} &= \frac{1}{4}\left(u_1^{(n+1)} + 19.7 + u_7^{(n)} + u_5^{(n)}\right)\\
u_5^{(n+1)} &= \frac{1}{4}\left(u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)}\right)\\
u_6^{(n+1)} &= \frac{1}{4}\left(u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21\right)\\
u_7^{(n+1)} &= \frac{1}{4}\left(0 + u_4^{(n+1)} + 8.7 + u_8^{(n)}\right)\\
u_8^{(n+1)} &= \frac{1}{4}\left(u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)}\right)\\
u_9^{(n+1)} &= \frac{1}{4}\left(u_8^{(n+1)} + u_6^{(n)} + 12.8 + 17\right)
\end{aligned}$$

First iteration: (put $n = 0$)

$$u_1^{(1)} = 7.79, u_2^{(1)} = 13.64, u_3^{(1)} = 12.84, u_4^{(1)} = 6.61, u_5^{(1)} = 11.88$$
$$u_6^{(1)} = 16.09, u_7^{(1)} = 6.61, u_8^{(1)} = 11.06, u_9^{(1)} = 12.238$$

Second iteration: (put $n = 1$)

$$u_1^{(2)} = 7.84, u_2^{(2)} = 16.64, u_3^{(2)} = 17.83, u_4^{(2)} = 6.58, u_5^{(2)} = 11.84$$
$$u_6^{(2)} = 16.23, u_7^{(2)} = 6.58, u_8^{(2)} = 11.19, u_9^{(2)} = 14.30$$

Third iteration: (put $n = 2$)

$$u_1^{(3)} = 7.83, u_2^{(3)} = 13.637, u_3^{(3)} = 17.86, u_4^{(3)} = 6.56, u_5^{(3)} = 11.90$$
$$u_6^{(3)} = 16.27, u_7^{(3)} = 6.61, u_8^{(3)} = 11.23, u_9^{(3)} = 14.32.$$

Fourth iteration: (put $n = 3$)

$$u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94$$
$$u_6^{(4)} = 16.28, u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$$

Fifth iteration: (put $n = 4$)

$$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95$$
$$u_6^{(5)} = 16.29, u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$$

Example 3. *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in figure 7.*

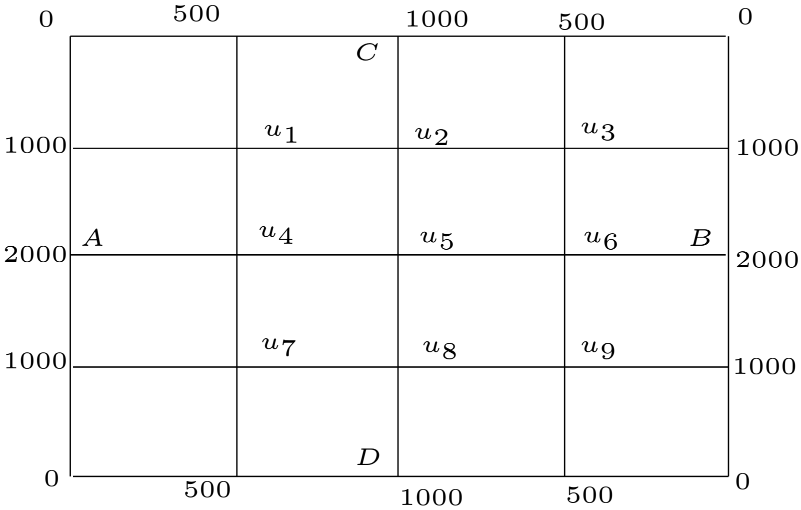

Figure 7

Solution Let $u_1, u_2, u_3, \ldots, u_9$ be the values of u at the interior mesh-points. Since the boundary value of u are symmetrical about AB,

$\therefore u_7 = u_1, u_8 = u_2, u_9 = u_3$.

Also the values of u being symmetrical about CD.

$u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4 and u_5.

Now we find their initial values in the following order:

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \qquad (Std.Formula) \tag{21}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \qquad (Diag.Formula) \tag{22}$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \qquad (Std.Formula) \tag{23}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \qquad (Std.Formula) \tag{24}$$

We carry out the iteration process using the formulae:

$$u_1^{(n+1)} = \frac{1}{4}\left(1000 + u_2^{(n)} + 500 + u_4^{(n)}\right) \tag{25}$$

$$u_2^{(n+1)} = \frac{1}{4}\left(2u_1^{(n+1)} + 1000 + u_5^{(n)}\right) \quad \because u_3 = u_1 \tag{26}$$

$$u_4^{(n+1)} = \frac{1}{4}\left(2000 + u_5^{(n)} + 2u_1^{(n+1)}\right) \quad \because u_7 = u_1 \tag{27}$$

$$u_5^{(n+1)} = \frac{1}{4}\left(2u_4^{(n+1)} + 2u_2^{(n+1)}\right) \quad \because u_2 = u_8,\ u_4 = u_6 \tag{28}$$

First iteration: (put $n = 0$)

$$\begin{aligned}
u_1^{(1)} &= \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032 \\
u_2^{(1)} &= \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164 \\
u_4^{(1)} &= \frac{1}{4}(2000 + 1500 + 1032 + 1125) = 1414 \\
u_5^{(1)} &= \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301
\end{aligned}$$

Second iteration: (put $n = 1$)

$$\begin{aligned}
u_1^{(2)} &= \frac{1}{4}(1000 + 1164 + 500 + 1414) = 1020 \\
u_2^{(2)} &= \frac{1}{4}(1020 + 1032 + 1000 + 1301) = 1088 \\
u_4^{(2)} &= \frac{1}{4}(2000 + 1301 + 1020 + 1032) = 1338 \\
u_5^{(2)} &= \frac{1}{4}(1338 + 1414 + 1088 + 1164) = 1251
\end{aligned}$$

Third iteration: (put $n = 2$)

$$\begin{aligned}
u_1^{(3)} &= \frac{1}{4}(1000 + 1088 + 500 + 1338) = 982 \\
u_2^{(3)} &= \frac{1}{4}(982 + 1020 + 1000 + 1251) = 1063 \\
u_4^{(3)} &= \frac{1}{4}(2000 + 1251 + 982 + 1020) = 1313 \\
u_5^{(3)} &= \frac{1}{4}(1313 + 1338 + 1063 + 1088) = 1201
\end{aligned}$$

Fourth iteration: (put $n = 3$)

$$\begin{aligned}
u_1^{(4)} &= \frac{1}{4}(1000 + 1063 + 500 + 1313) = 969 \\
u_2^{(4)} &= \frac{1}{4}(969 + 982 + 1000 + 1201) = 1038 \\
u_4^{(4)} &= \frac{1}{4}(2000 + 1201 + 969 + 982) = 1288 \\
u_5^{(4)} &= \frac{1}{4}(1288 + 1313 + 1038 + 1063) = 1176
\end{aligned}$$

Fifth iteration: (put $n = 4$)

$$\begin{aligned}
u_1^{(5)} &= \frac{1}{4}(1000 + 1038 + 500 + 1288) = 957 \\
u_2^{(5)} &= \frac{1}{4}(957 + 969 + 1000 + 1176) \approx 1026 \\
u_4^{(5)} &= \frac{1}{4}(2000 + 1176 + 957 + 969) \approx 1276 \\
u_5^{(5)} &= \frac{1}{4}(1276 + 1288 + 1026 + 1038) = 1157
\end{aligned}$$

Similarly,

$$u_1^{(6)} = 951, \quad u_2^{(6)} = 1016, \quad u_4^{(6)} = 1266, \quad u_5^{(6)} = 1146$$
$$u_1^{(7)} = 946, \quad u_2^{(7)} = 1011, \quad u_4^{(7)} = 1260, \quad u_5^{(7)} = 1138$$
$$u_1^{(8)} = 943, \quad u_2^{(8)} = 1007, \quad u_4^{(8)} = 1257, \quad u_5^{(8)} = 1134$$
$$u_1^{(9)} = 941, \quad u_2^{(9)} = 1005, \quad u_4^{(9)} = 1255, \quad u_5^{(9)} = 1131$$
$$u_1^{(10)} = 940, \quad u_2^{(10)} = 1003, \quad u_4^{(10)} = 1253, \quad u_5^{(10)} = 1129$$
$$u_1^{(11)} = 939, \quad u_2^{(11)} = 1002, \quad u_4^{(11)} = 1252, \quad u_5^{(11)} = 1128$$
$$u_1^{(12)} = 939, \quad u_2^{(12)} = 1001, \quad u_4^{(12)} = 1251, \quad u_5^{(12)} = 1126$$

Thus there is negligible difference between the values obtained in the 11^{th} and 12^{th} iterations.

Hence $u_1 = 939, u_2 = 1001, u_4 = 1251$ and $u_5 = 1126$.

Solution of one dimensional heat equation (Explicit method)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{29}$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm^2/sec)

Schmidt method

Consider a rectangular mesh in the $x - t$ plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u_{i,j+1} - u_{i,j}}{k} \\
\text{and} \quad \frac{\partial^2 u}{\partial x^2} &= \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}
\end{aligned}$$

substituting these in (29), we obtain

$$\begin{aligned} u_{i,j+1} - u_{i,j} &= \frac{kc^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] \\ \text{or} \quad u_{i,j+1} &= \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j} \end{aligned} \tag{30}$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j+1$ and j and is therefore, called a $2-$level formula. In schematic form the formula given in (30) is shown in figure 8 below. Hence (30) is called the Schmidt explicit formula which is valid only for $0 \leq \alpha \leq \frac{1}{2}$. For

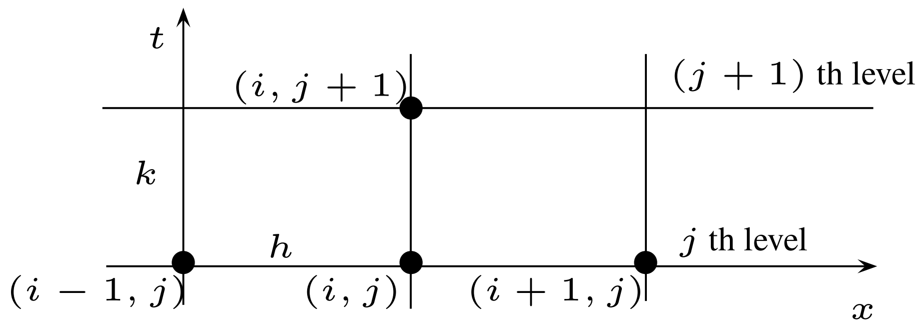

Figure 8

$\alpha > \frac{1}{2}$ the method is divergent and unstable.

Note: In particular when $\alpha = \frac{1}{2}$, the formula (30) reduces to

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \tag{31}$$

which shows that the values of u at x_i at time t_{j+1} is the mean of the u values at x_{i-1} and x_{i+1} at time t_j. This relation, known as **Bendre-Schmidt recurrence** relation, gives the values of u at the internal mesh points with the help of boundary conditions.

Example 4. *Find the values of $u(x,t)$ satisfying the parabolic equation $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the points $x = i$: $i = 0, 1, 2, \ldots, 8$ and $t = \frac{1}{8}j : j = 0, 1, 2, \ldots, 5$.*

Solution Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$. $\therefore$ We have Bendre-Schmidt's recurrence relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$$

Now since $u(0,t) = 0 = u(8,t)$ $\therefore$ $u_{0,j} = 0$ and $u_{8,j} = 0$ for all values of j, i.e. the entires in the first and last column are zero. Since

$$\begin{aligned} u(x,0) &= 4x-\frac{1}{2}x^2 \\ u_{i,0} &= 4i-\frac{1}{2}i^2 \\ &= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5, \text{ for } i=0,1,2,3,4,5,6,7 \text{ at } t=0 \end{aligned}$$

These are the entries of the first row. Putting $j=0$ in the formula, we have
$u_{i,1}=\frac{1}{2}(u_{i-1,1}+u_{i+1,0})$
Taking $i=1,2,\ldots,8$ successively, we get

$$\begin{aligned} u_{1,1} &= \frac{1}{2}(u_{0,0}+u_{2,0})=\frac{1}{2}(0+6)=3 \\ u_{2,1} &= \frac{1}{2}(u_{1,0}+u_{3,0})=\frac{1}{2}(3.5+7.5)=5.5 \\ u_{3,1} &= \frac{1}{2}(u_{2,0}+u_{4,0})=\frac{1}{2}(6+8)=7 \\ u_{4,1} &= 7.5,\ u_{5,1}=7,\ u_{6,1}=5.5,\ u_{7,1}=3. \end{aligned}$$

These are the entries in the second row. Putting $j=1$ in the formula, the entries of the third row are given by

$$u_{1,2}=\frac{1}{2}(u_{i-1,1}+u_{i+1,1})$$

Similarly putting $j=2,3,4$ successively in formula, the entries of the fourth, fifth and sixth rows are obtained. Hence the values of $u_{i,j}$ are as given in the following table:

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Example 5. *Solve the equation* $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ *subject to the conditions* $u(x,0)=\sin\pi x$, $0\le x\le 1$; $u(0,t)=u(1,t)=0$, *using Schmidt method. Carryout computations for two levels, taking* $h=1/3,\ k=1/36$.

Solution: Here $c^2=1,\ h=1/3,\ k=1/36$ so that $\alpha=kc^2/h^2=1/4$.
Also $u_{1,0}=\sin\pi/3=\sqrt{3}/2,\ u_{2,0}=\sin 2\pi/3=\sqrt{3}/2$ and all boundary values are zero as shown in figure 9.

x / t	0	1/3	2/3	1
0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0
1/36	0	0.6495	0.6495	0
2/36	0	0.4871	0.4871	0

From Schmidt recurrence relation, we have

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

becomes $\quad u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right]$

For $i = 1, 2, j = 0$:

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = \frac{1}{4}\left[0 + 2 \times \sqrt{3}/2 + \sqrt{3}/2\right] = 0.65$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = \frac{1}{4}\left[\sqrt{3}/2 + +2 \times \sqrt{3}/2 + 0\right] = 0.65$$

For $i = 1, 2, j = 1$:

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.49$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.49$$

Solution of one dimensional Wave equation (Explicit method)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{32}$$

subject to the initial conditions : $u = f(x), \frac{\partial u}{\partial t} = g(x), 0 \le x \le 1 \text{at}\, t = 0$ (33)

and the boundary conditions : $u(0,\, t) = \phi(t), u(1,\, t) = \psi(t)$ (34)

Consider a rectangular mesh in the $x - t$ plane spacing h along x direction and k along time t direction. Denoting a mesh point $(x,\, t) = (ih,\, jk)$ as simply $i,\, j$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,\,j} - 2u_{i,\,j} + u_{i+1,\,j}}{h^2} \tag{35}$$

and
$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,\,j-1} - 2u_{i,\,j} + u_{i,\,j+1}}{h^2} \tag{36}$$

Replacing the derivatives in the given equation (32) by their above approximations, we obtain

$$u_{i,\,j-1} - 2u_{i,\,j} + u_{i,\,j+1} = \frac{c^2k^2}{h^2}(u_{i-1,\,j} - 2u_{i,\,j} + u_{i+1,\,j})$$

or $u_{i,\,j+1} = 2(1-\alpha^2c^2)u_{i,j} + \alpha^2c^2(u_{i-1,\,j} + u_{i+1,\,j}) - u_{i,\,j-1}$ (37)

where $\alpha = k/h$

Now replacing the derivative in (33) by its central difference approximation, we get

$$\begin{aligned} \frac{u_{i,j+1} - u_{i,j-1}}{2k} &= \frac{\partial u}{\partial t} = g(x) \\ u_{i,j+1} &= u_{i,j-1} + 2kg(x) \text{ at } t = 0 \\ u_{i,1} &= u_{i,-1} + 2kg(x) \text{ for } j = 0 \quad (38) \\ \text{Also initial condition } u &= f(x) \text{ at } t = 0 \text{ becomes } u_{i,-1} = f(x) \quad (39) \\ \text{Combining (38) and (39), we have} \quad & u_{i,1} = f(x) + 2kg(x) \quad (40) \end{aligned}$$

Also the boundary condition (34) gives $u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$.

Hence the explicit form in (37) gives the values of $u_{i,j+1}$ at the $(j+1)$th level when the nodal values at $(j-1)$th and jth levels are known from equations (39) and (40) as shown in figure 9.

Note 1. The coefficient of $u_{i,j}$ in the formula (37) will vanish if $\alpha c = 1$ of $k = h/c$. Then the formula (37) reduces to the simple form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad (41)$$

Note 2. For $\alpha = 1/c$, the solution of (37) is stable and coincides with the solution of (32)
For $\alpha < 1/c$, the solution is stable but inaccurate.
For $\alpha > 1/c$, the solution is unstable.

Note 3. The formula (37) converges for $\alpha \leq 1$ i.e. $k \leq h$.

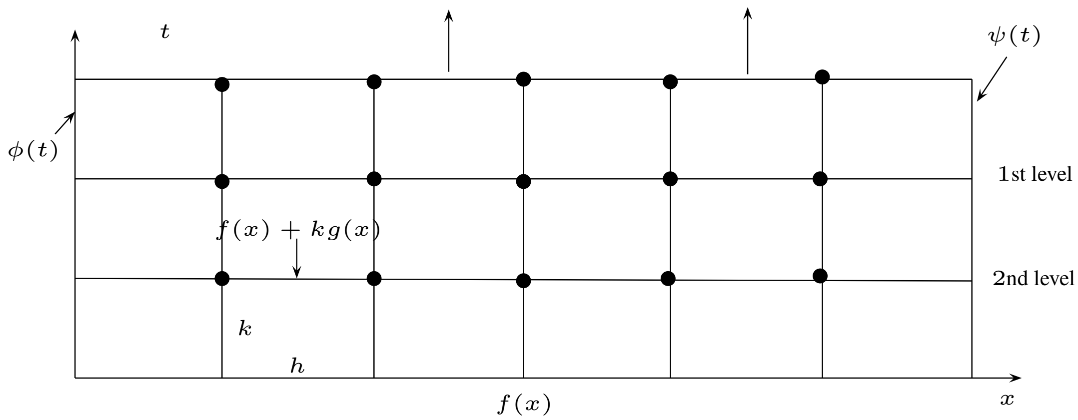

Figure 9

Example 6. *Evaluate the pivotal values of the equation* $u_{tt} = 16u_{xx}$, *taking* $\Delta x = 1$ *upto* $t = 1.25$. *The boundary conditions are* $u(0,t) = u(5,t) = 0$, $u_t(x,0) = 0$ *and* $u(x,0) = x^2(5-x)$.

Solution Here $c^2 = 16$

$\therefore$ The difference equation for the given equation is

$$u_{i,\,j+1} = 2(1-16\alpha^2)u_{i,j} + 16\alpha^2(u_{i-1,\,j} + u_{i+1,\,j}) - u_{i,\,j-1} \tag{42}$$

where $\alpha = k/h$.
Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, i.e. $k = h/4 = 1/4$.
$\therefore$ the formula in (42) reduces to

$$u_{i,\,j+1} = u_{i-1,\,j} + u_{i+1,\,j} - u_{i,\,j-1} \tag{43}$$

which gives a convergent solution (since $k/h < 1$). Its solution coincides with the solution of the given differential equation.
Now since $u(0,t) = u(5,t) = 0$, $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j
i.e. the entries in the first and last columns are zero.
Since $u_{x,0} = x^2(5-x)$
$\therefore u_{i,0} = i^2(5-i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.
These are the entries for the first row.
Finally since $u_{t(x,0)} = 0$ becomes

$$\therefore \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0, \text{when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \tag{44}$$

Thus the entries of the second row are the same as those of the first row.

Putting $j = 0$ in (43)

$$\begin{aligned} u_{i,1} &= u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}, \\ \text{or } u_{i,1} &= \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \end{aligned}$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$\begin{aligned} u_{1,1} &= \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0+12) = 6 \\ u_{2,1} &= \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4+18) = 11 \\ u_{3,1} &= \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12+16) = 14 \\ u_{4,1} &= \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18+0) = 9 \end{aligned}$$

These are the entries of the second row.
Putting $j = 1$ in (43), we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$\begin{aligned}
u_{1,2} &= u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7 \\
u_{2,2} &= u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8 \\
u_{3,2} &= u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2 \\
u_{4,2} &= = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2
\end{aligned}$$

These are the entries of the third row.
Similarly putting $j = 2, 3, 4$ successively in (43), the entries of the fourth, fifth and sixth rows are obtained.
Hence the values of $u_{i,j}$ are as shown in the table below:

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	-2	0
3	0	2	-2	-8	-7	0
4	0	-9	-14	-11	-6	0
5	0	-16	-18	-12	-4	0

Example 7. *The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$, with boundary conditions $u = 0$ at $x = 0, t > 0$ and $u = 0$ at $x = 4, t > 0$ and initial conditions $u = x(4 - x)$ and $\dfrac{\partial u}{\partial t} = 0,\ 0 \le x \le 4$. Solve this equation numerically for one half period of vibration, taking $h = 1$ and $k = 1/2$.*

Solution: Here $h/k = 2 = c$.
$\therefore$ The difference equation for the given equation is

$$u_{i,\,j+1} = u_{i-1,j} + u_{i+1,\,j} - u_{i,\,j-1} \tag{45}$$

which gives a convergent solution ($k < h$).
Now since $u_{0,t} = u_{4,t} = 0$,
$\therefore u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of j. i.e. the entries in the first and last columns are zero.
Since $u_{x,0} = x(4 - x)$,

$$\therefore u_{i,0} = i(4 - i) = 3, 4, 3 \text{ for } i = 1, 2, 3 \text{ at } t = 0.$$

These are the entries of the first row.

Also $u_{t(x,0)} = 0$ becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \tag{46}$$

Putting $j = 0$ in (45)

$$\begin{aligned} u_{i,1} &= u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1} \text{ (46)}, \\ \text{or } u_{i,1} &= \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \end{aligned} \tag{47}$$

Taking $i = 1, 2, 3$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{1,0}) = 2; u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3; u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 2$$

These are the entries of the 2^{nd} row.
Putting $j = 1$ in (45), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$
Taking $i = 1, 2, 3$, successively, we get

$$\begin{aligned} u_{1,2} &= u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0 \\ u_{2,2} &= u_{1,1} + u_{2,1} - u_{2,0} = 2 + 2 - 4 = 0 \\ u_{3,2} &= u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0 \end{aligned}$$

These are the entries if the 3^{rd} row and so on.
Now the equation of the vibrating string of length l is $u_{tt} = c^2 u_{xx}$.
$\therefore$ Its period of vibration $= \frac{2l}{c} = \frac{2\times4}{2} = 4$sec. $[\because l = 4$ and $c = 2]$ This shows that we have to compute $u_{(x,t)}$ upto $t = 2$
i.e. similarly we obtain the values of $u_{i,2}(4^{th})$ row and $u_{i,3}(5^{th}$ row).
Hence the values of $u_{i,j}$ are as shown in the table below:

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	-3	-2	0
4	0	-3	-4	-3	0