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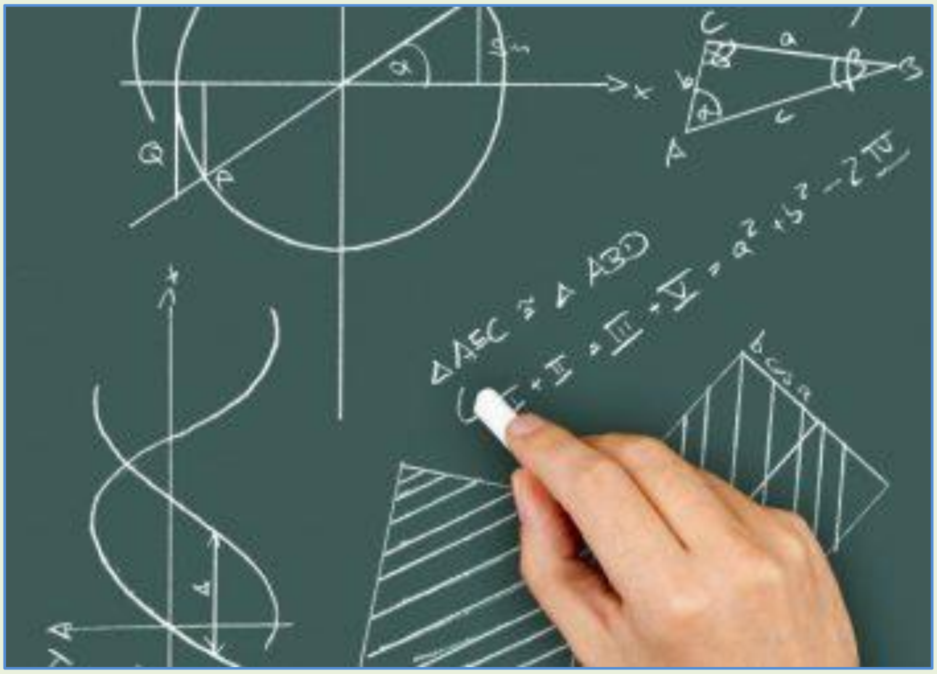
# HANDBOOK OF MATHEMATICS
# FOR
# SECOND YEAR B.E. PROGRAM



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# HANDBOOK OF MATHEMATICS

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# SECOND YEAR B.E. PROGRAM

# Contents

# TRIGONOMETRY

## Basic Functions

- $sin\,\theta = \frac{Opposite\ Side}{Hypotenuse}$
- $cos\,\theta = \frac{Adjacent\ Side}{Hypotenuse}$
- $tan\,\theta = \frac{Opposite\ Side}{Adjacent\ Side} = \frac{\sin\theta}{\cos\theta}$
- $sec\,\theta = \frac{Hypotenuse}{Adjacent\ Side} = \frac{1}{\cos\theta}$
- $cosec\,\theta = \frac{Hypotenuse}{Opposite\ Side} = \frac{1}{\sin\theta}$
- $cot\,\theta = \frac{Adjacent\ Side}{Opposite\ Side} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

## Identities

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$

- $\sin(\pi - x) = \sin x$
- $\cos(\pi - x) = -\cos x$
- $\tan(\pi - x) = -\tan x$
- $\sin(\pi + x) = -\sin x$
- $\cos(\pi + x) = -\cos x$
- $\tan(\pi + x) = \tan x$
- $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
- $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$
- $\tan\left(\frac{3\pi}{2} - x\right) = \cot x$

- $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$
- $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$
- $\tan\left(\frac{3\pi}{2} + x\right) = -\cot x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
  $= 2\cos^2 x - 1$
- $\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$
- $\cos^2 x + \sin^2 x = 1$
- $\sec^2 x - \tan^2 x = 1$
- $\mathrm{cosec}^2\, x - \cot^2 x = 1$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$

- $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
- $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

# BASIC CALCULUS

## Differentiation

| $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ |
|---|---|---|---|
| $a$ | $0$ | $a^x$ | $a^x \log_e a$ |
| $x^n, n \neq -1$ | $nx^{n-1}$ | $e^{ax}$ | $ae^{ax}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ | $\log_e x$ | $\frac{1}{x}$ |
| $\sin x$ | $\cos x$ | $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos x$ | $-\sin x$ | $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan x$ | $\sec^2 x$ | $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\operatorname{cosec} x$ | $-\cot x \operatorname{cosec} x$ | $\operatorname{cosec}^{-1} x$ | $-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| $\sec x$ | $\tan x \sec x$ | $\sec^{-1} x$ | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ | $\cot^{-1} x$ | $-\frac{1}{1+x^2}$ |
| $\sinh x = \frac{e^x - e^{-x}}{2}$ | $\cosh x$ | $\sinh^{-1} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $\cosh x = \frac{e^x+e^{-x}}{2}$ | $\sinh x$ | $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ | $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\operatorname{cosech} x$ | $-\coth x \operatorname{cosech} x$ | $\operatorname{cosech}^{-1} x$ | $-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$ |
| $\operatorname{sech} x$ | $-\tanh x \operatorname{sech} x$ | $\operatorname{sech}^{-1} x$ | $-\frac{1}{\lvert x\rvert\sqrt{1-x^2}}$ |
| $\coth x$ | $-\operatorname{cosech}^2 x$ | $\coth^{-1} x$ | $\frac{1}{1-x^2}$ |

## Rules of differentiation

- $\frac{d}{dx}(fg) = gf' + fg'$
- $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf'-fg'}{g^2}$
- $\frac{d}{dx}\big(f(t)\big) = \frac{d}{dt}\big(f(t)\big)\frac{dt}{dx}$

## Integration

| $f(x)$ | $\int f(x)dx$ | $f(x)$ | $\int f(x)dx$ |
|---|---|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1}$ | $\frac{1}{x}$ | $\log_e x$ |
| $e^{ax}$ | $\frac{e^{ax}}{a}$ | $\log_e x$ | $x(\log_e x - 1)$ |
| $a^x$ | $\frac{a^x}{\log_e a}$ | $\operatorname{cosec} x$ | $\log_e(\operatorname{cosec} x - \cot x)$ |
| $\sin x$ | $-\cos x$ | $\sec x$ | $\log_e(\sec x + \tan x)$ |
| $\cos x$ | $\sin x$ | $\cot x$ | $\log_e \sin x$ |
| $\tan x$ | $\log_e \sec x$ | $\sec^2 x$ | $\tan x$ |
| $\sinh x$ | $\cosh x$ | $\operatorname{cosec}^2 x$ | $-\cot x$ |
| $\cosh x$ | $\sinh x$ | $\tanh x$ | $\log_e \cosh x$ |
| $\frac{1}{\sqrt{a^2 - x^2}}$ | $sin^{-1}\left(\frac{x}{a}\right)$ | $\frac{1}{\sqrt{a^2 + x^2}}$ | $sinh^{-1}\left(\frac{x}{a}\right)$ |
| $\frac{1}{a^2 + x^2}$ | $\frac{1}{a} tan^{-1}\left(\frac{x}{a}\right)$ | $\frac{1}{\sqrt{x^2 - a^2}}$ | $cosh^{-1}\left(\frac{x}{a}\right)$ |
| $\frac{1}{a^2 - x^2}$ | $\frac{1}{2a}\log_e\left(\frac{a+x}{a-x}\right)$ | $\frac{1}{x^2 - a^2}$ | $\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)$ |
| $\sqrt{a^2 - x^2}$ | $\frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 sin^{-1}\left(\frac{x}{a}\right)\right]$ | $e^{ax}\sin bx$ | $\frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$ |
| $u(x)v(x)$ | $u\int v\,dx - \int\left[\frac{du}{dx}\left[\int v\,dx\right]dx\right]$ | $e^{ax}\cos bx$ | $\frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$ |

# COMPLEX ANALYSIS

**Algebra of complex numbers:**

- If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$ is non-negative real number.
- $z = x + iy$ is represented by a point $P(x, y)$ in the $XY$ plane, $x - axis$ is real axis, $y - axis$ is imaginary axis, plane is complex plane.
- $z = re^{i\theta}$ is the polar form of complex number $z$ where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$.
- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the distance between the points $z_1$ and $z_2$ in complex plane.
- $|z - z_0| = R$ represents complex equation of circle with centre $z_0$ and radius $R$.
- $|z - z_0| < R$ represents the region with in, but not on, a circle of radius $R$ centred at the point $z_0$, the point $z_0$ is said to be interior point.
- $|z - z_0| \leq R$ represents the region with in, and on, a circle of radius $R$ centred at the point $z_0$.
- $|z - z_0| > R$ represents the region outside the circle with centre $z_0$ and radius $R$.

**Cauchy-Riemann (C-R) equations:**

- In Cartesian form: If $f(z) = u(x, y) + i\, v(x, y)$ is differentiable at $z = x + iy$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- In polar form: If $f(z) = u(r, \theta) + i\, v(r, \theta)$, then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

**Harmonic Functions:** A function $\phi$ is said to be a harmonic function if it satisfies Laplace equation $\nabla^2\phi = 0$.

- In Cartesian form $\phi(x, y)$ is harmonic if $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.
- In polar form $\phi(r, \theta)$ is harmonic if $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$.

**Taylor theorem:** Taylor series expansion for the function $f(z)$ about the point $z = a$ is

$$f(z) = f(a) + (z - a)f'(a) + (z - a)^2\frac{f''(a)}{2!} + \cdots$$

**Maclaurin theorem:** Maclaurin series for the function $f(z)$ about the point $z = 0$ is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + \cdots$$

**Binomial Expansion:** If $|x| < 1$, then

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

**Laurent's theorem:** Let $c_1$and $c_2$ are concentric circles cantered at "$a$", then Laurent's series of $f(z)$ about the point $z = a$,

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z - a)^n},$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,dw$ $\quad b_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{-n+1}}\,dw$ for $n = 0, 1, 2 \ldots$

**Determination of poles:** If $f(z) = \frac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic and not zero at the point '$a$', then '$a$' is a pole of order $m$ of $f(z)$. The poles of $f(z)$ may be obtained by solving the equation $\frac{1}{f(z)} = 0$.

**Residue:** The coefficient of $\frac{1}{z-a}$ in the Laurent's expansion of $f(z)$ is the Residue of $f(z)$ at the pole $z = a$.

**Determination of a residue:** If '$a$' is a pole of order m $\geq 1$ of $f(z)$, then residue of $f(z)$ at '$a$' is $\frac{1}{(m-1)!}\left[\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\right]$.

**Cauchy's residue theorem:** Let $C$ be a simple closed curve and $f(z)$ be analytic within and on $C$ except at a finite number of poles $a_1, a_2, \ldots, a_n$ which lie inside $C$ , then

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots + R_n)$$

where $R_1, R_2 \ldots R_n$ are the residues of $f(z)$ at $a_1, a_2, \ldots, a_n$ respectively.

## PARTIAL DIFFERENTIAL EQUATIONS

**Lagrange's linear equation:** The first order linear partial differential equation of the form $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R$, where $P$, $Q$ and $R$ are functions of $x, y, z$ is known as Lagrange's Linear equation.

**Subsidiary/Auxiliary equation:** The equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is known as the subsidiary/ auxiliary equation of as Lagrange's Linear equation $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R$.

**One-dimensional wave equation:** $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$ the phase speed, $T$ is the tension, and $\rho$ density of the string.

**One-dimensional heat equation:** $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{\kappa}{s\rho}$ the thermal diffusivity, $\kappa$ thermal conductivity, s specific heat and ρ density of the material of the body.

**Two-dimensional Laplace equation:** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

## NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

**Laplace equation:** $u_{xx} + u_{yy} = 0$

- Standard 5- point formula: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$
- Diagonal 5- point formula: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]$.

**One-dimensional heat equation:** $u_t = c^2 u_{xx}$

- Schmidt formula: $u_{i,j+1} = \alpha\left(u_{i-1,j} + u_{i+1,j}\right) + (1-2\alpha)u_{i,j}$ where $\alpha = \frac{kc^2}{h^2}$.

- Bendre-Schmidt relation: $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$ when $\alpha = \frac{1}{2}$.

**One-dimensional wave equation:** $u_{tt} = c^2 u_{xx}$

- Explicit formula: $u_{i,j+1} = \beta^2\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1-\beta^2)\, u_{i,j} - u_{i,j-1}$, where $\beta^2 = \frac{c^2k^2}{h^2}$.
- For $\beta = 1$ and $k = \frac{h}{c}$, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$.

The above scheme is used with standard initial and boundary conditions**.**

## CALCULUS OF VARIATION

**Euler's equation:** A necessary condition for the functional $I = \int_{x_1}^{x_2} f(x,y,y')\, dx$ to be an extremum is that $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

**Alternate forms:**

- $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$
- $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\,\frac{\partial f}{\partial y'}\right) = 0$
- $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial (y')^2} y'' = 0$

**Cases of Euler's Equation:**

- If $f$ is only the function of $y'$, then Euler's equation will be**:** $\frac{df}{dy'} = c$ where $c$ is an arbitrary constant.
- If $f$ is independent of $y$, then Euler's equation will be**:** $\frac{\partial f}{\partial y'} = c$ where $c$ is an arbitrary constant.
- If $f$ is independent of $y'$ then Euler's equation will be**:** $\frac{\partial f}{\partial y} = 0$.
- If $f$ is independent of $x$ and $y$ then Euler's equation will be**:** $y''\,\frac{\partial^2 f}{\partial {y'}^2} = 0$.
- If $f$ does not contain $x$ then Euler's equation will be**:** $f - y'\,\frac{\partial f}{\partial y'} = c$ where $c$ is an arbitrary constant.

**Cartesian coordinate system:** $u_1 = x, u_2 = y, u_3 = z$

Element of arc length $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

**Cylindrical coordinate system:** $u_1 = r, u_2 = \theta\,, u_3 = z$

Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + (dz)^2}$

**Spherical coordinate system:** $u_1 = r\,, u_2 = \theta$ and $u_3 = \phi$.

Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2}$.

# LAPLACE TRANSFORM

**Gamma function:**

- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, (n > 0)$
- $\Gamma(1) = 1$
- $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$
- $\Gamma(n+1) = \begin{cases} n\Gamma(n), & n > 0 \\ n!, & n \text{ postive integer} \end{cases}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n) = \frac{\Gamma(n+1)}{n}, (n < 0, \neq -1, -2, \dots)$

**Beta Function:**

- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$
- $\beta(m,n) = \beta(n,m)$
- $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$
- $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
- $\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$

**Laplace transform of $f(t)$:** $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

**Transform of elementary functions:**

- $L(e^{at}) = \frac{1}{s-a}, \quad s > a$
- $L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}, s > |a|$
- $L(\sin at) = \frac{a}{s^2+a^2}, \quad s > 0$
- $L(\cosh at) = \frac{s}{s^2-a^2}, \quad s > |a|$
- $L(\cos at) = \frac{s}{s^2+a^2}, \quad s > 0$
- $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$
- $L[H(t-a)] = \frac{e^{-as}}{s}$, where $H$ is Heaviside unit step function

**Properties of Laplace transform:**

- $L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$.
- If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, where $a$ is a positive constant.
- Let a be any real constant then $L[e^{at} f(t)] = F(s-a)$
- If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, 3, \dots$
- If $L[f(t)] = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds$.
- If $L[f(t)] = F(s)$, then $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$
- If $L[f(t)] = F(s)$, then $L\int_0^t f(t) dt = \frac{1}{s} F(s)$
- If $f(t)$ is a periodic function of period $T$, then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$.
- If $L\{f(t)\} = F(s)$, then $L[f(t-a) H(t-a)] = e^{-as} F(s)$
- If $f(t)$ is a continuous function at $t = a$, then $\int_0^\infty f(t)\delta(t-a) dt = f(a)$, where $\delta(t-a)$ is unit impulse function.

**Inverse Laplace transform of $F(s)$ using Convolution theorem:** If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, then $L^{-1}[F(s)G(s)] = \int_0^t f(u) g(t-u) du = f(t) * g(t)$.

# FOURIER SERIES

**Fourier series of $f(x)$ in the interval $(a, a+2l)$:**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where Fourier coefficients $a_0, a_n, b_n$ are given by

$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx,$ $a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$ $n = 1, 2, 3, \dots$ and
$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx,$ $n = 1, 2, 3, \dots$

**Complex Fourier Series of $f(x)$ in the interval $(a, a+2l)$:**

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$, where $c_n = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$ , $n = 0, \pm 1, \pm 2, \pm 3, \dots$

**Relation between Fourier and complex Fourier coefficients:**

$a_0 = 2c_0,\ a_n = c_n + c_{-n},\ \ b_n = i(c_n - c_{-n})$

**Half-range Fourier series:**

- Sine series of $f(x)$ in $(0, l)$: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where
  $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$
- Cosine series of $f(x)$ in $(0, l)$: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$, where
  $a_0 = \frac{2}{l}\int_0^l f(x)dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

**Harmonic Analysis:** Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \dots, y_m$ corresponding to a given set of equi-spaced values $x_0, x_1, x_2, \dots, x_m$ in $(a, a + 2l)$, Fourier series of $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where $a_0$, $a_n$ and $b_n$ are computed using the formulae:

$$a_0 = 2\frac{\sum y}{m}, \quad a_n = 2\frac{\sum y\cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2\frac{\sum y\sin\left(\frac{n\pi x}{l}\right)}{m}.$$

- Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \dots, y_m$ corresponding to a given set of equi spaced values $x_0, x_1, x_2, \dots, x_m$ in $(0, l)$, Half range cosine Fourier series and Half range sine Fourier series of $f(x)$ are respectively:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \cdots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots,$$

where $a_0$, $a_n$ and $b_n$ are computed from the table by using the formulae:

$$a_0 = 2\frac{\Sigma y}{m}, \quad a_n = 2\frac{\Sigma y\cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2\frac{\Sigma y\sin\left(\frac{n\pi x}{l}\right)}{m}$$

# FOURIER TRANSFORMS

**Complex Fourier transform or Fourier transform of $f(x)$:**

- Fourier transform of $f(x)$: $\hat{f}(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$, provided the integral exists.
- Inverse Fourier transform of $\hat{f}(\alpha)$: $F^{-1}[\hat{f}(\alpha)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-i\alpha x}d\alpha$.

**Fourier sine and cosine transforms:**

- Fourier sine transform of $f(x)$: $\hat{f}_s(\alpha) = F_s[f(x)] = \int_0^{\infty} f(x)\sin\alpha x\ dx$
- Inverse Fourier sine transform $\hat{f}_s(\alpha)$: $F_s^{-1}[\hat{f}_s(\alpha)] = \frac{2}{\pi}\int_{-\infty}^{\infty} \hat{f}_s(\alpha)\sin\alpha x\ d\alpha$
- Fourier cosine transform of $f(x)$: $\hat{f}_c(\alpha) = F_c[f(x)] = \int_0^{\infty} f(x)\cos\alpha x\, dx$
- Inverse Fourier cosine transform of $\hat{f}_s(\alpha)$**:** $F_c^{-1}[\hat{f}_s(\alpha)] = \frac{2}{\pi}\int_{-\infty}^{\infty} \hat{f}_c(\alpha)\cos\alpha x\, d\alpha$

**Relation between Fourier sine and cosine transform:**

- $F_s[xf(x)] = -\frac{d}{d\alpha}F_c[f(x)]$ and $F_c[xf(x)] = \frac{d}{d\alpha}F_s[f(x)]$

**Properties of Fourier transforms:**

- For any two functions $f(x)$ and $\phi(x)$ (whose Fourier transforms exist) and any two constants $a$ and $b$, $F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant $a$, $F[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant $a$,
  (i) $F[f(x-a)] = e^{i\alpha a}\hat{f}(\alpha)$
  (ii) $F[e^{iax}f(x)] = \hat{f}(\alpha + a)$
- $F[f'(x)] = -isF[f(x)]$ and $F[f''(x)] = -s^2F[f(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then
  (i) $F[f(x)\cos ax] = \frac{1}{2}[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)]$ and
  (ii) $F[f(x)\sin ax] = \frac{1}{2}[\hat{f}(\alpha + a) - \hat{f}(\alpha - a)]$, where $'a'$ is a real constant.
- If $\hat{f}_s(\alpha)$ and $\hat{f}_c(\alpha)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then
  (i) $F_s[f(x)\cos ax] = \frac{1}{2}[\hat{f}_s(\alpha + a) + \hat{f}_s(\alpha - a)]$,
  (ii) $F_c[f(x)\sin ax] = \frac{1}{2}[\hat{f}_s(\alpha + a) - \hat{f}_s(\alpha - a)]$,
  (iii) $F_s[f(x)\sin ax] = \frac{1}{2}[\hat{f}_c(\alpha - a) + \hat{f}_c(\alpha + a)]$.

**Convolution theorem:** *If* $F[f(x)] = \hat{f}(\alpha)$ and $F[g(x)] = \hat{g}(\alpha)$, then

$$F^{-1}[\hat{f}(\alpha)\hat{g}(\alpha)] = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

**Parseval's identity:**

- If the Fourier transform of $f(x)$ and $g(x)$ are $F(\alpha)$ and $G(\alpha)$ , respectively, then
  (i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha \;=\; \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$
  (ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha \;=\; \int_{-\infty}^{\infty} |f(x)|^2 dx$

- If Fourier cosine and sine transform of $f(x)$ and $g(x)$ are $F_c(\alpha)$, $G_c(\alpha)$ and $F_s(\alpha)$, $G_s(\alpha)$ respectively, then
  (i) $\frac{2}{\pi}\int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty f(x)g(x)dx$
  (ii) $\frac{2}{\pi}\int_0^\infty F_s(\alpha)G_s(\alpha)d\alpha = \int_0^\infty f(x)g(x)dx$
  (iii) $\frac{2}{\pi}\int_0^\infty [F_c(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2 dx$
  (iv) $\frac{2}{\pi}\int_0^\infty [F_s(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2 dx$

## LINEAR ALGEBRA

**Basic transformation matrices:**

- Stretch matrix/dilation matrix: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where $a$ is real constant.
- Rotation matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- Projection matrix: $\begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$
- Reflection matrix: $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

**Gram-Schmidt process:** If $\{x_1, x_2, \dots, x_p\}$ is a basis for a subspace $W$ of vector space $R^n$, then corresponding orthogonal basis $\{v_1, v_2, \dots, v_p\}$ for $W$, where

$$v_1 = x_1 \text{ and } v_i = x_i - \left(\frac{x_i \cdot v_1}{v_1 \cdot v_1}\right)v_1 - \left(\frac{x_i \cdot v_2}{v_2 \cdot v_2}\right)v_2 - \cdots - \left(\frac{x_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}}\right)v_{i-1} \text{ for } i = 2,3,4,\dots,p.$$

## STATISTICS

**Moments for ungrouped data:**

- The $r^{th}$ moment about origin: $\mu_r' = \frac{1}{n}\sum_{i=1}^n x_i^r$, where $r = 1, 2, 3, \dots$ , $x_1, x_2, \cdots x_n$ are $n$ observations
- The $r^{th}$ central moment: $\mu_r = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^r$, where $r = 1, 2, 3, \cdots$ and $\bar{x}$ is mean

**Moments for grouped data:**

- The $r^{th}$ moment about origin: $\mu_r' = \frac{1}{N}\sum_{i=1}^n f_i x_i^r$, $r = 1, 2, 3, \cdots$, where observations $x_1, x_2, \dots, x_n$ are the mid points of the class-intervals and $f_1, f_2, \dots, f_n$ are their corresponding frequencies and N = $\sum_{i=1}^n f_i$
- The $r^{th}$ central moment: $\mu_r = \frac{1}{N}\sum_{i=1}^n f_i(x_i - \bar{x})^r$, $r = 1, 2, 3 \cdots$ & $\bar{x}$ is mean
- The $r^{th}$ moment about any point A: $\mu_r' = \frac{1}{N}\sum_{i=1}^n f_i(x_i - A)^r$, $r = 1, 2, 3 \cdots$

**Relation between raw moments (about origin or any point) and central moments:**

- $\mu_r = \mu_r' - {}^rC_1\, \mu_{r-1}'\, \mu_1' + {}^rC_2\, \mu_{r-2}'\, \mu_1'^2 + \dots + (-1)^r \mu_1'^r$, r = 1, 2, 3 …

- $\mu'_r = \mu_r + {}^rC_1\mu_{r-1}\,\mu'_1 + {}^rC_2\,\mu_{r-2}{\mu'_1}^2 + \ldots + {\mu'_1}^r$

**Measures of kurtosis:** $\beta_2 = \frac{\mu_4}{\mu_2^2}$

**Measures of Skewness (Karl Pearson's coefficient):** $S_k = \frac{\sqrt{\beta_1}\,(\beta_2+3)}{2(5\beta_2-6\beta_1-9)}$, where $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

**Fitting of a straight line $y = a + bx$:** The normal equations for estimating the values of a and $b$ are

$$\sum y = n\mathrm{a} + b\sum x,\ \sum xy = a\sum x + b\sum x^2$$

**Fitting of a second-degree equation (quadratic) $y = a + bx + cx^2$:** The normal equations for estimating the values of $a$ , $b$ , $c$ are

$$\sum y = n\mathrm{a} + b\sum x + c\sum x^2,$$
$$\sum xy = \mathrm{a}\sum x + b\sum x^2 + c\sum x^3,$$
$$\sum x^2y = \mathrm{a}\sum x^2 + b\sum x^3 + c\sum x^4.$$

**Correlation coefficient (Karl Pearson correlation coefficient):**

- $r = \frac{\sum(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$, where $\sigma_x^2 = \frac{\sum(x-\overline{x})^2}{n}$ variance of the $x$ series, $\sigma_y^2 = \frac{\sum(y-\bar{y})^2}{n}$ variance of the $y$ series,
- $\overline{x} = \frac{\sum x}{n}$ → Mean of the $x$ series $\overline{y} = \frac{\sum(x-\overline{x})^2}{n}$ → mean of the $y$ series.
- $r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{\{n\sum x^2 - (\sum x)^2\}\{n\sum y^2 - (\sum y)^2\}}}$.

**Rank correlation coefficient $r_s$ (Spearman's rank correlation coefficient):**

- If $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be the ranks of $n$ individuals in characteristics A and B respectively, then $r_s = 1 - \frac{6\sum_{i=1}^{n} {d_i}^2}{n(n^2-1)}$, where $d_i$ is difference between ranks assigned in characteristics A and B. and n is number of pairs of data.
- Rank correlation coefficient for tied ranks: $r_s = 1 - \frac{6\left[\sum_{i=1}^{n} {d_i}^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$, where $m_1, m_2, \ldots$ are number of repetitions of ranks.

**Linear regression:**

- Regression line of $y$ on $x$: $y - \bar{y} = b_{yx}(x - \overline{x})$, where
$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(x-\overline{x})^2} = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - (\sum x)^2}$$
- Regression line of $x$ on $y$: $x - \overline{x} = b_{xy}(y - \overline{y})$, where
$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(y-\overline{y})^2} = \frac{n\sum xy - \sum x\sum y}{n\sum y^2 - (\sum y)^2}$$
- Angle between two lines of regression: $\tan\theta = \frac{1-r^2}{r}\ \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$.

**Multiple linear regression:** Fitting of a multiple linear regression model: $y = \beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_kx_k$ for $n$ sets of data $(y_i, x_{ij})$, where $i = 1, \ldots, n$ and $j = 1, \ldots, k$. The normal equations for estimating the values of $\beta_0, \beta_1, \beta_2, \ldots, \beta_k$ are

$$n\beta_0 + \beta_1\sum_{i=1}^{n} x_{i1} + \beta_2\sum_{i=1}^{n} x_{i2} + \cdots \beta_k\sum_{i=1}^{n} x_{ik} = \sum_{i=1}^{n} y_i,$$

$$\beta_0\sum_{i=1}^{n} x_{i1} + \beta_1\sum_{i=1}^{n} x_{i1}^2 + \beta_2\sum_{i=1}^{n} x_{i1}x_{i2} + \cdots \beta_k\sum_{i=1}^{n} x_{i1}x_{ik} = \sum_{i=1}^{n} x_{i1}y_i,$$

⋮

$$\beta_0 \sum_{i=1}^{n} x_{ik} + \beta_1 \sum_{i=1}^{n} x_{ik}x_{i1} + \beta_2 \sum_{i=1}^{n} x_{ik}x_{i2} + \cdots \beta_k \sum_{i=1}^{n} x_{ik}^2 = \sum_{i=1}^{n} x_{ik}y_i.$$

**Multivariate data (Sample):**

| **Dimensions** | **Sample** | **Sample Mean** | **Sample variance (unbiased) / Sample Covariance** |
|---|---|---|---|
| 1-dimensional | $x_1 \dots . x_n$ | $\bar{x} = \sum_{i=1}^{n} x_i$ | $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ |
| p-dimensional | p-random variable vector $X = \begin{bmatrix} X_1 \\ . \\ . \\ . \\ Xp \end{bmatrix}$ where $x_1, x_2, \dots, x_n \in R^p$ <br> n-dimensional data matrix $X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ & . & & \\ & . & & \\ & . & & \\ x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix}_{n \times p}$ | Sample Mean: $\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$, j = 1, 2, ...,p. <br> and <br> Sample Mean Vector: $\bar{x} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i1} \\ . \\ . \\ . \\ \frac{1}{n}\sum_{i=1}^{n} x_{ip} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ . \\ . \\ . \\ \bar{x}_p \end{bmatrix}$ | Sample Variance: $s_{jj} = s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ij} - \bar{x}_j\right)^2$, j = 1, 2, ...,p and i = j. <br> and <br> Sample Covariance: $s_{jk} = s_{kj} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ij} - \bar{x}_j\right)\left(x_{ik} - \bar{x}_k\right)$, $1 \leq k, j \leq p$ and $j \neq k$. <br> Sample Covariance Matrix: $S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1p} \\ s_{21} & s_{22} & \dots & s_{2p} \\ & . & & \\ & . & & \\ & . & & \\ s_{p1} & s_{p2} & \dots & s_{pp} \end{bmatrix}_{p \times p}$ <br> $= \frac{1}{n-1}\sum_{i=1}^{n}\begin{bmatrix} \left(x_{i1} - \bar{x}_1\right)^2 \dots \left(x_{i1} - \bar{x}_1\right)\left(x_{ip} - \bar{x}_p\right) \\ . \\ . \\ . \\ \left(x_{ip} - \bar{x}_p\right)\left(x_{i1} - \bar{x}_1\right) \dots \left(x_{ip} - \bar{x}_p\right)^2 \end{bmatrix}_{p \times p}$ |

**Multivariate data (Population):**

| **Dimensions** | **Population** | **Population Mean** | **Population variance / Population Covariance** |
|---|---|---|---|
| 1-dimensional | $x_1 \dots . x_n$ | $\mu = \sum_{i=1}^{n} x_i$ | $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$ |

| | | | |
|---|---|---|---|
| p-dimensional | p-random variable vector<br>$$X = \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_p \end{bmatrix}$$<br>where $x_1 \ldots . x_n \in R^p$<br><br>n-dimensional data matrix<br>$$X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ & & . & \\ & & . & \\ & & . & \\ x_{n1} & x_{n2} & \ldots & x_{np} \end{bmatrix}_{n \times p}$$<br>Here each column in the data matrix corresponds to a random variable $X_j$. | Population Mean:<br>$$\mu_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij},$$<br>j = 1, 2, ...,p.<br>and<br>Population Mean Vector:<br>$$\mu = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i1} \\ . \\ . \\ . \\ \frac{1}{n}\sum_{i=1}^{n} x_{ip} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ . \\ . \\ . \\ \mu_p \end{bmatrix}$$ | Population Variance:<br>$$\sigma_{jj} = \sigma_j^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_{ij} - \mu_j\right)^2,$$<br>j = 1, 2, ...,p and i = j.<br>and<br>Population Covariance:<br>$$\sigma_{jk} = \sigma_{kj} = \frac{1}{n}\sum_{i=1}^{n}\left(x_{ij} - \mu_j\right)\left(x_{ik} - \mu_k\right),$$<br>$1 \leq k, j \leq p$ and $j \neq k$.<br>Population Covariance Matrix:<br>$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2p} \\ & & . & \\ & & . & \\ & & . & \\ \sigma_{p1} & \sigma_{p2} & \ldots & \sigma_{pp} \end{bmatrix}_{p \times p}$$<br>$$= \frac{1}{n}\sum_{i=1}^{n} \begin{bmatrix} \left(x_{i1} - \mu_1\right)^2 \ldots \left(x_{i1} - \mu_1\right)\left(x_{ip} - \mu_p\right) \\ . \\ . \\ . \\ \left(x_{ip} - \mu_p\right)\left(x_{i1} - \mu_1\right) \ldots \left(x_{ip} - \mu_p\right)^2 \end{bmatrix}_{p \times p}$$ |

| Rank Correlation, Partial Correlation, Multiple Correlation | |
|---|---|
| In the case of tri variate data $X_1, X_2$ and $X_3$ | i) Coefficient of partial correlation of $X_1$ and $X_2$, keeping $X_3$ constant:<br>$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$<br>ii) Coefficient of partial correlation of $X_1$ and $X_3$, keeping $X_2$ constant:<br>$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{23}^2}}$$<br>iii) Coefficient of partial correlation of $X_2$ and $X_3$, keeping $X_1$ constant:<br>$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{13}^2}}$$<br>**Note**: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$. |

<table>
<tr><td>In the case of tri variate data: $X_1, X_2$ and $X_3$</td><td>
i. Multiple correlation coefficient of $X_1$ on $X_2$ and $X_3$: $R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$<br>
ii. Multiple correlation coefficient of $X_2$ on $X_1$ and $X_3$: $R_{2.13} = \sqrt{\frac{r_{21}^2 + r_{23}^2 - 2r_{21}r_{23}r_{13}}{1 - r_{13}^2}}$<br>
iii. Multiple correlation coefficient of $X_2$ on $X_1$ and $X_3$: $R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{31}r_{32}r_{12}}{1 - r_{21}^2}}$<br><br>
Given variables $x, y$ and $z$, we define the multiple correlation coefficient.<br><br>
$R_{z.xy} = \sqrt{\frac{r_{zx}^2 + r_{zy}^2 - 2r_{zx}r_{zy}r_{yx}}{1 - r_{yx}^2}}$ and $r_{xy} = \frac{n\sum xy - \sum x \sum y}{\sqrt{\{n(\sum x^2) - (\sum x)^2\}\{n(\sum y^2) - (\sum y)^2\}}}$<br><br>
Note: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$.
</td></tr>
<tr><td colspan="2" align="center">Analysis of Variance (ANOVA)</td></tr>
</table>

One-way ANOVA Table

| Sources of variation | Degrees of freedom | Sum of squares | Mean sum of squares | $F_{cal}$ |
|---|---|---|---|---|
| Between classes | $k-1$ | SSB | MSSB | $F_{cal} = \frac{MSSB}{MSSW}$ |
| Within classes | $N-k$ | SSW | MSSW | |
| Total | $N-1$ | SST | | |

$k$: number of classes, $N = pq$ is number of rows × number pf columns, $n$: number of entries in each $X_i$

$T$: $Grand\ sum$: $\Sigma X_i$ , Correction factor $(C.F.) = \frac{T^2}{N}$

SSB: Sum of squares between classes = $\frac{(\Sigma X_i)^2}{n_i} - C.F.$

SST: Total sum of squares = $\Sigma X_i^2 - C.F.$

SSW: Sum of squares within classes = $SST - SSB$

MSSB: Mean sum of squares between classes = $\frac{SSB}{k-1}$

MSSW: Mean sum of squares within classes = $\frac{SSW}{N-k}$

**Conclusion:**

If $F_{cal} < F_{tab}$, there is NO significant difference, Null Hypothesis $H_0$ is accepted.

If $F_{cal} > F_{tab}$, there is a significant difference, Null Hypothesis $H_0$ is rejected and Alternative Hypothesis, $H_1$ is accepted.

Note: In one-way ANOVA, $F_{cal}$ can also be computed using columns.

Two-way ANOVA table

| Source of variation | Degrees of freedom | Sum of squares | Mean sum of squares | $F_{cal}$ |
|---|---|---|---|---|
| Between rows | $r-1$ | SSR | MSSR | $F_{cal\ (rows)} = \frac{MSSR}{MSSE}$ |
| Between columns | $c-1$ | SSC | MSSC | $F_{cal\ (columns)} = \frac{MSSC}{MSSE}$ |
| Error (residual) | $(r-1)(c-1)$ | SSE | MSSE | |
| Total | $rc-1$ | SST | | |

Number of levels of row factor = $r$, Number of levels of column factor = $c$

Total number of observations = $rc$ = (rows × columns)

Observations in $ij^{th}$ cell of table = $x_{ij}$ , Sum of c observations in $i^{th}$ row = $T_{Ri} = \Sigma_j x_{ij}$

Sum of $r$ observations in $j^{th}$ column = $T_{Cj} = \Sigma_i x_{ij}$

Sum of all observations = $T = \Sigma_i T_{Ri} = \Sigma_j T_{Cj}$

Correction factor = $C.F. = \frac{T^2}{rc}$

Sum of squares between rows: SSR = $\sum \frac{T_{Ri}^2}{c} - C.F.$

Sum of squares between columns: SSC = $\sum \frac{T_{Cj}^2}{r} - C.F.$

Total sum of squares: SST = $\Sigma_i \Sigma_j x_{ij}^2 - C.F.$

Error (residual) sum of squares: SSE = SST-SSR-SSC

MSSR = mean sum of squares between rows = $\frac{SSR}{r-1}$

MSSC = mean sum of squares between columns = $\frac{SSC}{c-1}$

MSSE = mean sum of squares of error = $\frac{SSE}{(r-1)(c-1)}$

**Conclusion:**

If $F_{cal} < F$ tab, there is no significant difference, null hypothesis $H_0$ is accepted.

If $F_{cal} > F$ tab, there is a significant difference, null hypothesis $H_0$ is rejected and alternative hypothesis, $H_1$ is accepted.

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# PROBABILITY AND RANDOM VARIABLES

**Probability of an event:** If a trial results in $n$ exhaustive, mutually exclusive and equally likely cases and $m$ of them are favourable to the happening of an event $E$ then the probability $'p'$ of happening of $E$ is given by $p = P(E) = \frac{m}{n}$

**Addition theorem:**

**For two events:** If $A$ and $B$ are any two events with respective probabilities $P(A)$ and $P(B)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

**For three events:** If $A, B$ and $C$ are any three events with respective probabilities $P(A), P(B)$ and $P(C)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

**Conditional probability**: Conditional probability of $B$, given $A$, denoted by $P(B|A)$, is defined by $P(A|B) = \frac{P(A \cap B)}{P(A)}$, provided $P(A) > 0$.

**Multiplication rule:** Suppose $A$ and $B$ are events in a sample space $S$ with $P(A) > 0$ multiplication rule is: $P(A \cap B) = P(A)\, P(B|A)$

**Bayes' theorem (rule):** If $B_1, B_2, \dots, B_n$ are mutually disjoint events with $P(B_i) \neq 0$ $(i = 1,2, \dots, n)$ then for any arbitrary event $A$ which is a subset of $\bigcup_{i=1}^{n} B_i$ such that $P(A) > 0$, then $\mathrm{P(B_i|A)} = \frac{\mathrm{P(B_i)P(A|B_i)}}{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{P(B_i)P(A|B_i)}}$.

**Discrete random variable:** Let $X$ be a discrete random variable. A function $p(x)$ is a probability mass function of the discrete random variable $X$ if $p(x) \geq 0, \forall x \in X$ and $\sum_x p(x) = 1$.

- Expectation, $E(X) = \sum_x xp(x)$
- If $Y = g(X)$, then $E(Y) = \sum_x g(x)p(x)$
- Variance, $Var(X) = E\left[\left(X - E(X)\right)^2\right] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- The cumulative distribution function, $F(t) = P(X \leq t) = \sum_{x \leq t} p(x)$

**Continuous random variable:** Suppose $X$ is a continuous random variable. A function $f(x)$ is called a probability density function of the continuous random variable $X$ if $f(x) \geq 0$ $\forall x \in X$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

- Expectation, $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$
- If $Y = g(X)$, then $E(Y) = \int_{-\infty}^{\infty} g(x)\, f(x)dx$
- Variance, $Var(X) = E\left[\left(X - E(X)\right)^2\right] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- Cumulative distribution function, $F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)dx$

- $P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b) = \int_a^b f(x)dx$

**Markov's inequality:** For any random variable $X$ with finite $E[X]$ and any $k > 0$, the probability that $X$ is at least $k$ times its expected value is at-most $\frac{1}{k}$. That is, $P\big[X \ge kE[X]\big] \le \frac{1}{k}$ or $P[X \ge k] \le \frac{E[X]}{k}$.

**Chebyshev's inequality:** Any random variable $X$ with expectation $\mu = E[X]$ and variance $\sigma^2 = Var[X]$ belongs to the interval $\mu \pm k = [\mu - k, \mu + k]$ with probability of at least $1 - \left(\frac{\sigma}{k}\right)^2$. That is, $P\{|X - \mu| \ge k\} \le \left(\frac{\sigma}{k}\right)^2 = \frac{\sigma^2}{k^2}$

**Joint probability mass function:** Suppose $X$ and $Y$ are two discrete random variables. A function $p(x,y)$ is called a joint probability mass function of $X$ and $Y$ if $p(x,y) \ge 0, \forall x \in X, y \in Y$ and $\sum_x \sum_y p(x,y) = 1$.

- Let $Z = g(X,Y)$. Expectation, $E[Z] = \sum_x \sum_y g(x,y)p(x,y)$
- The marginal distributions of $X$ alone and of $Y$ alone are: $g(x) = \sum_y p(x,y)$ and $h(y) = \sum_x p(x,y)$
- Covariance, $Cov(X,Y) = E(XY) - E(X)E(Y)$
- Correlation of $X$ and $Y$, $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
- For discrete random variables $X$ and $Y$ with joint pmf $p(x,y)$ and $x, y$ such that $g(x) > 0, h(y) > 0$, then the conditional probability mass functions are
  (i) $P(X = x|Y = y) = \frac{p(x,y)}{h(y)}$, (ii) $P(Y = y|X = x) = \frac{p(x,y)}{g(x)}$.
- If $X$ and $Y$ are independent, then $E(XY) = E(X)E(Y)$

**Joint probability density function:** Suppose $X$ and $Y$ are two continuous random variables. A function $f(x,y)$ is called a joint probability density function of $X$ and $Y$ if $f(x,y) \ge 0, \forall x \in X, y \in Y$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$.

- Let $Z = g(X,Y)$. Expectation, $E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dxdy$
- The marginal distributions of $X$ alone and of $Y$ alone are $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ and $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$
- Covariance, $Cov(X,Y) = E(XY) - E(X)E(Y)$
- Correlation of $X$ and $Y$, $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
- If $X$ and $Y$ are independent, then $E(XY) = E(X)E(Y)$
- The conditional distribution of the random variable $Y$ given that $X = x$ is $f(y|x) = \frac{f(x,y)}{g(x)}$, provided $g(x) > 0$
- The conditional distribution of the random variable $X$ given that $Y = y$ is $f(x|y) = \frac{f(x,y)}{h(y)}$, provided $h(y) > 0$.

# PROBABILITY DISTRIBUTIONS

**Bernoulli distribution:**

- The probability mass function is given by $f(x,p) = p^x(1-p)^{1-x}$ ; $x \in (0,1)$
- Mean, $\mu = p$
- Variance, $\sigma^2 = pq$
- Standard deviation, $\sigma = \sqrt{pq}$

**Binomial distribution:**

- The probability function of the binomial distribution is given by $b(x;n,p) = n_{C_x}\, p^x q^{n-x}$, where $x = 1,2,3,\cdots$, $p$ is the probability of success and $q = 1 - p$ is the probability of failure.
- Mean, $\mu = np$
- Variance, $\sigma^2 = npq$
- Standard deviation, $\sigma = \sqrt{npq}$

**Poisson distribution:**

- The probability function of the Poisson distribution is given by $p(x;\lambda) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, where $\lambda$ is the parameter of the Poisson distribution.
- Mean, $\mu = \lambda$
- Variance, $\sigma^2 = \lambda$
- Standard deviation, $\sigma = \sqrt{\lambda}$

**Geometric distribution:**

- Probability mass function is given by $P(X = x) = (1-p)^{x-1}p$, where $0 < p < 1$
- Mean, $\mu = 1/p$
- Variance, $\sigma^2 = \frac{1-p}{p^2}$
- Standard deviation $\sigma = \frac{\sqrt{1-p}}{p}$

**Exponential distribution:** A continuous random variable $X$ assuming non-negative values is said to have an exponential distribution with parameter $\lambda > 0$ , if its probability density function is given by

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

- Mean $\mu = \frac{1}{\lambda}$
- Variance $\sigma^2 = \left(\frac{1}{\lambda}\right)^2$
- Standard deviation $\sigma = \frac{1}{\lambda}$

**Uniform distribution:** The probability density function is $f(x) = \frac{1}{b-a}$ $for\ a \leq x \leq b$

- Mean $\mu = \frac{a+b}{2}$
- Variance $\sigma^2 = \frac{(b-a)^2}{12}$

- Standard deviation $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

**Normal distribution:** A random variable $X$ is said to have a normal distribution with parameters $\mu$ (called "mean") and $\sigma^2$ (called "variance") if its density function is given by the probability law:

$$f(x;\ \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\ exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \text{ for } -\infty < x < \infty, -\infty < \mu < \infty \text{ and } 0 < \sigma < \infty\,.$$

- Standard normal distribution or $z-$ distribution is given by

$$f(z;\ 0,1) = \frac{1}{\sqrt{2\pi}}\ exp\left[\frac{-z^2}{2}\right], \qquad -\infty < z < \infty,$$

- Cumulative standard normal distribution is

$$\phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}u^2}\ du.$$

Note: The value of the integral can be calculated by the Normal distribution table.

## SAMPLING THEORY

**Numerical summation of data:** If the $n$ observations in a sample are denoted by $x_1, x_2, \dots, x_n$, then

- Sample mean: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
- Sample variance: $s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$
- Standard deviation is the positive square root of the sample variance.
- If the population is finite with size $N$, then for the sampling distribution of $\bar{x}$:

Mean: $\mu_{\bar{x}} = \mu$

Variance: $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \times \frac{N-n}{N-1}$, if the sampling is without replacement

$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$, if the sampling is with replacement or population is infinite.

**Sampling distributions:** Let $X$ be a random variable of a population with mean $\mu$ and variance $\sigma^2$. If the random variables $X_1, X_2, \dots, X_n$ are a random sample on size $n$, then sample mean $\bar{X} = \frac{X_1+X_2+\cdots+X_n}{n}$

**Sampling distribution of means**

- Mean: $\mu_{\bar{X}} = \mu$
- Variance: $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$
- If the population is normal or sample size is sufficiently large, then the distribution of $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is approximately standard normal.

**Sampling distribution of a difference in sample means**

- Mean: $\mu_{(\bar{X}_1-\bar{X}_2)} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2$
- Variance: $\sigma^2_{(\bar{X}_1-\bar{X}_2)} = \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

- If the populations are normal or sample sizes are sufficiently large, then the distribution of $Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$ is approximately standard normal.

**Sampling distribution of proportions:**

- Mean $\mu_{\hat{p}} = p$
- Standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \left[\frac{p(1-p)}{n}\right]^{\frac{1}{2}}$
- The distribution of $Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$ is approximately standard normal if $n$ is large or $np \geq 5$ and $np(1-p) \geq 5$.

**Statistical decisions:**

- Levels of significance for two – tailed test

| Level of significance | Critical value | Acceptance region |
|---|---|---|
| 0.05 | $z_c = 1.96$ | $(-1.96, 1.96)$ |
| 0.01 | $z_c = 2.58$ | $(-2.58, 2.58)$ |

- Levels of significance for one– tailed test

| Level of significance | Critical value | | Critical region | |
|---|---|---|---|---|
| | Right-tailed test | Left-tailed test | Right-tailed test | Left-tailed test |
| 0.05 | $z_c = 1.645$ | $-z_c = -1.645$ | $(1.645, \infty)$ | $(-\infty, -1.645)$ |
| 0.01 | $z_c = 2.33$ | $-z_c = -2.33$ | $(2.33, \infty)$ | $(-\infty, -2.33)$ |

- For the standard normal variate $Z$, and the test value of $Z$ is $z_0$, then the P-value is determined as follows:
  (i) For left-tail test, the P-value is $P(Z \leq z_0)$
  (ii) For right-tail test, the P-value is $P(Z \geq z_0)$
  (iii) For two tail test, the P-value is $P(Z \leq -|z_0|) + P(Z \geq |z_0|) = 2\big(1 - \phi(|z_0|)\big)$

**Test of significance:**

**z-test**

- Test statistics:
  (i) $z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$, if $\sigma$ is known.
  (ii) $z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$, if $\sigma$ is unknown and $n$ is large.

**t-test:**

- Test statistic:
  $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$, if $\sigma$ is unknown.

**Chi – square ($\chi^2$) test:**

- If $s^2$ is the variance of a random sample of size $n$ taken from a normal population having variance $\sigma^2$, then the statistic

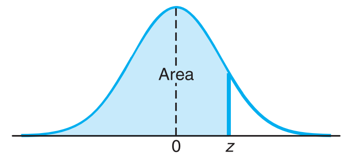
$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2}$$

is a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

- If $f_1, f_2, f_3, \ldots \ldots f_n$ are the observed frequencies and $e_1, e_2, e_3 \ldots \ldots e_n$ are the expected or theoretical frequencies. The statistic $\chi^2 = \sum_{i=1}^{n} \frac{(f_i - e_i)^2}{e_i}$

**F- test:**

- Test statistic: $F_0 = \frac{S_1^2}{S_2^2}$

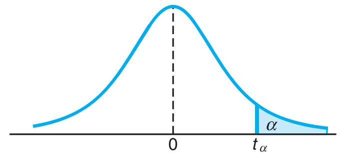



Areas under the Normal Curve

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **−3.4** | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| **−3.3** | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| **−3.2** | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| **−3.1** | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| **−3.0** | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| **−2.9** | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| **−2.8** | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| **−2.7** | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| **−2.6** | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| **−2.5** | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| **−2.4** | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| **−2.3** | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| **−2.2** | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| **−2.1** | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| **−2.0** | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| **−1.9** | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| **−1.8** | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| **−1.7** | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| **−1.6** | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| **−1.5** | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| **−1.4** | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| **−1.3** | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| **−1.2** | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| **−1.1** | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| **−1.0** | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| **−0.9** | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| **−0.8** | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| **−0.7** | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| **−0.6** | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| **−0.5** | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| **−0.4** | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| **−0.3** | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| **−0.2** | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| **−0.1** | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| **−0.0** | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

(continued) Areas under the Normal Curve

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

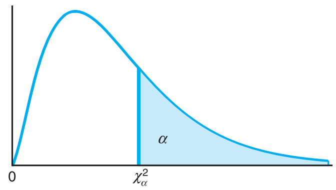



## Critical Values of the $t$-Distribution

| | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $v$ | **0.40** | **0.30** | **0.20** | **0.15** | **0.10** | **0.05** | **0.025** |
| **1** | 0.325 | 0.727 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 |
| **2** | 0.289 | 0.617 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 |
| **3** | 0.277 | 0.584 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 |
| **4** | 0.271 | 0.569 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 |
| **5** | 0.267 | 0.559 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 |
| **6** | 0.265 | 0.553 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 |
| **7** | 0.263 | 0.549 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 |
| **8** | 0.262 | 0.546 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 |
| **9** | 0.261 | 0.543 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 |
| **10** | 0.260 | 0.542 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 |
| **11** | 0.260 | 0.540 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 |
| **12** | 0.259 | 0.539 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 |
| **13** | 0.259 | 0.538 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 |
| **14** | 0.258 | 0.537 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 |
| **15** | 0.258 | 0.536 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 |
| **16** | 0.258 | 0.535 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 |
| **17** | 0.257 | 0.534 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 |
| **18** | 0.257 | 0.534 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 |
| **19** | 0.257 | 0.533 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 |
| **20** | 0.257 | 0.533 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 |
| **21** | 0.257 | 0.532 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 |
| **22** | 0.256 | 0.532 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 |
| **23** | 0.256 | 0.532 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 |
| **24** | 0.256 | 0.531 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 |
| **25** | 0.256 | 0.531 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 |
| **26** | 0.256 | 0.531 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 |
| **27** | 0.256 | 0.531 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 |
| **28** | 0.256 | 0.530 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 |
| **29** | 0.256 | 0.530 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 |
| **30** | 0.256 | 0.530 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 |
| **40** | 0.255 | 0.529 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 |
| **60** | 0.254 | 0.527 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 |
| **120** | 0.254 | 0.526 | 0.845 | 1.041 | 1.289 | 1.658 | 1.980 |
| **$\infty$** | 0.253 | 0.524 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 |

(continued) Critical Values of the $t$-Distribution

| | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $v$ | **0.02** | **0.015** | **0.01** | **0.0075** | **0.005** | **0.0025** | **0.0005** |
| **1** | 15.894 | 21.205 | 31.821 | 42.433 | 63.656 | 127.321 | 636.578 |
| **2** | 4.849 | 5.643 | 6.965 | 8.073 | 9.925 | 14.089 | 31.600 |
| **3** | 3.482 | 3.896 | 4.541 | 5.047 | 5.841 | 7.453 | 12.924 |
| **4** | 2.999 | 3.298 | 3.747 | 4.088 | 4.604 | 5.598 | 8.610 |
| **5** | 2.757 | 3.003 | 3.365 | 3.634 | 4.032 | 4.773 | 6.869 |
| **6** | 2.612 | 2.829 | 3.143 | 3.372 | 3.707 | 4.317 | 5.959 |
| **7** | 2.517 | 2.715 | 2.998 | 3.203 | 3.499 | 4.029 | 5.408 |
| **8** | 2.449 | 2.634 | 2.896 | 3.085 | 3.355 | 3.833 | 5.041 |
| **9** | 2.398 | 2.574 | 2.821 | 2.998 | 3.250 | 3.690 | 4.781 |
| **10** | 2.359 | 2.527 | 2.764 | 2.932 | 3.169 | 3.581 | 4.587 |
| **11** | 2.328 | 2.491 | 2.718 | 2.879 | 3.106 | 3.497 | 4.437 |
| **12** | 2.303 | 2.461 | 2.681 | 2.836 | 3.055 | 3.428 | 4.318 |
| **13** | 2.282 | 2.436 | 2.650 | 2.801 | 3.012 | 3.372 | 4.221 |
| **14** | 2.264 | 2.415 | 2.624 | 2.771 | 2.977 | 3.326 | 4.140 |
| **15** | 2.249 | 2.397 | 2.602 | 2.746 | 2.947 | 3.286 | 4.073 |
| **16** | 2.235 | 2.382 | 2.583 | 2.724 | 2.921 | 3.252 | 4.015 |
| **17** | 2.224 | 2.368 | 2.567 | 2.706 | 2.898 | 3.222 | 3.965 |
| **18** | 2.214 | 2.356 | 2.552 | 2.689 | 2.878 | 3.197 | 3.922 |
| **19** | 2.205 | 2.346 | 2.539 | 2.674 | 2.861 | 3.174 | 3.883 |
| **20** | 2.197 | 2.336 | 2.528 | 2.661 | 2.845 | 3.153 | 3.850 |
| **21** | 2.189 | 2.328 | 2.518 | 2.649 | 2.831 | 3.135 | 3.819 |
| **22** | 2.183 | 2.320 | 2.508 | 2.639 | 2.819 | 3.119 | 3.792 |
| **23** | 2.177 | 2.313 | 2.500 | 2.629 | 2.807 | 3.104 | 3.768 |
| **24** | 2.172 | 2.307 | 2.492 | 2.620 | 2.797 | 3.091 | 3.745 |
| **25** | 2.167 | 2.301 | 2.485 | 2.612 | 2.787 | 3.078 | 3.725 |
| **26** | 2.162 | 2.296 | 2.479 | 2.605 | 2.779 | 3.067 | 3.707 |
| **27** | 2.158 | 2.291 | 2.473 | 2.598 | 2.771 | 3.057 | 3.689 |
| **28** | 2.154 | 2.286 | 2.467 | 2.592 | 2.763 | 3.047 | 3.674 |
| **29** | 2.150 | 2.282 | 2.462 | 2.586 | 2.756 | 3.038 | 3.660 |
| **30** | 2.147 | 2.278 | 2.457 | 2.581 | 2.750 | 3.030 | 3.646 |
| **40** | 2.123 | 2.250 | 2.423 | 2.542 | 2.704 | 2.971 | 3.551 |
| **60** | 2.099 | 2.223 | 2.390 | 2.504 | 2.660 | 2.915 | 3.460 |
| **120** | 2.076 | 2.196 | 2.358 | 2.468 | 2.617 | 2.860 | 3.373 |
| $\infty$ | 2.054 | 2.170 | 2.326 | 2.432 | 2.576 | 2.807 | 3.290 |



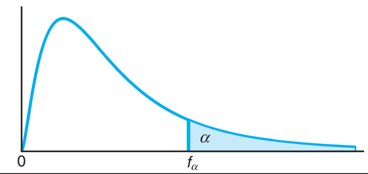

## Critical Values of the Chi-Squared Distribution

| | α | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **$v$** | **0.995** | **0.99** | **0.98** | **0.975** | **0.95** | **0.90** | **0.80** | **0.75** | **0.70** | **0.50** |
| **1** | $0.0^4393$ | $0.0^3157$ | $0.0^3628$ | $0.0^3982$ | 0.00393 | 0.0158 | 0.0642 | 0.102 | 0.148 | 0.455 |
| **2** | 0.0100 | 0.0201 | 0.0404 | 0.0506 | 0.103 | 0.211 | 0.446 | 0.575 | 0.713 | 1.386 |
| **3** | 0.0717 | 0.115 | 0.185 | 0.216 | 0.352 | 0.584 | 1.005 | 1.213 | 1.424 | 2.366 |
| **4** | 0.207 | 0.297 | 0.429 | 0.484 | 0.711 | 1.064 | 1.649 | 1.923 | 2.195 | 3.357 |
| **5** | 0.412 | 0.554 | 0.752 | 0.831 | 1.145 | 1.610 | 2.343 | 2.675 | 3.000 | 4.351 |
| **6** | 0.676 | 0.872 | 1.134 | 1.237 | 1.635 | 2.204 | 3.070 | 3.455 | 3.828 | 5.348 |
| **7** | 0.989 | 1.239 | 1.564 | 1.690 | 2.167 | 2.833 | 3.822 | 4.255 | 4.671 | 6.346 |
| **8** | 1.344 | 1.647 | 2.032 | 2.180 | 2.733 | 3.490 | 4.594 | 5.071 | 5.527 | 7.344 |
| **9** | 1.735 | 2.088 | 2.532 | 2.700 | 3.325 | 4.168 | 5.380 | 5.899 | 6.393 | 8.343 |
| **10** | 2.156 | 2.558 | 3.059 | 3.247 | 3.940 | 4.865 | 6.179 | 6.737 | 7.267 | 9.342 |
| **11** | 2.603 | 3.053 | 3.609 | 3.816 | 4.575 | 5.578 | 6.989 | 7.584 | 8.148 | 10.341 |
| **12** | 3.074 | 3.571 | 4.178 | 4.404 | 5.226 | 6.304 | 7.807 | 8.438 | 9.034 | 11.340 |
| **13** | 3.565 | 4.107 | 4.765 | 5.009 | 5.892 | 7.041 | 8.634 | 9.299 | 9.926 | 12.340 |
| **14** | 4.075 | 4.660 | 5.368 | 5.629 | 6.571 | 7.790 | 9.467 | 10.165 | 10.821 | 13.339 |
| **15** | 4.601 | 5.229 | 5.985 | 6.262 | 7.261 | 8.547 | 10.307 | 11.037 | 11.721 | 14.339 |
| **16** | 5.142 | 5.812 | 6.614 | 6.908 | 7.962 | 9.312 | 11.152 | 11.912 | 12.624 | 15.338 |
| **17** | 5.697 | 6.408 | 7.255 | 7.564 | 8.672 | 10.085 | 12.002 | 12.792 | 13.531 | 16.338 |
| **18** | 6.265 | 7.015 | 7.906 | 8.231 | 9.390 | 10.865 | 12.857 | 13.675 | 14.440 | 17.338 |
| **19** | 6.844 | 7.633 | 8.567 | 8.907 | 10.117 | 11.651 | 13.716 | 14.562 | 15.352 | 18.338 |
| **20** | 7.434 | 8.260 | 9.237 | 9.591 | 10.851 | 12.443 | 14.578 | 15.452 | 16.266 | 19.337 |
| **21** | 8.034 | 8.897 | 9.915 | 10.283 | 11.591 | 13.240 | 15.445 | 16.344 | 17.182 | 20.337 |
| **22** | 8.643 | 9.542 | 10.600 | 10.982 | 12.338 | 14.041 | 16.314 | 17.240 | 18.101 | 21.337 |
| **23** | 9.260 | 10.196 | 11.293 | 11.689 | 13.091 | 14.848 | 17.187 | 18.137 | 19.021 | 22.337 |
| **24** | 9.886 | 10.856 | 11.992 | 12.401 | 13.848 | 15.659 | 18.062 | 19.037 | 19.943 | 23.337 |
| **25** | 10.520 | 11.524 | 12.697 | 13.120 | 14.611 | 16.473 | 18.940 | 19.939 | 20.867 | 24.337 |
| **26** | 11.160 | 12.198 | 13.409 | 13.844 | 15.379 | 17.292 | 19.820 | 20.843 | 21.792 | 25.336 |
| **27** | 11.808 | 12.878 | 14.125 | 14.573 | 16.151 | 18.114 | 20.703 | 21.749 | 22.719 | 26.336 |
| **28** | 12.461 | 13.565 | 14.847 | 15.308 | 16.928 | 18.939 | 21.588 | 22.657 | 23.647 | 27.336 |
| **29** | 13.121 | 14.256 | 15.574 | 16.047 | 17.708 | 19.768 | 22.475 | 23.567 | 24.577 | 28.336 |
| **30** | 13.787 | 14.953 | 16.306 | 16.791 | 18.493 | 20.599 | 23.364 | 24.478 | 25.508 | 29.336 |
| **40** | 20.707 | 22.164 | 23.838 | 24.433 | 26.509 | 29.051 | 32.345 | 33.66 | 34.872 | 39.335 |
| **50** | 27.991 | 29.707 | 31.664 | 32.357 | 34.764 | 37.689 | 41.449 | 42.942 | 44.313 | 49.335 |
| **60** | 35.534 | 37.485 | 39.699 | 40.482 | 43.188 | 46.459 | 50.641 | 52.294 | 53.809 | 59.335 |

(continued) Critical Values of the Chi-Squared Distribution

| | α | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **$v$** | **0.30** | **0.25** | **0.20** | **0.10** | **0.05** | **0.025** | **0.02** | **0.01** | **0.005** | **0.001** |
| **1** | 1.074 | 1.323 | 1.642 | 2.706 | 3.841 | 5.024 | 5.412 | 6.635 | 7.879 | 10.827 |
| **2** | 2.408 | 2.773 | 3.219 | 4.605 | 5.991 | 7.378 | 7.824 | 9.210 | 10.597 | 13.815 |
| **3** | 3.665 | 4.108 | 4.642 | 6.251 | 7.815 | 9.348 | 9.837 | 11.345 | 12.838 | 16.266 |
| **4** | 4.878 | 5.385 | 5.989 | 7.779 | 9.488 | 11.143 | 11.668 | 13.277 | 14.860 | 18.466 |
| **5** | 6.064 | 6.626 | 7.289 | 9.236 | 11.070 | 12.832 | 13.388 | 15.086 | 16.750 | 20.515 |
| **6** | 7.231 | 7.841 | 8.558 | 10.645 | 12.592 | 14.449 | 15.033 | 16.812 | 18.548 | 22.457 |
| **7** | 8.383 | 9.037 | 9.803 | 12.017 | 14.067 | 16.013 | 16.622 | 18.475 | 20.278 | 24.321 |
| **8** | 9.524 | 10.219 | 11.030 | 13.362 | 15.507 | 17.535 | 18.168 | 20.090 | 21.955 | 26.124 |
| **9** | 10.656 | 11.389 | 12.242 | 14.684 | 16.919 | 19.023 | 19.679 | 21.666 | 23.589 | 27.877 |
| **10** | 11.781 | 12.549 | 13.442 | 15.987 | 18.307 | 20.483 | 21.161 | 23.209 | 25.188 | 29.588 |
| **11** | 12.899 | 13.701 | 14.631 | 17.275 | 19.675 | 21.920 | 22.618 | 24.725 | 26.757 | 31.264 |
| **12** | 14.011 | 14.845 | 15.812 | 18.549 | 21.026 | 23.337 | 24.054 | 26.217 | 28.300 | 32.909 |
| **13** | 15.119 | 15.984 | 16.985 | 19.812 | 22.362 | 24.736 | 25.471 | 27.688 | 29.819 | 34.527 |
| **14** | 16.222 | 17.117 | 18.151 | 21.064 | 23.685 | 26.119 | 26.873 | 29.141 | 31.319 | 36.124 |
| **15** | 17.322 | 18.245 | 19.311 | 22.307 | 24.996 | 27.488 | 28.259 | 30.578 | 32.801 | 37.698 |
| **16** | 18.418 | 19.369 | 20.465 | 23.542 | 26.296 | 28.845 | 29.633 | 32.000 | 34.267 | 39.252 |
| **17** | 19.511 | 20.489 | 21.615 | 24.769 | 27.587 | 30.191 | 30.995 | 33.409 | 35.718 | 40.791 |
| **18** | 20.601 | 21.605 | 22.760 | 25.989 | 28.869 | 31.526 | 32.346 | 34.805 | 37.156 | 42.312 |
| **19** | 21.689 | 22.718 | 23.900 | 27.204 | 30.144 | 32.852 | 33.687 | 36.191 | 38.582 | 43.819 |
| **20** | 22.775 | 23.828 | 25.038 | 28.412 | 31.410 | 34.170 | 35.020 | 37.566 | 39.997 | 45.314 |
| **21** | 23.858 | 24.935 | 26.171 | 29.615 | 32.671 | 35.479 | 36.343 | 38.932 | 41.401 | 46.796 |
| **22** | 24.939 | 26.039 | 27.301 | 30.813 | 33.924 | 36.781 | 37.659 | 40.289 | 42.796 | 48.268 |
| **23** | 26.018 | 27.141 | 28.429 | 32.007 | 35.172 | 38.076 | 38.968 | 41.638 | 44.181 | 49.728 |
| **24** | 27.096 | 28.241 | 29.553 | 33.196 | 36.415 | 39.364 | 40.270 | 42.980 | 45.558 | 51.179 |
| **25** | 28.172 | 29.339 | 30.675 | 34.382 | 37.652 | 40.646 | 41.566 | 44.314 | 46.928 | 52.619 |
| **26** | 29.246 | 30.435 | 31.795 | 35.563 | 38.885 | 41.923 | 42.856 | 45.642 | 48.290 | 54.051 |
| **27** | 30.319 | 31.528 | 32.912 | 36.741 | 40.113 | 43.195 | 44.140 | 46.963 | 49.645 | 55.475 |
| **28** | 31.391 | 32.620 | 34.027 | 37.916 | 41.337 | 44.461 | 45.419 | 48.278 | 50.994 | 56.892 |
| **29** | 32.461 | 33.711 | 35.139 | 39.087 | 42.557 | 45.722 | 46.693 | 49.588 | 52.335 | 58.301 |
| **30** | 33.530 | 34.800 | 36.250 | 40.256 | 43.773 | 46.979 | 47.962 | 50.892 | 53.672 | 59.702 |
| **40** | 44.165 | 45.616 | 47.269 | 51.805 | 55.758 | 59.342 | 60.436 | 63.691 | 66.766 | 73.403 |
| **50** | 54.723 | 56.334 | 58.164 | 63.167 | 67.505 | 71.420 | 72.613 | 76.154 | 79.490 | 86.660 |
| **60** | 65.226 | 66.981 | 68.972 | 74.397 | 79.082 | 83.298 | 84.58 | 88.379 | 91.952 | 99.608 |

Critical Values of the $F$-Distribution




$f_{0.05}(v_1, v_2)$

| $v_2$ \ $v_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 161.45 | 199.50 | 215.71 | 224.58 | 230.16 | 233.99 | 236.77 | 238.88 | 240.54 |
| **2** | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| **3** | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| **4** | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| **5** | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| **6** | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| **7** | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| **8** | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| **9** | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| **10** | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| **11** | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| **12** | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| **13** | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| **14** | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| **15** | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| **16** | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| **17** | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| **18** | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| **19** | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| **20** | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| **21** | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| **22** | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| **23** | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| **24** | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| **25** | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| **26** | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| **27** | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| **28** | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| **29** | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| **30** | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| **40** | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| **60** | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| **120** | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 |
| **∞** | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

| | $f_{0.05}(v_1, v_2)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | | | | | | | | | |
| $v_2$ | **10** | **12** | **15** | **20** | **24** | **30** | **40** | **60** | **120** | $\infty$ |
| **1** | 241.88 | 243.91 | 245.95 | 248.01 | 249.05 | 250.10 | 251.14 | 252.20 | 253.25 | 254.31 |
| **2** | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| **3** | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| **4** | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| **5** | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.36 |
| **6** | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| **7** | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| **8** | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| **9** | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| **10** | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| **11** | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| **12** | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| **13** | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| **14** | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| **15** | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| **16** | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| **17** | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| **18** | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| **19** | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| **20** | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| **21** | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| **22** | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| **23** | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| **24** | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| **25** | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| **26** | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| **27** | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| **28** | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| **29** | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| **30** | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| **40** | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| **60** | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| **120** | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| $\infty$ | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

| | $f_{0.01}(v_1, v_2)$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | | | | | | | | |
| $v_2$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | 4052.18 | 4999.50 | 5403.35 | 5624.58 | 5763.65 | 5858.99 | 5928.36 | 5981.07 | 6022.47 |
| **2** | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 |
| **3** | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 |
| **4** | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 |
| **5** | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 |
| **6** | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 |
| **7** | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 |
| **8** | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 |
| **9** | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 |
| **10** | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 |
| **11** | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.89 | 4.74 | 4.63 |
| **12** | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 |
| **13** | 9.07 | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 |
| **14** | 8.86 | 6.51 | 5.56 | 5.04 | 4.69 | 4.46 | 4.28 | 4.14 | 4.03 |
| **15** | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 |
| **16** | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 |
| **17** | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 |
| **18** | 8.29 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 |
| **19** | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 |
| **20** | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 |
| **21** | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 |
| **22** | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 |
| **23** | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 |
| **24** | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.50 | 3.36 | 3.26 |
| **25** | 7.77 | 5.57 | 4.68 | 4.18 | 3.85 | 3.63 | 3.46 | 3.32 | 3.22 |
| **26** | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 |
| **27** | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 |
| **28** | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 |
| **29** | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 |
| **30** | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 |
| **40** | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 |
| **60** | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 |
| **120** | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.79 | 2.66 | 2.56 |
| $\infty$ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 |

(continued) Critical Values of the $F$-Distribution

| | $f_{0.01}(v_1, v_2)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | | | | | | | | | |
| $v_2$ | **10** | **12** | **15** | **20** | **24** | **30** | **40** | **60** | **120** | $\infty$ |
| **1** | 6055.85 | 6106.32 | 6157.28 | 6208.73 | 6234.63 | 6260.65 | 6286.78 | 6313.03 | 6339.39 | 6365.86 |
| **2** | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50 |
| **3** | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13 |
| **4** | 14.55 | 14.37 | 14.20 | 14.02 | 13.93 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46 |
| **5** | 10.05 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| **6** | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| **7** | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| **8** | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| **9** | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| **10** | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| **11** | 4.54 | 4.40 | 4.25 | 4.10 | 4.02 | 3.94 | 3.86 | 3.78 | 3.69 | 3.60 |
| **12** | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| **13** | 4.10 | 3.96 | 3.82 | 3.66 | 3.59 | 3.51 | 3.43 | 3.34 | 3.25 | 3.17 |
| **14** | 3.94 | 3.80 | 3.66 | 3.51 | 3.43 | 3.35 | 3.27 | 3.18 | 3.09 | 3.00 |
| **15** | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| **16** | 3.69 | 3.55 | 3.41 | 3.26 | 3.18 | 3.10 | 3.02 | 2.93 | 2.84 | 2.75 |
| **17** | 3.59 | 3.46 | 3.31 | 3.16 | 3.08 | 3.00 | 2.92 | 2.83 | 2.75 | 2.65 |
| **18** | 3.51 | 3.37 | 3.23 | 3.08 | 3.00 | 2.92 | 2.84 | 2.75 | 2.66 | 2.57 |
| **19** | 3.43 | 3.30 | 3.15 | 3.00 | 2.92 | 2.84 | 2.76 | 2.67 | 2.58 | 2.49 |
| **20** | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| **21** | 3.31 | 3.17 | 3.03 | 2.88 | 2.80 | 2.72 | 2.64 | 2.55 | 2.46 | 2.36 |
| **22** | 3.26 | 3.12 | 2.98 | 2.83 | 2.75 | 2.67 | 2.58 | 2.50 | 2.40 | 2.31 |
| **23** | 3.21 | 3.07 | 2.93 | 2.78 | 2.70 | 2.62 | 2.54 | 2.45 | 2.35 | 2.26 |
| **24** | 3.17 | 3.03 | 2.89 | 2.74 | 2.66 | 2.58 | 2.49 | 2.40 | 2.31 | 2.21 |
| **25** | 3.13 | 2.99 | 2.85 | 2.70 | 2.62 | 2.54 | 2.45 | 2.36 | 2.27 | 2.17 |
| **26** | 3.09 | 2.96 | 2.81 | 2.66 | 2.58 | 2.50 | 2.42 | 2.33 | 2.23 | 2.13 |
| **27** | 3.06 | 2.93 | 2.78 | 2.63 | 2.55 | 2.47 | 2.38 | 2.29 | 2.20 | 2.10 |
| **28** | 3.03 | 2.90 | 2.75 | 2.60 | 2.52 | 2.44 | 2.35 | 2.26 | 2.17 | 2.06 |
| **29** | 3.00 | 2.87 | 2.73 | 2.57 | 2.49 | 2.41 | 2.33 | 2.23 | 2.14 | 2.03 |
| **30** | 2.98 | 2.84 | 2.70 | 2.55 | 2.47 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| **40** | 2.80 | 2.66 | 2.52 | 2.37 | 2.29 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| **60** | 2.63 | 2.50 | 2.35 | 2.20 | 2.12 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| **120** | 2.47 | 2.34 | 2.19 | 2.03 | 1.95 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| $\infty$ | 2.32 | 2.18 | 2.04 | 1.88 | 1.79 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |