UNIT – II

COMPLEX ANALYSIS

Topic Learning Outcomes:

At the end of this unit, student will be able to

1. Understand the concept of analytic functions, C-R equations, and harmonic functions.
2. Construct analytic functions using Milne-Thomson method.
3. Determine harmonic conjugates, orthogonal trajectories of family of curves, potential functions and stream functions in fluid flow.
4. Develop Taylor's and Laurent's series of a complex function in the given region.
5. Identify the singularities and poles of a complex function.
6. Evaluate the integral of a complex function over a simple closed curve using Cauchy's theorem and Residue theorem.

Complex analysis is the theory of functions of a complex variable. It is the branch of mathematical analysis that investigates functions of complex variable. The term complex number came from Carl Friedrich Gauss in 1831 from the idea of geometric representation of complex numbers. Gauss mentions the theorem that was known later as Cauchy's theorem. Augustin-Luis Cauchy (1789-1857) was a revolutionary in Mathematics and a highly original founder of modern complex function theory who discovered and rediscovered countless amazing results in the area of complex analysis along with constructing the set of complex numbers in 1847.Some of these results that will be emphasized are Cauchy's Integral Theorem and Residue Theorem. The residue calculus is an important tool in evaluating definite integrals, summing series, and discovering integral expressions for the roots of equations and the solutions of differential equations. It is also useful calculating real definite integrals, integrals involving sines and cosines, improper integrals.

Complex analysis is useful in many branches of Mathematics, including algebraic geometry, number theory and analytic combinatory. It is also useful in physics including hydro dynamics, thermodynamics and particularly quantum mechanics. Complex analysis has many applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering. In control theory knowing the location of poles and zeros gives the conclusion about the stability of the system.

Complex equations and their graphs are used to visualize electrical & fluid flow in the real world. Murray R Spiegel described complex analysis as "one of the beautiful as well as useful branches of Mathematics".

Basic Concepts:

- If x, y are real numbers & $i = \sqrt{-1}$ then $z = x + iy$ is a complex number whose real part is x & imaginary part is y.
- Two complex numbers $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2,\ y_1 = y_2$.
- $z = x + iy$ is equal to zero if $x = 0$ & $y = 0$.
- If α is a real number & $z = x + iy$ is complex number then $\alpha z = \alpha x + i\alpha y$ is a complex number.
- If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$, then
  - $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
  - $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
  - $\frac{z_1}{z_2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x^2+y^2}$
  - $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$
- If $z = x + iy$ then $\bar{z} = x - iy$ is called complex conjugate of z.
- If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$ is non-negative real number.
- $z = x + iy$ is represented by a point $P(x, y)$ in the XY plane, $x -$ axis is real axis, $y -$ axis is imaginary axis, plane is complex plane.
- $z = re^{i\theta}$ is the polar form of complex number z where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}(^y/_x)$.
- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the distance between the points z_1&z_2 in complex plane.
- $|z - z_0| = R$ represents complex equation of circle with centre z_0 & radius R.
- $|z - z_0| < R$ represents the region with in, but not on, a circle of radius R centred at the point z_0 , the point z_0 is said to be interior point.
- $|z - z_0| \leq R$ represents the region with in, and on, a circle of radius R centred at the point z_0.
- $|z - z_0| > R$ represents the region outside the circle with centre z_0 & radius R.

Neighbourhood of a point:

A neighbourhood of a point z_0 in the complex plane is the set of all points z such that $|z - z_0| < \delta$, where δ is a small positive real number. Geometrically, a neighbourhood of a point z_0 is the set of all points inside a circle having z_0 as the centre & δ as the radius.

Function of a Complex Variable:

Let S be a set of complex numbers. If to each complex number z in S there corresponds a unique complex number w according to some rule, then $w = f(z)$ is called a complex function of z defined on S.

In Cartesian form $w = f(z) = u(x, y) + i\, v(x, y)$
In polar form $w = f(z) = u(r, \theta) + i\, v(r, \theta)$

Example:

$$f(z) = z^2$$
$$f(z) = (x + iy)^2$$
$$= (x^2 - y^2) + i(2xy)$$
$$= u + iv$$

$\therefore u = x^2 - y^2$, $v = 2xy$

In polar form, $f(z) = \left(re^{i\theta}\right)^2 = r^2e^{2i\theta} = r^2(\cos 2\theta + i\sin 2\theta)$

$\therefore u = r^2 cos2\theta, \quad v = r^2 sin2\theta.$

Analytic function:

A function $f(z)$ is said to be analytic at a point z_0 , if *f(z)* is differentiable not only at z_0 but at every point in some neighborhood of z_0.

- A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.
- An analytic function is also known as holomorphic, regular and monogenic function.

Entire Function:

A function which is analytic everywhere (for all z in the complex plane) is known as entire function.

Example: Polynomials, rational functions, $e^z, sinz, cosz$ are entire.

A fundamental result of complex analysis is the C-R equations which gives the conditions, a function must satisfy in order for a complex generalization of the derivative, the so-called complex derivative, to exist. When the complex derivative is defined ‘everywhere’, the function is said to be analytic.

Cauchy Riemann (C-R) Equations in Cartesian form:

If $f(z) = u(x, y) + i\, v(x, y)$ is differentiable at $z = x + iy$ then at this point the first order partial derivatives of u and v exist and satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the Cauchy- Riemann equations.

Note: These conditions are only necessary for $f(z)$ to be analytic. The sufficient conditions for $f(z)$ to be analytic are that $u, v, u_x, u_y, v_x \& v_y$ are continuous.

Example: (i) $f(z) = e^z = e^x(cosy + isiny)$

$$u(x, y) = e^x cosy, \quad v(x, y) = e^x siny$$
$$u_x = e^x cosy, \qquad u_y = -e^x siny$$
$$v_x = e^x siny, \qquad v_y = e^x cosy$$

Partial derivatives u_x, v_x, u_y, v_y are continuous & C-R equations $u_x = v_y$ & $u_y = -v_x$ are satisfied for all x & y.

$\therefore f(z) = e^z$ is analytic everywhere.

Harmonic Functions:

A function $\emptyset$ is said to be a Harmonic function if it satisfies Laplace equation $\nabla^2\emptyset = 0$.

In Cartesian form $\emptyset(x, y)$ is Harmonic if $\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$.

Consequences of Analytic function:

1. The real and imaginary part of an analytic function are harmonic. If $f(z) = u + iv$ is an analytic function, then u & v are harmonic conjugate function to each other.

Note: The converse of the above result is not true i.e., we can give examples of functions like $u = e^x cosy$ & $v = 2xy$, satisfying Laplace equation but not satisfying C-R equations.

Example: Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$
$$u_x = 3x^2 - 3y^2 + 6x \qquad ; \; u_{xx} = 6x + 6$$
$$u_y = -6xy - 6y \qquad ; \; u_{yy} = -6x - 6$$
$$\therefore \; u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$$

Hence u is harmonic.

2. If $f(z) = u + iv$ is analytic function, then the family of curves $u(x, y) = u_0$ & $v(x, y) = v_0$, u_0 & v_0 being constants, intersect each other orthogonally.

Note: Converse of the above result is not true.

Problems:

1. Show that analytic function with constant real part is constant.

Proof: Given $f(z) = u + iv$ is analytic

Given u is constant = k

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0$$

Since f is an analytic function $\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0$

$\Rightarrow$ v is constant

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i(0) = 0$$

$\Rightarrow f(z)$ is constant.

2. Prove that analytic function with constant modulus is constant.

Proof: $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2} = c \quad \text{(given)}$$

$$|f(z)|^2 = u^2 + v^2 = c^2$$

differentiating w.r.t x & y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$uu_x + uu_y = 0 \qquad \dots\dots(i)$$

And

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

Using C-R equations

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \qquad \dots\dots.(ii)$$

Squaring and adding (i) and (ii)

$$\Rightarrow \; (u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow |f'(z)|^2 = 0$$

$$\Rightarrow f(z) = \text{constant}$$

3. If $f(z)$ is analytic function , show that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

Proof: let $f(z) = u + iv$ be analytic.

$$\therefore \quad |f(z)| = \sqrt{u^2 + v^2} \;\text{ or }\; |f(z)|^2 = u^2 + v^2$$

differentiate w.r.t. x partially twice we get

$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2[u\,u_{xx} + u_x^2 + v\,v_{xx} + v_x^2]$----------------(i)

Similarly, we can also get

$$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u\,u_{yy} + u_y^2 + v\,v_{yy} + v_y^2\right]$$----------------(ii)

Adding (i) and (ii) we have,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2\left[u\left(u_{xx} + u_{yy}\right) + v\left(v_{xx} + v_{yy}\right) + u_x^2 + v_x^2 + u_y^2 + v_y^2\right]$$

$$\text{Since } f(z) = u + iv \text{ is analytic}, u \text{ and } v \text{ are harmonic. hence}$$

$$u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$$

Further we also have *C-R* equations $v_y = u_x\,, u_y = -v_x$

using these results in the R.H.S, we have

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2[u.0 + v.0 + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2]$$

$$i.e., \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2[2u_x^2 + 2v_x^2] = 4[u_x^2 + v_x^2] \qquad --- \text{(iii)}$$

But $\qquad f'(z) = u_x + iv_x$

$$\therefore \quad |f'(z)| = \sqrt{u_x^2 + v_x^2} \;\; or \; |f'(z)|^2 = u_x^2 + v_x^2$$

Using this in R.H.S of (iii) we have $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$

4. If $f(z)$ is a regular function of z show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = |f'(z)|^2$$

Proof: let $f(z) = u + iv$ be regular (analytic)function.

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2 + v^2}}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]$$

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \frac{1}{\sqrt{u^2 + v^2}}(u^2u_x^2 + v^2v_x^2 + 2uv\,u_xv_x)$$

Similarly $\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{1}{\sqrt{u^2+v^2}}\left(u^2u_y^2 + v^2v_y^2 + 2uv\,u_yv_y\right)$

Adding $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \frac{1}{u^2+v^2}[(u^2u_x^2 + v^2v_x^2 + 2uv\,u_xv_x) + \left(u^2u_y^2 + v^2v_y^2 + 2uv\,u_yv_y\right)]$

Since $f(z) = u + iv$ is analytic we have C-R equations $v_y = u_x\,, u_y = -v_x$,using these in the second bracket of the R.H.S we have,

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2$$
$$= \frac{1}{u^2+v^2}[(u^2u_x^2 + v^2v_x^2 + 2uv\,u_xv_x) + (u^2v_x^2 + v^2u_x^2 - 2uv\,u_xv_x)]$$
$$= \frac{1}{u^2+v^2}[u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2)]$$
$$= \frac{1}{u^2+v^2}(u_x^2+v_x^2)(u^2+v^2)$$
$$= u_x^2 + v_x^2$$
$$= |f'(z)|^2$$

Construction of Analytic Function Using Milne-Thomson Method

Working procedure:

- Given u or v as functions x, y we find $u_x\,, u_y$ or $v_x\,, v_y$ and consider $f'(z) = u_x + iv_x$
- Given u, we use $C - R$ equation $v_x = -u_y$ or given v we use $u_x = v_y$ so that
- $f'(z) = u_x - iu_y$ or $f'(z) = v_y + iv_x$
- We substitute the expression for the partial derivatives in R.H.S and then put $x = z$, $y = 0$ to obtain $f'(z)$ as a function of z
- Integrating w.r.t z, we get $f(z)$

Applications:

Complex potential, steam and potential functions:

In a fluid flow, the analytic function $w(z) = \emptyset(x,y) + i\psi(x,y)$ is known as complex potential, $\emptyset(x,y)$is known as velocity potential, $\psi(x,y)$ is known as stream function. The velocity potential function and stream function are harmonic functions that satisfy Laplace equation. The equipotential lines $\emptyset(x,y) = c$, the stream lines $\psi(x,y) = d$ cut orthogonally.

In the study of electrostatics and gravitational fields, the curves $\emptyset(x,y) = c$and $\Psi(x,y) = d$ are called equipotential lines and lines of force respectively.

In heat flow problems, the curves $\emptyset(x,y) = c$ and $\psi(x,y) = d$ are known as isothermals and heat flow lines respectively.

Problems:

1. If $w(z) = u + iv$ is the complex potential function. Construct analytic function $w(z)$ whose potential function is $u = log\sqrt{x^2 + y^2}$. Also find the flux function v

Solution: $u = log\sqrt{x^2 + y^2} = log(x^2 + y^2)^{1/2} = \frac{1}{2}log(x^2 + y^2)$

$$\therefore u_x = \frac{1}{2}.\frac{1}{x^2 + y^2}.2x = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{1}{2}.\frac{1}{x^2 + y^2}.2y = \frac{y}{x^2 + y^2}$$

Consider $f'(z) = u_x + iv_x$ but $v_x = -u_y$ ($C - R$ equation)

$$\therefore\ f'(z) = u_x - iu_y = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \qquad \ldots\ldots(1)$$

Putting $x = z$ and $y = 0$,

$$f'(z) = \frac{z}{z^2 + 0} - i0 = \frac{1}{z} \qquad \therefore\ f(z) = u_x + iu_y = \int \frac{1}{z}dz + c$$

i.e., $w(z) = logz + c$, where c is complex constant.

Separating real and imaginary parts, $v = tan^{-1}\frac{y}{x}$.

2. Find the analytic function $f(z)$ whose imaginary part is $e^x(xsin\,y + y\cos y)$

Solution: Let $v = e^x(xsin\,y + y\cos y)$

$$\therefore v_x = e^x(sin\,y) + (xsiny + y\cos y)e^x \qquad \text{[by product rule]}$$

i.e. $$v_x = e^x(sin\,y + xsiny + y\cos y) \qquad \ldots\ldots(i)$$

also $$v_y = e^x(x\,cos\,y - y\,siny + \cos y) \qquad \ldots\ldots(ii)$$

Consider $f'(z) = u_x + iv_x$ but $u_x = v_y$ ($C - R$ equation)

i.e., $f'(z) = v_y + iv_x = e^x(x\,cos\,y - y\,siny + \cos y) +$

$ie^x(sin\,y + xsiny + y\cos y)$

Putting $x = z$ and $y = 0$,

$$f'(z) = e^z(z + 1) \qquad \text{(since } sin\,0 = 0, cos\,0 = 1)$$

$$\therefore\ f(z) = \int (z + 1)e^z dz + c$$

Integrating by parts,

$$f(z) = (z+1)e^z - \int e^z.1dz + c = (z+1)e^z - e^z + c$$

i.e. $f(z) = ze^z + c.$

3. Find the analytic function whose real part is $\frac{x^4-y^4-2x}{x^2+y^2}$ hence determine v.

Solution: Given $u = \frac{x^4-y^4-2x}{x^2+y^2}$

$$\therefore u_x = \frac{(x^2+y^2)(4x^3-2) - (x^4-y^4-2x)2x}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(-4y^3) - (x^4-y^4-2x)2y}{(x^2+y^2)^2}$$

Consider $f'(z) = u_x + iv_x$ but $v_x = -u_y$ ($C-R$ equation)

$\therefore \quad f'(z) = u_x - iu_y$

Putting $x = z$ and $y = 0$,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

i.e., $f'(z) = \frac{z^2(4z^3-2)-(z^4-2z)2z}{(z^2)^2} - i(0)$

$$= \frac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4} = \frac{2z^5 + 2z^2}{z^4}$$

i.e., $f'(z) = 2\frac{z^5}{z^4} + 2\frac{z^2}{z^4} = 2z + \frac{2}{z^2}$

$$\therefore \quad f(z) = \int \left(2z + \frac{2}{z^2}\right) dz + c = z^2 - \frac{2}{z} + c$$

Thus $f(z) = z^2 - \frac{2}{z} + c$

To find v, separate the R.H.S of $f(z)$ into real and imaginary parts.

i.e., $u + iv = (x+iy)^2 - \frac{2}{x+iy} + c$

$$= (x^2 + i^2y^2 + 2xiy) - \frac{2(x-iy)}{(x+iy)(x-iy)} + c$$

$$= (x^2 - y^2) + 2xiy - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \left[x^2 - y^2 - \frac{2x}{x^2+y^2}\right] + i\left[2xy + \frac{2y}{x^2+y^2}\right] + c$$

$$u + iv = \left[\frac{x^4 - y^4 - 2x}{x^2 + y^2}\right] + i\left[\frac{2x^3y + 2xy^3 + 2x}{x^2 + y^2}\right] + c$$

Equating real and imaginary parts we observe that the real part u is same as in the given problem and the required $v = \frac{2x^3y+2xy^3+2x}{x^2+y^2} + c$.

3. Determine the analytic function $f(z) = u + iv$ Given that

$u = e^{2x}(xcos2y - ysin2y)$.

Solution: $u = e^{2x}(xcos2y - ysin2y)$

$$u_x = e^{2x}.cos2y + 2e^{2x}(xcos2y - ysin2y)$$

$$= e^{2x}(cos2y + 2xcos2y - 2ysin2y)$$

$$u_y = e^{2x}(-2xsin2y - 2ycos2y - sin2y)$$

$$= -e^{2x}(2xsin2y + 2ycos2y + sin2y)$$

Consider $f'(z) = u_x + iv_x = u_x - iu_y$ $(by\ C - R$ equation)

Putting $x = z$ and $y = 0$,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

i.e., $f'(z) = e^{2z}(1 + 2z)$

$$\therefore \quad f(z) = \int (1 + 2z)e^{2z}dz$$

$$\therefore \quad f(z) = (1 + 2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} = \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

Thus $f(z) = ze^{2z} + c$

Also $f(z) = u + iv = (x + iy)e^{2(x+iy)} + c$

$$= e^{2x}(x + iy)(cos2y + i\ sin2y) + c$$

i.e., $f(z) = e^{2x}(xcos2y - y\ sin2y) + e^{2x}(x\ sin2y + y\ cos2y) + c$

4. Find the analytic function $f(z)$ whose real part is $\frac{sin2x}{\cosh 2y - \cos 2x}$ and hence find the imaginary part.

Solution: Given

$$u = \frac{sin2x}{\cosh 2y - \cos 2x}$$

$$\therefore u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cos\text{h } 2y - \cos 2x)^2}$$

$$u_y = \frac{-\sin 2x\,(2\sin\text{h } 2y)}{(\cos\text{h } 2y - \cos 2x)^2}$$

Consider $f'(z) = u_x + iv_x = u_x - iu_y$ $(by\ C - R$ equation)

Putting $x = z$ and $y = 0$,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

i.e.,

$$f'(z) = \frac{(1 - cos2z)(2cos2z) - 2sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\,(2\cos^2 2z + \sin^2 2z)}{(1 - cos2z)^2}$$

$$= \frac{-2(1 - \cos 2z)}{(1 - cos2z)^2} = \frac{-2}{(1 - cos2z)} = \frac{-2}{2sin^2 z}$$

Thus $f'(z) = -cosec^2 z$ $\Rightarrow$ $f(z) = cotz + c$

Separate $cotz = \cot\,(x + iy)$ into real and imaginary parts to determine the imaginary part v

Consider $f(z) = cotz + c$

i.e., $u + iv = \cot(x + iy) = \frac{\cos(x+iy)}{\sin(x+iy)} + c$

$$= \frac{\cos(x + iy)\sin\,(x + iy)}{\sin(x + iy)\sin\,(x + iy)} + c$$

$$= \frac{\frac{1}{2}[\sin(x - iy + x + iy) + \sin\,(x - iy - x - iy)]}{\frac{1}{2}[\cos(x + iy - x + iy) - \cos\,(x + iy + x - iy)]} + c$$

$$= \frac{\sin 2x + \sin\,(-2iy)}{\cos(2iy) - \cos\ 2x} = \frac{\sin 2x - i\sin\ h2y}{\cos h2y - \cos\ 2x} + c$$

Thus $u + iv = \left[\frac{\sin 2x}{\cos h2y - cos2x}\right] + i\left[\frac{-sinh2y}{cosh2y - cos2x}\right] + c$

(It may be observed that the real part u is the given problem)

The required imaginary part $v = \frac{-sinh2y}{cosh2y-cos2x} + c$.

5. If $f(z) = \phi + i\psi$ represents complex potential of a two dimensional fluid flow where ϕ is velocity potential, ψ is stream function. the velocity potential ϕ for given stream function $\psi = (x^2 - y^2) + \frac{x}{x^2+y^2}$.

Solution: $\psi_x = 2x + \frac{(x^2+y^2)1-x.2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\psi_y = -2y + \frac{(x^2+y^2)0 - x.2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Consider $f'(z) = \phi_x + i\psi_x$ but $\phi_x = \psi_y$ $(by\ C - R$ equation)

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [\psi_y]_{(z,0)} - i[\psi_x]_{(z,0)}$$

i.e. $f'(z) = 0 + i\left(2z + \frac{-z^2}{(z^2)^2}\right) = i\left(2z - \frac{1}{z^2}\right)$

$$\therefore f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c = i\left(z^2 + \frac{1}{z}\right) + c$$

Thus $f(z) = i\left(z^2 + \frac{1}{z}\right) + c$

To find ϕ,separate the R.H.S into real and imaginary parts

i.e., $\phi + i\Psi = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$

$$= i\left\{(x^2 + i^2y^2 + 2xiy) + \frac{x-iy}{(x+iy)(x-iy)}\right\} + c$$

$$= i\{(x^2 - y^2) + 2xiy\} + i\left\{\frac{x-iy}{x^2+y^2}\right\} + c$$

$$= i(x^2 - y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$$

i.e. $\phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c.$

equating the real and imaginary parts we observe that the imaginary part $\Psi$is same as the given problem and the required $\phi = -2xy + \frac{y}{x^2+y^2} +$ c.

6. In a two dimensional fluid flow, the complex potential is w(z) = u + iv show that velocity potential $u = e^x(x\cos y - y\sin y)$ is harmonic and find its harmonic conjugate.

Solution: $u = e^x(x\cos y - y\sin y)$

$$u_x = e^x.\cos y + (x\cos y - y\sin y)e^x$$

$$i.e., \quad u_x = e^x(\cos y + x\cos y - y\sin y)$$

Now $\quad u_{xx} = e^x.\cos y + (\cos y + x\cos y - y\sin y)e^x$

$i.e., \quad u_{xx} = e^x(2\cos y + x\cos y - y\sin y) \quad \ldots..(i)$

Also $\quad u_y = e^x(-x\sin y - [y\cos y + \sin y])$

$$= -e^x(x\sin y + y\cos y + \sin y)$$

Now $u_{yy} = -e^x(x\cos y + [-y\sin y + \cos y] + \cos y\ )$

$i.e., \quad u_{yy} = -e^x(2\cos y + x\cos y - y\sin y) \quad \ldots..(ii)$

$(i) + (ii)$ gives $u_{xx} + u_{yy} = 0 \quad \therefore$ u is harmonic.

To find $f'(z)$

consider $f'(z) = u_x + iv_x$, but $v_x = -u_y$ (C − R equation)

i.e., $f'(z) = u_x - iu_y$ putting $x = z$ and $y = 0$ then
integrating & seperating real & imaginary parts,

$$v = e^x(x\sin y + y\cos y) + c.$$

7. Find *f(z)* given that $u - v = x^3 + 3x^2y - 3xy^2 - y^3$.
Solution: Partially differentiating w. r. t. x and y

$u_x - v_x = 3x^2 + 6xy - 3y^2$ ________________(1)

$u_y - v_y = 3x^2 - 6xy - 3y^2$ ________________(2)

By C − R equation, the second equation reduces to

$-v_x - u_x = 3x^2 - 6xy - 3y^2$ ________________(3)

(1) +(3) gives $v_x = 3y^2 - 3x^2$

(1) - (3) gives $u_x = 6xy$

consider $f'(z) = u_x + iv_x = 6xy - i(3x^2 - 3y^2)$

putting $x = z$ and $y = 0$

$f'(z) = 0 - i(3z^2)$

Thus $f(z) = -i(z^3) + c$

Exercise:

1. Find the orthogonal trajectories of the following family curves.
   (i) $x^4 + y^4 - 6x^2y^2 =$ constant.
   (ii) $xy + e^{-x}cosy = c$

Ans: (i) $x^3y - xy^3$ = constant (ii) $y^2 - x^2 - 2e^{-x}siny = c$

2. Construct the analytic function whose imaginary part is
   (i) –sinx sinhy (ii) $e^x[(x^2 - y^2)cosy - 2xysiny]$

Ans: (i) cosz + c (ii) $ie^z z^2 + c$

3. Construct the analytic function whose real part is $e^{x^2-y^2}cos2xy$

Ans: $e^{z^2} + C$

4. In a two dimensional fluid flow if the stream function is given by $\tan^{-1}\frac{y}{x}$ determine the velocity potential.
5. The Laplace equation governs steady heat conduction in a rectangular plate. Prove that the temperature function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$satisfies Laplace equation and determine the corresponding analytic function u+ iv.
6. If the potential function is log $(x^2 + y^2)$, find the flux function and the complex potential function.
7. An electrostatic field in the xy-plane is given by the potential function $\emptyset = x^2 - y^2$, find the stream function.

Equation of a curve in complex plane:

A pair of equations of the form x = x(t), y = y(t), where 't' is a parameter represents a curve C in the x-y plane. The equation z = z(t) = x(t) + i y(t) represents the equation of the curve C in the complex form, where 't 'varies over an appropriate interval.

Example: $z = a\,e^{it}, 0 \le t \le 2\pi$ represents equation of a circle centred at origin, radius a.

Simple curve:

The curve C is said to be simple if it does not intersect itself.

The curve C given by $z(t) = x(t) + iy(t)$, $a \le t \le b$ is simple if $z(t_1) \ne z(t_2)$ for any two different values t_1 and t_2 in (a, b).

Example: circle, semicircle etc.,

Simple closed curve:

A curve is said to be a simple closed curve if it is simple and the end points coincide.

Example: circle, triangle, rectangle etc.,

Smooth curve:

A curve is said to be smooth if there exists a unique tangent at each of points(differential curves)

A smooth curve does not contain sharp corners.

Example: Arc of a circle is a smooth curve. Triangle is not a smooth curve.

Contour:

A continues chain of finite number of smooth curves is called a contour.

Example:

Circle is contour (single smooth curve.)

Triangle is contour (chain of three lines)

Rectangle is a contour (chain of four lines)

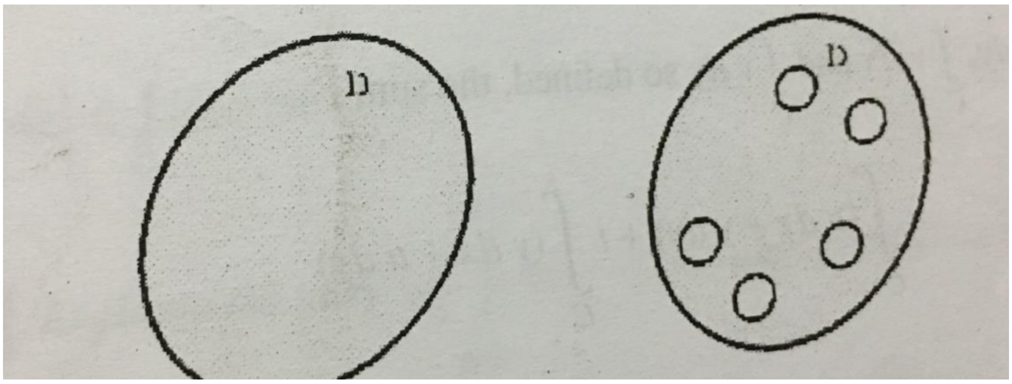
Positively oriented curve:

A curve is called a positive orientated if any point z(t) on the curve $z(t) = x(t) + iy(t), \quad a \le t \le b$ varies from the initial point to the terminal point as t varies from a to b.

Simply connected region:

A region D is said to be simply connected if every simple closed curve lying entirely in D can be shrunk a point without crossing the boundary of D.

A region which is not simply connected is called a multiply connected (region having holes).



Line integral of a complex function:

The line integral of a complex function *f(z)* from z_1 to z_2 along a curve C is defined by

$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$, where C is called the path of integration.

Taylor's Theorem:

If a complex function f(z) is analytic at all points inside and on the circle C: $|z-a| = r$ .Then at each point inside C, we have

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + \cdots = f(a) + \sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$$

the above series is called Taylor's series expansion for the function f(z) about the point z = a.

In particular, if a = 0, $f(z) = f(0) + zf'(0) + z^2\frac{f''(a)}{2!} + \cdots = f(0) + \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$ is called Maclaurin's series for the function f(z).

Examples:

1. Exponential series : $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\ldots$
2. Logarithmic series: $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3}\ldots$
3. Sine series: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}\ldots$
4. Cosine series: $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}\ldots$

Problems:

1. Expand $f(z) = \frac{z-1}{z+1}$ in Taylor's series about the point **(i)** z = 0 **(ii)** z = 1.

Solution: $f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$

(i) Taylor's series expansion of f(z) about the point z = 0 is

$f(z) = f(0) + \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$ ------ (1)

$f(0) = -1, f^n(z) = -2\left(\frac{(-1)^n n!}{(z+1)^{n+1}}\right), f^n(0) = 2(-1)^{n+1}n!$

Substituting in (1),

$\frac{z-1}{z+1} = -1 + \sum_{n=1}^{\infty}2\frac{z^n}{n!}(-1)^{n+1}(n!) = -1 + \sum_{n=1}^{\infty}2(-1)^{n+1}z^n$

(ii) Taylor's series expansion of f(z) about the point z = 1 is

$f(z) = f(1) + \sum_{n=1}^{\infty}\frac{f^{(n)}(1)}{n!}(z-1)^n$ ----(2)

$f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}, f(1) = 0$

$$f^n(z) = -2\left(\frac{(-1)^n n!}{(z+1)^{n+1}}\right), f^n(1) = -2\frac{(-1)^n n!}{2^{n+1}} = \frac{(-1)^{n+1}n!}{2^n}$$

Substituting in (2),

$\frac{z-1}{z+1} = \sum_{n=1}^{\infty}\frac{(z-1)^n}{2^n}(-1)^{n+1}$

2. Find the Taylor's series expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point z = i.

Solution: $f(z) = \frac{2z^3+1}{z^2+z} = (2z-2) + \frac{2z+1}{z^2+z} = 2(z-1) + \frac{1}{z} + \frac{z}{z+1}$

By Taylor series expansion of f(z) about the point z = i is

$f(z) = f(i) + \sum_{n=1}^{\infty} \frac{f^{(n)}(i)}{n!}(z-i)^n$ ------ (1)

$$f(i) = \frac{i}{2} - \frac{3}{2}$$

$$f'(z) = 2 - \frac{1}{z^2} + \frac{1}{(z+1)^2}, f'(i) = 3 + \frac{i}{2}$$

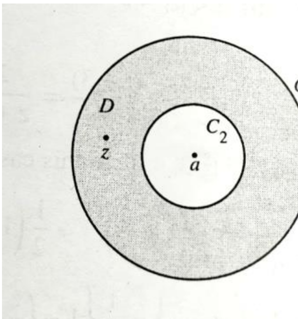
$$f^n(z) = \frac{(-1)^n\, n!}{z^{n+1}} + \frac{(-1)^n\, n!}{(z+1)^{n+1}}, n \geq 2, f^n(i) = (-1)^n n! \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}, n \geq 2$$

Substituting in (1),

$$f(z) = \frac{i}{2} - \frac{3}{2} + 3 + \frac{i}{2}(z-i) + \sum_{n=2}^{\infty}(-1)^n n! \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

Laurent's Theorem:

A complex function f(z) is analytic inside and on the boundary of the annular region D bounded by two concentric circles C_1 and C_2 centred at a with C_1 as outer circle and C_2 as inner circle, then for all z in D,



$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ ------ (1)

The above series is known as Laurent's series of f(z) about the point z = a, where

$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0,1,2,3 \ldots.$ ----- (2)

$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1,2,3 \ldots.$ ----- (3)

Note:

1. The first part in the R.H.S of (1) is called the ANALYTIC part of f(z) and the second part is called the PRINCIPAL part of f(z).
2. R.H.S of (1) is a power series when it contains non-negative powers of (z-a).
3. The evaluation of the coefficients a_n, b_n by using (2) and (3) is more complex. Usually a rational function is expanded by using known expansions like Binomial, Exponential and Logarithmic etc.,

Problems:

1. Obtain the power series expansion of the function $f(z) = \frac{z^2-1}{z^2+5z+6}$ in the following regions.

(i) $|z| < 2$ **(ii)** $2 < |z| < 3$ **(iii)** $|z| > 3$

Solution: $f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{5z+7}{z^2+5z+6} = 1 - \frac{5z+7}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

(i) $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1 \text{and} \left|\frac{z}{3}\right| < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{2(1+{}^{z}/_{2})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left\{1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + - - -\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{2}\right)^3 - - -\right\}$$

(ii) $2 < |z| < 3 \Rightarrow \left|\frac{2}{z}\right| < 1 \text{and} \left|\frac{z}{3}\right| < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + - - -\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{2}\right)^3 - - -\right\}$$

(iii) $|z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1 \text{and} \left|\frac{2}{z}\right| < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{z(1+{}^{3}/_{z})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + - - -\right\} - \frac{8}{z}\left\{1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + - -\right\}$$

2. Find the power series expansion of $f(z) = \frac{z}{(z^2+1)(z^2+4)}$ in the following regions.

(i) $|z| < 1$ **(ii)** $1 < |z| < 2$ **(iii)** $|z| > 2$

Solution: $f(z) = \frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3}\left\{\frac{1}{z^2+1} - \frac{1}{z^2+4}\right\}$

(i) $|z| < 1 \Rightarrow |z|^2 < 1 \text{and} \left|\frac{z}{2}\right|^2 < 1$

$$f(z) = \frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3}\left\{\frac{1}{z^2+1} - \frac{1}{4(1+(z/2)^2)}\right\}$$

$$= \frac{z}{3}(1+z^2)^{-1} - \frac{1}{4}\left[1+(z/2)^2\right]^{-1}$$

$$= \frac{z}{3}(1 - z^2 + z^4 - - -) - \frac{z}{12}\left\{1 - \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^4 - \left(\frac{z}{2}\right)^6 + - - -\right\}$$

(ii) $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1 \text{and} \left|\frac{1}{z}\right|^2 < 1, \left|\frac{z}{2}\right| < 1 \text{and} \left|\frac{z}{2}\right|^2 < 1$

$$f(z) = \frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3}\left\{\frac{1}{z^2(1+\frac{1}{z^2})} - \frac{1}{4(1+\left(\frac{z}{2}\right)^2)}\right\}$$

$$= \frac{1}{3z}\left[1+\frac{1}{z^2}\right]^{-1} - \frac{z}{12}\left[1+\left(\frac{z}{2}\right)^2\right]^{-1}$$

$$= \frac{1}{3}\left[\frac{1}{z}\left\{1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3 + - -\right\} - \frac{z}{4}\left\{1 - \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^4 - \left(\frac{z}{2}\right)^6 + - -\right\}\right]$$

(iii) $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1 \text{and} \left|\frac{2}{z}\right|^2 < 1, \left|\frac{1}{z}\right| < 1 \text{and} \left|\frac{1}{z}\right|^2 < 1$

$$f(z) = \frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3}\left\{\frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{z^2\left(1+\frac{4}{z^2}\right)}\right\}$$

$$= \frac{z}{3}\left\{\frac{1}{z^2}\left[1+\frac{1}{z^2}\right]^{-1}\right\} - \frac{1}{z^2}\left[1+\frac{4}{z^2}\right]^{-1}$$

$$= \frac{1}{3z}\left[\left\{1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 - - -\right\} - \left\{1 - \frac{4}{z^2} + \left(\frac{4}{z^2}\right)^2 - - -\right\}\right]$$

3. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for

(i) $0 < |z+1| < 2$, **(ii)** $|z+1| > 2$

Solution: $f(z) = \frac{1}{(z+1)(z+3)}$

Let $u = z+1$, then

(i) $0 < |z+1| < 2 \Rightarrow |u| < 2 \text{ or } \left|\frac{u}{2}\right| < 1$

$$f(z) = \frac{1}{u(u+2)} = \frac{1}{2u(1+\frac{u}{2})} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$=\frac{1}{2u}\left\{1-\left(\frac{u}{2}\right)+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+---\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{u}-\frac{1}{2}+\frac{u}{2^2}-\frac{u^2}{2^3}+---\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{z+1}-\frac{1}{2}+\left(\frac{z+1}{2^2}\right)+\frac{1}{2}\left(\frac{z+1}{2}\right)^2----\right\}$$

(ii) $|z+1|>2 \Longrightarrow |u|>2\,,\left|\frac{2}{u}\right|<1$

$$f(z)=\frac{1}{u(u+2)}=\frac{1}{u^2(1+\frac{2}{u})}=\frac{1}{u^2}(1+\frac{2}{u})^{-1}$$

$$=\frac{1}{u^2}\left\{1-\left(\frac{2}{u}\right)+\left(\frac{2}{u}\right)^2-\left(\frac{2}{u}\right)^3+---\right\}$$

$$=\frac{1}{(z+1)^2}-\frac{2}{(z+1)^3}+\frac{2^2}{(z+1)^4}-\frac{2^3}{(z+1)^5}+----$$

4. Expand $f(z)=\frac{2z^2-3z+4}{(z-1)(z+2)^2}$ in Laurent's series valid for

(i) $1<|z|<2$, **(ii)** $|z+1|>2$

Solution: $f(z)=\frac{2z^2-3z+4}{(z-1)(z+2)^2}=\frac{1}{3}\frac{1}{(z-1)}+\frac{5}{3}\frac{1}{(z+2)}-\frac{6}{(z+2)^2}$

(i) $1<|z|<2 \Rightarrow \left|\frac{1}{z}\right|<1\text{and}\left|\frac{z}{2}\right|<1$

$$f(z)=\frac{1}{3z}\frac{1}{(1-{}^{1}/_{z})}+\frac{5}{3}\frac{1}{2\,(1+{}^{z}/_{2})}-\frac{6}{4\left(1+{}^{z}/_{2}\right)^2}$$

$$=\frac{1}{3z}\left[1-{}^{1}/_{z}\right]^{-1}+\frac{5}{6}\left[1+{}^{z}/_{2}\right]^{-1}-\frac{3}{2}\left[1+{}^{z}/_{2}\right]^{-2}$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\frac{5}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}z^n-\frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2^n}z^n$$

(ii) $|z+1|>2, let\ u=z+1 so\ that\ |u|>2\ or\ \left|\frac{2}{u}\right|<1\ and\ \left|\frac{1}{u}\right|<1$

$$f(z)=\frac{1}{3}\frac{1}{(u-2)}+\frac{5}{3}\frac{1}{(u+1)}-\frac{6}{(u+1)^2}$$

$$=\frac{1}{3u}\frac{1}{(1-{}^{2}/_{u})}+\frac{5}{3}\frac{1}{u(1+{}^{1}/_{u})}-\frac{6}{u^2\left(1+{}^{1}/_{u}\right)^2}$$

$$=\frac{1}{3u}\left[1-{}^{2}/_{u}\right]^{-1}+\frac{5}{3u}\left[1+{}^{1}/_{u}\right]^{-1}-\frac{6}{u^2}\left[1+{}^{1}/_{u}\right]^{-2}$$

$$=\frac{1}{3u}\sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^n+\frac{5}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{u^{n+1}}-6\sum_{n=0}^{\infty}\frac{(-1)(n+1)}{u^{n+2}}$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\frac{2}{(z+1)^{n+1}}+\frac{5}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z+1)^{n+1}}-6\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z+1)^{n+2}}$$

5. Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ in Laurent's series about the point z = 1.

Solution: Let u = z-1

$$f(z) = \frac{e^{2(u+1)}}{u^3} = \frac{e^2}{u^3} e^{2u} = \frac{e^2}{u^3} \sum_{n=0}^{\infty} \frac{(2u)^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{2^n u^{n-3}}{n!} = e^2 \sum_{n=0}^{\infty} \frac{2^n (z-1)^{n-3}}{n!}.$$

Singular point:

A point 'a' at which a complex function f(z) fails to be analytic is called singular point of f(z).

Example: z = 0 is a singular point for f(z)=1/z.

Singularities of an analytic function:

Isolated Singular point:

A singular point 'a' of f(z) is said to be an Isolated singular point if there exists a neighbourhood of 'a' which contains no other singular point of f(z).

Example: z = 0 is a Isolated singular point of f(z) =1/z.

Removable Singularity:

Laurent's expansion of f(z) about the point'a' is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

If all the negative powers of (z-a) [Principal part] in the above series are zero, then such a singularity is called removable singularity. The singularity can be removed by defining f(z) such that it becomes analytic at z = a. If $\lim_{z \to a} f(z)$ exists finitely, then z = a is a removable singularity.

Example: z = 0 is a removable singularity of $f(z) = \frac{z-\sin z}{z^2}$.

Zeros:

A point 'a' is called a zero of an analytic function f(z) if f(a) =0

Pole:

Laurent's expansion of f(z) about 'a' is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

If the principal part terminates at n = m where $m \geq 1$ so that $b_{m+1} = b_{m+2} = \cdots = 0$, then f(z) is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

Then the point z = a is called a pole of order m of f(z).

Pole of order one called simple pole, pole of order two is called double pole, pole of order three is called triple pole.

Determination of poles:

If $f(z)=\frac{\emptyset(z)}{(z-a)^m}$ where $\emptyset(z)$ is analytic and not zero at the point 'a', then 'a' is a pole of order m of f(z). The poles of f(z) may be obtained by solving the denominator of f(z)=0. i,e (equating the denominator of f(z) to zero).

Examples:

1. $f(z)=\frac{1}{\cos z-\sin z}, z=\frac{\pi}{4}$ is a simple pole of f(z).
2. $f(z)=\frac{\cos z}{(z-1)^3}, z=1$ is a triple pole of f(z).
3. $f(z)=\frac{z^2}{(z+1)^2(z^2+1)}, z=-1$ is a double pole and $z=\pm i$ are simple poles.

Essential Singularity: Laurent's expansion of f(z) about 'a' is

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n+\sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}$$

If the number of negative powers of (z –a) in the above series [principal part] is infinite, then z = a is called an essential singularity. In such case $\lim_{z\to a} f(z)$ does not exist.

Example: If $f(z)=(z+1)\sin\frac{1}{z-2}=1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+$ ... Since there are infinite number of terms in the negative powers of (z-2), z = 2 is an essential singularity.

Problems:

Find the nature of the singularities of the following functions:

(i) $f(z)=\frac{1}{1-e^z}$ **(ii)** $f(z)=\frac{e^{2z}}{(z-1)^4}$

Solution:

(i) $f(z)=\frac{1}{1-e^z}$ f(z) has simple pole at $z=2\pi i$.

(ii) $f(z)=\frac{e^{2z}}{(z-1)^4}=e^2\frac{1}{(z-1)^4}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{3(z-1)}+\frac{2}{3}+\frac{4}{15}(z-1)+\cdots$

Since there are 4 terms containing negative powers of (z-1), z = 1 is a pole of 4th order.

Residue: The coefficient of $\frac{1}{z-a}$ in the Laurent's expansion of f(z) is called Residue of f(z) at the pole z = a.

Determination of a residue:

If 'a' is a pole of order $m\geq 1$ of f(z) then residue of f(z) at 'a' is $R=\frac{1}{(m-1)!}\left[\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\right]$

Note:

1. If m = 1(simple pole), Residue = $\lim_{z\to a}(z-a)f(z)$
2. If m = 2(double pole), Residue = $\frac{1}{1!}\lim_{z\to a}\frac{d}{dz}\{(z-a)^2 f(z)\}$

3. If m = 3(triple pole), Residue = $\frac{1}{2!}\lim_{z\to a}\frac{d^2}{dz^2}\{(z-a)^3 f(z)\}$

Problems:

For the following functions find the poles and residues at each pole.

1. $f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z+1)(z-2)}$

Solution: z = -1, z = 2 are simple poles.

Residue at z = -1 is $\lim_{z\to -1}(z+1)f(z) = \frac{1}{3}$

Residue at z = 2 is $\lim_{z\to 2}(z-2)f(z) = \frac{5}{3}$

2. $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Solution: z = 1 double pole, z = -2 simple pole.

Residue at z = 1 is $\lim_{z\to 1}\frac{d}{dz}\{(z-1)^2 f(z)\} = \frac{5}{9}$

Residue at z = -2 is $\lim_{z\to -2}(z+2)f(z) = \frac{4}{9}$

3. $f(z) = \frac{ze^z}{(z-1)^3}$

Solution: z = 1 is a triple pole.

Residue at z = 1 is $\frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\{(z-1)^3 f(z)\} = \frac{3e}{2}$

Cauchy's Residue theorem:

Let C be a simple closed curve and f(z) be analytic within and on C except at a finite number of poles $a_1, a_2, \ldots, a_n$ which lie inside C ,then

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots + R_n)$$

Where $R_1, R_2 \ldots R_n$ are the residues of f(z) at $a_1, a_2, \ldots a_n$ respectively.

Problems:

1. Using Cauchy's residue theorem, evaluate the integral $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz$ where C is the circle $|z| = 3$.

Solution: Given $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$

z = -1 and z = 2 are simple poles both lie inside C

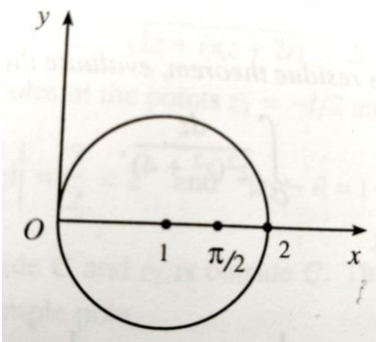
Residue at z = -1 is $\lim_{z\to -1}(z+1)f(z) = \frac{e^{-2}}{-3} = R_1$

Residue at z = 2 is $\lim_{z\to 2}(z-2)f(z) = \frac{e^{4}}{3} = R_2$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}dz = 2\pi i(R_1 + R_2) = 2\pi i\left[\frac{e^{-2}}{-3} + \frac{e^{4}}{3}\right]$$

2. Using Cauchy's residue theorem, evaluate the integral $\int_C \frac{z\cos z}{(z-\pi/2)^3}dz$ where C is the circle
$|z-1| = 1.$

Solution: Given $f(z) = \frac{z\cos z}{(z-\pi/2)^3}, z = \frac{\pi}{2}$ is a triple pole which lie inside the circle C

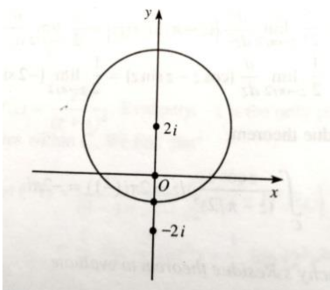


Residue at $z = \frac{\pi}{2} = \frac{1}{2!}\lim_{z\to\frac{\pi}{2}}\frac{d^2}{dz^2}\left\{\left(z-\pi/2\right)^3 f(z)\right\} = -1 = R_1$

$$\int_C \frac{z\cos z}{(z-\pi/2)^3}dz = 2\pi i R_1 = -2\pi i.$$

3. Evaluate the integral $\int_C \frac{1}{z^2(z^2+4)}dz$ where C is the circle $|z-2i| = 3$ using Cauchy's residue theorem.

Solution: Given $f(z) = \frac{1}{z^2(z^2+4)}$, z = 0 double pole, z = 2i &-2i are simple poles.



$z = 0$ & $z = 2i$ lie inside the circle $|z - 2i| = 3$

Residue at $z = 0$ is $\lim_{z\to 0}\frac{d}{dz}\{z^2 f(z)\} = 0 = R_1$

Residue at $z = 2i$ is $\lim_{z\to 2}(z - 2i)f(z) = \frac{i}{16} = R_2$

$\int_C \frac{1}{z^2(z^2+4)}dz = 2\pi i(R_1 + R_2) = -\frac{\pi}{8}$.

4. Evaluate the integral $\int_C \tan z\, dz$ where C is the circle $|z| = 2$ Using Cauchy's residue theorem.

Solution: $f(z) = \tan z$ has simple poles at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

$z = \pm\frac{\pi}{2}$ lie inside the circle.

Residue at $z = \frac{\pi}{2}$ is $\lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)f(z) = -1 = R_1$

Residue at $z = -\frac{\pi}{2}$ is $\lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right)f(z) = -1 = R_2$

$\int_C \tan z dz = 2\pi i(R_1 + R_2) = 2\pi i(-1 - 1) = -4\pi i$

Exercise:

1. If C is the circle $|z - a| = r$ (constant), prove the following:

(i) $\int_C \frac{dz}{z-a} = 2\pi i$ (ii) $\int_C (z - a)^n dz = 0$, where n is an integer $\neq -1$.

2. If C is the circle $|z| = 1$, verify Cauchy's theorem for **(i)** $f(z) = z^3$, **(ii)** $f(z) = ze^{-z}$.

3. Find the Laurent's series expansion of

(i) $f(z) = \frac{1}{z^2-4z+3}$ about $z = 0$ in the region $1 < |z| < 3$.

(ii) $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$.

(iii) $f(z) = \frac{7z-2}{(z+1)(z-2)z}$ about $z = -1$.

Ans:

(i) $\boldsymbol{f(z)} = -\sum_{n=1}^{\infty}\left(\frac{z^{n-1}}{2.3^n} + \frac{1}{z^n}\right)$

(ii) $\boldsymbol{f(z)} = \sum_{n=0}^{\infty}\left(\frac{3^n}{(z+2)^{n+1}} + \frac{(z+2)^n}{5^{n+1}}\right) + \frac{1}{z+2}$

(iii) $f(z) = -\frac{3}{z+1} - \frac{5}{3} - \left(1 + \frac{2}{3^2}\right)(z+1) - \left(1 + \frac{2}{3^3}\right)(z+1)^2 - \left(1 + \frac{2}{3^4}\right)(z+1)^3 \ldots$

4. Determine the poles of the following functions and the residues at each pole:

(i) $\frac{\sin z}{z^2}$ **(ii)** $\frac{z^2-2z}{(z+1)^2(z^2+4)}$

Ans: (i) z = 0 is a pole of order 2 and Residue = 1

(ii) z = -1 is a double pole , Residue = -14/25,

$z = \pm 2$ is simple pole. Residue $\frac{7+i}{25}, \frac{7-i}{25}$

5. Evaluate the following integrals using Cauchy's residue theorem.

(i) $\int_C \frac{1-2z}{z(z-1)(z-2)} dz, C: |z| = 1.5$ (ii) $\int_C \frac{z+4}{z^2+2z+5} dz, C: |z+1-i| = 2$

(iii) $\int_C \frac{2z+1}{(2z-1)^2} dz, C: |z| = 1$ (iv) $\int_C \frac{dz}{(z^2+4)^2} dz, C: |z-i| = 2$

Ans: (i) $3\pi i$ (ii) $\frac{\pi}{2(3+2i)}$ (iii) πi (iv) $\frac{\pi}{16}$

Video Links:

https://www.youtube.com/watch?v=ie9hvxq_I54

https://www.youtube.com/watch?v=0Won5Vs_65E

https://www.youtube.com/watch?v=zQ1IxVLi8SA

https://www.youtube.com/watch?v=bIY6ahHVgqA

https://www.youtube.com/watch?v=c8oDYqoMiGA