



UNIT-I LAPLACE TRANSFORM

Topic Learning Objectives:

Upon Completion of this unit, students will be able to:

- Understand the existence, uniqueness and basic concepts of Laplace transform (LT).
- Determine the Laplace transform of elementary functions.
- Describe the properties of the Laplace transform such as linearity, time shifting, scaling, differentiation in the s-domain, division by t, differentiation and integration in the time domain.
- Develop the Laplace transform of periodic functions, Heaviside (unit step) function, Dirac Delta function and its applications.

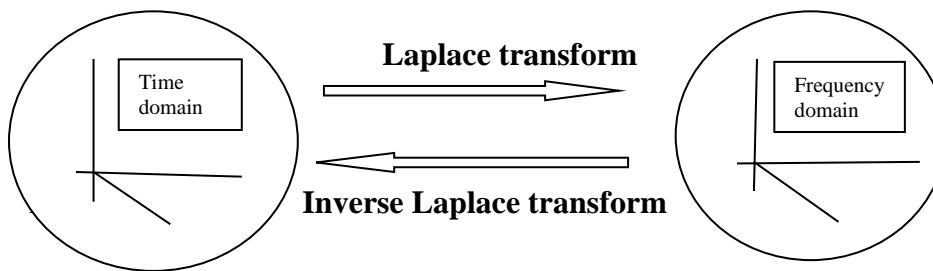
Transforms are used in Science and Engineering as a tool for simplifying analysis and look at data from different angle.

The Laplace Transform method is a technique for solving linear differential equations with initial conditions. It is commonly used to solve electrical circuit and systems problems.

The purpose of Laplace transformation is to solve different differential equations. There are a number of methods to solve homogeneous and non-homogeneous equations but Laplace transform comes in to use when to solve the equations that cannot be solved by any of the regular methods developed.

The Laplace transform converts integral and differential equations into algebraic equations. Although it is a different and beneficial alternative of variations of parameters and undetermined coefficients, the transform is most advantageous for input force functions that are piecewise, periodic or pulsive.

In simple words, Laplace transform converts time domain signal into frequency domain, which is depicted more clearly in below figure:



Examples:



The problem of solving the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$ with conditions $y(0) = y'(0) = 1$ is an initial value problem.

The problem of solving the equation $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x$ with $y(1) = 1$, $y(2) = 3$ is called boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. It possesses a powerful set of properties for analysis of signals and systems. Besides, this technique may also be employed to find the solution of certain integral.

The transform is named after the French Mathematician Pierre-Simon, marquis de Laplace (1749 – 1827).



Pierre-Simon, marquis de Laplace was a prominent French mathematical physicist and astronomer of the 19th century, who made crucial contributions in the arena of planetary motion by applying Sir Isaac Newton's theory of gravitation to the entire solar system. His work regarding the theory of probability and statistics is considered pioneering and has influenced a whole new generation of mathematicians. Laplace heavily contributed in the development of differential equations, difference equations, probability and statistics. His work was important to the development of engineering, mathematics, statistics, physics, astronomy and philosophy.

Definition:

Let $f(t)$ be a real valued function defined for all $t \geq 0$ and s be a parameter, real or complex. Then the integral

$$\int_0^\infty e^{-st} f(t) dt$$

Is said to be the **Laplace transform** of f , provided the integral converges and is denoted by $L\{f(t)\}$.

$$\text{Thus } L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \quad (1)$$

Observe that the value of the integral on the right hand side of (1) depends on s . Hence $L\{f(t)\}$ is a function of s denoted by $F(s)$ or $\bar{f}(s)$.

$$\text{Thus } L\{f(t)\} = F(s) \quad (2)$$



In relation (2), $f(t)$ is called the Inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$.

$$\text{Thus } L^{-1}[F(s)] = f(t). \quad (3)$$

Suppose $f(t)$ is defined as follows:

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases}$$

Note that $f(t)$ is piecewise continuous. The Laplace transform of $f(t)$ is defined as

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^a e^{-st} f_1(t) dt + \int_a^b e^{-st} f_2(t) dt + \int_b^{\infty} e^{-st} f_3(t) dt \end{aligned}$$

Note:

In a practical situation the variable t represents time and s represents frequency. Hence the Laplace transform converts time domain into the frequency domain.

Uses of Laplace Transformation in Control System

Laplace transform is useful mathematical tool to explain the integrals in the interval from 0 to infinity. It is also used for analyzing and designing the analog signals.

Also we know that control systems mainly deals with analog systems. So usually it is obligatory to take integral from 0 to infinity unless initial conditions are not provided. Laplace transform converts time domain into frequency domain which makes evaluation easy.

We can get the time response of the given system by taking inverse Laplace transform that is ratio of Laplace of output to the Laplace of input.

Examples:

1. Find $L\{f(t)\}$ given $f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}$

Solution:

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \int_0^3 e^{-st} t dt + \int_3^{\infty} 4e^{-st} dt.$$

Integrating the terms on the RHS to get $L\{f(t)\} = \frac{1}{s} e^{-3s} + \frac{1}{s^2} (1 - e^{-3s})$

2. Find $L\{f(t)\}$ given $f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Solution:

$$\begin{aligned} L\{f(t)\} &= \int_0^{\pi} e^{-st} f(t) dt + \int_{\pi}^{\infty} e^{-st} f(t) dt = \int_0^{\pi} e^{-st} \sin 2t dt \\ &= \left[\frac{e^{-st}}{s^2 + 4} \{-s \sin 2t - 2 \cos 2t\} \right]_0^{\pi} = \frac{2}{s^2 + 4} [1 - e^{-\pi s}] \end{aligned}$$

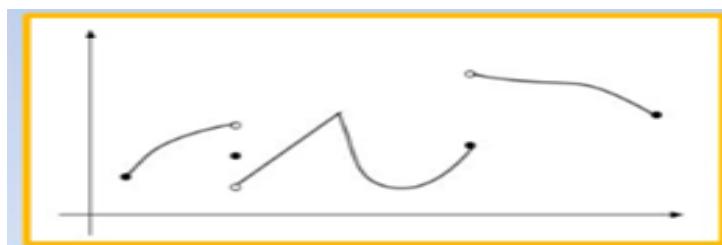
Existence and Uniqueness of Laplace Transform

The sufficient condition for existence of Laplace transform is based on the concepts of Piecewise continuous function and exponential order.

Piecewise Continuous function

If an interval $[a, b]$ can be partitioned by a finite number of points $a_0 = t_0 < t_1 < t_2 < \dots < t_n = b$ such that

- (i) f is continuous on each sub interval (t_i, t_{i+1}) .
- (ii) $\left| \lim_{t \rightarrow t_i^+} f(t) \right| < \infty \quad \forall i = 0, 1, 2, 3, \dots, n-1$
- (iii) $\left| \lim_{t \rightarrow t_{i+1}^-} f(t) \right| < \infty \quad \forall i = 1, 2, 3, \dots, n.$



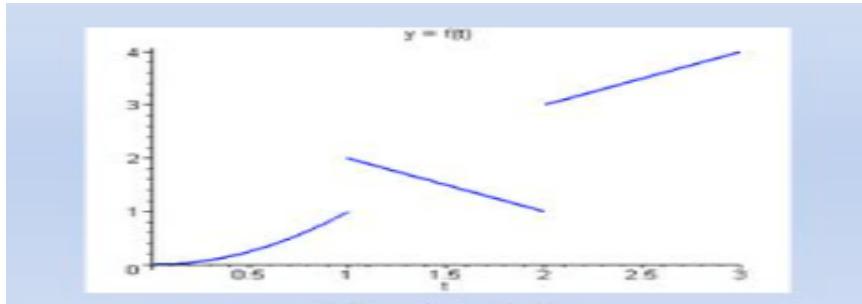
Then the function f is piecewise continuous or f is piecewise continuous on $[a, b]$ if it is continuous there except for a finite number of jump discontinuities.

Examples:

1. The function defined by

$$f(t) = \begin{cases} t^2 & 0 \leq t \leq 1 \\ 3 - t & 1 < t \leq 2 \\ t + 1 & 2 < t \leq 3 \end{cases}$$

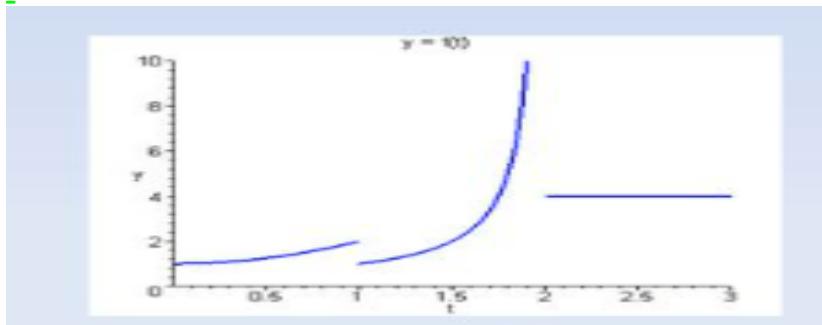
is piecewise continuous in $[0, 3]$.



2. The function defined by

$$f(t) = \begin{cases} t^2 + 1 & 0 \leq t \leq 1 \\ \frac{1}{2-t} & 1 < t \leq 2 \\ 4 & 2 < t \leq 3 \end{cases}$$

is not a piecewise continuous in $[0, 3]$.



Exponential order

A function $f(t)$ is said to be of exponential order c , if there exist constants M and c such that $|f(t)| \leq M e^{ct}$ for sufficiently large t .

Example:

Any Polynomial is of exponential order. This is clear from the fact that

$$e^{at} \equiv \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} = t^n \leq \frac{n!}{a^n} e^{at}, \text{ but } f(t) = e^{t^2} \text{ is not of exponential order.}$$

Sufficient condition for the existence of Laplace transform

Let $f(t)$ be a piece-wise continuous function in $[0, \infty)$ and is of exponential order then Laplace transform $F(s)$ of $f(t)$ exists for $s > c$, where c is a real number that depends on $f(t)$.

Uniqueness of Laplace transform

Let $f(t)$ and $g(t)$ be continuous functions such that $F(s) = G(s)$ for all $s > k$ then $f(t) = g(t)$ at all t .



Transform of elementary functions

1. Let 'a' be a constant then

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty = \frac{1}{s-a},$$

$$\text{Thus } L(e^{at}) = \frac{1}{s-a}, s > a$$

$$\text{In particular when } a = 0 \text{ to get } L(1) = \frac{1}{s}, s > 0.$$

$$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2} \int_0^\infty e^{-st} [e^{at} + e^{-at}] dt$$

$$= \frac{1}{2} \int_0^\infty [e^{-(s-a)t} + e^{-(s+a)t}] dt$$

$$L(\cosh at) = \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty = \frac{s}{s^2 - a^2}$$

$$\text{Thus, } L(\cosh at) = \frac{s}{s^2 - a^2}, s > |a|,$$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{a}{s^2 - a^2}, s > |a|$$

$$\text{Thus, } L(\sinh at) = \frac{a}{s^2 - a^2}, s > |a|$$

$$2. L(\sin at) = \int_0^\infty e^{-st} \sin at dt$$

Suppose $s > 0$ then integrate by using the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$L(\sin at) = \frac{a}{s^2 + a^2}, s > 0.,$$

$$3. L(\cos at) = \int_0^\infty e^{-st} \cos at dt$$

Suppose $s > 0$ and integrate by using the formula



$$\int e^{ax} \cos b x dx = \frac{e^{ax}}{a^2+b^2} [a \cos b x + b \sin b x]$$

$$L(\cos a t) = \frac{s}{s^2+a^2}, s > 0.$$

4. Let n be a constant, which is a non-negative real number or a negative non-integer

$$\text{Then } L(t^n) = \int_0^\infty e^{-st} t^n dt$$

$$\text{Let } s > 0 \text{ and put } st = x \text{ then } L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

The integral $\int_0^\infty e^{-x} x^n dx$ is called gamma function of (n+1) denoted by $\Gamma(n+1)$.

$$\text{Thus } L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular if n is a non-negative integer then $\Gamma(n+1) = n!$

$$\text{Hence } L(t^n) = \frac{n!}{s^{n+1}}.$$

Table of Laplace Transform

<u>f(t)</u>	F(s)
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$\cosh at$	$\frac{s}{s^2-a^2}, s > a $
$\sinh at$	$\frac{a}{s^2-a^2}, s > a $
$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
t^n	$\frac{n!}{s^{n+1}}, s > 0$ n is a positive integer
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}, s > 0$ $n \neq -1, -2, -3, \dots$



Region of Convergence (RoC)

Laplace transform $F(s)$ of a function $f(t)$ converges provided that the limit $\lim_{R \rightarrow \infty} \int_0^R f(t)e^{-st} dt$ exists. The Laplace transform converges absolutely if the integral $\int_0^\infty |f(t)e^{-st}| dt$ exists. The set of values of s for which $F(s)$ exists is known as the region of convergence.

Examples:

Evaluate: (i) $L(\sin 3t \sin 4t)$ (ii) $L(\cos^2 4t)$ (iii) $L(\sin^3 2t)$.

Solution:

$$\begin{aligned} \text{(i)} \quad L(\sin 3t \sin 4t) &= L\left[\frac{1}{2}(\cos t - \cos 7t)\right] \\ &= \frac{1}{2}[L(\cos t) - L(\cos 7t)] \quad (\text{linearity property}) \\ &= \frac{1}{2}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 49}\right] = \frac{24s}{(s^2 + 1)(s^2 + 49)}. \end{aligned}$$

$$\text{(ii)} \quad L(\cos^2 4t) = L\left[\frac{1}{2}(1 + \cos 8t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 64}\right].$$

$$\text{(iii)} \quad \text{We have } \sin^3 t = \frac{1}{4}(3\sin t - \sin 3t)$$

$$\sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$$

$$\text{So } L(\sin^3 2t) = \frac{1}{4}\left[\frac{6}{s^2 + 4} - \frac{6}{s^2 + 36}\right] = \frac{48}{(s^2 + 4)(s^2 + 36)}$$

4. Find $L(\cos t \cos 2t \cos 3t)$.

Solution:

$$\cos 2t \cos 3t = \frac{1}{2}[\cos 5t + \cos t]$$

$$\begin{aligned} \text{So } \cos t \cos 2t \cos 3t &= \frac{1}{2}[\cos 5t \cos t + \cos^2 t] \\ &= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t] \end{aligned}$$

$$\text{Thus } L(\cos t \cos 2t \cos 3t) = \frac{1}{4}\left[\frac{s}{s^2 + 36} + \frac{s}{s^2 + 16} + \frac{1}{s} + \frac{s}{s^2 + 4}\right].$$

5. Find $L(\cosh^2 2t)$.

Solution:



$$\text{Using } \cosh^2 t = \frac{1 + \cosh 2t}{2}$$

$$\text{For } \cosh^2 2t = \frac{1 + \cosh 4t}{2}$$

$$\text{Thus } L(\cosh^2 2t) = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 - 16} \right].$$

$$6. \text{ Evaluate (i) } L(\sqrt{t}) \text{ (ii) } L\left(\frac{1}{\sqrt{t}}\right) \text{ (iii) } L(t^{-\frac{3}{2}}).$$

Solution:

$$\text{We have } L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$(i) \text{ For } n = \frac{1}{2}, \quad L(t^{\frac{1}{2}}) = \frac{\Gamma\left(\frac{1}{2} + 1\right)}{s^{3/2}}$$

$$\text{Since } \Gamma(n+1) = n\Gamma(n), \text{ we have } \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\text{Thus } L(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.$$

$$(ii) \text{ For } n = -\frac{1}{2} \text{ to get } L(t^{-\frac{1}{2}}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$(iii) \text{ For } n = -\frac{3}{2} \text{ to get } L(t^{-\frac{3}{2}}) = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}} = \frac{-2\sqrt{\pi}}{s^{-\frac{1}{2}}} = -2\sqrt{\pi s}.$$

$$7. \text{ Evaluate: (i) } L(t^2) \text{ (ii) } L(t^3).$$

Solution:

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$(i) \text{ For } n = 2, \quad L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}.$$

$$(ii) \text{ For } n = 3, \quad L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}.$$

$$8. \text{ Find the Laplace transform of } a^{kt}$$

Solution:



$$a^{kt} = e^{\log a^{kt}}$$

$$a^{kt} = e^{kt \log a}$$

$$L\{a^{kt}\} = \frac{1}{s - a \log k}$$

Exercise:

Find the Laplace transform of the following functions:

- (i) $\sin(3t + 4)$ (ii) $\cos 2t \sin 3t$ (iii) $\sin t \sin 2t \sin 3t$ (iv) $\cos^3 t$ (v) $(\sin t - \cos t)^2$

Answers:

$$(i) \frac{3\cos 4 + s \sin 4}{s^2 + 4} \quad (ii) \frac{1}{2} \left(\frac{5}{s^2 + 25} + \frac{1}{s^2 + 1} \right) \quad (iii) \frac{1}{4} \left(\frac{2}{s^2 + 4} + \frac{4}{s^2 + 16} - \frac{6}{s^2 + 36} \right)$$

$$(iv) \frac{s(s^2 + 7)}{(s^2 + 1)(s^2 + 9)} \quad (v) \frac{1}{s} - \frac{2}{s^2 + 4}$$

Properties of Laplace transform

1. Linearity

For any two functions $f(t)$ and $g(t)$ whose Laplace transform exists and any two constants a and b we have

$$L[a f(t) + b g(t)] = a L\{f(t)\} + b L\{g(t)\}.$$

Proof

By definition

$$\begin{aligned} L[a f(t) + b g(t)] &= a \int_0^\infty e^{-st} f(t) dt + b \int_0^\infty e^{-st} g(t) dt \\ &= a L\{f(t)\} + b L\{g(t)\} \end{aligned}$$

In particular

$$\text{for } a = b = 1, L[f(t) + g(t)] = L\{f(t)\} + L\{g(t)\} \text{ and}$$

$$\text{for } a = -b = 1, L[f(t) - g(t)] = L\{f(t)\} - L\{g(t)\}.$$

The linearity of the Laplace transform follows from its definition as an integral and the fact that integration is a linear operation.

2. Scaling

If $L\{f(t)\} = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, where a is a positive constant.

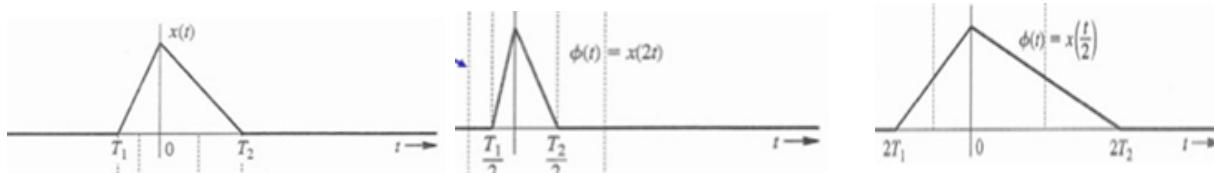
Proof

By definition $L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$ _____ (1)

Put $at = x$ the expression (1) becomes,

$$L\{f(at)\} = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Scaling in time introduces the inverse scaling in s. (It depends on the sign of 'a')



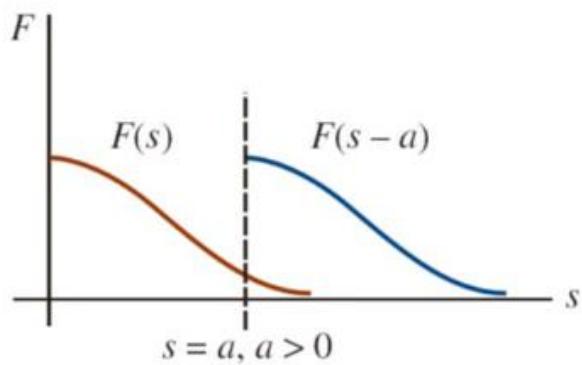
3. s- domain shift (First shifting property)

Let a be any real constant then $L[e^{at}f(t)] = F(s-a)$

Proof

$$\begin{aligned} \text{By definition } L[e^{at}f(t)] &= \int_0^\infty e^{-st} [e^{at}f(t)] dt = \int_0^\infty e^{-(s-a)t} f(t) dt \\ &= F(s-a) \end{aligned}$$

Multiplication by an exponential in time introduces a shift in frequency s to the Laplace transform of $f(t)$ i.e Laplace transform of $e^{at}f(t)$ can be written down directly by changing s to $s-a$ in $F(s)$.



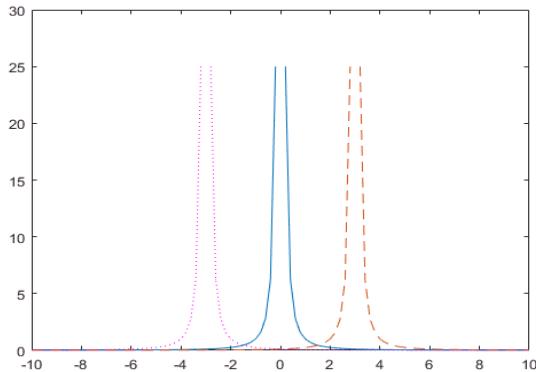
Example:

If $f(t) = t$, then $F(s) = \frac{1}{s^2}$

If $f(t) = e^{3t} t$ then $F(s) = \frac{1}{(s-3)^2}$ (Shifting towards right)



If $f(t) = e^{-3t} t$ then $F(s) = \frac{1}{(s+3)^2}$ (Shifting towards left)



Examples:

- Find $L[e^{-3t} (2\cos 5t - 3\sin 5t)]$.

Solution:

Consider $f(t) = 2\cos 5t - 3\sin 5t$

$$L\{f(t)\} = L[(2\cos 5t - 3\sin 5t)]$$

$$F(s) = 2 \frac{s}{s^2 + 25} - \frac{3(5)}{s^2 + 25}$$

Given $L[e^{-3t} (2\cos 5t - 3\sin 5t)]$

$$= 2 \frac{(s+3)}{(s+3)^2 + 25} - \frac{15}{(s+3)^2 + 25} \quad (\text{s- domain shift})$$

$$= \frac{2s+9}{s^2 + 6s + 34}$$

- Find $L[\cosh at \sinh at]$.

Solution:

$$\begin{aligned} L[\cosh at \sinh at] &= L\left[\frac{(e^{at} + e^{-at})}{2} \sin at\right] \\ &= \frac{1}{2} \left[\frac{a}{(s-a)^2 + a^2} + \frac{a}{(s+a)^2 + a^2} \right] \\ &= \frac{a(s^2 + 2a^2)}{[(s-a)^2 + a^2][(s+a)^2 + a^2]} \end{aligned}$$



3. Find $L(\cos ht \sin^3 2t)$.

Solution:

$$\begin{aligned} \text{Given } L\left[\left(\frac{e^t + e^{-t}}{2}\right)\left(\frac{3\sin 2t - \sin 6t}{4}\right)\right] \\ = \frac{1}{8}[3L(e^t \sin 2t) - L(e^t \sin 6t) + 3L(e^{-t} \sin 2t) - L(e^{-t} \sin 6t)] \\ = \frac{1}{8}\left[\frac{6}{(s-1)^2 + 4} - \frac{6}{(s-1)^2 + 36} + \frac{6}{(s+1)^2 + 4} - \frac{6}{(s+1)^2 + 36}\right] \\ = \frac{3}{4}\left[\frac{1}{(s-1)^2 + 4} - \frac{1}{(s-1)^2 + 36} + \frac{1}{(s+1)^2 + 24} - \frac{1}{(s+1)^2 + 36}\right]. \end{aligned}$$

4. Find $L(e^{-4t} t^{-5/2})$.

Solution:

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Put } n = -5/2, L(t^{-5/2}) = \frac{\Gamma(-3/2)}{s^{-3/2}} = \frac{4\sqrt{\pi}}{3s^{-3/2}} \text{ Change } s \text{ to } s+4.$$

$$\text{Therefore } L(e^{-4t} t^{-5/2}) = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}.$$

Exercise:

Find the Laplace transform of the following functions.

$$(i) e^{-2t} \cos^2 2t \quad (ii) e^{2t} \sin 3t \cos 2t$$

Solutions:

$$(i) \frac{(s+2)^2 + 8}{(s+2)((s+2)^2 + 16)} \quad (ii) \frac{(s-2)^2 + 13}{((s-2)^2 + 25)((s-2)^2 + 1)}$$

Differentiation in the s -domain

If $L\{f(t)\} = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, 3, \dots$

Suppose that n is positive integer by definition



$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Differentiating 'n' times on both sides w.r.t. s,

$$\frac{d^n}{ds^n} F(s) = \frac{\partial^n}{\partial s^n} \int_0^\infty e^{-st} f(t) dt$$

Performing differentiation under the integral sign,

$$\frac{d^n}{ds^n} F(s) = \int_0^\infty (-t)^n e^{-st} f(t) dt$$

Multiplying on both sides by $(-1)^n$,

$$(-1)^n \frac{d^n}{ds^n} F(s) = \int_0^\infty (t^n f(t)) e^{-st} dt = L[t^n f(t)], \text{ by definition}$$

Thus, $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$. This is the transform of $t^n f(t)$.

Also $L^{-1}\left[\frac{d^n}{ds^n} F(s)\right] = (-1)^n t^n f(t)$.

Differentiation in s-domain corresponds to multiplication by $-t$ in the time domain

In particular for $n = 1$, $L[t f(t)] = -\frac{d}{ds} F(s)$,

for $n = 2$, $L[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$ etc.

Transform of $\frac{f(t)}{t}$

By definition, $F(s) = \int_0^\infty e^{-st} f(t) dt$

Therefore $\int_s^\infty F(s) ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t) dt \right] ds = \int_0^\infty f(t) \left[\int_s^\infty e^{-st} ds \right] dt$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t} \right]_s^\infty dt = \int_0^\infty e^{-st} \left[\frac{f(t)}{t} \right] dt = L\left(\frac{f(t)}{t}\right)$$

Thus $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds$.

Examples:



1. Find $L(t^2 \sin 3t)$.

Solution:

$$L(\sin 3t) = \frac{3}{s^2 + 9}$$

$$\text{So that } L(t^2 \sin 3t) = \frac{d^2}{ds^2} \left(\frac{3}{s^2 + 9} \right) = -6 \frac{d}{ds} \frac{s}{(s^2 + 9)^2} = \frac{18(s^2 - 3)}{(s^2 + 9)^3}.$$

2. Find $L[t e^{-t} \sin 4t]$.

Solution:

$$L(\sin 4t) = \frac{4}{s^2 + 16}$$

$$\text{So that } L(t \sin 4t) = -\frac{d}{ds} \frac{4}{s^2 + 16} = \frac{8s}{(s^2 + 16)2}.$$

By s – shifting property

$$\text{So that } L[t e^{-t} \sin 4t] = \frac{8(s+1)}{(s+1)^2 + 16} = \frac{8(s+1)}{(s^2 + 2s + 17)^2}.$$

$$3. \text{ Find } L\left(\frac{e^{-t} \sin t}{t}\right).$$

Solution:

$$L(\sin t) = \frac{1}{s^2 + 1}$$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = [\tan^{-1}(s)]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

By s – shifting property

$$\text{Hence } L\left(\frac{e^{-t} \sin t}{t}\right) = \cot^{-1}(s+1).$$

$$4. \text{ Find } L\left(\frac{\sin t}{t}\right) \text{ and hence evaluate } L\left(\frac{\sin at}{t}\right).$$

Solution:



$$\text{We have } L(\sin t) = \frac{1}{s^2 + 1}$$

$$\begin{aligned}\text{So } L\{f(t)\} &= L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = [\tan^{-1} s]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = F(s).\end{aligned}$$

$$\begin{aligned}\text{Consider } L\left(\frac{\sin at}{t}\right) &= a L\left(\frac{\sin at}{at}\right) = a Lf(at) \\ &= a \left[\frac{1}{a} F\left(\frac{s}{a}\right) \right] \text{(scaling)} \\ &= \cot^{-1}\left(\frac{s}{a}\right).\end{aligned}$$

$$5. \text{ Find } L\left[\frac{\cos at - \cos bt}{t}\right].$$

Solution:

$$\text{We have } L[\cos at - \cos bt] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

$$\begin{aligned}\text{So that } L\left[\frac{\cos at - \cos bt}{t}\right] &= \int_s^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds = \frac{1}{2} \left[\log\left(\frac{s^2 + a^2}{s^2 + b^2}\right) \right]_s^\infty \\ &= \frac{1}{2} \left[Lt \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right) - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right) \right]_{s \rightarrow \infty} \\ &= \frac{1}{2} \left[0 + \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right) \right] = \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right).\end{aligned}$$

Exercise:

Find the Laplace transform of the following functions.

$$(i) t \sin 3t \cos 2t \quad (ii) t^2 e^{-t} \cos t \quad (iii) \frac{1 - e^{-2t}}{t} \quad (iv) \frac{e^{-at} - e^{-bt}}{t} \quad (v) \frac{\sin^2 t}{t}.$$

Answers:

$$(i) \frac{5s}{(s^2 + 25)^2} + \left(\frac{s}{(s^2 + 1)^2} \right) \quad (ii) \frac{2(s+1)(s^2 + 2s - 2)}{((s+1)^2 + 1)^3} \quad (iii) \log\left(\frac{s+2}{s}\right) \quad (iv) \frac{1}{2} \log\left(\frac{\sqrt{s^2 + 4}}{s^2}\right).$$



6. Prove that $\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$.

Solution:

$$\text{By } \int_0^\infty e^{-st} t \sin t dt = L(t \sin t) = -\frac{d}{ds} L(\sin t) = -\frac{d}{ds} \left[\frac{1}{s^2 + 1} \right] = \frac{2s}{(s^2 + 1)^2}$$

$$\text{Put } s = 3 \text{ to get } \int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$$

Exercise:

Evaluate the following integrals using Laplace transforms:

$$\text{(i) } \int_0^\infty te^{-2t} \sin 3t dt \quad \text{(ii) } \int_0^\infty e^{3t} t^3 \cos t dt \quad \text{(iii) } \int_0^\infty \left(\frac{e^{-3t} - e^{-6t}}{t} \right) dt \quad \text{(iv) } \int_0^\infty \frac{e^{-t} \sin \sqrt{3}t}{t} dt$$

Answers:

$$(i) \frac{12}{169} \quad (ii) \frac{168}{10^4} \quad (iii) \log 2 \quad (iv) \frac{\pi}{3}$$

Differentiation in the time domain

If $L\{f(t)\} = F(s)$, then $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$.

Consider

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt \\ &= \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t) dt \quad (\text{integration by parts}) \\ &= \left| L_t(e^{-st} f(t) - f(0)) \right|_{t \rightarrow \infty} + s L\{f(t)\} \\ &= 0 - f(0) + s L\{f(t)\} \end{aligned}$$

Thus

$$L\{f'(t)\} = s L\{f(t)\} - f(0)$$

Similarly

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

In general

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$



Integration property

If $L\{f(t)\} = F(s)$, then $L \int_0^t f(t)dt = \frac{1}{s}F(s)$

Let $\phi(t) = \int_0^t f(t)dt$ then $\phi(0) = 0$ and $\phi'(t) = f(t)$

$$\begin{aligned}\text{Now } L\{\phi(t)\} &= \int_0^\infty e^{-st} \phi(t) dt = \left[\phi(t) \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \phi'(t) \frac{e^{-st}}{-s} dt \\ &= (0 - 0) + \frac{1}{s} \int_0^\infty f(t) e^{-st} dt\end{aligned}$$

Thus $L \int_0^t f(t)dt = \frac{1}{s}F(s)$

Examples:

1. By using the Laplace transform of $\sin at$, find the Laplace transform of $\cos at$.

Solution:

$$\text{Let } f(t) = \sin at, \text{ then } L\{f(t)\} = \frac{a}{s^2 + a^2}$$

We note that $f'(t) = a \cos at$

Taking Laplace transforms, $L\{f'(t)\} = L(a \cos at) = aL(\cos at)$

$$\begin{aligned}\text{Or } L\{\cos at\} &= \frac{1}{a} L\{f'(t)\} = \frac{1}{a} [sL\{f(t)\} - f(0)] \\ &= \frac{1}{a} \left[\frac{sa}{s^2 + a^2} - 0 \right].\end{aligned}$$

$$\text{Thus } L\{\cos at\} = \frac{s}{s^2 + a^2}.$$

$$2. \text{ Given } L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}}, \text{ show that } L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}.$$

Solution:

$$\text{Let } f(t) = 2\sqrt{\frac{t}{\pi}}, \text{ given } L[f(t)] = \frac{1}{s^{3/2}}$$

$$\text{Note that } f'(t) = \frac{2}{\sqrt{\pi}} \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$$

$$\text{Taking Laplace transform to get } L\{f'(t)\} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$

$$\text{Hence } L\left[\frac{1}{\sqrt{\pi t}}\right] = Lf'(t) = sLf(t) - f(0) = s\left(\frac{1}{s^{3/2}}\right) - 0$$



$$\text{Thus } L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}.$$

$$3. \text{ Find } L\left\{\int_0^t \left(\frac{\cos at - \cos bt}{t}\right) dt\right\}.$$

Solution:

$$\text{Here } L\{f(t)\} = L\left(\frac{\cos at - \cos bt}{t}\right) = \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)$$

$$\text{Using the result } L\left\{\int_0^t f(t) dt\right\} = \frac{1}{s} L\{f(t)\}$$

$$L\left\{\int_0^t \left(\frac{\cos at - \cos bt}{t}\right) dt\right\} = \frac{1}{2s} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right).$$

$$4. \text{ Find } L\left\{\int_0^t te^{-t} \sin 4t dt\right\}.$$

Solution:

$$L[te^{-t} \sin 4t] = \frac{8(s+1)}{(s^2 + 2s + 17)^2}$$

$$\text{Thus } L\left\{\int_0^t te^{-t} \sin 4t dt\right\} = \frac{8(s+1)}{s(s^2 + 2s + 17)^2}.$$

Exercise:

Find the Laplace transform of the following functions.

$$(i) L\left\{\int_0^t e^t \frac{\sin t}{t} dt\right\} \quad (ii) L\left\{\int_0^t e^t \cosh t dt\right\} \quad (iii) L\left\{\int_0^t t^2 \sin at dt\right\}$$

Answers:

$$(i) \frac{1}{s} \cot^{-1}(s-1) \quad (ii) \frac{(s-1)}{s(s^2-2s)} \quad (iii) \frac{2a(3s^2-a^2)}{s(s^2+a^2)^3}$$

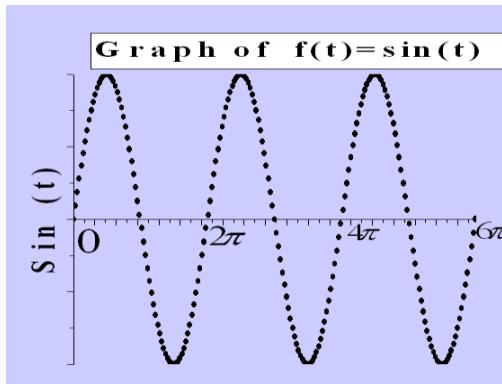
Periodic function

Definition

A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t) = f(t + nT)$ where $n = 1, 2, 3, \dots$. The graph of the periodic function repeats itself in equal intervals.

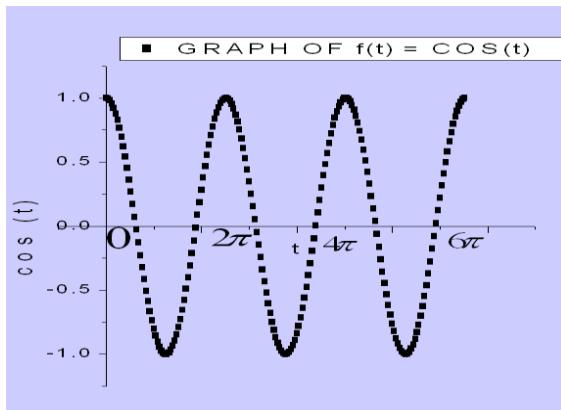
For example $\sin t, \cos t$ are periodic functions of period 2π since $\sin(t + 2n\pi) = \sin t$, $\cos(t + 2n\pi) = \cos t$.

The graph of $f(t) = \sin t$ is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

The graph of $f(t) = \cos t$ is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

Laplace Transform of a periodic function

Let $f(t)$ be a periodic function of period T then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$.

Proof:

$$\begin{aligned} \text{By definition we have } L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-su} f(u) du \\ &= \int_0^T e^{-su} f(u) du + \int_T^{2T} e^{-su} f(u) du + \dots + \int_{nT}^{(n+1)T} e^{-su} f(u) du + \dots + \infty \end{aligned}$$



$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u) du$$

Let us set $u = t + nT$, then $L\{f(t)\} = \sum_{n=0}^{\infty} \int_{t=0}^T e^{-s(t+nT)} f(t+nT) dt$

Here $f(t+nT) = f(t)$, by periodic property

$$\begin{aligned} \text{Hence, } L\{f(t)\} &= \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-st} f(t) dt \\ &= \left[\frac{1}{1-e^{-sT}} \right]_0^T \int e^{-st} f(t) dt, \text{ identifying the above series as a geometric series.} \end{aligned}$$

$$\text{Thus } L\{f(t)\} = \left[\frac{1}{1-e^{-sT}} \right]_0^T \int e^{-st} f(t) dt.$$

Examples

1. For the periodic function $f(t)$ of period 4, defined by $f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$, find $L\{f(t)\}$.

Solution:

Here period of $f(t) = T = 4$

$$\begin{aligned} \text{By definition } L\{f(t)\} &= \left[\frac{1}{1-e^{-sT}} \right]_0^T \int e^{-st} f(t) dt = \left[\frac{1}{1-e^{-4s}} \right]_0^4 \int e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-4s}} \left[\int_0^2 3te^{-st} dt + \int_2^4 6e^{-st} dt \right] \\ &= \frac{1}{1-e^{-4s}} \left[3 \left\{ \left[t \left(\frac{e^{-st}}{-s} \right) \right]_0^2 - \int_0^2 1 \cdot \frac{e^{-st}}{-s} dt \right\} + 6 \left(\frac{e^{-st}}{-s} \right)_2^4 \right] \\ &= \frac{1}{1-e^{-4s}} \left[\frac{3(1-e^{-2s} - 2se^{-4s})}{s^2} \right]. \end{aligned}$$

$$\text{Thus } L\{f(t)\} = \frac{3(1-e^{-2s} - 2se^{-4s})}{s^2(1-e^{-4s})}$$

2. A periodic function of period $\frac{2\pi}{\omega}$ is defined by

$$f(t) = \begin{cases} E \sin \omega t & 0 \leq t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega} \end{cases}$$



where E and ω are positive constants. Show that $L\{f(t)\} = \frac{E\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}$.

Solution:

$$\begin{aligned} \text{Here } T &= \frac{2\pi}{\omega}. \text{ Therefore } L\{f(t)\} = \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_0^{2\pi/\omega} e^{-st} f(t) dt = \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_0^{\pi/\omega} E e^{-st} \sin \omega t dt \\ &= \frac{E}{1 - e^{-s(2\pi/\omega)}} \left[\frac{e^{-st}}{s^2 + \omega^2} \{-s \sin \omega t - \omega \cos \omega t\} \right]_0^{\pi/\omega} \\ &= \frac{E}{1 - e^{-s(2\pi/\omega)}} \frac{\omega(e^{-s\pi/\omega} + 1)}{s^2 + \omega^2} \\ &= \frac{E\omega(1 + e^{-s\pi/\omega})}{(1 - e^{-s\pi/\omega})(1 + e^{-s\pi/\omega})(s^2 + \omega^2)} \\ &= \frac{E\omega}{(1 - e^{-s\pi/\omega})(s^2 + \omega^2)}. \end{aligned}$$

3. A periodic function $f(t)$ of period $2a$, $a > 0$ is defined by

$$f(t) = \begin{cases} E, & 0 < t \leq a \\ -E, & a < t \leq 2a. \end{cases}$$

$$\text{Show that } L\{f(t)\} = \frac{E}{s} \tanh\left(\frac{as}{2}\right).$$

Solution:

$$\begin{aligned} \text{Here } T &= 2a. \text{ Therefore } L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2as}} \left[\int_0^a E e^{-st} dt + \int_a^{2a} -E e^{-st} dt \right] \\ &= \frac{E}{s(1 - e^{-2as})} \left[(1 - e^{-sa}) + (e^{-2as} - e^{-as}) \right] \\ &= \frac{E}{s(1 - e^{-2as})} \left[(1 - e^{-as})^2 \right] = \frac{E(1 - e^{-as})^2}{s(1 - e^{-as})(1 + e^{-as})} \\ &= \frac{E}{s} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right] = \frac{E}{s} \tanh\left(\frac{as}{2}\right). \end{aligned}$$

Exercise:

- Find $L\{f(t)\}$ of periodic function given $f(t) = \begin{cases} t & 0 \leq t \leq a \\ 2a - t & a < t \leq 2a \end{cases}$ and $f(2a + t) = f(t)$.

- Find $L\{f(t)\}$ given

$$f(t) = \begin{cases} 1 & 0 < t < \frac{a}{2} \\ -1 & \frac{a}{2} < t < a \end{cases}$$

$$f(2a + t) = f(t).$$

Solution:

1. $\frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$ 2. $\frac{1}{s} \tanh\left(\frac{as}{4}\right)$

Unit Impulse function

Phenomena of an impulsive nature, such as the action of forces or voltages over short interval of time, arise in various applications, for instance, if a mechanical system is hit by a hammer blow, an aeroplane makes ‘hard’ landing, a ship is hit by a single high wave, hit a tennis ball by a racket and so on. These problems are modeled by ‘**Dirac’s delta function**’ (unit impulse function) $\delta(t)$ and efficiently solved by Laplace transform.

The unit impulse function may be regarded as the limiting form of the function

$$\delta_\varepsilon(t - a) = \begin{cases} \frac{1}{\varepsilon}, & a < a + \varepsilon \\ 0, & \text{otherwise} \end{cases} \quad \text{as } \varepsilon \rightarrow 0.$$

As $\varepsilon \rightarrow 0$, the width of the strip shown in fig becomes smaller and smaller, the height of the strip increases indefinitely in such a way that the area remains unity in $(\varepsilon, \frac{1}{\varepsilon})$.

Thus the unit impulse function or Dirac delta function $\delta(t - a)$ is defined as

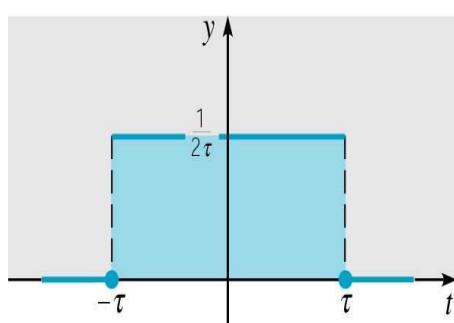
$$\delta(t - a) = \begin{cases} \infty & \text{at } t = a \\ 0 & \text{for } t \neq a. \end{cases}$$

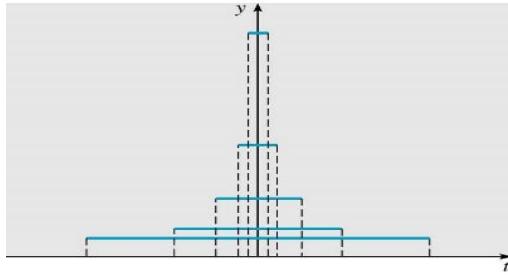
Such that $\int_0^\infty \delta(t - a) dt = 1$ ($a \geq 0$).

The unit impulse function $\delta(t)$ is defined to have the properties

$$\delta(t) = 0 \text{ for } t \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1$$

The unit impulse function is usually called the **Dirac delta function**.





Laplace Transform of Unit Impulse function

If $f(t)$ be a continuous function at $t = a$, then $\int_0^\infty f(t)\delta(t-a)dt = f(a)$

Proof:

$$\int_0^\infty f(t)\delta_\varepsilon(t-a)dt = \int_a^{a+\varepsilon} f(t)\frac{1}{\varepsilon}dt = (a + \varepsilon - a)f(\tau)\frac{1}{\varepsilon} = f(\tau)$$

As $\varepsilon \rightarrow 0$, $\int_0^\infty f(t)\delta(t-a)dt = f(a)$

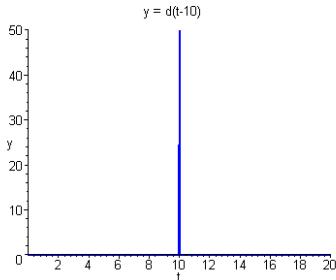
Taking $f(t) = e^{-st}$ we get,

$$\int_0^\infty e^{-st}\delta(t-a)dt = e^{-as}$$

ie., $L[\delta(t-a)] = e^{-as}$.

In particular, if $a = 0$, $L[\delta(t)] = 1$.

For example when $a = 10$, $L\{\delta(t-10)\} = e^{-10s}$.



Video links:



<https://www.youtube.com/watch?v=6MXMDrs6ZmA>

<https://www.youtube.com/watch?v=wnnnv4wt-Lw>

<https://www.youtube.com/watch?v=N-zd-T17uiE>

<https://www.youtube.com/watch?v=l7nzLD3t4Uc>