A   A manual computation for a simple initial configuration

Consider the one-die chip-clearing game with three fields and outcome probabilities $p_1, p_2, p_3 \geq 0$ satisfying $p_1 + p_2 + p_3 = 1$. Player A starts with two chips on field 1 and none elsewhere, while player B starts with one chip on field 3 and none elsewhere. We denote this initial configuration by

$$(200, 001) \equiv \big((2, 0, 0), (0, 0, 1)\big),$$

where the first tuple represents A's remaining chips and the second tuple represents B's remaining chips.

Proposition 1. *For the initial configuration* $(200, 001)$ *one has*

$$P_A(200, 001) = \left(\frac{p_1}{p_1 + p_3}\right)^2,$$

$$P_B(200, 001) = \frac{p_3(2p_1 + p_3)}{(p_1 + p_3)^2},$$

$$P_U(200, 001) = 0.$$

Proof. Let $P(200, 001)$ denote $P_A(200, 001)$. From state $(200, 001)$ a single roll yields three possibilities: with probability p_1 a chip is removed from A, leading to $(100, 001)$; with probability p_2 no chip is removed and the state remains unchanged; with probability p_3 the only chip of B is removed, leading to $(200, 000)$, which is a losing state for A.

Therefore,

$$P(200, 001) = p_1 P(100, 001) + p_2 P(200, 001) + p_3 P(200, 000).$$

Since $P(200, 000) = 0$, the recursion reduces to

$$(1 - p_2)\, P(200, 001) = p_1\, P(100, 001).$$

Recalling that $1 - p_2 = p_1 + p_3$, we obtain

$$P(200, 001) = \frac{p_1}{p_1 + p_3}\, P(100, 001).$$

Similarly,

$$P(100, 001) = p_1 P(000, 001) + p_2 P(100, 001) + p_3 P(100, 000),$$

where $P(000,001) = 1$ and $P(100,000) = 0$. Thus,

$$P(100,001) = \frac{p_1}{p_1 + p_3}.$$

Substitution yields

$$P_A(200,001) = \left(\frac{p_1}{p_1 + p_3}\right)^2.$$

Since both players cannot finish on the same roll in this configuration, $P_U(200,001) = 0$, and the stated expression for $P_B(200,001)$ follows from $P_A + P_B = 1$. $\qquad\square$

Remark 1. An equivalent interpretation is obtained by conditioning on the first roll that removes a chip from either player. Rolls landing on empty fields may be ignored. Conditional on a chip being removed, the probability that A removes the next chip equals $p_1/(p_1 + p_3)$. In the configuration $(200,001)$, player A must succeed in this event twice before B removes its single chip, which yields the same squared expression for $P_A(200,001)$.