EITQ

Counting Problems, Matchgate Quantum Computing

Renaud Vilmart

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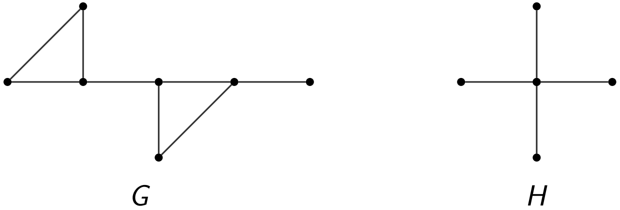

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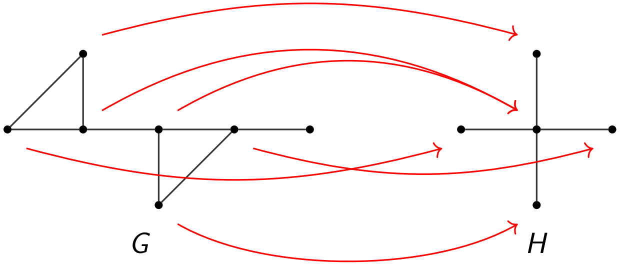

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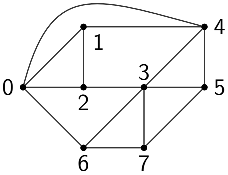
E.g. 

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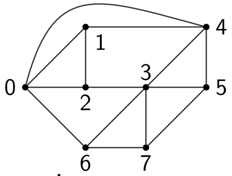
E.g. 

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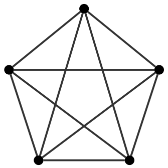
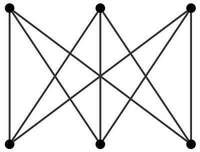
Two very important non-planar graphs:

$K_5 :=$ 

$K_{3,3} :=$ 

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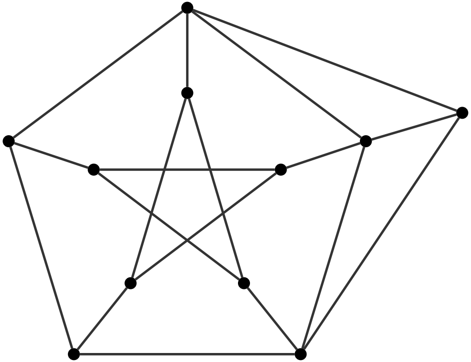
  - Planarity is **P**.

Example: Planarity?

Is the following graph planar?

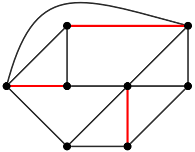


Matchings and Perfect Matchings

Definition

With $G = (V, E)$ a graph:

- Matching: $M \subseteq E$, s.t. $\forall e_i, e_j \in M,\ i \neq j \implies e_i \cap e_j = \emptyset$.



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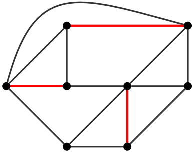

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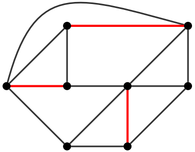

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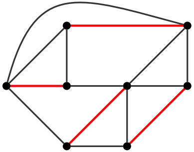
- Perfect Matching: A matching M that contains all vertices, i.e. $\bigcup_{e \in M} e = E$.



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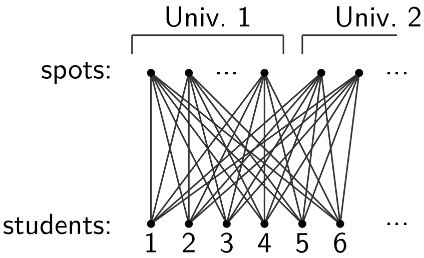

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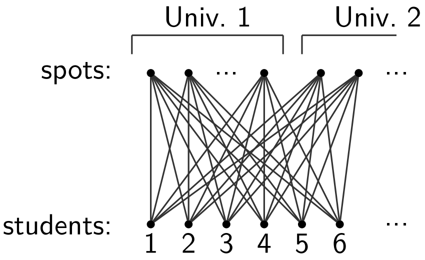

Aparté:

Particular case of a bipartite graph.

Suppose $|students| < |spots|$. A maximum cardinality matching covers all students iff $\forall W \subseteq \text{students},\ |W| \leq |N(W)|$.

Proof of above gives an algorithm.

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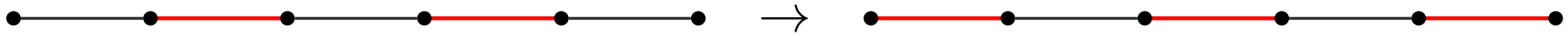

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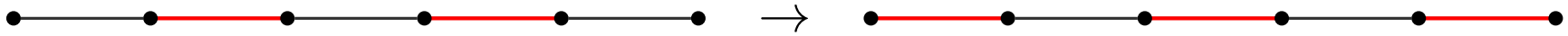

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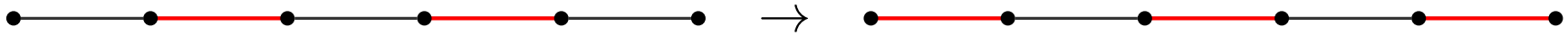

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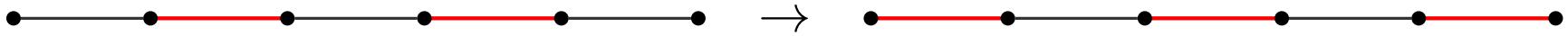

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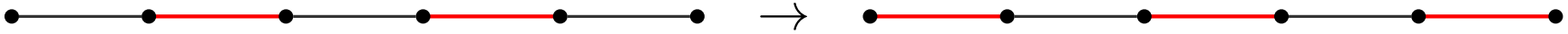

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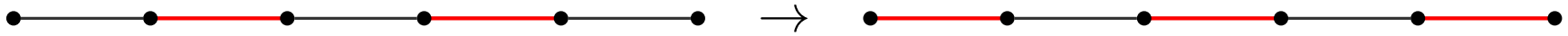

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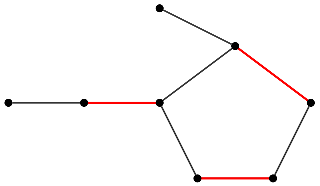

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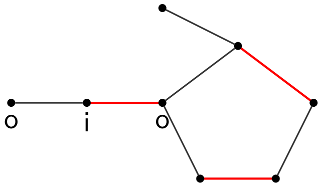

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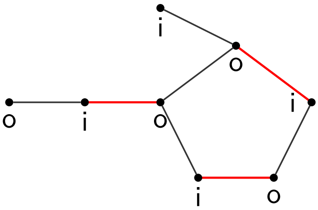

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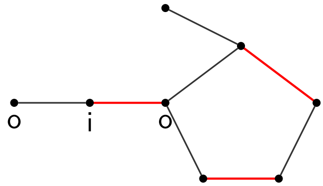

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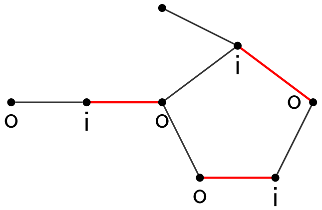

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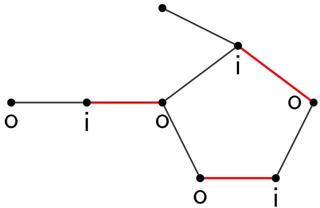

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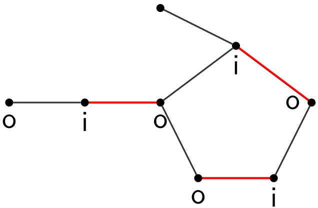

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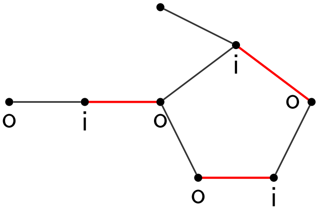

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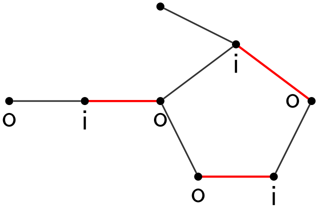

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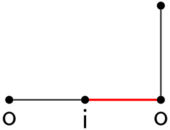

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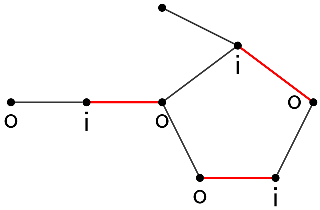

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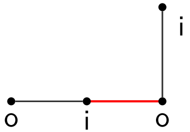

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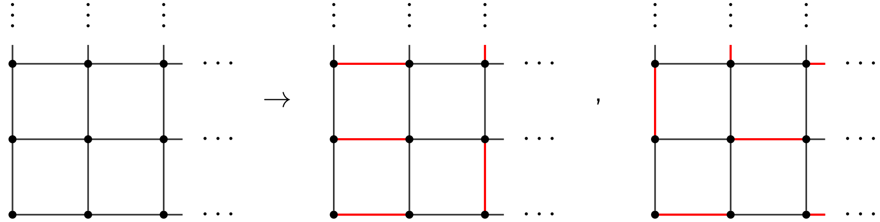
- Molecules laid out on a lattice. How many ways to “dimerise” them?



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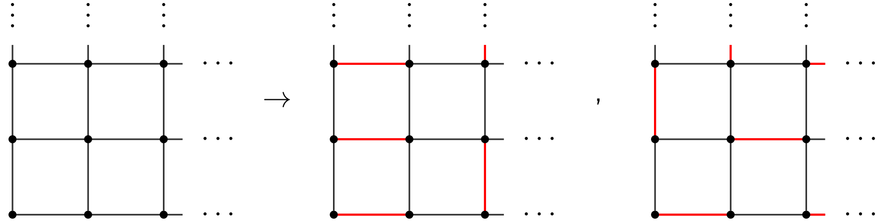

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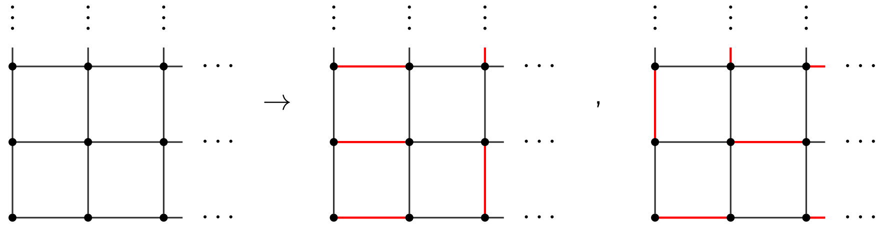
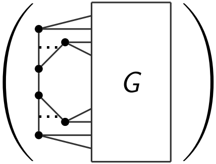
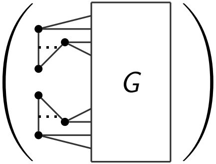
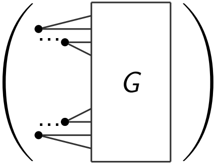


- $\equiv$ counting perfect matchings in the lattice.
- Requires studying complexity of counting problems

A Natural (non-efficient) Way to Count Perfect Matchings

#PM 

= #PM 

+ #PM 

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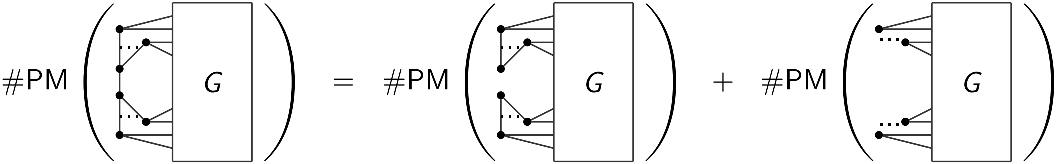
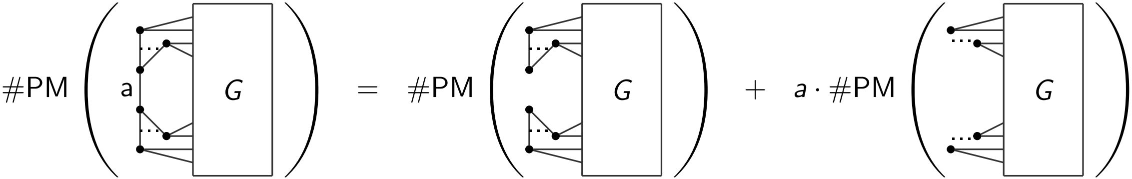


With weights on edges ($w : E \to R$):



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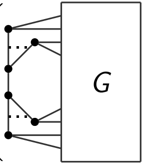

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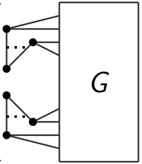

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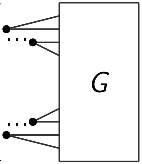

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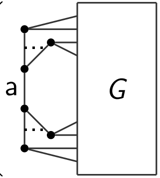

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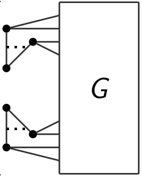

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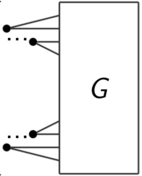

)

$\#\mathrm{PM}_R$: counting perfect matchings with weights in R.

Complexity of Counting Problems

- **#P** problems: functions $f : \{0,1\}^* \to \mathbb{N}$ such that there exists a polynomial-time non-deterministic Turing machine M such that $f(x)$ is the number of accepting paths in M's computation of x.

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- **#P-complete**: **#P** problems such that for all other **#P** problems f, there is a polynomial-time counting reduction ($\leqslant_{\#}$) from f to that problem

Polynomial-Time Counting Reduction

Given two problems X and Y, we say that X reduces to Y ($X \leqslant_{\#} Y$) if there exist polynomial-time computable functions f and g such that the following diagram commutes:

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$\#$3-SAT is **#P-complete**: the Cook-Levin construction preserves the number of solutions (i.e. $g = id$)

Proving #PM is **#P-complete**

- $\#\text{SAT} \leqslant_{\#} \text{PERM}_{\{-1,0,1,2,3\}}$
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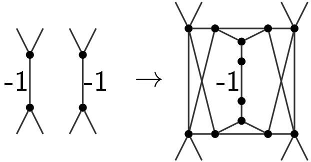

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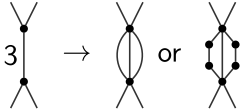

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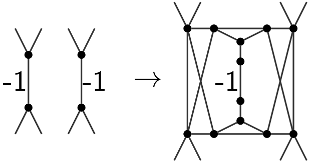

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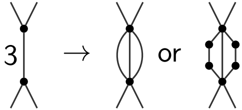

More details in the TD sheet.

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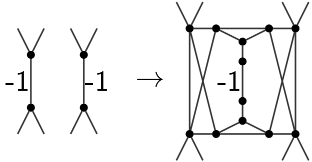

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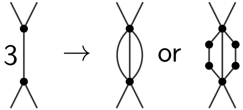

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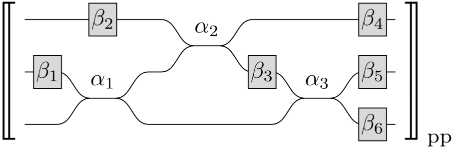
While PM $\in$ **P**, #PM $\in$ **#P-complete**! (other example: #2SAT)

Permanent for Linear Optics

n-mode linear optical circuit: composed of phase shifters and beam splitters.

Single-photon semantics: $n \times n$ unitary:



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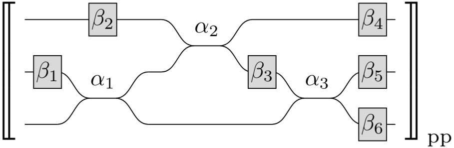

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Then, many-photon semantics $\mathcal{U}$ such that:

$$\langle \ell_1, \dots, \ell_n | \, \mathcal{U} \, | k_1, \dots, k_n \rangle = \mathrm{Perm} \left[\begin{array}{cc} & k_1 \\ \ell_1 & \begin{pmatrix} u_{0,0} & \cdots & u_{0,0} & u_{0,1} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \\ u_{0,0} & \cdots & u_{0,0} & u_{0,1} & \cdots \\ u_{1,0} & \cdots & u_{1,0} & u_{1,1} & \cdots \\ \vdots & & \vdots & \vdots & \end{pmatrix} \end{array} \right]$$

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  - Rewriting of tensors
- (Planar) matchgates: family of quantum circuits efficiently simulable

Matchgate Computations

Let $\quad A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$

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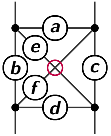
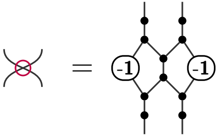
planar matchgate: 

with 

Tensors in Matchgates

Everything with 2-dimensional indices:

$$\left(\textcircled{r} \right)^i_j = r^i \delta^i_j = \begin{cases} 1 & \text{if } i = j = 0 \\ r & \text{if } i = j = 1 \\ 0 & \text{if } i \neq j \end{cases}$$

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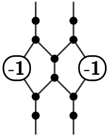
 $=$ 

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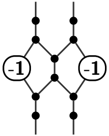
 $=$ 

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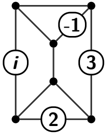
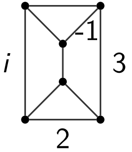


computes the number of perfect matchings in 

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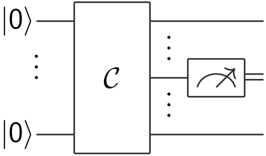

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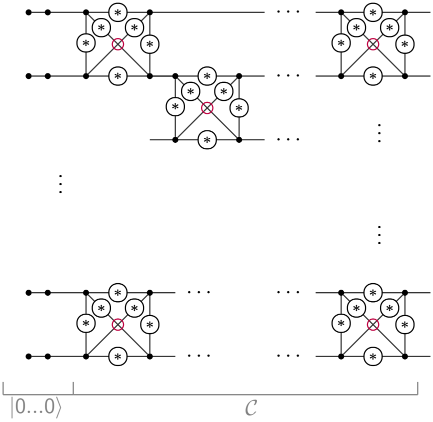

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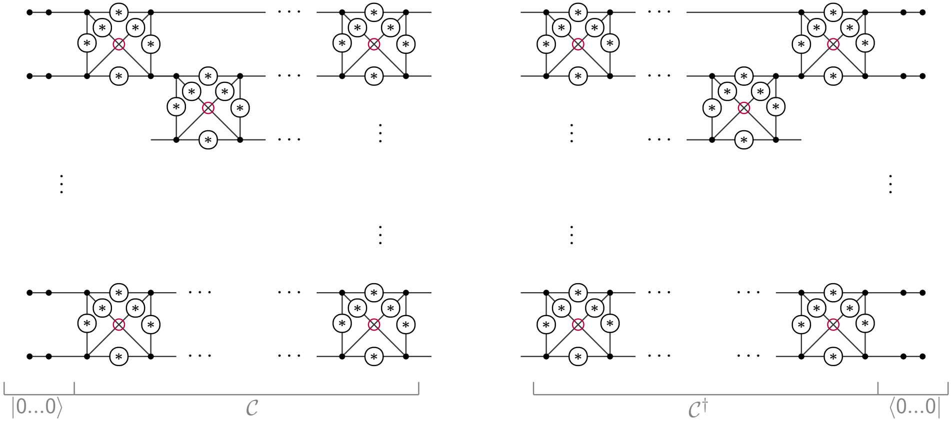

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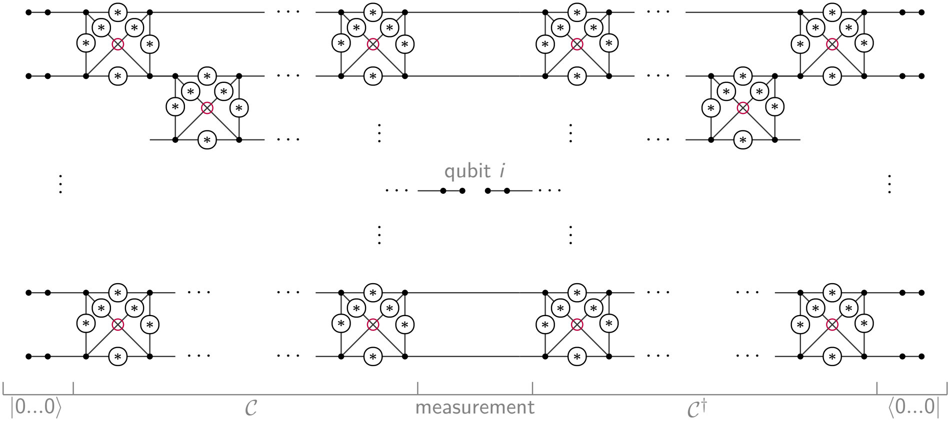

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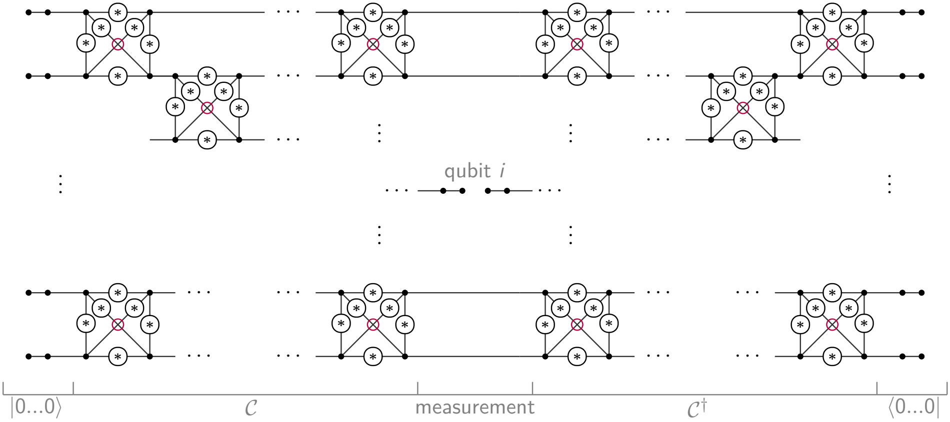
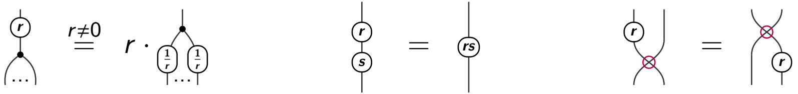


Interpretation in $\mathcal{M}_{1\times1}(\mathbb{C}) \cong \mathbb{C}$: a scalar, the probability we are looking for!
Alternatively: graph with complex edge weights.

Simulating Planar Matchgates via Rewriting

- Weights can be moved around



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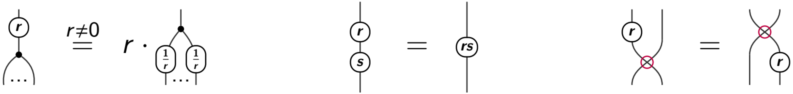

- Fermionic swaps  are dealt with like swaps

Simulating Planar Matchgates via Rewriting

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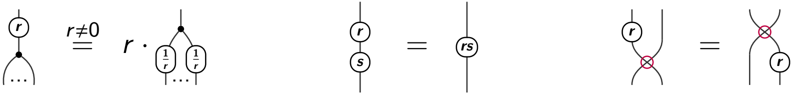

- Fermionic swaps  are dealt with like swaps

- Few rewrites

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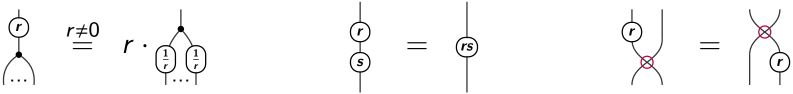

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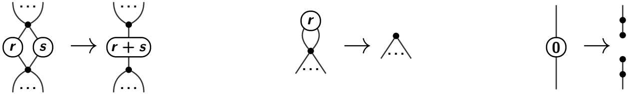

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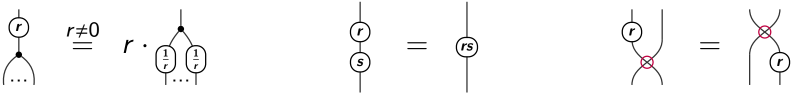

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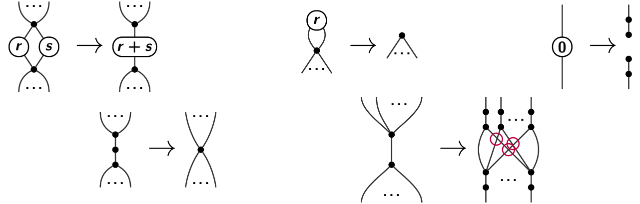

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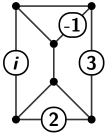

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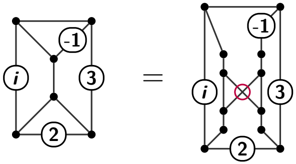

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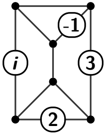

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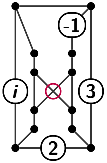

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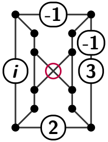

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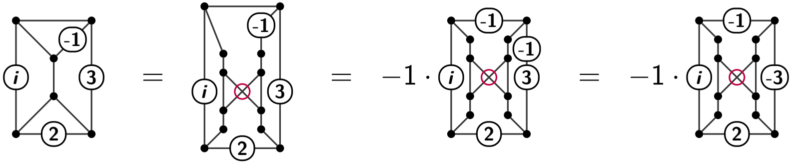

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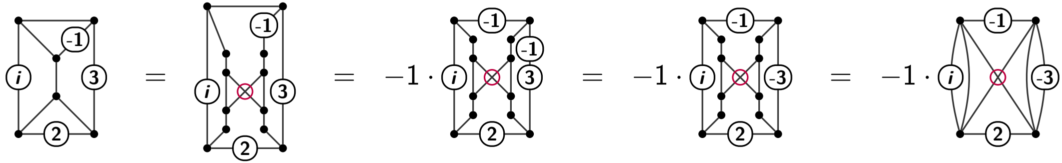

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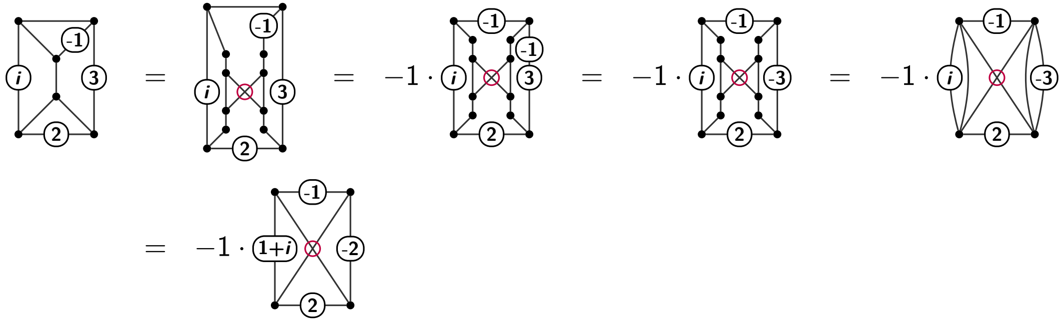

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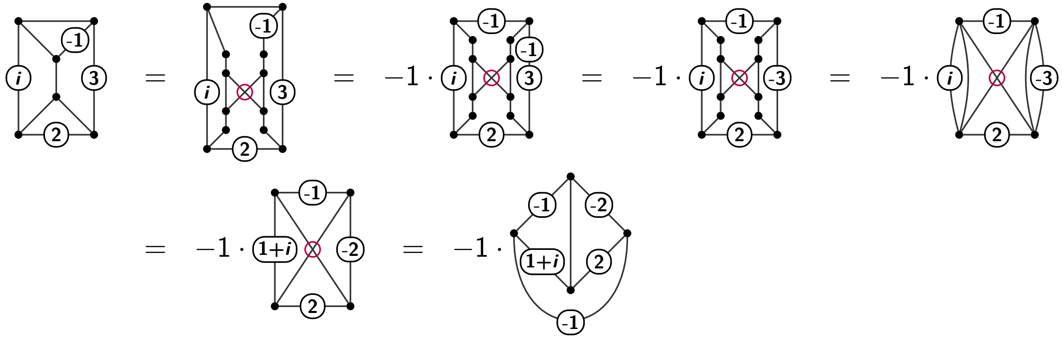

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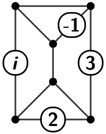

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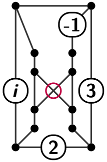

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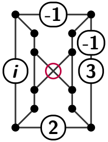

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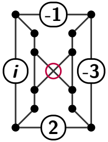

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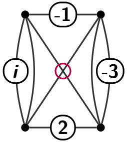

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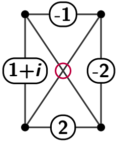

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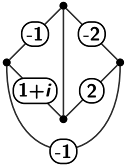

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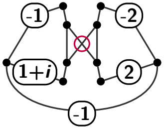

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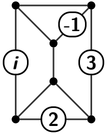

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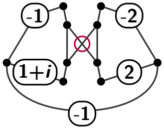

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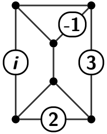

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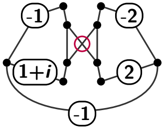

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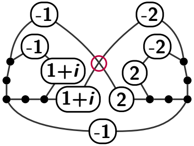

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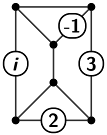

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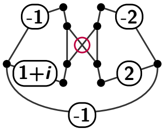

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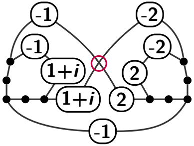

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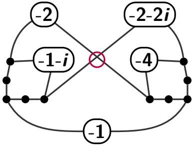

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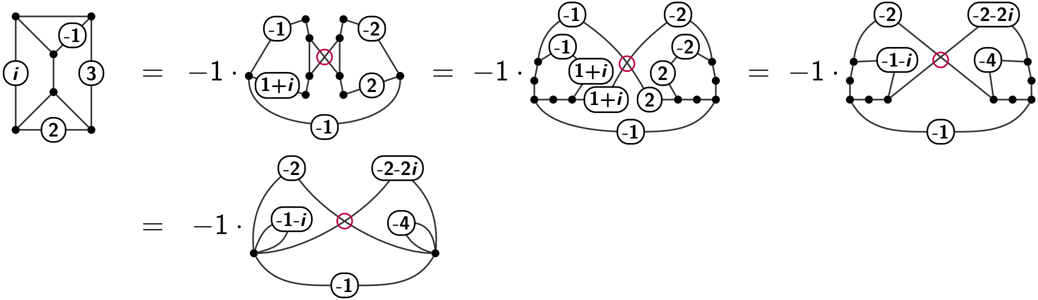

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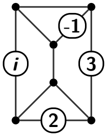

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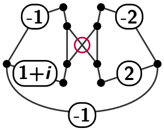

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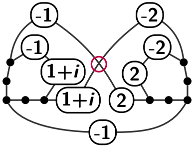

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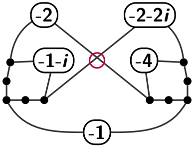

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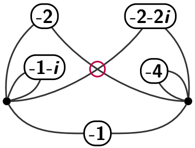

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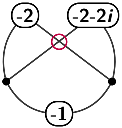

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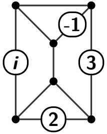

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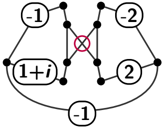

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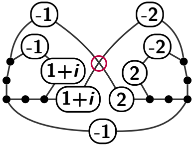

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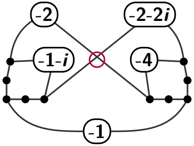

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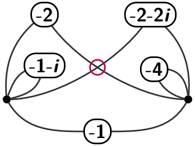

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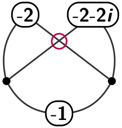

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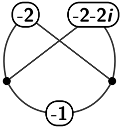

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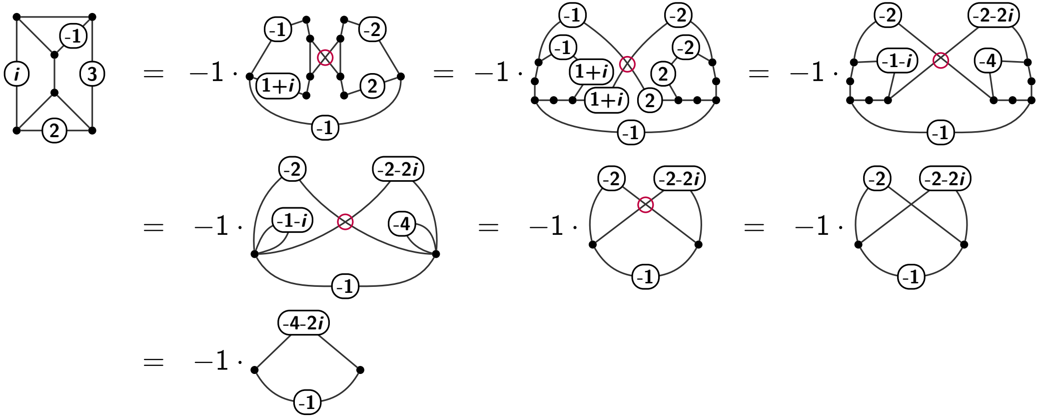
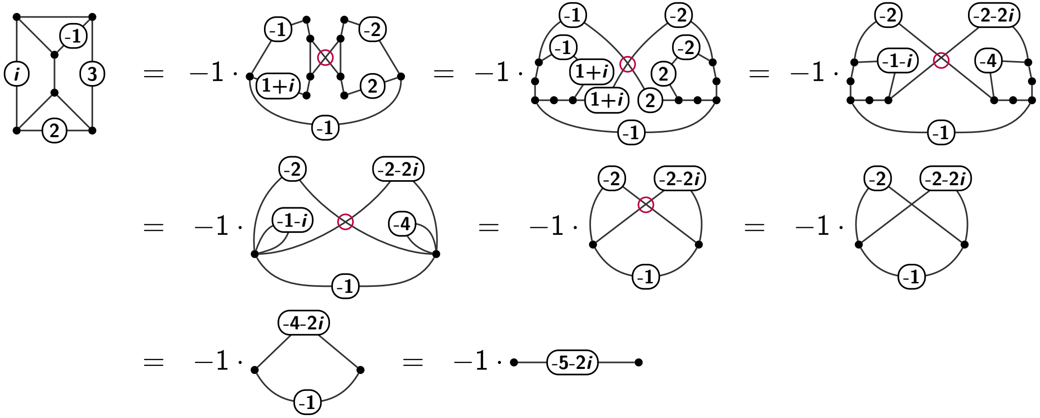
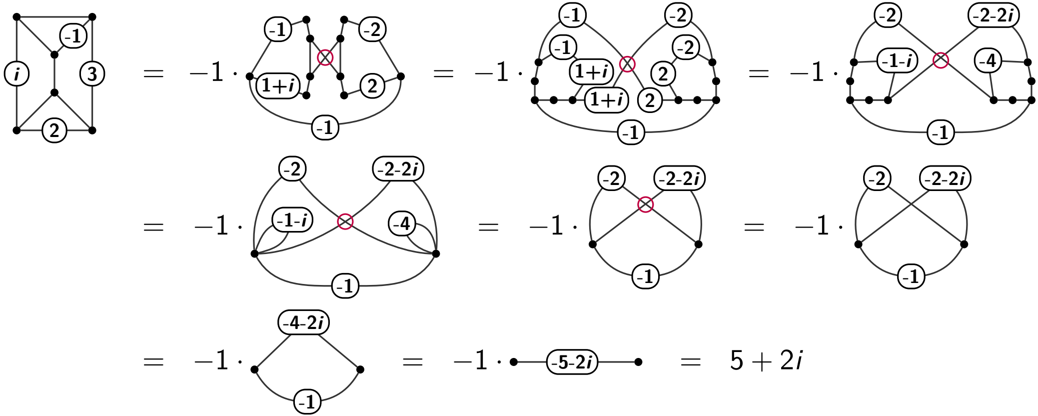

-1
i
3
2
= −1 ·
-1
-2
1+i
2
-1
= −1 ·
-1
-2
-1
-2
1+i
2
1+i
2
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-4-2i
-1

Example



-1
i
3
2
= −1 ·
-1
-2
1+i
2
-1
= −1 ·
-1
-2
-1
-2
1+i
2
1+i
2
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-4-2i
-1
= −1 ·
-5-2i

Example



i
-1
3
2
= −1 ·
-1
-2
1+i
2
-1
= −1 ·
-1
-2
-1
-2
1+i
2
1+i
2
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1-i
-4
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-2
-2-2i
-1
= −1 ·
-4-2i
-1
= −1 ·
-5-2i
= 5 + 2i

Recap

- Deciding existence of solution to SAT problem: **NP-complete**

Recap

- Deciding existence of solution to SAT problem: **NP-complete**
- Counting solutions to SAT problem: **#P-complete**

Recap

- Deciding existence of solution to SAT problem: **NP-complete**
- Counting solutions to SAT problem: **#P-complete**

- Deciding existence of perfect matching: **P**

Recap

- Deciding existence of solution to SAT problem: **NP-complete**
- Counting solutions to SAT problem: **#P-complete**

- Deciding existence of perfect matching: **P**
- Counting perfect matchings: **#P-complete**

Recap

- Deciding existence of solution to SAT problem: **NP-complete**
- Counting solutions to SAT problem: **#P-complete**

- Deciding existence of perfect matching: **P**
- Counting perfect matchings: **#P-complete**

- Counting perfect matchings in planar graph: **FP**

Recap

- Deciding existence of solution to SAT problem: **NP-complete**
- Counting solutions to SAT problem: **#P-complete**

- Deciding existence of perfect matching: **P**
- Counting perfect matchings: **#P-complete**

- Counting perfect matchings in planar graph: **FP**
- Deciding planarity: **P**