QDCS : Digrammatic Calculus and Error Correction

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TD 5

1 Small CSS Codes

Question 1. For the following pairs of X and Z parity check matrices, give the CSS code that they define, if it exists:

1. $H_Z = \begin{pmatrix}1\end{pmatrix}, \quad H_X = \begin{pmatrix}0\end{pmatrix}$
2. $H_Z = \begin{pmatrix}0\end{pmatrix}, \quad H_X = \begin{pmatrix}1\end{pmatrix}$
3. $H_Z = \begin{pmatrix}1 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}0 & 1\end{pmatrix}$
4. $H_Z = \begin{pmatrix}1 & 1\end{pmatrix}, \quad H_X = \begin{pmatrix}0 & 1\end{pmatrix}$
5. $H_Z = \begin{pmatrix}1 & 1 & 1\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$
6. $H_Z = \begin{pmatrix}0 & 0 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$
7. $H_Z = \begin{pmatrix}0 & 0 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$

Question 2. For the above codes, what are their length, dimension and minimal distance?

Question 3. For the above CSS codes, build the state as a ZX-diagram if it is maximal, otherwise complete the parity check matrices as you see fit, then build the encoder as a ZX-diagram.

2 CSS Code States

Let H_X and H_Z be two parity check matrices, for codes C_X and C_Z respectively. Let $\left|z + C_X^\perp\right\rangle := \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} |z + x\rangle$ for $z \in C_Z$.

Question 1. Let $u \in C_X^\perp$. Show that $X^u \left|z + C_X^\perp\right\rangle = \left|z + C_X^\perp\right\rangle$.

Question 2. Let $u \in C_Z^\perp$. Show that $Z^u \left|z + C_X^\perp\right\rangle = \left|z + C_X^\perp\right\rangle$.

3 The Steane Code

The Steane code is defined as the CSS code with:

$$H_X = H_Z = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & 1 & 1\end{pmatrix}$$

Question 1. Check that this indeed defines a CSS code.

Question 2. How many logical qubits does it encode?

Question 3. What is its minimal distance?
Hint: See if you can't conclude from the respective minimal distances of the image and the kernel of these matrices.

Question 4. Complete H_X and H_Z to get a maximal CSS code (as you see fit)

Question 5. Build the encoder for the Steane code

4 Surface Code

Consider the 3×3 surface code from the course.

Question 1. Complete the CSS code into a maximal CSS code.

Question 2. Provide a ZX-diagram implementing the obtained code. Deduce an encoder of the surface code.