QMI / QDCS : ZX-Calculus

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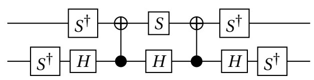
TD6: Clifford & Full ZX-calculus

1 Clifford circuit

Consider the following circuit:



Question 1. Turn the circuit into a ZX-diagram.

Question 2. Put the diagram in graph-like form.

Question 3. Put the diagram in Clifford normal form.

Answer: Done in class.

2 Stabilisers

Question 1. Show that the group generated by the stabilisers $YYZZ$, $ZXYX$, $-IXIX$, $-XXZI$ does not contain $-IIII$. Build the ZX state whose stabilisers are generated by the above.

Answer: All the Pauli strings commute, and none gave a $\pm i$ as factor, so $-IIII$ is not part of the group. Using $Y = iXZ$, the stabilisers can be grouped in the parity matrix as follows:

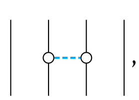
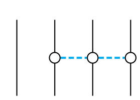
$$\left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

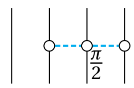
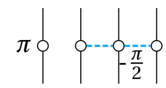
We can then simplify it as follows:

$$\overset{R_3 \mathrel{+}= R_0}{\equiv} \left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \overset{R_1 \mathrel{+}= R_2}{\equiv} \left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

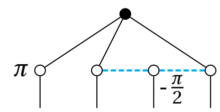
$$\overset{R_2 \mathrel{+}= R_0}{\equiv} \left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \overset{\substack{R_0 \mathrel{+}= R_3 \\ R_1 \mathrel{+}= R_3 \\ R_2 \mathrel{+}= R_3}}{\equiv} \left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

where in the last matrix we've highlighted the (only) pivot column of H_Z. The elements in that column for H'_Z are zero, and H_X without that column is the identity. We now want to build the state in such a way that the Pauli strings turn into other Pauli strings, where the scalar powers and H'_Z become 0. To get rid of the 1 in the first row, we need to apply a CZ between qubits 1 and 2 (if we start counting at 0). Continuing on, we get:

$\rightsquigarrow$  $,\left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$ $\rightsquigarrow$  $,\left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$

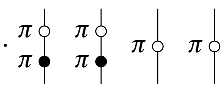
$\rightsquigarrow$  , $\left(\begin{array}{c|cccc|cccc} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$ $\rightsquigarrow$  , $\left(\begin{array}{c|cccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$

We are left with the stabilisers of a CSS state. Applying the construction of CSS states from the H_Z matrix, we get, as a final state:

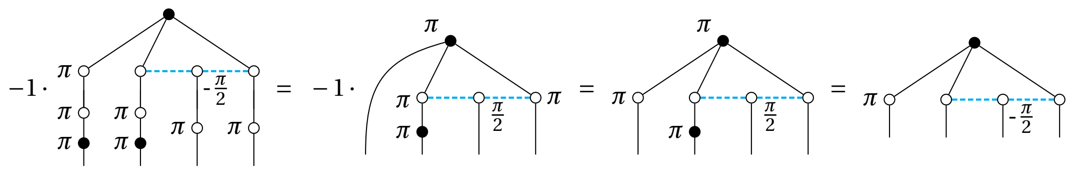


Question 2. Graphically check that $YYZZ$ stabilises the state.

Answer: Again using the fact that $Y = iXZ$, the stabiliser $YYZZ$ can be written in ZX as:

$-1 \cdot$ 

Applying it to the state, and using the usual rules of ZX, gets us:



We indeed have a state $|\psi\rangle$ such that $(YYZZ)|\psi\rangle = |\psi\rangle$.

3 Measurements of the GHZ state

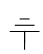
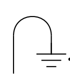
Recall the family of so-called GHZ states, which can be built inductively as: $|\text{GHZ}_1\rangle = H|0\rangle$, and $|\text{GHZ}_{n+1}\rangle = (I^{\otimes n-1} \otimes \text{CNot}) \circ (|\text{GHZ}_n\rangle \otimes |0\rangle)$.

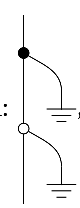
Question 1. Show by induction that $|\text{GHZ}_n\rangle$ is simply a $0 \to n$ phase-0 Z-spider, up to a global scalar. Specify that scalar.

Question 2. Recall a ZX representation of the scalar 0. Show using variables that, when measuring all qubits in the Z basis, outcomes where the output bits are not identical, have probability 0.

Answer: Done in class

4 Consecutive Measurements

Question 1. Define the "upside-down" trace as:  := . This is hence a quantum state. Compute its interpretation. Explain why it can be understood as complete noise.

Question 2. Compute the interpretation of the diagram: , and show it is equivalent to tracing out the input state and outputting complete noise.

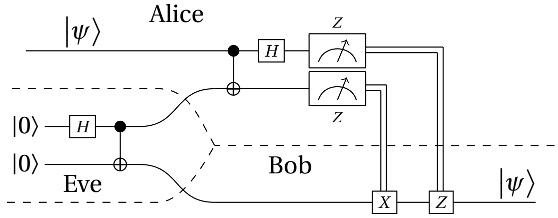
Question 3. Show the equality between the diagrams of the previous question, but graphically this time, in two different ways:

1. directly using the axioms of the partial trace
2. by going through the CPM construction

Answer: Done in class.

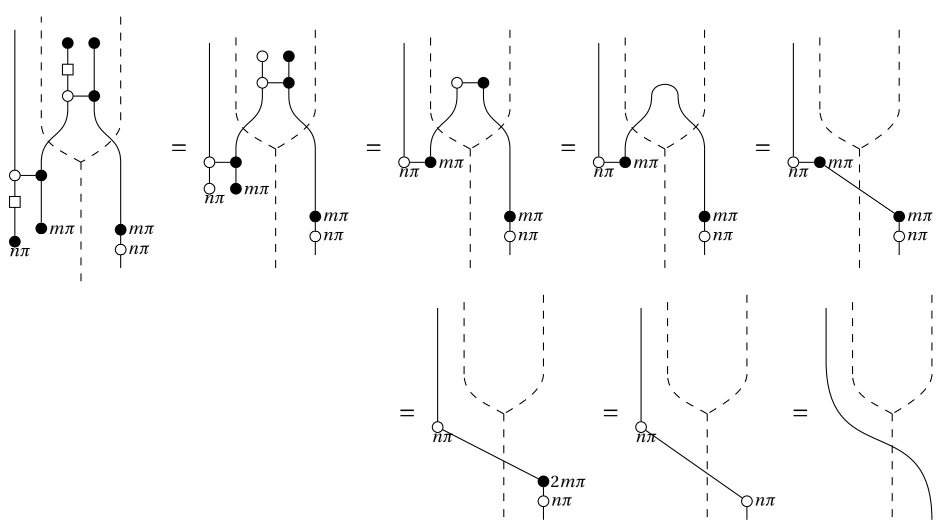
5 Quantum Teleportation

The problem is the following: Two parties, Alice and Bob, can communicate, but only classically, i.e. they can exchange classical bits, but not qubits. Alice has a qubit in an unknown state $|\psi\rangle$ which she wishes to send to Bob. If no further assumption is made, the problem can be shown to be impossible to solve: Alice cannot send her qubit to Bob through a classical channel. Measurement would destroy the state and only give him (very) partial information.
The problem is however shown to be solvable if Alice and Bob initially share an entangled pair of qubits (Alice has the first one, and Bob the second one). In the following, we can imagine that this state is prepared by a third party called Eve, who splits the pair between Alice and Bob before the protocol really starts. Then, when Alice wants to send her qubit to Bob, she entangles it with her half of the pair (using a CNot), then measures her two qubits in the appropriate basis. Doing this disturbs the system but the disturbance depends on the measurement results. In this case, the errors introduced by Alice's measurements can be corrected by Bob, provided he has access to the measurement results (which are classical). Alice hence sends them to Bob. After correction, the qubit is flawlessly transmitted.
The precise protocol (where measurements use the observable Z) is given in circuit form as follows:



Question 1. Translate the quantum teleportation protocol in ZX using variables for the measurements. Show that the whole diagram is equivalent to the identity (or more exactly to a direct 1-qubit communication from Alice to Bob).

Answer:



Remark: if we take the scalars into account, we should end up with $\frac{1}{2}$, which amounts to a $\frac{1}{4}$ probability... It should be understood as the fact that the 4 possible assignments of a and b (corresponding to 4 different outcomes), each have probability $\frac{1}{4}$ of occurring. They hence all do add up to 1. The corrections make it so that all measurement outcomes in the end have the same effect on the diagram, which allows us to rid it of the variables, but they are still technically "there".

Question 2. Translate the quantum teleportation protocol in ZX using partial traces for the measurements. Show that the whole diagram is equivalent to the identity.

Answer: