Quantum Computation

ZX calculus — reference sheet

(December 2022)

Niel de Beaudrap

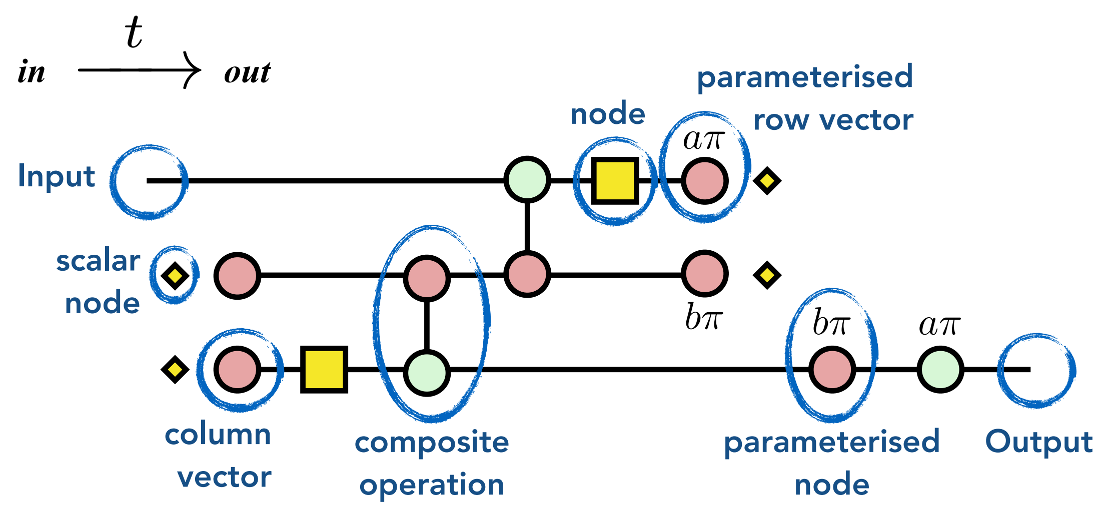
This reference sheet is intended to present the most important (i.e., the most often used) rewrites for the ZX calculus, in the 'well-tempered' normalisation [arXiv:2006.02557].

This set of rewrites is incomplete: there are more rewrites, which you can use (and which are necessary to use) to be able to show when any two diagrams represent the same state or operator. The rules presented here are intended to cover essentially any 'simple' calculation in the ZX calculus.

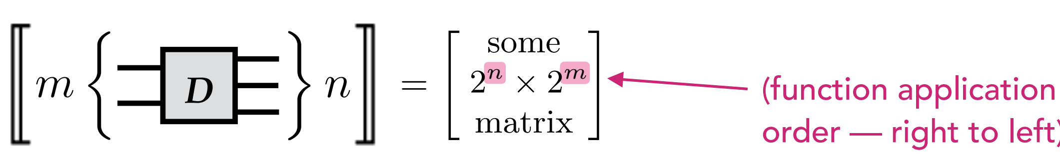
Using diagrammatic notation

The meaning of a diagram

The ZX calculus uses diagrams to represent sequences of transformations on column vectors (and on tensor products of column vectors)

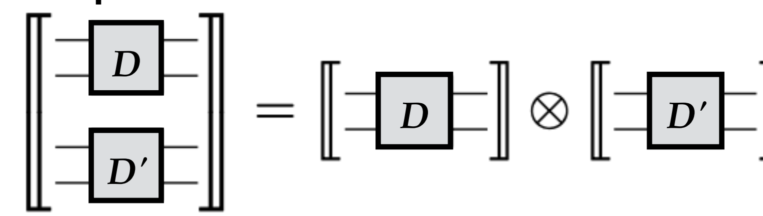


Each diagram represents a matrix, described through a "semantic map" $[\![\quad]\!] : (\text{diagrams}) \to (\text{matrices})$

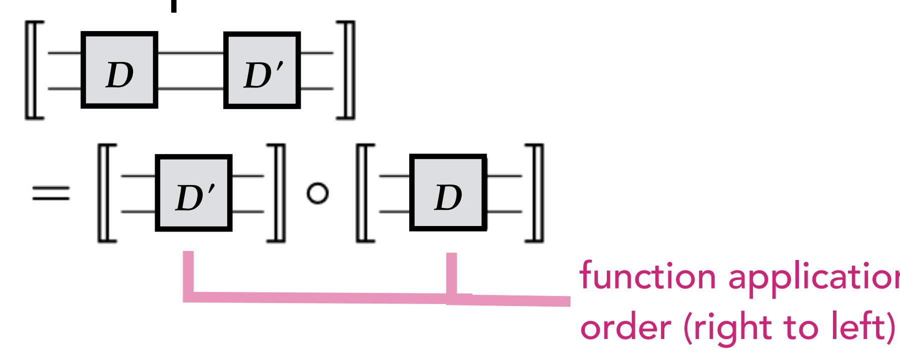


Composing diagrams

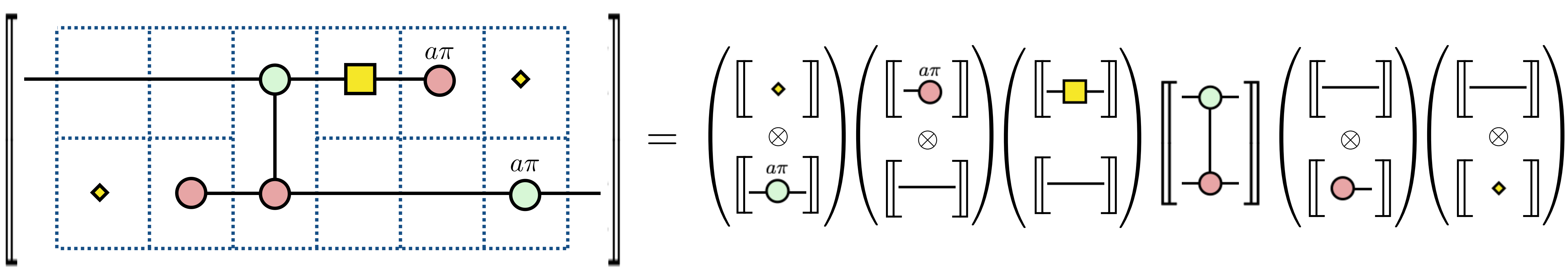
- In parallel ...



- In sequence ...

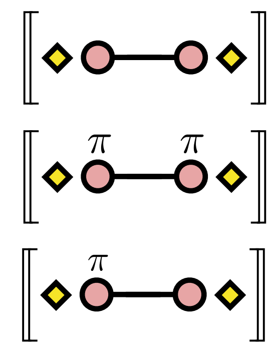


We multiply the matrices / vectors in the same ***sequence*** as in the diagram — ***but*** from right to left (as this is how we transform column vectors)



Essential ZX for circuits

Regarding normalisation: to allow for simple rewrite rules, we define the parametrised nodes "θ (red node)", "θ (green node)", etc. to have normalisation > 1. We use the "◆" scalar node as a normalising factor, to allow us to easily represent unit vectors.

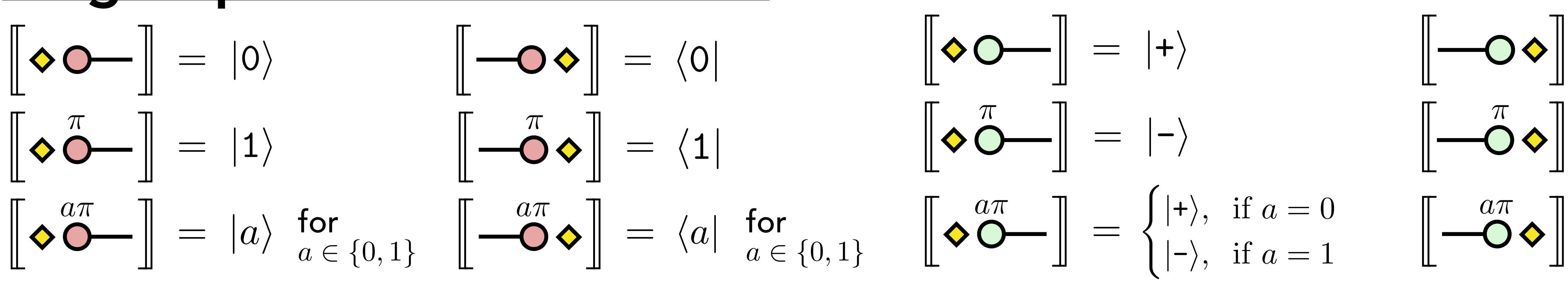


$= \langle 0|0\rangle = 1 = \langle +|+\rangle =$

$= \langle 1|1\rangle = 1 = \langle -|-\rangle =$

$= \langle 0|1\rangle = 0 = \langle +|-\rangle =$

Single-qubit states and tests

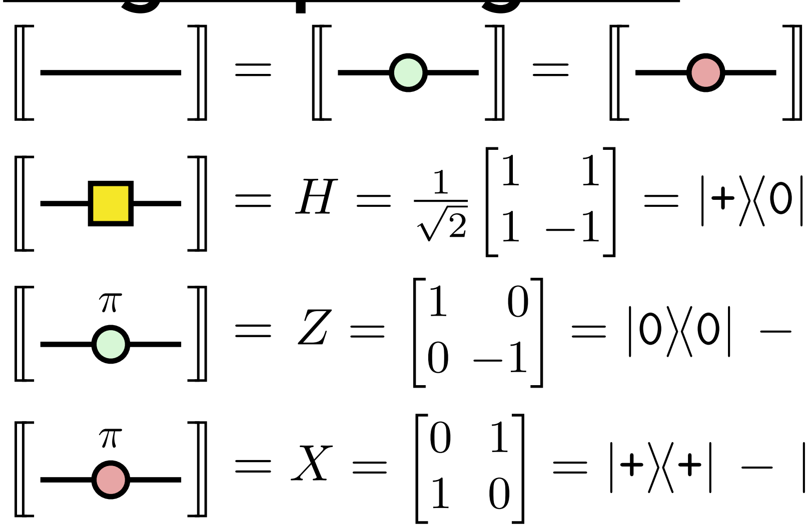


$= |0\rangle$ $= \langle 0|$ $= |+\rangle$ $= \langle +|$

$= |1\rangle$ $= \langle 1|$ $= |-\rangle$ $= \langle -|$

$= |a\rangle$ for $a \in \{0,1\}$ $= \langle a|$ for $a \in \{0,1\}$ $= \begin{cases} |+\rangle, & \text{if } a = 0 \\ |-\rangle, & \text{if } a = 1 \end{cases}$ $= \begin{cases} \langle +|, & \text{if } a = 0 \\ \langle -|, & \text{if } a = 1 \end{cases}$

Single-qubit gates



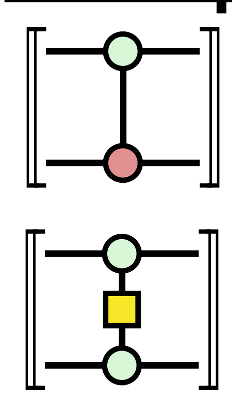
$= \mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$

$= H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle\langle 0| + |-\rangle\langle 1|$

$= Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

$= X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle +| - |-\rangle\langle -|$

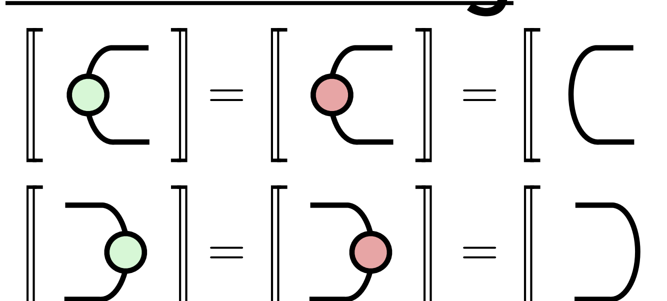
Two-qubit controlled gates



$= \mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$

$= CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z$

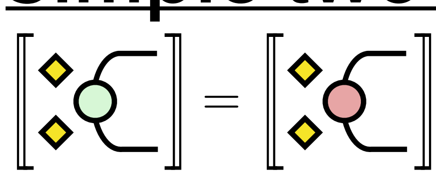
Wire bending



$= |0,0\rangle + |1,1\rangle$

$= \langle 0,0| + \langle 1,1|$

Simple two-qubit entangled state



$= |\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|0,0\rangle + |1,1\rangle \big)$

Essential ZX simplifications

Only the connectivity matters

Because the vectors $|0\rangle$, $|+\rangle$, $|1\rangle$, and $|-\rangle$ (the vectors which we use to define the red and green nodes) all have real coefficients, we may change input wires to output wires and vice-versa when these are composed with a 'cup' or a 'cap' (a wire which bends back on itself).

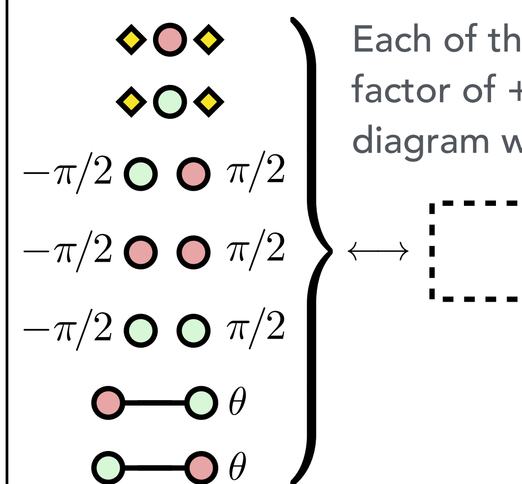
As "only the connectivity matters", all of the rewrites below also hold true if they are flipped horizontally or vertically. For example, following the 'fusion' rewrites on the next page, we have:

Angles

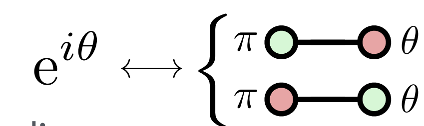
Both red and green nodes (with any # inputs or outputs): angles are evaluated mod 2π. If no angle is noted, the default angle is 0.

Diagrams for "global phases"

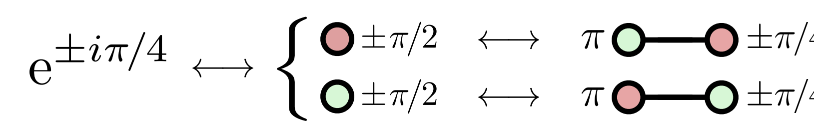
It is not uncommon for authors who use ZX to ignore scalar factors. When describing a quantum state, this may be ***safely*** done, when the scalar factor is a complex number with norm (or modulus) equal to 1 — sometimes called a "global phase factor".



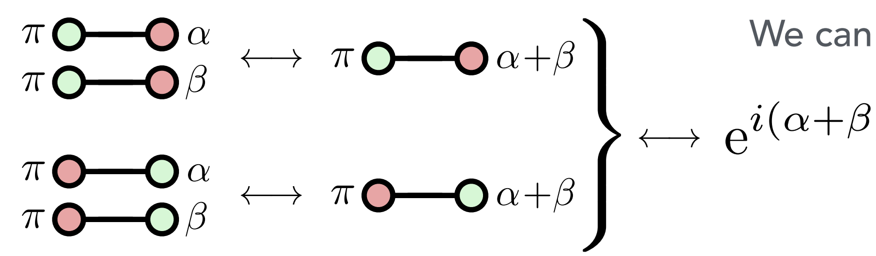
Each of these groups of 2–3 nodes, represent a scalar factor of +1, and may be safely deleted from the diagram without any change in its meaning



Two-node diagrams of this sort represent global phases

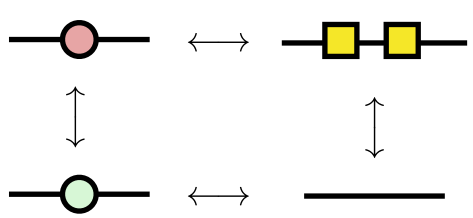


These specific scalars may be denoted by a single node



We can also simplify diagrams by accumulating scalar factors together

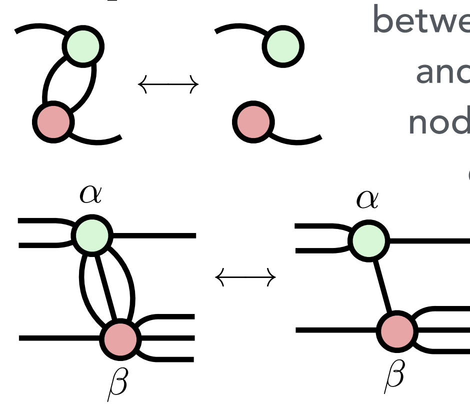
Identity



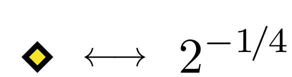
Phase-free degree-2 red and green nodes are identity operations; and Hadamards are self-inverse.

Hopf Law

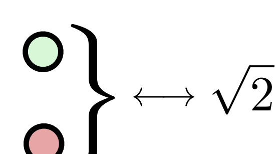
Parallel edges between a red and a green node can be canceled in pairs



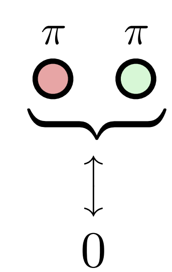
More scalar factors



This represents a scalar factor < 1, which we use for renormalising factors



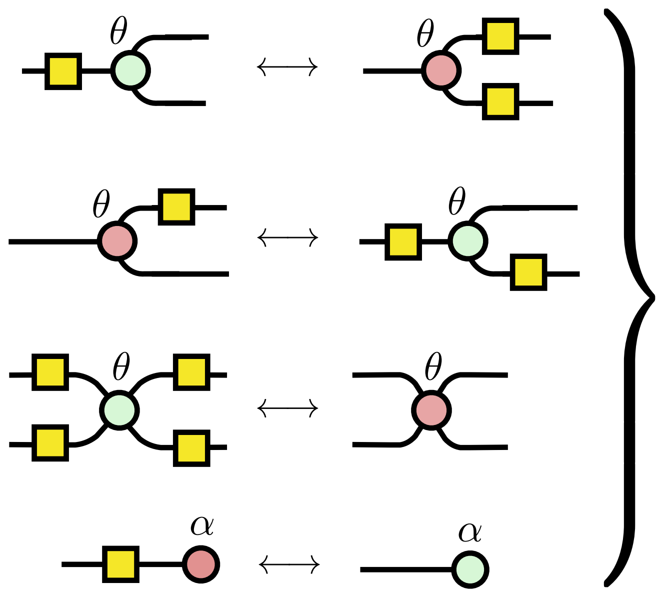
Each of these nodes represent a scalar factor > 1, and so affect the normalisation



each of these nodes represent a scalar of 0: the larger diagram represents a zero vector or operator

Essential ZX transformations

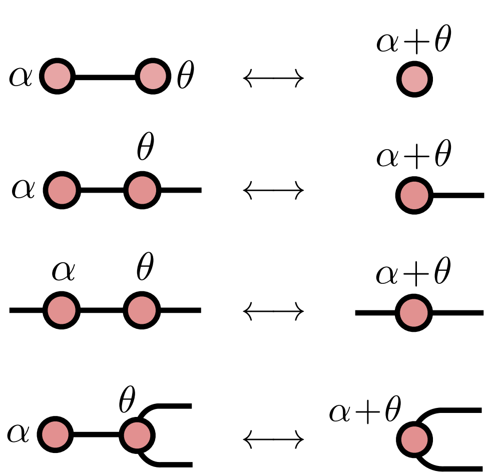
Colour change rules



A red or green node with Hadamards on some/all of its wires, can be replaced with a colour-switched node with Hadamards on the subset of its wires which did not have Hadamards on them

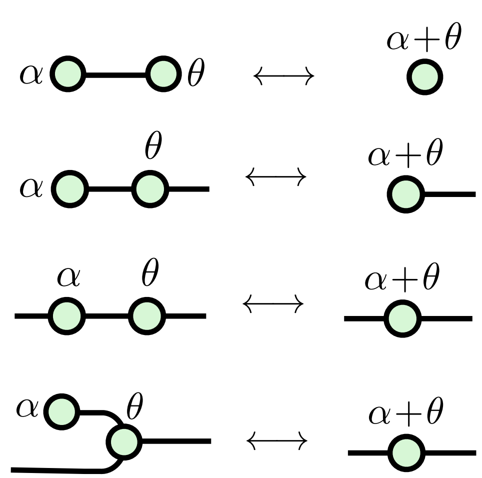
Every single ZX rewrite remains correct if we change every red node to a green node, and vice versa. This is precisely because they are related through Hadamad operations in the way described above.

Fusion rules



Red nodes fuse (adding their angles)

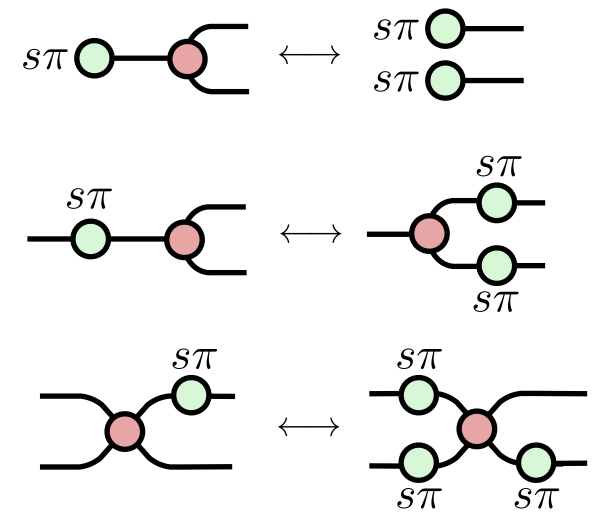
etc.



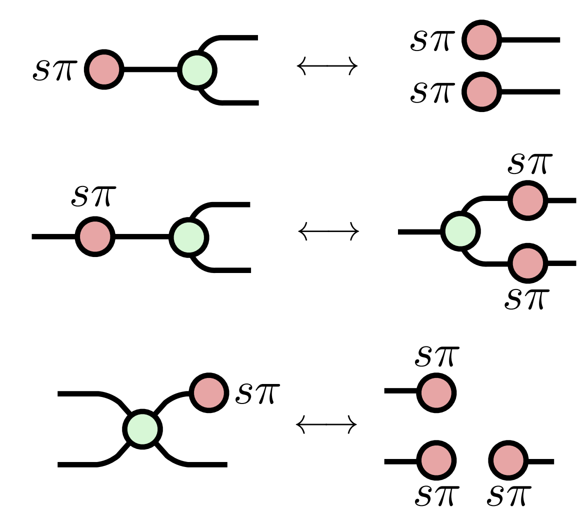
Green nodes fuse (adding their angles)

etc.

Copying rules

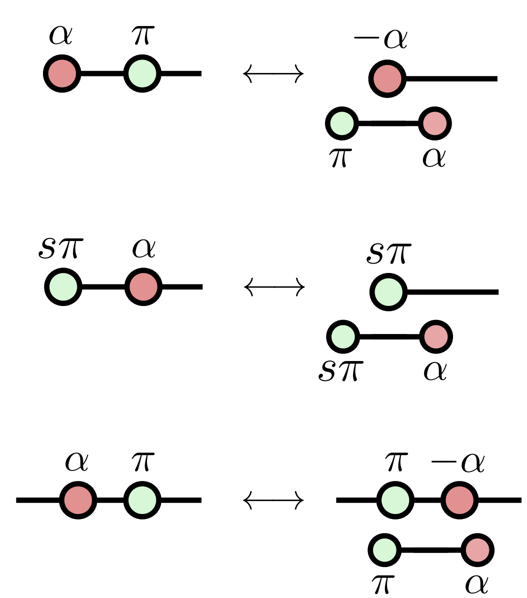


'Green data' — states, tests, and rotations with angle in $\pi\mathbb{Z}$ — are copied by phase-free red nodes

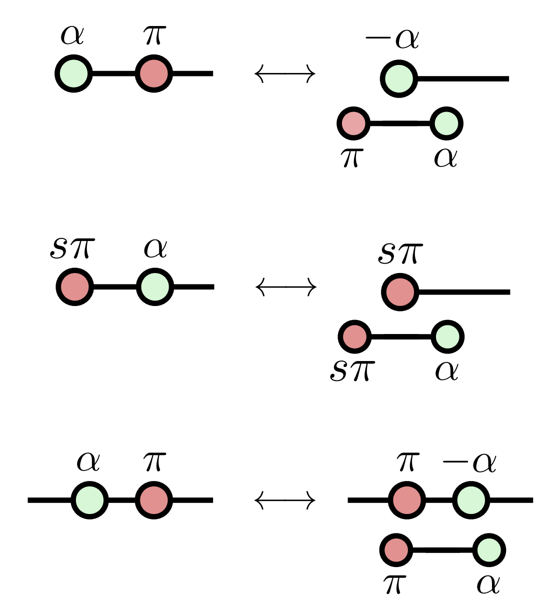


'Red data' — states, tests, and rotations with angle in $\pi\mathbb{Z}$ — are copied by phase-free green nodes

π-phase commutation & absorbtion



Up to a scalar factor, 'green π phases' can commute past, absorb, or be absorbed by a red α phase, with a change in sign in α if it isn't absorbed

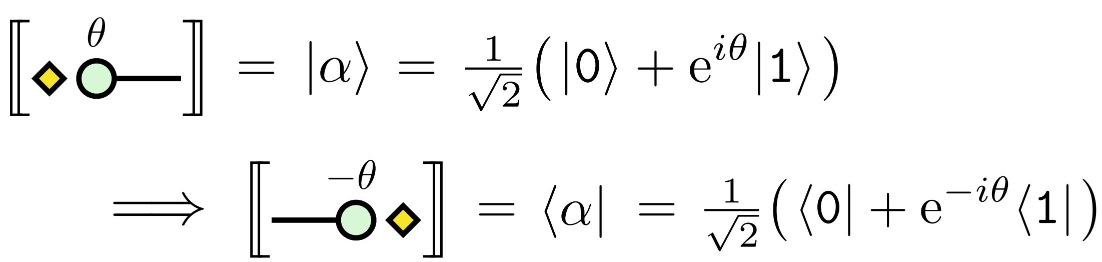


Up to a scalar factor, 'red π phases' can commute past, absorb, or be absorbed by a green α phase, with a change in sign in α if it isn't absorbed

More ZX rewrites

Adjoints

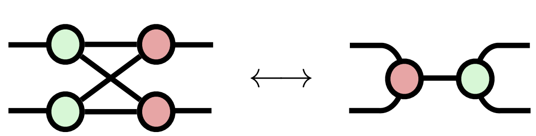
To take the adjoint (also known as the Hermitian conjugate) of any diagram — e.g., to represent the inverse of a unitary matrix U, or to represent a measurement outcome $\langle\alpha|$ which corresponds to a quantum state $|\alpha\rangle$ — we flip the diagram horizontally (which yields the transpose of the original diagram) and then negate all of the angles of the nodes (which takes the complex conjugate), e.g.:



Loops

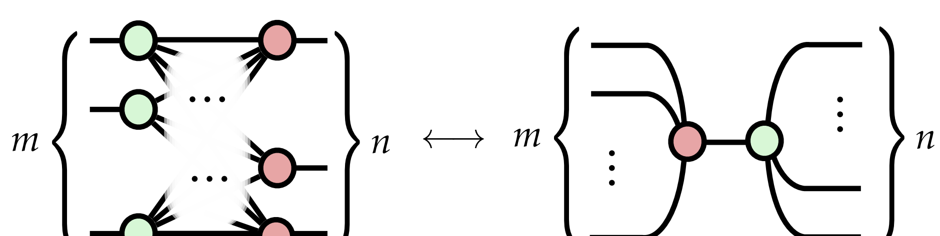
Each closed loop on any red or green node can be replaced by an isolated phase-free node (denoting a scalar factor of $\sqrt{2}$)

Bialgebra rule



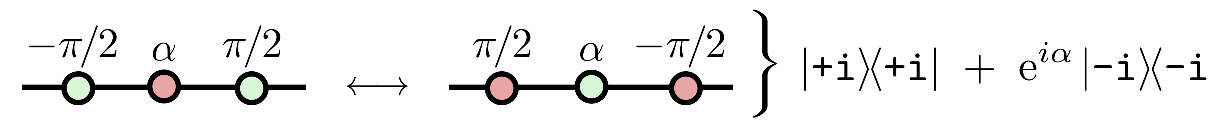
The bialgebra rule applies to sub-diagrams of phase-free red and green nodes, and describes the complementary relationship between the two bases $|0\rangle/|1\rangle$ and $|+\rangle/|-\rangle$

In the general form shown here, each green node is connected to each red node



Y-axis states and rotations

The states on the Y-axis, are the only states that can be represented by either red nodes or green nodes, but require different angles to do so (and they are also related by a global scalar factor)



Rotations about the Y-axis involve interplay between X-axis rotations and Z-axis rotations, which can take multiple equivalent forms

±π/2 rotations and Hadamards