QDCS : Digrammatic Calculus and Error Correction

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TD 5

1 Small CSS Codes

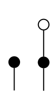
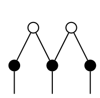
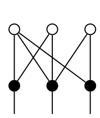
Question 1. For the following pairs of X and Z parity check matrices, give the CSS code that they define, if it exists:

1. $H_Z = \begin{pmatrix}1\end{pmatrix}, \quad H_X = \begin{pmatrix}0\end{pmatrix}$
2. $H_Z = \begin{pmatrix}0\end{pmatrix}, \quad H_X = \begin{pmatrix}1\end{pmatrix}$
3. $H_Z = \begin{pmatrix}1 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}0 & 1\end{pmatrix}$
4. $H_Z = \begin{pmatrix}1 & 1\end{pmatrix}, \quad H_X = \begin{pmatrix}0 & 1\end{pmatrix}$
5. $H_Z = \begin{pmatrix}1 & 1 & 1\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$
6. $H_Z = \begin{pmatrix}0 & 0 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$
7. $H_Z = \begin{pmatrix}0 & 0 & 0\end{pmatrix}, \quad H_X = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$

Question 2. For the above codes, what are their length, dimension and minimal distance?

Question 3. For the above CSS codes, build the state as a ZX-diagram if it is maximal, otherwise complete the parity check matrices as you see fit, then build the encoder as a ZX-diagram.

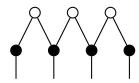
Answer: Using the Z-X normal form every time:

1. 
2. 
3. 
4. Not a CSS code
5. 
6. 

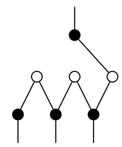
7. This CSS code is not maximal, we can complete it as follows:

$$\tilde{H}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \tilde{H}_X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(We can easily check that $\tilde{H}_Z \tilde{H}_X^T = 0$). This yields the following diagram:



which is finally turned into an encoder by bending the rightmost output (that represents the column we have added to H_X and H_Z) back up:



2 CSS Code States

Let H_X and H_Z be two parity check matrices, for codes C_X and C_Z respectively. Let $\left|z + C_X^\perp\right\rangle := \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} |z + x\rangle$ for $z \in C_Z$.

Question 1. Let $u \in C_X^\perp$. Show that $X^u \left|z + C_X^\perp\right\rangle = \left|z + C_X^\perp\right\rangle$.

Answer:

$$X^u \left|z + C_X^\perp\right\rangle = \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} X^u |z + x\rangle = \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} |z + x + u\rangle = \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x' \in C_X^\perp} |z + x'\rangle = \left|z + C_X^\perp\right\rangle$$

Equations are obtained in order as follows:

- by distributing the product over the sum
- using the property from the class
- by noticing that if x visits all elements of $C_X^\perp$ once, then $x' := x + u$ also visits them all once since $u \in C_X^\perp$
- by definition

Question 2. Let $u \in C_Z^\perp$. Show that $Z^u \left|z + C_X^\perp\right\rangle = \left|z + C_X^\perp\right\rangle$.

Answer:

$$\begin{aligned} Z^u \left|z + C_X^\perp\right\rangle &= \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} Z^u |z + x\rangle = \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} (-1)^{\langle u|z+x\rangle} |z + x\rangle \\ &= \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} (-1)^{\langle u|z\rangle + \langle u|x\rangle} |z + x\rangle = \frac{1}{\sqrt{|C_X^\perp|}} \sum_{x \in C_X^\perp} |z + x\rangle = \left|z + C_X^\perp\right\rangle \end{aligned}$$

The only subtlety here is to realise that since $u \in C_Z^\perp$ and $z \in C_Z$, we have $\langle u|z\rangle = 0$; and $x \in C_X^\perp \subseteq C_Z$ which also implies $\langle u|x\rangle = 0$.

3 The Steane Code

The Steane code is defined as the CSS code with:

$$H_X = H_Z = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Question 1. Check that this indeed defines a CSS code.

Answer: Checking that $H_X H_Z^T = 0$ is direct (each pair of rows have exactly two 1s in common).

Question 2. How many logical qubits does it encode?

Answer: It is easy to check that the matrices are full rank. Since the length of the code is $n = 7$, its dimension is $k = 7 - 3 - 3 = 1$.

Question 3. What is its minimal distance?
Hint: See if you can't conclude from the respective minimal distances of the image and the kernel of these matrices.

Answer: Since $H_X = H_Z$, the formula from the course gives us $d = d_{\min}(\ker(H_X) \setminus \mathrm{Im}(H_Z))$.

- We can check quickly that all the non-zero elements in $\mathrm{Im}(H_Z)$ have weight 4: each row has weight 4, adding a row to another doesn't change the weight of the result, and adding together all 3 rows again gives a weight 4 element.
- We can then check that the non-zero elements in $\ker(H_X)$ have at least weight 3, and that some element u in there exactly has weight 3:
  - $d_{\min}$ can't be 1, as there is no 0-column
  - $d_{\min}$ can't be 2, as there are no 2 columns that are equal
  - Element $u := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$ is in $\ker(H_X)$ and has weight 3.

Since all non-zero elements of $\mathrm{Im}(H_Z)$ have weight 4, $u \in \ker(H_X) \setminus \mathrm{Im}(H_Z)$.

Warning: since $S = \ker(H_X) \setminus \mathrm{Im}(H_Z)$ is not a priori a vector space, we can't use $\min_{x \neq y \in S}(d(x, y)) = \min_{x \neq 0 \in S}(w(x))$. However, removing elements from $\ker(H_X)$ cannot reduce its distance, which means $d_{\min}(\ker(H_X) \setminus \mathrm{Im}(H_Z))$ is 3 or larger. But we know it is 3 because $u \in \ker(H_X) \setminus \mathrm{Im}(H_Z)$. Hence $d = 3$.

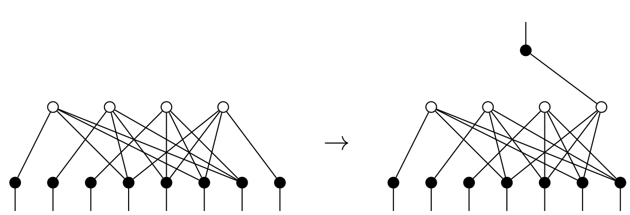
Question 4. Complete H_X and H_Z to get a maximal CSS code (as you see fit)

Answer: We can complete the matrices as follows to get a maximal CSS code:

$$\tilde{H}_X = \tilde{H}_Z = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Question 5. Build the encoder for the Steane code

Answer: We can use the Z-X normal form again to create the generating state of the maximal CSS code, then bend the last wire back up to get the encoder:



4 Surface Code

Consider the 3×3 surface code from the course.

Question 1. Complete the CSS code into a maximal CSS code.

Answer: The matrices can be completed as follows (we can easily check the validity of the CSS code):

$$\tilde{H}_X := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \tilde{H}_Z := \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Question 2. Provide a ZX-diagram implementing the obtained code. Deduce an encoder of the surface code.

Answer: Let's use $\tilde{H}_X$ and the Z-X normal form, and let's directly give the encoder (with the rightmost wire bent back up). The construction is more easily seen by keeping the grid topology that was used to define the code: each vertex becomes an X-spider, each X-face becomes a Z-spider that connects the corresponding qubits. We hence get: