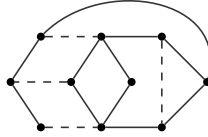# EITQ TD: Counting Complexity

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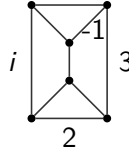
1 Playing with perfect matchings

Question 1. Suppose we have found the following matching (in dashed lines), in the following graph:

Can you find an augmenting path? Does the graph have a perfect matching?

Question 2. How many (weighted) perfect matchings does the following graph have?

2 Detecting Cycles

Let $G = (V, E)$ be a **directed** graph.

Question 1. Consider a vertex $v \in V$ with only outgoing edges. How many cycles can go through this node? Same question for a vertex with only incoming edges.

Question 2. Suppose all the vertices in G have at least 1 incoming edge, and at least 1 outgoing edge. Does G necessarily have a cycle?

Question 3. From the previous observations, suggest an algorithm to detect the presence of a cycle in a directed graph. What is the complexity of this algorithm?

3 Permanent and Cycle Covers

Let $G = (V, E, w)$ be a weighted **directed** graph, and M_G an adjacency matrix of G. A cycle cover of G is a subset C of edges E of G, such that for each vertex $v \in V$, there is exactly one edge $(v, \star) \in C$ and exactly one edge $(\star, v) \in C$.

We call #CC the function that computes the (weighted) number of cycle covers of a graph. Our goal in this exercise is to prove that:
$$\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$$

Recall the definition of the permanent:

$$\mathrm{PERM}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)}$$

for all $n \times n$ matrices $M = (m_{i,j})$, with S_n the set of permutations over $\{1, ..., n\}$.

Question 1. Show the Laplace expansion formula: for any i:

$$\mathrm{PERM}(M) = \sum_{j=1}^{n} m_{i,j}\, \mathrm{PERM}(M_i^j)$$

where M_i^j is matrix M without row i and column j (called minor of M).

Question 2. Consider the outgoing edges of a vertex v of G. Reasoning on the (number of) cycles that go through v, determine a way to express the number of cycle covers of G from the number of cycle covers of graphs built from G, with one fewer vertex.

Question 3. Relate this with the Laplace expansion formula on M_G, and show that $\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$.

4 Permanent and Bipartite Perfect Matchings

Let $G = (V_0, V_1, E, w)$ be a weighted **bipartite** graph with $|V_0| = |V_1|$, and M_G be a biadjacency matrix of G. Our goal here is to prove that:
$$\#\mathrm{PM}(G) = \mathrm{PERM}(M_G)$$

Question 1. Considering a vertex $v \in V_0$, and reasoning on the different ways to cover v by a perfect matching, determine a way to compute $\#\mathrm{PM}(G)$ from the number of perfect matchings of simpler graphs derived from G.

Question 2. Relate this to the Laplace expansion formula on M_G, and show that $\#\mathrm{PM}(G) = \mathrm{PERM}(M_G)$.