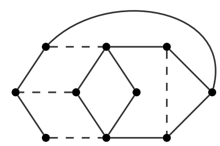
EITQ TD: Counting Complexity

Renaud Vilmart
renaud.vilmart@inria.fr

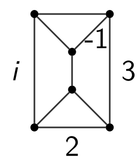
1 Playing with perfect matchings

Question 1. Suppose we have found the following matching (in dashed lines), in the following graph:



Can you find an augmenting path? Does the graph have a perfect matching?

Question 2. How many (weighted) perfect matchings does the following graph have?



2 Detecting Cycles

Let $G = (V, E)$ be a **directed** graph.

Question 1. Consider a vertex $v \in V$ with only outgoing edges. How many cycles can go through this node? Same question for a vertex with only incoming edges.

These vertices cannot be part of a cycle.

Question 2. Suppose all the vertices in G have at least 1 incoming edge, and at least 1 outgoing edge. Does G necessarily have a cycle?

It does. Start building a path from a vertex v. At every step, take a random neighbour of the last vertex in the path, and add it to the path. If that vertex was already in the path, a cycle has been found, the algo can stop. Otherwise continue. As there are finitely many vertices, we will eventually reach a vertex that was already explored.

Question 3. From the previous observations, suggest an algorithm to detect the presence of a cycle in a directed graph. What is the complexity of this algorithm?

Remove leaves or co-leaves as long as the graph has some. At the end, if the graph is empty, it initially had no cycle. If vertices remain, the graph had a cycle.

3 Permanent and Cycle Covers

Let $G = (V, E, w)$ be a weighted **directed** graph, and M_G an adjacency matrix of G. A cycle cover of G is a subset C of edges E of G, such that for each vertex $v \in V$, there is exactly one edge $(v, \star) \in C$ and exactly one edge $(\star, v) \in C$.

We call #CC the function that computes the (weighted) number of cycle covers of a graph. Our goal in this exercise is to prove that:

$$\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$$

Recall the definition of the permanent:

$$\mathrm{PERM}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)}$$

for all $n \times n$ matrices $M = (m_{i,j})$, with S_n the set of permutations over $\{1, ..., n\}$.

Question 1. Show the Laplace expansion formula: for any i:

$$\mathrm{PERM}(M) = \sum_{j=1}^{n} m_{i,j}\,\mathrm{PERM}(M_i^j)$$

where M_i^j is matrix M without row i and column j (called minor of M).

First notice that for a given i, in each term there is a single j such that $m_{i,j}$ is a factor. We can hence partition the terms into those that have $m_{i,1}$ as a factor, those that have $m_{i,2}$ as a factor, ... Consider the terms with $m_{i,j}$, which we can factorise with

$$m_{i,j} \sum_{\substack{\sigma \in S_n \\ \sigma(i)=j}} \prod_{k \neq i} m_{k,\sigma(k)} = m_{i,j}\,\mathrm{PERM}(M_i^j)$$

Question 2. Consider the outgoing edges of a vertex v of G. Reasoning on the (number of) cycles that go through v, determine a way to express the number of cycle covers of G from the number of cycle covers of graphs built from G, with one fewer vertex.

Let $(v, u_1), ..., (v, u_n)$ be the n outgoing edges from v. The number of cycles that go through v are the number of cycles that go through edge (v, u_1) plus the nb of cycles that go through (v, u_2) plus ... plus the nb of cycles that go through (v, u_n). The number of cycles that go through (v, u_i) is the number of cycles in the graph obtained by: 1) removing incoming edges of u_i except (v, u_i), 2) contracting edge (v, u_i) and 3) removing edges (v, u_j) for $j \neq i$. We can hence compute the number of cycle covers in G as a sum of cycle covers in smaller graphs obtained from G.

Question 3. Relate this with the Laplace expansion formula on M_G, and show that $\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$.

When expressed in terms of adjacency matrices, the above decomposition is exactly the Laplace expansion formula. The base case being true ($\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$ when G contains a single vertex), this proves the result by induction.

4 Permanent and Bipartite Perfect Matchings

Let $G = (V_0, V_1, E, w)$ be a weighted **bipartite** graph with $|V_0| = |V_1|$, and M_G be a biadjacency matrix of G. Our goal here is to prove that:

$$\#\mathrm{PM}(G) = \mathrm{PERM}(M_G)$$

Question 1. Considering a vertex $v \in V_0$, and reasoning on the different ways to cover v by a perfect matching, determine a way to compute $\#\mathrm{PM}(G)$ from the number of perfect matchings of simpler graphs derived from G.

Let $\{u_1, ..., u_n\}$ be the neighbours of v. The different perfect matchings of G can be partitioned into the ones that contain (v, u_1), the ones that contain (v, u_2), ..., the ones that contain (v, u_n). The number of perfect matchings that contain (v, u_i) is exactly the number of perfect matchings in the graph obtained by removing vertices v and u_i (and associated edges). Notice that the obtained graph is still bipartite. We can hence compute the number of perfect matchings in G as a sum of perfect matchings in smaller bipartite graphs obtained from G.

Question 2. Relate this to the Laplace expansion formula on M_G, and show that $\#\mathrm{PM}(G) = \mathrm{PERM}(M_G)$.

Similarly as the cycle covers case, when expressed in terms of biadjacency matrices, the above decomposition is exactly the Laplace expansion formula. The base case being true ($\#\mathrm{CC}(G) = \mathrm{PERM}(M_G)$ when G contains only 2 vertices), this proves the result by induction.