

Minimality in Finite-Dimensional ZW-Calculi

Anonymous

Abstract

The ZW-calculus is a graphical language capable of representing 2-dimensional quantum systems (qubit) through its diagrams, and manipulating them through its equational theory. We extend the formalism to accommodate finite dimensional Hilbert spaces beyond qubit systems.

First we define a qudit version of the language, where all systems have the same arbitrary finite dimension d , and show that the provided equational theory is both complete – i.e. semantical equivalence is entirely captured by the equations – and minimal – i.e. none of the equations are consequences of the others. We then extend the graphical language further to allow for mixed-dimensional systems. We again show the completeness and minimality of the provided equational theory.

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1 Introduction

Graphical languages for quantum computations are a product of the categorical quantum mechanics program [1, 15] devoted to studying the foundations of quantum mechanics through the prism of category theory. These graphical languages come in different flavours, depending on which generators are used to build the diagrams (graphical representations of the quantum operators), and critically, displaying different kinds of interactions between said generators. The ZX-calculus describes the interaction between two complementary bases [13], the ZW-calculus, the interaction between the two “spiders” derived from the “GHZ” and “W” states, the only two fully entangled tripartite states up to SLOCC-equivalence [14], and the ZH-calculus the interaction between the same GHZ-state inferred spider and a spider obtained by generalising the Hadamard gate [3].

The equations that describe these interactions form “equational theories”, that define syntactic equivalence classes of diagrams, that are also semantically equivalent. When the syntactic equivalence matches perfectly the semantical one (i.e. when two diagrams represent the same quantum operator iff they can be turned into one another), we say that the equational theory is *complete*. Complete equational theories have been found for the aforementioned graphical languages, betimes for restrictions of them [2, 4, 10, 11, 24, 26, 29, 31, 46].

As is customary for the computer science part of the community, the focus was largely set onto *qubit* systems, i.e. systems where the base quantum system is 2-dimensional, yet this is enough to get applications in optimisation [5, 33], quantum error correction [18, 28, 45], verification [22, 27], simulation [34–36]... However, physics allow for qudit systems (where the base quantum system is d -dimensional with $d > 2$) and even infinite dimensional systems. Several attempts have hence been made to go beyond the qubit case [7, 25, 47], but it was only recently that a complete equational theory was found for d -dimensional (i.e. qudit) systems [41] and later for finite dimensional systems (so-called “qufinite”, i.e. for the category **FdHilb**) [49]. The results were obtained by generalising both the ZX and the ZW calculi and mixing them together. The W-node in particular allows for a neat intuitive (and unique)



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normal form for the diagrams. Satisfying the necessary conditions for every diagram to be normalisable then yields a complete equational theory. However, we believe that the ones obtained in [41, 49] are far from being *minimal*, due in particular to the presence of the generators from both the ZX and the ZW calculi.

From a foundational perspective, it can be enlightening to know if an equation is a defining property of quantum systems (and hence necessary), or on the contrary if it is derivable from more fundamental properties (see e.g. [16, 21]). The redundancy in the equational theory may also cause issues when trying to explore the space of equivalent diagrams, or to transport the completeness result to other diagrammatic languages for qudit systems (every equation has to be proven in the new language, hence the fewer the better); or when trying to generalise further, e.g. to the **FdHilb** setting.

We argue here that the ZW-calculus is enough to get a natural normal form (akin to that of [41]) even in the qudit version, and provide an elegant equational theory that we show to be *complete*, resorting to the normal form instead of transporting the completeness result from [41], for the reason described above. We also show that the equational theory is minimal, meaning that none of the equations can be derived from the others, hence avoiding the aforementioned redundancy in the presentation.

We then adapt diagrams and the equational theory of the graphical language to accommodate all finite dimensional Hilbert spaces (**FdHilb**), in a way that requires no additional generator and only one new equation. Here again we prove the completeness and the minimality of the equational theory, by leveraging that of the qudit setting.

The paper is split into two parts, Section 2 and Section 3, devoted respectively to the **Qudit_d** version, and to the **FdHilb** version. In the **Qudit_d** version, diagrams and their interpretation are introduced in Section 2.1 and the equational theory is introduced and discussed in Section 2.2. We then show its minimality in Section 2.3 and its completeness in Section 2.4. In the **FdHilb** version, diagrams and their interpretation are introduced in Section 3.1, and the equational theory is introduced and shown to be complete in Section 3.2. All missing proofs are provided in the appendix.

The Dirac Notation

All the upcoming diagrams can be given an interpretation as a linear map, in the appropriate category. In quantum information, it is usual to express such linear maps using the so-called Dirac notation. The current section hence serves as a gentle introduction to this notation.

Let $d \geq 2$. In the d -dimensional Hilbert space \mathbb{C}^d , the canonical basis

$$\left\{ (1 \ 0 \ \dots \ 0)^\top, (0 \ 1 \ \dots \ 0)^\top, \dots, (0 \ 0 \ \dots \ 1)^\top \right\}$$

is usually denoted $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ (with $(\cdot)^\top$ being the transpose). All 1-qudit systems have states that live in \mathbb{C}^d and that can hence be represented by linear combinations of the elements of this basis: $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + \dots + a_{d-1} |d-1\rangle$ (notice that the “ket” notation $|\cdot\rangle$ is used for states in general, not only basis elements).

To combine systems, we use the tensor product (Kronecker product): $(\cdot \otimes \cdot)$ which is a fairly standard operation on linear maps. In particular, the overall state obtained by composing two 1-qudit systems in respective states $|\psi\rangle$ and $|\varphi\rangle$ is simply $|\psi\rangle \otimes |\varphi\rangle$. Notice that $\{|i\rangle \otimes |j\rangle\}_{0 \leq i < d, 0 \leq j < d'}$ forms a basis of $\mathbb{C}^d \otimes \mathbb{C}^{d'} \simeq \mathbb{C}^{d \times d'}$. It is customary to write $|\psi, \varphi\rangle$ to abbreviate $|\psi\rangle \otimes |\varphi\rangle$.

The “bra” notation $\langle \cdot |$ is used to represent the dagger (the conjugate transpose) of a state, i.e. $\langle \psi | = |\psi\rangle^\dagger = |\psi\rangle^\top$. The choice of the “bra-ket” notation is such that composing

a bra with ket forms the bracket, the usual inner product in \mathbb{C}^d : $\langle \psi | \circ | \varphi \rangle = \langle \psi | \varphi \rangle$. Linear combinations of kets and bras of the canonical basis can be used to represent any linear map of the correct dimensions, e.g. the 1-qudit identity: $id = \sum_{k=0}^{d-1} |k\rangle\langle k|$.

2 ZW-Calculus for Qudit Systems

In this section, we introduce a graphical language for quantum systems that all have the same fixed dimension d ($d \geq 2$): qudit systems.

2.1 Diagrams of ZW_d and their Interpretation

First, we need to introduce the mathematical objects at the heart of the graphical language – the diagrams – and what they represent.

2.1.1 The Diagrams

As is traditional for graphical languages for finite-dimensional quantum systems, we work with a \dagger -compact prop [37,44,50]. Categorically speaking, this is a symmetric, compact closed monoidal category generated by a single object, endowed with a contravariant endofunctor that behaves well with the symmetry and the compact structure. The following explains some of these concepts in more detail.

Let us denote ZW_d the \dagger -compact prop generated by:

- the Z-spiders $\begin{array}{c} \cdots \\ \vdots \\ \textcircled{r} \\ \vdots \\ \cdots \end{array} : n \rightarrow m$, for $r \in \mathbb{C}$ and $n, m \geq 0$
- the W-nodes $\begin{array}{c} | \\ \vdots \\ \textcircled{} \\ \vdots \\ \cdots \end{array} : 1 \rightarrow n$, for $n \geq 0$
- the state $|1\rangle \begin{array}{c} \bullet \\ | \end{array} : 0 \rightarrow 1$
- the global scalars $r : 0 \rightarrow 0$, for $r \in \mathbb{C}$
- the swap $\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} : 2 \rightarrow 2$ representing the symmetry of the prop
- the cup $\cup : 2 \rightarrow 0$ and cap $\cap : 0 \rightarrow 2$ representing the compact structure
- and the identity $| : 1 \rightarrow 1$.

All these generators can be composed sequentially and in parallel, as follows:

$$\begin{array}{c} \cdots \\ \boxed{D_1} \\ \cdots \\ \boxed{D_2} \\ \cdots \end{array} := \begin{array}{c} \cdots \\ \boxed{D_2} \\ \cdots \end{array} \circ \begin{array}{c} \cdots \\ \boxed{D_1} \\ \cdots \end{array} \quad \text{and} \quad \begin{array}{c} \cdots \\ \boxed{D_1} \\ \cdots \end{array} \begin{array}{c} \cdots \\ \boxed{D_2} \\ \cdots \end{array} := \begin{array}{c} \cdots \\ \boxed{D_1} \\ \cdots \end{array} \otimes \begin{array}{c} \cdots \\ \boxed{D_2} \\ \cdots \end{array}$$

The symmetry and the compact structure satisfy the following identities:

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = || \quad \begin{array}{c} \diagup \quad \diagdown \\ \boxed{D} \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \cdots \\ \boxed{D} \\ \cdots \end{array} \quad \cap = \cup \quad \cup = | = \cup$$

This compact structure in particular allows us to define the "upside-down" version of the

$$\text{generators, for instance: } \begin{array}{c} \cdots \\ \cup \end{array} := \begin{array}{c} \cdots \\ \cap \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ | \end{array} := \begin{array}{c} \bullet \\ | \end{array}$$

The \dagger functor is defined inductively as:

$$\begin{aligned} (D_2 \circ D_1)^\dagger &= D_1^\dagger \circ D_2^\dagger & \left(\begin{array}{c} \cdots \\ \textcircled{r} \\ \cdots \end{array} \right)^\dagger &= \begin{array}{c} \cdots \\ \textcircled{\bar{r}} \\ \cdots \end{array} & \text{with the other generators} \\ (D_1 \otimes D_2)^\dagger &= D_1^\dagger \otimes D_2^\dagger & & \text{being mapped to their} \\ (r)^\dagger &= \bar{r} & & \text{upside-down version.} \end{aligned}$$

123 Notice that thanks to the identities satisfied by the \dagger -compact prop, the \dagger -functor is involutive.

124 As will be made clearer in what follows, in \mathbf{ZW}_d , $d \geq 2$ represents the dimension of the
 125 "base" quantum system, called qudit. As this d will be fixed in the following, we may forget
 126 to specify it. For convenience, we define an empty white node as a parameter-1 Z-spider:
 127 $\bigcirc := \bigcirc$ and give the $0 \rightarrow 1$ W-node a special symbol, akin to that of $|1\rangle$ (as its
 128 interpretation, as we will see later, is merely $|0\rangle$): $\uparrow := \uparrow$. We generalise the ket symbol
 129 inductively as follows (for $2 \leq k < d$): $\uparrow^{k+1} := \uparrow^1 \uparrow^k$. These last symbols can be given
 130 an upside-down definition using the compact structure as was done for $|1\rangle$ and the W-node.

131 2.1.2 The Interpretation

132 The point of the diagrams of the \mathbf{ZW}_d is to represent quantum operators on multipartite
 133 d -dimensional systems. The way those are usually specified is thanks to the category \mathbf{Qudit}_d .
 134 This forms again a symmetric \dagger -compact prop, where the base object is $1 := \mathbb{C}^d$, and
 135 morphisms $n \rightarrow m$ are linear maps $\mathbb{C}^{d^n} \rightarrow \mathbb{C}^{d^m}$. The symmetry and the compact structure
 136 correspond to their counterparts in \mathbf{ZW}_d , they will be stated out in the following, as part of
 137 the interpretation of \mathbf{ZW}_d diagrams. The \dagger functor is the usual \dagger of linear maps in \mathbb{C} .

138 We may hence interpret diagrams of the \mathbf{ZW}_d -calculus thanks to the functor $[\cdot] : \mathbf{ZW}_d \rightarrow$
 139 \mathbf{Qudit}_d inductively defined as follows:

$$\begin{array}{ll}
 [D_2 \circ D_1] = [D_2] \circ [D_1] & \left[\begin{array}{c} \dots \\ \text{r} \\ \dots \end{array} \right] = \sum_{k=0}^{d-1} r^k \sqrt{k!}^{n+m-2} |k^m\rangle \langle k^n| \\
 [D_1 \otimes D_2] = [D_1] \otimes [D_2] & \left[\begin{array}{c} \uparrow \\ \dots \\ n \end{array} \right] = \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} \sqrt{\binom{k}{i_1, \dots, i_n}} |i_1, \dots, i_n\rangle \langle k| \\
 \left[\begin{array}{c} | \\ | \end{array} \right] = \sum_k |k\rangle \langle k| & \left[\begin{array}{c} \uparrow \\ \dots \\ n \end{array} \right] = |1\rangle \\
 \left[\begin{array}{c} \times \end{array} \right] = \sum_{k, \ell} |\ell, k\rangle \langle k, \ell| & [r] = r \\
 \left[\begin{array}{c} \cap \end{array} \right] = \left[\begin{array}{c} \cup \end{array} \right]^\dagger = \sum_k |k, k\rangle &
 \end{array}$$

141 where $\binom{k}{i_1, \dots, i_n} = \frac{k!}{i_1! \dots i_n!}$ is a multinomial coefficient. Notice that the interpretation
 142 of the $0 \rightarrow 1$ W-node is simply: $\left[\begin{array}{c} \uparrow \end{array} \right] = \sum_{\substack{k \in \{0, \dots, d-1\} \\ 0=k}} \sqrt{\binom{k}{0}} |k\rangle = |0\rangle$, and that of the black

143 node symbol k for $k < d$ is $|k\rangle$ up to renormalisation: $\left[\begin{array}{c} \uparrow^k \end{array} \right] = \sqrt{\binom{k}{1, \dots, 1}} |k\rangle = \sqrt{k!} |k\rangle$. The
 144 presence of $\sqrt{}$ on the coefficients is not particularly relevant, and is simply an artefact
 145 of us maintaining some symmetry between generators and their dagger. Indeed, we want
 146 $\left[\begin{array}{c} \uparrow^k \\ \uparrow^k \end{array} \right] = k!$, and for that the coefficient $k!$ needs to be split between both nodes, resulting
 147 in either an asymmetric presentation or a square root (see Appendix A for an equivalent
 148 semantics, without any $\sqrt{}$ but asymmetric instead).

149 Notice also that the interpretation of the Z-spider differs from more usual generalisations
 150 of its qubit counterpart, because of the $\sqrt{k!}^{n+m-2}$ which depends on the degree of the spider.
 151 While it makes the interpretation of the diagrams slightly more complicated, it allows us – as
 152 will be stated later – to quite conveniently generalise equations from the qubit ZW-calculus,
 153 and hence have a simpler equational theory. It will be shown in the following (Corollary 9),
 154 that the above set of generators makes for a universal calculus, i.e. any linear map of \mathbf{Qudit}_d

155 can be represented by a \mathbf{ZW}_d -diagram.

156 To gain intuition about the upcoming equations between diagrams, it can be useful to
 157 *semantically* decompose a diagram into sums of simpler ones¹. To do so, it can be convenient
 158 to understand $|k\rangle$ as a bunch of k indistinguishable particles:

$$\begin{aligned}
 \left[\begin{array}{c} \bullet^k \\ \vdots \\ \bullet^k \end{array} \right] &= \sum_{i_1 + \dots + i_n = k} \binom{k}{i_1, \dots, i_n} \left[\begin{array}{c} \bullet^{i_1} \dots \bullet^{i_n} \\ \vdots \\ \bullet^{i_1} \dots \bullet^{i_n} \end{array} \right] \quad (1) \\
 \left[\begin{array}{c} \bullet^k \quad \bullet^\ell \\ \vdots \\ \bullet^k \quad \bullet^\ell \end{array} \right] &= \begin{cases} \left[\begin{array}{c} \bullet^{k+\ell} \\ \vdots \\ \bullet^{k+\ell} \end{array} \right] & \text{if } k + \ell < d \\
 \bar{0} & \text{if } k + \ell \geq d \end{cases} \quad (2) \\
 \left[\begin{array}{c} \bullet^k \\ \vdots \\ \bullet^k \end{array} \right] &= r^k \left[\begin{array}{c} \bullet^k \dots \bullet^k \\ \vdots \\ \bullet^k \dots \bullet^k \end{array} \right] \quad (3) \\
 \left[\begin{array}{c} \bullet^k \\ \vdots \\ \bullet^\ell \end{array} \right] &= k! \langle k | \ell \rangle \quad (4) \\
 \left[\begin{array}{c} | \\ | \end{array} \right] &= \sum_{k=0}^{d-1} \frac{1}{k!} \left[\begin{array}{c} \bullet^k \\ \vdots \\ \bullet^k \end{array} \right] \quad (5)
 \end{aligned}$$

160 Equation (1) explains how the W-node spreads the k “particles” that enter it following a
 161 multinomial distribution. Equation (2) shows that the $2 \rightarrow 1$ W-node takes two bunches of
 162 particles k and ℓ and regroups them into one, and yields the null state if $k + \ell$ exceeds the
 163 “capacity” (i.e. the dimension) of a single wire. This will be proven graphically (Lemma 29)
 164 from the upcoming equational theory (Figure 1). When $k + \ell < d$, the fact that there is no
 165 additional scalar is due to the rescaling of the k -dots to represent $\sqrt{k!} |k\rangle$. This rescaling also
 166 makes the “copy” more natural: The Z-spider $1 \rightarrow n$ copies any bunch of k particles entering
 167 it, yielding global scalar r^k in the process, as is shown by Equation (3). This will again be
 168 proven graphically (Lemma 26) from the equational theory. The rescaling, however, forces
 169 Equation (4). Finally, it can be useful to decompose the identity as a linear combination of
 170 products of kets and bras as is done in Equation (5).

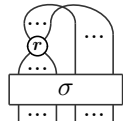
171 2.2 Equational Theory

172 With the above interpretation of the \mathbf{ZW}_d , different diagrams may yield the same linear map.
 173 All axioms of symmetric \dagger -compact props in particular preserve the interpretation. More
 174 generally, we may want to relate together *all* diagrams that have the same semantics. This
 175 is done through an equational theory, i.e. a set of equations that can be applied locally in a
 176 diagram without changing the semantics of the whole.

177 2.2.1 Equations of the \mathbf{ZW}_d -Calculus

178 On top of the axioms of symmetric \dagger -compact props, we assume some conventional equations
 179 about the topology of the generators, which should align with the symmetries of the symbols
 180 used to depict them. The Z-spider does not distinguish between any of its connections: it is
 181 “flexsymmetric” [9], meaning that we can interchange any of its legs without changing the



182 semantics. Graphically, for any permutation of wires σ :

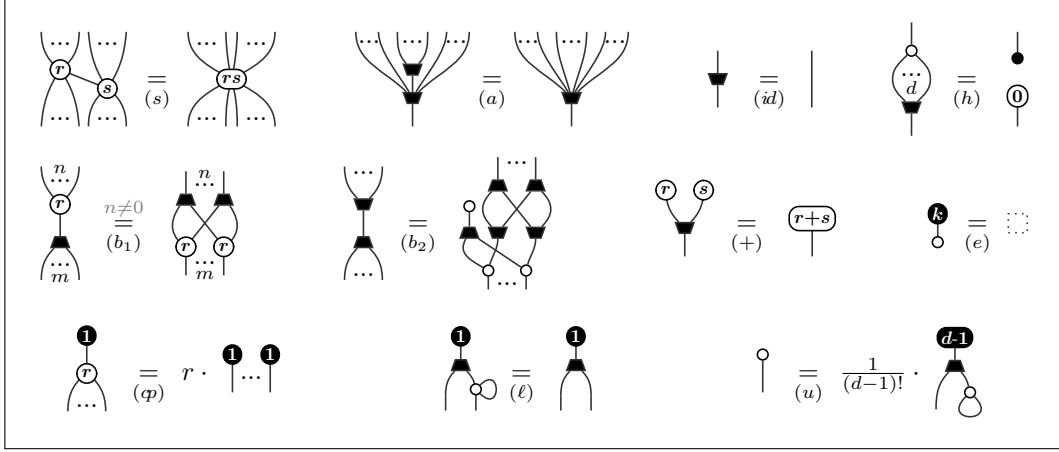


183 other hand, the binary W-node is only co-commutative, which, together with the upcoming
 184 Equation (a), means that all the outputs of the n -ary W-node can be exchanged, i.e. for any

¹ Notice that here, such decompositions are merely semantical. The upcoming completeness is only interested in equivalence between single diagrams.

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185 permutation of wire σ :  = . With all this in place, we can give the core of the
 186 equational theory, in Figure 1. When diagram D_1 can be turned into diagram D_2 using the
 187 rules of ZW_d , we write $\text{ZW}_d \vdash D_1 = D_2$.



■ **Figure 1** Equational theory ZW_d for the qudit **ZW**-calculus.



188 ► **Remark 1.** In this framework, we can tensor global scalars together $r \otimes s$, which graphically
 189 could be confused with their product rs . This is actually unambiguous in the equational
 190 theory, as, using Equation (q):

$$191 \quad \text{ZW}_d \vdash r \otimes s \stackrel{(q)}{=} r \cdot \begin{array}{c} \bullet \\ \circ \\ s \end{array} \stackrel{(q)}{=} \begin{array}{c} \bullet \\ \circ \\ r \end{array} \stackrel{(s)}{=} \begin{array}{c} \bullet \\ \circ \\ rs \end{array} \stackrel{(q)}{=} rs$$

192

193 Moreover, using Equation (e), one can easily show that global scalar 1 is the empty diagram
 194 (Lemma 20). Scalar multiplication is assumed to be automatically applied, and scalar 1 is
 195 assumed to be automatically removed in the following after Lemma 20 is proven.

196 All rules up to (e) are fairly standard generalisations of rules of the qubit **ZW**-calculus
 197 (with (b_2) being inspired from [41] to avoid using a *fermionic swap*²). The non-conventional
 198 \sqrt{k}^{n+m-2} coefficients in the interpretation of the Z-spider seem to be necessary for Equations
 199 (s), (b_1) and $(+)$ to all work. Notice that this makes the Z-spider *non-special*, meaning

200 that:  \neq . Equation (l) however gives a context in which that inequality becomes an
 201 equality. Finally, Equation (u) shows how a $0 \rightarrow 1$ Z-spider can be obtained by distributing
 202 $d - 1$ “particles” over two paths, and erasing (or post-selecting) adequately one of the two
 203 paths.

204 ► **Remark 2.** Thanks to the compact structure and its interaction with the generators of
 205 the language, all upside-down version of the equations of Figure 1 are derivable, by simply
 206 deforming the diagrams to get the actual axiom. For instance, the upside-down version of

² The fermionic swap, introduced in [24], is a generator that has most – but not all – good properties of the actual swap. The qubit version of ZW uses the fermionic swap, but this generator loses some properties when going in larger dimensions. By not using it here, we avoid having to axiomatise it.

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derived from the other equations, but we do not pinpoint for which parameters the equation is necessary or not. Nevertheless:

► **Theorem 4.** *All equations in ZW_d are necessary, hence ZW_d is minimal.*

Several arguments in the proof require to distinguish between Z-spiders that have a non-trivial influence on the semantics of the diagrams, from those that essentially have no effect on the semantics. This is captured by the following definition:

► **Definition 5** (Effective Z-spider). *Let z be a Z-spider in diagram D . For $x \in \mathbb{C}$, we denote by $D[z * x]$ the diagram obtained by multiplying the parameter of z by x in D . We say that z is effective if: $\forall x \in \mathbb{C} \setminus \{1\}, \forall \lambda \in \mathbb{C}, \llbracket D[z * x] \rrbracket \neq \lambda \llbracket D \rrbracket$*

We need a similar notion for determining whether a W-node could essentially just be an identity from its input to one of its outputs, the other outputs merely bearing a $|0\rangle$:

► **Definition 6** (Trivial W-node and sole effective output).

Let D be a diagram with a W-node. Let's call 'a' one of its outputs, as shown in Equation (6) below. We say that edge 'a' is the sole effective output of the W-node if Equation (7) is verified:

$$\begin{array}{c} \text{Diagram (6): A box labeled } D \text{ with multiple inputs and outputs. One output is labeled } a. \text{ This is equal to a diagram } D' \text{ where the W-node is represented by a spider with one output } a \text{ and other outputs connected to a box.} \end{array} \quad (6)$$

$$\begin{array}{c} \text{Diagram (7): A diagram with a W-node and a box. The W-node has one output } a \text{ and other outputs connected to a box. This is equal to a diagram where the W-node is a spider with one output } a \text{ and other outputs connected to a box.} \end{array} \quad (7)$$

If a W-node has a sole effective output, we say that it is trivial.

We can now prove Theorem 4:

Proof. We consider each of these equations individually:

- (s) When applying the transformation that turns all Z-spider parameters and global scalars to their real part ($r \mapsto \text{Re}(r)$), Equation (s) is the only one that is not preserved.
- (a) It is the only equation permitting to create non-trivial W-nodes with arity $> d$ from a diagram that only has non-trivial W-nodes with arity $\leq d$.
- (id) It is the only equation that can create nodes connected to boundaries from a node-free diagram.
- (q) It is the only equation that can create a non-real scalar from a diagram with only real scalars.
- (e) It is the only equation that can create generators out of empty diagrams.
- (ℓ) Let $\varpi := e^{i \frac{\pi}{d-1}}$ be the first $(2d-2)$ -th root of unity. Consider the \dagger -compact monoidal functor (i.e. functor that preserves compositions, symmetry and compact structure) that maps the generators as follows:

$$\begin{array}{c} \text{Diagram (1)} \mapsto \varpi \text{ Diagram (1)} \quad \text{Diagram (2)} \mapsto \text{Diagram (2)} \quad \text{Diagram (3)} \mapsto \text{Diagram (3)} \end{array}$$

When $d > 2$, all equations but (ℓ) are preserved by this functor. We can make the argument work when $d = 2$ by choosing any ϖ such that $\varpi^2 \neq 1$ and by working up to colinearity.

- (u) Take the interpretation that maps Diagram (1) (and subsequently all Diagram (k)) to Diagram (1) . All rules hold (up to colinearity) except (u).

- (+) Take the interpretation that maps all the Z-spider parameters (and the global scalars) to their absolute value ($r \mapsto |r|$). This interpretation preserves all equations except (+).
- (b_1) Consider diagrams as graphs, and define an “effective Z-path” in the diagram as a path 1) that goes from a boundary to another boundary, and 2) that only goes through effective Z-spiders or through trivial W-nodes, through their sole effective output. All equations except (b_1) preserve the existence of effective Z-paths.
- (b_2) Consider diagrams as graphs, and define a “W-path” in the diagram as a path 1) that goes from a boundary to another boundary, 2) which cannot use two outputs of a W-node (if it goes through a W-node, it has to use the input edge) and 3) that does not go through a Z-spider. All equations, except (b_2), preserve the existence of a W-path. (b_2) is the only equation that can bring the number of W-paths from non-zero to zero (which is done by adding a Z-spider on the path).
- (h) To each wire in a diagram D , we associate a number $0 \leq k < d$ (or more graphically we annotate each wire by some number k). The procedure to do so is as follows:

1. annotate all wires with $d - 1$
2. rewrite the annotations using the following rules, until a fixed point is reached:

$$\begin{array}{c}
 \text{300} \quad = \quad a \begin{array}{c} \bullet \\ | \\ \hline \end{array} \xrightarrow{a \neq 0} 0 \begin{array}{c} \bullet \\ | \\ \hline \end{array} \quad = \quad a \begin{array}{c} \bullet \\ | \\ \hline \end{array} \xrightarrow{a \geq 1} 1 \begin{array}{c} \bullet \\ | \\ \hline \end{array} \quad = \quad b_1 \begin{array}{c} a \\ \cap \\ \dots \\ b_n \end{array} \rightarrow \begin{array}{c} \min(a, \sum b_i) \\ \cap \\ \dots \\ \min(a, b_1) \quad \min(a, b_n) \end{array} \\
 \text{301} \quad = \quad \begin{array}{c} a_0 \dots a_k \\ \cap \\ \dots \\ a_{k+1} \dots a_q \end{array} \xrightarrow{a} \begin{array}{c} a \dots a \\ \cap \\ \dots \\ a \dots a \end{array} \quad \text{with } a := \min(a_i)
 \end{array}$$

This simple procedure obviously terminates, as a step is only applied if at least one of the annotations is decreased. By considering inputs and outputs of D (which are the only wires that can be guaranteed to remain during rewrites with ZW_d), we can check that Equation (h) is the only one that can modify the outcome of the procedure. ◀

2.4 Completeness

Completeness of an equational theory with respect to a semantics is the fundamental property that ensures that semantical equivalence of diagrams is entirely captured by the equational theory. Minimality is worthless without some form of completeness, as it is extremely simple to design minimal, but not complete, equational theories. For instance, the empty equational theory (that contains no axiom), is minimal but clearly not complete for the qudit ZW-diagrams. We hence show in this section that we indeed have completeness.

2.4.1 Normal Form and Universality

The usual way to prove completeness is to show that any diagram can be put in a normal form, and that this normal form is unique and similar for all equivalent diagrams. As is customary in a category that is compact-closed, we can focus on states, as there is an isomorphism between operators and states [1]:

$$\begin{array}{c}
 \text{318} \quad \begin{array}{c} \dots \\ | \\ \boxed{D} \\ | \\ \dots \end{array} = \begin{array}{c} \dots \\ | \\ \boxed{D} \\ | \\ \dots \end{array} \begin{array}{c} \dots \\ | \\ \dots \end{array} =: \begin{array}{c} \boxed{\lceil D \rceil} \\ | \\ \dots \end{array}
 \end{array}$$

Proving completeness requires a fair amount of diagrammatic derivations, especially when starting from a minimal equational theory, to get enough material to define a normalisation strategy. These lemmas and their proofs are offset to the appendix, pages 20 to 30.

► **Definition 7.** We define $\mathcal{N} : \mathbf{Qudit}_d \rightarrow \mathbf{ZW}_d$ as the functor that maps any n -qudit state

323 $|\psi\rangle = r_0 |0...0\rangle + r_1 |0...01\rangle + \dots + r_i |x_1^i...x_n^i\rangle + \dots$
to the diagram on the right.

We say of any diagram in the image of \mathcal{N} that it is in normal form.

This construction is a direct generalisation of the normal form of the qubit ZW-diagrams [24], which is also considered in [25] in the context of q -arithmetic. It creates a diagram whose interpretation is the starting state:

327 ► **Proposition 8.** $\forall |\psi\rangle \in \mathbf{Qudit}_d[0, n], \llbracket \mathcal{N}(|\psi\rangle) \rrbracket = |\psi\rangle.$

As a simple consequence of this proposition, any qudit operator can be represented by a diagram of \mathbf{ZW}_d :

330 ► **Corollary 9** (Universality). $\forall f \in \mathbf{Qudit}_d[n, m], \exists D_f \in \mathbf{ZW}_d[n, m], \llbracket D_f \rrbracket = f.$

Since we defined the normal form as the image of a map from the semantics, any diagram
can only be associated to a unique normal form.

333 2.4.2 Completeness

Our goal now is to show that any diagram can be put in normal form. To do so, we show that all generators can be put in normal form, and that all compositions of diagrams in normal form can be put in normal form.

337 We start by showing the latter for the tensor product:

338 **► Proposition 10.** *The spatial composition of diagrams in normal form can be put in normal*
339 *form, i.e. $\text{ZW}_d \vdash \mathcal{N}(v_1) \otimes \mathcal{N}(v_2) = \mathcal{N}(v_1 \otimes v_2)$.*

When turning arbitrary operators into states, the sequential composition turns into the application of cups \cup onto pairs of outputs of the state, as:

$$\begin{array}{c}
\vdots \\
\boxed{D_1} \\
\vdots \\
\boxed{D_2} \\
\vdots
\end{array}
=
\begin{array}{c}
\boxed{\lceil D_2 \rceil} \boxed{\lceil D_1 \rceil} \dots \\
\vdots \quad \vdots \quad \vdots
\end{array}
\stackrel{\text{Prop. 10}}{=}
\begin{array}{c}
\boxed{\lceil D_2 \rceil \otimes \lceil D_1 \rceil} \dots \\
\vdots \quad \vdots \quad \vdots
\end{array}$$

344 ► **Proposition 11.** *The diagram obtained by applying a cup \cup to two outputs of a diagram*
 345 *in normal form can be put in normal form.*

Then we move on to showing that all the generators can be put in normal form. To do
so, the following lemma will prove useful:

348 ► **Lemma 12.** *The diagram obtained by applying \cup to two outputs of a normal form can*
349 *be put in normal form.*

350 ► **Proposition 13.** *All generators of the \mathbf{ZW}_d -calculus can be put in normal form.*

351 Putting all the latter results together, we can show the completeness of the language:

352 **► Theorem 14** (Completeness for Qudit Systems). *The language is complete: for any two*
353 *diagrams D_1 and D_2 of the \mathbf{ZW}_d -calculus:*

$$354 \quad \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \text{ZW}_d \vdash D_1 = D_2$$

Proof. By Proposition 13, any generator of the language can be put in normal form. Thanks to Propositions 10 and 11, compositions of diagrams in normal form can be put in normal form. As a consequence, any diagram can be put in normal form. By uniqueness of this normal form, if two diagrams share the same semantics, they can be rewritten into the same diagram. This proves completeness. ◀

3 Finite Dimensional Hilbert Spaces

In the previous setting, all systems are required to be d -dimensional for some fixed d . Here we relax that constraint, which allows us to go “mixed-dimensional” and to represent morphisms of **FdHilb**³.

FdHilb is the strict symmetric monoidal \dagger -compact category of finite dimensional Hilbert spaces [1]. Its objects are tensor products of finite dimensional Hilbert spaces $\mathbb{C}^d (d \in \mathbb{N} \setminus \{0\})$, and its morphisms are linear maps between them. The symmetry and the compact structure are naturally extended from that of **Qudit** _{d} .

In this new setting, we will be able to represent *all* morphisms of **FdHilb**, at the cost of annotating the wires of the diagrams to keep track of their dimensions. Instead of the dimension itself, we rather annotate the wire with its dimension -1 , i.e. with the largest k such that $|k\rangle$ is allowed on the wire. We call such k the *capacity* of the wire. This makes the bookkeeping a little bit less tedious.

3.1 Diagrams and Interpretation

We also require the following constraints for the capacities around each generator:

- All capacities around a Z-spider are the same
- The input capacity of the W-node must be larger than each of its outputs

The first constraint follows from the fact that Z-spiders in ZW can be seen as a generalisation of graph edges – more precisely they can be seen as hyperedges. Hence the whole hyperedge should have a single capacity. The second constraint simply comes from the fact that a larger capacity on the outputs of a W-node will never be used, so we might as well prevent it. When considering $1 \rightarrow 1$ W-nodes, which represent projections, this restriction allows us to see at a glance which side has the largest dimension.

The first restriction further allows us to put the capacity annotation on the Z-spider rather than on all its legs, making annotating diagrams less cumbersome.

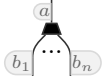
We now work with a \dagger -compact symmetric monoidal category, but which is not a prop anymore. Our base objects are \mathbb{C}^d for $d \in \mathbb{N} \setminus \{0\}$. Every pair of objects can be composed with \otimes to form a third object, with \otimes being associative, and with the tensor unit I being $I := \mathbb{C}^1$. We work with a strict monoidal category, so we consider $I \otimes \mathbb{C}^d = \mathbb{C}^d = \mathbb{C}^d \otimes I$. To simplify notations, we represent objects $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_n}$ by a list of capacities $\langle d_1 - 1, d_2 - 1, \dots, d_n - 1 \rangle$ (the tensor product simply becomes the concatenation of lists). The tensor unit is represented by $\langle \rangle$. Since $\langle 0 \rangle = \langle \rangle$, we forbid 0 capacities on the wires.


In this new setting, the generators are generalised as follows:

- Z-spiders $\begin{array}{c} \begin{array}{c} \vdots \\ n \\ \vdots \end{array} \\ \textcircled{a} \\ \begin{array}{c} \vdots \\ m \\ \vdots \end{array} \end{array} : \langle \overbrace{a, \dots, a}^n \rangle \rightarrow \langle \overbrace{a, \dots, a}^m \rangle$ with $r \in \mathbb{C}$ and $a \geq 1$

³ Technically, the skeleton of **FdHilb**, i.e. where all d -dimensional Hilbert spaces are identified with the canonical representative \mathbb{C}^d . We take the liberty in this paper to name **FdHilb** this skeleton.




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394 ■ W-nodes  : $\langle a \rangle \rightarrow \langle b_1, \dots, b_n \rangle$ with $a \geq \max_{1 \leq i \leq n} (b_i)$ and $b_i \geq 1$

395 ■ state $|1\rangle$  : $\langle \rangle \rightarrow \langle a \rangle$ with $a \geq 1$



396 ■ global scalars $r : \langle \rangle \rightarrow \langle \rangle$ with $r \in \mathbb{C}$

397 with the symmetry and the compact structure being generalised to

398 ■  : $\langle a, b \rangle \rightarrow \langle b, a \rangle$ ■  : $\langle \rangle \rightarrow \langle a, a \rangle$ and  : $\langle a, a \rangle \rightarrow \langle \rangle$,

399 (again with $a, b \geq 1$) and the identity to  : $\langle a \rangle \rightarrow \langle a \rangle$ ($a \geq 1$).

400 Diagrams can still be composed together both sequentially and in parallel. The sequential
401 composition prevents us from composing diagrams with unmatched objects (e.g. two Z-spiders
402 with different capacities in sequence). These diagrams are called **ZW_f**-diagrams, and are
403 graphical representations of the morphisms of the \dagger -compact symmetric monoidal category
404 **FdHilb** (the dagger functor can be given in **ZW_f** in a similar way as **ZW_d**).

405 The compact structure still allows us to define upside-down versions of the W-node and the
406 kets. Again, we give the $\langle \rangle \rightarrow \langle a \rangle$ W-node a special symbol (for $a \geq 1$):  := , and we

407 generalise the ket symbol inductively as follows (for $k \geq 1$ and $a > k$):  := .

408 The interpretation of these diagrams is now a monoidal functor $\llbracket \cdot \rrbracket : \mathbf{ZW}_f \rightarrow \mathbf{FdHilb}$
409 inductively defined as:

$$\begin{aligned} \llbracket D_2 \circ D_1 \rrbracket &= \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket & \llbracket \text{sym} \rrbracket &= \sum_{k=0}^a \sum_{\ell=0}^b |\ell, k\rangle \langle k, \ell| \\ \llbracket D_1 \otimes D_2 \rrbracket &= \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket & \llbracket \text{cap} \rrbracket &= \sum_{k=0}^a r^k \sqrt{k!}^{n+m-2} |k^m\rangle \langle k^n| \\ \llbracket \text{id} \rrbracket &= \sum_{k=0}^a |k\rangle \langle k| & \llbracket \text{state } |1\rangle \rrbracket &= |1\rangle \\ \llbracket \text{cup} \rrbracket &= \llbracket \text{cap} \rrbracket^\dagger = \sum_{k=0}^a |k, k\rangle & \llbracket r \rrbracket &= r \\ \llbracket \text{W-node} \rrbracket &= \sum_{\substack{0 \leq k_i \leq b_i \\ k_1 + \dots + k_n \leq a}} \sqrt{\binom{k_1 + \dots + k_n}{k_1, \dots, k_n}} |k_1, \dots, k_n\rangle \langle k_1 + \dots + k_n| \end{aligned}$$

411 By composition, one can check that, for $k > 1$ and $a \geq k$: $\llbracket \text{ket } k \rrbracket = \sqrt{k!} |k\rangle$.

412 Notice that we use the same notation for the interpretation of **ZW_d**-diagrams, and for the
413 interpretation of the **ZW_f**-diagrams. Which interpretation we are referring to should be clear
414 from the context.

415 3.2 Complete Equational Theory

416 We once again equip the language with an equational theory **ZW_f**, defined in Figure 2.
417 This equational theory only slightly differs from the one for qudit systems in Figure 1. It
418 is interesting to notice that 1) the associativity of the W-node is broken down into two
419 equations (a) and (o), whose choice depends on the capacities involved, 2) the W-bialgebra
420 equation (b₂) does not need a context anymore, but instead side conditions on the capacities,
421 3) we managed to remove Equation (e), 4) we now need an equation (i) that states that a
422 $|1\rangle$, when “injected” into a larger dimensional Hilbert space, is still a $|1\rangle$.

We also notice the existence of an interesting equation, that we did not include in Figure 2 as it turns out to be derivable; and which states that a $0 \rightarrow 1$ Z-state can be “copied” by the

W-node, as follows:

$$\begin{array}{c} a+b \\ \circ \\ a \quad b \end{array} = \begin{array}{c} a \\ \circ \end{array} \begin{array}{c} b \\ \circ \end{array}.$$

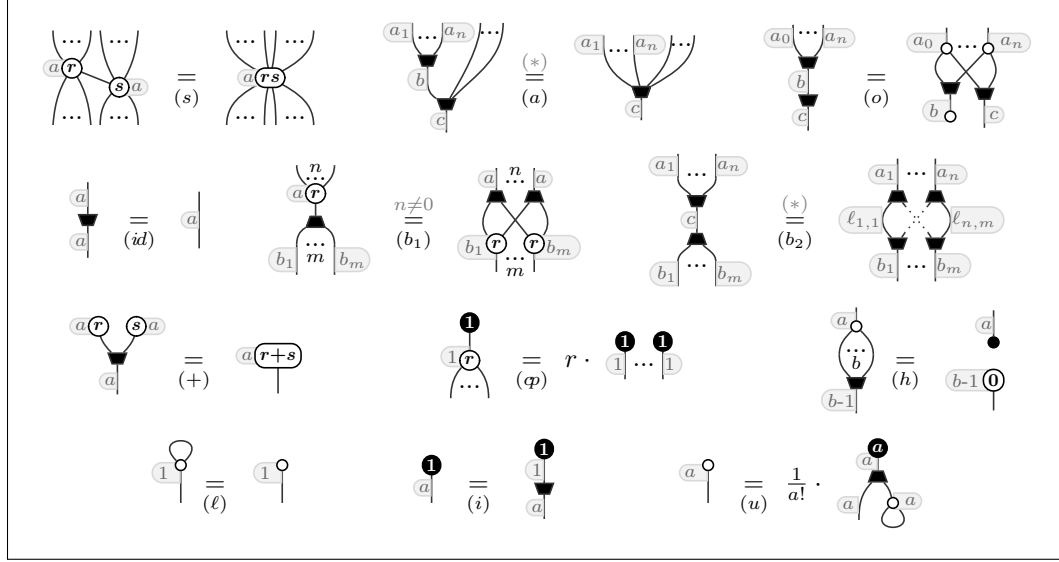


Figure 2 Equational theory \mathbf{ZW}_f for the finite-dimensional \mathbf{ZW} -calculus. In (a), we require that $b = c$ or $b \geq \sum_i a_i$; and in (b₂) that $c \geq \min(\sum a_i, \sum b_i)$ on the lhs, and that $\ell_{ij} = \min(a_i, b_j)$ on the rhs.

The category \mathbf{Qudit}_d is a full subcategory of \mathbf{FdHilb} , and as such there is an obvious inclusion functor $\mathbf{Qudit}_d \xrightarrow{i_d} \mathbf{FdHilb}$. This inclusion transports to the ZW-calculi: we can turn any \mathbf{ZW}_d -diagram into a \mathbf{ZW}_f -diagram through ι_d in such a way that the following

$$\begin{array}{ccc} \mathbf{ZW}_d & \xrightarrow{\iota_d} & \mathbf{ZW}_f \\ \downarrow [\![\cdot]\!] & & \downarrow [\![\cdot]\!] \\ \mathbf{Qudit}_d & \xrightarrow{i_d} & \mathbf{FdHilb} \end{array}.$$

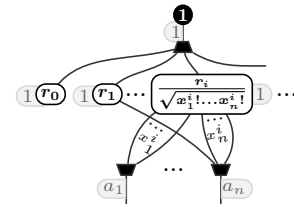
The functor ι_d simply takes a \mathbf{ZW}_d -diagram and annotates all its wires with $d-1$.

We show that the present equational theory is complete. To do so, we need to adapt the notion of normal form from qudit systems (and we again use the map/state duality to focus on states rather than arbitrary morphisms):

Definition 15. We define $\mathcal{N} : \mathbf{FdHilb} \rightarrow \mathbf{ZW}_f$ as the functor that maps any n -ary state $|\psi\rangle \in \mathbf{FdHilb}[\langle \cdot \rangle, \langle a_1, \dots, a_n \rangle]$:

$$|\psi\rangle = r_0 |0\dots 0\rangle + r_1 |0\dots 01\rangle + \dots + r_i |x_1^i \dots x_n^i\rangle + \dots \quad \mathcal{N}(|\psi\rangle) =$$

to the diagram on the right. We say of any diagram in the image of \mathcal{N} that it is in normal form.



We can once again show that \mathcal{N} builds a diagram that represents $|\psi\rangle$:

Lemma 16. $\forall |\psi\rangle \in \mathbf{FdHilb}[0, \langle a_1, \dots, a_n \rangle], [\![\mathcal{N}(|\psi\rangle)]\!] = |\psi\rangle$

We hence get universality of the language as a direct consequence:

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438 ► **Corollary 17** (Universality of \mathbf{ZW}_f).

439 $\forall f \in \mathbf{FdHilb}[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \exists D_f \in \mathbf{ZW}_f[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \llbracket D_f \rrbracket = f$

440 Most of the arguments given for the minimality of \mathbf{ZW}_d can be adapted to arguments for
441 the necessity of the equations of \mathbf{ZW}_f , and the few remaining equations can be given a new
442 argument as explained in the appendix. We hence have:

443 ► **Theorem 18.** *The equational theory \mathbf{ZW}_f is minimal.*

444 Using the normal form, we can then leverage the completeness from the qudit ZW-calculus
445 to get the similar result in the current setting:

446 ► **Theorem 19** (Completeness for Finite Dimensional Systems). *The language is complete: for
447 any two diagrams D_1 and D_2 of the \mathbf{ZW}_f -calculus:*

448
$$\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \mathbf{ZW}_f \vdash D_1 = D_2$$

449 **Proof.** The right-to-left implication (soundness) is again a straightforward verification. The
450 other is proven in its entirety in Appendix C, and uses the completeness of \mathbf{ZW}_d for qudit
451 systems. The idea is to show that i) we can turn both D_1 and D_2 into a diagram with a high
452 enough capacity d everywhere (except boundaries), and that ii) all the equations of \mathbf{ZW}_d can
453 be proven in \mathbf{ZW}_f (through ι_d). ◀

454 4 Related Work

455 4.1 Qudit Framework

456 The first and only result to date of a complete equational theory for a graphical language
457 describing qudit systems comes from the “ZXW-calculus” [41]. There, the authors start
458 from a qudit version of the ZX-calculus and most probably end up requiring a W-node in
459 the definition of a normal form, and hence in the equational theory leading to completeness.
460 We argue here that we can get a complete equational theory purely inside the ZW-calculus.
461 By keeping the number of generators as low as possible, we also end up with few, intuitive
462 equations in the equational theory.

463 The W-node we used is a different generalisation of the qubit W-node than the one used
464 in [41]. The version we used offers two advantages with respect to the aims of the paper.
465 First, it allows to use a single parameter in the Z-spiders (which aligns with the spirit of
466 keeping things as minimal as possible) and to sum such parameters together, while the other
467 version requires to have $(d - 1)$ -sized lists of coefficients as parameters in order to get a
468 $(+)$ -like rule to sum coefficients together. Second, it allows us to define $|k\rangle$ (up to a scalar) as
469 a composition using only $|1\rangle$ and the W-node. Again, this lowers the number of generators,
470 as all the $|k\rangle$ (for $k > 1$) become syntactic sugar.

471 Focussing on ZW-calculus is not a new idea. The first ever completeness proof for
472 qubit graphical languages was in the (qubit) ZW-calculus, introduced in [14] and tweaked
473 and made complete in [24, 26]. The ZW-calculus noticeably has very nice combinatorial
474 properties different from those of its counterparts, which in particular allows for a very
475 natural notion of normal form. It is hence not suprising that some attempts were made to get
476 a complete equational theory of qudit systems purely in ZW. There have then been tentative
477 generalisations for qudit systems, in particular in [25] where q -arithmetic is used, and in [48]
478 where the W-node is generalised in a different way (and that we encounter in [41]). It is
479 to be noted that our two main generators are essentially the same as in [25], except with

usual arithmetic instead of q -arithmetic. While some equations are sound with respect to the q -arithmetic semantics, others are truly specific to the standard arithmetic. Adapting the results of the present paper to q -arithmetic semantics hence seems non-trivial. Other presentations for qudit systems have also been proposed (without proof of completeness) in [17, 42]. Finally, complete presentations for fragments of qudit quantum mechanics can be found e.g. in [8, 39].

Another system we are close to is **QPath** [19]. Our W-node is merely the “triangle” node of **QPath** that we truncated to a finite dimension⁴, and we generalised their “line weight” to an n -ary Z-spider. The degree-2 Z-spider furthermore has exactly the same interpretation as the line weight. While in **QPath** the triangle nodes satisfy a bialgebra, this is not the case when truncating to finite dimension. Here we could either resort to define a “fermionic swap” that would replace the usual swap in the bialgebra (as in [25] and [48]), or give a context in which the bialgebra works (as is done in [41]). While such a “fermionic swap” exists in our setting, it does not have all the nice properties of the qubit fermionic swap, that in particular allow us to see it as a quasi-proper swap. Instead we went with the latter solution, which as it turns out works in our setting, despite the W-node having a different interpretation from that of [41], and we end up with Equation (b_2).

4.2 Finite Dimensional Framework

Another complete presentation of a graphical language for **FdHilb** was announced recently before the first version of the current paper [49]. This one builds upon the aforementioned ZXW-calculus, and introduces a new generator that takes two systems, of dimensions a and b , and builds a system of dimension $a \times b$. Our approach builds upon **ZW_d**, the qudit version of the ZW-calculus from Section 2 and hence starts with fewer generators and equations. As a consequence, the graphical language for **FdHilb** we end up with has fewer equations as well. Moreover, we did not require a new generator, and simply promoted the qudit W-node to work with any mix of dimensions in a natural manner, which was enough to provide us with universality.

A version of the so-called algebraic ZX-calculus for **FdHilb** was recently provided and shown to be complete [40]. The proof of completeness for their graphical language was obtained by transporting the property from the **ZW_f**-calculus of the first version of the current paper to the ZX-calculus, through a system of translations between the two languages.

5 Conclusion

In this paper, we explored the potential for a minimal yet complete diagrammatic language for quantum mechanics beyond qubit systems. This starts with a well-chosen generalisation of the generators of the ZW-calculus, allowing us to have few and intuitive equations. For both qudit systems and finite dimensional systems, we showed that the diagrams are universal, and that the equational theories are both minimal and complete for their respective interpretation.

⁴ The idea of truncating this tensor has also been considered in [20] during a translation between graphical languages.

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691 A Asymmetric Presentation for Qudit Systems

692 In this section, we give an alternative semantics for the ZW_d -diagrams, which breaks the
 693 up/down symmetry of the generators. On the one hand, the dagger-functor becomes less
 694 natural; on the other hand, the combinatorics associated to the diagrams becomes simpler
 695 (except for the cap):

$$\begin{aligned}
 \llbracket D_2 \circ D_1 \rrbracket_{\downarrow} &= \llbracket D_2 \rrbracket_{\downarrow} \circ \llbracket D_1 \rrbracket_{\downarrow} \\
 \llbracket D_1 \otimes D_2 \rrbracket_{\downarrow} &= \llbracket D_1 \rrbracket_{\downarrow} \otimes \llbracket D_2 \rrbracket_{\downarrow} \\
 \llbracket \mid \rrbracket_{\downarrow} &= \sum_k |k\rangle\langle k| \\
 \llbracket \times \rrbracket_{\downarrow} &= \sum_{k,\ell} |\ell, k\rangle\langle k, \ell| \\
 \llbracket \cup \rrbracket_{\downarrow} &= \sum_k k! \langle k, k| \\
 \llbracket \cap \rrbracket_{\downarrow} &= \sum_k \frac{1}{k!} |k, k\rangle
 \end{aligned}
 \quad
 \begin{aligned}
 \llbracket \begin{array}{c} n \\ \vdots \\ r \\ \vdots \\ m \end{array} \rrbracket_{\downarrow} &= \sum_{k=0}^{d-1} r^k k!^{n-1} |k^m\rangle\langle k^n| \\
 \llbracket \begin{array}{c} \mid \\ \vdots \\ n \end{array} \rrbracket_{\downarrow} &= \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} \binom{k}{i_1, \dots, i_n} |i_1, \dots, i_n\rangle\langle k| \\
 \llbracket \begin{array}{c} \vdots \\ \cup \end{array} \rrbracket_{\downarrow} &= \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} |k\rangle\langle i_1, \dots, i_n| \\
 \llbracket \begin{array}{c} \bullet \\ \mid \end{array} \rrbracket_{\downarrow} &= |1\rangle \quad \text{and} \quad \llbracket \begin{array}{c} \mid \\ \bullet \end{array} \rrbracket_{\downarrow} = \langle 1| \\
 \llbracket r \rrbracket_{\downarrow} &= r
 \end{aligned}$$

697 As a direct consequence, we have:

$$\begin{aligned}
 \llbracket \begin{array}{c} \bullet \\ \mid \end{array} \rrbracket_{\downarrow} &= |k\rangle \quad \text{and} \quad \llbracket \begin{array}{c} \mid \\ \bullet \end{array} \rrbracket_{\downarrow} = k! \langle k|
 \end{aligned}$$

700 This semantics being equivalent to $\llbracket - \rrbracket_{\downarrow}$, the equational theory of ZW_d -diagrams remains
 701 universal, sound and complete in this setting.

B Lemmas and Proofs For Qudit

In this section, we give the proofs necessary for the completeness of the qudit setting. To get there, we also provide a set of useful derivable equations:

► Lemma 20.

$$ZW_d \vdash 1 = \text{[empty box]}$$

► Lemma 21.

$$ZW_d \vdash \text{[circle with dot]} = \text{[vertical line]}$$

► Lemma 22.

$$ZW_d \vdash \text{[diagram with two spiders]} = \text{[diagram with two spiders and crossings]}$$

► Lemma 23.

$$ZW_d \vdash \text{[diagram with spider and dot]} = \text{[diagram with two dots]}$$

► Lemma 24.

$$ZW_d \vdash \text{[diagram with two spiders]} = \text{[empty box]}$$

► Lemma 25.

$$ZW_d \vdash \text{[diagram with spider and 0]} = \text{[diagram with dot]}$$

► Lemma 26.

$$ZW_d \vdash \text{[diagram with spider and k]} = r^k \cdot \text{[diagram with spider and n]}$$

► Lemma 27.

$$ZW_d \vdash \text{[diagram with two spiders]} = \text{[diagram with spider and r+s]}$$

► Lemma 28.

$$ZW_d \vdash \text{[diagram with spider and loop]} = \text{[diagram with spider and dot]}$$

► Lemma 29.

$$ZW_d \vdash \text{[diagram with two spiders]} = \begin{cases} \text{[diagram with spider and k+l]} & \text{if } k+l < d \\ 0 \cdot \text{[diagram with dot]} & \text{if } k+l \geq d \end{cases}$$

► Lemma 30.

$$ZW_d \vdash \text{[diagram with spider and loop]} = \text{[diagram with two dots]}$$

► Lemma 31.

$$ZW_d \vdash \text{[diagram with spider and loop]} = \text{[diagram with spider and loop]}$$

► Lemma 32.

$$ZW_d \vdash \text{[diagram with spider and loop]} = \text{[diagram with spider and loop]}$$

where on the right-hand-side, there are as many Z-spiders as there are ways to decompose n as a sum of m natural numbers: $n = i_1 + \dots + i_m$.

► Lemma 33.

$$ZW_d \vdash \text{[diagram with two spiders]} = \text{[diagram with spider and loop]}$$

► Lemma 34. With $0 < k < d$:

$$ZW_d \vdash \text{[diagram with spider and k]} = \text{[diagram with spider and k]}$$

► Lemma 35. If $0 < k < d$:

$$ZW_d \vdash k! \cdot \text{[diagram with spider]} = \text{[diagram with spider and k]}$$

► Lemma 36. If $k \neq \ell$:

$$ZW_d \vdash \text{[diagram with two spiders]} = k! \quad \text{and} \quad ZW_d \vdash \text{[diagram with two spiders]} = 0$$

► Lemma 37.

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{r} \end{array} = r \cdot \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{1/r} \end{array}$$

► Lemma 38. If $0 \leq k < d$:

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{\dots} \\ \downarrow \\ \textcircled{k} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{} \end{array}$$

► Lemma 39. If $0 \leq k < d$, we have:

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{k} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{k!} \end{array}$$

753

► Lemma 40. The bialgebra between W -nodes can be used in the following context, if $0 \leq k_i < d$ for all i :

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{\dots} \\ \downarrow \\ \textcircled{k_1 \dots k_n} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{\dots} \\ \downarrow \\ \textcircled{k_1 \dots k_n} \end{array}$$

758

As we have quite a number of them, we provide in Figure 3 a small (automatically generated) graph of all dependencies between lemmas, propositions and theorems, so as to convince the reader that there is no circular proof. Equations from the equational theories are not displayed in the graph, they are assumed to be available at any point. The graph also includes the dependencies for the proofs of Section 3.

Proof of Lemma 20.

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(e)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(q)} \end{array} = 1$$

Proof of Lemma 21.

$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(s)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(id)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(b_2)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(id)} \end{array}$$

Proof of Lemma 22.

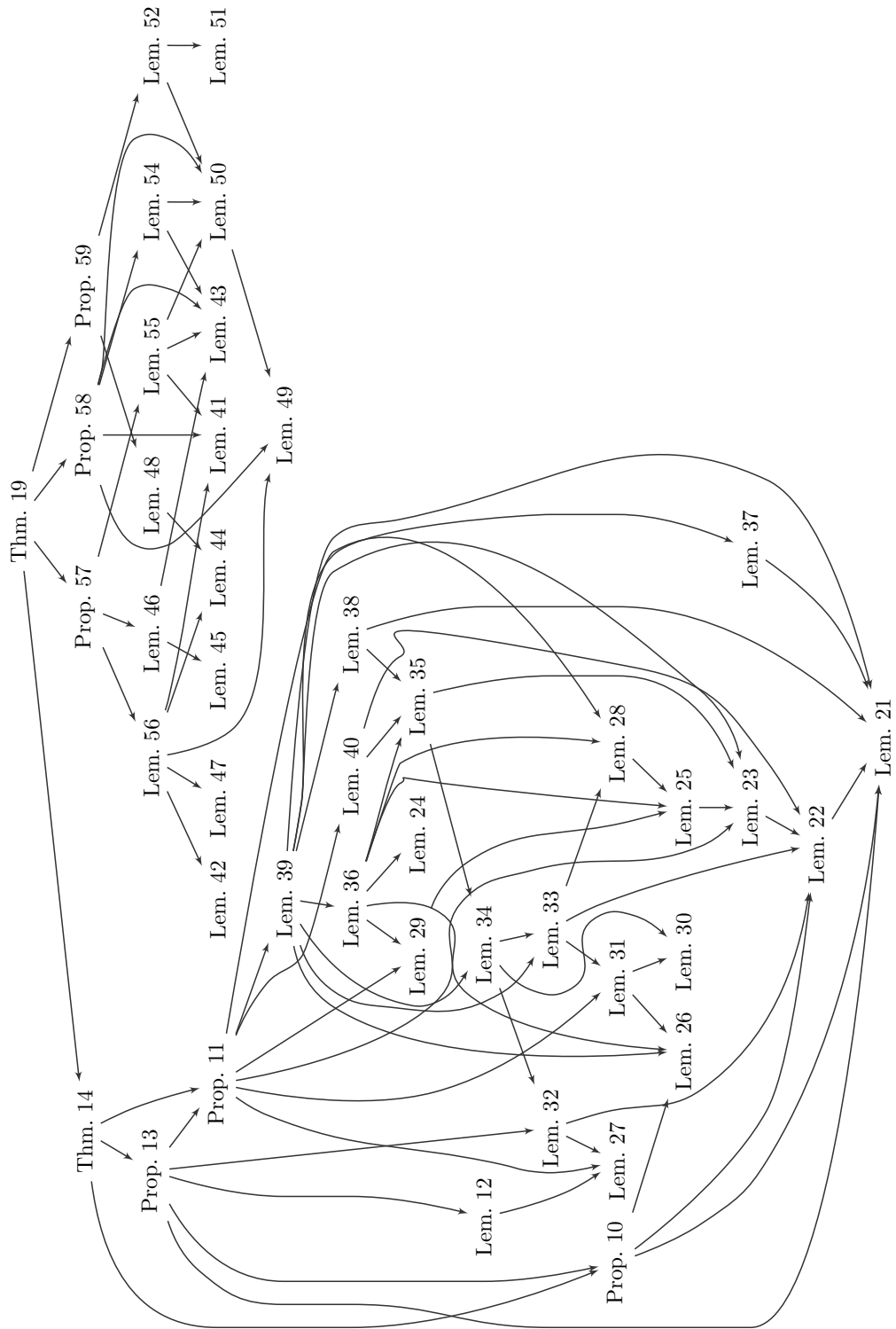
$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(b_2)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(s)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(b_1)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(s)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(s)} \end{array}$$

Proof of Lemma 23.

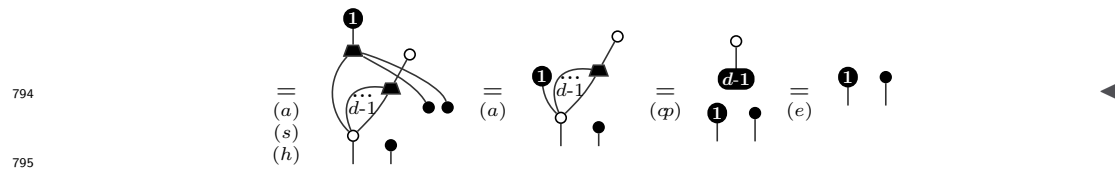
$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(e)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(id)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(a)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(id)} \end{array}$$

Proof of Lemma 24.

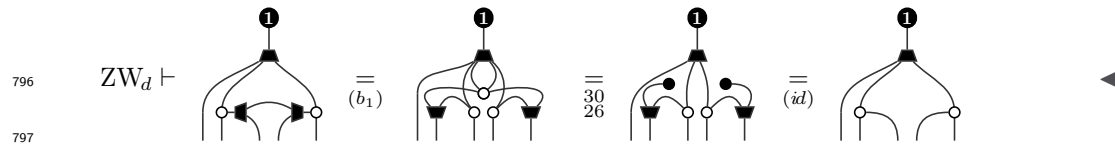
$$ZW_d \vdash \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(q)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(s)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(id)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(a)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(a)} \end{array} = \begin{array}{c} \textcircled{1} \\ \downarrow \\ \textcircled{(q)} \end{array}$$



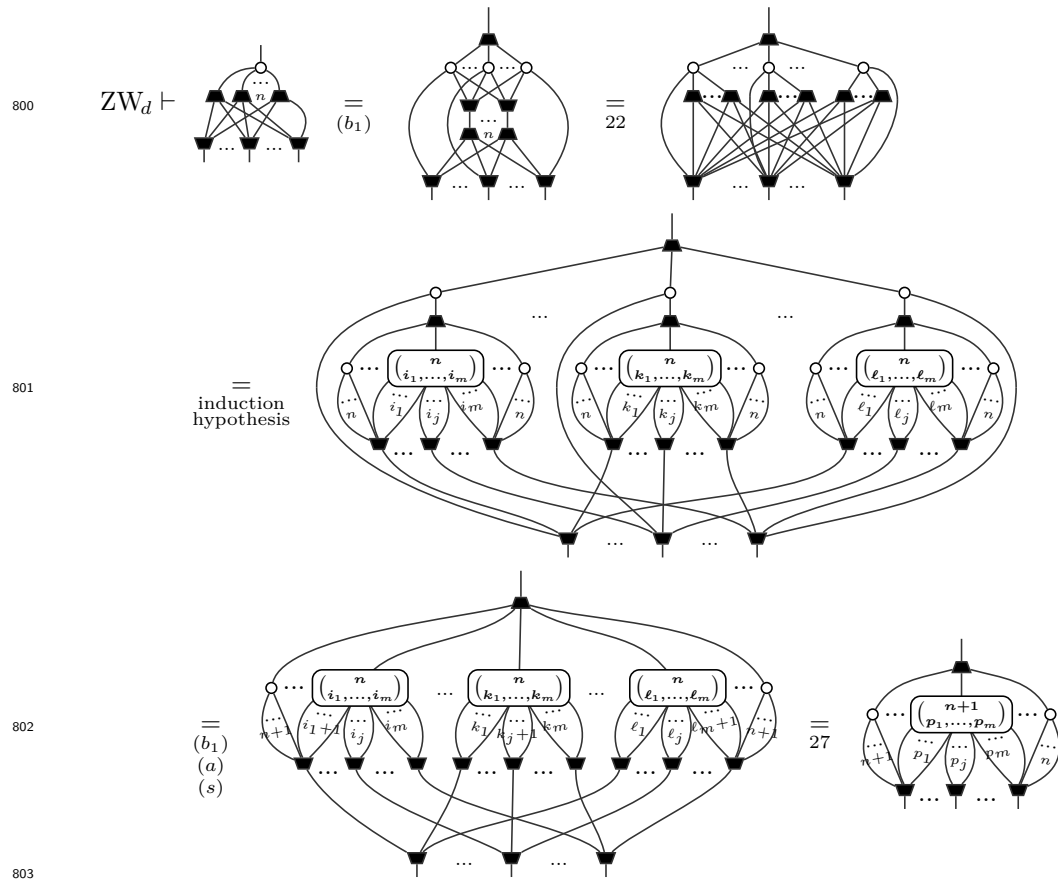
■ **Figure 3** Dependencies between lemmas, propositions and theorems.



Proof of Lemma 31.

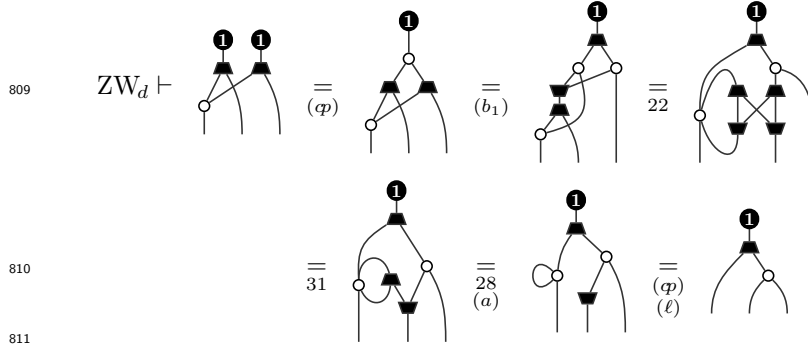


Proof of Lemma 32. By induction on n . The base case $n = 1$ is direct. Suppose we have proven the case n , then we can prove the case $n + 1$ as follows:

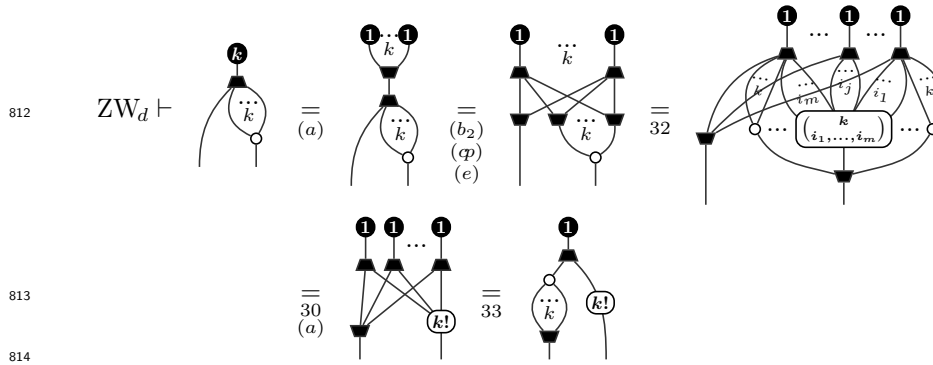


where $\sum_j i_j = \sum_j k_j = \sum_j \ell_j = n$ for all j , and $\sum_j p_j = n + 1$. The last equation is obtained by summing together all white nodes with the same connections. The nodes with connections (p_1, \dots, p_m) are exactly the ones carrying parameters $\binom{n}{p_1-1, \dots, p_m}, \dots, \binom{n}{p_1, \dots, p_m-1}$, which sum to $\binom{n+1}{p_1, \dots, p_m}$ thanks to the generalised Pascal formula. Assuming $\binom{n}{\dots, -1, \dots} = 0$, the formula still holds when some of the p_i are zero. \blacktriangleleft

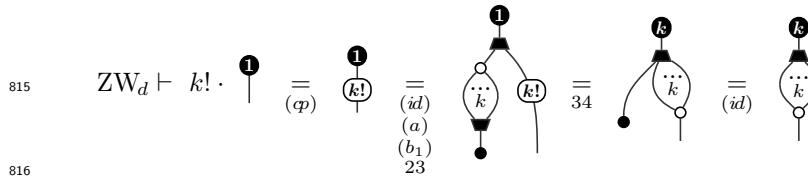
Proof of Lemma 33.



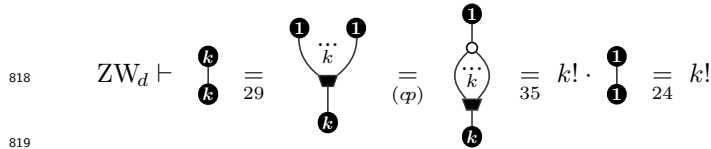
Proof of Lemma 34.



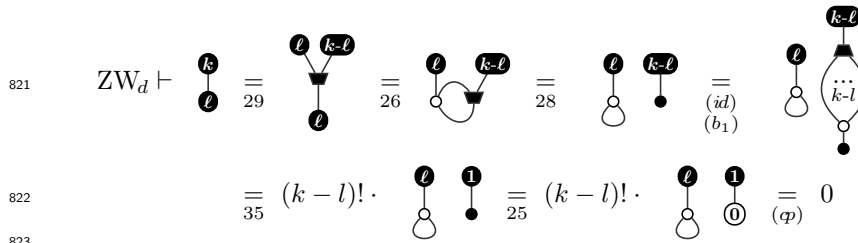
Proof of Lemma 35.



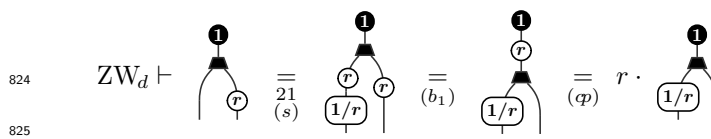
Proof of Lemma 36. First:



820 Then, supposing w.l.o.g. that $k > \ell$:



Proof of Lemma 37.



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826 **Proof of Lemma 38.** First notice that the equation derives from (b_1) when $k = 0$. Then,
 827 for $0 < k < d$:

$$\begin{array}{c}
 828 \quad \text{ZW}_d \vdash \text{Diagram 1} \stackrel{35}{=} \frac{1}{k!} \cdot \text{Diagram 2} \stackrel{(b_1)}{=} \frac{1}{k!} \cdot \text{Diagram 3} \stackrel{(s)}{=} \frac{1}{k!} \cdot \text{Diagram 4} \\
 829 \quad \stackrel{(a)}{=} \frac{1}{k!} \cdot \text{Diagram 5} \stackrel{(b_1)}{=} \frac{1}{k!} \cdot \text{Diagram 6} \stackrel{35}{=} \text{Diagram 7} \quad \blacktriangleleft
 \end{array}$$

831 **Proof of Lemma 39.** If $k = 0$, the equation derives from Lemma 26 and Lemma 23. Else, if
 832 $k < d$:

$$\begin{array}{c}
 833 \quad \text{ZW}_d \vdash \text{Diagram 1} \stackrel{(b_1)}{=} \text{Diagram 2} \stackrel{33}{=} \text{Diagram 3} \stackrel{(s)}{=} \text{Diagram 4} \\
 834 \quad \stackrel{(a)}{=} \text{Diagram 5} \stackrel{(b_1)}{=} \text{Diagram 6} \stackrel{26}{=} \text{Diagram 7} \stackrel{(s)}{=} \text{Diagram 8} \\
 835 \quad \stackrel{38}{=} \text{Diagram 9} \stackrel{37}{=} \text{Diagram 10} \stackrel{33}{=} \text{Diagram 11} \stackrel{(s)}{=} \text{Diagram 12} \quad \blacktriangleleft
 \end{array}$$

837 **Proof of Lemma 40.** To apply (b_2) , we would need to have k_i connections between the
 838 leftmost W-node and the i -th W-node of the bottom of the bialgebra, but so far we only
 839 have one. We can get more, in the following way:

$$\begin{array}{c}
 840 \quad \text{ZW}_d \vdash \text{Diagram 1} \stackrel{35}{=} \frac{1}{(d-1)!} \text{Diagram 2} \stackrel{(b_1)}{=} \frac{1}{(d-1)!} \text{Diagram 3} \\
 841 \quad \stackrel{(s)}{=} \text{Diagram 4}
 \end{array}$$

842 as $k_i < d$, we now have enough connections to apply (b_2) . Doing so and undoing the
 843 transformations on the left part, we get:

$$\begin{array}{c}
 844 \quad \text{ZW}_d \vdash \frac{1}{(d-1)!} \text{Diagram 1} \stackrel{22}{=} \frac{1}{(d-1)!} \text{Diagram 2} = \text{Diagram 3} \quad \blacktriangleleft
 \end{array}$$

845 **Proof of Proposition 8.** First notice that:

$$\begin{aligned}
 846 \quad \left[\begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ \dots \quad \dots \quad \dots \end{array} \right] &= |1, 0, \dots, 0\rangle + |0, 1, \dots, 0\rangle + \dots + |0, 0, \dots, 1\rangle \\
 847 \quad &= \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] + \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] + \dots + \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 848 \quad &
 \end{aligned}$$

849 In a given term, each $|0\rangle$ will merely cancel the Z-spider connected to it, and bring no
 850 contribution to the resulting state. Every contribution is brought by the $|1\rangle$. The number of
 851 parallel edges dictates to which basis state the contribution will go to. Consider for instance
 852 the i th term:

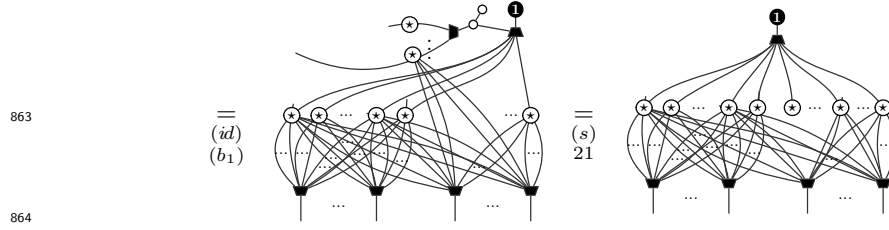
$$\begin{aligned}
 853 \quad \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] &= \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 &= \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 854 \quad &= \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 855 \quad &= r_i |x_1^i, \dots, x_n^i\rangle \\
 856 \quad &
 \end{aligned}$$

857 We do indeed recover the i th term in $|\psi\rangle$. We have something similar for all terms, so in the
 858 end $[\mathcal{N}(|\psi\rangle)] = |\psi\rangle$. \blacktriangleleft

859 **Proof of Proposition 10.** Putting two diagrams in normal form side by side, we get:

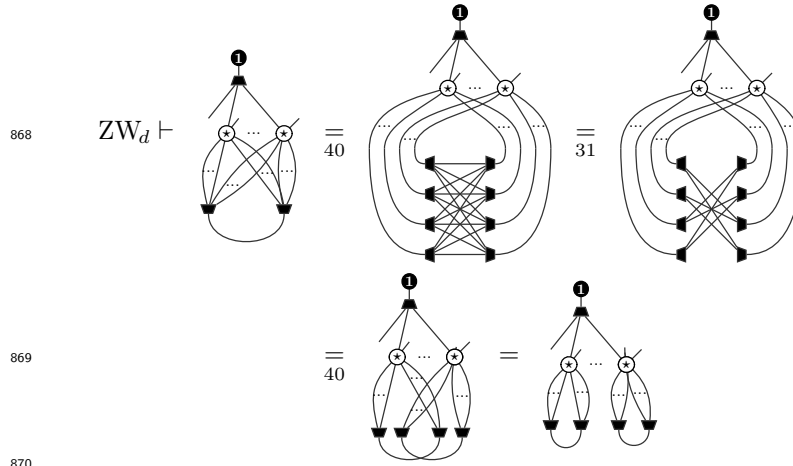
$$\begin{aligned}
 860 \quad ZW_d \vdash & \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = 26 \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = (b_1) \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 861 \quad &= 22 \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = (b_1) \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \\
 862 \quad &= (b_1) \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] = (s) \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right] \left[\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right]
 \end{aligned}$$

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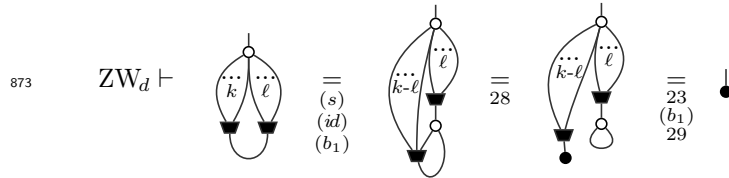


865 The diagram obtained at the end of this process is indeed in normal form. ◀

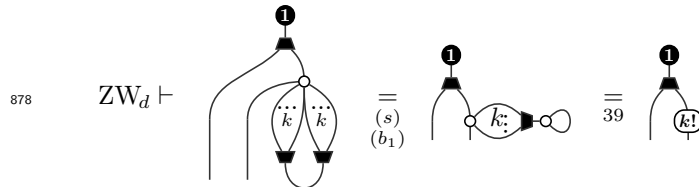
866 **Proof of Proposition 11.** First, we can show that the cup can “distribute” to each white
867 node in the normal form:



871 We can then treat each white node independently. If a white node had k connections with
872 the first output, and ℓ with the second one, with $k \neq \ell$, then it can simply be removed as:

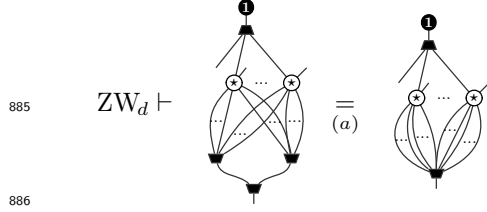


875 This \bullet then copies through the white node (with (b_1)) and gets absorbed by the top and
876 bottom W-nodes of the normal form (by (id)). If $k = \ell$, however the “cup gadget” disappears
877 and the white node gets a new parameter, as:



880 After the last two simplifications, it is possible that two white nodes end up with exactly
881 the same connections. It is then possible to merge them (performing the sum of their
882 parameters), using Lemma 27. After doing all these simplifications, the diagram is again in
883 normal form. ◀

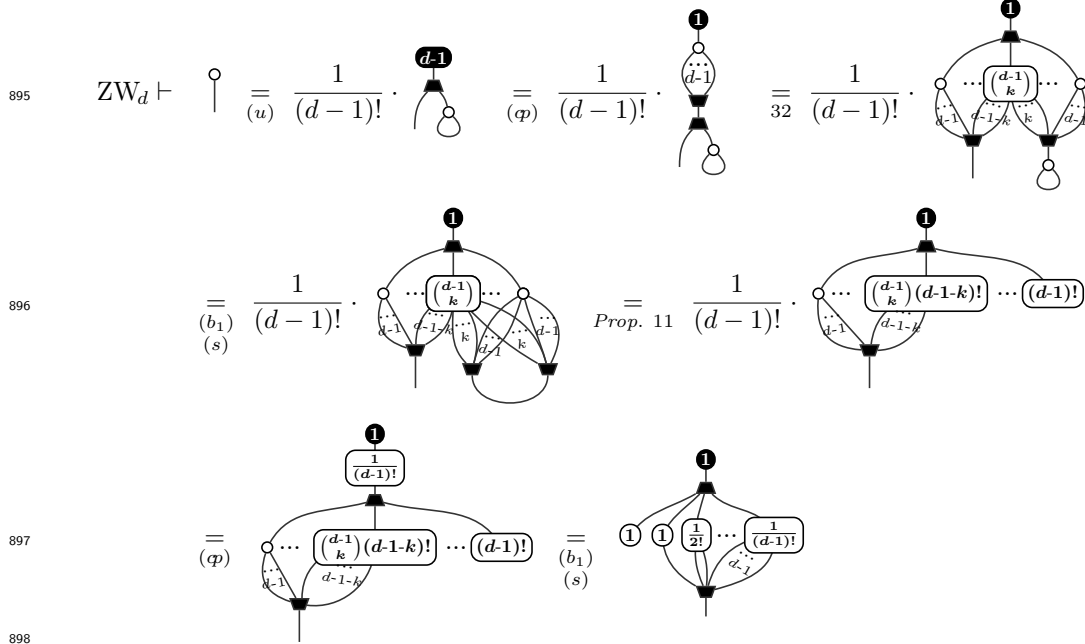
884 **Proof of Lemma 12.** By simply using (a), we get:



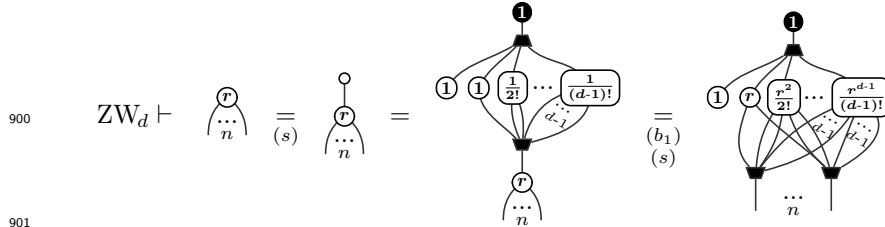
887 then, if a white node ends up with more than $d - 1$ connections with the resulting W-node,
 888 we can remove it by (h) (again the \bullet generated by the rule copies through the white node
 889 and gets absorbed by the top and bottom W-nodes of the normal form, and white nodes with
 890 the same connections can be merged with Lemma 27). If two nodes end up with the same
 891 connections, they can be merged using Lemma 27. We end up with a diagram in normal
 892 form. \blacktriangleleft

893 **Proof of Proposition 13.** We deal with each generator one at a time:

894 \equiv : First, if $n = 1$:




899 Then, if $n > 1$:


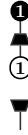


902 If $n = 0$, then \textcircled{r} can be obtained by compositions: $\textcircled{r} = \textcircled{r} \circ$. It can hence be turned

903 in normal form by Propositions 10 and 11.


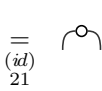
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904 $=$  : First, if $n = 0$:

905 $ZW_d \vdash$  $=$ 

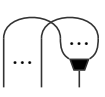
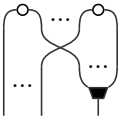
906

907 Then, when $n = 1$:



908 $ZW_d \vdash$  $=$ 


909



910 which can be put in normal form thanks to the previous item. For the general case, thanks
911 to:

912 $ZW_d \vdash$  $=$ 


913

914 the n -ary W-node can be seen as applying  to pairs of outputs (until only one is left) of
915 the normal form of  (notice that the diagram obtained by swapping 2 outputs of
916 a normal form is directly in normal form). Thanks to Lemma 12, the obtained diagram can
917 be put in normal form.

918 $=$  : we simply have

919 $ZW_d \vdash$  $=$ 

920

921 By Proposition 10 and Lemma 12, all the  can be put in normal form as well. ◀

922 C Lemmas and Proofs for FdHilb

923 In this section, for ease of notation, we will omit the box over the capacity annotations. First
924 we give more details about the minimality result:

925 **Proof of Theorem 18.** The argument given for the necessity of Equation (b_2) now works for
926 the necessity of Equation (o) , and that of Equation (e) now works for Equation (b_2) . Namely,
927 Equation (b_2) is the only that can create a non-empty diagram from an empty diagram. A
928 less artificial argument for Equation (b_2) is that it is the only equation that can create a
929 capacity $> k$ from a diagram whose capacities are all $\leq k$ for $k \geq 1$. Moreover:
930 Equation (i) is the only equation that can create a $|1\rangle$ with capacity $a \neq 1$, from a diagram
931 whose $|1\rangle$ s are all on capacity 1.

932 We can reformulate the argument of Equation (a) as: It is the only equation permitting to
933 create non-trivial W-nodes with arity ≥ 3 from a diagram where all non-trivial W-nodes
934 have arity ≤ 2 .

935 In the argument of Equation (ℓ), one can take for ϖ any complex number such that $\varpi^2 \neq 1$,
 936 and by working up to colinearity, as is done initially for $d = 2$.

937 In the argument of Equation (h), we can instantiate the protocol by annotating the wires
 938 with their capacities, then continue with the protocol as explained in the initial argument.

939 All the other arguments work right off the bat for their mixed-dimensional counterpart, hence
 940 the result of minimality. ◀

941 To get to the proof of completeness, we again require some lemmas beforehand.

► **Lemma 41.**

942
$$ZW_f \vdash \begin{array}{c} a \\ | \\ \circ \end{array} = \begin{array}{c} a \\ | \end{array}$$

943

Proof.

944
$$ZW_f \vdash \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(s)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(id)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(a)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(b_1)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(o)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(a)}{=} \begin{array}{c} a \\ | \\ \circ \end{array} \stackrel{(id)}{=} \begin{array}{c} a \\ | \end{array}$$

945

► **Lemma 42.**

946
$$ZW_f \vdash 1 = \begin{array}{c} \bullet \\ | \\ \circ \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

947

948 **Proof.** First, let us prove:

949
$$ZW_f \vdash 1 \stackrel{(\varphi)}{=} \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(s)}{=} \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(\varphi)}{=} 1 \cdot \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(\varphi)}{=} 1 \otimes 1$$

950

951 Then:

952
$$ZW_f \vdash 1 = 1 \otimes 1 \stackrel{(\varphi)}{=} 1 \cdot \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(id)}{=} 1 \cdot \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(a)}{=} 1 \cdot \begin{array}{c} \bullet \\ | \\ \circ \end{array} \stackrel{(u)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(\ell)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

953

954 Finally:

955
$$ZW_f \vdash \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(\varphi)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{(\ell)}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

956

► **Lemma 43.** If $b \geq a$:

958
$$ZW_f \vdash \begin{array}{c} a \\ | \\ \bullet \end{array} \stackrel{(b_2)}{=} \begin{array}{c} a \\ | \\ \bullet \end{array} \stackrel{(id)}{=} \begin{array}{c} a \\ | \end{array}$$

959

► **Lemma 44.**

960
$$ZW_f \vdash \begin{array}{c} \circ \\ | \\ a \end{array} = \begin{array}{c} \bullet \\ | \\ a \end{array}$$

961

Proof.

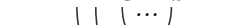
962
$$ZW_f \vdash \begin{array}{c} \circ \\ | \\ a \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet \\ | \\ a \end{array} \stackrel{(h)}{=} \begin{array}{c} \bullet \\ | \\ a \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \bullet \\ | \\ a \end{array} \stackrel{(a)}{=} \begin{array}{c} \bullet \\ | \\ a \end{array}$$

963

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964 ► **Lemma 45.** *Equation (o) can be generalised to:*

965 $ZW_f \vdash$


966 

Proof.

967 $ZW_f \vdash$

968

► **Lemma 46.**

969 $ZW_f \vdash$ 

970

971 **Proof.** With n the smallest between the number of inputs and the number of outputs:

[illegible]

► **Lemma 47.**

975 $\text{ZW}_f \vdash$

976

Proof. The proof is the same as that of Lemma 32, where we can use Lemma 46 in place of (b_2) . \blacktriangleleft

► **Lemma 48.**

$$\text{ZW}_f \vdash \text{Diagram 1} = \text{Diagram 2}$$

981 **Proof.** The case where there are no outputs is dealt with

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{r} \end{array} \stackrel{a}{=} \begin{array}{c} \textcircled{0} \\ | \\ \textcircled{r} \end{array} \stackrel{a}{=} \begin{array}{c} \textcircled{s} \end{array} \stackrel{a}{=} \begin{array}{c} \textcircled{0} \\ | \\ \textcircled{s} \end{array} \stackrel{a}{=} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \stackrel{a}{=} \begin{array}{c} \textcircled{b_2} \end{array}$$

984 The general case is a direct application of (b_1) with $m = 0$.

► **Lemma 49.**

$$\text{ZW}_f \vdash \text{fork}_a^k = \text{fork}_a^{1 \dots 1} = \text{fork}_a^{a \dots a}$$

Proof.

987

$\mathrm{ZW}_f \vdash \begin{array}{c} k \\ | \\ a \end{array} = \begin{array}{c} \quad \quad \quad 1 \quad 1 \\ \quad \quad \quad \diagdown \quad / \\ \quad \quad \quad \bullet \\ \quad \quad \quad / \quad \backslash \\ 1 \quad 1 \\ | \quad | \\ 1 \quad 2 \\ | \quad | \\ \bullet \quad \bullet \\ / \quad \backslash \\ 1 \quad 1 \\ | \quad | \\ a \quad k-1 \end{array}$

(a)

$= \begin{array}{c} 1 \quad \cdots \quad 1 \\ | \quad \quad | \\ \bullet \quad \bullet \\ | \quad | \\ a \quad a \end{array}$

(a)

$= \begin{array}{c} 1 \quad 1 \\ | \quad | \\ \bullet \quad \bullet \\ | \quad | \\ a \quad a \end{array}$

(i)

$= \begin{array}{c} 1 \quad \cdots \quad 1 \\ | \quad \quad | \\ \bullet \quad \bullet \\ | \quad | \\ a \quad a \end{array}$





988

989

► **Lemma 50.** *If $b \geq a \geq k \geq b_i$:*

$$\begin{array}{l} \text{991} \\ \text{992} \end{array} \quad \text{ZW}_f \vdash \frac{\bullet^k_a}{\downarrow_b} = \frac{\bullet^k}{\downarrow_b} \quad \text{and} \quad \text{ZW}_f \vdash \frac{\bullet^k_a}{\downarrow_{b_1 \dots b_n}} = \frac{\bullet^k}{\downarrow_{b_1 \dots b_n}}$$

Proof.

993 $ZW_f \vdash$  $\stackrel{=}{=}_{49}$  $\stackrel{=}{=}_{(a)}$  $\stackrel{=}{=}_{49}$  \vdash

995

$$\begin{array}{c} 996 \\ 997 \end{array} \quad \text{ZW}_f \vdash \quad \begin{array}{c} \bullet^k \\ \text{---} a \\ \text{---} b_1 \quad \text{---} b_n \end{array} = \begin{array}{c} \bullet^k \\ \text{---} a \\ \text{---} b_1 \quad \text{---} b_n \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet^k \\ \text{---} b_1 \quad \text{---} b_n \\ \text{---} b_1 \quad \text{---} b_n \end{array} \stackrel{(id)}{=} \begin{array}{c} \bullet^k \\ \text{---} b_1 \quad \text{---} b_n \end{array}$$

997

► **Lemma 51.**

$$\text{ZW}_f \vdash \begin{array}{c} \circ^a \\ \blacktriangle \\ | \\ 1 \end{array} = \begin{array}{c} \circ \\ | \\ 1 \end{array}$$





































































999

1000 **Proof.** First, we have:

$\begin{array}{c} \text{ZW}_f \vdash \\ (b_1) \\ = 43 \\ = (\ell)_{41} \end{array}$

1002

1003 Then:

1004 $\text{ZW}_f \vdash$  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  =  = =

1005

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► **Lemma 52.**

$$\begin{array}{c}
 1006 \\
 1007
 \end{array}
 \quad
 \text{ZW}_f \vdash
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 3} \\
 \text{Diagram 4}
 \end{array}$$

Proof.

$$\begin{array}{c}
 1008 \\
 1009 \\
 1010
 \end{array}
 \quad
 \text{ZW}_f \vdash
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2} \\
 \text{Diagram 3} \\
 \text{Diagram 4} \\
 \text{Diagram 5}
 \end{array}
 \begin{array}{c}
 = \\
 =_{50} \\
 =_{(b_2)} \\
 =_{(b_1)} \\
 =_{51}
 \end{array}
 \begin{array}{c}
 \text{Diagram 6} \\
 \text{Diagram 7} \\
 \text{Diagram 8} \\
 \text{Diagram 9} \\
 \text{Diagram 10}
 \end{array}
 \begin{array}{c}
 = \\
 =_{(a)}
 \end{array}
 \begin{array}{c}
 \text{Diagram 11} \\
 \text{Diagram 12}
 \end{array}$$

► **Definition 53.** With $d \geq a$, we define the “ a -restricted Z-spider” as:

$$\begin{array}{c}
 1012 \\
 1013
 \end{array}
 \quad
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2}
 \end{array}
 := \frac{1}{a!} \cdot \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \left(= \frac{1}{a!} \cdot \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right)$$

Notice that the a -restricted Z-spider by default only uses capacity d on all its wires.

► **Lemma 54.** With $d \geq a$:

$$\begin{array}{c}
 1016 \\
 1017
 \end{array}
 \quad
 \text{ZW}_f \vdash
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2}
 \end{array}
 =_{49}
 \begin{array}{c}
 \text{Diagram 3} \\
 \text{Diagram 4}
 \end{array}$$

Proof.

$$\begin{array}{c}
 1018 \\
 1019 \\
 1020
 \end{array}
 \quad
 \text{ZW}_f \vdash
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2} \\
 \text{Diagram 3} \\
 \text{Diagram 4} \\
 \text{Diagram 5} \\
 \text{Diagram 6} \\
 \text{Diagram 7} \\
 \text{Diagram 8} \\
 \text{Diagram 9}
 \end{array}
 \begin{array}{c}
 = \\
 =_{(u)} \\
 =_{43} \\
 =_{(b_1)} \\
 =_{(b_2)} \\
 = \\
 =_{50} \\
 =_{50} \\
 =_{a!}
 \end{array}
 \begin{array}{c}
 \text{Diagram 10} \\
 \text{Diagram 11} \\
 \text{Diagram 12} \\
 \text{Diagram 13} \\
 \text{Diagram 14} \\
 \text{Diagram 15} \\
 \text{Diagram 16} \\
 \text{Diagram 17} \\
 \text{Diagram 18}
 \end{array}$$

► **Lemma 55.**

$$1021 \quad ZW_f \vdash \begin{array}{c} \textcircled{1} \\ a \end{array} = \begin{array}{c} \textcircled{1} \\ a \end{array}$$

Proof.

$$1023 \quad ZW_f \vdash \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(50)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(s)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array}$$

1026 ► **Lemma 56.** *With $a \geq k$:*

$$1027 \quad ZW_f \vdash \begin{array}{c} \textcircled{k} \\ a \end{array} = k! \quad \text{and} \quad ZW_f \vdash \begin{array}{c} \textcircled{k} \\ a \end{array} = \boxed{}$$

1029 **Proof.** First:

$$1030 \quad ZW_f \vdash \begin{array}{c} \textcircled{k} \\ a \end{array} \stackrel{(49)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(\varphi)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array} \stackrel{(47)}{=} \begin{array}{c} \textcircled{1} \\ a \end{array}$$

$$1034 \quad ZW_f \vdash \begin{array}{c} \textcircled{k} \\ a \end{array} = \frac{1}{k!} \cdot \begin{array}{c} \textcircled{k} \\ a \end{array} \stackrel{(49)}{=} \frac{1}{k!} \cdot \begin{array}{c} \textcircled{k} \\ a \end{array} \stackrel{(s)}{=} \frac{1}{k!} \cdot \begin{array}{c} \textcircled{k} \\ a \end{array} = 1 \stackrel{(42)}{=} \boxed{}$$

1036 ► **Proposition 57.** *All equations of ZW_d are derivable from ZW_f , i.e.*

$$1037 \quad \forall d \geq 2, ZW_d \vdash D_1 = D_2 \implies ZW_f \vdash \iota_d(D_1) = \iota_d(D_2)$$

1038 **Proof.** First, notice that the result is obviously true for all axioms of compact-closed props.
 1039 Then, it is enough to show the result for the equations in Figure 1, as all equations provable
 1040 with ZW_d derive from them. Equations (s), (id), (a), (b₁), (+), (h), (φ) and (u) are directly
 1041 translated to an equation of ZW_f through ι_d . Then, (b₂) is exactly Lemma 46, (ℓ) is Lemma 55,
 1042 and (e) is Lemma 56. ◀

1043 ► **Proposition 58.** *Let $D \in ZW_f$ be a diagram (state) with its largest capacity being $d - 1$.
 1044 Then, there exists $D_d \in ZW_d$ such that:*

$$1045 \quad ZW_f \vdash \begin{array}{c} \boxed{D} \\ a_1 \dots a_n \end{array} = \begin{array}{c} \boxed{\iota_d(D_d)} \\ a_1 \dots a_n \end{array}$$

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Proof. Let us write $\delta := d - 1$ for the maximum capacity in D . Using Lemma 43, we can “force” capacity δ on all wires:

$$\text{ZW}_f \vdash a \Big| \stackrel{43}{=} \begin{array}{c} a \\ \downarrow \\ \delta \\ \downarrow \\ a \end{array}$$

Every original generator in D is then surrounded by W-nodes (connected to them by their output). These can be turned into purely qudit systems as follows, first for Z-spiders with degree ≥ 1 :

$$\text{ZW}_f \vdash \begin{array}{c} \delta \quad \delta \\ \swarrow \quad \searrow \\ a \quad r \\ \swarrow \quad \searrow \\ \delta \quad \delta \end{array} \stackrel{(s)}{=} \begin{array}{c} \delta \quad \delta \\ \swarrow \quad \searrow \\ a \quad r \\ \swarrow \quad \searrow \\ \delta \quad \delta \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \delta \\ \swarrow \quad \searrow \\ \delta \quad r \end{array} \stackrel{54}{=} \begin{array}{c} n \\ \vdots \\ r | a \\ \vdots \\ m \end{array}$$

The case of the Z-spider with no leg has to be dealt with differently:

$$\text{ZW}_f \vdash a \circlearrowleft \stackrel{(s)}{=} a \circlearrowleft \stackrel{43}{=} \begin{array}{c} a \\ \downarrow \\ \delta \\ \downarrow \\ r \end{array} \stackrel{54}{=} \begin{array}{c} | a \\ \vdots \\ r | a \end{array}$$

The case of the W-node is given by:

$$\text{ZW}_f \vdash \begin{array}{c} \delta \\ \downarrow \\ a \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(a)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \\ \delta \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(a)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \\ \delta \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{41}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \\ \delta \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{41}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \\ \delta \\ \swarrow \quad \searrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array}$$

The case of $\begin{array}{c} k \\ \vdots \\ r | a \\ \vdots \\ m \end{array}$ is dealt with Lemmas 50 and 49. Finally, using (s) with 54, all “ a -restricted”

Z-spiders $\begin{array}{c} n \\ \vdots \\ r | a \\ \vdots \\ m \end{array}$ can be turned into diagrams that only use capacity δ . Doing so for all

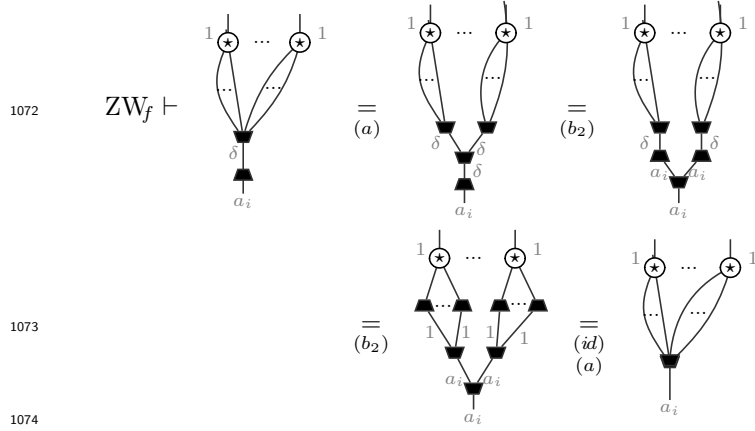
generators, we consume all the W-nodes created at the beginning of the proof through Lemma 43, except the ones that are connected to the outputs of the diagram. We hence end

up in the form $\begin{array}{c} \boxed{\iota_d(D_d)} \\ \downarrow \quad \dots \quad \downarrow \\ a_1 \quad \dots \quad a_n \end{array}$. ◀

► **Proposition 59.** Let D_d be a ZW_d -state in (qudit) normal form. Then $\begin{array}{c} \boxed{\iota_d(D_d)} \\ \downarrow \quad \dots \quad \downarrow \\ a_1 \quad \dots \quad a_n \end{array}$ can be put in (mixed-dimensional) normal form.

Proof. First, we may use Lemma 52 to turn all “internal” capacities in $\iota_d(D_d)$ into 1. $\iota_d(D_d)$ is hence technically in mixed-dimensional normal form. It remains to remove the W-nodes $\langle \delta \rangle \rightarrow \langle a_i \rangle$ at the bottom of the diagram. This can be done as follows, considering each

1071 output individually:



1075 Finally, we may remove Z-spiders connected k times to output a_i when $k > a_i$ using Equation
 1076 (h) followed by Lemma 48 and Equation (id). ◀

1077 **Proof of Theorem 19.** We can now show that any ZW_f -state D can be put in normal
 1078 form. First, use Proposition 58 to turn all but the outputs of the diagram into a qudit
 1079 diagram. Using Proposition 57 and Theorem 14, we can turn the qudit-part of that diagram
 1080 into qudit normal form. Finally, using Proposition 59, the whole diagram can be put in
 1081 mixed-dimensional normal form.

1082 If two ZW_f -states D_1 and D_2 are semantically equivalent ($\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$), then they can
 1083 both be turned into normal form. By uniqueness of the normal form, the two diagrams are
 1084 equal, i.e. $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \implies ZW_f \vdash D_1 = D_2$. ◀