QMI/QDCS : ZX-Calculus

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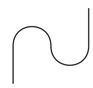
TD2: Diagrammatic approach & Phase-free ZX

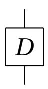
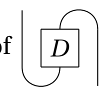
1 Cups and caps

Question 1. Write the following diagram as a composition of identities, cups and caps: .

Answer: Done in class.

Question 2. Compute its interpretation, and check it is the identity. What can we say about the mirrored version of the diagram?

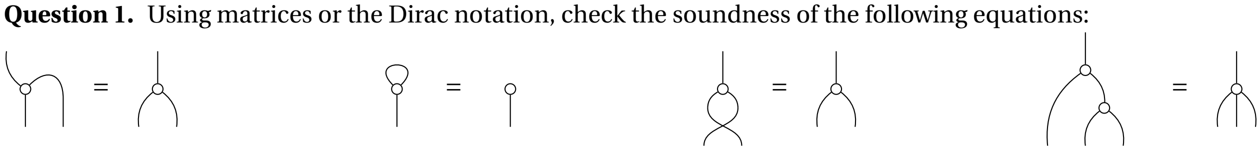
Answer: Done in class.

Question 3. Let  be a diagram representing an arbitrary 2×2 matrix M. Compute the interpretation of . What operation is applied on M?

Answer: Done in class.

2 Only connectivity matters

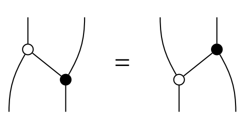
Question 1. Using matrices or the Dirac notation, check the soundness of the following equations:



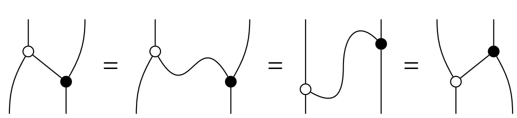
Answer: Assuming all the sum variables take their values in $\{0,1\}$, to avoid clutter:

$$\left(\sum_{i,j}|i,j\rangle\langle i,i,j|\right)\circ\left(\sum_{k,l}|k,l,l\rangle\right)=\sum_i |i,i\rangle\langle i| \qquad \left(\sum_i |i\rangle\langle i,i|\right)\circ\left(\sum_j |j,j\rangle\right)=\sum_i |i\rangle$$

$$\left(\sum_{i,j}|j,i\rangle\langle i,j|\right)\circ\left(\sum_k |k,k\rangle\langle k|\right) \qquad \left(\sum_{i,j}|i,j,j\rangle\langle i,j|\right)\circ\left(\sum_k |k,k\rangle\langle k|\right)=\sum_i |i,i,i\rangle\langle i|$$

Question 2. Show diagrammatically that: . What is the interpretation of these diagrams?

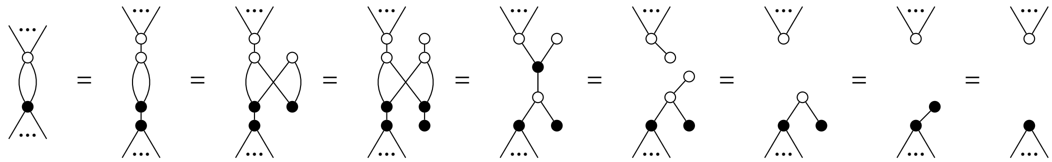
Answer: Using the axioms in detail:



With the result on diagram deformations, this becomes a triviality.

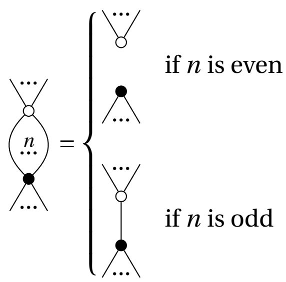
3 Important derived equations

Question 1. Consider the following derivation:

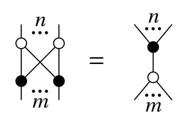


The resulting equation is called the "Hopf law". ⚠Only between a Z-spider and an X-spider.

1. Explain at each step what transformation(s) has been applied (up to diagram deformation).
2. Scalars have been ignored here. What should be the scalar of the last diagram?
3. From the Hopf law, show the following equation:



Answer: Done in class.

Question 2. By an induction, show that we can generalise the bialgebra rule to: 

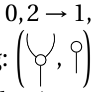
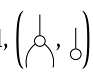
up to a global scalar. What is this scalar?

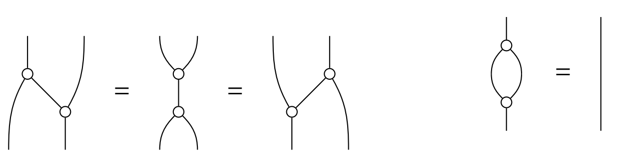
Hint: the diagram on the left is to be understood as a diagram with n inputs, each connected to a Z-spider, m outputs, each connected to an X-spider, and with an edge between each pair of Z- and X-spider. I.e. the Z- and X-spiders form a complete bipartite graph. The diagram on the right only has 2 spiders.

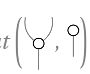
Answer: Done in class.

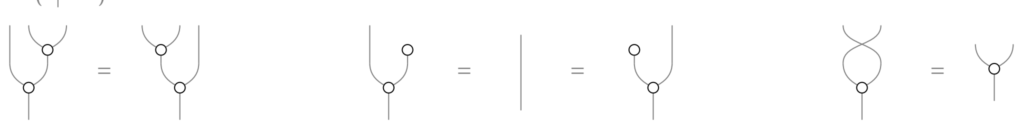
4 Spiders' backstory

In this exercise, we want to recover the spider structure from simpler, more atomic assumptions. Here, we are only allowed to use spiders with type $0 \to 1, 1 \to 0, 2 \to 1, 1 \to 2$.

The base assumptions are the following:  is a commutative monoid,  is a cocommutative comonoid (same but upside-down), subject to the following additional equations:

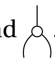


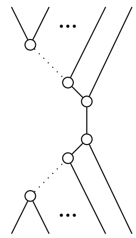
Hint: That  *is a commutative*[1] *monoid means they satisfy the following equations:*



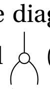
The cocommutative comonoid satisfy the exact same equations but upside-down.

[1] Commutativity is required here to get the proper result. It was mistakenly omitted in the first version.

Question 1. Let D be a **connected** diagram, composed only of  and . Show that the diagram can be rearranged (using the previously shown equations) in the following form:



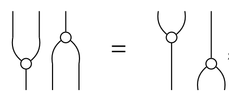
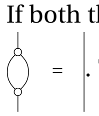
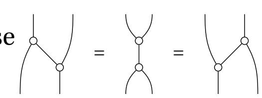
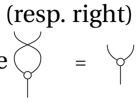
Hint: you can start by considering a topological order of the Z-spiders, and show that we can transform the diagram so that the topological order has first only $2 \to 1$ *spiders, then only* $1 \to 2$ *spiders.*

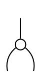
Answer: Reading the diagram from top to bottom, and ignoring the swaps, we can build a topological order of the generators  and  (a topological order is not unique, but satisfies the condition that if a generator is below another in the graph, then it must be after in the order). Let's write $\top_i$ to denote that the node with id i is a $2 \to 1$ spider, and $\bot_j$ to denote that the node with id j is a $1 \to 2$ spider. We first aim to rewrite the diagram such that the associated order has all $\top$s first, and all $\bot$s at the end.

To do so, we consider the following metric: the sum, for all $\bot$s, of the number of $\top$s that are after in the topological order:

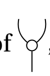
$$\sum_j \left|\{i \mid \top_i > \bot_j\}\right|$$

Consider the cases where a $\bot$ is just before a $\top$:

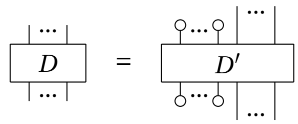
- If they are not connected, they can slide past each other directly: , effectively changing the order so that the $\top$ is now just before the $\bot$.
- If both the outputs of the $\bot$ are connected to the inputs of the $\top$, then we can directly get rid of them using . This reduces the number of "conflicts": number of $\bot$s before $\top$s.
- If the left (resp. right) output of $\bot$ is connected to the right (resp. left) input of $\top$, we can use  to swap their order.
- Finally, if the left (resp. right) output of $\bot$ is connected to the left (resp. right) input of $\top$ (or both at the same time), we can use  and the axioms of the swap to turn the pattern in one of the above cases.

In any of the above cases, the considered metric decreases, until it reaches 0: we have then put all the  at the top, and all the  at the bottom. Since the starting diagram was connected, and none of these rewrites changes connectivity, the obtained diagram is still connected. This means there is only and exactly one wire between the top part and the bottom part.

It is then a common result of commutative monoids, using associativity and commutativity, that any n-fold product on a sequence of elements can be reordered and written with e.g. parentheses stacked to the left. Similarly for cocommutative commonoids (by symmetry). We hence get the above form.

Question 2. Try to extend the previous result to connected diagrams composed of , ,  and .

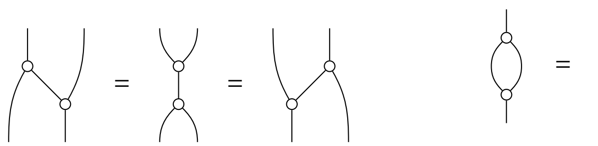
Answer: With D a connected diagram using the above generators, using the wire axioms, we can pull the s to the top left, and the to the bottom left:



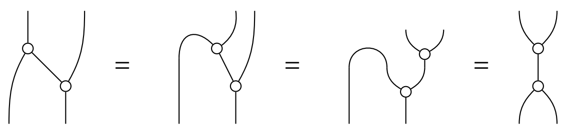
Notice that D' still has to be connected, and is now only composed of and . We can hence reuse the previous result on D'. The s being neutral elements, they can all be removed using = = . The only edge case is when D has 0 input, in which case exactly one remains. A similar reduction can be applied to the bottom part.

In this extended version, we can hence get to the same form as in the previous question (up to the special cases with 0 inputs and/or outputs).

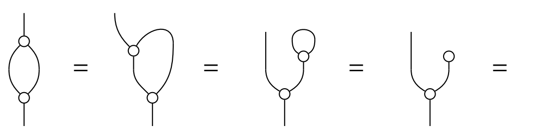
Question 3 (Bonus)**.** Assuming only the monoid and comonoid structures, as well as the two first equations of Question 2.1, show that we can recover:



Answer: For the first equation:



and similarly for the left-right symmetric version. For the second equation:



With this last question, we see that the equations that needed to be introduced to relate the monoid and comonoid, when considered up to diagram deformation and loop removal, boil down to monoid axioms (associativity and neutral element).