QMI : ZX-Calculus

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TD1: Reminders, linear map manipulations

1 Some linear algebra

Question 1. Show the following properties of the tensor product:

1. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
2. The tensor product of two unitaries is a unitary, similarly for isometries
3. The tensor product of two Hermitians is a Hermitian
4. The tensor product of two projections is a projection

Question 2. Show the following properties of the (partial) trace:

1. $\mathrm{tr}(I_A) = \dim(A)$ with I_A the identity on A
2. $\mathrm{tr}(A + \lambda B) = \mathrm{tr}(A) + \lambda\,\mathrm{tr}(B)$
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
4. $\mathrm{tr}_{A\otimes B} = \mathrm{tr}_A \circ \mathrm{tr}_B = \mathrm{tr}_B \circ \mathrm{tr}_A \quad \text{and} \quad \mathrm{tr} = \mathrm{tr} \circ \mathrm{tr}_A$
5. $\mathrm{tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$
6. $\mathrm{tr}([A, B]) = 0$

Question 3. Let U be a unitary. Show the following properties of unitary maps:

1. Inner product preservation: $\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$
2. Norm preservation: $\|Ux\| = \|x\|$
3. Trace preservation: $\mathrm{tr}(U\rho U) = \mathrm{tr}(\rho)$

Question 4. Let M be normal. By the spectral theorem, $M = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$. Show the following:

1. M is Hermitian iff $\lambda_i \in \mathbb{R}$
2. M is a projector iff $\lambda_i \in \{0, 1\}$
3. M is positive-definite iff $\lambda_i > 0$
4. M is involutive iff $\lambda_i \in \{-1, 1\}$
5. M is unitary iff $|\lambda_i| = 1$

Question 5. Let A be a normal matrix, and f be analytic on $\mathbb{C}$. Show that $f(A)$ can be obtained by diagonalising A and applying f on the eigenvalues of A in the diagonalisation.

2 Measurements in the pure picture

Question 1. Let $M_i := |i\rangle\langle i|$.

1. Show that $\{M_i\}_{i\in\{0,1\}}$ forms a complete single-qubit measurement.
2. When measuring $|\psi\rangle := \alpha\,|0\rangle + \beta\,|1\rangle$ with $\{M_i\}_{i\in\{0,1\}}$, what states do we get, and with what probability?
3. Define $|+\rangle := H\,|0\rangle$ and $|-\rangle := H\,|1\rangle$. Define $M_+ := |+\rangle\langle +|$ and $M_- := |-\rangle\langle -|$. Show that $\{M_+, M_-\}$ forms a complete single-qubit measurement.
4. Show that a measurement with $\{M_+, M_-\}$ directly followed by a measurement with $\{M_0, M_1\}$ completely destroys the information from the qubit.

Question 2.

1. Show that $\{\langle i|\}_{i\in\{0,1\}}$ forms a complete single-qubit measurement.
2. What is the difference with the previous measurement?
3. Show that $\{\langle 0| \otimes Z, \langle 1| \otimes X\}$ forms a measurement.
4. How do you interpret it?

3 Teleportation in the mixed picture

Recall the Bell basis:

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad \left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Question 1. Show that $(\langle\Phi^+| \otimes I) \circ (I \otimes |\Phi^+\rangle) = \frac{1}{2}I$.

Question 2. Show that $|\Phi^+\rangle = (I \otimes Z)\,|\Phi^-\rangle = (I \otimes X)\,|\Psi^+\rangle = (I \otimes ZX)\,|\Psi^-\rangle$.

Question 3. Show that $\mathcal{M} = \Big\{\, \langle\Phi^+| \otimes I,\ \langle\Phi^-| \otimes Z,\ \langle\Psi^+| \otimes X,\ \langle\Psi^-| \otimes ZX, \Big\}$ is a measurement.

The teleportation protocol works as follows: We start with a (mixed) state ρ. We then create a Bell pair $|\Psi^+\rangle$, so we get $\rho \otimes |\Psi^+\rangle\langle\Psi^+|$. We then apply the measurement $\mathcal{M}$ above. We should get ρ as a resulting state.

Question 4. What is the state obtained after application of the measurement? Show that it indeed simplifies to ρ.

Teaser: once we get enough theory of ZX, this protocol can be verified more easily diagrammatically.

4 Circuit & matrix manipulation

Question 1. $|\Phi^+\rangle$ is a particular case of a family of states, indexed by the number of qubits they are defined on, called *GHZ states*. They can be built inductively by $|\mathrm{GHZ}_1\rangle = H\,|0\rangle$, and $|\mathrm{GHZ}_{n+1}\rangle = (I^{\otimes n-1} \otimes \mathrm{CNot}) \circ (|\mathrm{GHZ}_n\rangle \otimes |0\rangle)$.
Show that $|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0...0\rangle + |1...1\rangle)$. (Hint: use the Dirac notation.)

Question 2. Show by matrix computation that $|0\rangle \circ \langle 0| = |0\rangle \otimes \langle 0| = \langle 0| \otimes |0\rangle$. Generalise and show that $|\psi\rangle \circ \langle\phi| = |\psi\rangle \otimes \langle\phi| = \langle\phi| \otimes |\psi\rangle$ for any states $|\psi\rangle$ and $|\phi\rangle$ using properties of the tensor.