QDCS : Diagrammatic Calculus and Error Correction

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TD 6

1 Stabilisers and Normalisers

Question 1. Show that any subgroup of $\mathcal{P}_n$ that does not contain $-I \otimes ... \otimes I$ is abelian.
Hint: First show that any element in the subgroup is involutive.

Question 2. Let $S \in \mathcal{N}(\mathcal{G})$ and $|\psi\rangle \in \mathcal{C}(\mathcal{G})$. Show that $S|\psi\rangle \in \mathcal{C}(\mathcal{G})$.

Question 3. Let $S \in \mathcal{P}_n \setminus \mathcal{N}(\mathcal{G})$ and $|\psi\rangle \in \mathcal{C}(\mathcal{G})$. Show that $S|\psi\rangle \in \mathcal{C}(\mathcal{G})^\perp$.

2 Syndrome Decoding

Let $\mathcal{G}$ be an admissible subgroup of $\mathcal{P}_n$, and $S \in \mathcal{G}$. We define $S_1 := \frac{I+S}{2}$ and $S_{-1} := \frac{I-S}{2}$.

Question 1. Show that $S = S_1 - S_{-1}$ defines a projective measurement.

Question 2. Show that for all $|\psi\rangle \in \mathcal{C}(\mathcal{G})$: $S_1|\psi\rangle = |\psi\rangle$ and $S_{-1}|\psi\rangle = 0$.

Suppose an error $E \in \mathcal{P}_n$ occurs on the codestate $|\psi\rangle \in \mathcal{C}(\mathcal{G})$, and we apply the above projective measurement on the system.

Question 3. Show that if E commutes with S then outcome -1 has probability 0. If instead E does not commute with S, show that outcome 1 has probability 0.

3 A Small Stabiliser Code

Let $\mathcal{G} := \langle ZXX, XZX, XXZ\rangle$.

Question 1. Is $\mathcal{G}$ admissible? What are its length and dimension?

Question 2. Build a ZX encoder for the state that spans $\mathcal{C}(\mathcal{G})$.

Suppose we had instead $\mathcal{G} := \langle ZXX, XZX\rangle$.

Question 3. Is $\mathcal{G}$ still admissible? What are its length and dimension?

Question 4. Complete the stabiliser tableau as you see fit, then create a ZX encoder for this code.

4 5-Qubit Code

Let $\mathcal{G}$ be the subgroup of $\mathcal{P}_5$ generated by the following tableau:

$$\begin{pmatrix} I & Z & X & X & Z \\ Z & I & Z & X & X \\ X & Z & I & Z & X \\ X & X & Z & I & Z \end{pmatrix}$$

Question 1. Is $\mathcal{G}$ admissible? What are its length and dimension?

Question 2. We can encode the logical Z operator as $ZZZZZ$ and the logical X operator as $XXXXX$. From this, can we deduce a way to complete the code into a maximal one?

Question 3. Build a ZX encoder for the original code, using the above completion.

Question 4. Show that the minimal distance of this code is 3.