QMI : ZX-Calculus

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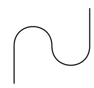
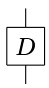
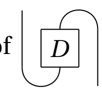
TD2: Diagrammatic approach & Phase-free ZX

1 Cups and caps

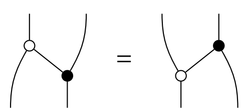
Question 1. Write the following diagram as a composition of identities, cups and caps: .

Question 2. Compute its interpretation, and check it is the identity. What can we say about the mirrored version of the diagram?

Question 3. Let  be a diagram representing an arbitrary 2×2 matrix M. Compute the interpretation of . What operation is applied on M?

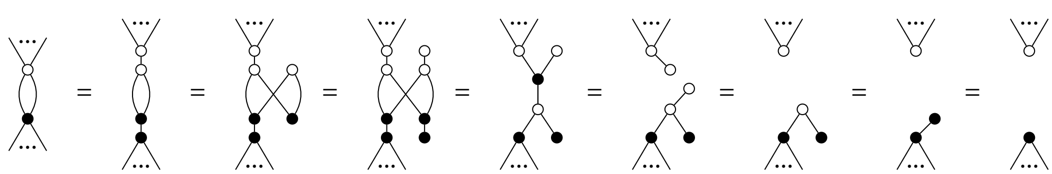
2 Only connectivity matters

Question 1. Using matrices or the Dirac notation, check the soundness of the following equations:

Question 2. Show diagrammatically that: . What is the interpretation of these diagrams?

3 Important derived equations

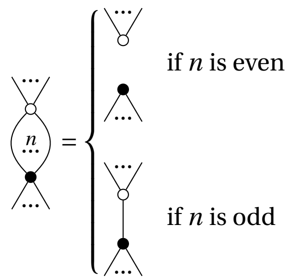
Question 1. Consider the following derivation:

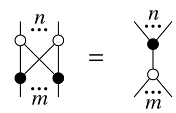


The resulting equation is called the "Hopf law". ⚠Only between a Z-spider and an X-spider.

1. Explain at each step what transformation(s) has been applied (up to diagram deformation).
2. Scalars have been ignored here. What should be the scalar of the last diagram?

3. From the Hopf law, show the following equation:

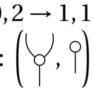
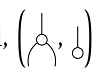


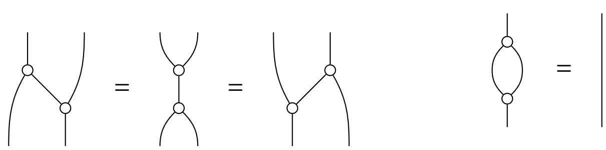
Question 2. By an induction, show that we can generalise the bialgebra rule to:  up to a global scalar. What is this scalar?

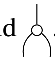
Hint: the diagram on the left is to be understood as a diagram with n inputs, each connected to a Z-spider, m outputs, each connected to an X-spider, and with an edge between each pair of Z- and X-spider. I.e. the Z- and X-spiders form a complete bipartite graph. The diagram on the right only has 2 spiders.

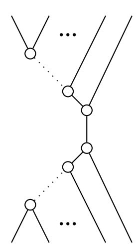
4 Spiders' backstory

In this exercise, we want to recover the spider structure from simpler, more atomic assumptions. Here, we are only allowed to use spiders with type $0 \to 1, 1 \to 0, 2 \to 1, 1 \to 2$.

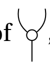
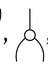
The base assumptions are the following:  is a monoid,  is a comonoid (same but upside-down), subject to the following additional equations:



Question 1. Let D be a **connected** diagram, composed only of  and . Show that the diagram can be rearranged (using the previously shown equations) in the following form:



Hint: you can start by considering a topological order of the Z-spiders, and show that we can transform the diagram so that the topological order has first only $2 \to 1$ spiders, then only $1 \to 2$ spiders.

Question 2. Try to extend the previous result to connected diagrams composed of , ,  and .

Question 3 (Bonus)**.** Assuming only the monoid and comonoid structures, as well as the two first equations of Question 2.1, show that we can recover: