EITQ TD: Advanced Complexity Theory

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1 SAT is NP-Complete

Question 1. Write a (non-deterministic) automaton that recognises a word $w = (w_1, ..., w_n)$ on the alphabet $\Sigma = \{0, 1\}$ iff the word ends with a 1.

We consider a finite set of Boolean variables $x_1, \dots x_n$. A SAT formula is obtained through the following grammar:

$$f_1, f_2 ::= x_i \mid \neg f_1 \mid f_1 \wedge f_2 \mid f_1 \vee f_2$$

A **literal** ℓ is a formula of the form x_i or $\neg x_i$, where $\neg$ stands for the negation. A **clause** is a formula of the form $\bigvee_i \ell_i$ where the ℓ_i are literals and $\vee$ stands for the Boolean "or". A **conjunction** of clauses is a formula of the form $\bigwedge_j c_j$ where the c_j are clauses and $\wedge$ stands for the Boolean "and". The problem SAT is as follows: given a formula find if there is an assignment to true or false of the variables such that the formula evaluates to true.

We want first to build a SAT formula that encodes the automaton and the input word w (and we want that process to work for any automaton and word w). We assume the automata we consider always terminate after $p(n)$ steps where n is the size of w.

We define variables $t_{i,k}$ $(1 \leq i \leq n+1,\ 1 \leq k \leq p(n))$ that are true iff at step k, the next letter to read is w_i. $t_{n+1,k}$ means we are done reading the word.

Question 2. Write a formula φ_0 ensuring that exactly one of $\{0, 1, \epsilon\}$ is read at each step.

We define variables $Q_{q,k}$ that are true iff we are at state q at step k.

Question 3. Write a formula φ_1 ensuring that we are at a single state at each step.

Question 4. Write a formula φ_2 stating which are the initial states.

Question 5. Write a formula φ_3 stating that we must finish on an accepting state.

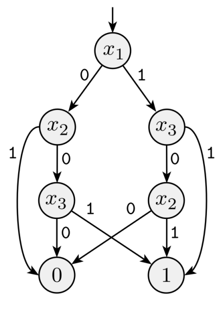
Question 6. Write a formula φ_4 encoding the transitions of the automaton.

Question 7. Justify why we can assume $|Q| = \mathcal{O}(1)$ (where $|Q|$ is the number of states of the automaton). What is then the size of the formula $\varphi := \varphi_0 \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$ w.r.t. n?

Justify why φ encodes the automaton into a SAT problem.

Question 8. What would we need to modify to make this work for a turing machine? Assuming it can be done, with a formula of size polynomial in n the size required on the tape to encode the input, what does that tell us about the complexity of the problem SAT?

2 Branching Programs



A branching program (BP) is a directed acyclic graph, where nodes are labelled by variables from V, except for two output nodes, labelled respectively by 0 and 1. The root node has no incoming edge. Each non-output node has two outgoing edges, labelled respectively 0 and 1. They represent boolean functions. For instance, the BP on the right represents the boolean function represented by the formula:

$$(\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (x_1 \wedge \neg x_3 \wedge x_2) \vee (x_1 \wedge x_3).$$

Question 1. Build a BP representing the following formulas:

- x_1
- $\neg x_1$
- $x_1 \wedge x_2$
- $x_1 \vee x_2$

Question 2. Suppose we have two BPs representing resp. formulas φ_0 and φ_1. Give a BP representing each of the following formulas:

- $\neg\varphi_0$
- $\varphi_0 \wedge \varphi_1$
- $\varphi_0 \vee \varphi_1$

Apply this construction on the formula above. Do we end up with the same branching program?

We denote by eqBP the problem of deciding the equivalence of two given BPs (i.e. if they represent the same boolean function).

Question 3. Considering that UNSAT is **coNP-complete**, show that eqBP is **coNP-complete**. Do you think eqBP is in **BPP**?

A read-once branching program (ROBP) is a branching program with the additional constraint: on each path from the root to an output node, each variable is queried at most once. We want to prove here that eqROBP (deciding the equivalence of two ROBP) is in **BPP**.

Question 4. Suppose we have two *real* variables y_1 and y_2. Find a real-valued polynomial P such that $P(y_1, y_2)$ gives the same value as $y_1 \wedge y_2$ when restricting to boolean values. Same question with $\neg y_1$ and $y_1 \vee y_2$.

Question 5. Given an ROBP, how do we build a real-valued polynomial P that behaves like the boolean function that it represents when restricting to boolean values? What polynomial do you obtain from the BP above? What is the degree of P in general (for each variable), and is the size of P polynomial in the number of variables?

We assume the following lemma, which should be somewhat intuitive:

Lemma 1. *Let $\mathcal{F}$ be a finite field with f elements, and $P \in \mathcal{F}[X_1, ..., X_m]$ non-zero, where each variable has degree $\leq d$. If $a_1, ..., a_m$ are selected at random in $\mathcal{F}$, then:*

$$\Pr[p(a_1, ..., a_m) = 0] \leq \frac{md}{f}$$

Question 6. Using this lemma, prove that eqROBP is in **BPP**.

Further Information on Branching Programs

It is not known whether eqROBP is in **P** or not.

A binary decision diagram (BDD) is a ROBP where variables always appear in the same order. Checking the equivalence of two BDDs with the same variable ordering is in **P**.

3 3-COLOUR is NP-complete

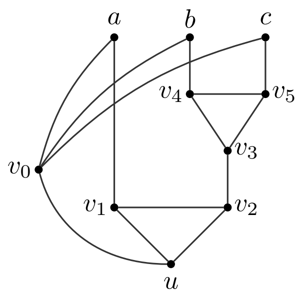
Here we assume that 3-SAT is **NP-complete** (proven in the next exercise with the additional result that SAT $\preccurlyeq_P$ 3-SAT). Given a graph $G = (V, E)$, a k-colouring of the graph is an assignment $C : V \to [\![0, k-1]\!]$ such that $(u, v) \in E \implies C(u) \neq C(v)$, i.e. there are no adjacent vertices with the same colour. We mean to show in this exercise that 3-COLOUR (deciding if a graph admits a 3-colouring) is **NP-complete**.

Question 1. Show that 3-COLOUR $\in$ **NP**.

We then move on to show that 3-SAT $\preccurlyeq_P$ 3-COLOUR. Hence, consider a SAT formula φ_0. We want to build a graph from φ_0 that is 3-colourable iff φ_0 is satisfiable.

Question 2. Start building a graph with three nodes T, F and B all connected together. For each variable p_i in φ_0, create two vertices labelled p_i and $\neg p_i$. How do you connect them to the rest of the graph to make sure that one colour corresponds to evaluating to True, and another colour to evaluating to False, consistently?

Question 3. Consider the following piece of graph:



Show that the only valid 3-colourings are the ones where $\exists x_i \in \{a, b, c\},\ C(x_i) = C(u)$.

Question 4. Using the above "gadget" for every clause in φ_0, complete the graph so that it has a 3-colouring iff φ_0 is satisfiable.

Question 5. Show that 2-COLOUR (the problem of deciding if a given graph admits a 2-colouring) is in **P**.