# QMI/QDCS : ZX-Calculus

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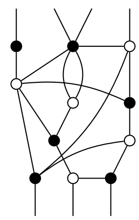
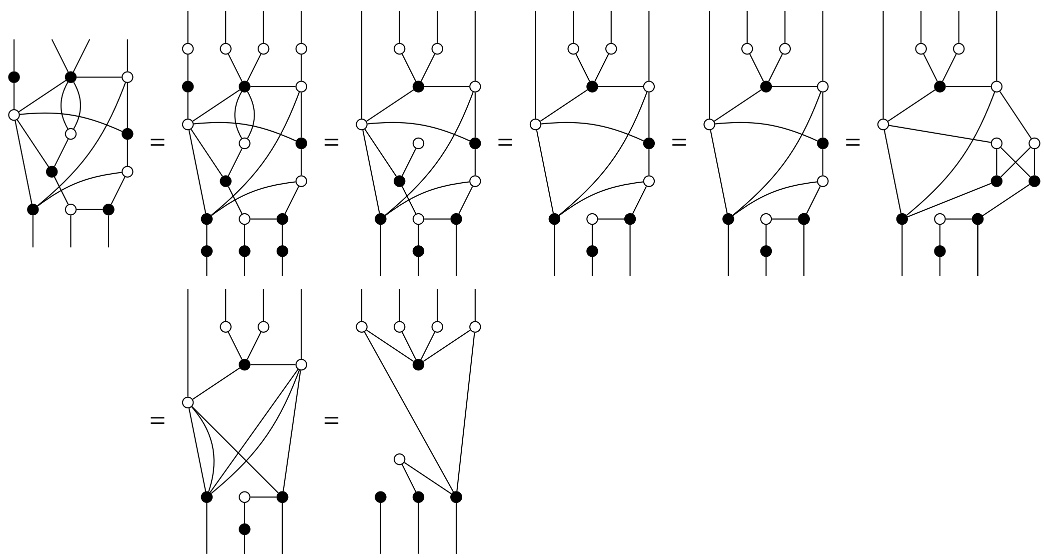
TD3: Phase-free normal form & CSS stabilisers

## 1 Normal Form

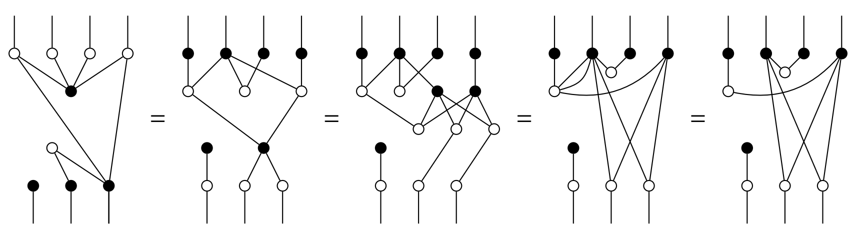
**Question 1.** Put the following diagram in Z-X normal form, then put it in X-Z normal form (ignore the overall scalars):



**Answer:** First, in the Z-X normal form:

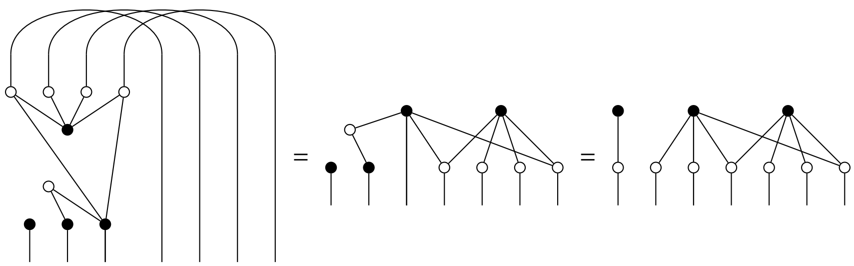


To put it in the X-Z normal form, we can either start from the first diagram, or from the one in Z-X normal form:



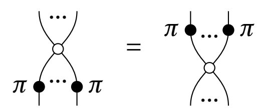
**Question 2.** Bend all input wires into outputs. Put the resulting diagram in Z-X and X-Z normal form.

**Answer:** Since all previous diagrams are equal up to deformation, we can once again choose which to start with. It is then easy to put them in either the Z-X or the X-Z normal form:



## 2 Interaction of spiders and Paulis

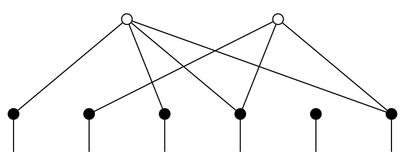
**Question 1.** Show the following equation (using the equational theory of ZX):




*I.e. up to diagram deformation, we can complement the X gates around a Z-spider.*

## 3 Stabilisers

Consider the following ZX state:

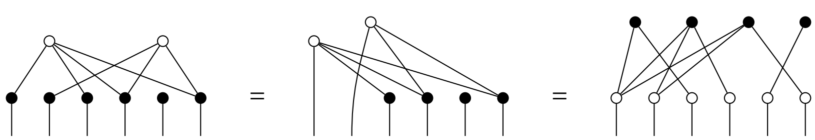


**Question 1.** Provide a generating set of its $X$ stabilisers. Express the state in the canonical basis.

**Answer:** Its stabilisers are $XIXXIX$ and $IXIXIX$. The associated state is hence $|000000\rangle + |101101\rangle + |010101\rangle + |111000\rangle$.

**Question 2.** Turn the diagram in $X - Z$ normal form. What are its $Z$ stabilisers? Check with the $X$ stabilisers that you have gathered **all** stabilisers (or a generating set for them). Express the state in the diagonal basis.

**Answer:**



The Z-stabilisers are then: $ZIZIII$, $ZZIZII$, $ZZIIIZ$ and $IIIIZI$. We can express the stabilisers in the parity matrices:

$$H_X = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The rows of $H_X$ are in row-echelon form, so $H_X$ is full rank. $H_Z$ can be put in row echelon form as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

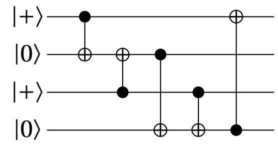
showing that $H_Z$ is also full rank. The sum of the 2 ranks is 6, which is the number of qubits. The two sets of stabilisers hence do generate the whole stabiliser group of the above state.

Up to renormalisation, the state can be expressed in the diagonal basis as:

$$|{+}{+}{+}{+}{+}{+}\rangle + |{-}{+}{-}{+}{+}{+}\rangle + |{-}{-}{+}{-}{+}{+}\rangle + |{+}{-}{-}{-}{+}{+}\rangle + |{-}{-}{+}{+}{+}{-}\rangle + |{+}{-}{-}{+}{+}{-}\rangle + |{+}{+}{+}{-}{+}{-}\rangle + |{-}{+}{-}{-}{+}{-}\rangle +$$
$$|{+}{+}{+}{+}{-}{+}\rangle + |{-}{+}{-}{+}{-}{+}\rangle + |{-}{-}{+}{-}{-}{+}\rangle + |{+}{-}{-}{-}{-}{+}\rangle + |{-}{-}{+}{+}{-}{-}\rangle + |{+}{-}{-}{+}{-}{-}\rangle + |{+}{+}{+}{-}{-}{-}\rangle + |{-}{+}{-}{-}{-}{-}\rangle$$

# 4 Circuit

**Question 1.** What are the stabilisers of the following circuit:




**Answer:** Done in class.