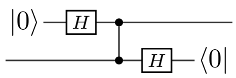
EITQ TD: Quantum(-related) complexity

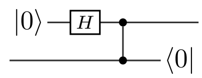
Renaud Vilmart
renaud.vilmart@inria.fr

November 6, 2024

1 Unitarity Checking is coNP-hard

Question 1. Compute the operators implemented by the following circuits with postselections:

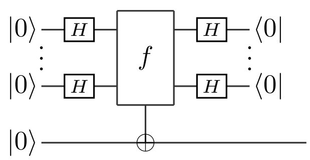




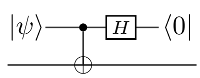
Are they unitary?

Let f be a SAT formula on n variables, and s be the number of variable assignments that satisfy f. We first want to encode the s into a quantum state $|\psi\rangle$ (using postselections).

Question 2. Show that the following circuit implements $|\psi\rangle := \frac{1}{N}\big((2^n - s)|0\rangle + s|1\rangle\big)$:



Question 3. What operator is implemented by the following circuit:

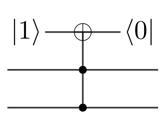


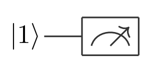
Check that it is unitary iff $s = 0$ or $s = 2^n$.

Question 4. Using the fact that UNSAT is a **coNP-complete** problem, conclude on the hardness of checking unitarity of circuits with postselections.
Hint: don't forget to deal with the $s = 2^n$ case.

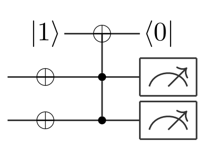
2 The Power of a Single Postselection

Question 1. Explain the behaviour of the following circuits (compute what they implement if you need), and provide an equivalent simplified circuit for each:





Question 2. Use the above to show that the following circuit is equivalent to 2 postselections in $\langle 0|$:



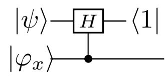
Question 3. Show that having a single postselection is as strong as having arbirarily many postselections.

3 PP $\subseteq$ PostBQP

Let f be a SAT formula on n variables, and s be the number of variable assignments that satisfy f. The majority problem asks whether $s < 2^{n-1}$ or $s \geq 2^{n-1}$.

We assume we know how to create state $|\psi\rangle$ from f as in Question 2 from Exercise 1.

Question 1. Let $|\varphi_x\rangle$ be the arbitrary state $\frac{1}{N}(|0\rangle + x|1\rangle)$ where N is a noralisation factor. Compute the state created by the following circuit:



We then reason w.r.t. the value of s.

Question 2. If $s < 2^{n-1}$, explain how to pick i that maximises $\langle +|\varphi_{2^i}\rangle$. Notice that for such i, if $|\varphi_{2^i}\rangle = \alpha|0\rangle + \beta|1\rangle$, then α is within $\beta/2$ and 2β. Infer a lower bound on $\langle +|\varphi_{2^i}\rangle$.

Question 3. If $s \geq 2^{n-1}$, show that for all $i \in [\![-n, n]\!]$, $|\langle +|\varphi_{2^i}\rangle| \leq \frac{1}{\sqrt{2}} \simeq 0.707$.

Question 4. Using the above, come up with a polynomial time algorithm that decides the majority problem. Conclude on the interaction between **PP** and **PostBQP**.

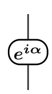
4 Simulation via Perfect Matchings

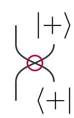
We want to show here how to turn any quantum circuit composed of the usual gates and postselections: $\mathcal{G} = \{H, P(\alpha), \text{CNot}, |0\rangle, \langle 0|\}_{\alpha\in\mathbb{R}}$, which implement respectively:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\ 0 & e^{i\alpha}\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \qquad \begin{pmatrix}1\\ 0\end{pmatrix} \qquad \begin{pmatrix}1 & 0\end{pmatrix}$$

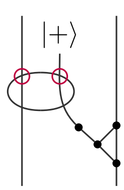
Question 1. Remind how we implement $|0\rangle$ and $\langle 0|$.

Let's first assume that our tensor networks allow the $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ state.

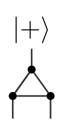
Question 2. Show that 

implements $P(\alpha)$. Same question with 

that implements $\sqrt{2}\cdot H$ and



that implements $2 \cdot \text{CNot}$.

Question 3. Show that 

implements $\sqrt{2}|++\rangle$. Explain how we can get rid of all but 1 occurrence of $|+\rangle$.

Question 4. Suppose we have a circuit C immplementaing U and we want to compute the amplitude $\langle 0...0|U|0...0\rangle$. Explain how we can turn this quantity into a tensor network with a single $|+\rangle$.

What is the number of perfect matchings in a graph with an odd number of vertices? Noticing that $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, explain how we can get rid of the last $|+\rangle$.