QDCS : Diagrammatic Calculus and Error Correction

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Exam 3

1 A CSS Code

Question 1. Recall the definition of the CSS code built from two classical parity check matrices H_X and H_Z. Is there a constraint on the matrices?

Answer: See the course.

We define the following two states: $|\psi_+\rangle := \frac{|000\rangle+|111\rangle}{\sqrt{2}}$ and $|\psi_-\rangle := \frac{|000\rangle-|111\rangle}{\sqrt{2}}$.

Question 2. Give two Z-Pauli strings (i.e. of the form Z^u) that stabilise both $|\psi_+\rangle$ and $|\psi_-\rangle$.

Answer: ZZI and IZZ work.

Question 3. Deduce 6 Z-Pauli strings that stabilise both $|\psi_+\rangle^{\otimes 3}$ and $|\psi_-\rangle^{\otimes 3}$.

Answer: All tensor products of one of the two above strings **OR** III work. This gives $3^3 = 27$ combinations, out of which we can pick 6 independent strings. Warning: considering only combinations of the above two strings (without III) gives 8 combinations, out of which we **can't** pick 6 independent ones.

Here are 6 strings that are independent:

$$ZZI\,III\,III \qquad IZZ\,III\,III \qquad III\,ZZI\,III \qquad III\,IZZ\,III \qquad III\,III\,ZZI \qquad III\,III\,IZZ$$

Question 4. What is the effect of $X \otimes X \otimes X$ on $|\psi_+\rangle$? Same question with $|\psi_-\rangle$.

Answer: $|\psi_+\rangle \mapsto |\psi_+\rangle$ and $|\psi_-\rangle \mapsto -|\psi_-\rangle$

Question 5. Deduce two X-Pauli strings that stabilise both $|\psi_+\rangle^{\otimes 3}$ and $|\psi_-\rangle^{\otimes 3}$.

Answer: We need to cancel out the -1s that appear from $|\psi_-\rangle^{\otimes 3}$, which can be done by applying XXX on two of the three states, i.e. the following Pauli strings work:

$$XXX\,XXX\,III \qquad III\,XXX\,XXX$$

Question 6. Deduce two $\mathbb{F}_2$-matrices H_X and H_Z such that:

$$\mathrm{CSS}(H_X, H_Z) = \mathrm{span}_{\mathbb{C}}\left\{|\psi_+\rangle^{\otimes 3} ; |\psi_-\rangle^{\otimes 3}\right\}$$

What are the length and dimension of that code?

Answer:

$$H_X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Length is $n = 9$ (number of columns), and dimension is $k = n - \mathrm{rk}(H_X) - \mathrm{rk}(H_Z) = 9 - 2 - 6 = 1$.

Question 7. Give a weight-3 undetectable X-error for that code (i.e. an X^u such that $X^u \left|\varphi\right\rangle \in \mathrm{CSS}(H_X, H_Z)$ for $\left|\varphi\right\rangle \in \mathrm{CSS}(H_X, H_Z)$ but $X^u \left|\varphi\right\rangle \neq \left|\varphi\right\rangle$).

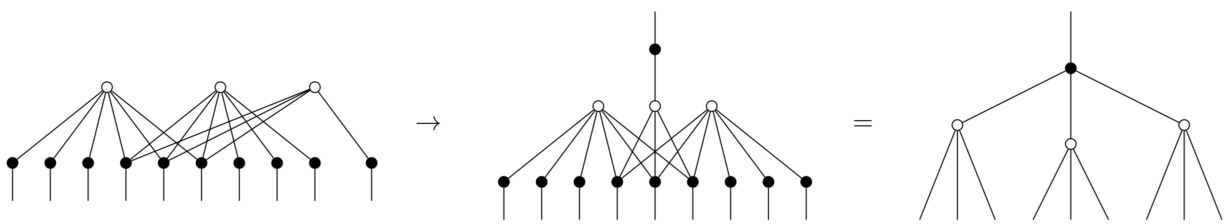
Answer: One can check that $XXX\,III\,III$ modifies a codeword into a different (in general) codeword, and it has weight 3.

Question 8. Complete this code into a minimal CSS code as you see fit. Build a ZX encoder for the initial code. Simplify it.

Answer: We decide to complete the CSS code as follows:

$$H'_X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad H'_Z = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

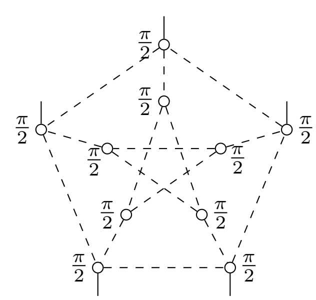
One can check that we still have $H_Z H_X^\intercal = 0$. The associated encoder is, using H'_X:



The equality comes from the application of 1/ a generalised bialgebra (3,3), and 2/ the removal of $1 \to 1$ spiders with no parameters.

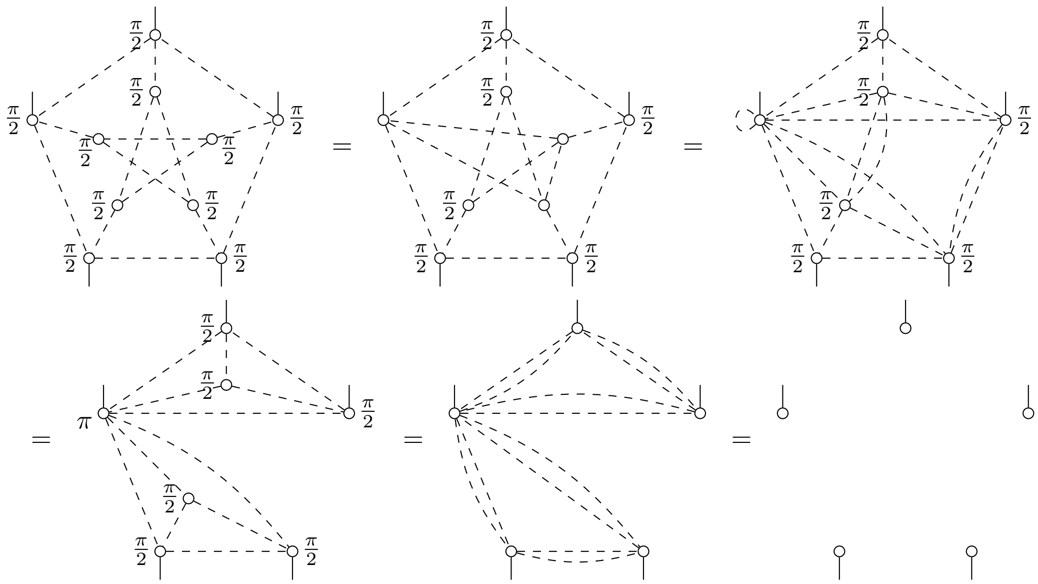
2 Clifford ZX-Diagram

Question 1. By using the algorithm from the course, remove all internal spiders of the following Clifford diagram (where the dashed edges represent an edge with an H gate on them):



Hint: be careful with potential self-loops that can happen when applying a pivoting/bialgebra

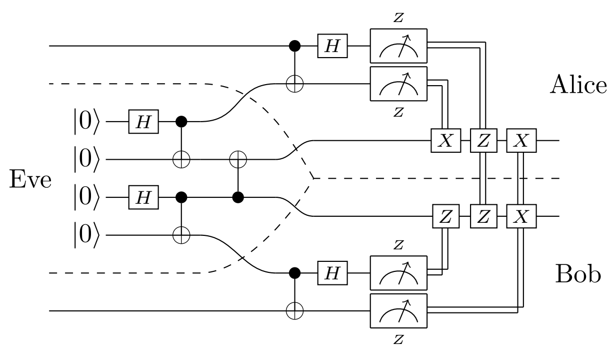
Answer:



First and fourth equalities are local complementations, second equality is a pivoting, and third and fifth equalities are "obvious" simplifications (Hopf law and Hadamard self-loop).

3 Gate Teleportation in ZX

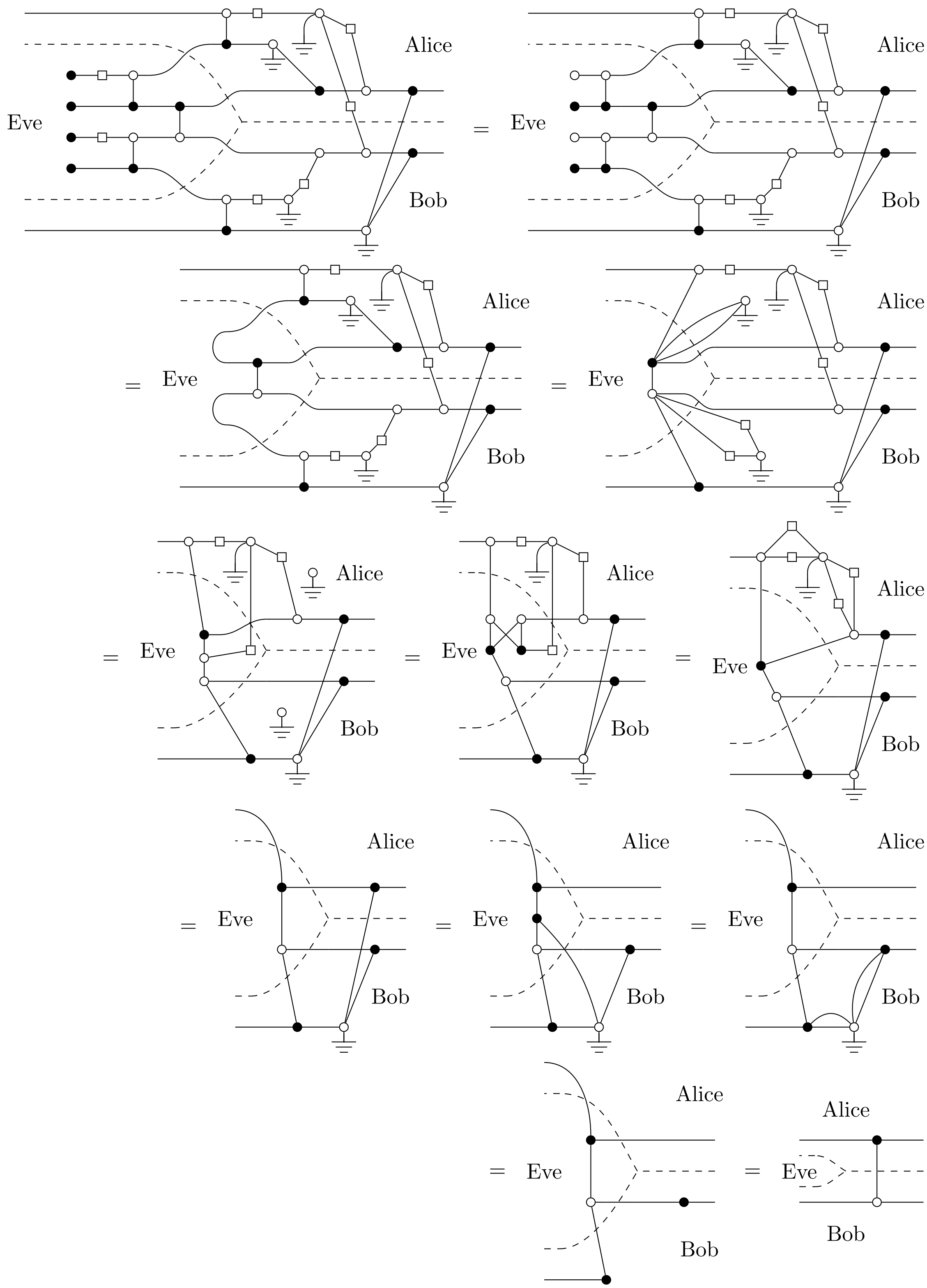
Alice and Bob both possess a qubit, and they would like to apply a CNot gate between the two, but they can only communicate classical information. The problem can be solved if they initially share an entangled state (created by Eve in the following circuit). This entangled state, and the rest of the protocol are represented below:



Question 1. Turn the above protocol into a ZX-diagram with discards (not variables).

Question 2. By rewriting the diagram, show that Eve has applied a CNot gate between Alice's qubit and Bob's qubit.
Hint: Apply the algorithm from the course, simply consider the discards as outputs and get rid of them when they get disconnected from the rest of the diagram.

Answer: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