QDCS : Calcul quantique avancé et codes correcteurs

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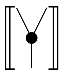
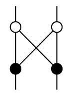
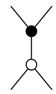
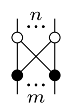
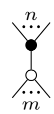
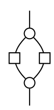
TD 2

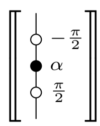
Disclaimer : In the following TD, overall non-null scalars can be ignored, and some of them will be. Equality will then be taken to be up to colinearity.

1. (a) Compute . What does it do on classical inputs?

   (b) Show that the bialgebra equation  $=$  is sound (i.e. that both sides have the same interpretation).

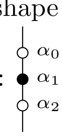
   (c) By an induction, show that it can be generalised to  $=$ .

   (d) Show that  $=$  (you can either show it is sound (easy) or derive it from the axioms of ZX (harder)), and deduce that  $=$ .

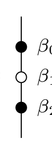
2. (a) Show that  $= \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$.

   (b) Using $\alpha \leftarrow \frac{\pi}{2}$, give a ZXZ decomposition of H. This is often taken to be a full-fledged axiom of ZX.

   (c) Prove that H can be given another similar decomposition, by negating all phases and swapping colours.

   (d) Recalling the shape of 1-qubit unitaries (from the previous TD), show that any 1-qubit unitary can be written:  for some α_i.

   (e) Show that if U_0 and U_1 are unitary, then $U_1 U_0$ is unitary.

   (f) Considering U unitary and HUH, show that for any α_i, there exists β_i such that  $=$  (we do not ask for the relation between the angles). This can be understood as a special case of the so-called *Euler angles*.

3. (a) Recall the MBQC implementation of H. What do we get if we perform the measurement in the basis $(|+_\alpha\rangle, |-_\alpha\rangle)$ where $|\pm_\alpha\rangle := |0\rangle \pm e^{i\alpha}|1\rangle$?

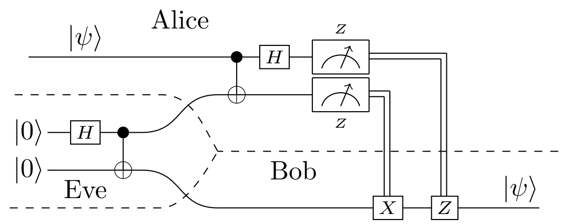
   (b) Deduce an MBQC implementation of $R_Z(\alpha)$.

(c) Again changing one measurement basis from the previous implementation, show that $R_X(\beta)R_Z(\alpha)$ has an MBQC implementation that uses only 2 measurements.

(d) Find an MBQC implementation of the CNot (hint: you can use the fact that CNot $= (I \otimes H)\,\mathrm{CZ}(I \otimes H)$).

(e) Deduce that any unitary can be implemented deterministically in MBQC.

4. Quantum Teleportation:
The problem is the following: Two parties, Alice and Bob, can communicate, but only classically, i.e. they can exchange classical bits, but not qubits. Alice has a qubit in an unknown state $|\psi\rangle$ which she wishes to send to Bob. If no further assumption is made, the problem can be shown to be impossible to solve: Alice cannot send her qubit to Bob through a classical channel. Measurement would destroy the state and only give her (very) partial information.
The problem is however shown to be solvable if Alice and Bob initially share an entangled pair of qubits (Alice has the first one, and Bob the second one). In the following, we can imagine that this state is prepared by a third party called Eve, who splits the pair between Alice and Bob before the protocol really starts. Then, when Alice wants to send her qubit to Bob, she entangles it with her half of the pair (using a CNot), then measures her two qubits in the appropriate basis. Doing this disturbs the system but the disturbance depends on the measurement results. In this case, the errors introduced by Alice's measurements can be corrected by Bob, provided he has access to the measurement results (which are classical). Alice hence sends them to Bob. After correction, the qubit is flawlessly transmitted.
The precise protocol (where measurements use the observable Z) is given in circuit form as follows:

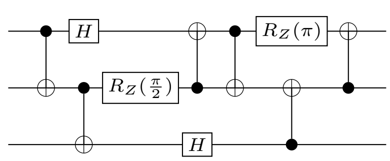


(a) Translate the quantum teleportation protocol in ZX

(b) Show that Eve's part can be reduced to $\frown$. This is the good ol' EPR pair.

(c) Show that the whole diagram is equivalent to the identity (or more exactly to a direct 1-qubit communication from Alice to Bob).

5. Consider the following circuit:



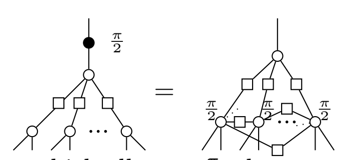
(a) Turn this circuit into a ZX-diagram.

(b) Turn all black nodes to white ones by a change of basis.

(c) Use H-involution whenever necessary.

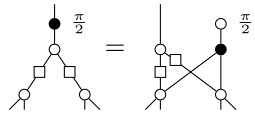
(d) Merge all adjacent white nodes together with the spider fusion rule. You should now have an open graph state (annotated by angles).

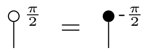
6. Graph states local complementation. We want to show the following identity:



where on the RHS there is a H gate linking every pair of the bottom white dots, which all get a $\frac{\pi}{2}$-phase. We will prove the result by induction on n the number of bottom white nodes.

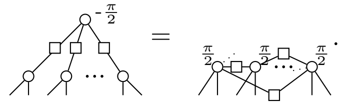
(a) Show the case $n = 1$.

(b) Show that 

(c) Show that  . Deduce the local complementation for $n = 2$.

(d) Prove the inductive part (you will probably need the H involution, the change of basis, the $n = 2$ case, and the generalised bialgebra of exo. 1c)

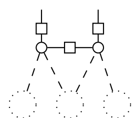
(e) A useful application of local complementation is the following identity:

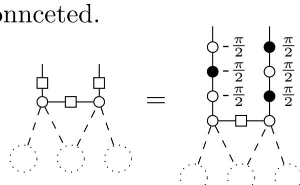
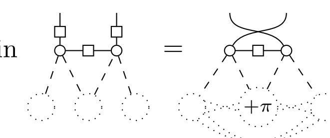


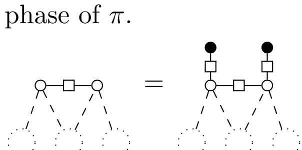
Prove it from the previous results.

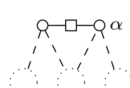
(f) Show that the local complementation and the previous identity both hold when their angles are negated.

7. Graph state pivoting:

(a) We want to simplify  , where the bottom three blobs represent respectively the neighbourhood of the first node, the common neighbourhood, and the neighbourhood of the second node. As we consider graph states, these blobs should be understood as white nodes connected to the node through Hs if the node is linked to the blob. Nothing prevents the nodes from all three blobs to be connceted.

Starting with  , apply local complementation (three times) until all $\pm\frac{\pi}{2}$-phases are removed from the two top wires. We should obtain  , where the dashed edges between the blobs represent complementation, and the common neighbours get an additional phase of π.

(b) Starting with  , apply the pivoting and simplify. We should be left with fewer nodes.

(c) Start a similar derivation with  . It should not simplify as easily. Show however that if $\alpha = k\frac{\pi}{2}$, we can simplify to the point where we are left with one fewer nodes than what we started with.

8. Circuit optimisation and Gottesman-Knill Theorem.

(a) Take the graph state obtained at the end of exo. 5, and try to reduce it using local complementation and pivoting. Can you extract a circuit from this simplified diagram?

(b) Clifford diagrams are ZX-diagrams where all phases are multiples of $\frac{\pi}{2}$. They are a generalisation of Clifford circuits, generated by $\langle \text{CNot}, H, R_Z(\frac{\pi}{2}) \rangle$. Show that any Clifford diagram can be reduced to a point where the number of nodes with arity $\neq 2$ is not larger than the number of inputs/outputs. (Hint: use pivoting and local complementation on *internal nodes* i.e. nodes of arity ≥ 3 that are not connected to an input/output through binary nodes). This essentially shows that Clifford circuits/diagrams are efficiently simulable by classical means (Gottesman-Knill theorem).