The Solow Model And Its Empirical Implications

Vitalii Rudko

December 2020

Abstract

The aim of this short assay is to describe the classical textbook Solow model and review its most fundamental empirical study [2]. It will be shown that Solow model indeed explains some cross country differences of output per capita growth. At the same time the empirical study confronts some of the model's assumptions.

1 Introduction

The Solow model was introduced by two independent authors Robert Solow and Trevor W. Swan in 1956. Robert Solow continued his worked on development of the model, thus it is broadly known as just Solow model. He eventually received a Nobel Prize for his contributions to development of economic theory. It is worth to mention, that before the advent of the Solow model, the Harrod-Domar model was the most popular approach. However, it was not the best place to start, as some assumptions were flawed [3]. Therefore Solow model is regarded as the first fundamental neo-classical growth model, which appeared in response to the Keynesian Harrod-Domar model.

2 The Solow Model

As in case of every theory, it is useful to start with main assumptions. We consider a closed economy, with a unique final good. The economy consists of many identical households, so the whole economy may be seen as a representative consumer [3]. The technology is free and publicly available. The rate of technological progress is constant and denoted as g. The markets, which constitute the economy, are competitive. The labor supply is equal to the population of all households, and it is not dependent on price. The working population grows at the constant rate n. The households also possess all of the capital stock in the economy, which they rent to firms. The price of produced goods is normalized to unity. The capital depreciates with time, in other words machines, buildings, other tools eventually wear out. We denote the rate of depreciation as δ. The investments equal savings, and it is assumed, that households save the portion of an output at the constant rate of s, and $1 - s$ is consumed.

The derivation of the model is conducted in the way suggested by D. Romer [4]. We will consider the model in continuous time for convenience. Let us begin with the Cobb-Douglas production function (1), where $K(t)$ denotes capital, $L(t)$ denotes labor, $A(t)$ is a level of technological progress, the product of $L(t)$ and $A(t)$ is an amount of effective labor, and the subscript t stands for a time period.

$$Y(t) = F(K(t), A(t)L(t)) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}, \qquad 0 < \alpha < 1. \tag{1}$$

The function is twice differentiable at points $K(t)$ and $L(t)$. It is concave and strictly positive. It implies diminishing marginal output if we increase $K(t)$ and $L(t)$, ceteris paribus. Besides, it satisfies Inada conditions, so the marginal output is very high for a very small initial capital stock and vise versa. Also, it is important to underline that increasing $A(t)$ in couple of times will reduce the amount of reacquired labor at the same rate given the output remains the same. The function generates constant returns to scale at $K(t)$ and $L(t)$, therefore

$$F(xK(t), xA(t)L(t)) = xY(t). \tag{2}$$

If we set $x = \frac{1}{A(t)N(t)}$, then we get the function of capital per effective labor

$$\frac{Y(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) = f\left(\frac{K(t)}{A(t)L(t)}\right). \tag{3}$$

We have assumed earlier that technological progress and labor grow at constant rates, thus

$$\dot{L}(t) = nL(t), \tag{4}$$

$$\dot{A}(t) = gA(t). \tag{5}$$

It implies that the variables grow exponentially, so they can be expressed as

$$L(t) = L(0)e^{nt}, \tag{6}$$

$$A(t) = A(0)e^{gt}. \tag{7}$$

Another important element of the model is the capital accumulation equation

$$\dot{K}(t) = I(t) - \delta K(t), \tag{8}$$

where $I(t)$ is investment, and according to the assumptions

$$I(t) = sY(t). \tag{9}$$

Now, suppose we denote output per effective worker as

$$k(t) = \frac{K(t)}{A(t)L(t)}. \tag{10}$$

Applying the chain rule to (10), and using the following facts (3), (4), (5), (8), (9), (10) we finally get the formula which explains dynamics of output per effective worker

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t). \tag{11}$$

Let k^* be a point at which output per effective worker reaches a steady state. It implies $\dot{k}(t)$ must be equal to zero, since output per effective worker is constant at this point, therefore

$$k^* = \left(\frac{s}{(n + g + \delta)}\right)^{\frac{1}{(1-\alpha)}}. \tag{12}$$

3 Empirical Implications

This section reviews the empirical findings of a regression analysis of the standard Solow model, which were described and studied in the famous article written by N. G. Mankiw, D. Romer and D. N. Weil [2].

First of all, it is helpful to describe a regression model and the set of assumptions adopted by N. G. Mankiw, D. Romer and D. N. Weil. So, we start with the substitution of (12) into (1) and taking logs of both sides, which gives us a steady state income per capita

$$\ln\left(\frac{Y(t)}{L(t)}\right) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln(s) - \frac{\alpha}{1-\alpha}\ln(n+g+\delta). \qquad (13)$$

N. G. Mankiw, D. Romer and D. N. Weil assumed that g and δ are constant, since the technological progress is not country specific and δ is just hard to estimate, and it is simply not the fact it varies across the countries. On the other hand, $A(0)$ is assumed to be dynamic, since it reflects not just the level of technological development at the starting point, but also other factors. Thus,

$$\ln A(0) = \alpha + \epsilon \qquad (14)$$

where α is a constant and ϵ is a random error. Substituting (14) into (13) yields

$$\ln\left(\frac{Y}{L}\right) = a + \frac{\alpha}{1-\alpha}\ln(s) - \frac{\alpha}{1-\alpha}\ln(n+g+\delta) + \epsilon. \qquad (15)$$

We assume that saving rate and population growth are independent of country specific shocks ϵ, thus the OLS method is applicable to (15). It is true, because the model assumes constant elasticity of output. Since the share of capital α in output is roughly speaking one third, the elasticity of output per capital with respect to saving rate must be approximately 0.5, and with respect to $n+g+\delta$ around -0.5. N. G. Mankiw, D. Romer and D. N. Weil used that fact in order to test their empirical results; if estimated parameters deviate by sign and value substantially fro theoretical ones, then some of the above assumptions do not hold.

N. G. Mankiw, D. Romer and D. N. Weil used the data for 1960 - 1985 from the Real National Accounts constructed by Summers and Heston. It was divided into three samples: the first one includes 98 countries that are not dependent on oil production, the second group of 75 countries characterized by small populations and relatively poor data quality, the last one consists of 22 OECD countries. The saving rate and population growth were taken as averages, whereas $g+\delta$ was assumed to be equal to 0.05. For more detail, please refer to the original text [2].

The figure 1 presents the summary of results of unrestricted and restricted regression. In the second case we calculate an implied α by restricting the first parameter to be equal the second one. In the unrestricted regression, we

ESTIMATION OF THE TEXTBOOK SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n + g + \delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\bar{R}^2$	0.59	0.59	0.01
s.e.e.	0.69	0.61	0.38
Restricted regression:			
CONSTANT	6.87	7.10	8.62
	(0.12)	(0.15)	(0.53)
$\ln(I/GDP) - \ln(n + g + \delta)$	1.48	1.43	0.56
	(0.12)	(0.14)	(0.36)
$\bar{R}^2$	0.59	0.59	0.06
s.e.e.	0.69	0.61	0.37
Test of restriction:			
p-value	0.38	0.26	0.79
Implied α	0.60	0.59	0.36
	(0.02)	(0.02)	(0.15)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

Figure 1: Summary.
Reprinted from: N. G. Mankiw, D. Romer, D. N. Weil (1992). *A contribution to the empirics of economic growth*, The Quarterly Journal of Economics, 107, 414.

observe parameters with adequate signs, as well as low standard errors of prediction, except the OECD group, where those errors are high, implying that these parameters are not statistically significant. Adjusted R^2 is high for the first two groups, and very low for the OECD countries. What is interesting implied alphas derived from the restricted model show that the proportion of capital in growth is higher than initially assumed one, one third, but only for the first two groups. In general it means, that the capital contributes far more in developing countries than other factors. The important conclusion of that part of the study is that easily observable variables explain far more variation of output per capita across the countries, than unobservable variables like technical progress. This contradicts with assumptions of the textbook Solow model.

The method reviewed in this short essay is the basic one, more sophisticated methods like the Augmented Solow model [2] or dynamic panel GMM approach [1] can be found in the cited references.

References

[1] D. K. Das (2013). *Empirical Estimation of the Solow Growth Model: A Panel Approach*, Master's thesis, Lund University, Lund, Sweden.

[2] N. G. Mankiw, D. Romer, D. N. Weil (1992). *A contribution to the empirics of economic growth*, The Quarterly Journal of Economics, 107, 407 - 437.

[3] D. Acemoglu (2007). *Introduction to Modern Economic Growth*, Manuscript, Massachusetts Institute of Technology, Massachusetts, MA.

[4] D. Romer (2012). *Advanced Macroeconomics*, Fourth edition, McGraw-Hill, New York, NY.

[5] R. Kiedrowski (2019). *Lecture 11. The Solow – Swan growth model*, Lecture notes, Poznań University of Economics and Business, Poznań, Poland.