The St. Petersburg Paradox

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Abstract

The purpose of this assay is to introduce St. Petersburg paradox to a reader, who is not familiar with the topic. It covers the most popular solutions to the problem. Also, It presents results of an experiment, which was conducted to illustrate the problem better.

The paradox was proposed by Nicolaus Bernoulli. He described the problem in his letter to Pierre Reymond de Montmort, on September 9, 1713. Later this year Monmort published his correspondence with Bernoulli in his second edition of "Essay d'analyse sur les jeux de hazard", p. 402 [1]. However, the paradox itself was named after a scientific journal, Papers of the Imperial Academy of Science in Petersburg, where his cousin Daniel Bernoulli suggested a solution to the problem in a paper "Exposition of a New Theory on the Measurement of Risk"[1]. published in 1738. The paradox remains to be a popular topic in academic circles until these days. So, a modern version is often presented as follows

St Petersburg Paradox. *A player has to pay a fee to participate in a game. A fair coin tossed repeatedly until it falls heads. The player wins a prize worth* $\mathbf{\$2^n}$*, where* $\boldsymbol{n}$ *is a number of the final toss. What is a 'fair' fee to enter the game?*

If we calculate the expected value of the prize we will get a sum close to infinity. Let $\mathbf{v}$ denote the prize, then

$$E(v) = \lim_{n\to\infty} \sum_{i=1}^{n} (2^n)(\frac{1}{2^n}) = 1 + 1 + 1 + 1 + .. = \infty \tag{1}$$

[1]Translation from Latin.

Therefore, according to the result from equation(1), we should be willing to pay any amount of money to play this game. It seems unreasonable, since the probability of wining a big prize is pretty small $\frac{1}{2^n} = 0$ as $n \to \infty$. That also implies, that a player almost surely can win only finite amount of money. So the first solution that was proposed by N.Bernoulli, was just to ignore infinitesimal probabilities[2].

I decided to conduct an experiment[2] to show why N.Bernoulli argument notwithstanding can be pretty reasonable. Let me introduce some additional notation first. Let **K** denote total number of simulations, and $\mathbf{v_k}$ is the value of the **k**th prize. After simulating a large number of games in Python [3], I obtained the results which are presented in the table 1.

K	Mean	σ	Mode	$v_k \leq \$4$
10000	\$14.59	\$157.5	\$2	75.3%
100000	\$20.84	\$1767.04	\$2	75.18%
1000000	\$17.45	\$620.77	\$2	74.98%

Table 1: The results of my experiment

Here we can observe a very interesting pattern the mean and the standard deviation are not consistent, which is not a surprise, but in every sample 75% of all prizes are less or equal to \$4. That can be easily explained by the fact, that probability of occurrence the \$2 prize is 50%, and the probability of occurrence \$4 prize is 25%. Therefore we can calculate the fee, which would cover for example 99.9% of outcomes. I used the very basic property of the probability measure to calculate it.

$$1 - \frac{1}{2^n} \geq 99.9\% \tag{2}$$

$$n \geq 9.97 \tag{3}$$

Thus the expected value of a prize for $n = 10$ is no more than \$10.

Now we will discuss another solution mentioned earlier. In his paper, D.Bernoulli implemented the concept of marginal utility in valuation of a risky proposition. He suggested this method as a solution to the paradox.

[2]Similar to G.L.L. Buffon's child play.

[3]The code is placed in the appendix.

Notably, Gabriel Cramer came up with almost the same idea in his letter to, the cousin, N.Bernoulli a few years earlier[1].

In general, *the marginal utility defined as the additional gain of pleasure emerged from any increase in one's wealth.* In other words, a rich man gain less pleasure than a poor man form the same amount of money. D.Bernoulli also assumed that *no matter how small increase in wealth it will always result in an increase in utility.* Then he formulated the following important hypothesis:

$$\Delta U = b\frac{\Delta w}{w} \tag{4}$$

Where **U** denotes utility, **w** is wealth, and **b** represents some constant. In words, it states that *a small change in utility is inversely proportionate to the relative change in wealth.* Further I will step away from Bernoulli derivation of utility function without the big harm to the main idea.

$$\frac{\Delta U}{\Delta w} = b\frac{1}{w} \tag{5}$$

Since **U** is a dependent variable and **w** is independent, thus if we integrate both sides, we will obtain the following result[4]

$$U(w) = b \ln w + C \tag{6}$$

We assume that our constant (**C**) is equal to 0.Then the expected utility for our case is calculated as

$$E(U) = \lim_{n\to\infty} \sum_{i=1}^{n} (b \ln 2^n)(\frac{1}{2^n}) \tag{7}$$

Since we are looking for the 'fair' fee, we want to asses it in \$, therefore let **x** denote the 'fair' fee. It also implies that **x** has an utility

$$U(x) = b \ln x \tag{8}$$

[4] Just to clarify, D.Bernoulli used *log*() function for his calculations and set the constant (**C**) to $-\log(\alpha)$, where α is initial possessions. At last, his formula looks like this

$$U(w) = b \log \frac{w}{\alpha}$$

Now we are stating that the expected utility of participating in the game is equal to the utility lost from paying the fee. This is classical assumption of a fair game. We cancel out **b**s and it results in

$$\ln x = \lim_{n\to\infty} \sum_{i=1}^{n} (\ln 2^n)(\frac{1}{2^n}) \tag{9}$$

$$= \ln 2 \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{n}{2^n}) \tag{10}$$

It is left to calculate the right-hand infinite sum. Let this sum be denoted as **S**, then we have

$$S = 2S - S = 1 + \left(\frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{n-1}}\right) + \frac{n}{2^n} \tag{11}$$

$$= 1 + \left(\frac{1}{2}\right) \lim_{n\to\infty} \left(\frac{1 - (\frac{1}{2})^{n-2}}{1 - \frac{1}{2}} - \frac{n}{2^n}\right) \tag{12}$$

$$= 1 + 1 = 2 \tag{13}$$

Finally, if we insert (12) into (10) and get rid of ln(), we will get the result

$$x = 4 \tag{14}$$

That means a reasonable man should not pay more than $4 to play this game. Gilbert W. Basset claims that this solution is unsatisfied, because we always can modify the game, such that the prize will grow faster and eventually ln() becomes unbounded [4].

One more interesting solution was suggested by William Feller. Let's start with assumptions of a 'fair game' given by him in his work 'An Introduction to Probability Theory and Its Applications'

i A player posses an unlimited capital. (This assumption helps to avoid the gambler's ruin problem)

ii A player does not have a privilege to stop the game.

iii The number of trials must be fixed in advance.

iv The game can be favorable, if expected gain is greater than a fee, or unfavorable to a player, otherwise. This differs from classical assumption, when a game is called 'fair', if the expected net profit of each side is equal to zero.

v The expected value and variance of player's net profit must be finite.

As we can see the Petersburg game is not 'fair', because it already does not satisfy (iii) and (v) assumptions.

In the same book W.Feller proposed the following solution to the problem. Since expected value of a gain does not exists, then we cannot keep fees constant. However, the game becomes more or less 'fair' if the ratio of accumulated gain to accumulated fees approaches 1. The accumulated gains here are defined as the sum of all gains after some number of played games. The accumulated fees are defined likewise. Therefore the game is 'fair' under the condition, that for every $\epsilon > 0$

$$P\left\{\left|\frac{S_n}{e_n} - 1\right| > \epsilon\right\} \to 0 \tag{15}$$

Where $\mathbf{S_n}$ is accumulated gain, and $\mathbf{e_n}$ is accumulated fees. Further W.Feller proved that if we take $e_n = n \log_2 n$, then (14) holds and the weak law of large numbers applies. In other words, our fee for **n**th game will be equal to $e_n - e_{n-1} = \log_2 n$. The only problem is that **n** must be large enough. Nevertheless, W.Feller was pretty sure that it resolves the paradox.

After so many years the paradox is still a source of controversy. It is a good example of the difficulty of the risk measure, even seemingly so simple one.

References

[1] Daniel Bernoulli. *Exposition of a New Theory on the Measurement of Risk*. Econometrica, Vol. 22, No. 1 (Jan., 1954), pp. 23-36

[2] Gabor Szekely, Donald Richards. *The St. Petersburg Paradox and the Crash of High-Tech Stocks in 2000*. The American Statistician, Vol. 58 (Feb., 2004), pp. 225-231

[3] Jesse Albert Garcia. *A Bit About the St. Petersburg Paradox*. March 20, 2013

[4] Gilbert W. Basset. *The St. Petersburg paradox and bounded utility.* History of Political Economy, 19:4, Duke University Press, 1987, pp. 517-523.

[5] William Feller. *An Introduction to Probability Theory and Its Applications.* Vol. I, 3d edition, New York Wiley, 1968.

Appendix

```
import random as r
import numpy as np
from scipy import stats as st

#Here I define a supplementary function, which calculates
#the length of a random vector until
#the last toss when heads falls.

#INPUT: no argument.
#OUTPUT: length of a random vector.

def win_streak():

outcomes = [ ]
condition = 1

while condition != 0:

variable= r.randint(0,1)
outcomes.append(variable)

if variable == 0:
    condition = 0
else:
    condition = 1

return len(outcomes)
```

```
#This is the main function which simulates n games
#and returns descriptive statistics.

#INPUT: a number of games to be simulated.
#OUTPUT: an average prize amount, a standard deviation,
#        a mode and a proportion of prizes that are
#        less than 4$.

def simulation(n):

condition = 0
results = []

while condition<=n:

condition += 1
variable = win_streak()
prize = 2**variable
results.append(prize)

results2 = np.array(results)

mean = round(np.mean(results2), 2)
std = round(np.std(results2), 2)
mode = st.mode(results)
less = round(np.sum(results2 <= 4 )/n * 100, 2)


return mean, std, mode, less
```