MAT4240 PROJECT 1

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1 Exact Solution

Bernoulli equation $\dfrac{dy}{dt} - \dfrac{y}{a+t} = -y^3$

$$\rightarrow \quad y^{-3}y' - \frac{1}{a+t}y^{-2} = -1 \quad \rightarrow \quad (y^{-2})' + \frac{2}{a+t}y^{-2} = 2 \quad \rightarrow \quad (y^{-2}(t+a)^2)' = 2(t+a)^2$$

$$\rightarrow \quad y^{-2}(t+a)^2 = \frac{2}{3}(t+a)^3 + a^2 - \frac{2}{3}a^3 \quad \rightarrow \quad y^2 = \frac{(a+t)^2}{a^3 - \frac{2}{3}a^3 + \frac{2}{3}(t+a)^3}$$

$$\rightarrow \quad y = \frac{a+t}{\sqrt{a^3 - \frac{2}{3}a^3 + \frac{2}{3}(t+a)^3}}$$

2 Plot of Solutions for Different N

For multi-step methods, use RK4 to initialize the first three values.

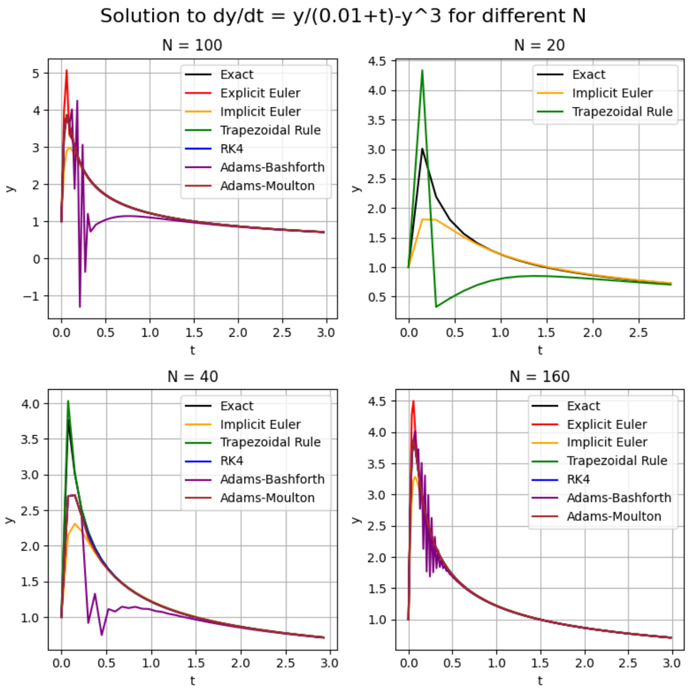


Figure 1: Plot of Solutions for Different N

3 Unstable Methods for Different N

When N=100, all six methods are stable.

When N=20, explicit Euler's method, four-step Adams-Bashforth method and four-step Adams-Moulton method are unstable. The program encounters the overflow and turns out runtime warning.

When N=40, only explicit Euler's method is unstable. The program encounters the overflow and turns out runtime warning.

When N=160, all six methods are stable

4 Max Error as a Function of N for Each Method

When N = 40, explicit Euler's method is unstable.

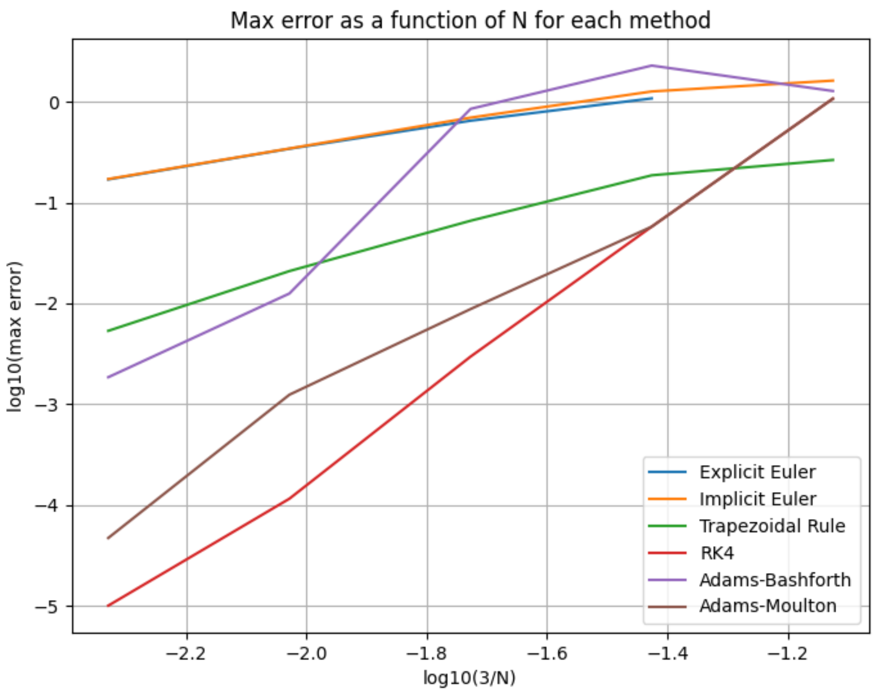
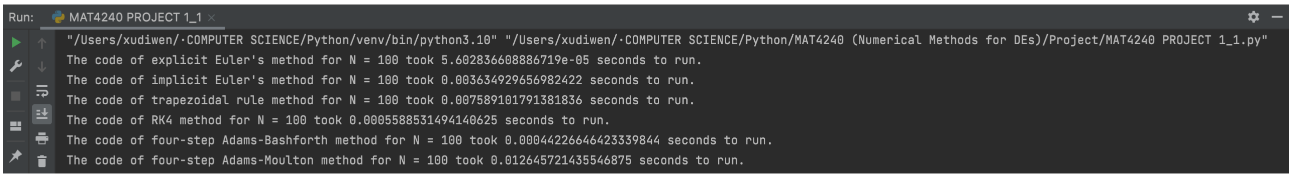


Figure 2: Max Error for Each Method

5 Comparison of Six Methods

Ease of use: Explicit Euler's method is the easiest one. Implicit Euler's method and trapezoidal rule method are more difficult than explicit Euler's method, because the two require solving the equation at each step. RK4 method is more complex to implement than the Euler's methods. The four-step Adams-Bashforth method is a multi-step method. It needs information from previous steps, so it's more complex to implement. The four-step Adams-Moulton method needs information from previous steps and require solving the equation at each step.

Speed of calculation: According to Figure 3, it shows the time of different methods for N = 100. Speed: Explicit Euler > Adams-Bashforth > RK4 > Implicit Euler > Trapezoidal Rule > Adams-Moulton. And, the speed of Adams-Bashforth and RK4 is quite close to each other.



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Run:  MAT4240 PROJECT 1_1
"/Users/xudiwen/·COMPUTER SCIENCE/Python/venv/bin/python3.10" "/Users/xudiwen/·COMPUTER SCIENCE/Python/MAT4240 (Numerical Methods for DEs)/Project/MAT4240 PROJECT 1_1.py"
The code of explicit Euler's method for N = 100 took 5.602836608886719e-05 seconds to run.
The code of implicit Euler's method for N = 100 took 0.003634929656982422 seconds to run.
The code of trapezoidal rule method for N = 100 took 0.007589101791381836 seconds to run.
The code of RK4 method for N = 100 took 0.0005588531494140625 seconds to run.
The code of four-step Adams-Bashforth method for N = 100 took 0.00044226646423339844 seconds to run.
The code of four-step Adams-Moulton method for N = 100 took 0.012645721435546875 seconds to run.
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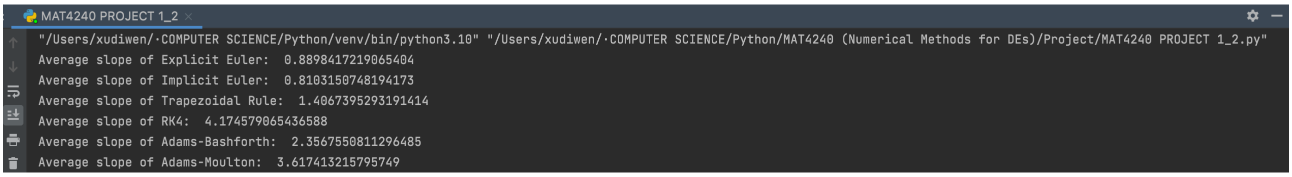
Figure 3: Time of Different Methods for N = 100

Accuracy of results: According to Figure 2, when N is large, accuracy: RK4 > Adams-Moulton > Adams-Bashforth > Trapezoidal Rule > Explicit Euler = Implicit Euler.

Apparent stability: According to Figure 4, average p: RK4 > Adams-Moulton > Adams-Bashforth > Trapezoidal Rule > Explicit Euler > Implicit Euler. Due to the error during solution of equations in implicit methods and other factors, p may become smaller than expected.



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MAT4240 PROJECT 1_2
"/Users/xudiwen/·COMPUTER SCIENCE/Python/venv/bin/python3.10" "/Users/xudiwen/·COMPUTER SCIENCE/Python/MAT4240 (Numerical Methods for DEs)/Project/MAT4240 PROJECT 1_2.py"
Average slope of Explicit Euler:  0.8898417219065404
Average slope of Implicit Euler:  0.8103150748194173
Average slope of Trapezoidal Rule:  1.4067395293191414
Average slope of RK4:  4.174579065436588
Average slope of Adams-Bashforth:  2.3567550811296485
Average slope of Adams-Moulton:  3.617413215795749
```

Figure 4: Average Slopes of Different Methods

Sensitivity to the stopping condition: According to Figure 5, the sensitivity varies for different N. Overall, using a relatively large tolerance 10^{-2} can avoid affecting accuracy and that enables the calculation go faster for three implicit methods. When N is getting larger, Adams-Moulton method becomes more sensitive than implicit Euler's method and trapezoidal rule method.

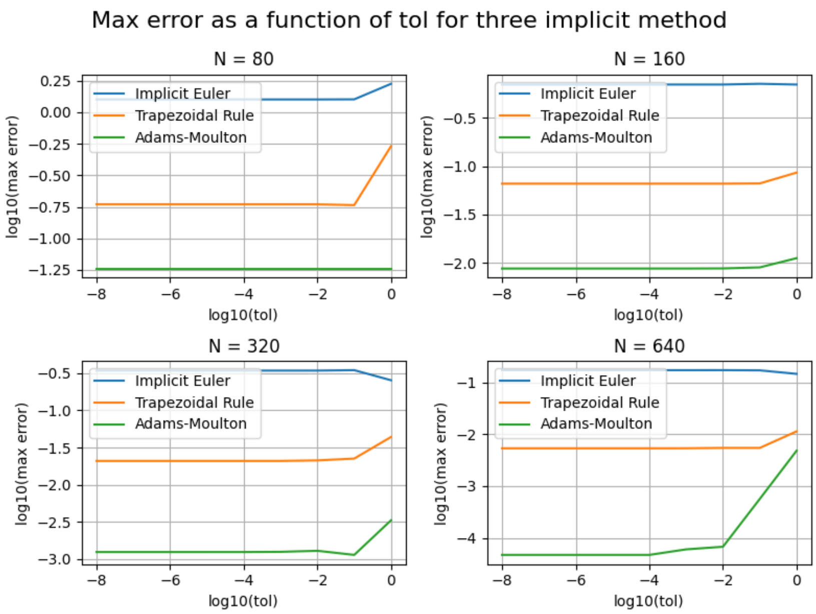

Figure 5: Sensitivity to the Stopping Condition

6 Conclusion

RK4 method is the best method among six methods to numerically solve the Bernoulli equation $\frac{dy}{dt} - \frac{y}{a+t} = -y^3$.

Because 1. not so difficult to use

2. median speed of calculation
3. give the most accurate result
4. has the most stable manner