2 Counting Techniques

2.1 Concepts

- Casework can be used to split a problem into easier problems. Whenever a problem has many different possibilities, try to break the problem into several cases.
- Sometimes, the outcomes which we don't want are easier to count that the outcomes we do want. In this case, we can subtract the undesired outcomes from the total set of outcomes. This technique is known as complementary counting.
- When an outcome can be constructed through multiple steps, we can find the total number of outcomes by multiplying the possible outputs of each step together. This is constructive counting.
- Often, problems involve restrictions that cause problems later on. It is generally best to deal with the most severe restrictions first.

2.2 Review Problems

1. How many pairs of positive integers (m, n) satisfy $m^2 + n < 30$? (Source: Introduction to Counting & Probability)
2. How many positive integers less than 100 can be written as the sum of two positive perfect cubes?
3. How many 4-letter permutations with at least one vowel can be constructed from the letters A, B, C, D, and E? (Source: Introduction to Counting & Probability)
4. A palindrome is an integer that reads the same forward and backward, such as 3663. What percent of the palindromes between 100 and 500 contain at least one 5? (Source: MathCounts 2009 State Countdown)
5. How many ways can we put 3 math books and 5 English books on a shelf if all the math books must stay together and all the English books must also stay together? (The math books are all different and so are the English books.) (Source: Introduction to Counting & Probability)

2.3 Review Problem Solutions

1. We proceed by casework. When $m = 1$, we have $n < 29$, giving us 28 values for n. Similarly,

$$m = 2, n < 26$$
$$m = 3, n < 21$$
$$m = 4, n < 14$$
$$m = 5, n < 5.$$

   This gives us an answer of $28 + 25 + 20 + 13 + 4 = \boxed{90}$

2. Note that any perfect cubes must be lower than $5^3 = 125$. We proceed by casework.
   - If our first perfect cube is 1^3, then the other can be $1^3, 2^3, 3^3$, or 4^3.
   - If our first perfect cube is 2^3, then the other can be $2^3, 3^3$, or 4^3. (Note that we avoid 1^3 to avoid counting the same pair twice.)
   - If our first perfect cube is 3^3, then the other can be 3^3 or 4^3.
   - If our first perfect cube is 4^3, then there are no other cubes which we have not already counted.

   Thus, our answer is $4 + 3 + 2 = \boxed{9}$.

3. We can use complementary counting by counting the number of 4-letter permutations that contain no vowels. Since there are 3 consonants to choose from, we can form 3^4 vowel-less permutations. Thus, our answer is $5^4 - 3^4 = \boxed{544}$.

4. Note that 5 cannot be the first or third digit, as that would create a number that is greater than 500. Thus, 5 must be the second digit. In this case, there are 4 palindromes between 100 and 500 that contain at least one 5, namely $151, 252, 353$, and 454. The total number of palindromes can be constructively counted with 4 options for the first and third digits (combined) and 10 options for the second digit: $4 \times 10 = 40$. Thus, 4 out of 40 palindromes, or $\boxed{\text{10 percent}}$, contain at least one 5.

5. There are 3! ways to arrange the math books and 5! ways to arrange the English books. Additionally, we can either put the math books to the left or to the right of the English books. Thus, there are $3! \times 5! \times 2 = \boxed{1440}$ ways to arrange the books.

2.4 Challenge Problems

1. How many squares of any size can be formed by connecting dots in a 4×4 grid?

2. A math club has 20 members and 3 officers: President, Vice President, and Treasurer. However, one member, Ali, has a huge crush on Brenda, and won't be an officer unless she is one too. Brenda is unaware of Ali's affection and doesn't care if he is an officer or not; she's perfectly happy to be an officer even if Ali isn't one. In how many ways can the club choose its officers? (Source: Introduction to Counting & Probability)

3. Derek's phone number, 336 - 7624, has the property that the three-digit prefix, 336, equals the product of the last four digits, $7 \times 6 \times 2 \times 4$. How many seven-digit phone numbers beginning with 336 have this property? (Source: MathCounts 2006 State Sprint)

4. The n members of a committee are numbered 1 through n. One of the members is designated as the "Grand Pooh-Bah." The n members sit in a row of n chairs, but no member with a number greater than the Grand Pooh-Bah may sit in the seat to the immediate right of the Grand Pooh-Bah. Suppose that the Grand Pooh-Bah is member number p, where $1 \leq p \leq n$. Find a formula, in terms of n and p, for the number of ways for the committee to sit. (Source: MathCounts)

5. Each of four students hands in a homework paper. Later the teacher hands back the graded papers randomly, one to each of the students. In how many ways can the papers be handed back such that every student receives someone else's paper? The order in which the students receive their papers is irrelevant. (Source: MathCounts 2007 National Countdown)

6. How many positive, three-digit integers contain at least one 3 as a digit but do not contain a 5 as a digit? (Source: MathCounts 2010 National Sprint)

7. Suppose we form all k-digit palindromes that consist only of 8's and 9's such that there is at least one of each digit. What is the smallest value of k such that there are at least 2004 numbers in the list? (Source: ARML)

2.5 Challenge Problem Solutions

1. There are nine 1×1 squares, four 2×2 squares, one 3×3 square, four $\sqrt{2}\times\sqrt{2}$ squares, and two $\sqrt{5}\times\sqrt{5}$ squares, for a total of $9+4+1+4+2=\boxed{20}$ squares.

2. There are $20\times19\times18$ ways to choose the officers without restrictions.

   The cases that we have to exclude are those where Ali is an officer but Brenda is not. We can count the number of these cases through constructive counting.

   There are 3 choices for the office that Ali will hold.

   There are 18 choices for the first remaining office (since Alie is already chosen and we're not allowed to choose Brenda).

   There are 17 choices for the last remaining office.

   So there are $3\times18\times17$ total choices for the 3 officers, provided that Ali must be one of the three and Brenda cannot be on of the three.

   To answer the original question, we subtract our excluded cases from the total number of permutations:
   $$(20\times19\times18)-(3\times18\times17)=\boxed{5922}.$$

3. We begin by factoring 336. $336=2^4\cdot3\cdot7$. Because we are looking for phone numbers, we want four single digits that will multiply to equal 336. Notice that 7 cannot be multiplied by anything, because $7\cdot2$ is 14, which is already two digits. So, one of our digits is necessarily 7. 3 can be multiplied by at most 2, and the highest power of 2 that we can have is $2^3=8$. Using these observations, it is fairly simple to come up with the following list of groups of digits whose product is 336 :
   $$1,6,7,8$$
   $$2,4,6,7$$
   $$2,3,7,8$$
   $$3,4,4,7$$
   For the first three groups, there are $4!=24$ possible rearrangements of the digits. For the last group, 4 is repeated twice, so we must divide by 2 to avoid overcounting, so there are $\frac{4!}{2}=12$ possible rearrangements of the digits. Thus, there are $3\cdot24+12=\boxed{84}$ possible phone numbers that can be constructed to have this property.

4. First, we choose the position that the Grand Pooh-Bah sits in. There are two cases:

   Case 1: Person p sits at the far right. There is no further restriction, and the other $n-1$ members can sit in $(n-1)!$ ways.

   Case 2: Person p sits anywhere other than the far right. There are $n-1$ choices for where person p sits, and $p-1$ choices for the person who sits to her immediate right (since it must be one of $1,2,3,\dots,p-1$). Then the remaining $n-2$ people can sit in the remaining seats in $(n-2)!$ ways.

   So the total number of seatings is $(n-1)!+(n-1)(p-1)(n-2)!$, which simplifies to $\boxed{p(n-1)!}$.

5. The first student can be handed any of 3 papers. Suppose the name of the person whose paper is given to that first student is "Bob". Then Bob can be handed any of the remaining 3 papers. Now the remaining students can each only be handed one paper, so our answer is $3 \cdot 3 \cdot 1 \cdot 1 = \boxed{9}$.

6. Let us consider the number of three-digit integers that do not contain 3 and 5 as digits; let this set be S. For any such number, there would be 7 possible choices for the hundreds digit (excluding $0, 3$, and 5), and 8 possible choices for each of the tens and ones digits. Thus, there are $7 \cdot 8 \cdot 8 = 448$ three-digit integers without a 3 or 5.

   Now, we count the number of three-digit integers that just do not contain a 5 as a digit; let this set be T. There would be 8 possible choices for the hundreds digit, and 9 for each of the others, giving $8 \cdot 9 \cdot 9 = 648$. By the complementary principle, the set of three-digit integers with at least one 3 and no 5s is the number of integers in T but not S. There are $648 - 448 = \boxed{200}$ such numbers.

7. If k is even, then the first $\frac{k}{2}$ digits determine the palindrome. There are two choices for each digit, so there are $2^{\frac{k}{2}}$ palindromes of length k.

   If k is odd, then the first $\frac{k+1}{2}$ digits determine the palindrome. Again, there are two choices for each digit, so there are $2^{\frac{k+1}{2}}$ palindromes of length k. Note that this is the same as the number of palindromes of length $k + 1$.

   But, we must also exclude the 2 palindromes that don't have at least one 8 and at least one 9. So we are looking for the smallest odd value of k such that $2^{\frac{k+1}{2}} - 2 \geq 2004$.

   We know that $2^{10} - 2 = 1022$ and $2^{11} - 2 = 2046$, so we must have $\frac{k+1}{2} \geq 11$, and the smallest such k is $\boxed{21}$.