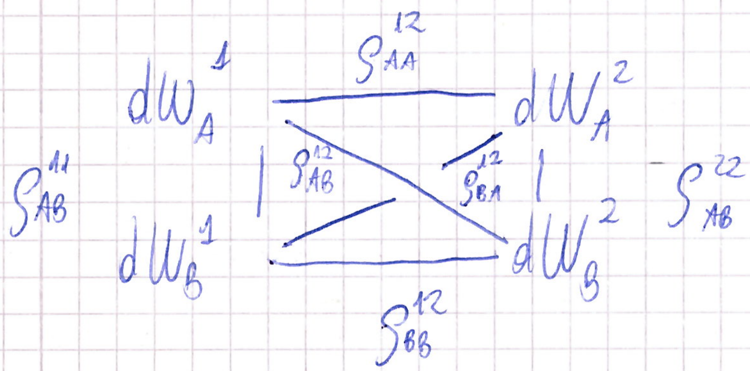
$$
\begin{cases}
d\log S_A = \left[r_A + \mu_A^0 + \mu_A^1 V_A\right]dt + \sqrt{V_A}\,dW_A^1 + \nu_A\,dN_A \\
dV_A = \left[\alpha_A - \beta_A V_A\right]dt + \sigma_A\sqrt{V_A}\,dW_A^2 \\
d\log S_B = \left[r_B + \mu_B^0 + \mu_B^1 V_B\right]dt + \sqrt{V_B}\,dW_B^1 + \nu_B\,dN_B \\
dV_B = \left[\alpha_B - \beta_B V_B\right]dt + \sigma_B\sqrt{V_B}\,dW_B^2
\end{cases}
$$



$s_A \equiv \log S_A$

$s_B \equiv \log S_B$

Let us assume that there is an exponentially affine form for the characteristic function.
It will be only a rough approximation

$$\phi(u_A, u_B, t, T) = E_t^Q\left[e^{u_A s_A + u_B s_B}\right] =$$

$$= e^{A(t,T) + \sum_{j=1}^{3} B_i^A(t,T)x_i^A + \sum_{j=1}^{3} B_i^B(t,T)x_i^B + C_A(t,T)V_A + C_B(t,T)V_B + u_A s_A + u_B s_B}$$

We can also consider the joint characteristic function for all the stochastic variables

$$\phi(\vec{u}, t, T) = E_t^Q\left[e^{\sum_{i=1}^{3}\omega_i^A x_i^A + \sum_{i=1}^{3}\omega_i^B x_i^B + \psi_A V_A + \psi_B V_B + \xi_A N_A + \xi_B N_B + u_A s_A + u_B s_B}\right] =$$

$$= e^{A(t,T) + \sum_{i=1}^{3} B_i^A x_i^A + \sum_{i=1}^{3} B_i^B x_i^B + C_A V_A + C_B V_B + \xi_A N_A + \xi_B N_B + u_A s_A + u_B s_B}$$

②

$$d\phi = \frac{\partial \phi}{\partial t}dt + \sum_{i=1}^{3}\frac{\partial \phi}{\partial x_i^A}dx_i^A + \sum_{i=1}^{3}\frac{\partial \phi}{\partial x_i^B}dx_i^B + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2 \phi}{\partial {x_i^A}^2}(dx_i^A)^2 + \frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2 \phi}{\partial {x_i^B}^2}(dx_i^B)^2 +$$

$$+ \sum_{i=1}^{3}\frac{\partial^2 \phi}{\partial x_i^A \partial x_i^B}(dx_i^A)(dx_i^B) \longleftarrow \text{if } A \neq B$$ (indices in two economies, pairwise correlated)

$$+ \frac{\partial \phi}{\partial V_A}dV_A + \frac{1}{2}\frac{\partial \phi}{\partial V_B}dV_B + \frac{1}{2}\frac{\partial^2 \phi}{\partial V_A^2}(dV_A)^2 + \frac{1}{2}\frac{\partial^2 \phi}{(\partial V_B)^2}(dV_B)^2 + \frac{\partial^2 \phi}{\partial V_A \partial V_B}dV_A\, dV_B$$

$$+ \frac{\partial \phi}{\partial s_A}ds_A + \frac{1}{2}\frac{\partial^2 \phi}{\partial s_A^2}(ds_A)^2 + \frac{\partial \phi}{\partial s_B}ds_B + \frac{1}{2}\frac{\partial^2 \phi}{\partial s_B^2}(ds_B)^2 + \frac{\partial^2 \phi}{\partial s_A \partial s_B}ds_A\, ds_B +$$

$$+ \frac{\partial^2 \phi}{\partial s_A \partial V_A}ds_A\, dV_A + \frac{\partial^2 \phi}{\partial s_B \partial V_B}ds_B\, dV_B + \frac{\partial^2 \phi}{\partial s_A \partial V_B}ds_A\, dV_B + \frac{\partial^2 \phi}{\partial s_B \partial V_A}ds_B\, dV_A$$

Jump $\rightarrow + \phi\left(\underset{s_A+v_A}{N_A+1}\right) - \phi(N_A) + \phi\left(\underset{s_B+v_B}{N_B+1}\right) - \phi(N_B)$

disregard joint events

$$\Delta F = e^{\sqcup + u_A(s_A+v_A) + g_A(N_t^A+1)} - e^{\sqcup + u_A s_A + g_A N_t^A}$$

$$= e^{\sqcup + u_A s_A + g_A N_A} \cdot \left(e^{u_A v_A + g_A} - 1\right)$$

$$= F\left(e^{u_A v_A + g_A} - 1\right)$$

$$\Rightarrow -\left(\mathring{A} + \sum_{j=1}^{3} \mathring{B}_i^A a_i^A + \sum_{i=1}^{3} \mathring{B}_i^B a_i^B + \mathring{C}_A V_A + \mathring{C}_B V_B\right) dt +$$

$$+ \sum_{i=1}^{3} B_i^A \left[\theta_i^A - k_i^A a_i^A\right] dt + \sum_{i=1}^{3} B_i^B \left(\theta_i^B - k_i^B a_i^B\right) dt +$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \left(B_i^A\right)^2 \sigma_i^{A2} a_i^A \, dt + \frac{1}{2} \sum_{i=1}^{3} \left(B_i^B\right)^2 \sigma_i^{B2} a_i^B \, dt + \underbrace{\sum_{i=1}^{3} B_i^A B_i^B \rho_{ii} \sigma_i^A \sigma_i^B \sqrt{a_i^A a_i^B} \, dt}$$

disregard

consider same economy

$$+ C_A \left(\alpha^A - \beta^A V_A\right) dt + C_B \left(\alpha^B - \beta^B V_B\right) dt +$$

$$+ \frac{1}{2} C_A^2 \sigma_A^2 V_A dt + \frac{1}{2} C_B^2 \sigma_B^2 V_B dt + C_A C_B \sigma_A \sigma_B \rho_{AB}^{22} \sqrt{V_A} \sqrt{V_B} \, dt$$

$$+ u_A \left(\eta_A + \mu_A^0 + \mu_A^1 V_A\right) dt + \frac{1}{2} u_A^2 V_A dt + u_B \left(\eta_B + \mu_B^0 + \mu_B^1 V_B\right) dt + \frac{1}{2} u_B^2 V_B \, dt$$

$$+ u_A u_B \rho_{AB}^{11} \sqrt{V_A} \sqrt{V_B} \, dt +$$

$$+ u_A C_A \rho_{AA}^{12} V_A \, dt + u_B C_B \rho_{BB}^{12} V_B \, dt + u_A C_B \sqrt{V_A V_B} \rho_{AB}^{12} dt + u_B C_A \sqrt{V_A V_B} \rho_{BA}^{12} dt$$

$$+ E\left[e^{u_A V_A + g_A} - 1\right] \left(\lambda_0^A + \lambda_1^A V_A\right) dt + E\left[e^{u_A V_B + g_B} - 1\right] \left(\lambda_0^B + \lambda_1^B V_B\right) dt$$

As a first order approximation consider all the Γ terms constant, at a fixed horizon (for instance steady state)

$$\Rightarrow \quad -\Big(\mathring{A} + \sum_{j=1}^{3} \mathring{B}_j^A \alpha_j^A + \sum_{j=1}^{3} \mathring{B}_j^B \alpha_j^B + \mathring{C}_A V_A + \mathring{C}_B V_B\Big) dt +$$

$$+ \sum_{i=1}^{3} B_i^A \big(\vartheta_i^A - k_i^A \underset{\wedge}{\alpha_i^A}\big) dt + \sum_{i=1}^{3} B_i^B \big(\vartheta_i^B - k_i^B \underset{\hat{\wedge}}{\alpha_i^B}\big) dt +$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \big(B_i^A\big)^2 \big(\sigma_i^A\big)^2 \underset{\wedge}{\alpha_i^A}\, dt + \frac{1}{2} \sum \big(B_i^B\big)^2 \big(\sigma_i^B\big)^2 \underset{\hat{\wedge}}{\alpha_i^B}\, dt +$$

$$+ C_A \big(\alpha^A - \beta^A \underset{\sim}{V_A}\big) dt + C_B \big(\alpha^B - \beta^B \underset{\approx}{V_B}\big) dt +$$

$$+ \frac{1}{2} C_A^2 \sigma_A^2 \underset{\sim}{V_A}\, dt + \frac{1}{2} C_B^2 \sigma_B^2 \underset{\approx}{V_B}\, dt + C_A C_B \sigma_A \sigma_B \rho_{AB}^{22}\ \Theta(h) +$$

$$+ u_A \Big(\sum \underset{\wedge}{\alpha_i^A} + \mu_A^0 + \mu_A^1 \underset{\sim}{V_A}\Big) dt + u_B \Big(\sum \underset{\hat{\wedge}}{\alpha_i^B} + \mu_B^0 + \mu_B^1 \underset{\approx}{V_B}\Big) dt +$$

$$+ \frac{1}{2} u_A^2 \underset{\sim}{V_A}\, dt + \frac{1}{2} u_B^2 \underset{\approx}{V_B}\, dt + u_A u_B \rho_{AB}^{11}\ \Theta(h)\, dt$$

$$+ u_A C_A \rho_{AA}^{12} \underset{\sim}{V_A}\, dt + u_B C_B \rho_{BB}^{12} \underset{\approx}{V_B}\, dt +$$

$$+ u_A C_B\ \Theta(h) \rho_{AB}^{12}\, dt + u_B C_A\ \Theta(h) \rho_{BA}^{12}\, dt +$$

$$+ E J_A \big(\lambda_0^A + \lambda_1^A \underset{\sim}{V_A}\big) dt + E J_B \big(\lambda_0^B + \lambda_1^B \underset{\approx}{V_B}\big) dt \quad = 0$$

$\Rightarrow$

$$
\left\{
\begin{aligned}
\mathring{B}_i^A &= \frac{1}{2}\left(B_i^A \sigma_i^A\right)^2 - K_i B_i^A + u_A \\
\mathring{B}_i^B &= \frac{1}{2}\left(B_i^B \sigma_i^B\right)^2 - K_i B_i^B + u_B \\
\mathring{C}_A &= \frac{1}{2} C_A^2 \sigma_A^2 - C_A\left(\beta_A - u_A \rho_{AA}^{12}\right) + u_A \mu_A^1 + \frac{1}{2} u_A^2 + E J_A \lambda_1^A \\
\mathring{C}_B &= \frac{1}{2} C_B^2 \sigma_B^2 - C_B\left(\beta_B - u_B \rho_{BB}^{12}\right) + u_B \mu_B^1 + \frac{1}{2} u_B^2 + E J_B \lambda_1^B \\
\mathring{A} &= \sum \theta_i^A B_i^A + \sum \theta_i^B B_i^B + \alpha_A C_A + \alpha_B C_B + C_A C_B \sigma_A \sigma_B \rho_{AB}^{22} \theta(h) + \\
&\quad + u_A \mu_A^0 + u_B \mu_B^0 + u_A u_B \rho_{AB}^{11} \theta(h) + \\
&\quad + u_A C_B \rho_{AB}^{12} \theta(h) + u_B C_A \rho_{BA}^{12} \theta(h) + \\
&\quad + E J_A \lambda_0^A + E J_B \lambda_0^B
\end{aligned}
\right.
$$