1. The Hilbert Scheme. Questions in algebraic geometry of the form:

"How many objects with property X are there?"

are interpreted in terms of intersection theory on moduli spaces. Algebraic geometers have been doing this implicitly for more than 150 years, but the tools to make the arguments rigorous are of a much more recent vintage. In the early 1960's, Grothendieck introduced a category-theoretic framework for considering moduli problems and constructed the Hilbert scheme, which is certainly the most fundamental of the moduli spaces. This is a non-trivial construction, and setting up the background in algebraic geometry will now occupy us for some time.

Schemes: Unlike algebraic varieties, schemes are constructed in such a way that closed subschemes "remember their equations." This leads to many useful properties, which explains the near-universal adoption of schemes throughout algebraic geometry. Here is a brief introduction.

A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X together with a sheaf of rings $\mathcal{O}_X$ with the property that all the stalks $\mathcal{O}_{X,x}$ are local rings (with maximal ideal written as m_x). A morphism of locally ringed spaces is also a pair:

$$f : X \to Y \quad \text{and} \quad f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

such that f is continuous and $f^{\#}$ induces a local homomorphism on stalks, i.e. $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ satisfies ${f^{\#}}^{-1}(m_x) = m_{f(x)}$.

Example. A differentiable (resp. complex) manifold with its sheaf of $\mathcal{C}^\infty$ (resp. holomorphic) functions is a locally ringed space, since the rings of germs of such functions are local. A differentiable (resp. holomorphic) map determines a morphism with $f^{\#} = f^*$, the pull-back map on functions.

Affine schemes are the analogues of open subsets of $\mathbf{R}^n$. Let A be a commutative ring with 1. Then:

Spec$(A) = \{$prime ideals in $A\}$ with the Zariski topology. In particular,

$U_f = \{$prime ideals not containing $f\}$ are a basis for the topology.

$\widetilde{A}$ is the "natural" sheaf of rings on Spec(A) satisfying $\widetilde{A}(U_f) \cong A_f$ and $\widetilde{A}_x \cong A_{\mathcal{P}_x}$ (where $\mathcal{P}_x \subset A$ is the prime ideal corresponding to $x \in \text{Spec}(A)$).

Given a ring homomorphism $\phi : B \to A$, the pair:

$f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ defined by $\mathcal{P}_{f(x)} = \phi^{-1}(\mathcal{P}_x)$ which satisfies $f^{-1}(U_g) = U_{\phi(g)}$ for all basic open sets, and

$f^{\#} : \widetilde{B} \to f_*\widetilde{A}$ deduced from the localized maps $\phi_g : B_g \to A_{\phi(g)}$

is a morphism of locally ringed spaces and the contravariant functor:

$$A \mapsto (\mathrm{Spec}(A), \widetilde{A}); \quad \phi \mapsto (f, f^{\#})$$

is fully faithful. Moreover, within the category of locally ringed spaces,

$$(U_f, \widetilde{A}|_{U_f}) \text{ is isomorphic to } (\mathrm{Spec}(A_f), \widetilde{A}_f)$$

so the basic open sets "are" themselves affine schemes.

A **scheme** is a locally ringed space that is locally affine (e.g. an open subset of an affine scheme). Fix a scheme S. Then an S**-scheme** is a scheme X together with a morphism $\pi : X \to S$. A morphism in the category of S-schemes is a map $f : X \to X'$ commuting with the given maps to S:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

It is of great importance that **products exist** in the category of S-schemes.

Projective space over S, denoted $\mathbf{P}^n_S$, is patched from projective spaces over the open affines $\mathrm{Spec}(A) \subset S$ as follows.

$\mathbf{P}^n_A = \{$homogeneous prime ideals in $A[x_0, ..., x_n]$ not containing $\langle x_0, ..., x_n\rangle\}$ with the Zariski topology. In particular, the

$U_f = \{x \in \mathbf{P}^n_A \mid f \notin \mathcal{P}_x\}$ for homogeneous f are a basis for the topology.

$\mathcal{O}_{\mathbf{P}^n_A}$ is the "natural" sheaf of rings on $\mathbf{P}^n_A$ with the property that for all homogeneous $f \in A[x_0, ..., x_n]$ of positive degree, $\mathcal{O}_{\mathbf{P}^n_A}(U_f) \cong A[x_0, ..., x_n]_{(f)}$ (the degree zero part of $A[x_0, ..., x_n]_f$) and for all $x \in \mathbf{P}^n_A$, the stalks satisfy $\mathcal{O}_{\mathbf{P}^n_A,x} \cong A[x_0, ..., x_n]_{(\mathcal{P}_x)}$ (defined similarly).

The fact that $\mathbf{P}^n_A$ (hence $\mathbf{P}^n_S$) is a scheme follows from isomorphisms $(U_f, \mathcal{O}_{\mathbf{P}^n_A}|_{U_f}) \cong (\mathrm{Spec}(A[x_0, ..., x_n]_{(f)}), \widetilde{A[x_0, ..., x_n]_{(f)}})$ for positive degree f's. $\mathbf{P}^n_S$ is an S-scheme via the canonical maps $\mathbf{P}^n_A \to \mathrm{Spec}(A)$.

A morphism of schemes $f : X \to Y$ is a **closed immersion** if f induces a homeomorphism onto a closed subset $Z \subset Y$ (with the induced topology) and $f^\#$ is surjective. **Closed subschemes** of Y are equivalence classes of closed immersions modulo isomorphisms. The closed subschemes of $\mathrm{Spec}(A)$ correspond to ideals $I \subset A$, and it is in this sense that closed subschemes remember their equations. An S-scheme X is **projective** over S if it is isomorphic to a closed subscheme of some $\mathbf{P}^n_S$.

Suppose $S = \mathrm{Spec}(k)$ where k is a field. An S-scheme (or "k-scheme") X is a **variety** if:

(a) X is covered by finitely many affine open sets $\mathrm{Spec}(A_i)$ with the property that each A_i is a finitely generated k-algebra.

(b) X is an irreducible topological space.

(c) $\mathcal{O}_X$ contains no nilpotent elements (i.e. X is reduced).

(d) $(X, \mathcal{O}_X)$ is "separated" (the analogue of Hausdorff), i.e. the diagonal

$$\Delta : X \to X \times X \text{ is a closed immersion}$$

It is a quasi-projective variety if, moreover, X is isomorphic to an open subset of a projective k-scheme (in which case conditions (a) and (d) are redundant).

If we limit ourselves to the case where k is algebraically closed, then Hilbert's Nullstellensatz can be used to link k-varieties to "classical" algebraic geometry via the equivalence of categories:

$$k\text{-varieties (as schemes)} \;\leftrightarrow\; \text{classical abstract varieties over } k$$

where a scheme X is sent to the subset of closed points and the sheaf $\mathcal{O}_X$ is sent to the sheaf of regular functions.

The **Zariski tangent space** T_x to a scheme X at a point $x \in X$ is the vector space $(m_x/m_x^2)^*$ over the residue field $k(x) = \mathcal{O}_{X,x}/m_x$. A variety X over an algebraically closed field k is **non-singular of dimension** d at x if $d = \dim(T_x) = \dim(\mathcal{O}_{X,x})$. X is nonsingular if it is nonsingular at all points.

If we further limit ourselves to the case $k = \mathbf{C}$, the field of complex numbers, then Chow's lemma (generalized by Serre in GAGA) links nonsingular projective varieties to complex manifolds via an equivalence of categories:

$$\text{nonsingular projective } \mathbf{C}\text{-varieties} \;\leftrightarrow\; \text{complex projective manifolds}$$

This completes (for now) our brief introduction to schemes.

Fine Moduli Spaces: If $\mathcal{C}$ is any category, and *Sets* is the category of sets, then an object X in $\mathcal{C}$ determines the contravariant "functor of points" h_X from $\mathcal{C}$ to *Sets* as follows:

(i) For any object S in $\mathcal{C}$,

$$h_X(S) = \{\text{morphisms } \phi : S \to X\}$$

(ii) For any morphism $f : S \to T$ in $\mathcal{C}$,

$$h_X(f) = f^* : h_X(T) \to h_X(S)$$

That is, each $\psi : T \to X$ is taken to $f^*\psi := \psi \circ f : S \to X$.

Ex 1.1. Check that h_X is indeed a functor.

We need one more level of abstraction. The (contravariant) functors from one category to another form a category. Let's denote by $\mathcal{C}/Sets$ the category of contravariant functors from $\mathcal{C}$ to *Sets*. Then the functor of points should be viewed as a functor $h : \mathcal{C} \to \mathcal{C}/Sets$ via:

$$h(X) = h_X \quad \text{and} \quad h(g : X \to Y) = g_* : h_X \to h_Y$$

That is, $g_*(\phi) = g \circ \phi$ and $(g_* \circ f^*) = f^*$, i.e. $g_*(f^*(\psi)) = f^*(g_*(\psi)) = g \circ \psi \circ f$.

Ex 1.2. (a) What are the morphisms in $\mathcal{C}/Sets$?

(b) Check that h is indeed a (covariant) functor.

Yoneda Lemma: h is fully faithful, i.e. for every pair X, Y of objects of $\mathcal{C}$, the map on morphisms is a bijection: $\{f : X \to Y\} \stackrel{h}{\leftrightarrow} \{F : h_X \to h_Y\}$

Ex 1.3. Prove this.

Corollary: If h_X is isomorphic to h_Y, then X is isomorphic to Y.

Proof: The bijection of the Lemma carries one isomorphism to the other.

Example. The "classical" geometric categories of topological spaces and manifolds have a unique (up to isomorphism) one-point object x which has the property that the morphisms $f : x \to X$ correspond to the points of X, giving rise to a natural bijection $h_X(x) = \{f : x \to X\} \leftrightarrow \{x \in X\}$ hence the terminology "functor of points."

Although the category of k-schemes resembles the categories of manifolds in many ways, the morphisms from Spec(k) to a k-scheme X only pick up the set of "k-rational" points. Even when k is algebraically closed, these only constitute the points corresponding to maximal ideals. Moreover, there are plenty of non-isomorphic one-point k-schemes, including the affine schemes Spec(A) where A is a finitely generated k-algebra which is a local Artinian ring, i.e. A is of the form:

$$k[x_1, ..., x_n]/\langle x_1, ..., x_n\rangle^m \to A \to 0$$

The "ring of dual numbers" is the simplest non-trivial example:

$$k[\epsilon] \cong k[x]/\langle x^2\rangle$$

An S-scheme X is said to **represent** a functor F from the category of S-schemes to the category of sets if F is isomorphic to h_X. In that case, X is called a **fine moduli space** for the moduli problem given by F.

We have now assembled the some of the definitions for Grothendieck's:

Theorem (Existence of the Hilbert Scheme): Fix a projective S-scheme X and a polynomial $P(n)$. Then the "Hilbert" functor $F_{X,P(n)}$ defined by:

$$F_{X,P(n)}(T) = \{\text{closed subschemes } Z \subset X \times T \text{ that are flat over } T \text{ with Hilbert polynomial } P(n)\},$$

$$F_{X,P(n)}(f) = \text{ "base extension by } f\text{"}$$

is represented by a **projective** S-scheme, denoted $H_{X,P(n)}$.

Obviously, we have some more background to fill in.

Reading: Hartshorne II.1-II.3.

2. Coherent Sheaves and Cohomology. Coherent sheaf cohomology encodes global properties of schemes. We will discuss some of the main results, particularly Serre's Theorems A and B. Serre's FAC probably remains the best reference, though it was written before the Grothendieck "revolution."

Coherent Sheaves: Quasi-coherent sheaves over a scheme "globalize" modules over a commutative ring. Locally free sheaves are the analogues of vector bundles, and invertible sheaves are the analogues of line bundles.

A sheaf $\mathcal{F}$ of abelian groups on a scheme X is a **sheaf of $\mathcal{O}_X$-modules** if each $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ in such a way that the module structures are compatible with the restriction maps.

Example. If M is a module over A, then there is a natural associated sheaf of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules, denoted by $\widetilde{M}$, such that:

$\widetilde{M}(U_f) \cong M_f$ on basic open sets and

$\widetilde{M}_x \cong M_{\mathcal{P}_x}$ for each $x \in \mathrm{Spec}(A)$.

A sheaf of $\mathcal{O}_X$-modules on an arbitrary scheme is **quasi-coherent** if it is locally of the form $\widetilde{M}$ on open subsets of the form $\mathrm{Spec}(A)$. It is **coherent** if, moreover, the modules M are each finitely generated A-modules. It is **locally free** if the stalks are each free $\mathcal{O}_{X,x}$-modules.

The assignment $M \mapsto \widetilde{M}$ with the natural map on morphisms is an equivalence of categories between A-modules and quasi-coherent sheaves over $\mathrm{Spec}(A)$. If A is Noetherian, it is an equivalence between finitely generated A-modules and coherent sheaves over $\mathrm{Spec}(A)$. Over an arbitrary scheme X, the category of quasi-coherent sheaves is closed under kernels, cokernels and images, and an extension of quasi-coherent sheaves is again quasi-coherent. The same is true for coherent sheaves if X is **Noetherian**, meaning that X is covered by finitely many Noetherian open affine subschemes.

If $i : X \to Y$ is a closed immersion, the **ideal sheaf** $\mathcal{I}_{X/Y}$ is the kernel:

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \stackrel{i^\#}{\to} i_*\mathcal{O}_X \to 0$$

and is locally of the form $\widetilde{I}$, where $I \subset A$ are the ideals corresponding to the closed subschemes $\mathrm{Spec}(A) \cap X$ of each open $\mathrm{Spec}(A) \subset Y$. This is therefore quasi-coherent, and coherent if A is Noetherian.

Conversely, if $\mathcal{I}$ is a quasi-coherent sub-sheaf of $\mathcal{O}_Y$, then there is a uniquely determined closed subscheme $X \subset Y$ such that $\mathcal{I} = \mathcal{I}_{X/Y}$.

If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. In the special case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $f_*(\widetilde{M}) = \widetilde{M}$, where the M on the right is viewed as a B-module via the associated ring homomorphism $\phi : B \to A$. It follows that the push-forward of a quasi-coherent sheaf is again quasi-coherent.

Given a sheaf $\mathcal{G}$ on Y, recall that the inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf on X associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$. If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then $f^{-1}\mathcal{G}$ is not a sheaf of $\mathcal{O}_X$-modules in any reasonable way, but $f^\#$ determines a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ and the **pull-back**:

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is rigged to be a sheaf of $\mathcal{O}_X$-modules. In the special case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $f^*(\widetilde{N}) = \widetilde{N \otimes_B A}$, and it follows that f^* commutes with tensor products (note that f_* does not!), and (quasi)-coherent sheaves pull back to (quasi)-coherent sheaves.

An S-scheme X is **separated** if $\Delta : X \to X \times_S X$ is a closed immersion. In that case, let $\mathcal{I}$ be the ideal sheaf of the diagonal. Then $\Omega_{X/S} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ is the **sheaf of relative differentials** of X over S. It is quasi-coherent by our discussion thus far, but this can be seen directly. Namely, for each $\mathrm{Spec}(B) \subset S$ and $\mathrm{Spec}(A) \subset X$ lying over $\mathrm{Spec}(B)$, the sheaf $\Omega_{X/S}$ restricts to $\widetilde{\Omega}_{A/B}$ on $\mathrm{Spec}(A)$, where $\Omega_{A/B}$ is the module of Kähler differentials for the associated map $\phi : B \to A$. If in addition S is Noetherian and X is an S-scheme **of finite type**, meaning that each $\pi^{-1}(\mathrm{Spec}(B))$ as above is covered by finitely many affines $\mathrm{Spec}(A)$ such that each A is a finitely generated B-**algebra**, then $\Omega_{X/S}$ is coherent because the $\Omega_{A/B}$ are finitely generated A-**modules** . A k-variety satisfies these assumptions, by definition, and if X is a non-singular k-variety, then $\Omega_X := \Omega_{X/\mathrm{Spec}(k)}$ is a locally free coherent sheaf of constant rank equal to the dimension of X, and vice-versa.

A morphism $f : X \to S$ is **proper** if X is separated and of finite type as an S-scheme and if in addition the projection maps $\pi_Y : X \times_S Y \to Y$ to all other S-schemes Y take closed sets to closed sets. Closed immersions and projective morphisms are proper, the latter by a theorem of Grothendieck.

A morphism $f : X \to S$ is **finite** if the inverse image of an affine open subscheme $\mathrm{Spec}(B) \subset S$ is always affine (call it $\mathrm{Spec}(A) \subset X$) and A is a finitely generated B-**module**. Finite morphisms are proper and quasi-finite, meaning that the inverse image of a point in S is always a finite set.

If $f : X \to S$ is proper and S is Noetherian, then coherent sheaves on X push forward to coherent sheaves on S (This is non-trivial to prove!) In particular, if $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X, then the space $\Gamma(X, \mathcal{F})$ of global sections is a finite-dimensional vector space over k.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a scheme X, then $\Gamma(X, \mathcal{F})$ is a $\Gamma(X, \mathcal{O}_X)$-module by definition. We say $\mathcal{F}$ is **spanned** by a collection $\{s_i\}$ of sections of $\mathcal{F}$ if the images of the s_i generate every stalk of $\mathcal{F}$. Alternatively, the s_i determine a map $\oplus \mathcal{O}_X \to \mathcal{F}$, and $\mathcal{F}$ is spanned by the sections exactly when this map of sheaves of $\mathcal{O}_X$-modules is surjective.

The **fiber** of $\mathcal{F}$ at x is the vector space:

$$\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

For example, the fiber $\Omega_X(x)$ of a proper k-scheme X at a k-rational point is the cotangent space m_x/m_x^2, so that the sheaf Ω_X can be seen as the cotangent sheaf, even if X is singular.

It follows from Nakayama's lemma that if a map $\mathcal{F} \to \mathcal{G}$ of coherent sheaves induces a surjective map $\mathcal{F}(x) \to \mathcal{G}(x)$ at a point $x \in X$, then the sheaf map is surjective on stalks at x (hence in a neighborhood of x). In particular, the fiber dimension of a coherent sheaf $\mathcal{F}$ is an upper-semi-continuous function on X and $\mathcal{F}$ is generated by sections $\{s_i\}$ if and only each fiber $\mathcal{F}(x)$ is generated by the sections.

A coherent sheaf $\mathcal{L}$ is **invertible** if it is locally free of rank one. The dual sheaf $\mathcal{L}^* := Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ inverts $\mathcal{L}$ in the sense that $\mathcal{L} \otimes \mathcal{L}^* \cong \mathcal{O}_X$ hence the terminology. Isomorphism classes of invertible sheaves form an abelian group, the **Picard group** $Pic(X)$, under tensor product.

Another Example. Suppose M is a graded module over $R := A[x_0, ..., x_n]$. There is a natural sheaf $\widehat{M}$ of $\mathcal{O}_{\mathbf{P}^n_A}$-modules on $\mathbf{P}^n_A$ with the property that $\widehat{M}|_{U_f} \cong \widetilde{M_{(f)}}$ for each "basic" open affine. This exhibits $\widehat{M}$ is a quasi-coherent sheaf, which is coherent if A is Noetherian and M is finitely generated.

In particular, if $M = \oplus M_e$ is a graded R-module, let $M(d)$ denote the "shifted" graded R-module with $M(d)_e = M_{d+e}$. In particular, the sheaves $\mathcal{O}_{\mathbf{P}^n_A}(d) := \widehat{R(d)}$ are invertible, and satisfy $\widehat{M(d)} \cong \widehat{M} \otimes_{\mathcal{O}_{\mathbf{P}^n_A}} \mathcal{O}_{\mathbf{P}^n_A}(d)$ for all graded modules M.

This time, the assignment $M \mapsto \widehat{M}$ does not give an equivalence between the categories of graded R-modules and of quasi-coherent sheaves on $\mathbf{P}^n_A$ because graded modules which are isomorphic in all sufficiently high degrees give rise to the same sheaves. This is the only ambiguity, however, and a sort of inverse is given by the twisted global sections:

$$\widehat{M} \mapsto \Gamma_* \widehat{M} := \bigoplus_{d \in \mathbf{Z}} \Gamma(\mathbf{P}^n_A, \widehat{M}(d))$$

If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathbf{P}^n_A$, then $\mathcal{F} \mapsto \Gamma_* \mathcal{F} \mapsto \widehat{\Gamma_* \mathcal{F}}$ yields a quasi-coherent sheaf which is isomorphic to $\mathcal{F}$. It follows for example that closed subschemes of $\mathbf{P}^n_A$ correspond to homogeneous ideals in R.

If S is a scheme, then the invertible sheaves $\mathcal{O}_{\mathbf{P}^n_A}(d)$ over open affines $\mathrm{Spec}(A) \subset S$ glue to an invertible sheaf $\mathcal{O}_{\mathbf{P}^n_S}(d)$. An invertible sheaf $\mathcal{L}$ on an S-scheme X is **ample** (or S-ample) if there is a closed immersion $f : X \to \mathbf{P}^n_S$ (so X is projective, in particular) and a positive integer d such that $\mathcal{L}^{\otimes d} \cong f^* \mathcal{O}_{\mathbf{P}^n_S}(1)$. $\mathcal{L}$ is **very ample** if d may be chosen to be 1.

Any S-morphism $f : X \to \mathbf{P}^n_S$ determines $n+1$ global sections $s_0, ..., s_n$ generating the invertible sheaf $\mathcal{L} := f^* \mathcal{O}_{\mathbf{P}^n_S}(1)$ on X. These sections are pulled back from the "coordinate functions" $x_0, ..., x_n$ on $\mathbf{P}^n_S$. Conversely, an invertible sheaf $\mathcal{L}$ on X together with $n+1$ generating sections $s_0, ..., s_n$ uniquely determine an S-morphism to $\mathbf{P}^n_S$.

Suppose X is a proper scheme over an algebraically closed field k with invertible sheaf $\mathcal{L}$ and generating sections $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ as above. Let $V \subset \Gamma(X, \mathcal{L})$ be the **linear series** (i.e. the subspace) generated by the s_i. Then the associated morphism $f : X \to \mathbf{P}^n_k$ is a closed immersion if V **separates points**, meaning that for each pair $p, q \in X$ of closed points there is an $s \in V$ such that $s \in m_p \mathcal{L}$ but $s \notin m_q \mathcal{L}$, and V **separates tangent directions**, meaning that for each closed point $p \in X$, the vector space $m_p \mathcal{L} / m_p^2 \mathcal{L}$ is spanned by the sections $s \in V$ that lie in $m_p \mathcal{L}$. This is proved using Nakayama's lemma again, and applies even to non-reduced schemes.

Serre's Theorem A: If X is a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ denote a very ample invertible sheaf on X. Then there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ on X so that $\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ is generated by its global sections whenever $d \geq d_{\mathcal{F}}$.

Very ampleness of a line bundle is difficult to detect without knowing the space of global sections. Ampleness, however, is a much easier notion to work with, as evidenced by the following refinement of Serre's Theorem A.

Criterion for Projectivity: Suppose A is Noetherian and X is a proper A-scheme with invertible sheaf $\mathcal{L}$. Then $\mathcal{L}$ is ample (hence X is projective) **if and only if** there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ so that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ is generated by its global sections whenever $d \geq d_{\mathcal{F}}$.

Cohomology: Global properties of coherent sheaves are encoded in their cohomology spaces. Some important vanishing theorems are listed in this brief introduction to the most important tool in algebraic geometry.

The **global section functor** Γ is left exact, meaning that if:

$$0 \to \mathcal{F}' \xrightarrow{f'} \mathcal{F} \xrightarrow{f''} \mathcal{F}'' \to 0$$

is a short exact sequence of sheaves of abelian groups, then:

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$$

is exact, but the last map may not be surjective. The cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are the "right derived functors" of the global section functor. In particular, $\mathrm{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ and given a short exact sequence as above, there are (functorial) connecting homomorphisms:

$$\delta^i : \mathrm{H}^i(X, \mathcal{F}'') \to \mathrm{H}^{i+1}(X, \mathcal{F}')$$

so that the following sequence is long-exact:

$$\ldots \xrightarrow{\mathrm{H}^i(f')} \mathrm{H}^i(X, \mathcal{F}) \xrightarrow{\mathrm{H}^i(f'')} \mathrm{H}^i(X, \mathcal{F}'') \xrightarrow{\delta^i} \mathrm{H}^{i+1}(X, \mathcal{F}') \xrightarrow{\mathrm{H}^{i+1}(f')} \ldots$$

A Noetherian scheme is a Noetherian space, meaning that it satisfies the descending chain condition on closed subsets. The maximal length of such a chain of irreducible closed subsets is the (Noetherian) dimension of the space. Hence the relevance of the following vanishing theorem of Grothendieck.

Vanishing Theorem: If X is a Noetherian space and $i > \dim(X)$, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ of abelian groups on X.

Another vanishing theorem of Serre gives the following:

Criterion for Affineness: If X is a Noetherian scheme, then the following are equivalent:

(i) X is affine.

(ii) $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > 0$.

(iii) $\mathrm{H}^1(X, \mathcal{I}) = 0$ for all coherent sheaves (of ideals) $\mathcal{I} \subset \mathcal{O}_X$.

On a Noetherian separated scheme X, an intersection of open affines is again affine. Using this fact, Serre's vanishing theorem allows us to conclude that the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on such an X can be computed as the Cech cohomology with respect to **any** open affine cover. For example, one can use the standard cover of $\mathbf{P}^n_A$ to check:

(a) $A[x_0, ..., x_n] \cong \Gamma_* \mathcal{O}_{\mathbf{P}^n_A} = \oplus \mathrm{H}^0(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d))$.

(b) $\mathrm{H}^i(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d)) = 0$ for all d and all $0 < i < n$.

(c) $\mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-n-1)) \cong A$.

(d) For each d, there is a perfect pairing of free A-modules:

$$\mathrm{H}^0(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(d)) \times \mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-d-n-1)) \to \mathrm{H}^n(\mathbf{P}^n_A, \mathcal{O}_{\mathbf{P}^n_A}(-n-1)) \cong A$$

Serre's Theorem B: If X is a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Then for each coherent sheaf $\mathcal{F}$, the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are all finitely generated A-modules, and there is a $d_{\mathcal{F}}$ such that $\mathrm{H}^i(X, \mathcal{F}(d)) = 0$ for each $d \geq d_{\mathcal{F}}$ and $i > 0$.

Once again, this is refined by replacing "very ample" with "ample."

Criterion for Projectivity (2): Suppose A is Noetherian and X is a proper A-scheme with invertible sheaf $\mathcal{L}$. Then $\mathcal{L}$ is ample **if and only if** there is a $d_{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$ so that $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ whenever $i > 0$ and $d \geq d_{\mathcal{F}}$.

Remark: Note that the criteria are independent of the Noetherian ring A. Thus ampleness, unlike very ampleness, only depends upon X, and not the map to $\mathrm{Spec}(A)$. I.e. ampleness is an "absolute" property of X.

Let X be a projective scheme over a field k. Then the **Hilbert function**:

$$d \mapsto \dim_k \Gamma(X, \mathcal{F}(d))$$

is polynomial in d for sufficiently large d. This polynomial $P_{\mathcal{F}}(d)$ is the **Hilbert polynomial** of $\mathcal{F}$. We can see the polynomial growth of the Hilbert function in one of two ways.

The **projection formula** states that if $f : X \to Y$ is a morphism of locally ringed spaces and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_Y$-modules, then:

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$$

If we apply this to the closed immersion $f : X \to \mathbf{P}^n_k$ with $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n_k}(d)$, we get:

$$\dim_k \Gamma(X, \mathcal{F}(d)) = \dim_k \Gamma(\mathbf{P}^n_k, f_*\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}^n_k}(d))$$

It follows from Serre's theorem A that the graded module $M := \Gamma_* \widehat{f_*} \mathcal{F}$ over $k[x_0, ..., x_n]$ is finitely generated, therefore the dimensions of the M_d grow as a polynomial in d for sufficiently large d. (See e.g. Atiyah-Macdonald)

Alternatively, we may define the **Euler characteristic** of $\mathcal{F}(d)$ by:

$$\chi(X, \mathcal{F}(d)) = \sum_{i=0}^{\dim(X)} (-1)^i \dim_k \mathrm{H}^i(X, \mathcal{F}(d))$$

Then $\chi(X, \mathcal{F}(d)) = \dim_k \Gamma(X, \mathcal{F}(d))$ for all sufficiently large d by Serre's theorem B. Thus it suffices (and gives us much more information!) to prove that the Euler characteristic $\chi(X, \mathcal{F}(d))$ is polynomial in d for all d, hence equal to the Hilbert polynomial. This follows from the fact that the Euler characteristic is additive on exact sequences. If $X_1, ..., X_m \subset X$ are the irreducible components of maximal dimension, then multiplication by any section $s \in \mathcal{O}_X(1)$ that is nonzero on each of these components gives rise to an exact sequence:

$$0 \to i_*\mathcal{N} \to \mathcal{F} \xrightarrow{\cdot s} \mathcal{F}(1) \to i_*\mathcal{Q} \to 0$$

where $\mathcal{N}$ and $\mathcal{Q}$ are coherent sheaves on the zero scheme $D \subset X$ of s. Since $\dim(D) = \dim(X) - 1$, and since tensoring by $\mathcal{O}_X(d)$ is exact, the polynomiality follows by induction on dimension, the projection formula, and the observation that cohomology commutes with closed immersions. We also see in this way that the degree of the Hilbert polynomial is bounded above by the dimension of X.

3. Castelnuovo-Mumford Regularity and Spectral Sequences. This section is devoted to the following refinement of Serre's theorems A and B:

Uniform Vanishing Theorem: Fix the following data:

an infinite field k and a projective k-scheme X,

a locally free coherent sheaf $\mathcal{E}$ on X and a polynomial $P(d)$.

Then there is a d_0 depending only on $X, \mathcal{E}$ and $P(d)$ such that for all coherent subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$ with Hilbert polynomial $\chi(X, \mathcal{F}(d)) = P(d)$ and for all $d \geq d_0$, $\mathcal{F}(d)$ is generated by its sections and $\mathrm{H}^i(X, \mathcal{F}(d)) = 0$ for all $i > 0$.

As we will see, this is critical to the construction of the Hilbert scheme. Mumford's proof of this theorem uses the notion of:

Castelnuovo Regularity: A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_k$ is m-regular if:

$$\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0$$

Examples: (a) $\mathcal{O}_{\mathbf{P}^n_k}$ is m-regular for $m \geq 0$, but not -1-regular.

(b) Suppose L is an invertible sheaf on a curve C such that:

(i) L is generated by it global sections,

(ii) $\mathrm{H}^1(C, L) = 0$, and

(iii) the multiplication map $\mathrm{Sym}^2\mathrm{H}^0(C, L) \to \mathrm{H}^0(C, L^2)$ is surjective.

Let $f : C \hookrightarrow \mathbf{P}^n_k$ be the closed immersion given by the complete linear series $\mathrm{H}^0(C, L)$. Then it follows from the exact sequences:

$$0 \to \mathcal{I}_C(d) \to \mathcal{O}_{\mathbf{P}^n_k}(d) \to f_* L^{\otimes d} \to 0$$

and (ii)-(iii) that $\mathcal{I}_C$ is 3-regular.

The Regularity Theorem: If $\mathcal{F}$ is m-regular, then:

(a) $\mathcal{F}$ is $m+1$-regular, and

(b) the multiplication map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \to H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Corollary: If $\mathcal{F}$ is an m-regular coherent sheaf on $\mathbf{P}^n_k$, then:

(a) $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(d)) = 0$ for all $i > 0$ and $d \geq m-1$ and

(b) $\mathcal{F}(d)$ is generated by global sections for all $d \geq m$.

Proof: Induction gives (a) immediately, and implies that the maps:

$$\mathrm{Sym}^e \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1)) \otimes \ \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(d)) \to \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(d+e))$$

are surjective for all $d \geq m$ and all $e > 0$. In particular, fix d and let e be large. By Serre's theorem B, the sheaf $\mathcal{F}(d+e)$ is generated by its sections, and it follows that $\mathcal{F}(d)$ is also generated by global sections.

Example: If L is a line bundle on a projective curve C satisfying (i)-(iii) of the example above, then $\mathrm{H}^1(C, L^d) = 0$ for all $d \geq 1$ and all multiplication maps $\mathrm{Sym}^d \mathrm{H}^0(C, L) \to \ \mathrm{H}^0(C, L^d)$ are surjective, i.e. the embedded curve is projectively normal. The corollary also tells us that $\mathcal{I}_C(3)$ is generated by its sections, hence C is an intersection of cubic hypersurfaces.

Mumford has a quick proof of the regularity theorem, but the funny-looking definition of regularity and the role of $\mathbf{P}^n_k$ seem to be better explained using spectral sequences, so we will prove it this way (following Mark Green).

Spectral Sequences: A **double complex** of abelian groups (or sheaves of abelian groups) is a collection $K_{p,q}$ of such groups (we'll assume $p, q \geq 0$), together with maps

$$d' : K_{p,q} \to K_{p+1,q} \text{ and } d'' : K_{p,q} \to K_{p,q+1}$$

satisfying $d' \circ d' = 0, d'' \circ d'' = 0$ and $d' \circ d'' + d'' \circ d' = 0$.

Given a double complex, one defines the associated **total complex**:

$$K_n := \bigoplus_{p+q=n} K_{p,q}, \ D = d' + d'' : K_n \to K_{n+1}$$

The basic homological algebra result asserts that the homology of the total complex:

$$E_n := \frac{\ker(D : K_n \to K_{n+1})}{\mathrm{im}(D : K_{n-1} \to K_n)}$$

filters in two different ways.

Starting from d': Let $E'^1_{p,q}$ be the homology of $K_{p,q}$ with respect to d'. Then d'' induces maps $d'_1 : E'^1_{p,q} \to E'^1_{p,q+1}$ satisfying $d'_1 \circ d'_1 = 0$. Inductively, if $E'^{r+1}_{p,q}$ is the homology of $E'^r_{p,q}$ with respect to $d'_r : E'^r_{p,q} \to E'^r_{p-r+1,q+r}$, then there are induced maps $d'_{r+1} : E'^{r+1}_{p,q} \to E'^{r+1}_{p-r,q+r+1}$.

Once $E'^r_{p,q} = E'^{r+1}_{p,q} = ...$, then we define $E'^\infty_{p,q} := E'^r_{p,q}$.

Starting from d'': The analog for d'' produces $d''_r : E''^r_{p,q} \to E''^r_{p+r,q-r+1}$ and we define $E''^\infty_{p,q}$ similarly.

The Spectral Sequence Theorem: There are two filtrations:

$$0 \subset F'_{n,n} \subset ... \subset F'_{n,q} \subset ... \subset F'_{n,0} = E_n \text{ and } 0 \subset ... \subset F''_{p,n} \subset ... \subset E_n$$

with subquotients $F'_{n,q}/F'_{n,q+1} \cong E'^\infty_{n-q,q}$ and $F''_{p,n}/F''_{p+1,n} \cong E''^\infty_{p,n-p}$.

Corollary: If $E'^\infty_{p,q} = 0$ for all p, q, then $E''^\infty_{p,q} = 0$ for all p, q.

Now suppose that $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to ... \to \mathcal{F}_m \to 0$ is a long **exact** sequence of coherent sheaves on a Noetherian separated scheme X. Take an open cover $\mathcal{U}$ of X by affines $U_0, ..., U_n$ and consider the double complex:

$$K_{p,q} = C^q(\mathcal{U}, \mathcal{F}_p) := \bigoplus_{i_0 < ... < i_q} \mathcal{F}_p(U_{i_0} \cap ... \cap U_{i_q})$$

with $(-1)^p d''$ the Cech coboundary map and d' induced from the sheaf maps.

Because the sequence of sheaves is exact, it follows that $E'^\infty_{p,q} = E'^1_{p,q} = 0$ for all p, q. So by the theorem, it must be true that $E''^\infty_{p,q} = 0$ as well. Since the terms $E''^1_{p,q} = \mathrm{H}^q(X, \mathcal{F}_p)$ are in general not zero, this gives us information about maps between cohomology groups of the $\mathcal{F}_p$.

Exercise: Apply this to a short exact sequence of sheaves to obtain the connecting homomorphisms from the maps d''_2.

The **Koszul complex** $\mathcal{K}_*$ is the exact sequence:

$$0 \to \wedge^{n+1} V \otimes_k \mathcal{O}(-n-1) \to \wedge^n V \otimes_k \mathcal{O}(-n) \to ... \to V \otimes_k \mathcal{O}(-1) \to \mathcal{O} \to 0$$

of locally free sheaves on $\mathbf{P}^n_k$, where V is the vector space $\mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1))$ with basis $x_0, ..., x_n$ and $\wedge^r V \otimes_k \mathcal{O}(-r) \to \wedge^{r-1} V \otimes_k \mathcal{O}(-r+1)$ are the contraction maps:

$$x_{i_1} \wedge ... \wedge x_{i_r} \otimes s \to \sum (-1)^{j-1} x_{i_1} \wedge ... \wedge \widehat{x_{i_j}} \wedge ... \wedge x_{i_r} \otimes x_{i_j} s$$

Proof of the Regularity Theorem: Let $\mathcal{K}_* \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{F}(m+1)$ be the complex of sheaves obtained by tensoring the terms of the Koszul complex by $\mathcal{F}(m+1)$. Because the terms in the Koszul complex are locally free, it follows that this complex is exact. One now computes: $E''^1_{n,0} = V \otimes_k \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{F}(m))$, $E''^1_{n+1,0} = \mathrm{H}^0(\mathbf{P}^n, \mathcal{F}(m+1))$ and d''_1 is the multiplication map. Furthermore, each $E''^1_{n-r,r} = \wedge^{r+1} V \otimes \mathrm{H}^r(\mathbf{P}^n, \mathcal{F}(m-r))$ that might eventually produce an $E''^r_{n-r,r}$ mapping to $E''^r_{n+1,0}$ under d''_r is already zero, since we assumed $\mathcal{F}$ was m-regular. Since all d''_r maps out of $E''^r_{n+1,0}$ are obviously zero as well, the vanishing of $E''^\infty_{n+1,0}$ forces the surjectivity of d''_1, giving us (b).

Similarly, the vanishings $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(m+1-i))$ are seen by considering the long exact sequences $\mathcal{K}_* \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{F}(m+1-i)$ for each $1 \le i \le n$.

Proof of the Uniform Vanishing Theorem: We prove the theorem first in case $X = \mathbf{P}^n_k$, $\mathcal{F}$ is locally free, and $\mathcal{E} = \oplus \mathcal{O}_{\mathbf{P}^n_k}(l)$ for some l. We'll work by induction on n, with the case $n = 0$ being trivial.

An injective map $i : \mathcal{F} \hookrightarrow \mathcal{E}$ of locally free sheaves need not be injective on all fibers, but it does need to be injective on fibers at all generic points, hence at all points $x \in U$ of a dense open set, by Nakayama's lemma. The converse also holds. A map of locally free sheaves on an integral scheme is injective if and only if it is injective on fibers at some point.

Now suppose $f : H \hookrightarrow \mathbf{P}^n_k$ is the hyperplane corresponding to the section $s \in \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(1))$. Then the following sequences are exact for all d:

$$(*)\ 0 \to \mathcal{F}(d-1) \xrightarrow{s} \mathcal{F}(d) \to f_*\mathcal{F}(d)|_H \to 0$$

where $\mathcal{F}|_H := f^*\mathcal{F}$. Moreover, if H passes through a point where i is injective on fibers, then $i|_H : \mathcal{F}|_H \to \mathcal{E}|_H \cong \oplus \mathcal{O}_H(l)$ is injective, as well. We will argue by induction using such an H.

Suppose $P(d) = \chi(\mathbf{P}^n, \mathcal{F}(d)) = \sum_{i=0}^n a_i \binom{d}{i}$. Then

$$\begin{aligned} \chi(H, \mathcal{F}|_H(d)) &= \textstyle\sum_{i=0}^n a_i \binom{d}{i} - a_i \binom{d-1}{i} \\ &= \textstyle\sum_{i=0}^{n-1} a_{i+1} \binom{d-1}{i} \end{aligned}$$

Thus the restrictions $\mathcal{F}|_H$ all have the same Hilbert polynomial and all are subsheaves of the fixed locally free sheaf $\mathcal{E}|_H = \oplus \mathcal{O}_H(l)$.

So we may assume that there is an N_0 independent of $\mathcal{F}$ such that

$$\mathrm{H}^i(H, \mathcal{F}_H(d)) = 0 \text{ for all } i > 0, d \geq N_0$$

From the exact sequence on cohomology associated to $(*)$, this gives us:

$$\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(N_0 - 1)) = \mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(N_0)) = \ldots \text{ forall } i \geq 2$$

and so by Serre's theorem B, we get $\mathrm{H}^i(\mathbf{P}^n_k, \mathcal{F}(d)) = 0$ for $i \geq 2, d \geq N_0 - 1$. To get the H^1 term, we use the cohomology sequences for $d \geq N_0$:

$$0 \to \mathrm{H}^0(\mathcal{F}(d-1)) \to \mathrm{H}^0(\mathcal{F}(d)) \xrightarrow{\rho_d} \mathrm{H}^0(\mathcal{F}_H(d)) \to$$

$$\to \mathrm{H}^1(\mathcal{F}(d-1)) \to \mathrm{H}^1(\mathcal{F}(d)) \to 0$$

Since all the higher cohomology of $\mathcal{F}_H(N_0)$ and higher twists vanishes, it follows in particular that $\mathcal{F}_H$ is $N_1 := N_0 + n - 1$ regular, so by the regularity theorem, the multiplication map:

$$\mathrm{H}^0(\mathcal{O}_H(1)) \otimes \mathrm{H}^0(\mathcal{F}_H(d)) \to \mathrm{H}^0(\mathcal{F}_H(d+1))$$

is surjective for $d \geq N_1$. Thus if the restriction map ρ_d is surjective for some $d \geq N_1$, then the restriction map ρ_{d+1} must also be surjective, as is evident from the following diagram:

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^n_k}(1)) \otimes \mathrm{H}^0(\mathcal{F}(d)) & \to & \mathrm{H}^0(\mathcal{F}(d+1)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\mathcal{O}_H(1)) \otimes \mathrm{H}^0(\mathcal{F}_H(d)) & \to & \mathrm{H}^0(\mathcal{F}_H(d+1)) \end{array}$$

On the other hand, $\dim_k \mathrm{H}^1(\mathcal{F}(d-1)) > \dim_k \mathrm{H}^1(\mathcal{F}(d))$ whenever ρ_d is not surjective, so the dimensions of $\mathrm{H}^1(\mathcal{F}(d))$ are strictly decreasing down to zero for $d \geq N_1 - 1$. In particular, **all** the higher cohomology of $\mathcal{F}(d)$ vanishes as soon as $d \geq N_2 := N_1 + \dim_k \mathrm{H}^1(\mathcal{F}(N_1 - 1))$. We now need to show that this can be bounded independent of $\mathcal{F}$.

Since $\mathrm{H}^i(\mathcal{F}(N_1 - 1)) = 0$ for $i \geq 2$, we have:

$$\begin{aligned} \dim_k \mathrm{H}^1(\mathcal{F}(N_1 - 1)) &= \dim_k \mathrm{H}^0(\mathcal{F}(N_1 - 1)) - \chi(\mathcal{F}(N_1 - 1)) \\ &\leq \dim_k \mathrm{H}^0(\mathcal{E}(N_1 - 1)) - P(N_1 - 1) \end{aligned}$$

which indeed only depends upon $P(d)$ and $\mathcal{E}$.

Not only do we have vanishing of higher cohomology when $d \geq N_2$, but we also see that $\mathcal{F}$ is $N_2 + 1$ regular. Hence the $\mathcal{F}(d)$ are generated by their sections for all $d \geq d_0 := N_2 + 1$ and we get uniform vanishing.

Next, suppose $\mathcal{F}$ is not assumed to be locally free. Then the proof above holds provided there is always a hyperplane $H \subset \mathbf{P}^n_k$ so that the sequences:

$$(**)\ 0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}_H(d) \to 0$$

are exact, and $i_H : \mathcal{F}_H \to E_H$ is injective.

For this, we need a result from commutative algebra. If $\mathcal{F}$ is a sheaf and $s \in \mathcal{F}(U)$ is a section over an open set U, then the **support** of s in U is the set of points $x \in U$ such that $0 \neq s_x \in \mathcal{F}_x$. Note that this is always closed. If $\mathcal{F}$ is a coherent sheaf over a Noetherian scheme, then we say $x \in X$ is an **associated point** of $\mathcal{F}$ if there exist an s and $x \in U$ as above so that x is in the support of s and moreover, the support of s is the closure(!) of x in U.

For example, the generic point(s) are the only possible associated points of a subsheaf of a locally free sheaf on a reduced scheme. The extra associated points of $\mathcal{O}_X$ on a non-reduced scheme X are called the **embedded primes**.

If $X =$ Spec(A) is affine and $\mathcal{F} = \widetilde{M}$, then it follows from the primary decomposition of the sub-module $0 \subset M$ (see Atiyah-MacDonald Ch. 4,7) that there are only finitely many associated points of $\mathcal{F}$, and that if $a \in A$ does not belong to the union of the corresponding prime ideals in A, then $am \neq 0$ for all $m \in M$. What we glean from this in general is:

$$\mathrm{A}(\mathcal{F}) := \{\text{associated points of } \mathcal{F}\} \text{ is a finite set}$$

Returning to the proof, notice that since $\mathcal{F}$ is a subsheaf of a locally free sheaf, $\mathcal{F}$ has only the generic point for an associated point, hence $(**)$ is exact for any choice of H. If we choose H to be disjoint from the finite set $\mathrm{A}(\mathcal{E}/\mathcal{F})$, then the equation for H is not a zero divisor in any stalk of $\mathcal{E}/\mathcal{F}$. Let $\mathcal{O}_x$ be the stalk of $\mathcal{O}_{\mathbf{P}^n_k}$ at x and s_x be the equation for H at x. Then on stalks at x the map i_H is the first map of the sequence:

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to \mathcal{E}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to \mathcal{E}_x/\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/s_x \to 0$$

which has kernel $\mathrm{Tor}^1_{\mathcal{O}_x}(\mathcal{E}_x/\mathcal{F}_x, \mathcal{O}_x/s_x)$. But since our choice of H guarantees that multiplication of $\mathcal{E}_x/\mathcal{F}_x$ by s_x has no kernel, it follows that this Tor is zero, and that i_H is injective.

Finally, to prove the theorem on any projective k-scheme $f : X \hookrightarrow \mathbf{P}^n_k$ and for any locally free $\mathcal{E}$ on X, we first note that by Serre's theorem A, we can always inject $\mathcal{E} \hookrightarrow \oplus\mathcal{O}_X(l)$ for some l. Thus, we may as well assume that $\mathcal{E} = \oplus\mathcal{O}_X(l)$. Then for any $\mathcal{F} \hookrightarrow \oplus\mathcal{O}_X(l)$, there is a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{K}_1 & = & \mathcal{K}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{K}_2 & \to & \oplus\mathcal{O}_{\mathbf{P}^n_k}(l) & \to & f_*\left(\oplus\mathcal{O}_X(l)/\mathcal{F}\right) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & f_*\mathcal{F} & \to & \oplus\mathcal{O}_X(l) & \to & f_*\left(\oplus\mathcal{O}_X(l)/\mathcal{F}\right) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Note that $\mathcal{K}_1$ and $\mathcal{K}_2$ are both subsheaves of $\oplus\mathcal{O}_{\mathbf{P}^n_k}(l)$ with Hilbert polynomials only depending upon $X, \mathcal{E}$ and $P(d)$. Thus uniform vanishing applies to both $\mathcal{K}_1$ and $\mathcal{K}_2$. From the exact sequences:

$$\to \mathrm{H}^i(\mathbf{P}^n_k, \mathcal{K}_2(d)) \to \mathrm{H}^i(X, \mathcal{F}(d)) \to \mathrm{H}^{i+1}(\mathbf{P}^n_k, \mathcal{K}_1(d)) \to$$

and the surjectivity:

$$\mathcal{K}_2(d) \to f_*(\mathcal{F}(d))$$

it follows that uniform vanishing holds for $\mathcal{F}$ with d_0 equal to the maximum of the values for $\mathcal{K}_1$ and $\mathcal{K}_2$. This completes the proof.

Remark: The only place we used the fact that $\mathcal{E}$ was locally free was in the assertion that there is an injection:

$$\mathcal{E} \hookrightarrow \oplus\mathcal{O}_X(l)$$

This injection is achieved by applying Serre's theorem A to the dual locally free sheaf $\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ to find a surjection $\oplus\mathcal{O}_{\mathbf{P}^n_k} \to \mathcal{E}^*(l)$, twisting by $\mathcal{O}_{\mathbf{P}^n_k}(-l)$ and then dualizing back. Thus it certainly suffices to assume that $\mathcal{E}^{**} \cong \mathcal{E}$ (i.e. $\mathcal{E}$ is **reflexive**), or even just that $\mathcal{E}$ injects into $\mathcal{E}^{**}$ (i.e. $\mathcal{E}$ is **torsion-free**), since $\mathcal{E}^{**}$ is always reflexive.

4. Flatness. Our first goal is to prove and generalize the following:

Properness Theorem: Let S be a nonsingular curve, and let $U \subset S$ be the punctured curve $S - s_0$. Then every family of projective schemes $X_U \subset \mathbf{P}^n_U$ with constant Hilbert polynomial $P(d)$ extends uniquely to a family $X_S \subset \mathbf{P}^n_S$ such that the Hilbert polynomial of the "limit" $X_{s_0} \subset \mathbf{P}^n_{k(s_0)}$ is also $P(d)$.

The fibers of an S-scheme $f : X \to S$ are $X_s := X \times_S \mathrm{Spec}(k(s)) \subset X$, where $\mathrm{Spec}(k(s)) \to S$ "is" the inclusion of the point $s \in S$. These fibers are the associated **family** of subschemes of X parametrized by S. Similarly, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules determines the family $\mathcal{F}_s := \mathcal{F}_s|_{X_s}$ of sheaves of $\mathcal{O}_{X_s}$-modules parametrized by S. Thus $\mathbf{P}^n_S$ is the family of projective spaces $\mathbf{P}^n_{k(s)}$ and if a closed immersion $X \subset \mathbf{P}^n_S$ is given, then the subschemes $X_s \subset \mathbf{P}^n_{k(s)}$ and sheaves $\mathcal{F}_s$ have Hilbert polynomials depending upon the point $s \in S$.

Grothendieck first recognized the usefulness of the algebraic notion of flatness in defining a "good" family of schemes, or of coherent sheaves.

$\mathcal{F}$ is **flat over** S **at** $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x)}$-module. If $\mathcal{F}$ is flat over S at every point of X, we say that $\mathcal{F}$ is **flat over** S. We say that the morphism f is flat if $\mathcal{O}_X$ is flat over S.

"Classical" smooth families are flat. That is, if $f : X \to Y$ is a morphism of nonsingular varieties over an algebraically closed field k, with the property that $\dim(X) - \dim(Y) = n$ and $\Omega_{X/Y}$ is locally free of rank n, then f is flat.

If Y is a Noetherian scheme and $f : X \to Y$ is a finite map, then a coherent sheaf $\mathcal{F}$ is flat over Y if and only if $f_*\mathcal{F}$ is locally free. This is a consequence of the fact that a finitely generated module over a Noetherian local ring is flat if and only if it is free.

For the properness theorem, the key property of flatness is the following:

Constancy of the Hilbert Polynomial: If A is a Noetherian integral domain, X is a projective A-scheme with very ample invertible sheaf $\mathcal{O}_X(1)$, and $\mathcal{F}$ is a coherent sheaf on X, then the following are equivalent:

(a) $\mathcal{F}$ is flat over $\mathrm{Spec}(A)$

(b) $\mathrm{H}^0(X, \mathcal{F}(d))$ is a locally free A-module, for all $d >> 0$

(c) The Hilbert polynomial $\chi(X_s, \mathcal{F}_s(d))$ is independent of $s \in \mathrm{Spec}(A)$.

Proof: (a) $\Rightarrow$ (b). Let $\mathcal{U}$ be an affine cover of X. By Serre's theorem B, the higher cohomology of $\mathcal{F}(d)$ vanishes for $d >> 0$, hence the following Cech complexes are **exact** for large d:

$$0 \to \mathrm{H}^0(X, \mathcal{F}(d)) \to C^0(\mathcal{U}, \mathcal{F}(d)) \to C^1(\mathcal{U}, \mathcal{F}(d)) \to ... \to C^n(\mathcal{U}, \mathcal{F}(d)) \to 0$$

But flatness of $\mathcal{F}$ implies that each term in the Cech complex is a flat A-module, hence $\mathrm{H}^0(X, \mathcal{F}(d))$ is also flat, and since it is finitely generated, it is locally free.

(b) $\Rightarrow$ (a). Let $f : X \to \mathbf{P}^n_A$ be the closed immersion. Then evidently $\mathcal{F}$ is flat over $\mathrm{Spec}(A)$ if and only if $f_*\mathcal{F}$ is flat over $\mathrm{Spec}(A)$. Let

$$M = \bigoplus_{d \geq d_0} \Gamma(\mathbf{P}^n_A, f_*\mathcal{F}(d))$$

where d_0 is chosen so that $\mathrm{H}^0(X, \mathcal{F}(d))$ is locally free for all $d \geq d_0$. Then M is flat, and $\widehat{M} = f_*\mathcal{F}$, hence $f_*\mathcal{F}$ is flat.

(b) $\Rightarrow$ (c). We will prove that for each $s \in S$ and all $d >> 0$, there are isomorphisms:

$$(*) \quad \mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A k(s) \tilde{\to} \mathrm{H}^0(X_s, \mathcal{F}_s(d))$$

Then (b) implies that $\chi(X_s, \mathcal{F}_s(d))$ is the rank of the locally free A-module $\mathrm{H}^0(X, \mathcal{F}(d))$ at s for large d, hence independent of s.

If $s \in \mathrm{Spec}(A)$ is a closed point, then given a presentation of $k(s) \cong A/m$:

$$A^n \to A \to k(s) \to 0$$

we get a presentation of $\mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A k(s)$ by tensoring with $\mathrm{H}^0(X, \mathcal{F}(d))$, or we can sheafify the given presentation and tensor with $\mathcal{F}$ over $\mathcal{O}_X$ to get a presentation of $\mathcal{F}_s(d)$. For $d >> 0$, this yields a presentation of $\mathrm{H}^0(X_s, \mathcal{F}_s(d))$ identical to the other.

In general, one uses a "flat base extension" to reduce to the case where s is a closed point. That is, construct the fiber product:

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_{\mathcal{P}_s}) & \to & \mathrm{Spec}(A) \end{array}$$

where $\mathcal{P}_s$ is the prime in A corresponding to s. Since localizing is exact, and each Cech complex on X localizes to a Cech complex on X', it follows that:

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_A A_{\mathcal{P}_s} \tilde{\to} \mathrm{H}^i(X', \mathcal{F}')$$

for all i and all coherent sheaves $\mathcal{F}$ on X ($\mathcal{F}'$ is the pull-back of $\mathcal{F}$ to X'). But now the fiber of X' over the closed point of $\mathrm{Spec}(A_{\mathcal{P}_s})$ is X_s, and we have reduced to the previous case.

(c) $\Rightarrow$ (a). Suppose $\chi(X_s, \mathcal{F}_s(d)) = \chi(X_\xi, \mathcal{F}_\xi(d))$ where $\xi \in S$ is the generic point. Then by $(*)$ and Serre's theorem B, we know that $\mathrm{H}^0(X, \mathcal{F}(d))$ has the same rank at s as it has at the generic point, for all large enough d. By Nakayama's lemma, this implies that for such large d, $\mathrm{H}^0(X, \mathcal{F}(d)) \otimes_A A_{\mathcal{P}_s}$ is a free $A_{\mathcal{P}_s}$-module. As in the proof of (b) $\Rightarrow$ (a), this implies that $\mathcal{F}$ is flat over S at all points mapping to s.

Corollary: Suppose S is a Noetherian connected scheme and $X \subset \mathbf{P}^n_S$ is a projective S-scheme which is flat over S. Then:

(a) The fiber dimensions $\dim(X_s)$ are constant

(b) The Euler characteristic $\chi(X_s, \mathcal{O}_{X_s})$ is constant, and

(c) The degree of $X_s \subset \mathbf{P}^n_s$ is constant.

Proof: (a)-(c) are obtained from the Hilbert polynomial as the degree, the constant term and the leading coefficient, respectively. Any two points on a connected scheme can be joined by a sequence of integral subschemes.

Remark: There are at least two good reasons to use flatness instead of the equivalent constancy of the Hilbert polynomial. First, flatness gives the "correct" generalization to projective schemes over schemes that are not reduced (this will be particularly important in deformation theory) and secondly, flatness is an easier notion to work with. As an example, we will use it to prove the properness theorem by making use of the particularly simple description of flatness over regular rings of dimension one.

Flatness over Nonsingular Curves 1: If $f : X \to S$ and S is irreducible and regular of dimension one, then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is flat over S if and only if f maps each associated point of $\mathcal{F}$ to the generic point of S.

E.g. If X is reduced, then $f : X \to S$ is a flat morphism if and only if every irreducible component of X dominates S.

Proof: If f is flat and $x \in X$ is a point with closed image $s = f(x) \in S$, then $\mathcal{O}_{S,s}$ is a DVR. Let $t \in m_s$ be a uniformizing parameter, which is in particular not a zero divisor in $\mathcal{O}_{S,s}$. Then $f^{\#}t \in m_x$ cannot be a zero divisor in the flat module $\mathcal{F}_x$. Hence x is not an associated point of $\mathcal{F}$.

Conversely, any $\mathcal{F}$ is flat at all points mapping to the generic point of S because the local ring at the generic point is a field(!) If $f(x) = s$ is closed and $\overline{\{x\}}$ does not contain an associated point of $\mathcal{F}$, then multiplication by the uniformizing parameter $f^{\#}t$ is injective on $\mathcal{F}_x$, and since t generates m_s, it follows that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$.

Remark: The previous result is certainly false if S is singular. For example, the normalization map $f : X \to S$ of a nodal curve is obviously not flat.

Flatness over Nonsingular Curves 2: If S is irreducible and regular of dimension one and $U \subset S$ is the complement of a closed point $s_0 \in S$, then "flat quotients uniquely extend across s_0". That is, given $f : X \to S$ and a coherent sheaf $\mathcal{E}$ on X, let $X_U = f^{-1}(U)$ and $\mathcal{E}_U = \mathcal{E}|_{X_U}$. Then a surjection $\mathcal{E}_U \to \mathcal{Q}$ to a coherent sheaf $\mathcal{Q}$ that is flat over U extends uniquely to a surjection $\mathcal{E} \to \overline{\mathcal{Q}}$ where $\overline{\mathcal{Q}}$ is coherent and flat over S.

Proof: Indeed, we will prove that $\mathcal{E} \to \overline{\mathcal{Q}}$ is universal in the sense that if $\mathcal{E} \to \mathcal{Q}'$ is another surjection extending $\mathcal{E}|_U \to \mathcal{Q}$, then there is a (unique) factorization $\mathcal{E} \to \mathcal{Q}' \to \overline{\mathcal{Q}}$

Note that if such a universal extension exists, then $\overline{\mathcal{Q}}$ is flat. Otherwise there would be a section of $\overline{\mathcal{Q}}$ with support lying over s_0 by Flatness over Curves 1, and the further quotient of $\overline{\mathcal{Q}}$ by the subsheaf generated by this section would violate universality. Similarly, given another $\mathcal{Q}'$, the kernel of the induced map $\mathcal{Q}' \to \overline{\mathcal{Q}}$ would be non-trivial, giving associated points in the fiber of f over s_0. Thus using Flatness over Curves 1 again, we see that the universal extension *if it exists* is the unique flat extension.

So we have to prove that such a universal extension exists. It obviously suffices to do this in case $S = \text{Spec}(B)$ is affine and $U = \text{Spec}(B_g)$ for some $g \in B$, but it also suffices to do this in case $X = \text{Spec}(A)$, since the extensions over affines will patch together by the universal property. In this case, $X_U = \text{Spec}(A_g)$, $\mathcal{E} = \widetilde{M}$, $\mathcal{E}_U = \widetilde{M}_g$, $\mathcal{Q}_U = \widetilde{N}$, and we are given $M_g \to N$, a surjective map of A_g-modules. We should take $\overline{N}$ to be the image of the A-module map $M \to M_g \to N$ and $\mathcal{Q}_S = \widetilde{\overline{N}}$. One then readily checks that $\widetilde{M} \to \widetilde{\overline{N}}$ is the universal extension of $\widetilde{M}_g \to \widetilde{N}$.

Example: In the case $\mathcal{E} = \mathcal{O}_X$, then all surjective maps are of the form $\mathcal{O}_X \to i_*\mathcal{O}_Z$, where $i : Z \hookrightarrow X$ is a closed immersion of S-schemes. The proof of Flatness over Curves 2 in this context shows that every closed subscheme $Y \subset X_U$ has a unique smallest "scheme-theoretic closure" $\overline{Y} \subset X$ and that if Y is flat over U, then $\overline{Y}$ is flat over S.

Putting the results of this section together, we now have a:

Proof of the Properness Theorem: If $\mathcal{E}$ is a coherent sheaf on $\mathbf{P}^n_S$ and a surjection $\mathcal{E}|_U \to \mathcal{Q}$ is given, such that the Hilbert polynomial of $\mathcal{Q}_s$ is constant for $s \in U$, then by the Constancy of the Hilbert Polynomial, $\mathcal{Q}$ is flat over U. By Flatness over Curves 2, there is a unique extension $\mathcal{E} \to \overline{\mathcal{Q}}$ such that $\overline{\mathcal{Q}}$ is flat over S, and by the Constancy result again, $\mathcal{Q}_{s_0}$ has the same Hilbert polynomial. When we apply this to the case $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n_S}$ and $\mathcal{Q} = i_*\mathcal{O}_{X_U}$ for the closed immersion $i : X_U \hookrightarrow \mathbf{P}^n_U$, we get the theorem.

The next result is a sort of Sard's theorem for flatness:

Generic Flatness: Given a reduced Noetherian scheme S, a morphism $f : X \to S$ of finite type, and a coherent sheaf $\mathcal{F}$ on X, then there is a non-empty open subset $U \subset S$ such that $\mathcal{F}_U$ is flat over U, where:

$$X_U := f^{-1}(U) = X \times_S U \quad \text{and} \quad \mathcal{F}_U := \mathcal{F}|_{X_U}$$

Proof: We may assume $S = \operatorname{Spec}(A)$ where A is an integral domain, replacing S by an open set. We may also assume f is dominant, otherwise generic flatness would be trivially true with $U = S - \overline{f(X)}$. More generally, we can always ignore locally closed subsets of X that do not dominate S by shrinking S if necessary.

Cover X by finitely many open affines $\operatorname{Spec}(B)$ and write $\mathcal{F}|_{\operatorname{Spec}(B)} = \widetilde{M}$. Then it suffices to prove generic flatness for each $\operatorname{Spec}(B)$ that dominates $\operatorname{Spec}(A)$. From the existence of a filtration of M:

$$0 \subset M_1 \subset M_2 \subset .... \subset M_n = M \text{ with } M_{i+1}/M_i \cong B/\mathcal{P}_i$$

it suffices to prove generic flatness for $M = B/\mathcal{P}$, and moreover for those $B/\mathcal{P}$ with the property that the map $A \to B/\mathcal{P}$ is injective, again shrinking S if necessary. Thus it suffices to prove the following:

Claim: If A is a Noetherian integral domain and B is an integral domain which is finitely generated as an A-algebra, then there is an $f \in A$ with the property that B_f is a free A_f-algebra.

Proof: Let n be the transcendence degree of B over A and let $K = K(A)$ be the field of fractions of A. Then the ring $K \otimes_A B$ is finitely generated as a K-algebra, hence by Noether Normalization it is integral over a polynomial ring $K[f_1, ..., f_n]$ with $f_1, ..., f_n \in B$. It does not follow that B is integral over $A[f_1, ..., f_n]$, but if B is generated by $b_1, ..., b_r$ as an A-algebra, then there are only finitely many denominators (in A) appearing in the minimal polynomials of the $b_i \in K \otimes_A B$ over $K[f_1, ..., f_n]$. If f is taken to be the product of the denominators, then it follows that B_f is integral over the polynomial ring $A_f[f_1, ..., f_n]$, hence of finite rank as an $A_f[f_1, ..., f_n]$-module. If $m = \mathrm{rk}(B_f)$, then there is an exact sequence of $A_f[f_1, ..., f_n]$-modules:

$$0 \to A_f[f_1, ..., f_n]^m \to B_f \to Q \to 0$$

where Q is a torsion module. Taking a filtration of Q as we did earlier with M, we return to the claim with B' an integral domain over A of smaller transcendence degree. By induction, we may therefore assume that Q_g is a free A_g-module, and it follows that B_{fg} is a free A_{fg}-module.

Corollary: In the same setting, there is a stratification:

$$S = \coprod V_i$$

of S by reduced, locally closed subschemes V_i such that $\mathcal{F}_{V_i}$ is flat, for all i.

Proof: Noetherian induction.

Corollary: If in addition $f : X \to S$ is a projective morphism, with closed immersion $X \subset \mathbf{P}^n_S$, then the Hilbert polynomials of the $\mathcal{F}_s$ vary in finite set.

Proof: Apply the constancy result to the components of the V_i strata.

This stratification is not quite good enough for our purposes. We need:

Flattening Stratifications: Given a projective morphism $f : X \to S$ over a Noetherian scheme S and a coherent sheaf $\mathcal{F}$ on X, there is a (unique) stratification:

$$S = \coprod S_i$$

of S by locally closed subschemes S_i such that $\mathcal{F}_{S_i}$ is flat over each S_i and more generally, given a morphism $g : T \to S$ and hence $\tilde{g} : X \times_S T \to X$, then $\tilde{g}^*\mathcal{F}$ is flat over T **if and only if** g factors through:

$$g : T \to \coprod S_i \to S$$

(Roughly, the S_i are the maximal thickenings of the V_i preserving flatness)

Proof: We may as well assume that $S = \mathrm{Spec}(A)$ since flattening stratifications for an open cover will glue by their universal property. And it clearly suffices to consider only affine $T = \mathrm{Spec}(B)$ as well. Given a morphism $g : T \to S$, we will write the fiber square as:

$$\begin{array}{ccc} X_B & \stackrel{\widetilde{g}}{\to} & X_A \\ \widetilde{f}\downarrow & & f\downarrow \\ \mathrm{Spec}(B) & \stackrel{g}{\to} & \mathrm{Spec}(A) \end{array}$$

Any Cech complex of A-modules:

$$0 \to \mathrm{H}^0(X_A, \mathcal{E}_A) \to C^0(\mathcal{U}, \mathcal{E}_A) \to \ldots \to C^n(\mathcal{U}, \mathcal{E}_A) \to 0$$

for a quasi-coherent sheaf $\mathcal{E}_A$ on X_A becomes a Cech complex for $\mathcal{E}_B := \widetilde{g}^*\mathcal{E}$ when we tensor it by B. It therefore follows from the right-exactness of the tensor product that there are always maps:

$$\mathrm{H}^i(X_A, \mathcal{E}_A) \otimes_A B \to \mathrm{H}^i(X_B, \mathcal{E}_B)$$

for all i. Moreover, if B is flat over A, then all these maps are isomorphisms. This generalizes an earlier "flat base extension" argument.

We will need the following two "base change" theorems:

Stable Base Change: If g is fixed and $d >> 0$, then:

$$\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A B \to \mathrm{H}^0(X_B, \mathcal{F}_B(d)) \text{ is an isomorphism}$$

Cohomology and Base Change: If $f : Y \to \mathrm{Spec}(B)$ is a projective morphism and $\mathcal{E}$ is a coherent sheaf on Y which is flat over $\mathrm{Spec}(B)$, let $t \in \mathrm{Spec}(B)$ be a point. Then:

(a) If $\phi^i(t) : \mathrm{H}^i(Y, \mathcal{E}) \otimes_B k(t) \to \mathrm{H}^i(Y_t, \mathcal{E}_t)$ is surjective, then it is an isomorphism and $\phi^i(t')$ is an isomorphism for all t' in a neighborhood of t.

(b) If $\phi^i(t)$ is surjective, then the following are equivalent:

(I) $\phi^{i-1}(t)$ is also surjective.

(II) $\mathrm{H}^i(Y, \mathcal{E})$ is locally free in a neighborhood of t.

Corollary: If $\mathrm{H}^1(Y_t, \mathcal{E}_t) = 0$ for all fibers, then $\mathrm{H}^0(Y, \mathcal{E})$ is locally free.

Now use the weak stratification coming from generic flatness to pull back $\mathcal{F}$ to coherent sheaves $\mathcal{F}_{V_i}$ on each $X_{V_i} := X \times_S V_i$ that are flat over V_i. If $\mathrm{Spec}(A_i) \subset V_i$ is an affine open set, then as in the earlier proof of (a) $\Rightarrow$ (b), the Cech complex for large d:

$$0 \to \mathrm{H}^0(X_{A_i}, \mathcal{F}_{A_i}(d)) \to C^0(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to \ldots \to C^n(\mathcal{U}, \mathcal{F}_{A_i}(d)) \to 0$$

is an exact sequence of flat A_i-modules. But because of flatness, if we tensor the sequence by $k(s)$ for $s \in \mathrm{Spec}(A_i)$, the sequence remains exact. Since it is then a Cech complex for $\mathcal{F}_s(d)$ on the fiber X_s, this shows that the higher cohomology of the $\mathcal{F}_s(d)$ for $s \in \mathrm{Spec}(A_i)$ all vanish for such d. Choosing d_0 large enough to work for all A_i simultaneously, we therefore obtain:

(i) $\mathrm{H}^i(X_s, \mathcal{F}_s(d)) = 0$ for all $s \in S$ and $d \geq d_0$.

Moreover, we can use the stable base change result to conclude that:

$$\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A A_i \tilde{\to} \mathrm{H}^0(X_{A_i}, \mathcal{F}_{A_i}(d) \text{ for all } A_i \text{ and } d \geq d_0$$

increasing d_0 if necessary. From the exact sequence above or the cohomology and base change result, we may then conclude:

(ii) $\mathrm{H}^0(X_A, \mathcal{F}_A(d)) \otimes_A k(s) \tilde{\to}\ \mathrm{H}^0(X_s, \mathcal{F}_s(d))$ for all $s \in S$ and $d \geq d_0$.

This is a uniform version of the result $(*)$ which we proved earlier.

The flattening stratification will now be a consequence of the following:

Claim: $\mathcal{F}_B$ is flat over $\mathrm{Spec}(B) \Leftrightarrow g^* f_* \mathcal{F}(d)$ is locally free for all $d \geq d_0$.

If $\mathcal{E}$ is a coherent sheaf on S and the fiber $\mathcal{E}(s)$ at $s \in S$ has rank e, then by Nakayama's lemma there is a presentation of $\mathcal{E}|_U$ in a neighborhood $s \in U$ of the form:

$$(**) \quad \mathcal{O}_U^f \stackrel{\psi_{ij}}{\to} \mathcal{O}_U^e \to \mathcal{E}|_U \to 0$$

The equations $\psi_{ij} = 0$ patch together to put a scheme structure S_e on the locally closed subset $V_e \subset S$ where $\mathcal{E}$ has constant rank e.

It is now easy to see that $g^*\mathcal{E}$ is locally free of rank e if and only if $g(T) \subset V_e$ (as sets) and $g^*\psi_{ij} = 0$ for all ψ_{ij}, which is to say if and only if $g : T \to S$ factors through S_e. Thus we obtain a locally free stratification $S = \coprod S_e$ associated to $\mathcal{E}$ and indexed by the rank of $\mathcal{E}$.

Now assume that the claim is true. Choose $n+1$ to exceed the degree of the Hilbert polynomial of each $\mathcal{F}_s$ in the family. Thus the ranks of the vector spaces $\mathrm{H}^0(X_s, \mathcal{F}_s(d_0)), ..., \mathrm{H}^0(X_s \mathcal{F}_s(d_0+n))$ determine the Hilbert polynomial of each $\mathcal{F}_s$ by (i) above. By (ii) above, these are the fibers of the sheaves $f_*\mathcal{F}(d_0), ..., f_*\mathcal{F}(d_0+n)$ since they are the sheafifications of the A-modules $\mathrm{H}^0(X_A, \mathcal{F}_A(d_0)), ..., \mathrm{H}^0(X_A, \mathcal{F}_A(d_0+n))$.

If we intersect the locally closed subschemes obtained in the locally free stratifications of each of $f_*\mathcal{F}(d_0), f_*\mathcal{F}(d_0+1), ..., f_*\mathcal{F}(d_0+n)$, we obtain a stratification $S = \coprod S_{P(d)}$ indexed by (distinct) Hilbert polynomials with the property that $g^*f_*\mathcal{F}(d_0), ..., g^*f_*\mathcal{F}(d_0+n)$ are simultaneously locally free if and only if $g : T \to S$ factors through $\coprod S_{P(d)}$. Further intersecting with the schemes obtained from the locally free stratifications of the rest of the sheaves $f_*\mathcal{F}(d_0+n+1), ...$ only serves to shrink the scheme structure on $S_{P(d)}$ leaving the same underlying reduced scheme $V_{P(d)}$. Since S is Noetherian, it follows that after finitely many such intersections, we obtain the limit scheme structure which according to the claim is the desired flattening stratification.

By the stable base change result, the maps:

$$g^*f_*\mathcal{F}(d) = \mathrm{H}^0(X, \mathcal{F}) \otimes_A B \to \mathrm{H}^0(X_B, \mathcal{F}_B(d)) = \widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$$

are isomorphisms for all $d >> 0$. Thus if the $g^*f_*\mathcal{F}(d)$ are all locally free for $d \geq d_0$, then so are the $\widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ for all $d >> 0$, and by the equivalence of conditions (b) and (a) in the constancy of the Hilbert polynomial, this imples that $\widetilde{g}^*\mathcal{F}$ is flat over T.

On the other hand, if $t \in \mathrm{Spec}(B)$ maps to $s \in \mathrm{Spec}(A)$, then by (i):

$$0 = \mathrm{H}^i(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \tilde{\to} \mathrm{H}^i(X_t, \mathcal{F}_t(d))$$

for all $d \geq d_0$ and all $i > 0$ because $\mathrm{Spec}(k(t))$ is flat over $\mathrm{Spec}(k(s))$. But these are the higher cohomologies of the fibers of $\widetilde{g}^*\mathcal{F}(d)$. Thus if $\widetilde{g}^*\mathcal{F}$ is flat over T, then by the corollary to cohomology and base change, it follows that $\widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ is locally free for all $d \geq d_0$. But now the map $g^*f_*\mathcal{F}(d) \to \widetilde{f}_*\widetilde{g}^*\mathcal{F}(d)$ induces the following isomorphisms fibers:

$$\mathrm{H}^0(X_s, \mathcal{F}_s(d)) \otimes_{k(s)} k(t) \tilde{\to} \mathrm{H}^0(X_t, \mathcal{F}_t(d))$$

using (ii) and cohomology and base change. Any map from a coherent sheaf to a locally free sheaf which is an isomorphism on fibers is an isomophism, by Nakayama's lemma, so the claim is proved.

5. Construction of the Hilbert Scheme: We'll start with a discussion of Grassmannians, then construct Grothendieck's quot schemes and Hilbert schemes using the results of the previous two sections.

Fix a field k. The Grassmannian $G(m,n)$ over k is a projective k-scheme which we may define as follows. Start with the "wedge" map:

$$f : \mathrm{Spec}(k[x_{i,j}]) = \mathbf{A}_k^{mn} \to \mathbf{A}_k^{\binom{n}{m}} = \mathrm{Spec}(k[x_J])$$

where $1 \le i \le m$, $1 \le j \le n$ and $J = (j_1, ..., j_m)$ for $1 \le j_1 < ... < j_m \le n$ (we'll call J a sorted m-tuple). Then f is defined by $f^*(x_J) = \det(x_{i,j_l})$ the determinant of the square submatrix of $(x_{i,j})$ obtained by removing all columns except those indexed by J. Define $\phi : \mathbf{A}_k^{mn} - - > \mathbf{P}_k^{\binom{n}{m}-1}$ to be the composition of the morphism f with the rational map to projective space. Thus ϕ is a morphism when restructed to the Zariski open subset $U := \mathbf{A}_k^{mn} - V(\{\det(x_{i,j_l})\})$ of matrices of "full rank."

If $I = (i_1, ..., i_m)$ is an (unsorted) m-tuple of integers with $1 \le i_l \le n$ and if $\sigma \in \Sigma_m$ is a permutation, let $\sigma(I) = (i_{\sigma(1)}, ..., i_{\sigma(m)})$. Note that there is at most one sorted m-tuple J in each orbit of Σ_m and exactly one if the i_l are distinct. It follows that we obtain well-defined $x_I \in \mathrm{H}^0(\mathbf{P}_k^{\binom{n}{m}-1}, \mathcal{O}(1))$ by setting:

$$x_I = \begin{cases} 0 & \text{if the } i_l \text{ are not distinct} \\ (-1)^{\mathrm{sgn}(\sigma)} x_J & \text{if } \sigma(I) = J, \text{ a sorted multi-index} \end{cases}$$

The **Plücker quadric** associated to a pair I, J of m-tuples and an integer $1 \le s \le m$ is given by:

$$Q_{I,J,s} := x_I x_J - \sum_{l=1}^{m} x_{(i_1,...,i_{s-1},j_l,i_{s+1},...,i_m)} x_{(j_1,...,j_{l-1},i_s,j_{l+1},...,j_m)}$$

The **Grassmannian** $G(m,n) \subset \mathbf{P}_k^{\binom{n}{m}-1}$ is the zero scheme of the set of all the Plücker quadrics.

Remark: These quadrics are redundant. For example, if $m = 2, n = 4$, then

$$G(2,4) = V(x_{(1,2)}x_{(3,4)} - x_{(1,3)}x_{(2,4)} + x_{(1,4)}x_{(2,3)})$$

i.e. $G(2,4) \subset \mathbf{P}_k^5$ is a nonsingular quadric hypersurface.

Proposition: $\phi : U \to \mathbf{P}_k^{\binom{n}{m}-1}$ factors through $G(m,n)$ and:

(a) $\phi : U \to G(m,n)$ is a locally trivial fiber bundle, with fiber $\mathrm{GL}(m,k)$. In particular, ϕ determines a bijection from the set of m-dimensional subspaces of k^n to the set of k-rational points of $G(m,n)$.

(b) Let $U_{x_J} = \mathbf{P}_k^{\binom{n}{m}-1} - V(x_J)$. Then $G(m,n) \cap U_{x_J} \cong \mathbf{A}_k^{m(n-m)}$. Hence $G(m,n)$ is a nonsingular projective variety over k of dimension $m(n-m)$.

Proof: ϕ is equivariant with respect to the "left multiplication" action $\mu : \mathrm{GL}(m,k) \times U \to U$ where $\mathrm{GL}(m,k) \subset \mathbf{A}_k^{m^2}$ is the open subscheme of invertible $m \times m$ matrices. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}(m,k) \times U & \xrightarrow{\mu} & U \\ \downarrow \pi & & \downarrow \phi \\ U & \xrightarrow{\phi} & \mathbf{P}_k^{\binom{n}{m}-1} \end{array}$$

This follows from the fact that if we multiply an $m \times n$ matrix A on the left by a square matrix M, then the determinants of the $m \times m$ submatrices of A are all multiplied by $\det(M)$.

Fix J and consider the closed subscheme $\mathbf{A}_k^{m(n-m)} \cong Z_J \subset U$ of $m \times n$ matrices whose square submatrix corresponding to J is the identity. Then the following are easily verified:

(i) $\mu :\ \mathrm{GL}(m,k) \times Z_J \to \phi^{-1}(U_{x_J})$ is an isomorphism.

(ii) $\phi^*(Q_{K,L,s}/x_J^2) \in I(Z_J)$ for all the Plücker quadrics.

(iii) $\phi|_{Z_J} : Z_J \to G(m,n) \cap U_{x_J}$ is an isomorphism.

The proposition now follows quickly. By (i) and (ii), and the fact that ϕ is μ-equivariant, it follows that $\phi^*(Q_{I,J,s}/x_J^2) = 0 \in I(\phi^{-1}U_{x_J})$ hence that ϕ factors through the Grassmannian.

Next, (i) and (iii) imply that $\phi : U \to G(m,n)$ is a locally trivial fibration. And again using the action, we see that given a k-rational point $x \in G(m,n)$, the closed points in the fiber $\phi^{-1}(x)$ are just the orbit of an $m \times n$ matrix A under the left action by $\mathrm{GL}(m,k)$. If we write the rows of A as vectors $v_1, ..., v_m$, then this identifies x with the subspace of k^n generated by $v_1, ..., v_m$, giving the bijection. This completes the proof of the Proposition.

Given a scheme T, consider exact sequences: $(*)\ 0 \to K \to \mathcal{O}_T^n \to Q \to 0$ where K, Q are locally free sheaves on T of ranks m and $n-m$, respectively. There is a natural equivalence relation on these. Namely, we'll say two such sequences $(*)$ and $(*')$ are equivalent if there is a commuting diagram:

$$\begin{array}{ccccccccc} (*)\ 0 & \to & K & \to & \mathcal{O}_T^n & \to & Q & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ (*')\ 0 & \to & K' & \to & \mathcal{O}_T^n & \to & Q' & \to & 0 \end{array}$$

where the vertical maps are isomorphisms.

Definition: Let $F_{m,n} \in Sch_k/Sets$ be the functor defined by:

$$F_{m,n}(T) := \{\text{equivalence classes of exact sequences } (*)\} \text{ with } F_{m,n}(f) = f^*$$

Theorem: The Grassmannian $G(m,n)$ represents the functor $F_{m,n}$.

Proof: First, we define a transformation of functors: $u : F_{m,n} \to h_{G(m,n)}$ to the functor of points of $G(m,n)$ which we'll then invert.

Given a sequence $(*)$ on T, choose an affine covering $T = \cup \mathrm{Spec}(A_i)$ such that K may be trivialized over each $\mathrm{Spec}(A_i)$. Then over each such affine, the map $K|_{\mathrm{Spec(A_i)}} \cong \mathcal{O}^m_{\mathrm{Spec(A_i)}} \to \mathcal{O}^n_{\mathrm{Spec(A_i)}}$ from $(*)$ is given by an $m \times n$ array of elements of A_i, determining a morphism $f_i : \mathrm{Spec}(A_i) \to \mathbf{A}_k^{mn}$. The fact that the cokernel of $K \to \mathcal{O}_T^n$ is also locally free implies that each f_i factors through U. These morphisms do not patch together, but the **compositions** $\phi \circ f_i$ (for the map $\phi : U \to G(m,n)$ defined earlier) do patch to a morphism $f : T \to G(m,n)$, and this morphism does not depend upon the trivializations or the sequence $(*)$ within its equivalence class.

Thus $u(T) : F_{m,n}(T) \to h_{G(m,n)}(T)$ is defined on the level of objects. One easily checks that u is a transformation of functors.

To invert u, we find the universal family on $G(m,n)$. The key is to construct the sub-bundle $K \hookrightarrow \mathcal{O}^n_{G(m,n)}$. To do this, we use the action μ again. Recall the open cover $W_J := G(m,n) \cap U_{x_J} \cong Z_J \cong \mathbf{A}_k^{m(n-m)}$ of the Grassmannian from (iii) of the proof of the proposition. Putting this together with (i) of the same proof, we obtain isomorphisms:

$$\rho_{I,J} : \mathrm{GL}(m,k) \times (W_I \cap W_J) \cong \phi^{-1}(W_I \cap W_J) \cong \mathrm{GL}(m,k) \times (W_I \cap W_J)$$

identifying $W_I \cap W_J$ with open subsets of Z_I on the left side and Z_J on the right. The morphism to $\mathrm{GL}(m,k)$ determined by $\rho_{I,J}$ descends to a morphism $g_{I,J} : W_I \cap W_J \to \mathrm{GL}(m,k)$ which satisfies the cocycle condition, defining the locally free sheaf K of rank m on $G(m,n)$.

The embeddings $W_J \cong Z_J \hookrightarrow \mathbf{A}_k^{mn}$ give maps $\mathcal{O}_{W_J}^m \to \mathcal{O}_{W_J}^n$ which glue together via the transition functions to an injective map $K \hookrightarrow \mathcal{O}_{G(m,n)}^n$. The cokernel is determined up to isomorphism, and is locally free. This is our universal family. Via the pull-back applied to this family, we obtain a transformation $v : h_{G(m,n)} \to F_{m,n}$.

It is immediate from the constructions that u and v are inverses(!)

Corollary: $G(m,n) \cong G(n-m,n)$.

Proof: We only need to find an isomorphism of functors $F_{m,n} \cong F_{n-m,n}$ by the Theorem. But such an isomorphism is obtained by dualizing the exact sequences $(*)$ and fixing an isomorphism $k^n \cong (k^n)^*$, which then gives simultaneous isomorphisms $\mathcal{O}_T^n \cong (\mathcal{O}_T^n)^*$ for all the schemes T, hence the desired bijection on equivalence classes of exact sequences.

Note: Of course this isomorphism coincides with the identification of a subspace of k^n of dimension m with a subspace of $(k^n)^*$ of dimension $n-m$ by taking the dual of the cokernel. But without the Theorem we cannot deduce an isomorphsm of **schemes** from this bijection of k-rational points.

Variations: The proof goes through without modification if we replace "k" by "A" (any commutative ring with 1) giving us Grassmannians over $\mathrm{Spec}(A)$. If S is any scheme with affine open cover $S = \cup \mathrm{Spec}(A_i)$, then the Grassmannians over $\mathrm{Spec}(A_i)$ patch together to give us a Grassmannian $G(m,n)$ over S. But we can do even better than this. If $\mathcal{E}$ is any locally free sheaf of rank n on S with trivializations over the $\mathrm{Spec}(A_i)$, then the Grassmannians over the affines $\mathrm{Spec}(A_i)$ patch together to give a "twisted" Grassmannian $G(m,\mathcal{E})$ representing the functor from schemes over S to sets:

$$F_{m,\mathcal{E}}(T) = \{\text{equivalence classes of sequences } (*)\}$$

where $(*) : 0 \to K \to \mathcal{E}_T \to Q \to 0$ are short exact sequences of locally free sheaves on T with $\mathrm{rk}(K) = m$ and $\mathcal{E}_T$ is the pull-back of $\mathcal{E}$ under $T \to S$.

The fibers of $G(m,\mathcal{E})$ are just the Grassmannians $G(m,n)$ over $k(s)$ and the corollary generalizes to an isomorphism $G(m,\mathcal{E}) \cong G(n-m,\mathcal{E}^*)$.

Fix a Noetherian scheme S, a projective S-scheme X, a coherent sheaf $\mathcal{E}$ on X, and a polynomial $P(d)$. Given a Noetherian S-scheme T, let $\mathcal{E}_T$ be the pull-back of $\mathcal{E}$ to $X \times_S T$. For coherent sheaf quotients $(*) : \mathcal{E}_T \to \mathcal{Q} \to 0$ on $X \times_S T$, we declare that $(*) \sim (*')$ if there is a commuting diagram:

$$\begin{array}{cccccc} (*) & \mathcal{E}_T & \to & \mathcal{Q} & \to & 0 \\ & \| & & \downarrow & & \\ (*') & \mathcal{E}_T & \to & \mathcal{Q}' & \to & 0 \end{array}$$

We now define a functor from (Noetherian) S-schemes to sets:

$F_{\mathcal{E},P(d)}(T) = \{$equivalence classes of quotients $(*)$ such that $\mathcal{Q}$ is flat over T and each $\mathcal{Q}_t$ has Hilbert polynomial $P(d)\}$

$F_{\mathcal{E},P(d)}(f) = \widetilde{f}^*$ where $\widetilde{f}$ is the "base extension"

$$\begin{array}{ccc} X \times_S U & \stackrel{\widetilde{f}}{\to} & X \times_S T \\ \downarrow & & \downarrow \\ U & \stackrel{f}{\to} & T \end{array}$$

given a morphism $f : U \to T$ of S-schemes.

Example! If $X = S$, $P(d) = n - m$ and $\mathcal{E}$ is locally free of rank n, then $F_{\mathcal{E},P(d)}$ is the functor $F_{m,\mathcal{E}}$ which was represented by the Grassmannian! After all, $\mathcal{Q}$ is flat over $T = X \times_S T$ with Hilbert polynomial $n - m$ if and only if $\mathcal{Q}$ is locally free of rank $n - m$(!)

Theorem (Grothendieck's Quot Scheme): The functor $F_{\mathcal{E},P(d)}$ is always represented by a projective scheme, which we denote by $\mathrm{Quot}(\mathcal{E}, P(d))$.

Proof: As in the earlier variations on the Grassmannian, we may as well assume $S = \mathrm{Spec}(A)$, since the quot schemes will patch. We prove first:

A Simple Case: Suppose $X = \mathbf{P}^m_A$ and $\mathcal{E} = \mathcal{O}^n_{\mathbf{P}^m_A}(l)$ for some l, n.

Let $\mathcal{K}$ be the kernel of $(*) : \mathcal{E}_T \to \mathcal{Q}$. Then $\mathcal{E}_T = \mathcal{O}^n_{\mathbf{P}^m_T}(l)$ is evidently flat over T, hence $\mathcal{K}$ is flat over T as well, with constant Hilbert polynomial $P'(d) := n\binom{m+l+d}{l+d} - P(d)$ on the fibers. Since the $\mathcal{K}_t \hookrightarrow \mathcal{O}^n_{\mathbf{P}^n_t}(l)$ are each coherent subsheaves of a fixed locally free sheaf for each t, uniform vanishing applies. Moreover, it follows from the proof of uniform vanishing in this special case that d_0 only depends upon $P(d), m, n$ and l and not upon the field k, hence it yields the following "more uniform" result:

There is an d_0 independent of $(*)$ and t such that:

$$\mathrm{H}^i(\mathbf{P}^m_t, \mathcal{K}_t(d)) = 0 \text{ for all } t \in T, i > 0, d \geq d_0$$

Remark: This holds for finite fields as well. If $k(t) = \mathbf{F}_q$, then vanishing under the flat base change $\mathrm{Spec}(\overline{\mathbf{F}}_q) \to \mathrm{Spec}(\mathbf{F}_q)$ to the (infinite) algebraic closure implies vanishing for $\mathbf{F}_q$.

By flatness, it follows that the sequences: $0 \to \mathcal{K}_t \to \mathcal{E}_t \to \mathcal{Q}_t \to 0$ are all exact, and uniform vanishing for $\mathcal{K}_t$ and $\mathcal{E}_t = \mathcal{O}^n_{\mathbf{P}^m_t}(l)$ implies uniform vanishing for $\mathcal{Q}_t$. It follows from cohomology and base change that:

$$0 \to \pi_*\mathcal{K}(d_0) \to \pi_*\mathcal{E}_T(d_0) \to \pi_*\mathcal{Q}(d_0) \to 0$$

are exact, and moreover that $\pi_*\mathcal{K}(d_0)$ and $\pi_*\mathcal{Q}(d_0)$ are locally free sheaves of ranks $P'(d_0)$ and $P(d_0)$ respectively, where $\pi : \mathbf{P}^n_T \to T$ is the projection. Finally, note that $\pi_*\mathcal{E}_T(d_0)$ is a trivial vector bundle, naturally isomorphic to the pull-back of $\mathrm{H}^0(\mathbf{P}^m_A, \mathcal{O}^n_{\mathbf{P}^m_A}(l + d_0))$ from $\mathrm{Spec}(A)$.

Hence $(*)$ determines a morphism to the Grassmannian over $\mathrm{Spec}(A)$:

$$T \to G = G(P'(d_0), \mathrm{H}^0(\mathbf{P}^m_A, \mathcal{O}^n_{\mathbf{P}^m_A}(l + d_0)))$$

That is, we have defined $F_{\mathcal{E},P(d)} \to h_G$ on the level of objects. I leave it to you to use flatness again to prove that this is a transformation of functors. Our next task is to find the "image."

Consider the cokernel:

$$\pi^*K \to \mathcal{E}_G(d_0) \to \mathcal{F} \to 0$$

where the first map is the composition $\pi^*K \to \pi^*\pi_*\mathcal{E}_G(d_0) \to \mathcal{E}_G(d_0)$ and K is the universal subbundle on the Grassmannian. Let $\mathrm{Quot}(\mathcal{E}, P(d)) \subset G$ be the term in the flattening stratification of G with respect to $\mathcal{F}$ over which $\mathcal{F}$ is flat with Hilbert polynomial $P(d_0 + d)$. I claim that the transformation of functors factors through $\mathrm{Quot}(\mathcal{E}, P(d))$ and that the "universal quotient": $(*)$ $\mathcal{E}_{\mathrm{Quot}(\mathcal{E},P(d))} \to \mathcal{F}(-d_0)|_{\mathbf{P}^m_{\mathrm{Quot}}}$ inverts the transformation, proving that this "Quot" scheme represents the functor $F_{\mathcal{E},P(d)}$. To see this, note that uniform vanishing implies that $\mathcal{K}_t(d_0)$ is generated by its global sections for each t, and by cohomology and base change again, it follows that the natural map: $\pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{K}(d_0)$ is surjective.

Thus if $T \to G$ is a morphism associated to a quotient $(*)$, then the cokernel above pulls back to:

$$\pi^*\pi_*\mathcal{K}(d_0) \to \mathcal{E}_T(d_0) \to \mathcal{Q}(d_0) \to 0$$

where the first map factors through $\mathcal{K}(d_0)$. This proves not only that the transformation factors through h_{Quot} but it also proves that the universal $(*)$ inverts this transformation, as desired.

So we have $\mathrm{Quot}(\mathcal{E}, P(d))$, given to us as a quasi-projective subscheme of a Grassmannian. To prove it is projective, the **valuative criterion** tells us that it suffices to prove that every morphism $\mathrm{Spec}(K) \to \mathrm{Quot} \to S$ from the fraction field of a discrete valuation ring R that extends to a morphism $\mathrm{Spec}(R) \to S$ also extends to a morphism $\mathrm{Spec}(R) \to \mathrm{Quot}$. In terms of the functor, this says that any flat quotient $(*)\mathcal{E}_K \to \mathcal{Q}$ on $\mathbf{P}^n_K$ extends uniquely to a flat quotient $(*)\mathcal{E}_R \to \overline{\mathcal{Q}}$ on $\mathbf{P}^n_R$. But this is exactly what the properness theorem (or rather its proof) gives us.

This completes the simple case.

Given a coherent sheaf $\mathcal{E}$ on a projective scheme $i : X \subset \mathbf{P}^m_A$, we may use Serre's theorem A to find a surjection $\mathcal{O}^n_X(l) \to \mathcal{E}$ giving rise to quotients: $\mathcal{O}^n_{\mathbf{P}^m_T}(l) \to i_{T*}\mathcal{E}_T$ for each S-scheme T, where $i_T : X_T \hookrightarrow \mathbf{P}^m_T$ is the induced closed immersion. In this way, flat quotients of $\mathcal{E}_T$ push forward under i_T to flat quotients of $\mathcal{O}^n_{\mathbf{P}^m_T}(l)$ and each $(*) : \mathcal{E}_T \to \mathcal{Q}$ determines a morphism $T \to \mathrm{Quot}(\mathcal{O}^n_{\mathbf{P}^m_A}(l), P(d))$ to the Quot scheme already constructed in the simple case. Let $\mathcal{G}$ be the kernel on $\mathbf{P}^m_{\mathrm{Quot}}$:

$$0 \to \mathcal{G} \to \mathcal{O}^n_{\mathbf{P}^m_{\mathrm{Quot}}}(l) \to i_{\mathrm{Quot}*}\mathcal{E}_{\mathrm{Quot}} \to 0$$

and choose $d_1 \geq d_0$ so that $\pi^*\pi_*\mathcal{G}(d_1) \to \mathcal{G}(d_1)$ is surjective. We have already seen that $\pi_*\mathcal{Q}(d_1)$ is locally free on Quot by cohomology and base change. It follows that the zero locus of $\pi_*\mathcal{G}(d_1) \to \pi_*\mathcal{Q}(d_1)$ is a closed subscheme $Z \subset \mathrm{Quot}(\mathcal{O}^n_{\mathbf{P}^m_A}(l), P(d))$. Over this subscheme, the universal quotient lifts to a quotient: $i_{Z*}\mathcal{E}_Z \to \mathcal{Q}|_{\mathbf{P}^m_Z}$ and it follows that $\mathcal{Q}|_{\mathbf{P}^m_Z}$ is the push-forward of a sheaf on X_Z which is the universal quotient for $\mathcal{E}_Z$. In other words, $\mathrm{Quot}(\mathcal{E}, P(d)) := Z$ represents $F_{\mathcal{E},P(d)}$.

Definition (The Hilbert Scheme): The Hilbert scheme $H_{X,P(d)}$ is the special case of the quot scheme representing the functor $F_{\mathcal{O}_X,P(d)}$.

Alternatively, the Hilbert scheme represents the functor:

$$F_{X,P(d)}(T) = \{\text{closed subschemes } Z \subset X \times_S T \text{ that are flat over } T \\ \text{with Hilbert polynomial } P(d)\}$$

via the identification of $i : Z \hookrightarrow X_T$ with the quotient $\mathcal{O}_{X_T} \to i_*\mathcal{O}_Z$.

Example (Mori): Suppose X is a projective scheme over Spec($\mathbf{Z}$) and $X_{\mathbf{F}_p}$ has a rational curve C (i.e. $\chi(C, \mathcal{O}_C) = 1$) of degree d for infinitely many primes p and fixed d. Then $X_{\mathbf{Q}}$ has a rational curve of degree d, as well. Indeed, the Hilbert polynomial of C is $P(d) = d + 1$, and since the Hilbert scheme $H_{X,d+1}$ is projective over Spec($\mathbf{Z}$), it must have only finitely many components. Thus it has a component whose image in Spec($\mathbf{Z}$) contains infinitely many primes. Such a component must dominate Spec($\mathbf{Z}$), and must therefore surject onto Spec($\mathbf{Z}$) by properness. A point in the fiber of such a component over Spec($\mathbf{Q}$) yields the desired rational curve.

Observation: Quot schemes "commute with base chage". That is, if $\mathcal{F}$ is a coherent sheaf on a T-scheme Y and we are given a morphism $S \to T$, then the Quot schemes fit into a fiber square:

$$\begin{array}{ccc} \mathrm{Quot}(\mathcal{F}_S, P(d)) & \to & \mathrm{Quot}(\mathcal{F}, P(d)) \\ \downarrow & & \downarrow \\ S & \to & T \end{array}$$

This follows from the theorem since the fiber product $\mathrm{Quot}(\mathcal{F}, P(d)) \times_T S$ represents the functor $F_{\mathcal{F}_S,P(d)}$.

Example (Maps to the Grassmannian): Suppose X is a projective k-scheme for some field k and suppose $P(d)$ is the Hilbert polynomial of a locally free sheaf Q of rank $n - m$ which is a quotient of a trivial bundle via $(*) : \mathcal{O}_X^n \to Q$. Then $(*)$ determines a k-rational point $q \in \mathrm{Quot}(\mathcal{O}_X^n, P(d))$ and also a morphism $f_q : X \to G(m, n)$. Moreover, the universal quotient $\mathcal{O}^n_{X\times \mathrm{Quot}} \to \mathcal{Q}$ is locally free in a neighborhood of $X \times \{q\}$ and it follows from properness that there is an open neighborhood $q \in U$ in the Quot scheme such that $\mathcal{Q}|_{X\times U}$ is locally free of rank $n-m$, hence all the points in U parametrize morphisms $X \to G(m, n)$. Thus we may think of the component(s) of the quot scheme containing U as a compactification of a space of maps from X to the Grassmannian.

Example (Another Look at the Grassmannian): Suppose $P(d)$ is the Hilbert polynomial of a linear subspace $V \subset \mathbf{P}_k^{n-1}$ of (projective) dimension $m-1$. Then $H_{\mathbf{P}_k^{n-1},P(d)} \cong G(m,n)$. Indeed, out of the universal subbundle $K \hookrightarrow \mathcal{O}_G^n$, one obtains an inclusion of projective bundles $P(K) \subset \mathbf{P}_k^{n-1} \times G$ which is a flat family over G. This gives a morphism $G(m,n) \to H_{\mathbf{P}_k^{n-1},P(d)}$. On the other hand, one can show that the only projective subschemes of $\mathbf{P}_k^{n-1}$ with Hilbert polynomial $P(d)$ are the linear subspaces, and that this morphism is an isomorphism.

Last Example (Hypersurfaces in Projective Space): The complete linear series $\mathbf{P}_k^{\binom{n+d}{n}-1}$ of hypersurfaces of degree d in $\mathbf{P}_d^n$ is a Hilbert scheme. Indeed, let $x_0, ..., x_n$ be the coordinates on $\mathbf{P}_k^n$ and let a_J for $J = (j_0, j_1, ..., j_n)$ satisfying $\sum j_i = d$ be the coordinates on $\mathbf{P}_k^{\binom{n+d}{n}-1}$. Then:

$$V(\sum_J a_J x_0^{j_0} x_1^{j_1} ... x_n^{j_n}) \subset \mathbf{P}_k^n \times \mathbf{P}_k^{\binom{n+d}{n}-1}$$

is the universal divisor, exhibiting $\mathbf{P}_k^{\binom{n+d}{n}-1}$ as the Hilbert scheme for hypersurfaces of degree d. Again, one checks that the only projective subscheme of $\mathbf{P}_k^n$ with the Hilbert polynomial of a hypersurface is in fact a hypersurface.

6. Deformation Theory I. Deformation theory expresses properties of moduli spaces in terms of cohomology groups by studying families over local rings, especially Artinian local rings. As our first foray into this subject, we will characterize the Zariski tangent space to the Quot scheme, give a criterion for smoothness and finally a dimension estimate.

Throughout this section, X will be a nonsingular projective variety over a field k, and the Quot scheme $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is therefore a projective scheme over k. Let q be a k-rational point of the Quot scheme and let $q : \mathcal{O}_X^n \to \mathcal{Q}$ be the corresponding quotient, with kernel $\mathcal{K}$ determined up to isomorphism.

Theorem 1: The Zariski tangent space to $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ at q is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ as a vector space over k.

Theorem 2: If $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}) = 0$ then $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is nonsingular at q.

Theorem 3: In any case,

$$\dim_q \mathrm{Quot}(\mathcal{O}_X^n, P(d)) \geq \dim_k(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})) - \dim_k(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}))$$

and equality implies $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is a local complete intersection at q.

Corresponding to the theorems are three "commutative algebra" lemmas which are valid for any k-rational point y in a k-scheme Y of finite type.

Lemma 1: The set of morphisms $f : \mathrm{Spec}(k[\epsilon]/\epsilon^2) \to Y$ extending $y \in Y$ may be naturally identified with the Zariski tangent space to Y at y.

Lemma 2: $\mathcal{O}_y$ is regular if and only if for every surjection $B \to A$ of local Artinian rings with residue field k, every element of $\mathrm{Hom}_k(\mathcal{O}_y, A)$ lifts to an element of $\mathrm{Hom}_k(\mathcal{O}_y, B)$.

Lemma 3: $\mathcal{O}_y$ is a quotient $f : R \to \mathcal{O}_y$ of a regular local k-algebra R, such that f induces an isomorphism on Zariski tangent spaces.

(In particular, the dimension of R is equal to $\dim_k(m_y/m_y^2)$.)

Proof of Theorem 1: The natural identification of Lemma 1 goes as follows. The morphisms f extending $y \in Y$ correspond to local homomorphisms $f^\# : \mathcal{O}_y \to k[\epsilon]/\epsilon^2$. These are equivalent to linear maps $m_y/m_y^2 \to \epsilon k$, hence to the Zariski tangent space. For maps to the Quot scheme, each such f is a quotient $\overline{q} : \mathcal{O}^n_{X_{k[\epsilon]/\epsilon^2}} \to \overline{\mathcal{Q}}$ extending the given quotient $q : \mathcal{O}_X^n \to \mathcal{Q}$. We thus need to characterize such quotients.

We first work locally. Suppose T is a finitely generated k-algebra and A is a Noetherian local ring with residue field k. Then a finitely generated module M over $T_A := A \otimes_k T$ is flat over A if and only if:

$$0 = \mathrm{Tor}_1^A(k, M) = \ker(m_A \otimes_A M \to M)$$

Now suppose $T_A^n \to \overline{Q}$ is a flat quotient of T_A-modules with kernel $\overline{K}$ which reduces to $T^n \to Q$ (with kernel K) modulo m_A. Then flatness of $\overline{Q}$ is equivalent to the exactness of the sequence of reductions modulo m_A: $0 \to k \otimes_A \overline{K} \to T^n \to Q \to 0$, i.e. to the condition that $\overline{K}$ reduce to K.

It is also equivalent, by the five lemma, to the condition that the left vertical map below be surjective (it is automatically injective)

$$\begin{array}{ccccccccc} 0 & \to & \ker & \to & m_A \otimes_A T_A^n & \to & m_A \otimes_A \overline{Q} & \to & 0 \\ & & f\downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{K} \cap m_A T_A^n & \to & m_A T_A^n & \to & m_A \overline{Q} & \to & 0 \end{array}$$

But the image of f is $m_A \overline{K}$, so this really says that:

$$m_A \overline{K} = \overline{K} \cap m_A T_A^n$$

is a criterion for flatness of $\overline{Q}$. One example of a flat $\overline{Q}$ is the trivial quotient:

$$(q \otimes 1) : T_A^n \to Q_A \quad \text{with kernel} \quad K_A = K + m_A K$$

which indeed satisfies $m_A K_A = m_A K = K_A \cap m_A T_A^n$.

Now set $A = k[\epsilon]/\epsilon^2$ and choose generators and relations for K:

$$R \xrightarrow{r} G \xrightarrow{g} T^n$$

That is, R and G are free R-modules, $\ker(g) = \mathrm{im}(r)$ and $\mathrm{im}(g) = K$. Note that if $\overline{q} : T_A^n \to \overline{Q}$ is a flat extension of q, then by Nakayama's Lemma, the generators lift to generators of $\overline{K}$ since $\overline{K}$ reduces to K modulo m_A. Likewise by flatness of $\overline{K}$, the relations lift to relations, giving a presentation of $\overline{K}$:

$$R_A \xrightarrow{\overline{r}} G_A \xrightarrow{\overline{g}} T_A^n$$

Conversely, let $\overline{r}$ and $\overline{g}$ be arbitrary lifts of r and g. Then:

Assertion 1: The map $(q \otimes 1) \circ \overline{g} \circ \overline{r} : R_A \to Q_A$ does not depend upon $\overline{r}$ and descends to a map of T-modules:

$$e(\overline{g}) : R \to \epsilon Q \text{ for } R = R_A/\epsilon R_A \text{ and } \epsilon Q \subset Q_A$$

which vanishes if and only if $\mathrm{coker}(\overline{g})$ is flat over A.

Proof: Since $\overline{g} \circ \overline{r}$ is zero modulo ϵ and $\epsilon^2 = 0$, it follows that the map descends as indicated. Since $(q \otimes 1) \circ \overline{g}$ is zero modulo ϵ, it does not depend upon the lift of $\overline{r}$. If $\mathrm{coker}(\overline{g})$ is flat, we've already seen that we can find a lift $\overline{r}$ so that $\overline{g} \circ \overline{r} = 0$. Thus $e(\overline{g}) = (q \otimes 1) \circ \overline{g} \circ \overline{r} = 0$ in that case.

For an arbitrary lift $\overline{g}$, let $\overline{K}$ be its image. Then for each $\overline{r}$, the image of $\overline{g} \circ \overline{r}$ is in $\overline{K} \cap \epsilon T_A^n$, and if $e(\overline{g}) = 0$, then it is additionally in $\epsilon K = \epsilon \overline{K}$. Thus to show that $\mathrm{coker}(\overline{g})$ is flat by the earlier criterion for flatness, we need to show that $\epsilon K = \overline{K} \cap \epsilon T_A^n$. To do this, it suffices to show that given $z \in \overline{K} \cap \epsilon T_A^n$ and a lift $\overline{r}$, then there is an $x \in R_A$ so that $\overline{g} \circ \overline{r}(x) - z \in \epsilon K$. Choose $y \in G_A$ so that $\overline{g}(y) = z$. Then the image of y in G is in the image of r, so we can find an $x \in R_A$ with the same image in G. But then $\overline{r}(x) - y \in \epsilon G$, hence $\overline{g} \circ \overline{r}(x) - z \in \epsilon K$ as desired.

Given a lift $\overline{g}$ satisfying $e(\overline{g}) = 0$, then $(q \otimes 1) \circ \overline{g}$ descends to a map:

$$\phi(\overline{g}) : K \to \epsilon Q$$

(it descends to $G \to \epsilon Q$ as above, and further to $\phi(\overline{g})$ because $e(\overline{g}) = 0$). Notice that this map only depends upon the image of $\overline{g}$, i.e. only upon the equivalence class of flat quotients determined by $\overline{g}$. Moreover, if $\phi(\overline{g}) = \phi(\overline{g}')$, then $\mathrm{im}(\overline{g} - \overline{g}') \subset \epsilon K \subset \overline{K} \cap \overline{K}'$, so the images are the same and thus $\phi(\overline{g})$ determines the image of $\overline{g}$. Finally, any $\phi : K \to \epsilon Q$ gives rise to a map $G_A \to Q_A$ which can be factored through a $\overline{g}$ satisfying $\phi(\overline{g}) = \phi$. Thus the $\overline{q} : T^n \to \overline{Q}$ have been identified with $\mathrm{Hom}_T(K, \epsilon Q)$.

It is easy to globalize this to X. Choose an open cover of X by affines $U_i = \mathrm{Spec}(T_i)$ over which the quotient $q : \mathcal{O}_X^n \to \mathcal{Q}$ is of the form $T_i^n \to Q_i$. Then an extension of q to a flat quotient $\overline{q} : \mathcal{O}_{X_A}^n \to \overline{\mathcal{Q}}$ gives local extensions of the quotients $T_i^n \to Q_i$ each of which corresponds to a $\phi_i \in \mathrm{Hom}_{T_i}(K_i, Q_i)$ via the local version above. The ϕ_i then patch together to a global element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ because they must agree on the overlaps. And conversely, an element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q})$ gives local quotients $T_{iA}^n \to \overline{Q}_i$ which patch to a global extension of q. This proves Theorem 1.

Proof of Theorem 2: Translating Lemma 2 into flat quotients, we need to show that any extension of q to a quotient $\overline{q}^A : \mathcal{O}^n_{X_A} \to \overline{\mathcal{Q}}^A$ which is flat over A further extends to a quotient $\overline{q}^B : \mathcal{O}^n_{X_B} \to \overline{\mathcal{Q}}^B$ which is flat over B. It suffices to prove this under the assumption that $m_B J = 0$ where:

$$0 \to J \to B \to A \to 0$$

(which is the case, for instance, when $B = k[\epsilon]/\epsilon^2$ and $A = k$)

Again, we will start locally. If a flat quotient $\overline{q}^A : T^n_A \to \overline{Q}^A$ is given. Then $\overline{q}^B : T^n_B \to \overline{Q}^B$ reducing to $\overline{q}^A$ modulo J is flat if and only if:

$$0 = \mathrm{Tor}_1^B(A, \overline{Q}^B) = \ker(J \otimes_B \overline{Q}^B \to \overline{Q}^B)$$

As before, this is equivalent to $\overline{K}^B$ reducing to $\overline{K}^A$ modulo J, and to:

$$J \otimes_k K = J\overline{K}^B = \overline{K}^B \cap JT^n_B$$

Also as before, we can start with a presentation of $\overline{K}^A$: $R_A \stackrel{\overline{r}^A}{\to} G_A \stackrel{\overline{g}^A}{\to} T^n_A$ and attempt to further lift to a presentation with flat $\mathrm{coker}(\overline{g})$.

Assertion 1: Suppose $R_B \stackrel{\overline{r}}{\to} G_B \stackrel{\overline{g}}{\to} T^n_B$ are arbitrary lifts of $\overline{r}^A$ and $\overline{g}^A$. Then $\overline{g} \circ \overline{r} : R_B \to T^n_B$ descends to a map from R to $J \otimes_k T^n$ and composing further with q, we obtain a map:

$$e(\overline{g}) : R \to J \otimes_k Q$$

which does not depend upon $\overline{r}$ and vanishes iff $\mathrm{coker}(\overline{g})$ is flat.

Proof: This basically is the same as the previous Assertion 1, except that the map $J \otimes_k T^n \to J \otimes_k Q$ does not descend from a trivial quotient extending q because the trivial quotient probably does not extend $\overline{q}^A$(!) Absent this, a couple of the arguments need to be slightly modified. If $\overline{r}$ and $\overline{r}'$ are two lifts of $\overline{r}^A$, then the image of $\overline{r} - \overline{r}'$ lies in $J \otimes_k G$, and the further image under $\overline{g}$ lies in $J \otimes_k K$, showing that $e(\overline{g})$ does not depend upon $\overline{r}$. Also, if $\overline{g} \circ \overline{r} = 0$, then it is clear that $e(\overline{g}) = 0$. The rest of the argument is the same.

Corollary: If the set of quotients $\overline{q}$ extending $\overline{q}^A$ is nonempty, then it is an affine space with underlying vector space $\mathrm{Hom}_T(K, Q) \otimes_k J$.

Proof: If a single quotient exists, denote it by $\overline{q}^B$, and use it to define:

$$\phi(\overline{g}) : K \to J \otimes_k Q$$

as before to be the descendant of $\overline{q}^B \circ \overline{g}$. Then as before, we get an identification of $\mathrm{Hom}_T(K, Q) \otimes_k J$ with the set of equivalence classes of flat quotients extending $\overline{q}^A$, and $\overline{q}^B$ corresponds to the zero element of this vector space. But the choice of $\overline{q}^B$ was not canonical (unlike the situation of Theorem 1) so we only get an affine space from this identification.

Next, we need to investigate the "obstruction" to the existence of a $\overline{q}^B$.

Assertion 2: Given $\overline{g}$, then the cokernel: $R \stackrel{r+e(\overline{g})}{\to} G + J \otimes_k Q \to E(\overline{g}) \to 0$ fits into an exact sequence:

$$0 \to J \otimes_k Q \to E(\overline{g}) \to K \to 0$$

and the corresponding extension class:

$$\epsilon(\overline{q}^A) \in \mathrm{Ext}^1(K, Q \otimes_k J)$$

only depends upon $\overline{q}^A$ and only vanishes if a flat $\overline{q}^B$ extending $\overline{q}^A$ exists.

Proof: Lots of homological algebra.

Now for the global version. Let $U_i = \mathrm{Spec}(T_i)$ be an open affine cover of X with intersections $U_i \cap U_j = \mathrm{Spec}(T_{ij})$. A global quotient $\overline{q}^A : \mathcal{O}^n_{X_A} \to \overline{\mathcal{Q}}^A$ gives rise to local quotients $\overline{q}^A_i : T^n_{i\ A} \to \overline{Q}^A_i$ hence to local extension classes $\epsilon(\overline{q}^A_i) \in \mathrm{Ext}^1_{T_i}(K_i, Q_i) \otimes_k J$ which agree on the overlaps, patching to a global extension class $\epsilon(\overline{q}^A) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{Q}) \otimes_k J$. From the local-to-global spectral sequence for Ext, it follows that there are maps:

$$\mathrm{H}^1(X, Hom(\mathcal{K}, \mathcal{Q})) \otimes_k J \hookrightarrow \mathrm{Ext}^1(\mathcal{K}, \mathcal{Q}) \otimes_k J \to \mathrm{H}^0(X, Ext^1(\mathcal{K}, \mathcal{Q})) \otimes_k J \to \ldots$$

If $\epsilon(\overline{q}^A)$ has zero image in $\mathrm{H}^0(X, Ext^1(\mathcal{K}, \mathcal{Q})) \otimes_k J$, then the local $\epsilon(\overline{q}^A_i)$ are zero, hence there are local lifts $\overline{g}_i$ giving rise to flat quotients $\overline{q}^B_i$. These local quotients may not patch to a global quotient, but their differences over $\mathrm{Spec}(T_{ij})$ may be interpreted as elements $\phi_{ij} \in \mathrm{Hom}_{T_{ij}}(K_{ij}, Q_{ij})$ which vanish if and only $\overline{q}^B_i$ patches with $\overline{q}^B_j$ by the corollary to Assertion 1. It is not hard to see that the ϕ_{ij} are a cocycle, i.e.

$$\{\phi_{ij}\} \in \mathrm{H}^1(X, Hom(\mathcal{K}, \mathcal{Q})) \otimes J$$

whose image in the $\mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$ is $\epsilon(\overline{q}^A)$. Thus if this global obstruction class is zero, then the cocycle is a coboundary, satisfying $\phi_{ij} = \phi_i - \phi_j$ for $\phi_i \in \mathrm{Hom}(K_i, Q_i) \otimes_k J$. We can use the ϕ_i to modify our $\overline{g}_i$ (again using the corollary to Assertion 1) to produce local quotients that do patch to a global quotient. Thus Theorem 2 is proved!

Remark: It is perfectly possible to have a proper subspace $V \subset \mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$ with the property that every $\epsilon(\overline{\mathcal{Q}}^A) \in V \otimes J$. We define:

$$\mathrm{Obst}(q) \subset \ \mathrm{Ext}^1(\mathcal{K},\mathcal{Q})$$

to be the smallest such subspace, called the obstruction space, and note that the previous proof shows that if the obstruction space is zero, then $\mathrm{Quot}(\mathcal{O}_X^n, P(d))$ is nonsingular at q. The converse is also easily seen to be true....if the Quot scheme is nonsigular, then global quotients lift, so local quotients lift, so the local obstruction classes are zero, and the resulting cocycles are zero because the local liftings patch to a global lift.

Proof of Theorem 3: Let $R \to \mathcal{O}_q$ be the surjective map of local rings from Lemma 3 and let I be the kernel. Then consider the exact sequence:

$$(*)\ 0 \to J \to B \to \mathcal{O}_q \to 0$$

where $B = R/m_R I$ and $J = I/m_R I$. Note that $Jm_B = 0$ and J is a finite dimensional vector space over k of dimension equal to the number of generators of I. A "universal" quotient $\overline{q} : \mathcal{O}^n_{X_{\mathcal{O}_q}} \to \overline{Q}$ over $\mathcal{O}_q$ exists, so we are in a position to use the machinery of the proof of Theorem 2. If we prove that $\dim_k(J) \leq \dim_k(\mathrm{Ext}^1(\mathcal{K},\mathcal{Q}))$ then we will have proved Theorem 3. In fact, we will prove that $\dim_k(J) \leq \dim_k(\mathrm{Obst}(q))$ which is somewhat better.

Here's the idea. Elements j^* of the dual space J^* push forward $(*)$ to:

$$0 \to k \to B' \to \mathcal{O}_q \to 0$$

giving rise to obstruction classes: $\epsilon_{j^*}(\overline{q}) \in \mathrm{Obst}(q)$. Such a class cannot be zero because if it were, then there would be a flat quotient over B' which, by the universal property of the Quot scheme, would yields a map $\mathcal{O}_q \to B'$ splitting the sequence above. But this would in turn imply that the Zariski tangent space of B' (hence of B, hence of R) is larger than that of $\mathcal{O}_q$, which contradicts our choice of R. Thus we have an injective map $J^* \to \mathrm{Obst}(q)$. It only remains to show that this map is k-linear to complete the proof.

Example (Quot Schemes on Smooth Curves) Let C be a nonsingular projective curve, and consider the Quot scheme: $\mathrm{Quot}(\mathcal{O}_C^n, P(d))$. Then $P(d)$ is either linear or constant, and we will treat the two cases separately.

(a) $P(d) = a$. Each quotient $\mathcal{O}_C^n \to \mathcal{Q}_Z$ is supported on a zero-dimensional subscheme $Z \subset C$ and each kernel $\mathcal{K}$ is locally free of rank n. Thus the Zariski tangent space to $\mathrm{Quot}(\mathcal{O}_C^n, e)$ at q is:

$$\mathrm{Hom}(\mathcal{K}, \mathcal{Q}_Z) = \mathrm{H}^0(X, \mathcal{K}^* \otimes \mathcal{Q}_Z) \text{ which has constant dimension } na$$

and the Quot scheme is nonsingular since $\mathrm{Ext}^1(\mathcal{K}, \mathcal{Q}) = \mathrm{H}^1(X, \mathcal{K}^* \otimes \mathcal{Q}_Z) = 0$

The Hilbert scheme is the familiar symmetric product:

$$\mathrm{Hilb}_{C,a} = \mathrm{Sym}^a(C)$$

and in general, there is a morphism $\mathrm{Quot}(\mathcal{O}_C^n, a) \to \mathrm{Sym}^a(C)$ defined as follows. The kernel $\mathcal{K} \to \mathcal{O}_{C\times \mathrm{Quot}}^n$ of the universal quotient over the Quot scheme is locally free (this is a consequence of flatness) and of rank n. One can check that the cokernel of the top exterior power $\wedge^n \mathcal{K} \to \mathcal{O}_{C\times \mathrm{Quot}}$ is a flat family of quotients of length a. This defines the morphism.

(b) $P(d) = (n-m)d + b$ where $0 < m < n$.

Then as before, $\mathcal{K}$ is locally free but this time $\mathcal{Q}$ is not supported on a zero-dimensional subscheme when q is a point of $\mathrm{Quot}(\mathcal{O}_C^n, (n-m)d + b)$. We have already discussed the fact that we should regard this scheme as a compactification of the space of maps of a fixed degree from C to $G(m,n)$. This space can be singular! However, using the surjection:

$$\mathrm{H}^1(C, \mathcal{K}^* \otimes \mathcal{O}_C^n) \to \mathrm{H}^1(C, \mathcal{K} \otimes \mathcal{Q}) = \mathrm{Ext}^1(\mathcal{K}, \mathcal{Q})$$

we see that if $\mathrm{H}^1(C, \mathcal{K}^*) = 0$, then the Quot scheme is nonsingular at q. This is always the case if $C = \mathbf{P}^1$ (every summand of $\mathcal{K}$ has non-positive degree) but may fail if $C \neq \mathbf{P}^1$.

Example: (Points on a Smooth Surface)

Let S be a nonsingular projective surface, and consider $H_{S,a}$ the Hilbert scheme of subschemes of length a of S. If $h \in H_{S,a}$ corresponds to a quotient:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$$

then it is relatively easy to see that:

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_Z, \mathcal{O}_Z) \text{ has dimension } 2a \text{ and } \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{I}_Z, \mathcal{O}_Z) \text{ has dimension } a$$

independent of $\mathcal{O}_Z$. On the other hand, as in the curve case, $(S^a - \Delta)/\Sigma_a$ (the complement of the diagonal quotiented by the symmetric group) is a nonsingular open subset of $H_{S,a}$ of dimension $2a$ corresponding to reduced subschemes. Unlike the curve case, this does not extend to an isomorphism from $\mathrm{Sym}^a(S)$. But this does prove, together with the dimension of the Zariski tangent space computed above, that the closure of this open subset is a nonsingular connected component of the Hilbert scheme. One can additionally prove that the Hilbert scheme is connected. Notice that this is an example where the obstruction space is zero but Ext^1 is not zero!

Example: (Local Complete Intersections)

If $Z \subset X$ is a local complete interesection, then:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{H}^1(Z, N_{Z/X})$$

where $N_{Z/X}$ is the normal bundle to Z in X. We can apply this to help us to analyze Hilbert schemes at such points. For example.

(a) If $C \subset \mathbf{P}^n$ is is a nonsingular curve of degree $e > 2g-2$, then it is a nonsingular point of the Hilbert scheme $H_{\mathbf{P}^n, ed+1-g}$. That is because the Euler sequence gives rise to a surjection:

$$\mathrm{H}^1(C, \mathcal{O}_C^{n+1}(1)) \to \ \mathrm{H}^1(C, N_{C/\mathbf{P}^n})$$

and the former space is zero by Riemann-Roch.

(b) Pairs of skew lines in $\mathbf{P}^3$ are nonsingular points of $H_{\mathbf{P}^3, 2d+1}$ giving a component of dimension $2\mathrm{dim}(G(2,4)) = 8$. On the other hand, a plane conic union a point is a nonsingular point of the same Hilbert scheme, giving a component, this time of dimension $5+3+3 = 11$ (dimensions of conics in $\mathbf{P}^2$, planes in $\mathbf{P}^3$ and points in $\mathbf{P}^3$ respectively). These two components meet along the locus of degenerate conics in $\mathbf{P}^2$ with a nilpotent in the singular locus. This gives an example of a Hilbert scheme with two components of different degrees.

(c) Projected canonical curves of genus 5. Let $C \subset \mathbf{P}^3 \subset \mathbf{P}^4$ be a nonsingular canonical genus 5 curve, projected from $\mathbf{P}^4$ and reembedded in $\mathbf{P}^4$. This is a point in the Hilbert scheme $H_{\mathbf{P}^4,10d-4}$. There are two components of this Hilbert scheme meeting along such points. One of them is the component (of dimension 36) consisting of embedded canonical curves (the general one of which is not projected) and the other (also of dimension 36) is the component consisting of curves of degree 10 and genus 5 which span a $\mathbf{P}^3$ in $\mathbf{P}^4$ (the general one of which is not canonical). This gives an example of a singular point of a Hilbert scheme which parametrizes a nonsingular embedded curve.

7. Intersection Theory I. Once a moduli space is constructed, the next step in "counting" is to convert enumerative questions into intersections. As our first foray into intersection theory, we will discuss Chern classes and the Riemann-Roch theorem. See Fulton's *Intersection Theory* for details.

Throughout, X will be a scheme of finite type over a field k.

The **cycle group** $Z_*(X)$ is the free abelian group on the set of closed subvarieties of X. It is graded by dimension, and a d-cycle α is **rationally equivalent to zero**, written $\alpha \sim 0$, if there exist finitely many subvarieties $W_i \subset X$ of dimension $d+1$ together with rational functions $r_i \in K(W_i)^*$ such that $\alpha = \sum[\mathrm{div}(r_i)]$, where $[\mathrm{div}(r_i)]$ is the divisor (of zeroes minus poles) associated to r_i. The set of d-cycles rationally equivalent to 0 is a subgroup of $Z_d(X)$, denoted $\mathrm{Rat}_d(X)$, and the **Chow group** of X is the graded group:

$$A_*(X) = \bigoplus_{d=0}^{n} Z_d(X)/\mathrm{Rat}_d(X).$$

Examples: (a) If $n = \dim(X)$, then $A_n(X)$ is the free abelian group on the components of X of dimension n.

(b) $A_*(\mathbf{A}_k^n) = A_n(\mathbf{A}_k^n) \cong \mathbf{Z}$.

(c) $A_d(\mathbf{P}_k^n) \cong \mathbf{Z}$, generated by linear subspaces, for all $0 \le d \le n$.

(d) $A_{n-1}(\mathbf{P}_k^n - S) \cong \mathbf{Z}/d\mathbf{Z}$ if S is an irreducible hypersurface of degree d.

(e) $A_0(C)$ is not finitely generated if C is a projective curve of genus > 0.

If $f : X \to Y$ is a **proper** morphism, then there is a functorial graded **proper push-forward** of Chow groups $f_* : A_d(X) \to A_d(Y)$ defined, on the level of closed subvarietes, by:

$$f_*[V] = \begin{cases} 0 \ \text{ if } \ \dim(f(V)) < \dim(V) \\ \\ [K(V) : K(f(V))]\,[f(V)] \text{ if } \dim(f(V)) = \dim(V) \end{cases}$$

Example: If X is proper over $\mathrm{Spec}(k)$, then the degree map:

$$\deg : A_0(X) \to \mathbf{Z} = A_0(\mathrm{Spec}(k))$$

is the proper push-forward to $\mathrm{Spec}(k)$.

If $f : X \to Y$ is a **flat** morphism of relative dimension n, then there is a functorial **flat pull-back** $f^* : A_d(Y) \to A_{d+n}(X)$ defined by setting $f^*[V] = [f^{-1}(V)]$ (the cycle associated to the scheme $f^{-1}(V)$). Flat pull-backs also commute with proper push-fowards in the following sense:

If $f : X \to Y$ is proper and $g : Y' \to Y$ is flat of relative dimension n, let:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & f \downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then $g^* f_* = f'_* g'^* : A_d(X) \to A_{d+n}(Y')$.

Suppose $i : Y \hookrightarrow X$ is a closed immersion, and $j : U \hookrightarrow X$ is the open immersion of the complement of Y, Then i is proper, j is flat of relative dimension 0, and the following sequence is exact:

$$A_*(Y) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \to 0$$

Another Example: If $f : E \to X$ is an affine bundle of rank n, then the flat pull-back:

$$f^* : A_d(X) \to A_{d+n}(E)$$

is surjective for all d (including $d < 0$, where $A_d(X) = 0$). If E is a vector bundle, then the f^* are injective, as well.

Suppose L is a **line bundle** on X. Then the **first chern class** $c_1(L)$ should be thought of as an operator:

$$\cap c_1(L) : A_d(X) \to A_{d-1}(X)$$

defined by $[V] \mapsto [\mathrm{div}(s)]$ where s is any (meromorphic) section of $L|_V$ other than the zero section. We let $c_1(L)^i$ be the i-fold iteration of $\cap c_1(L)$.

If E is a **vector bundle** of rank $e + 1$ on X, let $P(E)$ be the projective bundle, with (proper, flat) projection $p : P(E) \to X$ of relative dimension e and line bundle $\mathcal{O}_{P(E)}(1)$. The ***i*th Segre class** $s_i(E)$ is the operator

$$\cap s_i(E) : A_d(X) \to A_{d-i}(X); \quad \alpha \mapsto p_*(p^*\alpha \cap c_1(\mathcal{O}_{P(E)}(1))^{e+i})$$

where p^* is the flat pull-back and p_* is the proper push-forward.

The **Chern classes** $c_i(E)$ are defined by:

$$c(E)s(E) = (1 + c_1(E) + c_2(E) + ...)(1 + s_1(E) + s_2(E) + ...) = 1$$

where $c(E)$ and $s(E)$ are the **total** Chern and Segre classes.

Properties of Chern Classes:

(a) They commute. $(\alpha \cap c_i(E)) \cap c_j(F) = (\alpha \cap c_j(F)) \cap c_i(E)$.

(b) (Projection Formula) If $f : X' \to X$ is proper and $\alpha \in A_d(X')$, then:

$$f_*(\alpha \cap c_i(f^*E)) = f_*(\alpha) \cap c_i(E)$$

(c) (Flat Pull-Back) $f : X' \to X$ is flat and $\alpha \in A_k(X)$, then for all i,

$$f^*(\alpha \cap c_i(E)) = f^*(\alpha) \cap c_i(f^*E)$$

(d) (Geometric Interpretation) If $V \subset X$ is nonsingular and the zero scheme Z of a section $s \in \Gamma(V, E|_V)$ has codimension $e + 1$ in V, then

$$[V] \cap c_{e+1}(E) = [Z]$$

(e) (Vanishing) $c_i(E) = 0$ if $i > e + 1$.

(f) (Duals) $c_i(E^*) = (-1)^i c_i(E)$

(f) (Whitney Product Formula) Given $0 \to E' \to E \to E'' \to 0$, then:

$$c(E) = c(E')c(E'')$$

The **Chern roots** $\alpha_1, ..., \alpha_{e+1}$ are formal symbols, used for calculations involving Chern classes. That is, c_i is the ith symmetric polynomial σ_i in the α_j and more generally, any symmetric polynomial in the α_j corresponds to a polynomial in the σ_i, hence to a polynomial in the Chern classes. For example, the i**th Chern character** $ch_i(E)$ is defined by:

$$ch_i(E) = \sum_{j=1}^{e+1} \frac{\alpha_j^i}{i!}$$

The **total Chern character** is $ch(E) = (e + 1) + ch_1(E) + ch_2(E) + ...$

(g) For exact sequences as in (f), $ch(E) = ch(E') + ch(E'')$.

(h) (Tensor Product Formula) $ch(E' \otimes E'') = ch(E')ch(E'')$.

One can derive an expression for $ch(E)$ in terms of $c(E)$ recursively from:

Newton's Formula: Let $p_i = (i!)ch_i(E)$. Then $p_1 = c_1(E)$ and for all $i \geq 2$:

$$p_i = c_1(E)p_{i-1} - c_2(E)p_{i-2} + ... + (-1)^{i-2}c_{i-1}(E)p_1 + (-1)^{i-1}c_i(E)i.$$

Thus, for example,

$$ch_2(E) = \frac{1}{2}(c_1^2 - 2c_2) \text{ and } ch_3(E) = \frac{1}{6}(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E))$$

The **total Todd class** of E is defined by:

$$1 + td_1(E) + td_2(E) + ... = td(E) = \prod_{j=1}^{e+1} \frac{\alpha_j}{1 - \exp(-\alpha_j)}$$

and the first few terms are:

$$td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \frac{1}{24}(c_1(E)c_2(E)) + ...$$

(i) For exact sequences as in (f), $td(E) = td(E')td(E'')$.

Hirzebruch-Riemann-Roch: If X is a nonsingular projective variety, let TX be its tangent bundle and let $td(X) = td(TX)$. Then:

$$\chi(X, E) = \int_X ch(E).td(X) := \deg([X] \cap ch(E) \cap td(X))$$

Examples: (a) Curves. $td(C) = 1 - \frac{1}{2}K_C$, so:

$$\chi(X, E) = \int_C (e + 1 + c_1(E))(1 - \frac{1}{2}K_C) = \deg(E) + (e + 1)(1 - g)$$

where $\deg(E) = \int_C c_1(E)$ is the degree of E. Thus the Hilbert polynomial:

$$\chi(X, E(d)) = d(e + 1)\int_C c_1(\mathcal{O}_C(1)) + \deg(E) + (e + 1)(1 - g)$$

so the rank and degree of E are determined by the Hilbert polynomial.

(b) Surfaces. $td(S) = 1 - \frac{1}{2}K_S + \frac{1}{12}(K_S^2 + c_2(TX))$ so

$$\chi(X, \mathcal{O}_S) = \frac{1}{12}\int_S K_S^2 + c_2(TX)$$

For any vector bundle E on S:

$$\chi(X, E) = \frac{1}{2}\int_S (c_1^2(E) - 2c_2(E)) - \frac{1}{2}\int_S c_1(E).K_S + \chi(X, \mathcal{O}_S)$$

and the Hilbert polynomial of E is:

$$\frac{d^2}{2}(e+1)\int_S c_1^2(\mathcal{O}_S(1)) + d\int_S (c_1(\mathcal{O}_S(1)) - \frac{e+1}{2}K_S)\cdot c_1(E) + \chi(X, E)$$

The Hirzebruch-Riemann-Roch theorem generalizes in an important way.

The **Grothendieck ring** $K^0(X)$ is the free abelian group generated by the vector bundles on X modulo the subgroup generated by the relations: $[E] \sim [E'] + [E'']$ whenever $0 \to E' \to E \to E'' \to 0$. This is a commutative ring with 1 via the multiplication $[E] \cdot [F] := [E \otimes F]$ with $[\mathcal{O}_X] = 1$. The pull-back $f^*[E] := [f^*E]$ is well-defined and contravariant.

The **Grothendieck group** $K_0(X)$ is the free abelian group generated by the **coherent sheaves** on X modulo the same relation as in the Grothendieck ring. It is naturally a module over $K^0(X)$ via tensor product. In this case, the pull-back defined as above is only well-defined for flat morphisms f.

Example: (a) If X is nonsingular, then a theorem of Hilbert implies that every coherent sheaf on X has a finite resolution by vector bundles. Thus for such X, the Grothendieck group coincides with the Grothendieck ring! It follows that Hirzebruch-Riemann-Roch is valid for all coherent sheaves.

(b) On $\mathbf{P}_k^n$, every vector bundle may be resolved by line bundles. As a result, $K^0(\mathbf{P}^n) = K_0(\mathbf{P}^n)$ is generated (as a ring) by $[\mathcal{O}(1)]$ and $[\mathcal{O}(-1)]$.

In particular, the Koszul complex:

$$0 \to \wedge^{n+1}V \otimes \mathcal{O}(-n-1) \to ... \to \wedge^2 V \otimes \mathcal{O}(-2) \to V \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$$

is a relation, out of which we can express $[\mathcal{O}(-1)]$ in terms of $[\mathcal{O}(1)]$, and we obtain a surjective ring homomorphism:

$$\mathbf{Z}[x]/(1-x)^{n+1} \to K^0(\mathbf{P}^n);\ x \mapsto [\mathcal{O}(1)]$$

Definition: If X is equidimensional of dimension n, let

$$A^d(X) = A_{n-d}(X) \text{ and } A^d(X)_{\mathbf{Q}} = A^d(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and let $1 = [X] \in A^0(X)$.

Then properties (f)-(i) of Chern classes imply that the following maps:

$c_1 : K^0(X) \to A^1(X); [E] \mapsto [X] \cap c_1(E),$

$c : K^0(X) \to A^*(X); [E] \mapsto [X] \cap c(E),$

$ch : K^0(X)_{\mathbf{Q}} \to A^*(X)_{\mathbf{Q}}; [E] \mapsto [X] \cap ch(E),$

$td : K^0(X)_{\mathbf{Q}} \to A^*(X)_{\mathbf{Q}}; [E] \mapsto [X] \cap td(E)$

are well-defined, and compatible with (flat) pull-backs. The maps c_1 and ch preserve addition, while the other two convert addition into "multiplication." Note that the kernel of ch is an ideal in $K^0(X)$ (or $K^0(X)_{\mathbf{Q}}$) so we can think of ch as imposing the structure of a graded commutative ring with 1 on its image. When X is nonsingular, there is a multiplication defined on $A^*(X)$ via intersection theory, with respect to which ch is a ring homomorphism.

Back to our $\mathbf{P}^n$ example: The "chern character" ring homomorphism:

$$ch : \mathbf{Q}[x]/(1-x)^{n+1} \to \mathbf{Q}[h]/h^{n+1}$$

defined by:

$$ch(x) = \exp(h) = 1 + h + \frac{h^2}{2!} + ... \pmod{h^{n+1}}$$

is an **isomorphism**. The point is that:

$$1, \exp(h), \exp(2h), ..., \exp(nh)$$

are all linearly independent! In this case, we have the ring structure:

$$\mathbf{Q}[h]/h^{n+1} \quad \text{on} \quad A^*(\mathbf{P}^n)_{\mathbf{Q}}$$

and in particular, $\mathbf{Z}[x]/(1-x)^{n+1} \cong K^0(\mathbf{P}^n)$, because there can be no kernel.

Next, we introduce the proper push-forward on $K_0(X)$.

The push-forward of a coherent sheaf under a proper morphism remains coherent, but this operation is only left exact, so to obtain a homomorphism of Grothendieck groups, we will need to consider the right-derived functors. Specifically, if $f : X \to Y$ is a proper morphism and $\mathcal{F}$ is a coherent sheaf on X, then the sheaves $R^i f_* \mathcal{F}$ associated to the presheaves:

$$U \mapsto \mathrm{H}^i(f^{-1}(U), \mathcal{F})$$

are coherent, and are the right derived functors for the functor: $\mathcal{F} \rightsquigarrow f_* \mathcal{F}$. Thus, if we are given a short exact sequence: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, of coherent sheaves on X, then we obtain a (terminating) long exact sequence:

$$... \to R^{i-1} f_* \mathcal{F}'' \to R^i f_* \mathcal{F}' \to R^i f_* \mathcal{F} \to R^i f_* \mathcal{F}'' \to R^{i+1} f_* \mathcal{F}' \to ...$$

$R^i f_* \mathcal{F}$ restricts to $\mathrm{H}^i(X_A, \mathcal{F}_A)$ over open affines $\mathrm{Spec}(A) \subset Y$. It follows that many of the results obtained for these cohomology groups are also valid for the higher direct image sheaves. For instance:

Serre's Theorem B: If f is projective, then $R^i f_* \mathcal{F}(d) = 0$ for all $d >> 0$.

Flat Base Change: If f is any morphism and $g : Y' \to Y$ is flat, consider the fiber square:

$$\begin{array}{ccc} X' & \stackrel{\widetilde{g}}{\to} & X \\ \widetilde{f} \downarrow & & f \downarrow \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

Then:

$$g^* R^i f_*(\mathcal{F}) = R^i \widetilde{f}_*(\widetilde{g}^* \mathcal{F})$$

for all coherent sheaves $\mathcal{F}$ on X.

Thus if f is a proper morphism we obtain a well-defined proper push-forward on Grothendieck groups by taking the "Euler characteristic":

$$f_* : K_0(X) \to K_0(Y); \quad f_*[\mathcal{F}] := \sum (-1)^i [R^i f_* \mathcal{F}]$$

If f is proper in the fiber diagram above, then:

$$\widetilde{g}_* \widetilde{f}^* = f^* g_* : K_0(X) \to K_0(Y')$$

so that flat pull-backs commute with proper push-forwards.

The proper push-forward is functorial (this requires spectral sequences) and there is a projection formula. If $f : X \to Y$ is a proper morphism, $\mathcal{F}$ is a coherent sheaf on X, and E is a vector bundle on Y, then:

$$R^i f_*(f^*E \otimes \mathcal{F}) = E \otimes R^i f_*(\mathcal{F})$$

from which we obtain the projection formula:

$$f_*(f^*[E] \cdot [\mathcal{F}]) = [E] \cdot f_*[\mathcal{F}]$$

Grothendieck-Riemann-Roch: If $f : X \to Y$ is a projective morphism of nonsingular varieties over k and $\mathcal{F}$ is a coherent sheaf on X, then:

$$f_*\left(ch(\mathcal{F}) \cap td(X)\right) = ch(f_*[\mathcal{F}]) \cap td(Y)$$

In other words, $ch(\mathcal{F}) \cap td(X)$ commutes with proper push-forward.

Examples: (a) Hirzebruch-Riemann-Roch is the special case $Y = \mathrm{Spec}(k)$ since in that case, $f_*[\mathcal{F}] = \chi(X, \mathcal{F})$ and $td(Y) = 1$.

(b) If f is a closed immersion, let $N_{X/Y}$ be the normal bundle. Then:

$$f_*\left(ch(\mathcal{F}) \cap td^{-1}(N_{X/Y})\right) = ch(f_*\mathcal{F})$$

tells how to calculate Chern classes of $f_*\mathcal{F}$ in terms of Chern classes of $\mathcal{F}$.

The Chow Ring of the Grassmannian: On an equidimensional X with an exact sequence:

$$(*) \quad 0 \to K \to \mathcal{O}_X^n \to Q \to 0$$

and K, Q vector bundles of ranks m and $n - m$, we obtain some non-trivial relations among the Chern classes of K. Namely $c(K)c(Q) = c(\mathcal{O}_X^n) = 1$, hence $c_i(Q) = s_i(K)$ for all i and in particular, $s_i(K) = 0$ for all $i > n - m$. Using this, we obtain a map:

$$e : \mathbf{Z}[x_1, ..., x_m]/I \to A^*(X); \quad p(x_i) \mapsto [X] \cap p(c_i(K))$$

with I the ideal generated by the polynomials $s_{m-n+1}, ..., s_n$ defined by:

$$(1 + s_1 t + s_2 t^2 + ...)(1 + x_1 t + ... + x_e t^e) = 1$$

($s_{n+1}, s_{n+2}, ...$ are automatically in this ideal!)

This is a graded homomorphism if we let the degree of x_i be i.

Theorem: If $X = G(m,n)$ and $(*)$ is the universal short exact sequence, then e is an isomorphism of graded abelian groups.

(This gives $A^*(G(m,n))$ the structure of a graded commutative **ring**.)

Example: When $m = 1$, this gives:

$$\mathbf{Z}[x]/(-x)^n \cong A^*(G(1,n)) = A^*(\mathbf{P}_k^{n-1})$$

The minus sign is explained by the fact that $[\mathbf{P}_k^{n-1}] \cap c_1(K) = -H$.

The theorem may be proved in several steps.

Step 1: Properties of the graded ring $R(m,n) := \mathbf{Z}[x_1, ..., x_m]/\langle s_{n-m+1}, ..., s_n\rangle$.

The **Schur polynomial** of a partition $m \geq a_1 \geq ... \geq a_{n-m} \geq 0$ is:

$$\Delta_{a_1,\ldots,a_{n-m}} := \det \begin{pmatrix} x_{a_1} & x_{a_1+1} & \cdots & x_{a_1+n-m-1} \\ x_{a_2-1} & x_{a_2} & \cdots & x_{a_2+n-m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{a_{n-m}-(n-m)+1} & x_{a_{n-m}-(n-m)+2} & \cdots & x_{a_{n-m}} \end{pmatrix}$$

with the convention that $x_i = 0$ if $i < 0$ or $i > m$. Note that $\Delta_{a_1,\ldots,a_{n-m}}$ is homogeneous of degree $a_1 + ... + a_{n-m}$.

These Schur polynomials have the following remarkable properties:

(i) The $\Delta_{a_1,\ldots,a_{n-m}}$ freely generate $R(m,n)$.

Thus $R(m,n)$ is a free of rank $\binom{n}{m}$ and $R(m,n)_{m(n-m)} = \mathbf{Z}x_m^{n-m}$.

(ii) $\Delta_{a_1,\ldots,a_{n-m}}\Delta_{m-a_{n-m},\ldots,m-a_1} = x_m^{n-m}$ in $R(m,n)$ for all partitions.

Thus $R(m,n)$ satisfies "Poincaré duality". That is, the multiplications:

$$R(m,n)_d \times R(m,n)_{m(n-m)-d} \to R(m,n)_{m(n-m)} \cong \mathbf{Z}$$

are perfect pairings, inducing isomorphisms

$$R(m,n)_{m(n-m)-d} \cong \operatorname{Hom}(R(m,n)_d, \mathbf{Z}).$$

This is considerably more information than simply that $R(m,n)_d$ and $R(m,n)_{m(n-m)-d}$ have the same rank!

These facts may be proved directly. As it would take us too far afield, I will not spend class time on this. I encourage the interested student to go find a proof or better yet, to construct one.

Step 2:

$$\int_{G(m,n)} c_m^{n-m}(K) = (-1)^{m(n-m)}$$

The sections of the dual bundle K^* vanish along the (reduced) sub-Grassmannians $G(m,n-1) \subset G(m,n)$. A general section of $\oplus_{i=1}^{n-m} K^*$ will then vanish at a single reduced point, corresponding to a $G(m,m) \subset G(m,n)$. Thus $\int c_{m(n-m)}(\oplus K^*) = 1$ by the geometric realization of Chern classes and Step 2 follows from the Whitney formula and the formula for duals.

Corollary (of Steps 1 and 2): e is injective.

Proof: Suppose $e(P) = 0$ for some $0 \neq P \in R(m,n)$. We may as well assume $0 \neq P$ is homogeneous, say of degree d. Using the perfect pairing of Step 1, we know that there is a $Q \in R(m,n)_{m(n-m)-d}$ with the property that $0 \neq PQ \in R(m,n)_{m(n-m)}$, i.e. $PQ = rx_m^{n-m}$ for some nonzero $r \in \mathbf{Z}$. But then $e(PQ) \neq 0$, by Step 2, since $\int e(PQ) = \pm r$. But the kernel of e is clearly an ideal, so this contradicts $e(P) = 0$.

Step 3: A cellular decomposition of the Grassmannian.

Define a partial ordering on the sequences: $0 < i_1 < ... < i_m \leq n$:

$$I \preceq J \text{ if } i_k \leq j_k \text{ for all } k = 1, ..., m,$$

Recall the Plücker embedding $G(m,n) \subset \mathbf{P}_k^{\binom{n}{m}-1}$ with coordinates x_J. We define closed subsets $V_J := G(m,n) \cap V(\{x_I\}_{I \prec J})$ and (relatively) open subsets $Y_J = V_J - V(x_J)$.

The Y_J are clearly disjoint and cover $G(m,n)$. They are the isomorphic images of $m \times n$ matrices with 1's in the (k, j_k) positions and zeroes to the left. Thus, $Y_J \cong \mathbf{A}^{m(n-m)-\sum_{k=1}^{m}(j_k-k)}$. Moreover,

$$(\dagger) \quad \overline{Y_J} - Y_J = \cup_{I \succ J} Y_I$$

as the reader is invited to check.

Corollary (of all the Steps): e is surjective.

Proof: First of all, $A_*(G(m,n))$ is a quotient of a free group of rank $\binom{n}{m}$. This is seen by induction and the exact sequence on Chow groups, starting with the point $Y_{(n-m+1,\ldots,n)} \in G(m,n)$.

Assume we are given a closed union $Z \subset G(m,n)$ of finitely many Y_J. From (†) above and downward induction, there is a $Y = Y_I$ with the property that $Y \cup Z$ is also closed. Thus we then obtain an exact sequence:

$$A_*(Z) \to A_*(Z \cup Y) \to A_*(Y) \to 0$$

which is a split epimorphism with inverse $[Y] \mapsto [\overline{Y}]$, since the Y's are affine spaces. Proceeding in this way we fill the Grassmannian and untimately obtain a surjection:

$$\rho : \bigoplus_J \mathbf{Z}[\overline{Y_J}] \to A_*(G(m,n)).$$

By the previous corollary, we know that $A_*(G(m,n))$ contains a free abelian group of rank $\binom{n}{m}$, so it follows that ρ is an isomorphism.

We are not quite done yet(!). Namely, we have an injective map:

$$e : R(m,n) \to A^*(G(m,n))$$

and both sides are free abelian groups of the same rank. It follows that e embeds $R(m,n)$ as a lattice. Choose $\alpha \in A_d(G(m,n))$. Via the pairing

$$t_\alpha(P) := \operatorname{deg}(\alpha \cap P(c_i(K)))$$

we determine from α an element $t_\alpha \in \operatorname{Hom}(R(m,n)_d, \mathbf{Z})$.

By the perfect pairing of Step 1 and also using Step 2, there is an element $Q \in R(m,n)_{m(n-m)-d}$ such that $\int e(PQ) = t_\alpha(P)$ for every $P \in R(m,n)_d$. If we consider $\beta := e(Q)$ it follows that $t_\beta(P) = t_\alpha(P)$ for all $P \in R(m,n)_d$. Some multiple $r\alpha$ must be in the image, say $r\beta = e(S)$, since $R(m,n)$ embeds as a lattice in $A^*(G(m,n))$. But then $S - rQ$ pairs to zero with every element $P \in R(m,n)_d$. By Poincaré duality, it follows that $S = rQ$, and that $\alpha = \beta = e(Q)$. Thus e is surjective.

8. Hilbert Schemes of Points on Curves: We survey a few of the known properties of Hilbert schemes of points on curves.

If X is a projective scheme over $\mathbf{C}$, let $X^{[d]}$ denote the Hilbert scheme of subschemes of X of constant Hilbert polynomial d (i.e. of length d).

When C is a compact Riemann surface then $C^{[d]}$ is the symmetric product, a nonsingular complex projective variety of dimension d.

Theorem (Macdonald): The Betti numbers $b_i(C^{[d]}) = \dim \mathrm{H}^i(C^{[d]}, \mathbf{Q})$ are computed by the following generating function:

$$\sum_{d\geq 0}\sum_{i\geq 0} b_i(C^{[d]})t^i q^d = \frac{(1+tq)^{2g}}{(1-q)(1-t^2q)}$$

Given a coherent sheaf on $C^{[d]}$, there is a natural cycle map:

$$A^*(C^{[d]}) \to \mathrm{H}^{2*}(C^{[d]}, \mathbf{C})$$

i.e. Chern classes on $C^{[d]}$ determine cohomology classes. We may use the universal family to construct several of these.

Recall that the universal family is a subscheme:

$$\mathcal{D} \subset C^{[d]} \times C$$

which is flat over $C^{[d]}$ with fibers equal to the subschemes of C of length d. Let p and q be the projection maps to $C^{[d]}$ and C respectively.

It follows from cohomology and base change that $p_*q^*\mathcal{O}_C$ is locally free on $C^{[d]}$ of rank d. The **half-diagonal class** $\frac{1}{2}\delta$ is defined by:

$$\frac{1}{2}\delta := -c_1(p_*q^*\mathcal{O}_C)$$

One can show that $\delta \in A^1(C^{[d]})$ is represented by the "big diagonal"...the closed subvariety parametrizing schemes that contain a point of multiplicity two or more. This Chern class is present on any Hilbert scheme of points.

Choose a closed point $p_0 \in C$. Then there are induced closed immersions:

$$p_0 \in C \hookrightarrow C^{[2]} \hookrightarrow C^{[3]} \hookrightarrow ...$$

given by "addition of the point p_0." More accurately, when we take the quotient of the inclusion of products:

$$C^{d-1} \times \{p_0\} \subset C^d$$

by the action of the symmetric group, we get the desired closed immersion of symmetric products. We define:

$$x_{p_0} := [C^{[d-1]}] \in A^1(C^{[d]})$$

and note that

$$[C^{[d]}] \cap x_{p_0}^e = [C^{[d-e]}]$$

The classes $[C^{[d-1]}] \in A^1(C^{[d]})$ depend upon p_0 (when C has positive genus) but their images in $\mathrm{H}^2(C^{[d]}, \mathbf{C})$ do not. We will let x denote the image.

Finally, we may use the fact that $\mathcal{D} \subset C^{[d]} \times C$ is a Cartier divisor to define the **determinantal class** on $C^{[d]}$. We start first with the case $d = g - 1$. We define:

$$\theta := -c_1(p_! \mathcal{O}_{C^{[g-1]} \times C}(\mathcal{D})) = c_1(R^1 p_* \mathcal{O}(\mathcal{D})) - c_1(p_* \mathcal{O}(\mathcal{D}))$$

If we choose a "base divisor" $D_0 = p_1 + ... + p_g$ consisting of g distinct points, then the sequence:

$$0 \to \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\mathcal{D} + q^* D_0) \to \mathcal{O}_{q^{-1} D_0}(\mathcal{D} + q^* D_0) \to 0$$

pushes forward to a resolution:

$$0 \to p_* \mathcal{O}(\mathcal{D}) \to E \to F \to R^1 p_* \mathcal{O}(\mathcal{D}) \to 0$$

where E and F are vector bundles of rank g, and then:

$$\theta = c_1(F) - c_1(E)$$

We get well-defined cohomology classes $\theta \in \mathrm{H}^2(C^{[d]}, \mathbf{C})$ for all d by letting:

$$\theta := c_1(p_! \mathcal{O}(\mathcal{D} + q^* D_0))$$

for any divisor D_0 of degree $g - 1 - d$.

These three divisor classes on $C^{[d]}$ have **very** different properties.

(i) Each $x_{p_0} \in A^1(C^{[d]})$ is ample.

Proof: x_{p_0} pulls back to the ample divisor $\sum \pi_i^* p_0$ on the product C^d (where $\pi_i : C^d \to C$ is the projection). The cohomological criterion for ampleness then implies x_{p_0} is ample, because $C^d \to C^{[d]}$ is a finite map.

(ii) $\mathrm{H}^0(C^{[d]}, \mathcal{O}_{C^{[d]}}(\frac{n}{2}\delta)) = 0$ for odd n and is one-dimensional for even n.

(iii) θ is pulled back from an ample divisor on the Picard group. More precisely, each $\mathrm{Pic}^d(C)$ is an abelian variety of dimension g parametrizing the isomorphism classes of line bundles on C of degree d. There are Abel-Jacobi maps $C^{[d]} \to \mathrm{Pic}^d(C)$ for each d sending D to $\mathcal{O}_C(D)$. The fibers of the Abel-Jacobi maps are the linear series $|D|$. The Abel-Jacobi maps are surjective when $d \geq g$ by Riemann-Roch. Tensoring by $\mathcal{O}_C(p_0)$ gives isomorphisms $\mathrm{Pic}^d(C) \tilde{\to} \mathrm{Pic}^{d+1}(C)$ and a commuting diagram:

$$\begin{array}{ccccccccccc}
C & \hookrightarrow & C^{[2]} & \hookrightarrow & \cdots & \hookrightarrow & C^{[g]} & \hookrightarrow & \cdots & \hookrightarrow & C^{[2g-1]} \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
\mathrm{Pic}^1(C) & \tilde{\to} & \mathrm{Pic}^2(C) & \tilde{\to} & \cdots & \tilde{\to} & \mathrm{Pic}^g(C) & \tilde{\to} & \cdots & \tilde{\to} & \mathrm{Pic}^{2g-1}(C)
\end{array}$$

(degree $2g-1$ is significant because at that point and thereafter, the Abel-Jacobi map becomes a projective bundle)

On $\mathrm{Pic}^d(C) \times C$ there is a universal line bundle $\mathcal{L}$ with the property that when restricted to a fiber of the projection map p to $\mathrm{Pic}^d(C)$, $\mathcal{L}$ is isomorphic to the line bundle parametrized by the point of $\mathrm{Pic}^d(C)$. On $\mathrm{Pic}^{g-1}(C)$, we can define a $\theta \in A^1$ determinantally as earlier. Namely, we let:

$$\theta = -c_1(p_! \mathcal{L})$$

In this case, however, $p_* \mathcal{L} = 0$ and $R^1 p_* \mathcal{L}$ is supported (and generically of rank one along) the image of $C^{[g-1]}$. It follows that on $\mathrm{Pic}^{g-1}(C)$, θ is the push-forward of $[C^{[g-1]}]$ under the Abel-Jacobi map, and on $C^{[g-1]}$, θ is the first Chern class of the the normal sheaf to the Abel-Jacobi map. One can prove that θ (defined as before but on $\mathrm{Pic}^d(C)$) is ample on each $\mathrm{Pic}^d(C)$. Thus although θ is not ample on $C^{[d]}$ whenever the Abel-Jacobi map has positive-dimensional fibers, it is the case that a high multiple of θ is base-point free with an explicit image in projective space.

Theorem: The image of x and θ in $\mathrm{H}^2(C^{[d]}, \mathbf{Q})$ generate all the Chern classes of $C^{[d]}$, if C is "of general modulus."

Example:

$$\frac{1}{2}\delta = (d+g-1)x - \theta$$

One can similarly obtain formulas for the "smaller" diagonals ([ACGH]).

Another Example: If C is of genus g and general modulus, $d \leq 2g-2$, and r is a non-negative integer with the property that $(r+1)(g-d+r) \leq g$ then in particular, $r(g-d+r) \leq d$ and the "Brill-Noether" locus:

$$C_d^r := \{D \in C^{[d]} \mid \ \dim(|D|) \geq r\}$$

is nonempty and has codimension $r(d-g+r)$ in $C^{[d]}$.

(This is a deep result in Brill-Noether theory...see [ACGH])

The class $[C_d^r] \in H^{2r(g-d+r)}(C^{[d]}, \mathbf{Z})$ has been computed. It is:

$$[C_d^r] = \left(\prod_{i=0}^{r} \frac{i!}{(g-d+r+i-1)!}\right) \sum_{\alpha=0}^{r} (-1)^{\alpha} \frac{(g-d+r+\alpha-1)!}{\alpha!(r-\alpha)!} x^{\alpha}\theta^{r(g-d+r)-\alpha}$$

Finally, to compute numbers, one needs to know the intersection numbers of powers of x and of θ. These have also been computed. They are:

$$\int_{C^{[d]}} x^d = 1$$

$$\int_{C^{[d]}} \theta^d = \begin{cases} 0 & \text{if } \ d > g \\ \frac{g!}{(g-d)!} & \text{if } \ d \leq g \end{cases}$$

From this, for example, one can check that $[C_d^r]$ is not the zero class(!) by intersecting it with the appropriate power of x.