Quantum Geometric Langlands and S-duality

Robert Windesheim

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Robert Windesheim

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Betreuer:

Prof. Dr. Ilka Brunner

Dr. Ingmar Saberi

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Chapter 1

Introduction

The origin of the Langlands program lies in a series of conjectures relating representation theory to number theory proposed by Robert Langlands in the late '60s.[1] The overarching idea is the expression of the complicated problems of algebraic number theory in the more tractable language of representation theory and harmonic analysis. The Langlands program vastly generalizes many groundbreaking number-theoretical results containing them as corollaries. Since its original number-theoretic formulation, the correspondence has evolved to stretch across various fields. The most recent incarnation is (the categorical version of) the Geometric Langlands and the physical interpretations, which we will consider in this thesis.

Let us begin by giving a historical overview of the ideas that led to the geometric version of the program. The original number theoretical correspondence seeks to connect complex representations of (absolute) Galois groups of number fields $Gal(\bar{F}/F)$ to so-called automorphic representations of the general linear group with values in the (Adele rings of the) respective fields $GL_n(\mathbb{A}_F)$. Following this conjectural relationship proposed by Robert Langland, it was soon realized that the correspondence was difficult to approach in the language of number fields. This led to the application of the analogy between number fields and the better-understood function fields (an alternative approach is given by considering local fields). Function fields are fields of functions on some algebraic curve over some finite field $\mathbb{F}_q$. The analogy allows for the application

[1] The original conjectures were proposed in a letter to Andre Weil in 1967 [36].

of powerful geometric results related to algebraic curves. One can pass one step further from algebraic curves over finite fields, to curves over $\mathbb{C}$ (where the curves become Riemann surfaces). This threefold analogy is sometimes called Weil's Rosetta Stone and translates problems between the three different fields. When passing to the third column, Geometric Langlands emerges.

The (unramified) automorphic representations of $GL_n(\mathbb{A}_F)$ can be equated with certain functions on the double quotient $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n(O_F)$. In the function field analogy F is some function field over an algebraic curve X over $\mathbb{F}_q$, $\mathbb{A}_F$ is the corresponding Adele ring, and $O_F = \Pi_{x\in X} O_x$ where O_x can be viewed as the ring of formal Taylor series $\mathbb{C}[[t]]$. Via the Weil uniformization theorem this double quotient can be equated with the set of equivalence classes of vector bundles.[2] The functions modelling the automorphic representations then become certain sheaves on the space corresponding to this double quotient. When viewing X as a curve over $\mathbb{C}$ these special sheaves are so called D-modules. This gives a geometric interpretation of the "automorphic" side of the Langlands correspondence. The Hecke eigenfunctions, which play an important role in the number theoretical formulation as special automorphic representations, have geometric counterparts given by certain D-modules called Hecke eigensheaves.

On the Galois side of the Langlands correspondence one has complex representations of the Galois group manifesting as homomorphisms: $Gal(\bar{F}/F) \to GL_n$. Over $\mathbb{C}$ these turn out to be related to homomorphisms from the fundamental group of X to GL_n. The corresponding geometric space is that of local systems.[3] Sheaves on the moduli space of local systems then constitute the geometric version of the Galois side. Through this the number-theoretical correspondence should be interpreted geometrically as an equivalence between D-modules on the moduli space of vector bundles and sheaves on

[2]The idea is that any vector bundle can be trivialized away from finitely many points, the transition functions are given by elements of $GL_n(X \setminus \{x_1, ..., x_m\})$ which can be equated with $GL_n(F)$. The vector bundles can also be trivialized on formal disks around the points $\{x_1, ..., x_m\}$, the transition functions are then given by elements of $GL_n(\mathcal{O}_F)$. The overlaps can be viewed as punctured formal disks with transitions functions given by elements of $GL_n(\mathbb{A}_F)$. This leads to a the set of equivalence classes of vector bundles being fully defined by the double quotient $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n(O_F)$.

[3]We will explain this further on in detail, but local systems are heuristically given by vector bundles with flat connection.

the moduli space of local systems. This geometric equivalence was first conjectured by Laumon in 1987 [37]. Considering the categories corresponding to these objects one arrives at the categorical version of Geometric Langlands.

We can further pass from GL_n to a general reductive group G. Here something extraordinary happens, on the automorphic side, one has D-modules on the moduli of general principal G-bundles while, on the Galois side, the homomorphisms are now from $\pi_1(X)$ to the so-called Langlands dual group ${}^LG$.

This is also where physics comes into play. In 1970, Montonen and Olive had discovered a now-famous duality between $\mathcal{N} = 4$ supersymmetric quantum field theories in four dimensions [42]. The duality connects theories with gauge group G to theories with gauge group ${}^LG$. This was the original incarnation of S-duality and hinted towards a deep connection between the Langlands program and quantum field theory. Finally, in 2006 Kapustin and Witten gave a concrete description of the previously mysterious connection in their groundbreaking paper [34].

One can further generalize the Geometric Langlands correspondence by considering "quantum" deformation of the previously described objects. This leads to quantum Geometric Langlands.

We will give a modern introduction to the categorical formulation of the Geometric Langlands, touching on some of the more recent conceptualizations. An exposition of the further generalization to the quantum Geometric Langlands will follow. We refrain from using the original number-theoretical correspondence as motivation, as has been done in multiple great surveys [12, 11]. Instead, we give a viewpoint stemming from Pontryagin duality and the Fourier-Mukai transform which is perhaps more natural for a physically minded audience. After the mathematical side is illuminated we will briefly review S-duality, its six-dimensional origin, and topological twisting. We connect the mathematics with physics, explaining the Kapustin-Witten approach to Geometric Langlands and generalizations connecting to the quantum correspondence. This will lead us to a description of some of the cutting-edge developments connecting S-duality and quantum Geometric Langlands theory.

Chapter 2

Geometric Langlands

In this chapter, we will be introducing the categorical Geometric Langlands correspondence. The goal is to motivate the conjecture as a categorified Fourier transform, essentially realizing Geometric Langlands as a duality in a theory of categorical harmonic analysis. We will begin by introducing the Fourier-Mukai transform and proceed by arguing that the Geometric Langlands conjecture, in the abelian case, reduces to a special version of the Fourier-Mukai transform. This will allow us to establish the full Geometric Langlands conjecture as a possible non-abelian Fourier-Mukai transform between certain special categories on moduli of bundles. Finally, we describe an extension of the naive conjecture to a working equivalence. We assume familiarity with the main concepts of algebraic and differential geometry like varieties, schemes, and fiber bundles. Even though Geometric Langlands can be formulated over general fields k of characteristic 0, we focus on the case of $k = \mathbb{C}$.

Remark 1. *It needs to be noted that Geometric Langlands can be viewed in three different pictures: the Betti, the De Rham, and the Dolbeault (which we almost completely evade). These are all different formulations of the moduli problem of flat principal bundles on smooth complex algebraic curves. Analytically the three versions are equivalent. This essentially means that when viewing the moduli spaces as analytic spaces (spaces that are locally complex analytic varieties) and forgetting the extra structure all three are isomorphic. The Betti version is purely topological while the De Rham version includes* *information about the algebraic structure. The physical story of Kaputsin-Witten is oblivious to these different versions and only sees the analytic picture. We will mostly describe the De Rham side but try to mention when ambivalences appeared. The Betti version was introduced in [4].*

2.1 The Fourier-Mukai Transform

Let us begin by introducing the Fourier-Mukai transform and motivating its meaning. The classical Fourier transform gives an equivalence of function spaces on an abelian group and its Pontryagin dual group.[1] Analogously the Fourier-Mukai transform gives a categorified equivalence of function spaces of an abelian variety A and its dual $A^\vee$. Let us explain what is meant by this. For a detailed technical reference on the Fourier-Mukai transform see [32]. For a review of the connection to Pontryagin duality see [3].

An abelian variety A is a complex torus of dimension g, which is also a projective variety equipped with the abelian group structure inherited from the underlying torus. The group multiplication is given by the map $\mu : A \times A \to A$. Analogously to the characters forming the dual group in Pontryagin duality, the group of line bundles on A form the dual abelian variety $A^\vee$. This is a form of categorification by passing from the Pontryagin dual group $\hat{G} = Hom(G, U(1))$ to the moduli space of multiplicative line bundles $A^\vee = Hom(A, BU(1))$.[2] The group multiplication is now given by the tensor product of line bundles. This is related to the multiplication on A via the pullback of a line bundle along the projection to each of the factors $\pi_{1,2} : A \times A \to A$:

$$\mu^* L = \pi_1^* L \otimes \pi_2^* L \tag{2.1}$$

thus the line bundle L on A pulled back along the multiplication map is equivalent

[1] Recall that the Pontryagin dual group $\hat{G}$ is given by the group generated by the characters $\chi : G \to U(1)$ The characters give a basis for the space of functions on compact G. The Fourier transform on S^1, for example, would be $e^{i\pi n x}$ with $n \in \mathbb{Z}$ as the characters giving a "basis" for the space of $L^2(S^1)$ functions.

[2] $BU(1) = pt/U(1)$ is the classifying space considered as a one object groupoid with $Aut(BU(1)) = U(1)$.

to the tensor product of L pulled back along π_1 and π_2 respectively. This map is a categorification of the idea that Fourier transformation turns convolution of functions (on the group algebra) into the multiplication of characters. A restriction of μ to $x \in A$ gives a map $\mu_x^* L = V \otimes L$, where V is a single fiber at x. We can view V as the eigenvalue corresponding to L.

We can further define a line bundle $\mathcal{P}$ on the product $A \times A^\vee$ called the Poincare line bundle. $\mathcal{P}$ is given by a parametrization of line bundles on A by $A^\vee$. This means for $x \in A$ and $L \in A^\vee$, $\mathcal{P}_{(x,L)} = L_x$. Here the Poincare bundle models the role of the universal bicharacter in Pontryagin duality. In the example of the Fourier transform on $\mathbb{R}$ this would be the universal bicharacter given by $e^{i\pi xt}$.

To construct a categorification of the Fourier transform we need to recall that functions are well behaved under pullback, pushforward (integration), and multiplication. These operations have exact analogies in the abelian category of quasi-coherent sheaves[3] as direct image, inverse image, and tensor product. This gives us a direct analogy between the space of functions on a group and the category of quasi-coherent sheaves on an abelian variety. A categorical integral transform requires an enhancement to the derived category.[4] This enhancement allows for objects to be identified with their resolutions (as these are quasi-isomorphic to the object), but more importantly in our case, we need the natural inclusion of higher direct images in the derived category. These higher direct images allow for proper base change, which is necessary for the Fourier-Mukai transform to be formulated.

To see this we need one final construction given by defining two projections from

[3]Quasi-coherent sheaves on a ringed space $(X, \mathcal{O}_X)$ are locally presentable sheaves of $\mathcal{O}_X$-modules. Heuristically they can be viewed as vector bundles where the rank can vary across the base space. The abelian category of quasi-coherent sheaves is, therefore, an extension of the category of vector bundles, (which is not abelian) allowing for the kernel, cokernels, direct images etc. Also note that this is a monoidal category.

[4]Recall that the derived category of an abelian $\mathcal{A}$ category is the category of chain complexes of objects in $\mathcal{A}$ in which all quasi-isomorphisms are invertible.

$A \times A^\vee$ to A and $A^\vee$ respectively:

$$\begin{tikzcd} & \mathcal{P} \arrow[d] & \\ & A \times A^\vee \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & A^\vee \end{tikzcd}$$

Using these projections we can state Mukai's result.

Theorem 2.1.1 (Mukai)**.** *Let A be an abelian variety and $\mathcal{F} \in QCoh(A)$. Then there exists an equivalence of (bounded) derived categories of quasi-coherent sheaves given by the Fourier-Mukai functor:*

$$\phi_{FM} : \boldsymbol{QCoh}(A) \to \boldsymbol{QCoh}(A^\vee)$$

$$\mathcal{F} \mapsto \boldsymbol{R}p_*(\boldsymbol{L}q^*\mathcal{F} \otimes \mathcal{P})$$

Where $\boldsymbol{R}p_*$ and $\boldsymbol{L}q^*$ are the derived versions of the maps. The transform holds up to sign and cohomological shift.

We can view this functor as sending a line bundle L on A, seen as a locally constant quasi-coherent sheaf $\mathcal{F}$, to a skyscraper sheaf on $A^\vee$. The single stalk of the skyscraper sheaf is then exactly at the corresponding point $L \in A^\vee$. This is equivalent to the way delta functions are mapped to $e^{i\pi x}$ by the regular Fourier transform on $\mathbb{R}$ (delta functions are like skyscraper sheaves and exponentials are like line bundles in that they are characters of $\mathbb{R}$).

In a sense, we can view the Fourier-Mukai transform as a spectral decomposition of categories. Heuristically the skyscraper sheaves give a "basis" for $\mathbf{QCoh}(A)$ and the corresponding line bundles give a "basis" for $\mathbf{QCoh}(A^\vee)$ in the same way general functions on $\mathbb{R}$ can be written as an integral of delta functions or as an integral of exponentials (Fourier transform) equivalently.

2.2 Geometric Langlands for Abelian Groups

We will now motivate the abelian case of Geometric Langlands from the Fourier-Mukai transform. This version of the conjecture was proven by constructing a twisted version of the Fourier-Mukai transform [48, 38]. We follow [12] in this section. Let us begin by introducing some important concepts. Note that the abelian case is a version of geometric class field theory.

We begin by defining the central player of the Geometric Langlands correspondence in the abelian case. We take X to be a smooth complex curve.

Definition 2.2.1. *Let $LocSys_1(X)$ be the moduli space of rank one local systems on X. A rank one local system is parameterized by the pair (L, ∇), where L is a complex line bundle on X and ∇ is a holomorphically flat connection.*[5]

Remark 2. *It is important to note that we view $LocSys_1(X)$ as the moduli space of De Rham local systems (following the notation in [21]). Often Flat or Conn is used instead to denote De Rham local systems. For so called Betti local systems one sometimes uses $\mathcal{L}oc$. The difference essentially lies in that we consider a De Rham local system as a vector bundle on X with flat connection, while Betti local systems are maps $\pi_1(X) \to GL_n(\mathbb{C})$. These formulations are analytical equivalent via the Hilbert-Riemann correspondence, which maps connections to their monodromy.*

Now consider L as an element of the Jacobian variety (which is an abelian variety) of X. Recall that the Jacobian is the moduli space of degree 0 line bundles and because L supports a flat connection its first Chern class, and thus its degree, is 0. This allows us to realize $LocSys_1(X)$ as an affine fibration over $Jac(X)$. The fibers can be identified with the space of holomorphic one forms given by $H^0(X, \Omega^1_{dR})$, as every connection is can be determined uniquely as $\nabla' = \nabla + \omega$ with $\omega \in H^0(X, \Omega^1_{dR})$.

Let us consider a Fourier-Mukai like duality. We view $Jac(X)$ as the abelian variety A but instead of considering $A^{\vee}$[6] we pass to $LocSys_1(X)$ and only consider line

[5] A complex line bundle with holomorphically flat connections is equivalently a holomorphic line bundle with connection via the Kozul-Malgrange theorem.

[6] This would be $Jac(X)$ again as the Jacobian is a self dual variety.

bundles on $Jac(X)$ with flat connection. However, another alteration has to be made, quasi-coherent sheaves on $LocSys_1(X)$ do not correspond to quasi-coherent sheaves on $Jac(X)$ anymore. Instead, we should have quasi-coherent sheaves with flat connections on $Jac(X)$. An algebraic connection on an $\mathcal{O}_X$-module $\mathcal{F}$ is given by a k-linear homomorphism:

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X \tag{2.2}$$

where Ω^1_X is the cotangent sheaf, satisfying the Leibniz rule.[7] This is equivalent to specifying a map $\nabla : \mathcal{F} \otimes \Theta_X \to \mathcal{F}$ where Θ_X is the tangent sheaf.[8] A flat connection is defined by a connection with vanishing curvature thus by the following property: $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$ where $\xi, \eta \in \Theta_X$. Using this we can identify these "flat" quasi-coherent sheaves with objects called D-modules. For details see the classic reference on D-modules [25].

Definition 2.2.2. *Let D_X be the sheaf of differential operators on a smooth complex algebraic variety X. Then an algebraic D-module $\mathcal{D}$ is a quasi-coherent sheaf of D_X-modules.*[9]

Let us unravel this definition but first consider the following example of the most basic D-module.

Example. *Let $X = Spec\, k[x_1, ..., x_n]$ where k is of characterstic 0. Then we can view $\Gamma(X, D_X)$ as the nth Weyl algebra $A_n(k) = k[x_1, ..., x_n, \partial_1, ..., \partial_n]$. Here the variables x_i and ∂_i obey the relations $[\partial_i, x_j] = \delta_{ij}$[10] and $[\partial_i, \partial_j] = [x_i, x_j] = 0$.*

More generally we can view D_X, for a smooth complex variety X, as locally generated by the Weyl algebra. We interpret the x_i as local coordinates of $\mathcal{O}_X(U)$ and the ∂_i as vector fields meaning local coordinates of the tangent sheaf $\Theta_X(U)$ (here U

[7] Recall that this is equivalent to a connection on a vector bundle E on X (viewed as a sheaf of it's sections) which is specified by a k-linear map $\Gamma(E) \to \Gamma(E) \otimes \Omega^1_X$.

[8] For vector bundles this is the map $\nabla : \Gamma(E) \otimes \Gamma(TX) \to \Gamma(E)$.

[9] To be precise $\mathcal{D}$ is quasi-coherent when considered as a sheaf of O_X-modules.

[10] This can be interpreted as quantization of the cotangent bundle of X. It conveys the uncertainty between cotangent and spacial direction.

is a Zariski open subset of X). Therefore giving a general $\mathcal{O}_X$-module the structure of a D-module is equivalent to specifying an action of vector fields on sections extended to D_X via the Leibniz rule. This leads us to the main proposition and reason for our introduction of D-modules.

Proposition 2.2.1. *A (left) D_X-module is equivalent to a $\mathcal{O}_X$-module with flat connection.*

Proof. We specify the action of vectors fields on local sections of an $\mathcal{O}_X$-module $\mathcal{F}$. This is equivalent to defining a k-linear morphism of sheaves: $\nabla : \Theta_X \otimes \mathcal{F} \to \mathcal{F}$, $\xi \mapsto \nabla_\xi$ which satisfies the following properties:

1. $\nabla_{\xi f}(s) = f\nabla_\xi(s)$
2. $\nabla_\xi(fs) = f\nabla_\xi(s) + \xi(f)s$
3. $[\nabla_\xi, \nabla_\eta](s) = \nabla_{[\xi,\eta]}(s)$

where f and s are local sections of $\mathcal{O}_X$ and $\mathcal{F}$ respectively, and $\xi, \eta \in \Theta_X$.

It is apparent that specifying a flat connection is equivalent to defining the morphism above, where the flatness is just equivalent to property 3.[11] □

If we only consider D-modules that are coherent as $\mathcal{O}_X$-modules, we can view these directly as holomorphic vector bundles with flat connection. The reason for this is that a coherent sheaf with connection is inherently locally free and thus a vector bundle. This warrants our return to the case of holomorphic vector bundles with flat holomorphic connection. We are now also ready to describe a Fourier-Mukai like integral transform. Consider the projections of $Jac(X) \times LocSys_1(X)$ onto the two factors respectively:

$$\begin{tikzcd}
& \mathcal{P} \arrow[d] & \\
& Jac(X) \times LocSys_1(X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Jac(X) & & LocSys_1(X)
\end{tikzcd}$$

[11] Here the simplest example is just $\nabla_\xi(s) = \xi(s)$ which is locally given by the Weyl algebra.

Again we can define a Poincare like bundle $\mathcal{P}$ on $Jac(X) \times LocSys_1(X)$ called the "universal flat holomorphic line bundle". Its restriction can be viewed as a flat holomorphic bundle on $Jac(X)$. The integral transform then takes the form:

$$\Phi : \mathbf{QCoh}(LocSys_1(X)) \to \mathbf{DMod}(Jac(X)), \quad \mathcal{F} \mapsto \mathbf{R}p_*(\mathbf{L}q^*\mathcal{F} \otimes \mathcal{P}) \tag{2.3}$$

where **DMod** is the derived category of D-modules. If we consider the skyscraper sheaf $\mathcal{O}_{(L,\nabla)}$ at the point (L, ∇) of $LocSys_1(X)$, this transform can be interpreted as mapping $\mathcal{O}_{(L,\nabla)}$ to $(\mathcal{L}, \tilde{\nabla})$ (a holomorphic line bundle $\mathcal{L}$ with flat connection $\tilde{\nabla}$ on $Jac(X)$) considered as a D-module.

Laumon and Rothstein independently proved that this transform produces an equivalence of derived categories [48, 38]. They showed that the Fourier-Mukai transform acting on $LocSys_1(X)$, viewed as a unique affine bundle over $Jac(X)$ with fibers $H^0(X, \Omega^1_{dR})$, transforms the data in the fibers to a flat connection on coherent sheaves over $Jac(X)$, thereby giving the coherent sheaves over $Jac(X)$ a D-module structure.

Theorem 2.2.2 (Laumon and Rothstein)**.** *A twisted Fourier-Mukai transform induces an equivalence of (bounded) derived categories.*

$$\boldsymbol{DMod}(Jac(X)) \simeq \boldsymbol{QCoh}(LocSys_1(X))$$

the equivalence holds up to cohomological shift and sign.

This is the simplest case of the Geometric Langlands correspondence.

We can again consider the idea of a spectral decomposition of categories. Now special sheaves called Hecke eigensheaves, which are built exactly from the flat line bundles considered as D-modules, give a "basis" for $\mathbf{DMod}(Jac(X))$. In fact the flat line bundles are the degree 0 components of the Hecke eigensheaves. One the other side the skyscraper sheaves on $LocSys_1(X)$ give a "basis" for $\mathbf{QCoh}(LocSys_1(X))$.

Let us discuss what we mean by Hecke eigensheaf to a slightly higher degree of precision. Hecke eigensheaves are eigenobjects under the so called Hecke functors. Heuristically the Hecke eigensheaves are the "eigenvectors" of the Hecke functors with "eigenvalues" given by the local systems. These functors can be viewed as automorphisms of the derived category of D-modules on the moduli of bundles (in our case line bundles).

We will sketch their construction for the case of line bundles and return to the general case in the next section. Let $Pic^d(X)$ be the Picard variety parametrizing line bundles of degree d on X, $Jac(X) \times \mathbb{Z} = Pic(X)$. Then we can define the Hecke functor as:

$$H : \mathbf{DMod}(Pic(X)) \to \mathbf{DMod}(Pic(X) \times X) \tag{2.4}$$

This just amounts to the categorical pullback of the Abel-Jacobi map, for each $x \in X$:

$$\pi^d : Pic^d(X) \times X \to Pic^{d+1}(X), \quad (L, x) \mapsto L(x) \tag{2.5}$$

The Hecke eigensheaves are then exactly the D-modules $\mathcal{D}_E$ with corresponding local system E that obey:

$$H(\mathcal{D}_E) \simeq E \boxtimes \mathcal{D}_E \tag{2.6}$$

Here $\mathcal{F}_1 \boxtimes \mathcal{F}_2$, between two general sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X_1 and X_2 respectively, is called the external tensor product.[12] One can use the Hecke property to extend the equivalence from D-modules on $Jac = Pic^0$ to D-modules on on all components of Pic.

Until now, we considered line bundles, which essentially are principal bundles with structure group GL_1. Now the natural question arises: can we formulate this type of integral transform for principal G-bundles with general reductive algebraic groups G?

Laumon and prove that at least for $G = T$ such an integral transforms yields an equivalence [38]. However, the case for general reductive G remains wide open, conjectural formulations exist as we will see later. When moving beyond line bundles and thus the self-duality of Jac, we would have to consider a duality as we saw for the abelian variety in the Fourier-Mukai transform. This is where the Langlands dual group appears.

[12] This is essentially the tensor product of the $\mathcal{F}_1$ and $\mathcal{F}_2$ pulled back (along a projection to one of the factors) to the product space $X_1 \times X_2$. Notice how this construction already appeared implicitly in section 2.1.

Definition 2.2.3. *Let G be an algebraic reductive group then we can define its weight lattice. Then its Langlands dual group ^LG is the group with weight lattice and roots that are exactly the co-weight lattice and coroots of G.*

Example. *The Langlands dual group of GL_n is again GL_n, as we saw for GL_1. More interesting examples are given by $G = SU_n$ where $^LG = PSU_n$ or by $G = Sp_n$ where $^LG = SO_{2n+1}$.*

When we consider the full abelian case where $G = T$ we can schematically state Laumon's result:

Theorem 2.2.3 (Laumon [38]). *There is an equivalence of derived categories:*

$$\mathbf{DMod}(Bun_T(X)) \simeq \mathbf{QCoh}(LocSys_{^LT}(X))$$

where T is a torus and ^LT is its dual torus.

We will describe what is meant by these objects in the next section.

2.3 Geometric Langlands for general G

We pass to the full Geometric Langlands correspondence by considering principal G-bundles. This will, however, require defining some new generalizations of the structures we have already seen. The version of the Geometric Langlands correspondence we describe is called the categorical unramified version (unramified as X is taken to be smooth).

2.3.1 The Stacks Bun_G and $LocSys_G$

We first consider the automorphic side. When passing from the moduli space of line bundles to general principal G-bundles we lose the representability of the moduli space in terms of a variety or scheme. Part of the reason is that principal G-bundles have varying automorphism groups (unlike line bundles where the group is always $\mathbb{C}^*$). To construct an object parameterizing all G-bundles, while also including the information about the different automorphism groups, one needs to define the moduli stack of principal G-bundles.

Definition 2.3.1. *Let X be a smooth algebraic curve and G a reductive group then*

$$Bun_G(X) = Maps(X, BG)$$

where $BG = pt/G$ is the classifying space of G considered as a one object groupoid and $Maps(X,Y)$ is a prestack of Hom's between to spaces X and Y.

$Bun_G(X)$ is a categorical version of isomorphism classes of principal G-bundles viewed as homotopy classes of maps $X \to BG$. (Pre)stacks are higher categorifications of (pre)sheaves and can also be called 2-(pre)sheaves.[13] Thus we can also view $Maps(X, BG)$ as a presheaf of groupoids on X. $Bun_G(X)$ actually has more structure and can be considered as a smooth algebraic stack.[14] A guess for the automorphic side of the full picture would then correspond to the derived category D-modules on $Bun_G(X)$. It turns out that such a category exists and is well defined [23].

We now turn to the Galois side. Let us describe what happens when we want to generalize $LocSys_1$ to $LocSys_G$. We want something that captures the notion of a moduli space of G-bundles with flat connection.

The data of a flat connection on a fiber bundle on X gives the identification of nearby fibers in a homotopically trivial neighborhood $U \subset X$. This amounts to identifying fibers on points of the base space that are "infinitesimally close". For general varieties, we do not have a basis of homotopically trivial Zariski open subsets and therefore we need an algebraic notion of a flat connection if we want to move past the analytic viewpoint.[15] This is provided by the De Rham space which gives an identification of infinitesimally close points.[16] Thereby it transfers the idea of a flat connection into a change of the information of sheaves on a space (the underlying topological space is equivalent but the structure sheaf on space is different[17]). The naming stems from the

[13] These can be interpreted as (pre)sheaves taking values in a category.

[14] See [27] for what this means precisely.

[15] If X can be viewed as a complex analytic manifold we always have a notion of parallel transport. This is also why we have an analytic equivalence between Betti and De Rham local systems.

[16] The idea is based on Grothendieck's insight, that the data of an algebraic connection can be recovered as descent data on the De Rham space.

[17] Note how this is similar to considering the scheme with underlying space $Spec(k[x]/x^2)$ versus $Spec(k)$. The spaces are equivalent topologically as both consist of a point, the single prime ideal.

equivalence of the sheaf cohomology on the De Rham space to the algebraic De Rham cohomology of the respective space. A detailed definition and description of the de Rham space is given in [41]. We will provide a sketch of a definition:

Definition 2.3.2. *Let X be a variety or scheme and R a ring over k (characteristic 0 as always). Then we define the functor $X(R) = Hom(Spec(R), X)$. We take the quotient of R by its nilradical (the set of nilpotent ideals in R) R/I. Then the De Rham space is given by $X_{dR}(R) = Hom(Spec(R/I), X) = Hom(Spec(R), X_{dR})$. This is not generally representable as a scheme but always a functor of points.*[18]

Example. *If we take R to be the ring of dual numbers $k[\epsilon]/\epsilon^2$ and $X = \mathbb{A}^1$ then $X(k[\epsilon]/\epsilon^2)$ can be identified with tangent space of the affine line $T\mathbb{A}^1$. Then $R/I = k$ and $X_{dR}(R) = \mathbb{A}^1(k)$. The passage to the de Rham space rigidifies a space by removing the infinitesimal information.*

If X is smooth we can view X_{dR} as X quotient equivalence under infinitesimal closeness. We now have a purely algebraic way to express. the information of a connection. We can also extend the concept quasi-coherent sheaves (and D-modules) on varieties (or schemes) to general stacks, see [23] for the full picture.

These leads to a description of D-modules in a higher form of abstraction:

Theorem 2.3.1. *Let X be a variety, scheme or stack. For smooth X we have an equivalence of derived categories:*[19]

$$\boldsymbol{QCoh}(X_{dR}) = \boldsymbol{DMod}(X)$$

The automorphic side of Geometric Langlands should then be $\mathbf{QCoh}(Bun_G(X)_{dR}) = \mathbf{DMod}(Bun_G(X))$.

However the structure sheaf is different as it contains a nilpotent element, hence we have inequivalent schemes.

[18]Recall that a functor of R points, where R is a ring, of some scheme X is $Hom(Spec(R), X)$. These so called R points of a scheme are not necessarily a scheme.

[19]The left and side actually defines objects called (left) crystals for general X. These objects are powerful extensions of D-modules in that they work well for non-smooth X.

Let us also give a purely algebraic[20] definition of the stack of De Rham local systems for general G:

Definition 2.3.3. *Let X be a smooth algebraic curve and G a reductive group then*

$$LocSys_G(X) = Maps(X_{dR}, BG)$$

where X_{dR} is the De Rham space of X.

For the Galois side we now want to consider quasi-coherent sheaves on $LocSys_{^LG}(X)$, but this turns out to be more difficult then for $Bun_G(X)$.

Remark 3. *The Betti view only sees topological information and thus we can view Betti local systems as the stack $\mathcal{L}oc_G(X) = Maps(X_B, BG)$ where X_B is the Betti space which is roughly the fundamental groupoid of X. The Betti space is analytically equivalent to the De Rham space via a generalized Hilbert-Riemann correspondence [49].*

2.3.2 Some Derived Algebraic Geometry

When trying to define the derived category of quasi-coherent sheaves on $LocSys_G(X)$ we run into problems. It turns out that this stack has a non-trivial derived structure. To understand what this means we sketch the main idea of derived algebraic geometry. In derived algebraic geometry one replaces the usual ring with differentially graded algebras. This, for example, happens naturally when we have certain self-intersections of schemes (or more generally non-transverse intersections). We can view the resulting dg-scheme, written $(X, \mathcal{O}^\bullet)$, as an underlying topological space X (equivalent to that of a classical scheme) and a sheaf of dg-algebras. We recover the structure sheaf of the underlying classical scheme by taking the zeroth cohomology of the sheaf dg-algebras. The other cohomological degrees contain "hidden information" about the nature of the intersection. The most basic non-trivial derived scheme is given by:

Example. *Let us consider the self-intersection of a point on the affine line $Spec(k[x]) = \mathbb{A}^1$. With $pt = Spec(k)$ the intersection is of the form $pt \times_{\mathbb{A}^1} pt$. We can write this as the*

[20]Heuristically this is an algebraic version of the moduli space of principal G-bundles with flat connection.

spectrum of the derived tensor product of underlying dg-algebra $Spec(k[x]/x \otimes^{\mathbb{L}}_{k[x]} k[x]/x)$. *Let us resolve* $k[x]/x$ *via* $\xi k[x] \xrightarrow{d} k[x]$ *with* $d(\xi) = x, d(x) = 0$ *and* ξ *being a variable of cohomological degree -1.*

$$(\xi k[x] \xrightarrow{d} k[x]) \otimes_{k[x]} k[x]/x = (\xi k \xrightarrow{0} k)$$

The corresponding dg-scheme is of the form $(Spec(k), k[\xi])$. *We recover the classical scheme underlying this dg-scheme by taking* $H^0(k[\xi]) = k$. *This is just the classical structure sheaf of* $Spec(k)$.

See [39] For a great review of derived algebraic geometry.

Even though X is smooth, $LocSys_G(X)$ is not, instead it can be viewed as a derived intersection and therefore has extra cohomological information in nonzero degrees. The derived self-intersection of a point appears directly in the abelian case.

Example. *Already for the multiplicative group scheme* $\mathbb{G}_m$ *the stack takes the form* $LocSys_{\mathbb{G}_m} = LocSys_1 \times B\mathbb{G}_m \times pt \times_{\mathbb{A}^1} pt$. *Here* $B\mathbb{G}_m$ *gives the automorphisms while* $pt \times_{\mathbb{A}^1} pt$ *can be interpreted as glueing information [3].*

This extra derived structure is the reason we were only considering $LocSys_1$ in section 2.2. This is a good approximation because the duality maps quasi-coherent sheaves on $LocSys_1 \times B\mathbb{G}_m \times pt \times_{\mathbb{A}^1} pt$ to D-modules on $Jac \times \mathbb{Z} \times B\mathbb{G}_m$. This means that $LocSys_1$ is still mapped to Jac, while quasi-coherent sheaves on $B\mathbb{G}_m$ and $pt \times_{\mathbb{A}^1} pt$ are mapped to D-modules on $\mathbb{Z}$ and $B\mathbb{G}_m$ respectively.

A further caveat is that one cannot define the derived category of quasi-coherent sheaves properly on a derived stack. A passage to differentially graded categories is necessary to make things work. Dg-categories are an enhancement of regular derived categories of an abelian category in that they are more well behaved in many aspects like gluing or descent (see [18] for precise reasons). Heuristically dg-categories are categories where the morphisms between objects form differentially graded modules.[21] The simplest example is the dg-category with one object, which just a dg-algebra.

[21]Dg-catgories are best thought of in the context of ∞-categories. We can view dg-categories as so-called stable $(\infty, 1)$-categories under the stable Dold-Kan correspondence. See [40] for the classic reference on ∞-categories. This correspondence relies on the fundamental idea of ∞-categories having an entire topological space of morphisms.

2.3.3 The Naive Conjecture

We can now state the naive conjecture of Geometric Langlands as posed by Beilinson and Drinfeld [2] formulated in the dg-setting:

Conjecture. *We have an equivalence of dg-categories:*

$$\boldsymbol{QCoh}(LocSys_{\,^LG}(X)) \cong \boldsymbol{DMod}(Bun_G(X))$$

where X is a smooth complex algebraic curve and ^LG the Langlands dual group of G. The equivalence is given by a conjectural non-abelian categorical integral transform $\mathbb{F}$ associating to each Hecke eigensheaf a skyscraper sheaf on an irreducible local system.

The conjecture is naive in the sense that it was initially known to be false. Only in the case of $G = T$, stated in theorem 1.2.2, is this an equivalence. Note that this conjectural relationship can be formulated, without the use of dg-categories, as an equivalence of derived categories. However, as soon as one wants to construct a proof one needs the more well-behaved dg-enhancement.

2.3.4 The Hecke Category

Before moving to the final section discussing the reasons for this failure and a possible remedy, let us return to the idea of the spectral decomposition in this more general context. We define the global Hecke functors for the entire stack $Bun_G(X)$:

$$H_V : \mathbf{DMod}(Bun_G(X)) \longrightarrow \mathbf{DMod}(Bun_G(X) \times X) \tag{2.7}$$

where H_V is associated to each irreducible representation V of the Langlands dual group ^LG. On the Hecke eigensheaves the action takes the form:

$$H_V(\mathcal{D}_E) \simeq V_E \boxtimes \mathcal{D}_E \tag{2.8}$$

where $V_E = E \times_{\,^LG} V$ is the ^LG local system associated to E via $V \in \mathbf{Rep}(^LG)$. This amounts to a spectral decomposition of $\mathbf{DMod}(Bun_G(X))$ along $\mathbf{QCoh}(LocSys_{\,^LG}(X))$. The spectrum of the Hecke functors is approximately contained in $\mathbf{QCoh}(LocSys_{\,^LG}(X))$.

Slightly more precise is to say that the action of $\mathbf{Rep}({}^LG)$ on $\mathbf{DMod}(Bun_G(X))$ factors through the (monoidal) action of $\mathbf{QCoh}(LocSys_{{}^LG}(X))$ on $\mathbf{DMod}(Bun_G(X))$ uniquely [21]. Essentially the spectral decomposition amounts to simultaneously diagonalizing the Hecke functors.

The Hecke functors can be described as sitting in the Hecke category (which in turn is a categorification of the Hecke algebra). We can describe the Hecke category, denoted $\mathcal{H}_x$, for each $x \in X$ as the abelian category of D-modules on the double coset $G(O_x) \setminus G(K_x) / G(O_x)$. Here $O_x = \mathbb{C}[[x]]$ and $K_x = \mathbb{C}((x))$ are the rings of functions on the formal disk and punctured formal disk around x respectively and $G(R) = Hom(Spec(R), G)$ is just the functor of R points of G. We can identify $G(K_x)/G(O_x)$ with the affine Grassmanian $Gr_{G,x}$. The Hecke functors H_V are then described as elements of the entire Hecke category, roughly given by $\bigotimes_{x\in X} \mathcal{H}_x$.

The heuristic behind this definition comes from Weil's uniformization theorem mentioned in the introduction. Any G-bundle on X can be trivialized away from finitely many points of X, alternatively, it can be trivialized on the formal disks around those points. The transition functions are given by elements of $G(K_x)$. We can thus view $Gr_{G,x}$ as the moduli space of G-bundles on the formal disk around x, which are trivial away from x (and thereby an intrinsic symmetry of the moduli stack of G-bundles). Then $G(O_x) \setminus G(K_x) / G(O_x)$ is just the moduli space of pairs of G-bundles on the formal disk around x which are isomorphic away from x. If we now act with an element of this double coset space on an element of $P \in Bun_G(X)$ we modify P at the point x to get a new bundle P' which is isomorphic to P on $X \setminus x$, this is also called a Hecke modification. The affine Grassmannian hereby contains (for each x) the precise information for all such possible modifications.

This gives a complimentary way to construct the Hecke functors via the Hecke correspondence. Let $x \in X$ then $\mathcal{H}ecke_x$ is the Hecke stack. This is a $Gr_{G,x}$ fibration over $Bun_G \times Bun_G$. One can define two projections $h_{1,x}$ and $h_{2,x}$:

$$\begin{tikzcd} & \mathcal{H}ecke_x \arrow[dl, "h_{1,x}"'] \arrow[dr, "h_{2,x}"] & \\ Bun_G & & Bun_G \end{tikzcd}$$

The action of Hecke functors on $\mathcal{F} \in \mathbf{DMod}(Bun_G(X))$ for each $x \in X$ are then given by $H_{V,x}(\mathcal{F}) = h_{1,x}^*(h_{2,x*}\mathcal{F})$.

An important property is the monoidal equivalence between the category of representations of the Langlands dual group $\mathbf{Rep}(^LG)$ and the Hecke category:

$$\mathbf{DMod}(Gr_G)^{G(O_x)} = \mathbf{DMod}(G(O_x) \setminus G(K_x)\,/G(O_x)) \tag{2.9}$$

given by Tannakian reconstruction.[22] This identification is called the Geometric Satake equivalence and defines an action of $\mathbf{Rep}(^LG)$ on $\mathbf{DMod}(Bun_G(X))^{G(O_x)}$. As well as giving another motivation for the appearance of the Langlands dual group on the Galois side. Let us elucidate this in the simplest case.

Example. *For $GL(1)$ the Hecke stack is $Pic(X) \times \mathbb{Z}$, the affine Grassmanian is the fiber and isomorphic to $\mathbb{Z}$ (when $\mathbb{Z}$ is viewed as an ind-variety meaning an infinite disjoint union of points. D-modules on $Gr_{GL(1)}$ are vector spaces on the individual points of $\mathbb{Z}$. Therefore $\boldsymbol{DMod}(Gr_{GL(1)})^{G(O_x)}$ is just the category of $\mathbb{Z}$-graded vector spaces. This category in turn is equivalent to $\boldsymbol{Rep}(GL(1))$. Thus under the Geometric Satake equivalence $^LGL(1) = GL(1)$.*

The Satake equivalence is therefore another reason why ^LG local systems appear as eigenvalues of the action of the Hecke functors on Hecke eigensheaves.

We could further describe operators acting on $\mathbf{QCoh}(LocSys_{\,^LG}(X))$called Wilson[23] functors [11]. They send objects in $\mathbf{QCoh}(LocSys_{\,^LG}(X))$to sheaves on $LocSys_{\,^LG}(X)\times X$ that are quasi-coherent sheaves on $LocSys_{\,^LG}(X)$ and D-modules along X. Their action on skyscraper sheaves is then analogous to the action of Hecke functors on Hecke eigensheaves:

$$W_V(\mathcal{O}_E) = V_E \boxtimes \mathcal{O}_E \tag{2.10}$$

where $\mathcal{O}_E$ is a skyscraper sheaf on $E \in LocSys_{\,^LG}(X)$ and $V_E = E \times_{^LG} V$ is considered as a D-module on X. The conjectural non-abelian categorical Fourier transform should then intertwine the action of the "Wilson" functors and Hecke functors $\mathbb{F}(W_V) = H_V$.

Now let us discuss how the naive conjecture fails and how one could reformulate it.

[22]Here both categories need to be abelian.

[23]Note this naming is symbolic of the relation to the physical interpretation.

2.4 Beyond the Naive Conjecture

The problem with the naive Geometric Langlands conjecture is that the dg-category $\mathbf{QCoh}(LocSys_{{}^LG}(X))$ is too small.[24] Certain D-modules do not have corresponding objects in $\mathbf{QCoh}(Loc_{{}^LG}(X))$. The failure is essentially caused by the non-smoothness of the derived stack $LocSys_{{}^LG}(X)$. This manifests as certain coherent complexes in $\mathbf{QCoh}(LocSys_{{}^LG}(X))$ failing to be perfect.

Example. *Consider the ring of dual numbers $A = k[\epsilon]/\epsilon^2$. If we resolve the A-module k:*

$... \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} k$

then we have an infinite sequence of terms. Therefore the resolution of k is not perfect but still coherent.

A natural enlargement of dg-category **QCoh** would be **IndCoh** this is the inductive completion of the dg-category of coherent sheaves. We can imagine this as a category that contains all filtered colimits of objects in **Coh**. Essentially **IndCoh** gives an alternative way to enlarge **Coh**, that is compactly generated[25] by complexes of coherent sheaves. **QCoh**is instead compactly generated by perfect complexes (complexes of locally constant sheaves). However **IndCoh** turns out to be too large. This is where the nilpotent singular support, considered by Gaitsgory and Arinkin in [1], comes into action. Highly schematically the singular support condition asserts that a sheaf in **IndCoh** comes from $\mathbf{QCoh} \subset \mathbf{IndCoh}$ if and only if it obeys the nilpotent singular support condition. This condition implies that one can construct a map acting on objects of **IndCoh** that is nilpotent if and only if the object comes from **QCoh**. We can therefore extend **QCoh** to $\mathbf{IndCoh}_{\mathcal{N}}$, where $\mathcal{N}$ is the nilpotent singular support, and retain that the origin of the objects is **QCoh**. If we choose $\mathcal{N} = \{\emptyset\}$ we just recover **QCoh**. We now have complexes that are coherent but also always perfect.

[24] The failure of the equivalence has been directly constructed by Lafforgue in the case of $X = \mathbb{P}^1$ [35].

[25] A object in a category $\mathcal{C}$ is compact if its morphism commute with colimits. $\mathcal{C}$ is called compactly generated if at least one compact $C \in \mathcal{C}$ is always connected via a morphism to any object in the category.

This leads to the following refined conjecture.

Conjecture (Gaitsgory and Arinkin)**.** *We have the following equivalence of dg-categories*

$$\boldsymbol{IndCoh}_{\mathcal{N}}(LocSys_{\,^LG}(X)) \simeq \boldsymbol{DMod}(Bun_{\,G}(X))$$

where $\mathcal{N}$ is the nilpotent singular support and the symmetries of both sides intertwine.

We also have a compatibility of symmetries of the respective sides of the conjecture extended from the naive setting. An outline of a proof using this idea is given in [18] for the case of $GL(2)$. As one might have noticed, the correspondence as we have now formulated is somewhat asymmetric. It is this asymmetry that can be resolved by viewing the Geometric Langlands as a limiting case of the larger Quantum Geometric Langlands. This is what we will discuss in the next chapter.

Chapter 3

Quantum Geometric Langlands

The Geometric Langlands correspondence is the limiting case of a more symmetric quantum correspondence. In the so-called Quantum Geometric Langlands the symmetry is restored by passing from classical D-modules to their twisted counterparts. This conjectural equivalence was first proposed by Beilinson and Drinfeld in relation to their work on conformal field theory approach to Geometric Langlands [2]. We will begin by introducing what we mean by twisted D-modules but focus on the specific twisted D-modules we need. This will then lead to the definition of the quantum deformation of the correspondence. We follow [12] section 6.3. in this chapter.

Remark 4. *The Beilinson-Drinfeld perspective to Geometric Langlands gives the motivation for the quantum deformation. For a great review see [12] Part III. They construct D-modules on Bun_G twisted by $\mathcal{L}^{\otimes k}$, as the sheaves of conformal blocks of CFT's with an affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$ (at level k) symmetry.*[1] *At the critical level, when $k = h^\vee$ (where $h^\vee$ is the dual Coxeter number), they explicitly associate Hecke eigensheaves to a class of local systems called opers. The opers $Op_{{}^LG}(X)$ form a Lagrangian subspace of $LocSys_{{}^LG}(X)$ and can essentially be considered as (De Rham) local systems obeying special reduction conditions. In the abelian case an example for opers are the trivial flat line bundles. Beilinson and Drinfeld use this construction to prove part (roughly half) of Geometric Langlands for general G.*

[1] The affine Kac-Moody algebra is defined as the central extension $\mathbb{C}1 \to \hat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((x))$ and k is the eigenvalue of the action of 1 on a representation of $\hat{\mathfrak{g}}$.

3.1 D-modules Twisted by a Line Bundle

We are interested in the special class of differential operators twisted by a line bundle. For the classic reference on (twisted) D-modules see [25]. Let D_X be a sheaf of differential operators on a complex smooth variety X. Then we can define the sheaf of differential operators twisted by a line bundle $\mathcal{L}$ as:

$$D_X(\mathcal{L}, \mathcal{L}) = Hom(\mathcal{L} \otimes_{\mathcal{O}_X} D_X, \mathcal{L} \otimes_{\mathcal{O}_X} D_X) \tag{3.1}$$

We call this the $\mathcal{L}$-twisted sheaf of differential operators $D_{X,\mathcal{L}}$ [47]. The twisted differential operators become k-linear maps $d : \mathcal{L} \to \mathcal{L}$ and the sections of the structure sheaf $\mathcal{O}_X$ act on $\mathcal{L}$ through the $\mathcal{O}_X$ action. Thus a sheaf of $\mathcal{L}$ twisted differential operators can be described as the sheaf of differential operators acting on the sections of the line bundle $\mathcal{L}$. Choosing $\mathcal{L}$ as $\mathcal{O}_X$ recovers the notion of a sheaf of differential operators. Sheaves of differential operators are therefore just a special case of sheaves of twisted differential operators.

We can then define the $\mathcal{L}$-twisted D-modules as an extension of D-modules:

Definition 3.1.1. *Let $D_{X,\mathcal{L}}$ be the sheaf of $\mathcal{L}$-twisted differential operators. Then a twisted algebraic D-module $\mathcal{D}_{\mathcal{L}}$ is a quasi-coherent sheaf of $D_{X,\mathcal{L}}$-modules.*

This can be extended to differential operators twisted by an arbitrary complex tensor power of a line bundle, as we will see in the next section.

We also note that the construction of the category of $\mathcal{L}$ twisted (or more general twisted) D-modules can be formulated in the more abstract setting of viewing D-modules as quasi-coherent sheaves on the De Rham space. The twisting then becomes encoded in an object called the Picard groupoid of twistings.[2] For details on this construction see [24].

[2]The construction allows for viewing $\mathcal{L}$ twisted D-modules on a space X as the trivial $\mathbb{G}_m$ gerbe on X_{dR} together with a specification of further data.

3.2 The Quantum Deformation

We return to the stack $Bun_G(X)$. Let us now fix a non-degenerate pairing on the Lie algebra $\mathfrak{g}$, corresponding to the group G, denoted κ (the Killing form for example). The pairing identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ but is completely defined by the restriction to the Cartan subalgebra. The Langlands dual group is the group with dual root data and therefore κ equivalently defines a pairing on ${}^L\mathfrak{g}$.

We can associate κ the determinant line bundle $\mathcal{L}$ on $Bun_G(X)$, the multiples k of κ become the kth tensor power of the line bundle $\mathcal{L}^{\otimes k}$. This effectively means $\mathcal{L}$ models κ and $\mathcal{L}^{\otimes k}$ models multiples of κ. This line bundle, however, only exists for $k \in \mathbb{Z}$. This means as long as $k \in \mathbb{Z}$ sheaf of k-twisted D-modules is equivalent to the sheaf of differential operators acting on sections of $\mathcal{L}^{\otimes k}$. For $k \in \mathbb{C}$ this model fails but the the corresponding sheaf of sheaf of k-twisted differential operators is still well defined. This entire association works via the construction of D-modules as sheaves of conformal blocks, mentioned in Remark 4, where k also corresponds to the level of the CFT.

We can extend k by a point at infinity and thus have $k \in \mathbb{CP}^1$. As stated in Remark 4, $\mathcal{L}^{\otimes k}$-twisted D-modules relate to local systems when $k = h^\vee$. This is why, by considering a critical shift of k by $h^\vee$ we can ensure that D-modules become untwisted when corresponding to local systems. At $k = h^\vee$ the category of $\mathcal{L}^{\otimes k - h^\vee}$ twisted D-modules on $Bun_G(X)$ reduces to $\mathbf{DMod}(Bun_G(X))$. It is further conjectured that at $k = \infty$ one recovers the category $\mathbf{QCoh}(LocSys_{{}^LG})$ or $\mathbf{IndCoh}_{\mathcal{N}}(LocSys_{{}^LG})$ when taking singular support into account. This leads to an extension of the Geometric Langlands correspondence.

Conjecture ([20])**.** *There exists an equivalence of dg-categories*

$$\boldsymbol{DMod}^c(Bun_G(X)) \simeq \boldsymbol{DMod}^{-1/c}(Bun_{{}^LG}(X))$$

where c is the twisting parameter with absorbed critical shift.

This is called the quantum geometric Langlands as both sides are now some kind of quantization of the cotangent bundle $T^*Bun_G(X)$. In the abelian case this correspondence is a theorem [45]. Concretely one can construct an equivalence $\mathbf{DMod}^k(Jac(X)) \simeq$

$\mathbf{DMod}^{-\frac{1}{k}}(Jac(X)$. For the important cases where k becomes rational (e.g. $k = h^\vee$) the conjecture is more complicated as we will see later. One would also have to take singular support into account as $c = \infty$.

Chapter 4

S-duality from 6D

In order to explain the physical interpretations of Geometric Langlands one needs to introduce S-duality. The power of this duality lies in relating theories with strong coupling to theories with weak coupling in which calculations are often easier to perform. The duality was first proposed in the work of Montonen and Olive [42] which conjectured such a duality between four-dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theories. S-duality has since been conjecturally expanded to include dualities between certain string theories. We will focus on the original Montonen-Olive version of S-duality and explain its geometric origin from six dimensions. This will be followed by a description of the action of S-duality on a certain topological sector of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theories that play an important role in the physical interpretation of Geometric Langlands.

4.1 $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

We specify a maximally supersymmetric Yang-Mills theory on a four manifold X^4 with the gauge group G given by compact Lie group. This theory is usually derived by dimensional reduction[1] from ten-dimensional $\mathcal{N} = 1$ Yang-Mills see [34] for details.

[1]Via compactification on a six torus.

The bosonic part of the action is given by [2]:

$$\frac{1}{e^2}\int d^4x\, Tr\left(\frac{1}{2}\sum_{\mu,\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} + \sum_{\mu=0}^{3}\sum_{i=3}^{6} D_\mu\phi_i D^\mu\phi_i + \frac{1}{2}\sum_{i,j=1}^{6}[\phi_i,\phi_j]^2\right) \tag{4.1}$$

Here the curvature $F = \sum_{\mu,\nu}^{3} F_{\mu\nu}$ corresponds to some gauge field $A = \sum_{\mu}^{3} A_\mu$ which is a connection on the underlying principle G-bundle P. $D_\mu\phi_i$ is the covariant derivative of a scalar field and $[\phi_i, \phi_j]$ is the commutator of scalar fields.[3] Without changing the equations of motion, we can add a purely topological term to the action dependent on the theta angle θ and the second Chern class of P. This is given by:

$$\frac{\theta}{8\pi^2}\int_{M^4} Tr(F \wedge F) \tag{4.2}$$

The theta angle combines with the coupling constant into the parameter:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{4.3}$$

which takes values in the upper half plane. The angular nature of the parameter θ leads to an equivalence $\theta \simeq \theta + 2\pi$ which in turn produces a discrete symmetry $\tau \simeq \tau + 1$.

The 4d $\mathcal{N} = 4$ theory is superconformal and the R-symmetry is given by $Spin(6)$. On $\mathbb{R}^4$ space-time symmetry is naturally $Spin(4)$ (or $Spin(3,1)$ in Lorentz signature). The 16 supersymmetries are of the form $\bar{Q}^A_\alpha$ and $Q^A_{\dot{\alpha}}$ with $A = 1, ...4$ and $\alpha, \dot{\alpha} = 1, 2$ and also transform under $Spin(4) \times Spin(6)$(or Spin(3,1)$\times Spin(6)$).

4.2 S-Duality and the Langlands Dual Group

We already mentioned that τ lies in the upper half plane. Now recall that $SL(2,\mathbb{R})$ acts as a symmetry group of upper half plane. Therefore we have an $SL(2,\mathbb{R})$ action on τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{4.4}$$

[2]We will neglect the fermionic part throughout.

[3]The scalar fields are a result of the dimensional reduction on six circles.

with $a, b, c, d \in \mathbb{R}$ and $ab - bc = 1$.

Let us now state the S-duality conjecture first proposed by Goddard, Nuyts, Olive, and Montonen in [42] and [26] and investigate its properties.

Conjecture. *For $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on M^4 with gauge group G and coupling parameter τ, S-duality asserts the existence of a quantum equivalence of theories mapping $\tau \to -1/n_{\mathfrak{g}}\tau$ [4] and the gauge group G to its Langlands dual group ^LG (alternatively called GNO (Goddard-Nuyts-Olive) dual group coming from the physics side).*[5]

The $\tau \to -1/\tau$ symmetry together with the symmetry $\tau \to \tau + 1$ is an action of the group $SL(2, \mathbb{Z})$ as a discrete subgroup of $SL(2, \mathbb{R})$ on τ (for simply-laced G). These two symmetries even act as the generators of $SL(2, \mathbb{Z})$, often denoted S and T respectively.

Let us further state how S-duality should act on the 16 supersymmetries of the theory, see [34] for details again. We can define a phase $e^{i\phi/2} = |\tau|/\tau$, determined by computing the action of S-duality on the central charges, with which the supercharges multiply under S-duality:

$$Q^A_{\dot\alpha} \to e^{-i\phi/2}\, Q^A_{\dot\alpha} \qquad \bar{Q}^A_{\alpha} \to e^{i\phi/2}\, \bar{Q}^A_{\alpha} \tag{4.5}$$

This is a result of S-duality implying that the $SL(2, \mathbb{Z})$ action commutes with the $Spin(3, 1)$ and $Spin(6)$ action, making the duality a multiplication by the scalar phase.

4.3 The 6D Superconformal QFT

In 1978 Werner Nahm showed that six dimensions is the maximum in which a superconformal quantum field theory can exist [44]. It has since been conjectured (via string theory[6]) that such a six-dimensional theory exists as the worldvolume theory of the M5-brane. The theory is thought to be classified by an ADE choice of Lie algebra. The so-called six-dimensional (2,0)-model is believed to give rise to a multitude of dualities

[4] Where $n_{\mathfrak{g}}$ is the lacing number of the lie algebra of the gauge group G.

[5] This exchange of gauge groups is not exact if M^4 is not $\mathbb{R}^4$.

[6] Concretely IIB string theory compactified on $K3$.

between lower-dimensional CFT's and a vast array of corresponding mathematics (for an overview see [28]). Its superconformal symmetry group is given by the orthosymplectic group $OSp(2,6|2)$, the R-symmetry is accordingly $Sp(2)$. We follow [50] in this section.

Another peculiarity is that the theory is believed to not possess a classical Lagrangian and thereby only exist as a quantum theory. The reason for this can already be seen in the abelian case when $G = U(1)$. The field content of the theory is predicted to contain a self-dual three form $H = \star H$ with $H = dB$ in the abelian case.[7] The curvature term would look something like:

$$\int_{M^6} H \wedge \star H = \int_{M^6} H \wedge H \tag{4.6}$$

which is just vanishing.

Even though the general theory remains highly mysterious it can be studied via dimensional reduction on certain product spacetime manifolds. Of particular interest to S-duality (and thereby as we will later see Geometric Langlands) is (2,0) model on $T^2 \times M^4$. The compactification in the torus direction (in the limit of zero torus area) is believed to yield $\mathcal{N} = 4$ supersymmetric Yang-Mills.[8] Let us view $T^2 = S^1_1 \times S^1_2$ as the orthogonal product of two circles with radii R_1 and R_2 respectively. Then one can first compactify on R_1 followed by R_2 or vice versa, yielding equivalent theories. Usually one would assume that compactifying on the first circle yields a factor of $2\pi R_1$ after integrating over the circle fibers. Instead, something highly non-classical occurs, the 5d action [9] takes the form:

$$\frac{1}{2\pi R_1} \int_{M^4 \times S^1_2} Tr(F \wedge \star F) \ + \ ...$$

[7]Mathematically the abelian theory can be described by a bundle gerbe, which can roughly be interpreted as the total space of a principal $BU(1)$ 2-bundle. The 2-form B is then interpreted as a connection on this bundle gerbe.

[8]Here there is some subtly to which group gets reconstructed from the ADE choice of Lie algebra. This is believed to be related to the idea of relative field theory [10].

[9]The theory produced upon compactify the (2,0)-model is conjectured to be $\mathcal{N} = 5$ supersymmetric Yang-Mills.

The factor of $1/R_1$ appears due to the conformal invariance of the six-dimensional theory after dimensional reduction. It can be interpreted as a result of passing from a purely quantum theory without classical description to a theory that does have a description via a classical action. As we have a classical action in 5d we can now integrate over the circle fibers classically. This produces the following 4d action:

$$\frac{R_2}{R_1}\int_{M^4} Tr(F \wedge \star F) \ + \ ... \tag{4.7}$$

we can interpret the parameter τ as iR_2/R_1. The action of S-duality, mapping $\tau \rightarrow -1/\tau$ (we are restricted to the simply laced case), now becomes a reversal of the compactification order equating $iR_2/R_1 \rightarrow -iR_1/R_2$. This reversal together with the $\tau \rightarrow \tau + 1$ symmetry is equivalent to the natural action of $SL(2,\mathbb{Z})$ on T^2, thus the toroidal symmetry in six dimensions becomes, after compactification, an intrinsic symmetry of the four-dimensional theory.[10] The corresponding passage from the gauge group G to ${}^LG$ under S-duality can be attributed to a group of ADE type being equivalent to its Langlands dual group.[11]

One can extend this construction to obtain non-simply laced G in 5d from a ADE choice in six dimensions. This is done by performing a twisted compactification on S^1_1 [12] on the circle. Upon reducing classically on the second circle one then produces a $\mathcal{N} = 4$ theory in 4d with gauge group G. Alternatively one can first compactify on S^1_2 and then perform a twisted compactification on S^1_1. This yields a S-dual $\mathcal{N} = 4$ theory in 4d with gauge group ${}^LG$. For details see [5].

We will now examine S-duality more closely as it acts in a certain topologically protected sector of 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills, which is especially important

[10] To be precise $iR_2/R_1 = 1/e^2$. In order to describe τ one must consider general elliptic curves. The physical τ parameter then corresponds to the τ parameter the elliptic curve. As the moduli space of elliptic curves is naively $\mathbb{H}/SL(2,\mathbb{Z})$ two coupling constants related by an $SL(2,\mathbb{Z})$ action are equivalent.

[11] If a Lie algebra is of ADE type then its root data is equivalent to its dual root data thus making the corresponding group G equivalent to ${}^LG$.

[12] This involves compactification on a circle together with an automorphism on an ADE choice. The non-simply laced groups are then produced as a simply laced group together with an automorphism coming from six dimensions.

for Geometric Langlands.

4.4 Topologically Twisted $\mathcal{N} = 4$ Supersymmetric Yang-Mills

The idea of topological twisting has been widely discussed, for the original reference see [51], thus we will only describe it schematically. In essence, the topological twisting process begins by singling out a supercharge Q and passing to its cohomology. However, Q transforms as a spinor under the Lorentz group (or its double cover) so we need to redefine this action to allow Q to transform as a scalar (meaning trivially under the Lorentz group). We can then view Q as a BRST operator. This involves defining a twisting homomorphism which is a certain embedding of the Lorentz group into the R-symmetry group.[13] The theory then only includes observables that lie in the Q cohomology. If the energy-momentum tensor is Q exact the theory transforms trivially under translations and is considered topological.

In the case $\mathcal{N} = 4$ supersymmetric Yang-Mills on $\mathbb{R}^4$ we want to define a homomorphism $\rho : Spin(4) \to Spin(6)$. The homomorphism replaces $Spin(4)$ by an isomorphic subgroup $Spin(4)' \subset Spin(4) \times Spin(6)$. In this case, there are three distinct choices of twisting homomorphism.[14] The one considered by Kapustin and Witten in relation to Geometric Langlands uses the exceptional isomorphisms of Lie algebras $Spin(4) \cong SU(2)_L \times SU(2)_R$ and $Spin(6) \cong SU(4)$ by block diagonally embedding the two copies of $SU(2)$ into $SU(4)$:

$$\begin{pmatrix} SU(2)_L & 0 \\ 0 & SU(2)_R \end{pmatrix} \in SU(4) \tag{4.8}$$

This results in exactly one left-handed and one right-handed supersymmetry becoming

[13] This can be extended to a general smooth four manifold while still retaining the scalar Q.

[14] The twist related to Geometric Langlands. The Vafa-Witten twist and a twist similar to the Donaldson twist.

scalar under the new action of $Spin(4)'$. The resulting BRST operator is of the form:

$$Q_{KW} = uQ_l + vQ_r \tag{4.9}$$

where $u, v \in \mathbb{C}$ are relative scaling parameters. Equivalently we can define $Q_{KW} = Q_l + tQ_r$ where $t \in \mathbb{CP}^1$. Kapustin and Witten show that the action of this topologically twisted theory takes the form:

$$S = \{Q_{KW}, V\} + \frac{i\theta}{8\pi^2}\int_{M^4} Tr(F \wedge F) - \frac{1}{e^2}\left(\frac{t - t^{-1}}{t + t^{-1}}\right)\int_{M^4} Tr(F \wedge F) \tag{4.10}$$

where $\{Q_{KW}, V\}$ is a Q-exact term. Thus we derive a Lagrangian which is not dependent on the metric of the space time manifold. One can further combine the second and third term by defining the Kapustin-Witten canonical parameter taking values in $\mathbb{CP}^1$:

$$\Psi = \frac{i\theta}{8\pi^2} - \frac{1}{e^2}\frac{t - t^{-1}}{t + t^{-1}} = \frac{\tau + \bar{\tau}}{2} + \frac{\tau - \bar{\tau}}{2}\left(\frac{t - t^{-1}}{t + t^{-1}}\right) \tag{4.11}$$

upon which the theory is solely dependent. We thus have a $\mathbb{CP}^1$ family of topological field theories.

Let us now examine how S-duality acts on this topological sector. We already determined how S-duality acts on the 16 supercharges of the entire theory and we also know $Q_l \in \bar{Q}^A_\alpha$ and $Q_r \in Q^A_{\dot{\alpha}}$. This implies that in order to keep $Q_{KW} = uQ_l + vQ_r$ invariant u and v must transform by multiplication with $e^{-i\phi/2}$ and $e^{i\phi/2}$. Therefore the combined parameter transforms as $t \to \pm\frac{\tau}{|\tau|}t$ under S-duality.[15] The action of $SL(2, \mathbb{Z})$ on the canonical parameter Ψ is therefore equivalent to the action on τ:

$$S : \Psi \to \frac{1}{n_{\mathfrak{g}}\Psi} \quad T : \Psi \to \Psi + 1 \tag{4.12}$$

S-duality maps the topological sector (produced by the Kapustin-Witten twist) of 4d $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G and canonical parameter Ψ to the topological sector of 4d $\mathcal{N} = 4$ super Yang-Mills theory with gauge group ${}^LG$ and canonical parameter $\Psi^\vee = \frac{1}{n_{\mathfrak{g}}\Psi}$.

[15]The $\pm$ symmetry comes from the orientiablity of the space-time manifold.

We can now ask the question if the Kapustin-Witten twisted theory comes from some topological sector of the 6d (2,0) model? The 6d theory does not admit a topological twist on a general six manifold M^6 as the Lorentz group $Spin(6)$ is too large to be embedded into the R-symmetry group $Sp(2) \cong Spin(5)$. If however, we place the theory on a product manifold of the form $C \times M^4$, where C is some compact two manifold, we can topologically twist along M^4. The Lorentz group is then of the form $Spin(4) \times Spin(2)$ and one can define an embedding of $Spin(4)$ subgroup into the R-symmetry group $Spin(5)$. Using the exceptional isomorphisms of Lie groups one can diagonally embed:

$$Spin(4) \cong Sp(1) \times Sp(1) \to Sp(2) \cong Spin(5) \tag{4.13}$$

with $Sp(1) \cong SU(2)$. This embedding is equivalent to the embedding produced via the Kapustin-Witten twist homomorphism [43] as an embedding into the short roots of $Spin(5)$.

We can further investigate if the square zero supercharge $Q_{KW} = uQ_l + vQ_r$ in 4d has a corresponding supercharge in the 6d theory that is square zero in the (2,0)-algebra. For this, we use that the Poincare algebras of the 6d (2,0)-model is equivalent to the Poincare algebra of 4d $\mathcal{N} = 4$ theory. A rank 2 supercharge in the (2,0)-model on $M^4 \times \Sigma$ lies in $S_+^{(6d)} \otimes R$ where $R = \mathbb{C}^4$ (together with a symplectic form ω_R) and $S_+^{(6d)}$ is the S_+ of $Spin(6)$. If we reduce to four dimensions under the Kapustin-Witten twist S_+ reduces to 4d complex spin representation of $SO(4)$, which splits as $S_+ \oplus S_-$ (here $S_\pm$ are the fundamental representations of the two copies of $SL(2,\mathbb{C})$ in $SL(4,\mathbb{C})$). The 6d R reduces to $R_+ \oplus R_-$ and is identified with $S_+ \oplus S_-$ under the twist. Now $Q_{KW} \in (S_+ \oplus S_-) \otimes (R_+ \oplus R_-)$ or more precisely $Q_l = S_+ \wedge R_+$ and $Q_r = S_- \wedge R_-$. We know from [34] that under the bracket of 4d $\mathcal{N} = 4$ theory the Kapustin Witten super charge is square zero. By now computing the supercharge under the bracket of the 6d theory we can determine if the square zero condition is still obeyed. The bracket of 6d (2,0) algebra is given by wedging the $S_+^{(6d)}$part and contracting the R part with

ω_R [30]. Applying this bracket to Q_{KW} yields:

$$\{Q_{KW}, Q_{KW}\}_{(6d)} = u^2(S_+ \wedge S_+) + v^2(S_- \wedge S_-) \tag{4.14}$$

which is none vanishing. From this, we conclude that the Kapustin-Witten supercharge does not have a direct six dimensional counterpart but instead becomes square zero through the more complicated compactification.

Remark 5. *Nonetheless the 4d $\mathcal{N} = 4$ nilpotence variety*[16] *can be realized as fibration over $\mathbb{CP}^1$ [6]. The fibers are equivalent to the non-trivial union of the nilpotence varieties corresponding to supersymmetry algebras of the 6d (2,0)-model and 4d $\mathcal{N} = 4$ Yang-Mills theory respectively.*

[16]The variety parametrizing the set of square zero elements in the supersymmetry algebra.

Chapter 5

Quantum Geometric Langlands and S-duality

Let us now describe the relationship between the Geometric Langlands correspondence, its quantum deformation, and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We will give an overview of the approach taken by Kapustin and Witten in [34]. This will lead us to a discussion of the quantum Geometric Langlands in the physical setting. We will further discuss the shortcomings and possible extensions of the approach.

5.1 The Kaputsin-Witten Approach to Geometric Langlands

A four-dimensional topological quantum field theory associates to a four manifold M^4 a number, this is the partition function. To a three-manifold, the TQFT associates a vector space, the space of physical states. If we continue to the association with two manifolds, we see that the TQFT associates a category that is often considered to be the category of boundary conditions. This is a brief motivation of why it is natural to describe Geometric Langlands, a statement about 2d surfaces, in the language of 4d TQFT's.

Kapustin and Witten specifically look at the $\mathcal{N} = 4$ topological twisted field theory with gauge group G on the product $M^4 = \Sigma \times X$ where Σ and X are Riemann surfaces and X is compact and genus $g > 1$. They proceed to show that the theory becomes a family of topological sigma models with maps $\Phi : \Sigma \to \mathcal{M}_H(X, G)$.[1] One can identify $\mathcal{M}_H(X, G)$ with the Hitchin moduli space. This is the space of solutions to the Hitchins equations (modulo gauge transformations):

$$F_A - \phi \wedge \phi = 0 \quad d_A\phi = d_A \star \phi = 0 \tag{5.1}$$

here F is the curvature associated to a connection A on a semi-stable G-bundle E on X and $\phi \in H^0(X, ad(E))$. One can further define the moduli space of general semi-stable G-connections on semi-stable principal G-bundles on X modulo gauge transformations denoted $\mathcal{M}(X, G)$. This is the semi-stable locus of Bun_G.

$\mathcal{M}_H(X, G)$ has the structure of a hyper Kähler manifold of dimension $4(g-1)dim(G)$. This means that it is a Riemann manifold with three Kähler complex structures I, J, K satisfying the quaternion relations $I^2 = J^2 = K^2 = IJK = -Id$. This produces a $\mathbb{CP}^1$ family of complex structures. The Hitchin moduli space can further be viewed as a symplectic manifold, in symplectic structures ω_I, ω_J and ω_K depending on the choice of complex structure.

A general topological sigma model requires for a choice of two complex structures. In our case this means we have a pair $(J^-, J^+) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ on $\mathcal{M}_H(X, G)$. Two possible choices for a topological sigma model are the A-model ($J^- = -J^+$) and the B-model ($J^- = J^+$). A problem in connecting this with the $\mathbb{CP}^1$ family of topological $\mathcal{N} = 4$ supersymmetric theories is that the general sigma model is dependent on the complex structure of the surface X. To remedy this, Kapustin and Witten describe a $\mathbb{CP}^1$ subfamily of the $\mathbb{CP}^1 \times \mathbb{CP}^1$ family of complex structures exactly corresponding to the Ψ family which is also independent of the complex structure of X. This subfamily is parametrized by $(w_+, w_-) \in \mathbb{CP}^1$ and relates to the Kapustin-Witten twisting parameter t in the following way:

$$w_+ = -t \quad w_- = \frac{1}{t} \tag{5.2}$$

To recover Geometric Langlands one considers $t = \pm 1$ (while also taking $\theta = 0$) and its

[1]For the topologically twisted theory the volume of X is irrelevant.

S-dual $t = \pm i$. The canonical parameter then becomes $\Psi = 0$ and the S-dual parameter is $\Psi^{\vee} = \infty$. Kapustin and Witten show that the topological sigma model at $t = 1$ is an A-model with a symplectic structure ω_K corresponding to the complex structure K. At $t = i$ one gets a B-model with complex structure J. The respective target manifolds become $\mathcal{M}_H(X, G)$ with symplectic form ω_K (we will forth on denote this $\mathcal{M}_K(X, G)$) and $\mathcal{M}_{Flat}(X, {}^LG)$ which is the moduli space of semi-stable flat ${}^LG$-connections.[2] S-duality becomes mirror symmetry[3] between $\mathcal{M}_K(X, G)$ and $\mathcal{M}_{Flat}(X, {}^LG)$.

Remark 6. *At $t = 0$ one can recover $\mathcal{M}(X, G)$ in complex structure I. This can be identified with the moduli space of semi-stable Higgs bundles $Higgs_G^{s.s.}(X)$. A Higgs bundle is holomorphic G-bundle E with a holomorphic section ϕ of $ad(E)$. $Higgs_G^{s.s.}(X)$ can be identified with the semi-stable locus of the space $T^*\mathcal{M}(X, G)$.*

Now consider D-branes (which are roughly categories of boundary conditions) associated to the sigma models. One can argue that D-branes of the A-model (called A-branes) in complex structure K are given by a category that strictly contains the Fukaya category on the target.[4] The category of D-branes of the B-model (called B-branes) in complex structure J is then conjecturally equivalent to the derived category of quasi-coherent sheaves on $\mathcal{M}_{Flat}(X, {}^LG)$. Both of these categories are independent of the complex structure of X.

Kapustin and Witten then proceed to show that every A-brane on $\mathcal{M}_K(X, G)$ corresponds to a D-module on $\mathcal{M}(X, G)$. They consider a special brane called canonical isotropic branes (c.c. branes) $\mathcal{B}_{c.c.}$ with its defining condition making it not only an A-branes in the complex structure K but also in the complex structure I.[5] One can then

[2] This is not equivalent to $LocSys_{{}^LG}(X)$ which can be thought of as the moduli space of all connections.

[3] Mirror symmetry is an equivalence of A and B-model generated by two mirror Calabi-Yau manifolds as targets of the respective sigma models

[4] The Fukaya category is a A_∞-category, where the objects are Lagrangian submanifolds (endowed with a flat connection) of $\mathcal{M}_K(X, G)$.

[5] It can be equally viewed as brane on $\mathcal{M}_H(X, G)$, viewed as a real symplectic manifold (with symplectic form ω_K), or as a brane on $\mathcal{M}_H(X, G)$ in complex structure I viewed as a complex symplectic manifold.

describe the endomorphism algebra $Hom(\mathcal{B}_{c.c.}, \mathcal{B}_{c.c.})$[6] as a sheaf of non-commutative algebras (a sheaf of differential operators). Maps between any other A-brane $\mathcal{B}$ and the coisiotropic brane $Hom(\mathcal{B}_{c.c}, \mathcal{B})$ then become modules over the sheaf non-commutative algebra meaning D-modules. More precisely Kapustin and Witten identify with every A-brane, a D-module twisted by the square root of the canonical line bundle $K^{1/2}$ on $\mathcal{M}(X, G)$ (this will become important for quantum geometric Langlands).[7] It was further determined in [16] that D-module corresponding to the c.c. brane (given by the sheaf of differential operators on $\mathcal{M}(X, G)$) gets mapped, under S-duality, to the structure sheaf on the variety of opers (considered as a special B-brane on $\mathcal{M}(X, {}^L G)$).

We can sum up these relationships in the following diagram:

$$\begin{tikzcd}
 & \mathcal{N} = 4\ top.YM\ (\Psi, G, M^4) \arrow[d, "M^4 = X \times \Sigma"'] & \\
 & \mathcal{M}_H(X, G) \arrow[dl, "\Psi = 0"'] \arrow[dr, "\Psi = \infty"] & \\
\mathcal{M}_K(X, G) \arrow[rr, leftrightarrow, "mirror\ symmetry"'] \arrow[d] & & \mathcal{M}_{flat}(X, G) \arrow[d] \\
A - branes \arrow[d, leftrightarrow] & & B - branes \simeq QCoh(\mathcal{M}_{Flat}(X, G)) \\
DMod(Bun_G(X)) \arrow[urr, leftrightarrow, "GL?"] & &
\end{tikzcd}$$

Here the arrow denoted $GL?$ represents the approximation to Geometric Langlands via the Kapustin-Witten construction.

5.1.1 Wilson and t'Hooft Operators

An important property of the Geometric Langlands is that there are distinguished D-modules, the Hecke eigensheaves, that are supposed to give a "basis" for $\mathbf{DMod}(Bun_G(X))$. Let briefly describe how the corresponding Hecke (and the Wilson) functors are interpreted in the physical context.

[6] These Hom's are called strings between $\mathcal{B}_{c.c.}$ and $\mathcal{B}_{c.c.}$.

[7] This identification works by identifying c.c. branes on $Higgs_G^{s.s.}(X) = T^*\mathcal{M}(X, G)$.

Kapustin and Witten consider Wilson operators at $t = \pm i$ in the twisted $\mathcal{N} = 4$ theory. These take the form:

$$W_R(\gamma) = Tr_{^LR}(Hol(A \pm i\phi), \gamma) = Tr_{^LR}(P\,exp\left(\int_\gamma A \pm i\phi\right)) \tag{5.3}$$

where γ is some path on $M^4 = \Sigma \times X$, ^LR is some finite representation of ^LG and $A \pm \phi$ is a complexification of the connection. Inserting a Wilson operator into the partition function produces a particle along γ. S-duality maps the operators to t'Hooft operators. These are more difficult to describe as they can not be written as a functional of fields. Heuristically inserting a t'Hooft operator into the partition changes the fields (e.g. by adding singularity along the path γ).

The operators can be defined for the sigma model and act on boundary conditions $\mathcal{B}$ (meaning D-branes). By inserting them at a point $p \in X$ (with γ now equal to $\gamma_\Sigma \times p$ where γ_Σ is a curve on Σ) one can change the boundary condition of the sigma model. Thereby the Wilson and t'Hooft operators become endofunctors on the category of B-branes and A-branes respectively. One can further extend the analogy by describing the eigenobjects of these functors.

A general brane $\mathcal{B}$ is called eigenbrane for a line operator L inserted at $p \in X$ if it satisfies:

$$L_p\mathcal{B} = V_p \otimes \mathcal{B} \tag{5.4}$$

where V_p is a vector space. The vector spaces construct a flat line bundle over X where V_p is fiber over each point $p \in X$. A brane is called simultaneous eigenbrane for two line operators L_p and $\tilde{L}_p$ if $[L_p, \tilde{L}_p] = 0$. General branes that are simultaneous eigenbrane for all Wilson or t'Hooft operators are called electric or magnetic eigenbranes respectively. At $t = i$ one finds electric eigenbranes (these are certain B-branes) for the Wilson operators and S-duality then maps these to magnetic eigenbranes (certain A-branes) for the t'Hooft operators at $t = 1$. The electric eigenbranes identify with skyscraper sheaves while the magnetic eigenbranes identify with Hecke eigensheaves.

The essential difference is that the t'Hooft operators act on the category of A-branes on the semistable locus of $T^*Bun_G(X)$ while the Hecke functors act on the category of D-modules on the full stack $Bun_G(X)$.

5.2 Quantum Geometric Langlands from the Kapustin-Witten Construction

It is apparent that the parameter Ψ should have some connection to the twisting parameter k in Chapter 2. In [34] it is argued that taking θ to be nonzero and keeping $t = 1$ produces a quantum deformation for real Ψ. One defines:

$$\Psi = Re(\tau) = \frac{\theta}{2\pi} \tag{5.5}$$

This produces a family of A-models with target $\mathcal{M}(X, G)$ in complex structure K with an additional B-field after compactification on X. One can then construct D-modules via corresponding c.c. branes $\mathcal{B}^{\Psi}_{c.c.}$ and generic A-branes $\mathcal{B}^{\Psi}$ defined for all Ψ.

The D-modules associated to $Hom(\mathcal{B}^{\Psi}_{c.c.}, \mathcal{B}^{\Psi})$ are then D-modules twisted by $K^{1/2} \otimes \mathcal{L}^{\otimes\Psi}$. Here $K^{1/2}$ is the square root of the canonical bundle on $\mathcal{M}(X, G)$ and $\mathcal{L}$ is the determinant line bundle on $\mathcal{M}(X, G)$. S-duality then maps these twisted D-modules to D-modules twisted by $K^{1/2} \otimes \mathcal{L}^{\otimes\Psi^\vee}$ on $\mathcal{M}(X, {}^LG)$. It turns out that $K^{1/2}$ is isomorphic to $\mathcal{L}^{\otimes h^\vee}$ on $\mathcal{M}(X, G)$ [29]. This implies a physical interpretation of a certain part of the quantum correspondence. Concretely we have the approximation to quantum Geometric Langlands for the semi-stable locus of Bun_G where the parameter k is real.

A different approach to the quantum deformation is given by Kapustin in [33]. By taking $t = 0$ and $\theta = 0$, Ψ becomes equivalent to $Im(\tau)$. Upon compactification on X one now has an A-model with target $\mathcal{M}(X, G)$ in complex structure I. The construction now involves a special A-brane called the distinguished coisotropic brane or d.c. brane. Analogous to the procedure involving the c.c. brane, the d.c. brane allows for associating to any A-brane a twisted D-module $\mathcal{M}(X, G)$. The D-module are now twisted by $K^{1/2} \otimes \mathcal{L}^{\otimes -i\Psi} = \mathcal{L}^{\otimes -iIm(\tau) - h^\vee}$. S-duality thereby produces an approximation to quantum Geometric Langland for the semi-stable locus of Bun_G where the parameter k is imaginary.[8]

Notice that in the two cases stated above the quantum deformation parameter is

[8]It is perhaps important to note that no line operators like the Wilson and t'Hooft operators exist in the gauge theory at t=0.

not directly t dependent. This makes the twisting solely dependent on the geometry of the elliptic curve on which the 6d (2,0)-model is compactified. This in turn means that these two special cases do not approximate the entire $\mathbb{CP}^1$ family of possible quantum deformations.

5.3 Generalizations of the Kapustin-Witten Construction

In general, the Kapustin-Witten approach can only be viewed as an approximation to the whole Geometric Langlands. The Hitchin moduli space does not fully capture the stacky (and derived) nature of Bun_G and $LocSys_{^LG}$ and the category of branes is not well understood mathematically. In that sense the sigma model with target $\mathcal{M}(X, G)$ is only an approximation of the topologically twisted 4d $\mathcal{N} = 4$ theory compactified on a Riemann surface X. Furthermore, this physical story is only an analytic equivalence (as mentioned in Remark 1). Due to both sides being solely dependent on the topology of the underlying surface X it is perhaps natural to think of the Kapustin-Witten physic picture in the Betti framework. In [4] it is argued that the category of A-branes on Bun_G can be modelled by a certain dg-category of nilpotent sheaves denoted $\mathcal{S}hv_{\mathcal{N}}(Bun_G(X))$.

There are, however, other approaches that entirely circumvent the mathematically ill-defined category of branes. These approaches are connected in that they use the entire 4d $\mathcal{N} = 4$ theory to recover more complete information about the connection between S-duality and Geometric Langlands.

One novel construction is given by Elliot and Yoo [8]. They use derived algebraic geometry and the BV-formalism to directly construct derived (shifted symplectic) stacks modeling the space of solutions to the equations of motion of the full topologically twisted $\mathcal{N} = 4$ theory (compactified on Riemann surface). These are essentially derived resolutions of the critical locus of solutions to the equations of motion. This allows one to recover algebraic stacks corresponding to the respective sides of the Geometric Langlands. The approach is also used to explain the physical origin of the singular support condition [7]. There are however some problems that appear, for example when trying to extend the ideas to the quantum deformation.

A second approach was developed by Gaiotto, Frenkel, and others ([17, 13, 15]) to connect the Kapustin-Witten story to the work of Beilinson and Drinfeld (described in Remark 4). The idea is again to use the full topologically twisted $\mathcal{N} = 4$ theory to get a more complete picture of the relationship between Geometric Langlands and physics. The construction allows for recovering mathematical aspects of the (quantum) Geometric Langlands. We will focus on this approach and how it connects to the ideas of Kapustin and Witten and how it sheds light on the mathematical uncertainties of the quantum duality. We follow [13] in the next section.

5.3.1 The Approach via Junctions

The topologically twisted 4d $\mathcal{N} = 4$ theory T_G^Ψ (with gauge group G and canonical parameter Ψ) can be described via its category of 3d topological boundary conditions. Between the 3d boundary conditions, one can consider morphisms interpreted as 2d junctions. These junctions need to be holomorphic in order to exist between arbitrary boundary conditions.

One can study a compactification functor which takes objects of the 4d theory to objects of the theory compactified on a Riemann surface X, denoted $T_G^\Psi[X]$. The 3d boundary conditions are mapped to 1d boundary conditions under this functor. It is expected that the category of these 1d boundary conditions of $T_G^\Psi[X]$ (for irrational Ψ) is equivalent to $\mathbf{DMod}^c(Bun_G(X))$ with $c = \Psi$. We can extract more information out of the category of 1d boundary partly because it is strictly larger than the category of branes of the topological sigma model (which in it self only approximates $T[X]$). We want to construct such a compactification functor explicitly.

To each 3d boundary condition in the 4d theory one can associate a dg-category[9] of line defects $\mathcal{C}(T_G^\Psi, B) = \mathcal{C}_G^\Psi(B)$, defined in [9]. The conjecture is that the compactification functor should map the category of line defects to $\mathbf{DMod}^c(Bun_G(X))$. This allows for directly associating line defects of 3d boundary conditions to twisted D-modules.

[9] This is a chiral category [46] which is a categorification of a the chiral algebra.

We want to sketch the construction of this functor.

Let us first describe how S-duality acts on 3d boundary conditions. Gaiotto and Frenkel argue that one should consider an entire groupoid of dualities acting on T_G^Ψ. This duality groupoid does not only modify Ψ by the action of $SL(2,\mathbb{Z})$ via the generators S and T, but also maps G to ${}^LG$ under the generator S. Boundary conditions in T_G^Ψ are sent to boundary conditions of $T_G^{-1/n_{\mathfrak{g}}\Psi}$ under the S action.

Three of the basic boundary conditions in T_G^Ψ are Dirichlet, Neumann, and Nahm boundary conditions. Dirichlet and Neumann, for example, correspond to these respective boundary conditions for the gauge field.

The power of the approach is that the categories of line defects associated to these types of boundary conditions have precise mathematical descriptions. For us the Dirichlet boundary condition will be most important. Part of the reason for this importance lies in that only after specifying the Dirichlet boundary condition does one really specify the gauge group of the theory. It turns out that the category of line defects associated to the boundary conditions stemming from the duality action on the Dirichlet boundary condition B_D is believed to be equivalent to $\mathbf{DMod}^c(Gr_G)$, with $c = \Psi$ if $\Psi \in \mathbb{C}/\mathbb{Q}$ (see [14] for details). Under the S generator of the duality groupoid $\mathcal{C}_G^\Psi(B_D)$ is mapped to $\mathcal{C}_G^{1/n_{\mathfrak{g}}\Psi}(B_D)$. Therefore $\mathbf{DMod}^c(Gr_G)$ should be mapped to $\mathbf{DMod}^{-1/c}(Gr_G)$.

One can define the following functor in Geometric Langlands theory:

$$\pi_* : \mathbf{DMod}^c(Gr_G) \to \mathbf{DMod}^c(Bun_G(X)) \tag{5.6}$$

which is a pushforward between the two respective categories. Note that an analogous functor to $\mathbf{DMod}^c(Bun_G(X))$ exist for the mathematical descriptions of $\mathcal{C}(B)$ for Nahm and Neumann boundary conditions. The idea is that the map π_* can be realized explicitly via a version of the compactification functor. Quantum Geometric Langlands then descends (under the compactification functor) from the duality between $\mathbf{DMod}^c(Gr_G)$ and $\mathbf{DMod}^{-1/c}(Gr_G)$ for irrational c. The analogous should apply for the action of the compactification functor on the categories of line defects associated to the other types of boundary conditions. Let us proceed by describing this construction.

Consider two categories of line defects $\mathcal{C}(B_1)$, $\mathcal{C}(B_2)$ associated to two 3d boundary conditions B_1, B_2 one can define a functor associated to a junction $J_{12} : B_1 \to B_2$:

$$F^{T^\Psi_G}_{J_{12}} : \mathcal{C}(B_1) \boxtimes \mathcal{C}(B_2) \to V(J_{12}) - mod \tag{5.7}$$

where $V(J_{12}) - mod$ is a module over a vertex algebra of local operators associated to the junction J_{12}.[10]

For a specified Dirichlet boundary condition B_D and another arbitrary boundary condition B the functor becomes:

$$F^{T^\Psi_G}_{J_D} : \mathcal{C}^c_G(B_D) \boxtimes \mathcal{C}^c_G(B) \to V^c_G(J_D : B_D \to B) - mod \tag{5.8}$$

where $V^\Psi_G(J_D : B_D \to B)$ is believed to be a vertex algebra with an affine Kac-Moody algebra symmetry at level $c' = c - mn(G)$, where $n(G)$ is a specific Kac-Moody level associated to G and $m \in \mathbb{Z}$.

This connects the 4d gauge theory perspective with the Beilinson-Drinfeld approach mentioned in Remark 4. One can further use $F^{T^\Psi_G}_{J_D}$, under some assumptions (e.g the functor is fully faithful), to construct a localization functor:

$$\Delta_{c'} : V^c(J_D : B_D \to B) - mod \to \mathbf{DMod}^{c'}(Bun_G(X)) \tag{5.9}$$

one can then define the compactification functor acting on the category of line defects associated to an arbitrary boundary condition:

$$F^{T^\Psi_G, B}_c : \mathcal{C}^c_G(B) \to \mathbf{DMod}^c(Bun_G(X)) \tag{5.10}$$

as a composition of functors $F^{T^\Psi_G, B}_c = \Delta_{c'}|_{\mathcal{C}^c_G(B)} \circ \mathcal{L}^{\otimes mn(G)}$, where $\Delta_c|_{\mathcal{C}^c_G(B)}$ is the restriction of $\Delta_{c'}$ to the subcategory $\mathcal{C}^c_G(B)$ of $V^c(J_D : B_D \to B) - mod$.

This approach lends powerful tools for understanding quantum Geometric Langlands. The vertex algebras associated to junctions allows one to study the correspondence by composing junctions and via the action of the S-duality groupoid. Using the whole topologically twisted 4d theory allows one to recover the stacky nature of Geometric Langlands. The appearance of the Beilinson-Drinfeld approach also implies that

[10] This appearance of vertex algebras already hints towards a connection to the Beilinson-Drinfeld approach.

the 4d theory contains information about the algebraic/complex structure. Part of the reason is that we considered holomorphic junctions. A caveat, however, is that the case of rational c remains mysterious. There the category of 1d boundary conditions may contain boundary conditions which do not correspond to a D-module on $Bun_G(X)$ and vice versa.

5.3.2 A Short Outlook

Besides the continuous theta angle, one can add a discrete theta angle to the 4d theory.[11] Gaiotto and Frenkel conjecture that this extends S-duality entirely. The existence of a discrete theta angle actually expands the family of possible dualities, in the sense that the discrete theta angle adds theories that can not be reached by generators (T and S) of the duality groupoid. S-duality does not necessarily map a theory with discrete theta angle and gauge group G to a theory with gauge group ${}^LG$.[12]

The topological action corresponding to the discrete theta angle is classified by the group cohomology $H^4(BG, \mathbb{C}^*)$.[13] To $H^4(BG, \mathbb{C}^*)$ one can associate a gerbe that leads to a further twisting of $\mathbf{DMod}^c(Bun_G(X))$. Heuristically this means that we not only have a twist by a complex multiple of a line bundle corresponding to the killing form κ on Bun_G, but also a twisting by a gerbe corresponding to the discrete theta angle. This extra duality information implies that the map the between the twisting by the shifted Kac-Moody levels and the gerbe twisting is not surjective.

It is also belived that the extra information given by the discrete theta angle is related to something called metaplectic Langlands ([22, 19]), which can (very) roughly be considered as an alternative formulation of quantum Geometric Langlands. In [22] it is conjectured that metaplectic Langlands is equivalent to Geometric Langlands. One can imagine that the physical insight of the extension of the duality groupoid causes this conjecture to be problematic. To explore this further one would need to develop a

[11]This comes from a second (discrete) topological term that can be added to the action of the quantum field theory. For details see [31].

[12]The example provided by Gaiotto is $G = PSL_4$, which depending on the choice of theta angle can be mapped to SL_4 (the actual Langlands dual group), $SL_4/\mathbb{Z}_2$ or PSL_4.

[13]In the case topologically twisted theory.

dictionary between the metaplectic data and the physics.

Another interesting avenue for future exploration would be the connection between the approach via derived symplectic geometry and the junction vertex algebra constructions. Here one could hope to gain deeper insight into how the physics contains algebraic information. Ultimately many of the connections between (quantum) Geometric Langlands and the physical interpretations via S-duality remain opaque and warrant further investigation.

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Selbständigkeitserklärung

Ich versichere hiermit, die vorliegende Arbeit mit dem Titel

Die Quantendeformation der Geometrischen Langlands Korrespondenz und S-Dualität

selbständig verfasst zu haben und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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München, den 1. März 2022