MPHY0030 2020/21 Report

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1 Abstract

This task aims to design an image deformation algorithm based on Gaussian RBF Spline. The whole algorithm can be divided into four steps:

1) control points selection $\Rightarrow$ fixed control points.
2) control points deformation $\Rightarrow$ moved control points.
3) Gaussian RBF Spline fit on fixed and moved control points.
4) Gaussian RBF Spline evaluate on query points, namely all points of target image, according to fixed control points.

2 Algorithm Descriptions

2.1 Gaussian Spline

$\boldsymbol{Fit\ Stage}$ **:**

The main function of polynomial part is to promise the solvability of RBF registration equation. The equation has a property of the conditional positive definiteness, so polynomials are necessary to guarantee the non-singularity of the RBF matrix. But when RBF kernel is Gaussian which are positive definite, the RBF registration equation can be calculated without any polynomial part.

λ is to approximate the distance of the source landmarks to the target landmarks, it measures the smoothness of the resulting transformation. If λ is small, we obtain a solution with good approximation behavior (in the limit of $\lambda = 0$ we have an interpolating transformation). If λ is large, we obtain a very smooth transformation with little adaption to the local structure of the distortions. If information about the expected accuracy of the landmarks is missed, the $\mathbf{W^{-1}}$ could be ignored, the $\lambda\mathbf{W^{-1}}$ item becomes $\lambda\mathbf{I}$ under this condition.

After ignoring polynomial part and landmark localisation errors matrix, the RBF registration equation becomes $(\mathbf{K} + \lambda\mathbf{I})\boldsymbol{\alpha}_k = \mathbf{q}_k$. This problem can be converted to a least square problem. There are three main methods used to solve this kind of problem: pseudo inverse, SVD and Newton's method. I chose SVD as best linear algebra method.

Reasons:

1) SVD can avoid calculation difficulty problem of pseudo inverse method.
2) There is no need to set initial point and can save iteration time compared with Newton's method.

General calculation steps for SVD solving linear equation with square matrix are shown below.

Object: Solve $\mathbf{Ax} = \mathbf{b}$, where A is a square matrix.
Definitions and Properties:

1) Orthogonal matrix $\mathbf{U} \Rightarrow \mathbf{U}^{-1} = \mathbf{U}^T$
2) Any matrix can be singular value decomposed. $\mathbf{A} = \mathbf{U}\sum\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrix, $\sum$ is diagonal matrix.
3) $\|\mathbf{x}\|_2^2 = \|\mathbf{Ux}\|_2^2$ when U is orthogonal matrix.

Solution: The linear equation problem is a least square problem: $\mathbf{Ax} = \mathbf{b} \Rightarrow \min \|\mathbf{Ax} - \mathbf{b}\|_2^2$. According to definitions and properties above,
$\|\mathbf{Ax} - \mathbf{b}\|_2^2 = \left\|\mathbf{U}\sum\mathbf{V}^T\mathbf{x} - \mathbf{b}\right\|_2^2 = \left\|\sum\mathbf{V}^T\mathbf{x} - \mathbf{U^T b}\right\|_2^2$
Then $\min \|\mathbf{Ax} - \mathbf{b}\|_2^2 \Rightarrow \min \left\|\sum\mathbf{V}^T\mathbf{x} - \mathbf{U^T b}\right\|_2^2$. The solution is $\left\|\sum\mathbf{V}^T\mathbf{x} - \mathbf{U^T b}\right\|_2^2 = 0 \Rightarrow \sum\mathbf{V}^T\mathbf{x} - \mathbf{U^T b} = 0 \Rightarrow \mathbf{V}^T\mathbf{x} = \sum^{-1}\mathbf{U}^T\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{V}\sum^{-1}\mathbf{U}^T\mathbf{b}$

Therefore, the solution of the RBF registration equation, $(\mathbf{K} + \lambda\mathbf{I})\boldsymbol{\alpha}_k = \mathbf{q}_k$ is:
$\mathbf{K} + \lambda\mathbf{I} = \mathbf{U}_{RBF}\sum_{RBF}\mathbf{V}_{RBF}{}^T$,
$\boldsymbol{\alpha}_k = \mathbf{V}_{RBF}\sum_{RBF}{}^{-1}\mathbf{U}_{RBF}{}^T\mathbf{q}_k$.
Because the solution is obtained by linear multiplication, thus the equation can be solved one batch:
$\boldsymbol{\alpha} = \mathbf{V}_{RBF}\sum_{RBF}{}^{-1}\mathbf{U}_{RBF}{}^T\mathbf{q}$.

***Evaluate Stage* :**

1) In fit stage, K_{fit} is between fixed control points and fixed control points, $\boldsymbol{\alpha}$ is fitted according to K_{fit} and moved control points. In evaluate stage, $K_{evaluate}$ is between query points and fixed control points same as fit stage. The fixed control points are used to do interpolation, in detail, RBF result of each query point is controlled by all fixed control points and $\boldsymbol{\alpha}$ which is calculated in fit stage.
2) In evaluate stage, we do not need λ at all because λ is used to approximate registration equation by taking landmark localization errors into consideration. At evaluate stage, we do not solve transformation coefficients $\boldsymbol{\alpha}$ so λ should be ignored.

***σ Explanation* :**

Intuitively, the one dimension Gaussian radial basis function with different mean and variance is:

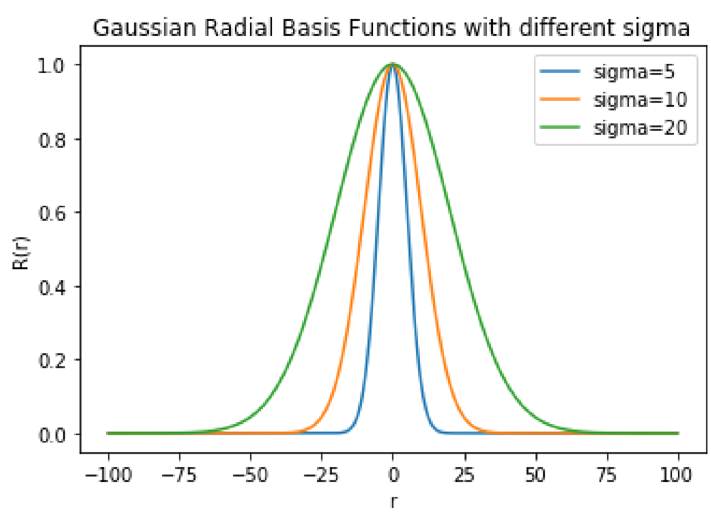

Figure 1: 1-D Gaussian Kernels

When calculating K, there are N Gaussian kernels applied to the image with different centers, where N is the number of control points. In this interpolation problem, the centers of these 1-D Gaussian radial basis function are on relevant control points and the direction of r axis is decided by direction along position of control point and query point. Therefore, σ controls the influence range of control points. In detail, for a large σ, the Gaussian kernel is smooth with wide influence range, however, a too large σ could cause small difference of RBF result between different query points, which makes model fail to fit $\boldsymbol{\alpha}$. In contrast, a small σ leads to narrow influence range and large difference of RBF result between different query points while a too small σ will cause high information loss rate. This will make interpolation result of each query point highly depend on one control point or even some query points cannot be controlled by any control points. Therefore, σ should be dependent on the interval between control points and should be carefully chosen.

$\boldsymbol{Vectorisation\ Strategies}$ **:**

For each query point, the distances from it to every control point need to be calculated. Based on this, a vectorisation strategy could be designed when calculating K matrix. Assume N is the number of control points and M is the number of query points. Repeat query points matrix $(M \times 3)$ N times to make its size become $M \times N \times 3 \Rightarrow \mathbf{Q}$. And repeat control points matrix $(N \times 3)$ M times to make its size also become $M \times N \times 3 \Rightarrow \mathbf{C}$. Make $\mathbf{r}_{i,j} = \|\mathbf{Q}_{i,j} - \mathbf{C}_{i,j}\|$, then $\mathbf{K} = R(\mathbf{r})$. Intuitively,

$$\begin{Bmatrix} \boldsymbol{R}_{1,1} & \cdots & \boldsymbol{R}_{1,N} \\ \vdots & \vdots & \vdots \\ \boldsymbol{R}_{M,1} & \cdots & \boldsymbol{R}_{M,N} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_1 \\ \vdots & \vdots & \vdots \\ \boldsymbol{q}_M & \cdots & \boldsymbol{q}_M \end{Bmatrix} - \begin{Bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_N \\ \vdots & \vdots & \vdots \\ \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_N \end{Bmatrix}$$

$$\begin{Bmatrix} \mathrm{r}_{1,1} & \cdots & \mathrm{r}_{1,N} \\ \vdots & \vdots & \vdots \\ \mathrm{r}_{M,1} & \cdots & \mathrm{r}_{M,N} \end{Bmatrix} = \begin{Bmatrix} \|\boldsymbol{R}_{1,1}\| & \cdots & \|\boldsymbol{R}_{1,N}\| \\ \vdots & \vdots & \vdots \\ \|\boldsymbol{R}_{M,1}\| & \cdots & \|\boldsymbol{R}_{M,N}\| \end{Bmatrix}$$

$$\begin{Bmatrix} \mathrm{K}_{1,1} & \cdots & \mathrm{K}_{1,N} \\ \vdots & \vdots & \vdots \\ \mathrm{K}_{M,1} & \cdots & \mathrm{K}_{M,N} \end{Bmatrix} = \begin{Bmatrix} \mathrm{R}(\mathrm{r}_{1,1}) & \cdots & \mathrm{R}(\mathrm{r}_{1,N}) \\ \vdots & \vdots & \vdots \\ \mathrm{R}(\mathrm{r}_{M,1}) & \cdots & \mathrm{R}(\mathrm{r}_{M,N}) \end{Bmatrix}$$

Where R(r) is radial basis function.

2.2 Free-form Deformation

$\boldsymbol{Control\ Points\ Selection}$ **:**

The six vertexes of the 3D image cuboid are selected and remaining points are uniformly sampled according to the number of control points in x,y, and z directions.

$\boldsymbol{Random\ Transform}$ **:**

$\mathbf{D}$ is a 3×3 matrix whose elements correspond to a normalisation distribution, in detail, $\mathbf{D}_{i,j} \sim \mathcal{N}(0, randomness/5)$. This matrix implements the ability to control the strength of the randomness. But to keep biophysically plausible transformation, a constraint needs to be added on $\mathbf{D}$. I chose truncated normalisation distribution, which means the elements of truncated $\mathbf{D}$, $\mathbf{TD}$, are truncated within $(-randomness/5, randomness/5)$.
Then the random transformation matrix $\mathbf{T} = \mathbf{I} + \mathbf{TD}$. When randomness is zero, $\mathbf{T} = \mathbf{I}$.
Fixed control points + random transformation $\Rightarrow$ moved control points. In theory, the fitted α can simulate the random transformation happened on control points, so interpolated query points are query points after doing random transformation. Therefore, the interpolated voxel coordinates, driven by the moved control points, represent biophysically plausible deformation.

$\boldsymbol{Detailed\ Steps}$ **:**

1) Select control points from image.
2) Generate random transformation.
3) Deform selected control points.
4) Fit α between fixed control points and moved control points by using RBF Spline.
5) Evaluate query points which are all points of image by using fit result and fixed control points.
6) Interpolate deformed query points into a image to get warped image.

3 Parameters Tuning

The table below contain corresponding randomness and warped results. $Figure_2$ and $Figure_3$ are warped images when the number of control points in each direction are 6, 6, 4 respectively. All images referred in this section are contained in Appendix.

Randomness	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Warped Image	Fig_2	Fig_2	Fig_2	Fig_2	Fig_2	Fig_3	Fig_3	Fig_3	Fig_3	Fig_3

Table 1: Corresponding Randomness and Warped Images

Then we fixed the random transform and tried some combinations of number of control points and Gaussian kernel parameter. All information are in the table below.

Number (x y z) / Sigma	3 3 3	4 4 4	5 5 5
40	Fig_4	Fig_5	Fig_6
35	Fig_4	Fig_5	Fig_6
30	Fig_4	Fig_5	Fig_6
25	Fig_4	Fig_5	Fig_6

Table 2: Corresponding Randomness and Warped Images

Summary of the effect of the three parameters:

1) Randomness controls the degree of image deformation.
2) Number of control points affects the interpolating precision.
3) Gaussian kernel parameter σ decides the effect range of control points(detailed description in 2.1). A too large σ will cause large area of black and a too small σ will generate spotted images.
4) The number of control points and the value of σ should be dependent on each other. A large amount of control points match a small σ and vice versa.

4 Appendix

Figure 2: Warped Images under Randomness from 0.1 to 0.5

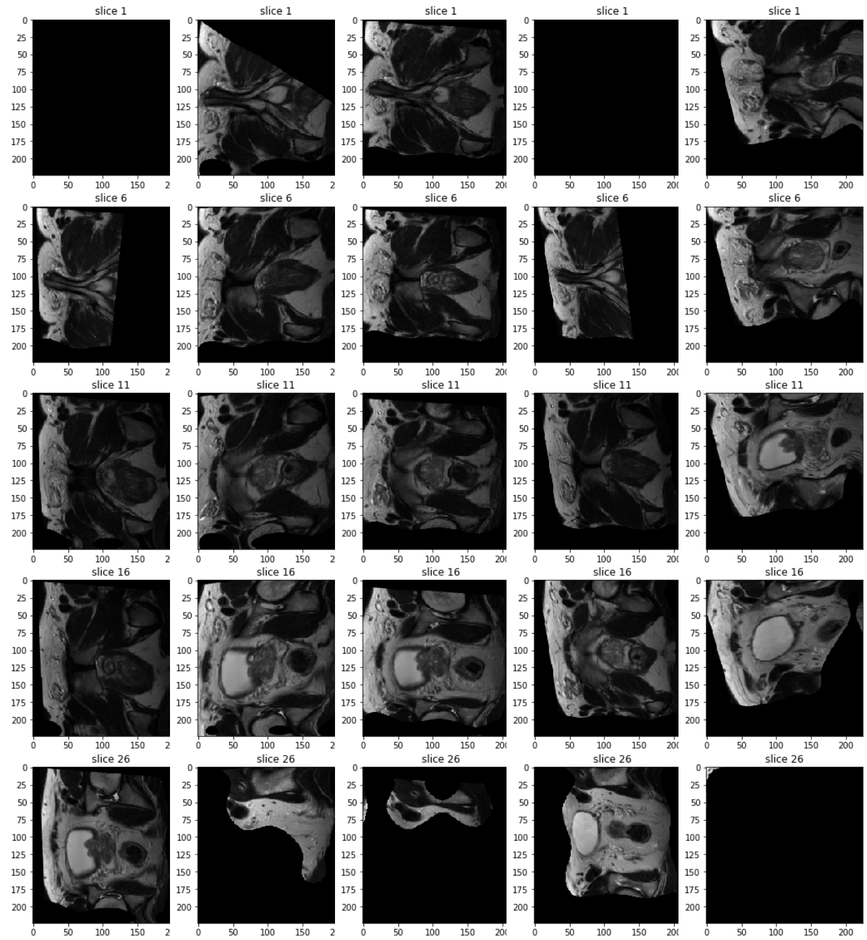

Figure 3: Warped Images under Randomness from 0.6 to 1.0

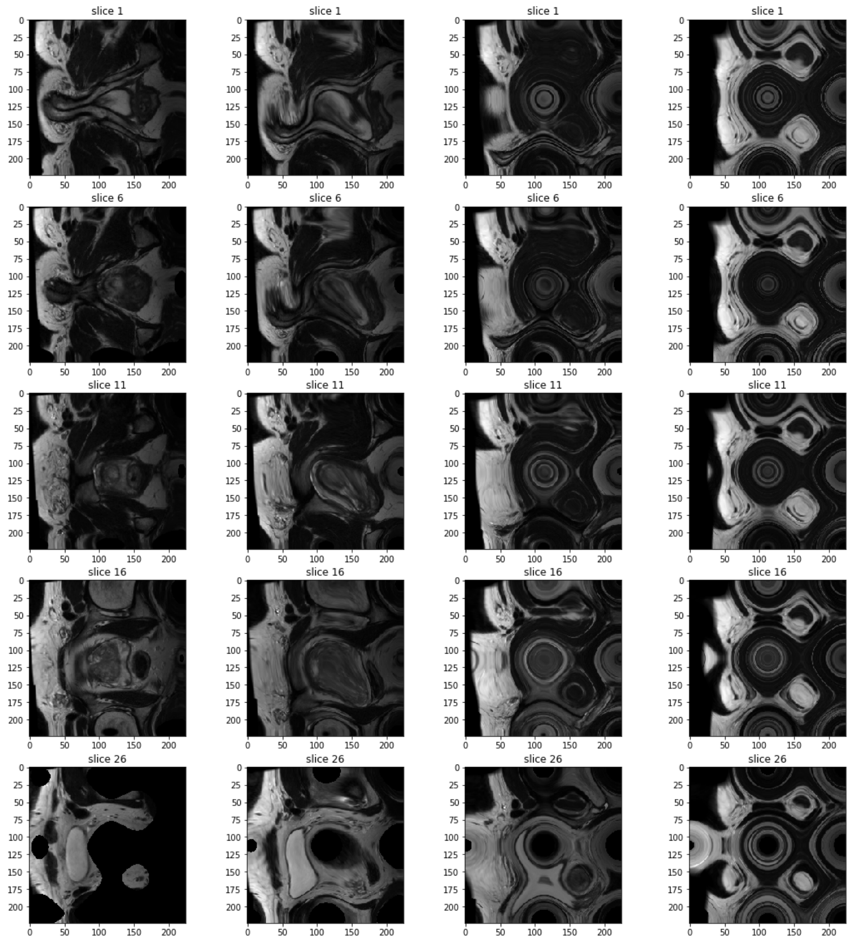

Figure 4: Warped Images with different sigma under NUM=3,3,3

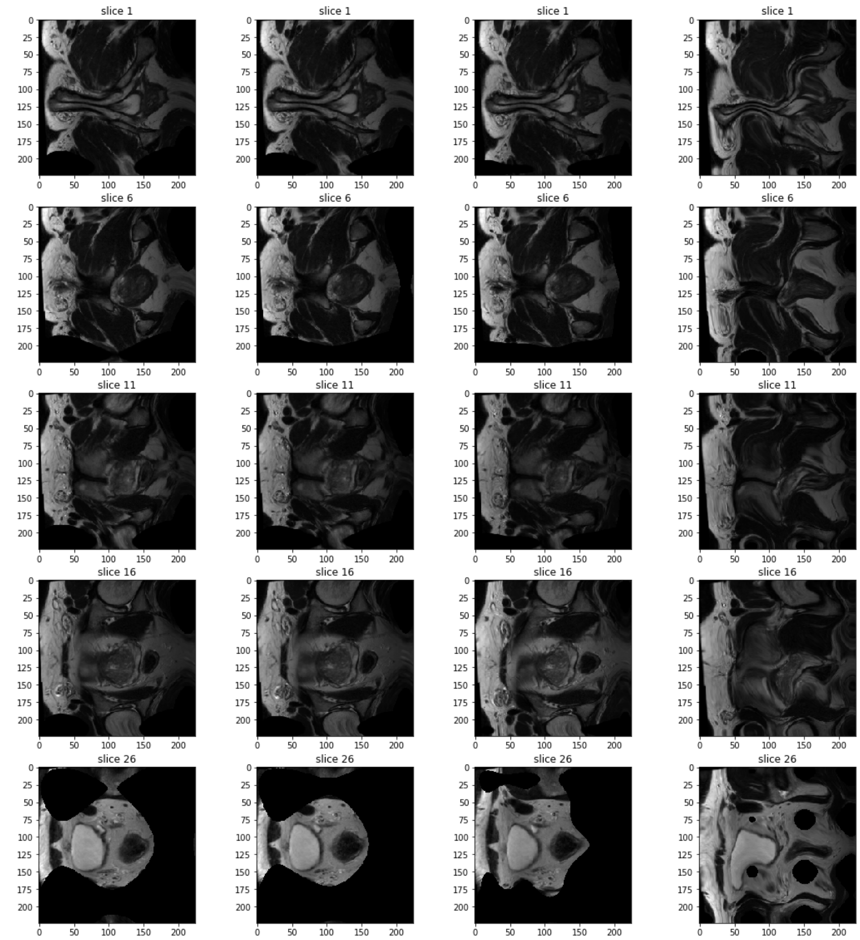

Figure 5: Warped Images with different sigma under NUM=4,4,4

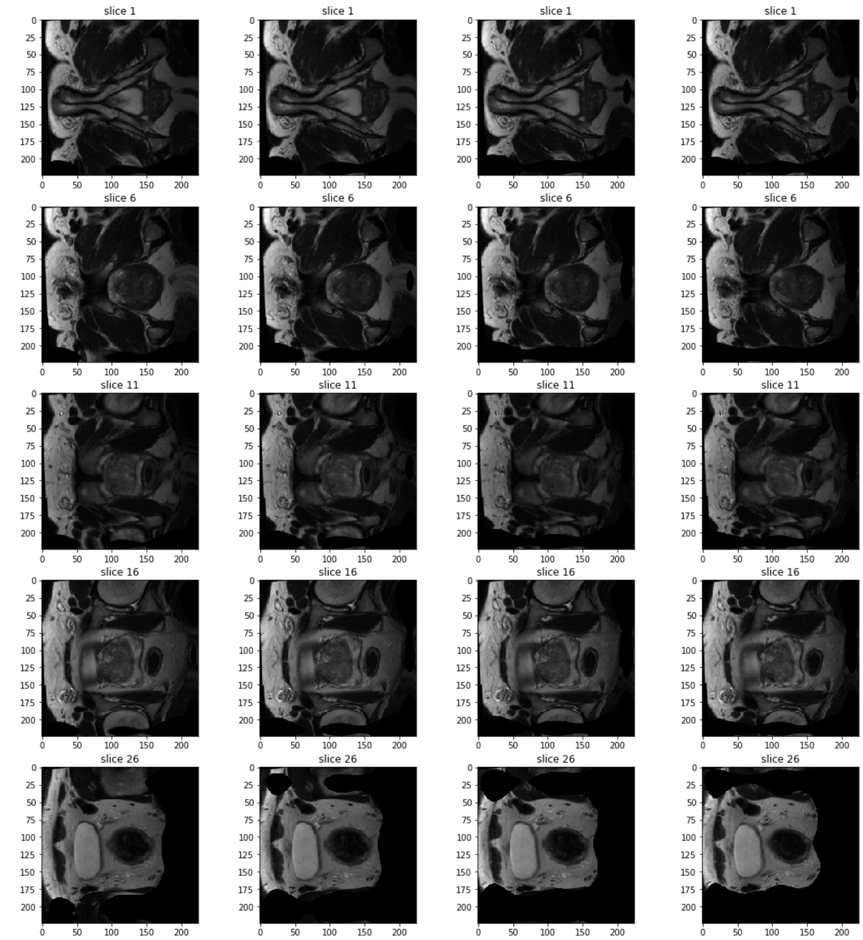

Figure 6: Warped Images with different sigma under NUM=5,5,5