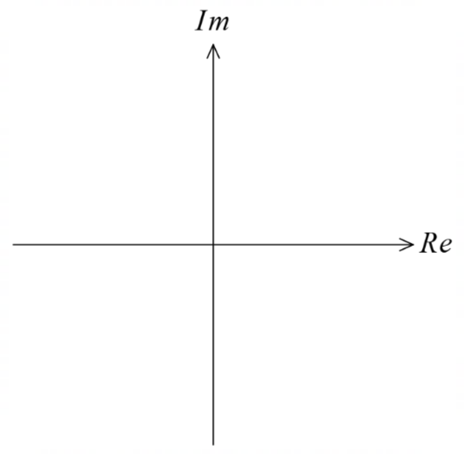
Question 10 **(9 marks)**

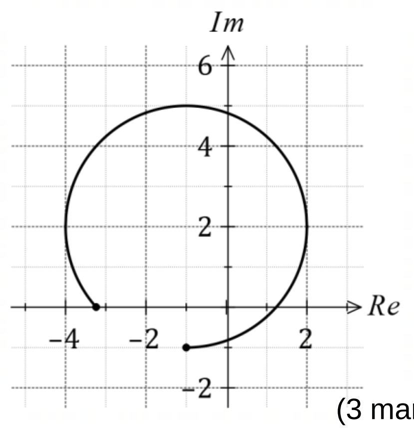
(a) Draw the subset of the complex plane determined by $|z+3| > |z-3i|$ on the axes below.

(3 marks)



(b) The circular arc in the diagram represents the locus of a complex number z.

Without using $Re(z)$ or $Im(z)$, write equations or inequalities in terms of z for the indicated locus.



(3 marks)

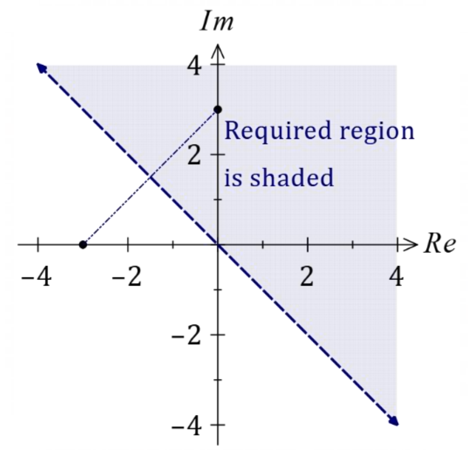
(c) Describe the subset of the complex plane determined by $|z-3| + |z+3i| = 3\sqrt{2}$.

(3 marks)

Question 10 **(9 marks)**

(a) Draw the subset of the complex plane determined by $|z+3| > |z-3i|$ on the axes below.

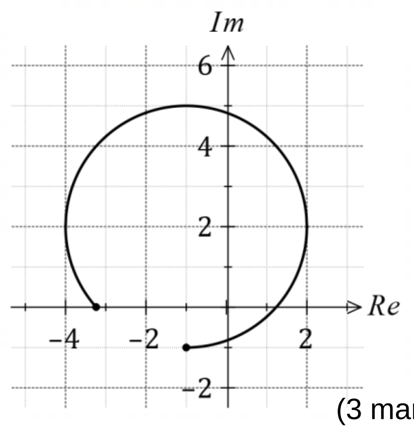
(3 marks)



Solution
See diagram
Specific behaviours
✓ indicates points in plane ✓ draws perp' bisector with dotted line ✓ shades correct region

(b) The circular arc in the diagram represents the locus of a complex number z.

Without using $Re(z)$ or $Im(z)$, write equations or inequalities in terms of z for the indicated locus.



(3 marks)

Solution
Circle has centre $-1+2i$ and radius 3. $$\|z-(-1+2i)\| = 3 \cap \left(-\frac{3\pi}{4} \le \arg z \le \pi\right)$$
Specific behaviours
✓ indicates correct centre and radius ✓ writes inequality for circle ✓ writes restriction for $\arg z$

(c) Describe the subset of the complex plane determined by $|z-3| + |z+3i| = 3\sqrt{2}$.

(3 marks)

Solution
Distance between 3 and $-3i$ in complex plane is $3\sqrt{2}$. Hence z must lie on the line segment between 3 and $-3i$ inclusive in the complex plane. Alternatively, when $z = x + iy$ then locus is $y = x - 3, 0 \le x \le 3$.
Specific behaviours
✓ indicates distance between points ✓ indicates subset is a line segment ✓ correct description that includes endpoints

Question 16 **(8 marks)**

(a) Determine all solutions to the equation $z^3 - 8i = 0$ in exact polar form. (3 marks)

(b) Consider the ninth roots of unity expressed in polar form $r \operatorname{cis} \theta$.

(i) Determine the roots for which $0 < \theta < \frac{\pi}{2}$. (2 marks)

(ii) Use all nine roots to show that $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0$.
(3 marks)

Question 16 **(8 marks)**

(a) Determine all solutions to the equation $z^3 - 8i = 0$ in exact polar form. (3 marks)

Solution
$z^3 = 8\,\text{cis}\left(\frac{\pi}{2}\right) \Rightarrow z = 2\,\text{cis}\left(\frac{\pi + 4n\pi}{6}\right), n = -1, 0, 1$ $z_1 = 2\,\text{cis}\left(-\frac{\pi}{2}\right), \quad z_2 = 2\,\text{cis}\left(\frac{\pi}{6}\right), \quad z_3 = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ expresses $8i$ in polar form   ✓ states one correct solution   ✓ states all correct solutions

(b) Consider the ninth roots of unity expressed in polar form $r\,\text{cis}\,\theta$.

(i) Determine the roots for which $0 < \theta < \frac{\pi}{2}$. (2 marks)

Solution
$z^9 = 1 = \text{cis}(2n\pi) \Rightarrow z = \text{cis}\left(\frac{2n\pi}{9}\right)$ where $n \in Z$.   Hence $z_1 = \text{cis}\left(\frac{2\pi}{9}\right), \quad z_2 = \text{cis}\left(\frac{4\pi}{9}\right).$
Specific behaviours
✓ general expression for roots   ✓ correct roots

(ii) Use all ten roots to show that $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0.$ (3 marks)

Solution
The nine roots are given by $z = \text{cis}\left(\frac{2n\pi}{9}\right), n = -4, -3, \ldots, 3, 4$, and the sum of these roots, and hence their real parts, will be 0: $\cos(0) + \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{6\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(-\frac{2\pi}{9}\right) + \cos\left(-\frac{4\pi}{9}\right) + \cos\left(-\frac{6\pi}{9}\right) + \cos\left(-\frac{8\pi}{9}\right) = 0$ But $\cos(-\theta) = \cos(\theta), \cos\left(\frac{6\pi}{9}\right) = -\frac{1}{2}$ and $\cos(0) = 1$. Hence $1 + 2\cos\left(\frac{2\pi}{9}\right) + 2\cos\left(\frac{4\pi}{9}\right) - 1 + 2\cos\left(\frac{8\pi}{9}\right) = 0$ $\therefore \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0$
Specific behaviours
✓ uses sum of real parts of all roots is 0 ✓ uses $\cos(-\theta) = \cos(\theta)$ and known values   ✓ simplifies to obtain required result

Question 18 **(9 marks)**

Let $u = \sqrt{3} + i$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{30}\right)$.

(a) Determine an exact value for

(i) $\arg(uv)$. (1 mark)

(ii) $|u + i|$. (1 mark)

(b) Let $w = \dfrac{u^4}{v^n}$, where n is a positive integer.
Determine the minimum value of n so that w is purely imaginary. (3 marks)

The modulus of complex number z is 1 and its argument is θ, where $-\pi < \theta \leq \pi$.

(c) Determine the value of θ for which

(i) $|u + z|$ is minimum. (1 mark)

(ii) $\arg(u + z)$ is maximum, where $-\pi < \arg(u + z) \leq \pi$. (3 marks)

Question 18 **(9 marks)**

Let $u = \sqrt{3} + i$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{30}\right)$.

(a) Determine an exact value for

(i) $\arg(uv)$. (1 mark)

Solution
$\arg(uv) = \arg u + \arg v = \frac{\pi}{6} + \frac{\pi}{30} = \frac{\pi}{5}$
Specific behaviours
✓ correct value

(ii) $|u + i|$. (1 mark)

Solution
$\lvert u + i\rvert = \left\lvert\sqrt{3} + 2i\right\rvert = \sqrt{3 + 4} = \sqrt{7}$
Specific behaviours
✓ correct value

(b) Let $w = \frac{u^4}{v^n}$, where n is a positive integer.
Determine the minimum value of n so that w is purely imaginary. (3 marks)

Solution
For $Re(w) = 0$ then $\arg w = \pm\frac{\pi}{2}$. $\arg w = 4\arg u - n\arg v = \frac{4\pi}{6} - \frac{n\pi}{30} = \frac{(20 - n)\pi}{30}$ $\frac{(20 - n)\pi}{30} = \frac{\pi}{2} \Rightarrow n = 5$
Specific behaviours
✓ expression for $\arg w$ ✓ indicates values of $\arg w$ for $Re(w) = 0$ ✓ correct value of n

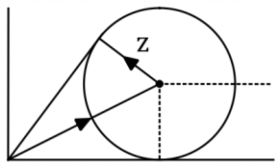
The modulus of complex number z is 1 and its argument is θ, where $-\pi < \theta \leq \pi$.

(c) Determine the value of θ for which

(i) $|u + z|$ is minimum. (1 mark)

Solution
For $|u + z|$ to be minimum, u and z must be parallel but in opposite direction. Hence $$\theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$$
Specific behaviours
✓ correct value

(ii) $\arg(u + z)$ is maximum, where $-\pi < \arg(u + z) \leq \pi$. (3 marks)

Solution	
Locus of $u + z$ is circle, centre u and radius 1. Maximum $\arg(u + z) = \frac{\pi}{3}$, and from geometric considerations this occurs when $\theta = \frac{5\pi}{6}$.	
Specific behaviours	
✓ sketch diagram (possibly seen in part(b)(i)) ✓ indicates z for maximum argument ✓ correct value	

Question 10 **(7 marks)**

(a) Solve the equation $32z^5 - i = 0$, giving exact solutions in the form $r \operatorname{cis} \theta, -\pi < \theta \leq \pi$.

(4 marks)

(b) One solution of the equation $z^n = 1$, where n is a positive integer, is $z = \operatorname{cis}\left({}^{11\pi}/_{13}\right)$. If N solutions of the equation satisfy $-{}^{\pi}/_{4} < \arg(z) < 0$, determine, with reasoning, the least value of N. (3 marks)

Question 10 **(7 marks)**

(a) Solve the equation $32z^5 - i = 0$, giving exact solutions in the form $r\operatorname{cis}\theta$, $-\pi < \theta \leq \pi$.

(4 marks)

Solution
$$z^5 = \frac{1}{32}i = \frac{1}{32}\operatorname{cis}\left(\frac{\pi}{2}\right)$$ $$z = \left(\frac{1}{32}\right)^{\frac{1}{5}}\operatorname{cis}\left(\frac{\pi + 4k\pi}{2 \times 5}\right), \qquad k \in \mathbb{Z}$$ $$z = \frac{1}{2}\operatorname{cis}\left(-\frac{7\pi}{10}\right), z = \frac{1}{2}\operatorname{cis}\left(-\frac{3\pi}{10}\right), z = \frac{1}{2}\operatorname{cis}\left(\frac{\pi}{10}\right), z = \frac{1}{2}\operatorname{cis}\left(\frac{\pi}{2}\right), z = \frac{1}{2}\operatorname{cis}\left(\frac{9\pi}{10}\right)$$
Specific behaviours
✓ writes in polar form $z^5 = \cdots$ with correct modulus ✓ determines correct argument ✓ states one correct solution ✓ states all correct solutions

(b) One solution of the equation $z^n = 1$, where n is a positive integer, is $z = \operatorname{cis}\left({}^{11\pi}/_{13}\right)$. If N solutions of the equation satisfy $-{}^{\pi}/_{4} < \arg(z) < 0$, determine, with reasoning, the least value of N.

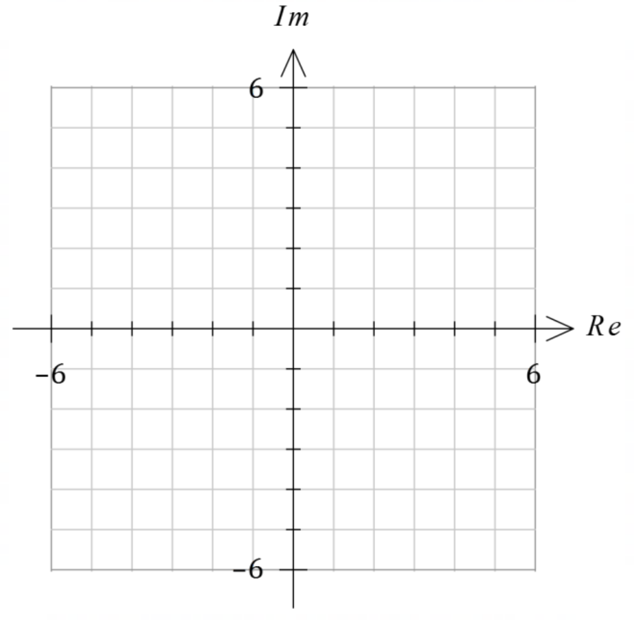
(3 marks)

Solution
Solutions to the equation must be of the form $z = \operatorname{cis}\left(\frac{2k\pi}{n}\right), k \in \mathbb{Z}$. Noting that before simplification the multiple of π will always be even, then the given solution can be written as $\operatorname{cis}\left(\frac{2\times 11\pi}{26}\right)$ and hence minimum value of $n = 26$. With this value of n and $-3 \leq k \leq -1$, then $-{}^{\pi}/_{4} < \arg(z) < 0$ and so least value of $N = 3$.
Specific behaviours
✓ indicates general solution for n^{th} roots of unity ✓ deduces value of n ✓ states correct number of solutions with required argument

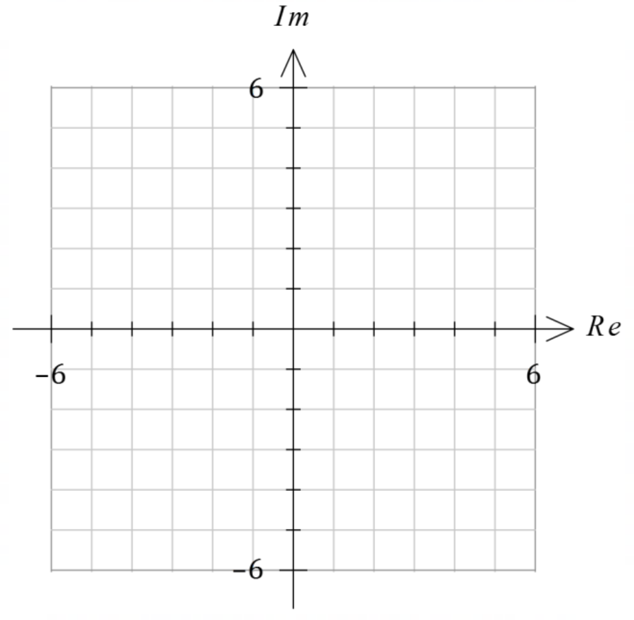
Question 14 **(8 marks)**

(a) On the Argand planes below sketch the locus of the complex number $z = x + iy$ given by

(i) $|z + 3 - 4i| = |z - 2 + i|$. (3 marks)



(ii) $|\bar{z} + 3i| \leq 3$. (3 marks)

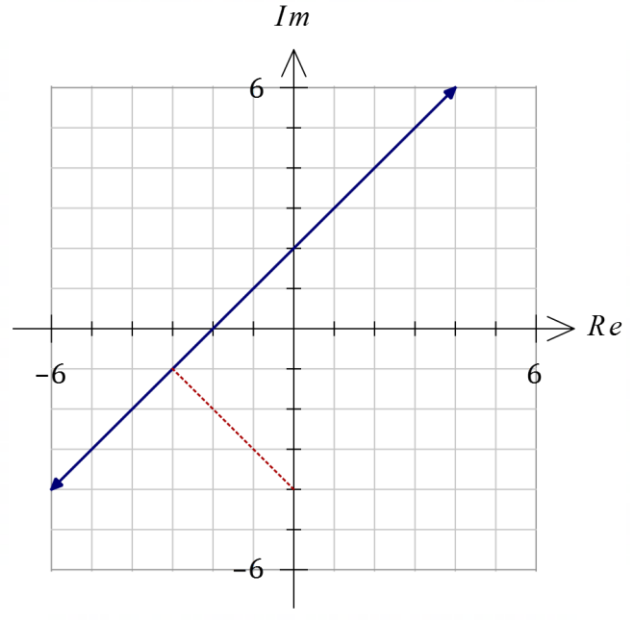


(b) For the locus $|z + 3 - 4i| = |z - 2 + i|$ in part (a), determine the minimum value for $|z + 4i|$. (2 marks)

Question 14 **(8 marks)**

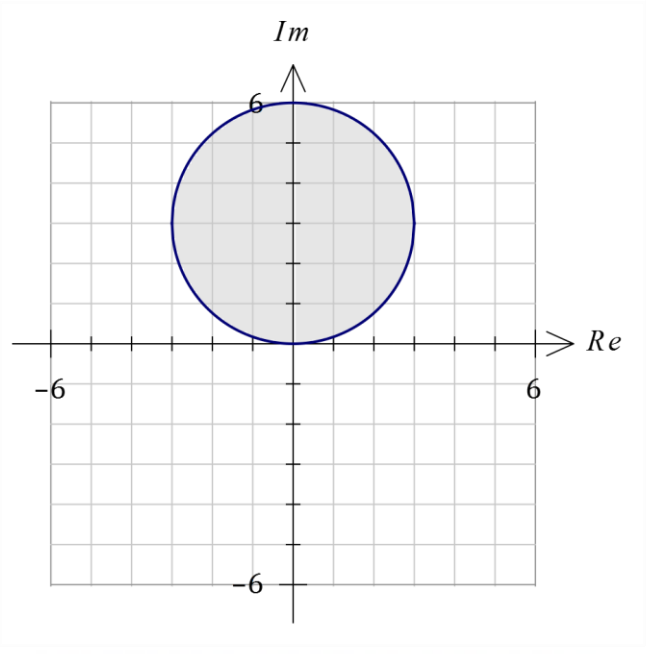
(a) On the Argand planes below sketch the locus of the complex number $z = x + iy$ given by

(i) $|z + 3 - 4i| = |z - 2 + i|$. (3 marks)



Solution
$\lvert z - (-3 + 4i)\rvert = \lvert z - (2 - i)\rvert$
See diagram.
Specific behaviours
✓ plots both points
✓ sketches perpendicular bisector
✓ correct axis intercepts

(ii) $|\bar{z} + 3i| \leq 3$. (3 marks)



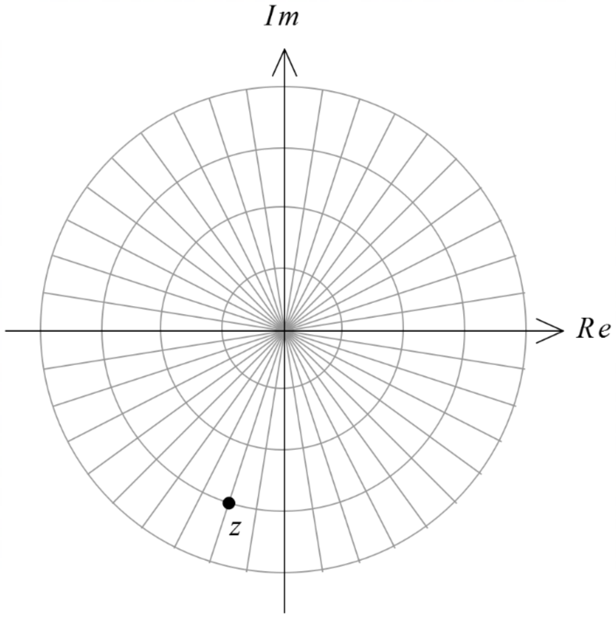
Solution
$\lvert x - (y - 3)i\rvert \leq 3$
$x^2 + (y - 3)^2 \leq 3^2$
See diagram.
Specific behaviours
✓ deals with conjugate
✓ indicates a shaded circle
✓ correct centre and radius

(b) For the locus $|z + 3 - 4i| = |z - 2 + i|$ in part (a), determine the minimum value for $|z + 4i|$. (2 marks)

Solution
Shortest distance from $z = -4i$ (on Im axis) to line. Hence minimum is $\sqrt{3^2 + 3^2} = 3\sqrt{2}$.
Specific behaviours
✓ indicates perpendicular distance to line
✓ correct minimum value

Question 17 **(5 marks)**

The complex number z is shown on the Argand diagram below and $w = \cos\left(-\frac{3\pi}{20}\right) + i\sin\left(-\frac{3\pi}{20}\right)$.

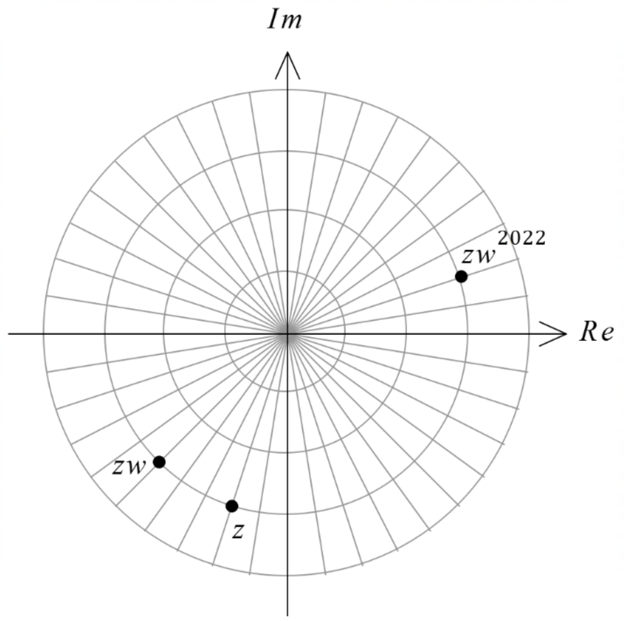


(a) Describe the geometric transformation performed by w when another complex number is multiplied by it, and plot and label zw on the Argand diagram. (2 marks)

(b) Plot and label the complex number zw^{2022} on the Argand diagram. (3 marks)

Question 17 **(5 marks)**

The complex number z is shown on the Argand diagram below and $w = \cos\left(-\frac{3\pi}{20}\right) + i\sin\left(-\frac{3\pi}{20}\right)$.



(a) Describe the geometric transformation performed by w when another complex number is multiplied by it, and plot and label zw on the Argand diagram. (2 marks)

Solution
w will rotate another complex number clockwise by $\frac{3\pi}{20}$ (27°) about the origin (or rotate $-\frac{3\pi}{20}$ about the origin).
Specific behaviours
✓ correctly describes transformation ✓ correctly locates zw on diagram

(b) Plot and label the complex number zw^{2022} on the Argand diagram. (3 marks)

Solution
$w^{2022} = \text{cis}\left(-\frac{6066\pi}{20}\right) = \text{cis}(-303.3\pi) = \text{cis}((304 - 303.3)\pi) = \text{cis}\left(\frac{14\pi}{20}\right)$ $\therefore \arg(zw^{2022}) = -\frac{12\pi}{20} + \frac{14\pi}{20} = \frac{2\pi}{20}$
Specific behaviours
✓ indicates correct argument of w^{2022} ✓ indicates correct argument of w^{2022} reduced to $-2\pi < \theta < 2\pi$ ✓ correctly locates zw^{2022} on diagram

Question 9 **(5 marks)**

The function $f(z)$ is of degree 4 and has factors $z - 4 - i$ and $z + 3i$.

(a) Determine $f(z)$ in the form $z^4 + az^3 + bz^2 + cz + d$, where $\{a, b, c, d\} \in \mathbb{R}$. (3 marks)

(b) Explain whether your answer to part (a) would change if the coefficients of the polynomial $f(z)$ were not restricted to real numbers. (2 marks)

Question 9 **(5 marks)**

The function $f(z)$ is of degree 4 and has factors $z - 4 - i$ and $z + 3i$.

(a) Determine $f(z)$ in the form $z^4 + az^3 + bz^2 + cz + d$, where $\{a, b, c, d\} \in \mathbb{R}$. (3 marks)

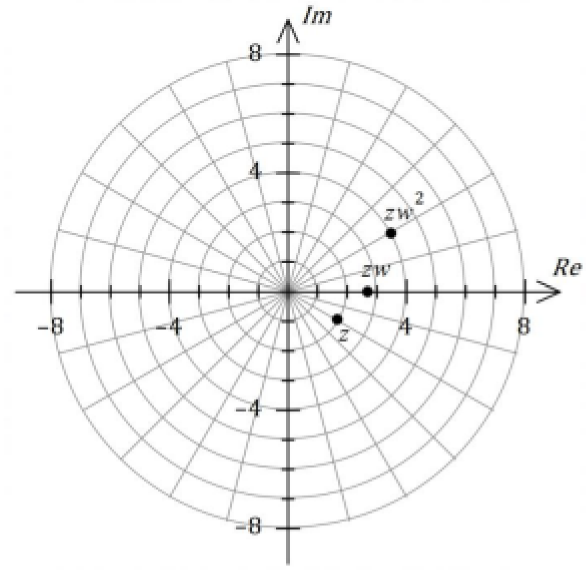
Solution
Two other factors must be $z - 4 + i$ and $z - 3i$. $(z - 4 - i)(z - 4 + i) = z^2 - 8z + 17$ $(z - 3i)(z + 3i) = z^2 + 9$ $(z^2 + 9)(z^2 - 8z + 17) = z^4 - 8z^3 + 26z^2 - 72z + 153$
Specific behaviours
✓ uses conjugate roots to obtain all factors ✓ indicates product of all factors ✓ correct $f(z)$

(b) Explain whether your answer to part (a) would change if the coefficients of the polynomial $f(z)$ were not restricted to real numbers. (2 marks)

Solution
When $\{a, b, c, d\} \in \mathbb{R}$ then there is a unique solution as the roots will be in conjugate pairs. Without this restriction there is an infinite number of choices for the other two factors and so answer would very likely be different.
Specific behaviours
✓ indicates unique solution for real coefficients ✓ indicates large number of possibilities otherwise

Question 15 **(8 marks)**

The complex numbers z, zw and zw^2 are represented on the Argand diagram below.



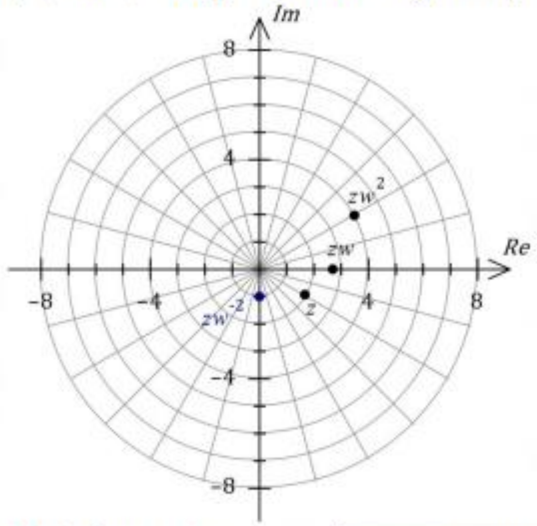
(a) Express z exactly in the form $a + bi$. (2 marks)

(b) Determine the modulus and argument of zw^5. (4 marks)

(c) Determine zw^{-2} and plot and label this point on the Argand diagram. (2 marks)

Question 15 **(8 marks)**

The complex numbers z, zw and zw^2 are represented on the Argand diagram below.



(a) Express z exactly in the form $a + bi$. (2 marks)

Solution
$z = 2\,\text{cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$
Specific behaviours
✓ polar form ✓ Cartesian form

(b) Determine the modulus and argument of zw^5. (4 marks)

Solution
$2 \times \lvert w\rvert^2 = 4 \Rightarrow \lvert w\rvert = \sqrt{2}$ $\arg(w) = \frac{\pi}{6}$ $zw^5 = 2\,\text{cis}\left(-\frac{\pi}{6}\right) \times \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)\right)^5$ $= 8\sqrt{2}\,\text{cis}\left(\frac{2\pi}{3}\right)$ Modulus: $8\sqrt{2}$, Argument: $\frac{2\pi}{3}$
Specific behaviours
✓ modulus of w ✓ argument of w (accept $\pm 2n\pi$) ✓ forms product ✓ states modulus and argument

(c) Determine zw^{-2} and plot and label this point on the Argand diagram. (2 marks)

Solution
$zw^{-2} = \text{cis}\left(-\frac{\pi}{2}\right) = -i$
Specific behaviours
✓ correct value in any form ✓ correctly plots point

Question 18 **(8 marks)**

(a) Determine, in the form $r \operatorname{cis} \theta$, the solution of the equation $z^4 + 625i = 0$ that lies in the third quadrant of the complex plane $(-\pi < \theta < -\frac{\pi}{2})$. (4 marks)

(b) Writing $5 - 12i = (a + bi)^2$, $\{a, b\} \in \mathbb{R}$, or otherwise, use an algebraic method that does not involve CAS to determine the square roots of $5 - 12i$. (4 marks)

Question 18 **(8 marks)**

(a) Determine, in the form $r \operatorname{cis} \theta$, the solution of the equation $z^4 + 625i = 0$ that lies in the third quadrant of the complex plane $(-\pi < \theta < -\frac{\pi}{2})$. (4 marks)

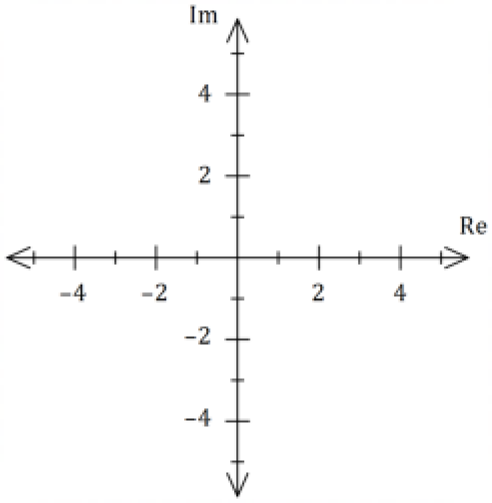
Solution
$z^4 = -625i = 625 \operatorname{cis}\left(-\frac{\pi}{2}\right)$ $z = 5 \operatorname{cis}\left(-\frac{\pi + 4n\pi}{2 \times 4}\right), n \in \mathbb{Z}$ $n = -1 \Rightarrow z = 5 \operatorname{cis}\left(-\frac{5\pi}{8}\right)$
Specific behaviours
✓ equation in polar form ✓ expression for roots ✓ indicates correct choice for n ✓ correct solution

(b) Writing $5 - 12i = (a + bi)^2$, $\{a, b\} \in \mathbb{R}$, or otherwise, use an algebraic method that does not involve CAS to determine the square roots of $5 - 12i$. (4 marks)

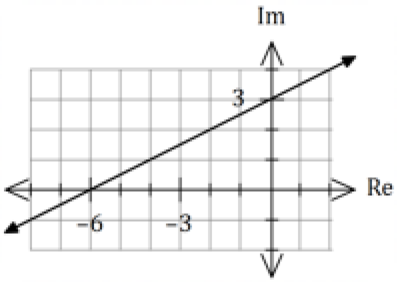
Solution
$(a + bi)^2 = a^2 - b^2 + 2abi$ Real parts: $a^2 - b^2 = 5 \quad ... (1)$ Imaginary parts: $2ab = -12 \quad ... (2)$ Also, $\lvert a + bi\rvert^2 = \lvert 5 - 12i\rvert \Rightarrow a^2 + b^2 = 13 \quad ... (3)$ From (1) and (3): $2a^2 = 18 \Rightarrow a = \pm 3$ From (2): $b = -12 \div 2(\pm 3) = \mp 2$ Hence square roots are $3 - 2i$ and $-3 + 2i$.
Specific behaviours
✓ equates real and imaginary parts ✓ equates moduli ✓ solves for one coefficient ✓ correct square roots

Question 20 **(8 marks)**

(a) Shade the region in the complex plane below that simultaneously satisfies $|z - 2i| \leq 3$ and $-\frac{\pi}{2} \leq \arg(z - 1) \leq \frac{\pi}{2}$. (4 marks)



(b) The locus of $|z + 2i| = |z + a + bi|$ in the complex plane is the straight line shown below, $\{a, b\} \in \mathbb{R}$.

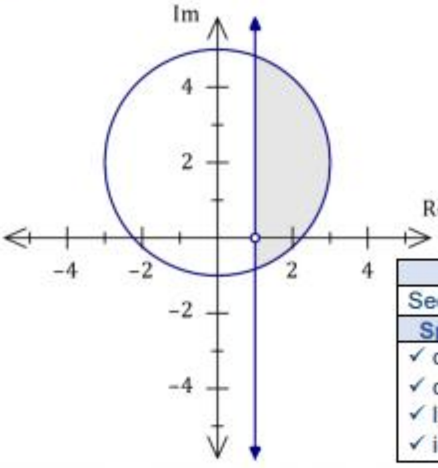


(i) State the value of constant a and the value of constant b. (2 marks)

(ii) Determine the minimum value of $|z|$ in exact form. (2 marks)

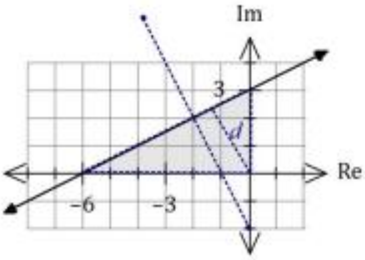
Question 20 **(8 marks)**

(a) Shade the region in the complex plane below that simultaneously satisfies $|z - 2i| \leq 3$ and $-\frac{\pi}{2} \leq \arg(z - 1) \leq \frac{\pi}{2}$. (4 marks)



Solution
See diagram
Specific behaviours
✓ circle $r = 3$
✓ centre of circle
✓ line $x = 1$
✓ identifies region

(b) The locus of $|z + 2i| = |z + a + bi|$ in the complex plane is the straight line shown below, $\{a, b\} \in \mathbb{R}$.



(i) State the value of constant a and the value of constant b. (2 marks)

Solution
$a = 4, \quad b = -6$
Specific behaviours
✓ value of a
✓ value of b

(ii) Determine the minimum value of $|z|$ in exact form. (2 marks)

Solution
Hypotenuse of triangle: $\sqrt{6^2 + 3^2} = 3\sqrt{5}$ Area of triangle: $A = \frac{1}{2}(6)(3) = \frac{1}{2}(3\sqrt{5})(d)$ Minimum $|z| = d = \frac{6\sqrt{5}}{5}$
Specific behaviours
✓ indicates length representing minimum $|z|$ ✓ exact value

Question 9 **(7 marks)**

(a) Determine the values of the real constant a and the real constant b given that $z - 4 + 2i$ is a factor of $z^3 + az + b$.

(4 marks)

(b) Clearly show that $2 + i$ is a root of the equation $z^3 - 7z^2 + 17z - 15 = 0$. (2 marks)

(c) State all three solutions of $z^3 - 7z^2 + 17z - 15 = 0$. (1 mark)

Question 9 **(7 marks)**

(a) Determine the values of the real constant a and the real constant b given that $z - 4 + 2i$ is a factor of $z^3 + az + b$.

(4 marks)

Solution
Let $z = 4 - 2i$, then $z^3 = 16 - 88i$ Hence $16 - 88i + 4a - 2ai + b = 0$ Re parts: $16 + 4a + b = 0$ Im parts: $-88 - 2a = 0$ Hence $a = -44, b = 160$
Specific behaviours
✓ identifies root and substitutes ✓ equates real and imaginary parts to zero ✓ solves for a ✓ correct values

(b) Clearly show that $2 + i$ is a root of the equation $z^3 - 7z^2 + 17z - 15 = 0$. (2 marks)

Solution
$z = 2 + i, 17z = 34 + 17i, 7z^2 = 21 + 28i, z^3 = 2 + 11i$ $z^3 - 7z^2 + 17z - 15 = 2 + 11i - 21 - 28i + 34 + 17i - 15$ $= 36 - 36 + 28i - 28i$ $= 0$
Specific behaviours
✓ shows expanded term for z^3 ✓ fully expands all terms and sums to zero

(c) State all three solutions of $z^3 - 7z^2 + 17z - 15 = 0$. (1 mark)

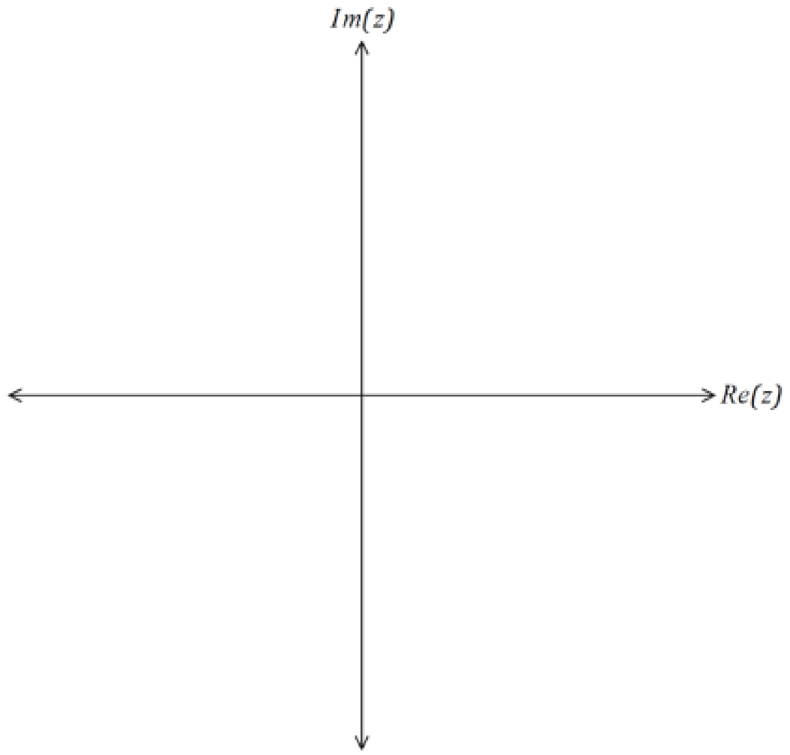
Solution
$z = 3, 2 + i, 2 - i$
Specific behaviours
✓ correct solutions

Question 12 **(9 marks)**

Let $w = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(a) Express w, w^2, w^3 and w^4 in the form $r \operatorname{cis} \theta$, $-\pi < \theta \leq \pi$. (2 marks)

(b) Sketch w, w^2, w^3 and w^4 as vectors on the Argand diagram below. (2 marks)



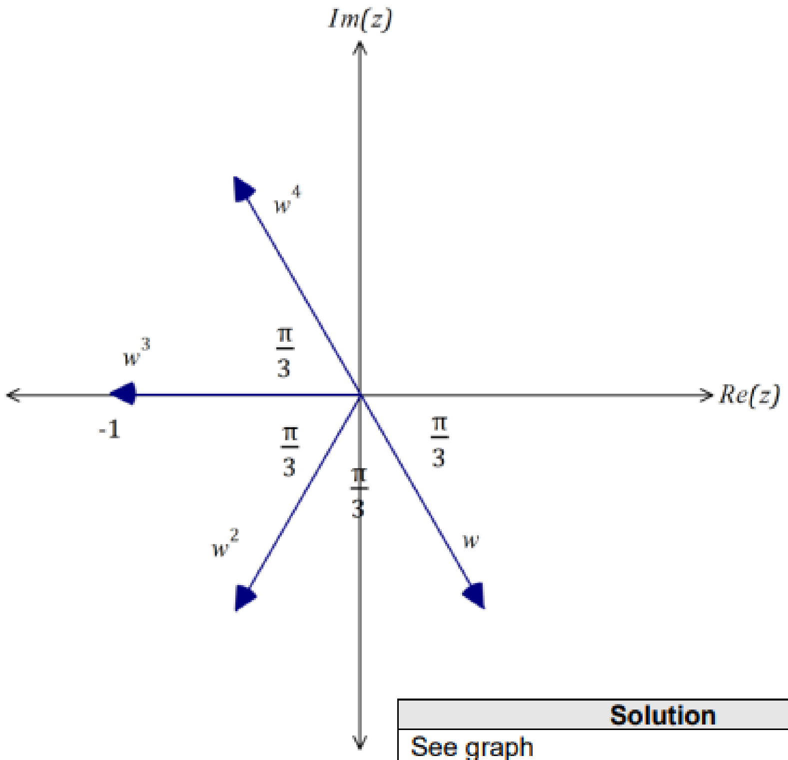
Question 12 **(9 marks)**

Let $w = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(a) Express w, w^2, w^3 and w^4 in the form $r \operatorname{cis}\theta$, $-\pi < \theta \le \pi$. (2 marks)

Solution
$w = \operatorname{cis}\left(-\frac{\pi}{3}\right), w^2 = \operatorname{cis}\left(-\frac{2\pi}{3}\right), w^3 = \operatorname{cis}(\pi), w^4 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$
Specific behaviours
✓ w correct ✓ all correct

(b) Sketch w, w^2, w^3 and w^4 as vectors on the Argand diagram below. (2 marks)



Solution
See graph
Specific behaviours
✓ 4 rays roughly as shown ✓ indicates scale and angle between rays

(c) Describe the transformation in the complex plane of any point z when it is multiplied by w.

(2 marks)

(d) Simplify

(i) $w + w^2 + w^3 + w^4 + w^5 + w^6$. (1 mark)

(ii) $w^1 + w^2 + w^3 + \cdots + w^{2018} + w^{2019}$. (2 marks)

(c) Describe the transformation in the complex plane of any point z when it is multiplied by w.

(2 marks)

Solution
Rotation about origin of $\frac{\pi}{3}$
Specific behaviours
✓ at least one element of transformation ✓ all three elements of transformation

(d) Simplify

(i) $w + w^2 + w^3 + w^4 + w^5 + w^6$. (1 mark)

Solution
0
Specific behaviours
✓ correct value

(ii) $w^1 + w^2 + w^3 + \cdots + w^{2018} + w^{2019}$. (2 marks)

Solution
$w + w^2 + w^3 + \cdots + w^{2016} = 0$ $w^{2017} + w^{2018} + w^{2019} = w + w^2 + w^3 = -1 - \sqrt{3}i$
Specific behaviours
✓ correct sum for $w + \cdots + w^{2016}$ ✓ correct value

Question 14 **(7 marks)**

(a) Solve the equation $z^5 - 32i = 0$, writing your solutions in polar form $r\,\text{cis}\,\theta$. (4 marks)

(b) Use your answers from (a) to show that $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0.$

(3 marks)

Question 14 **(7 marks)**

(a) Solve the equation $z^5 - 32i = 0$, writing your solutions in polar form $r \text{ cis } \theta$. (4 marks)

Solution
$z^5 = 32i$ $= 2^5 \text{ cis} \frac{\pi}{2}$ $z_n = 2 \text{ cis}\left(\frac{9\pi}{10} - \frac{4n\pi}{10}\right), n = 0,1,2,3,4$ $z_0 = 2 \text{ cis}\left(\frac{9\pi}{10}\right), z_1 = 2 \text{ cis}\left(\frac{\pi}{2}\right), z_0 = 2 \text{ cis}\left(\frac{\pi}{10}\right), z_0 = 2 \text{ cis}\left(\frac{-3\pi}{10}\right), z_0 = 2 \text{ cis}\left(\frac{-7\pi}{10}\right)$
Specific behaviours
✓ expresses in polar form ✓ states general solution ✓ states one correct solution in polar form ✓ states all correct solutions in polar form

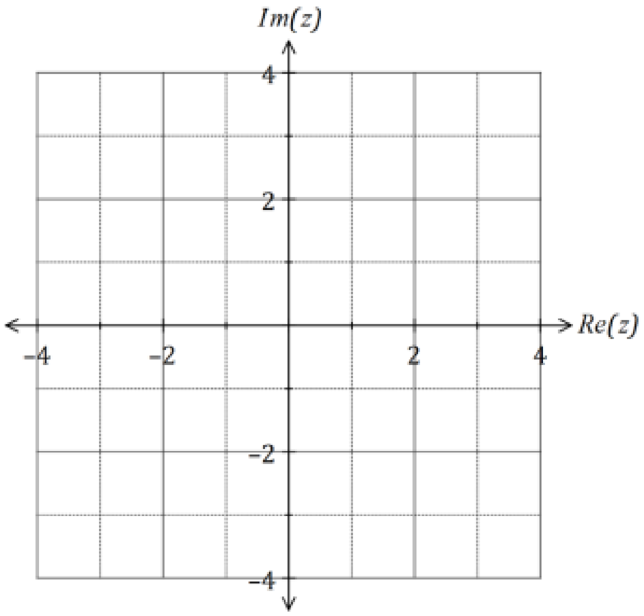
(b) Use your answers from (a) to show that $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0$. (3 marks)

Solution
Since $z_0 + z_1 + z_2 + z_3 + z_4 = 0$ then $\text{Re}(z_0 + z_1 + z_2 + z_3 + z_4) = 0$ $2\cos\left(\frac{9\pi}{10}\right) + 2\cos\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{10}\right) + 2\cos\left(-\frac{3\pi}{10}\right) + 2\cos\left(-\frac{7\pi}{10}\right) = 0$ But $\cos(-\theta) = \cos\theta$ and $\cos\frac{\pi}{2} = 0$ Hence $\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0$
Specific behaviours
✓ indicates that sum of roots is zero ✓ equates real part of sum of roots to zero ✓ states $\cos(-\theta) = \cos\theta$ and $\cos\frac{\pi}{2} = 0$ and simplifies

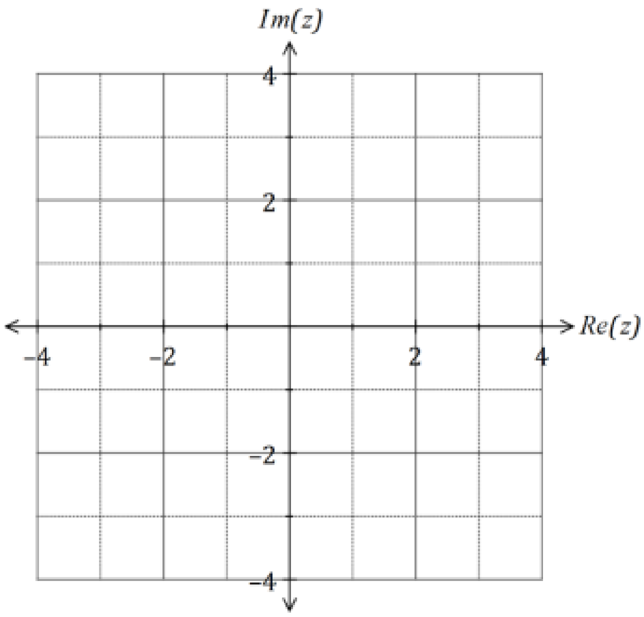
Question 21 **(6 marks)**

Sketch the locus of the complex number z given by

(a) $|z+1-i| \le |z-1+3i|$. (3 marks)



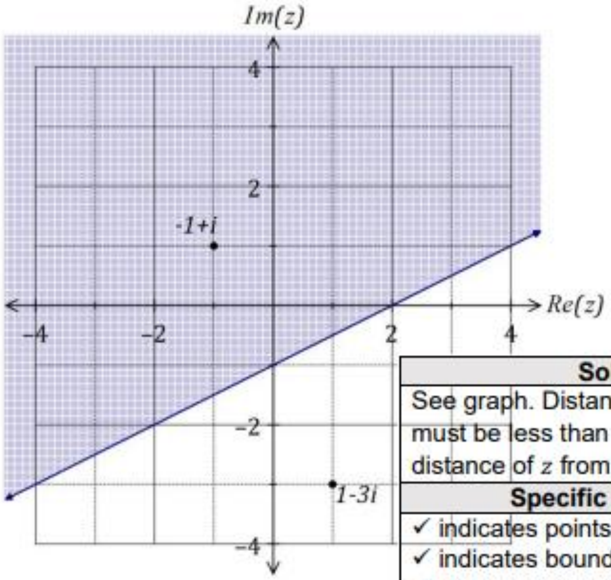
(b) $|z-2i| = |z| + 2$. (3 marks)



Question 21 **(6 marks)**

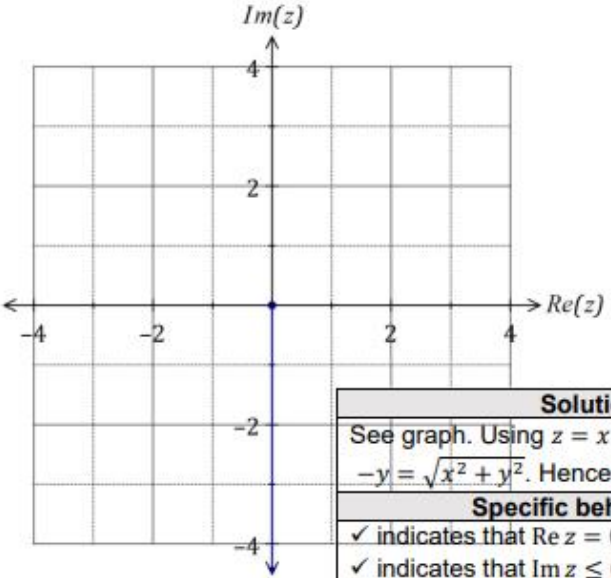
Sketch the locus of the complex number z given by

(a) $|z + 1 - i| \le |z - 1 + 3i|$. (3 marks)



Solution
See graph. Distance of z from $-1 + i$ must be less than or equal to the distance of z from $1 - 3i$
Specific behaviours
✓ indicates points ✓ indicates boundary ✓ indicates correct half plane

(b) $|z - 2i| = |z| + 2$. (3 marks)



Solution
See graph. Using $z = x + iy$ leads to $-y = \sqrt{x^2 + y^2}$. Hence $y \le 0$ and $x = 0$.
Specific behaviours
✓ indicates that $\mathrm{Re}\, z = 0$ ✓ indicates that $\mathrm{Im}\, z \le 0$ ✓ indicates correct ray that includes origin

Question 9 **(6 marks)**

Two complex numbers are $u = 1 - \sqrt{3}i$ and $v = 2\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$.

(a) Determine the argument of uv. (2 marks)

(b) Simplify $|u \times \bar{u} \times v^{-1}|$. (2 marks)

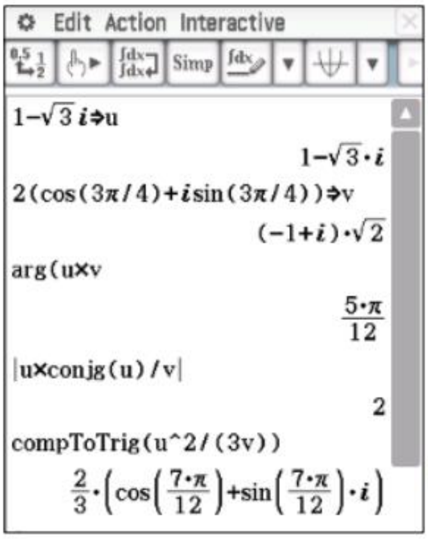
(c) Determine z in polar form if $3zv = u^2$. (2 marks)

Question 9 **(6 marks)**

Two complex numbers are $u = 1 - \sqrt{3}i$ and $v = 2\,\text{cis}\left(\frac{3\pi}{4}\right)$.

(a) Determine the argument of uv. (2 marks)

Solution
$\arg(u) = -\frac{\pi}{3}$ $\arg(uv) = -\frac{\pi}{3} + \frac{3\pi}{4} = \frac{5\pi}{12}$
Specific behaviours
✓ argument of u ✓ argument



(b) Simplify $|u \times \bar{u} \times v^{-1}|$. (2 marks)

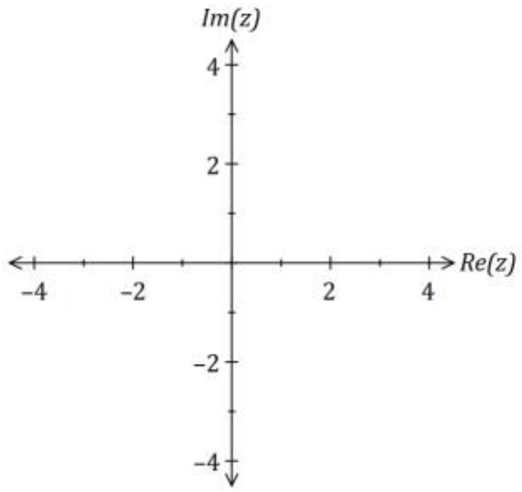
Solution
$\lvert u \times \bar{u}\rvert \times \left\lvert\frac{1}{v}\right\rvert = \lvert u\rvert^2 \times \left\lvert\frac{1}{v}\right\rvert = 4 \times \frac{1}{2}$ $= 2$
Specific behaviours
✓ evaluates $\lvert u \times \bar{u}\rvert$ ✓ correct magnitude

(c) Determine z in polar form if $3zv = u^2$. (2 marks)

Solution
$z = \frac{u^2}{3v} = \frac{1}{3} \times \frac{4\,\text{cis}\left(-\frac{2\pi}{3}\right)}{2\,\text{cis}\left(\frac{3\pi}{4}\right)}$ $z = \frac{2}{3}\text{cis}\left(\frac{7\pi}{12}\right)$
Specific behaviours
✓ indicates u^2 in polar form   ✓ simplifies z

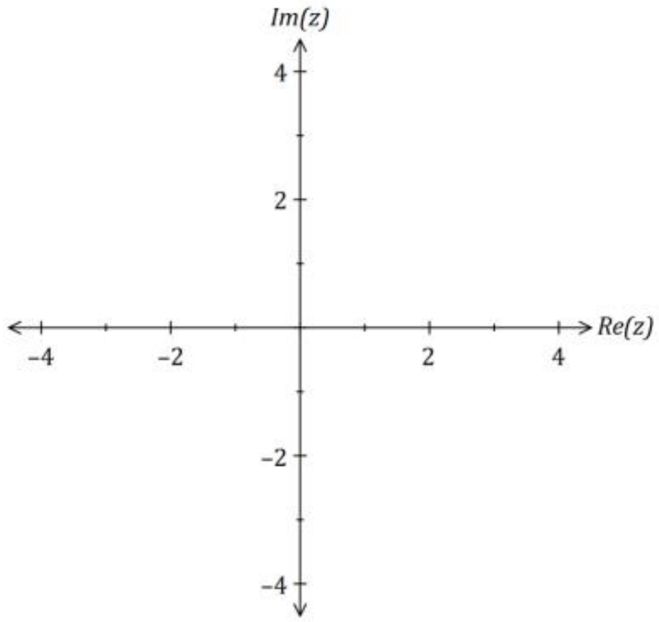
Question 12 **(9 marks)**

(a) On the Argand plane below, sketch the locus of $|z - 1 - i| = |z + 1 - 3i|$, where z is a complex number. (3 marks)



(b) Consider the three inequalities $|z + 2 + 2i| \leq 2$, $\arg(z) \geq -\frac{3\pi}{4}$ and $\mathrm{Re}(z) \leq -1$.

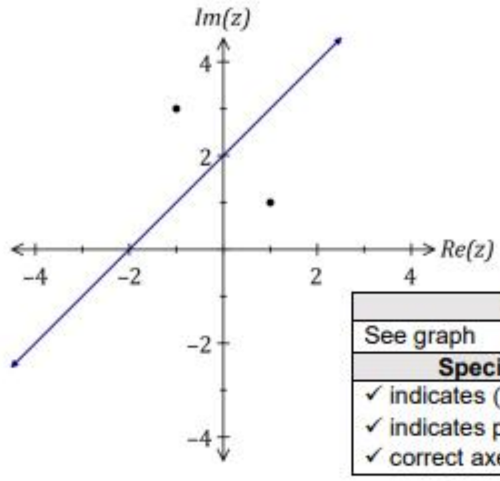
(i) On the Argand plane below, shade the region that represents the complex numbers satisfying these inequalities. (5 marks)



(ii) Determine the minimum possible value of $\mathrm{Re}(z)$ within the shaded region. (1 mark)

Question 12 **(9 marks)**

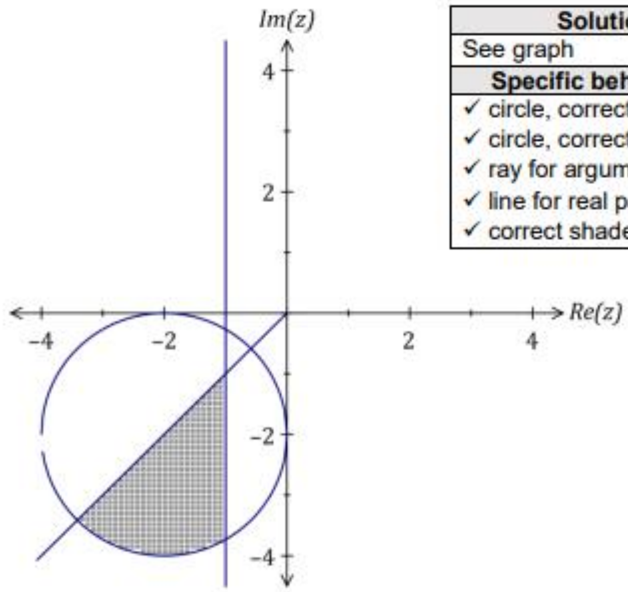
(a) On the Argand plane below, sketch the locus of $|z - 1 - i| = |z + 1 - 3i|$, where z is a complex number. (3 marks)



Solution
See graph
Specific behaviours
✓ indicates $(1, 1)$ and $(-1, 3)$
✓ indicates perpendicular bisector
✓ correct axes intercepts

(b) Consider the three inequalities $|z + 2 + 2i| \le 2$, $\arg(z) \ge -\frac{3\pi}{4}$ and $\operatorname{Re}(z) \le -1$.

(i) On the Argand plane below, shade the region that represents the complex numbers satisfying these inequalities. (5 marks)



Solution
See graph
Specific behaviours
✓ circle, correct centre
✓ circle, correct radius
✓ ray for argument
✓ line for real part
✓ correct shaded region

(ii) Determine the minimum possible value of $\operatorname{Re}(z)$ within the shaded region. (1 mark)

Solution
$-2 - 2\sin\frac{\pi}{4} = -2 - \sqrt{2}$
Specific behaviours
✓ correct value

Question 16 **(8 marks)**

(a) Let $r \operatorname{cis} \theta$ be a point in the complex plane. Determine, in terms of r and θ, the polar form of this point after it is reflected in the real axis and then rotated $-\frac{\pi}{3}$ about the origin.

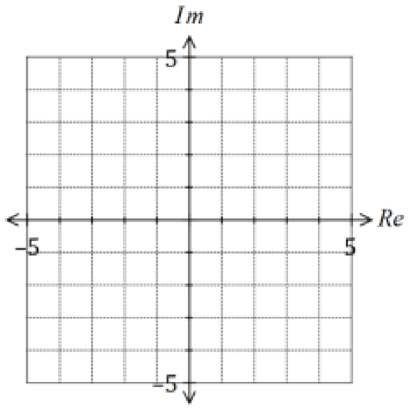
(2 marks)

(b) Let $f(w) = -i\overline{w} + 2 + 2i$.

(i) Complete the following table. (3 marks)

w	$2 + 2i$	$1 - 2i$	$-3 + i$
$f(w)$			

(ii) Sketch each point, w, and join it with a dotted line to its image, $f(w)$, on the diagram below. (1 mark)



(iii) Describe the geometric transformation that $f(w)$ represents. (2 marks)

Question 16 **(8 marks)**

(a) Let $r\,\text{cis}\,\theta$ be a point in the complex plane. Determine, in terms of r and θ, the polar form of this point after it is reflected in the real axis and then rotated $-\frac{\pi}{3}$ about the origin. (2 marks)

Solution
$z \to \bar{z} = r\,\text{cis}(-\theta)$ $\bar{z} \to r\,\text{cis}\left(-\theta - \frac{\pi}{3}\right)$
Specific behaviours
✓ conjugate   ✓ rotation

(b) Let $f(w) = -i\bar{w} + 2 + 2i$.

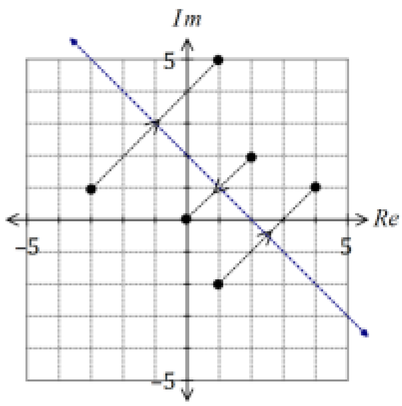
(i) Complete the following table. (3 marks)

w	$2+2i$	$1-2i$	$-3+i$
$f(w)$	0	$4+i$	$1+5i$

Solution
See table
Specific behaviours
✓✓✓ each point

(ii) Sketch each point, w, and join it with a dotted line to its image, $f(w)$, on the diagram below. (1 mark)

Solution
See diagram
Specific behaviours
✓ plots points



(iii) Describe the geometric transformation that $f(w)$ represents. (2 marks)

Solution
Reflection in the line $\text{Re}(z) + \text{Im}(z) = 2$
Specific behaviours
✓ reflection   ✓ line of reflection

Question 21 **(8 marks)**

(a) Consider the complex equation $z^5 = -(4+4i)$.

Solve the equation, giving all solutions in the form $r \operatorname{cis} \theta$ where $r > 0$ and $-\pi \leq \theta \leq \pi$.
(4 marks)

(b) One solution to the complex equation $z^5 = -9\sqrt{3}i$ is $z = \sqrt{3} \operatorname{cis}\left(-\frac{\pi}{10}\right)$.

Let u be the solution to $z^5 = -9\sqrt{3}i$ so that $\frac{\pi}{2} \leq \arg(u) \leq \pi$. Determine $\arg(u - \sqrt{3})$ in exact form.
(4 marks)

Question 21 **(8 marks)**

(a) Consider the complex equation $z^5 = -(4+4i)$.

Solve the equation, giving all solutions in the form $r \operatorname{cis} \theta$ where $r > 0$ and $-\pi \le \theta \le \pi$.

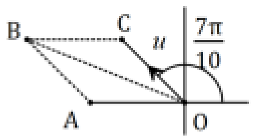
(4 marks)

Solution
$z^5 = 4\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$ $z_n = \sqrt{2} \operatorname{cis}\left(\frac{8k\pi - 19\pi}{20}\right), k = \{0, 1, 2, 3, 4\}$ $z_0 = \sqrt{2} \operatorname{cis}\left(-\frac{19\pi}{20}\right), z_1 = \sqrt{2} \operatorname{cis}\left(-\frac{11\pi}{20}\right), z_2 = \sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{20}\right), z_3 = \sqrt{2} \operatorname{cis}\left(\frac{5\pi}{20}\right), z_4 = \sqrt{2} \operatorname{cis}\left(\frac{13\pi}{20}\right)$ (Arguments in degrees: $-171°, -99°, -27°, 45°, 117°$)
Specific behaviours
✓ converts to polar form ✓ forms general expression for roots ✓ gives one correct root ✓ lists all correct roots

(b) One solution to the complex equation $z^5 = -9\sqrt{3}i$ is $z = \sqrt{3} \operatorname{cis}\left(-\frac{\pi}{10}\right)$.

Let u be the solution to $z^5 = -9\sqrt{3}i$ so that $\frac{\pi}{2} \le \arg(u) \le \pi$. Determine $\arg(u - \sqrt{3})$ in exact form.

(4 marks)

Solution	
 $OC = u, OA = -\sqrt{3}$ $\lvert OA \rvert = \lvert OC \rvert = \lvert AB \rvert = \lvert BC \rvert = \sqrt{3}$	$\arg(u) = -\frac{\pi}{10} + 2\left(\frac{2\pi}{5}\right) = \frac{7\pi}{10}$ $\angle COB = \frac{1}{2} \times \frac{3\pi}{10} = \frac{3\pi}{20}$ $\arg(u - \sqrt{3}) = \pi - \frac{3\pi}{20} = \frac{17\pi}{20}$
Specific behaviours	
✓ indicates argument of u ✓ sketch ✓ uses geometric property of rhombus ✓ correct argument	