Question 8 **(6 marks)**

A particle moves in space with position vector $\underset{\sim}{r}(t) = \begin{pmatrix} 2\cos(4t) \\ -3t \\ -2\sin(4t) \end{pmatrix}$ cm, where t is the time in seconds since its motion began.

(a) Determine the distance of the particle from its initial position after $\frac{\pi}{3}$ seconds. (3 marks)

(b) Show that the particle is moving with a constant speed. (3 marks)

Question 8 **(6 marks)**

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(a) Determine the distance of the particle from its initial position after $\frac{\pi}{3}$ seconds. (3 marks)

Solution
$\underset{\sim}{r}\left(\frac{\pi}{3}\right) - \underset{\sim}{r}(0) = \begin{pmatrix} -1 \\ -\pi \\ \sqrt{3} \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -\pi \\ \sqrt{3} \end{pmatrix}$ $\left\lvert \begin{pmatrix} -3 \\ -\pi \\ \sqrt{3} \end{pmatrix} \right\rvert = \sqrt{\pi^2 + 12} \approx 4.68 \text{ cm}$
Specific behaviours
✓ position vectors at $t = 0$ and required time   ✓ displacement vector   ✓ correct distance

(b) Show that the particle is moving with a constant speed. (3 marks)

Solution
Velocity vector: $\underset{\sim}{v}(t) = \frac{d}{dt}\left(\underset{\sim}{r}(t)\right) = \begin{pmatrix} -8\sin(4t) \\ -3 \\ -8\cos(4t) \end{pmatrix}$ Speed: $\left\lvert \underset{\sim}{v}(t) \right\rvert = \sqrt{(-8\sin(4t))^2 + (-3)^2 + (-8\cos(4t))^2}$ $= \sqrt{64\sin^2(4t) + 64\cos^2(4t) + 9}$ $= \sqrt{64 + 9}$ $= \sqrt{73} \approx 8.54 \text{ cm/s}$ Hence particle is moving with a constant speed.
Specific behaviours
✓ correct velocity vector   ✓ correct expression for magnitude of vector   ✓ simplifies magnitude to show constant

Question 12 **(7 marks)**

Relative to an origin O located on level ground, a projectile is launched from $\begin{pmatrix} 0 \\ 7.5 \end{pmatrix}$ m with an initial velocity of $\begin{pmatrix} 20 \\ 21 \end{pmatrix}$ m/s. The motion of the projectile is only affected by a constant acceleration of $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ m/s^2.

(a) Derive from the acceleration vector an expression for the position vector $\underset{\sim}{r}(t)$ of the projectile t s after its launch. (3 marks)

(b) Determine the distance travelled through the air by the projectile from when it is launched until the instant it reaches the ground, correct to the nearest 0.1 m. (4 marks)

Question 12 **(7 marks)**

Relative to an origin O located on level ground, a projectile is launched from $\begin{pmatrix}0\\7.5\end{pmatrix}$ m with an initial velocity of $\begin{pmatrix}20\\21\end{pmatrix}$ m/s. The motion of the projectile is only affected by a constant acceleration of $\begin{pmatrix}0\\-9.8\end{pmatrix}$ m/s^2.

(a) Derive from the acceleration vector an expression for the position vector $\underset{\sim}{r}(t)$ of the projectile t s after its launch. (3 marks)

Solution
$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,dt$ $= \begin{pmatrix}0\\-9.8t\end{pmatrix} + \underset{\sim}{c}$ and $\underset{\sim}{v}(0) = \begin{pmatrix}20\\21\end{pmatrix}$ $= \begin{pmatrix}20\\21-9.8t\end{pmatrix}$ $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt$ $= \begin{pmatrix}20t\\21t-4.9t^2\end{pmatrix} + \underset{\sim}{c}$ and $\underset{\sim}{r}(0) = \begin{pmatrix}0\\7.5\end{pmatrix}$ $= \begin{pmatrix}20t\\7.5+21t-4.9t^2\end{pmatrix}$
Specific behaviours
✓ antidifferentiates acceleration, shows constant   ✓ antidifferentiates velocity, shows constant   ✓ correctly uses initial conditions to evaluate constants

(b) Determine the distance travelled through the air by the projectile from when it is launched until the instant it reaches the ground, correct to the nearest 0.1 m. (4 marks)

Solution
Reaches ground level when vertical component of position is 0: $7.5 + 21t - 4.9t^2 = 0 \Rightarrow t = \dfrac{10\sqrt{3}+15}{7} \approx 4.6172\text{ s}$ Distance travelled: $d = \int_0^{4.6172} \left\lvert\underset{\sim}{v}(t)\right\rvert\,dt$ $= \int_0^{4.6172} \sqrt{20^2 + (21-9.8t)^2}\,dt$ $= 109.6\text{ m}$
Specific behaviours
✓ equation for time to reach ground level   ✓ obtains time to reach ground level   ✓ integral for distance travelled   ✓ correct distance travelled

Question 13 **(7 marks)**

At time t seconds, $t \geq 0$, the position vector $\underset{\sim}{r}(t)$ m of a particle is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2e^{-0.5t} - 3 \\ 1 - 4e^{-1.5t} \end{pmatrix}$$

(a) State the position vector of the point that the particle approaches as $t \to \infty$. (1 mark)

(b) Determine the speed of the particle when $t = 4$, correct to the nearest 0.001 m/s. (3 marks)

(c) Express the Cartesian equation for the path of the particle in the form $y = f(x)$. (3 marks)

Question 13 **(7 marks)**

At time t seconds, $t \geq 0$, the position vector $\underset{\sim}{r}(t)$ m of a particle is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 2e^{-0.5t} - 3 \\ 1 - 4e^{-1.5t} \end{pmatrix}$$

(a) State the position vector of the point that the particle approaches as $t \to \infty$. (1 mark)

Solution
$\lim\limits_{t\to\infty} \underset{\sim}{r}(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
Specific behaviours
✓ correct position vector

(b) Determine the speed of the particle when $t = 4$, correct to the nearest 0.001 m/s. (3 marks)

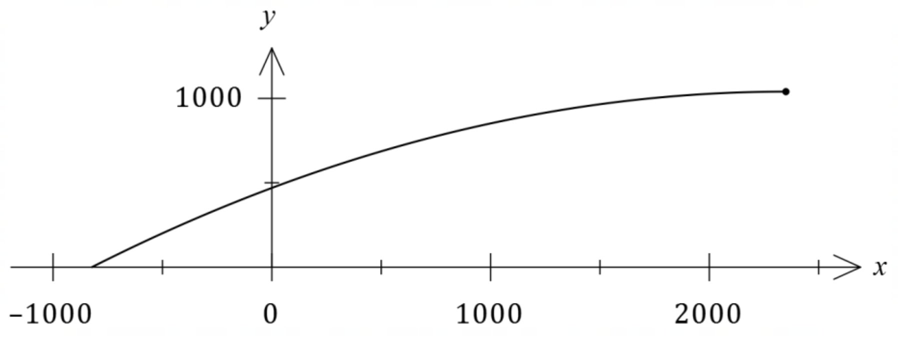
Solution
$\underset{\sim}{v}(t) = \begin{pmatrix} -e^{-0.5t} \\ 6e^{-1.5t} \end{pmatrix}$ $\underset{\sim}{v}(4) = \begin{pmatrix} -e^{-2} \\ 6e^{-6} \end{pmatrix}$ $\left\lvert\underset{\sim}{v}(4)\right\rvert = \sqrt{e^{-4} + 36e^{-12}}$ $= e^{-6}\sqrt{e^{8} + 36} \approx 0.136 \text{ m/s}$
Specific behaviours
✓ obtains velocity vector ✓ velocity vector at required time ✓ calculates magnitude of velocity

(c) Express the Cartesian equation for the path of the particle in the form $y = f(x)$. (3 marks)

Solution
$x = 2e^{-0.5t} - 3$ $e^{-0.5t} = \dfrac{x+3}{2}$ $\therefore e^{-1.5t} = \left(\dfrac{x+3}{2}\right)^3$ $y = 1 - 4e^{-1.5t}$ $= 1 - 4\left(\dfrac{x+3}{2}\right)^3$ $y = 1 - \dfrac{1}{2}(x+3)^3 \quad \text{where} \quad -3 < x \leq -1.$
Specific behaviours
✓ obtains expression for $e^{-0.5t}$ in terms of x ✓ obtains $y = f(x)$, simplification optional ✓ uses initial position and part (a) to state domain restriction

Question 15 **(10 marks)**

An aeroplane flying at a constant altitude releases a bomb at $2350\mathbf{i} + 1040\mathbf{j}$ with an initial velocity of $-220\mathbf{i}$. The path of the bomb is shown below.



Assume there is no wind in the region, air resistance can be ignored and the only acceleration acting on the bomb is $-10\mathbf{j}$ ms^{-2} due to gravity.

(a) Use the acceleration vector of the bomb to clearly deduce that its position vector at time t seconds after release is $\mathbf{r}(t) = (2350 - 220t)\mathbf{i} + (1040 - 5t^2)\mathbf{j}$. (3 marks)

(b) Determine the speed of the bomb 4 seconds after it is released. (2 marks)

Five seconds after the bomb is released, a projectile is launched from the origin with a speed of v_0 at an angle of elevation of $\theta°$ to intercept it at a height of 635 m.

The position vector of the projectile T seconds after its launch is

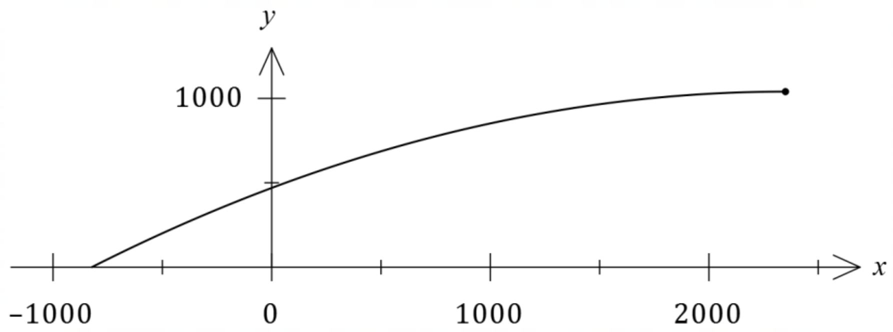
$$\mathbf{r}(T) = (v_0\cos(\theta)\,T)\mathbf{i} + (v_0\sin(\theta)\,T - 5T^2)\mathbf{j}.$$

(c) Determine the value of v_0 and the value of θ so that the projectile intercepts the bomb.

(5 marks)

Question 15 **(10 marks)**

An aeroplane flying at a constant altitude releases a bomb at $2350\mathbf{i} + 1040\mathbf{j}$ with an initial velocity of $-220\mathbf{i}$. The path of the bomb is shown below.



Assume there is no wind in the region, air resistance can be ignored and the only acceleration acting on the bomb is $-10\mathbf{j}$ ms^{-2} due to gravity.

(a) Use the acceleration vector of the bomb to clearly deduce that its position vector at time t seconds after release is $\mathbf{r}(t) = (2350 - 220t)\mathbf{i} + (1040 - 5t^2)\mathbf{j}$. (3 marks)

Solution
$\mathbf{v}(t) = \int \begin{pmatrix} 0 \\ -10 \end{pmatrix} dt = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \mathbf{c_1}$ $\mathbf{v}(0) = \begin{pmatrix} -220 \\ 0 \end{pmatrix} \Rightarrow \mathbf{c_1} = \begin{pmatrix} -220 \\ 0 \end{pmatrix}, \quad \therefore \mathbf{v}(t) = \begin{pmatrix} -220 \\ -10t \end{pmatrix}$ $\mathbf{r}(t) = \int \begin{pmatrix} -220 \\ -10t \end{pmatrix} dt = \begin{pmatrix} -220t \\ -5t^2 \end{pmatrix} + \mathbf{c_2}$ $\mathbf{r}(0) = \begin{pmatrix} 2350 \\ 1040 \end{pmatrix} \Rightarrow \mathbf{c_2} = \begin{pmatrix} 2350 \\ 1040 \end{pmatrix}, \quad \therefore \mathbf{r}(t) = \begin{pmatrix} 2350 - 220t \\ 1040 - 5t^2 \end{pmatrix}$
Specific behaviours
✓ correct integration of acceleration vector, with constant   ✓ clearly shows use of initial conditions to obtain velocity vector   ✓ repeats with velocity vector to obtain position vector

(b) Determine the speed of the bomb 4 seconds after it is released. (2 marks)

Solution
$\mathbf{v}(4) = \begin{pmatrix} -220 \\ -40 \end{pmatrix}$ $\therefore s = \sqrt{(-220)^2 + (-40)^2}$ $= 100\sqrt{5} \approx 223.6 \text{ m/s}$
Specific behaviours
✓ indicates velocity   ✓ calculates correct speed

Five seconds after the bomb is released, a projectile is launched from the origin with a speed of v_0 at an angle of elevation of $\theta°$ to intercept it at a height of 635 m.

The position vector of the projectile T seconds after its launch is

$$\mathbf{r}(T) = (v_0\cos(\theta)\,T)\mathbf{i} + (v_0\sin(\theta)\,T - 5T^2)\mathbf{j}.$$

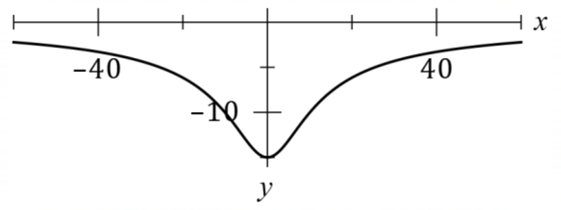
(c) Determine the value of v_0 and the value of θ so that the projectile intercepts the bomb.

(5 marks)

Solution
Bomb reaches 635 m when $$1040 - 5t^2 = 635 \Rightarrow t = 9\text{ s}$$ Horizontal position of bomb is $2350 - 220(9) = 370$ m. Projectile will travel for $T = 9 - 5 = 4$ seconds. Horizontal position of projectile $$v_0\cos(\theta)\,(4) = 370 \Rightarrow v_0\cos(\theta) = 92.5$$ Vertical position of projectile $$v_0\sin(\theta)\,(4) - 5(4^2) = 635 \Rightarrow v_0\sin(\theta) = 178.75$$ Hence $$\frac{v_0\sin(\theta)}{v_0\cos(\theta)} = \frac{178.75}{92.5} \Rightarrow \tan(\theta) = \frac{178.75}{92.5} \Rightarrow \theta \approx 62.64° \approx 1.093^r$$ And $$v_0 = 92.5 \div \cos 62.64° = 201.3\text{ m/s}$$
Specific behaviours
✓ calculates time of interception ✓ uses horizontal position of interception to form equation ✓ uses vertical position of interception to form equation ✓ solves for angle in degrees or radians ✓ solves for initial speed

Question 18 **(8 marks)**

The path of a small submersible moving below the surface of the sea (the x-axis) is shown in the diagram, where t is the time in seconds and $0 < t < 3\pi$.
The position vector of the submersible is



$$\mathbf{r}(t) = 9\cot\left(\frac{t}{3}\right)\mathbf{i} - 15\sin\left(\frac{t}{3}\right)\mathbf{j}\ \mathrm{m}.$$

(a) State, with reasoning, whether the submersible is moving from left to right or from right to left. (2 marks)

(b) Determine the Cartesian equation for the path of the submersible. (3 marks)

(c) Determine the distance travelled by the submersible when its depth below the surface is at least 7.5 metres, correct to the nearest centimetre. (3 marks)

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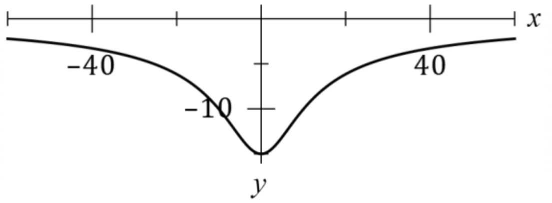


(a) State, with reasoning, whether the submersible is moving from left to right or from right to left. (2 marks)

Solution

$$\mathbf{v}(t) = -\,3\Big/\sin^2\left(\frac{t}{3}\right)\,\mathbf{i} - 5\cos\left(\frac{t}{3}\right)\mathbf{j}$$

The i-coefficient will always be negative and so submersible is moving from right to left.

Specific behaviours

✓ differentiates to obtain velocity vector
✓ states right to left, with reason

(b) Determine the Cartesian equation for the path of the submersible. (3 marks)

Solution

$$\frac{\cos^2 A}{\sin^2 A} + \frac{\sin^2 A}{\sin^2 A} = \frac{1}{\sin^2 A} \Rightarrow \cot^2 A + 1 = \frac{1}{\sin^2 A}$$

$$x = 9\cot\left(\frac{t}{3}\right) \Rightarrow \cot\left(\frac{t}{3}\right) = \frac{x}{9}, \qquad y = -15\sin\left(\frac{t}{3}\right) \Rightarrow \sin\left(\frac{t}{3}\right) = -\frac{y}{15}$$

$$\left(\frac{x}{9}\right)^2 + 1 = \left(-\frac{15}{y}\right)^2 \Leftrightarrow \left(\frac{x}{9}\right)^2 + 1 = \left(\frac{15}{y}\right)^2 \Leftrightarrow x^2y^2 + 81y^2 = 18\,225$$

Specific behaviours

✓ obtains suitable trigonometric identity
✓ sets $x = \cdots$, $y = \cdots$ and arranges equations for use with identity
✓ eliminates trigonometric terms and simplifies

(c) Determine the distance travelled by the submersible when its depth below the surface is at least 7.5 metres, correct to the nearest centimetre. (3 marks)

Solution

Depth is at least 7.5 m when

$$15\sin\left(\frac{t}{3}\right) = 7.5 \Rightarrow \frac{\pi}{2} \le t \le \frac{5\pi}{2}$$

Distance*:

$$d = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \sqrt{\left(-\frac{3}{\sin^2\left(\frac{t}{3}\right)}\right)^2 + \left(-5\cos\left(\frac{t}{3}\right)\right)^2}\, dt = 34.86\ \text{m}$$

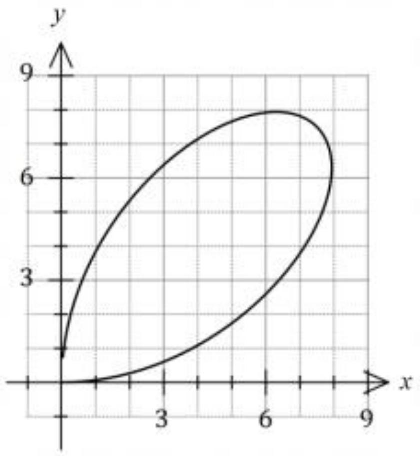
* *Will take* $15 \sim 25$ *seconds to evaluate using numerical integration*

Specific behaviours

✓ obtains correct time interval
✓ writes integral using magnitude of velocity
✓ obtains correct distance, with units

Question 13 **(9 marks)**

The path of a particle with position vector $\mathbf{r}(t) = \dfrac{15t}{1+t^3}\mathbf{i} + \dfrac{15t^2}{1+t^3}\mathbf{j}$ metres is shown below, where t is the time in seconds and $t \geq 0$.



(a) Determine the initial velocity of the particle. (2 marks)

(b) Determine the velocity of the particle at the instant, $t > 0$, when it is moving parallel to the x-axis. (2 marks)

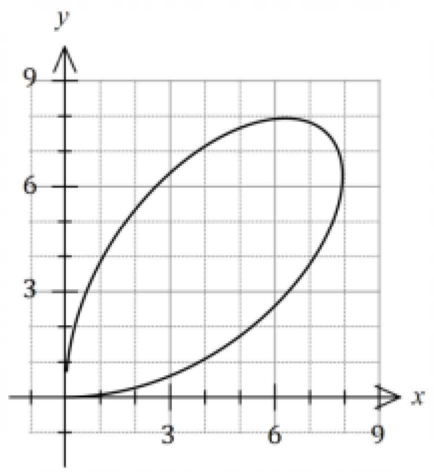
(c) Explain whether the particle will return to its initial position. (2 marks)

(d) Observing that $y - xt = 0$, show that the Cartesian equation for the path of the particle can be expressed in the form $x^3 + y^3 = kxy$ and state the value of the constant k.

(3 marks)

Question 13 **(9 marks)**

The path of a particle with position vector $\mathbf{r}(t) = \dfrac{15t}{1+t^3}\mathbf{i} + \dfrac{15t^2}{1+t^3}\mathbf{j}$ metres is shown below, where t is the time in seconds and $t \geq 0$.



(a) Determine the initial velocity of the particle. (2 marks)

Solution
$$\mathbf{v}(t) = \frac{15-30t^3}{(1+t^3)^2}\mathbf{i} + \frac{30t-15t^4}{(1+t^3)^2}\mathbf{j}$$ $$\mathbf{v}(0) = 15\mathbf{i}\ \text{m/s}$$
Specific behaviours
✓ expression for velocity ✓ initial velocity

(b) Determine the velocity of the particle at the instant, $t > 0$, when it is moving parallel to the x-axis. (2 marks)

Solution
Require **j**-coefficient of velocity to be zero: $$30t - 15t^4 = 0 \Rightarrow t = \sqrt[3]{2}$$ $$\mathbf{v}(\sqrt[3]{2}) = -5\mathbf{i}\ \text{m/s}$$
Specific behaviours
✓ solves for t ✓ velocity

(c) Explain whether the particle will return to its initial position. (2 marks)

Solution
The initial position is at the origin, so the answer is no. The particle will get very close as $t \to \infty$ but neither coefficient of the position vector will ever reach zero, apart from initially.
Specific behaviours
✓ indicates initial position ✓ explains will get close, but never returns

(d) Observing that $y - xt = 0$, show that the Cartesian equation for the path of the particle can be expressed in the form $x^3 + y^3 = kxy$ and state the value of the constant k.
(3 marks)

Solution
$t = \frac{y}{x}$ $x = \frac{15\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^3}$ $x\left(\frac{x^3 + y^3}{x^3}\right) = \frac{15y}{x}$ $x^3 + y^3 = \frac{15yx^3}{x^2}$ $x^3 + y^3 = 15xy \Rightarrow k = 15$
Specific behaviours
✓ expresses x or y term using $\frac{y}{x}$ ✓ correctly obtains expression containing $x^3 + y^3$ ✓ manipulates into required form and states value of k

Question 11 **(8 marks)**

Drone A and drone B move with constant velocities and relative to the origin O have initial positions $(-4, 22, 2)$ and $(5, 15, 3)$ respectively, where distances are in metres.

One second later, the position of A is $(-1, 20, 3)$ and the position of B is $(1, 14, 8)$.

(a) Determine a position vector relative to the origin for each drone after t seconds. (3 marks)

(b) Determine an expression for the distance between the two drones at any time t, $t \geq 0$. (3 marks)

(c) Determine the minimum distance between the drones. (2 marks)

Question 11 **(8 marks)**

Drone A and drone B move with constant velocities and relative to the origin O have initial positions $(-4, 22, 2)$ and $(5, 15, 3)$ respectively, where distances are in metres.

One second later, the position of A is $(-1, 20, 3)$ and the position of B is $(1, 14, 8)$.

(a) Determine a position vector relative to the origin for each drone after t seconds. (3 marks)

Solution
$\mathbf{v}_A = \begin{pmatrix} -1 \\ 20 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ 22 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_B = \begin{pmatrix} 1 \\ 14 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 15 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$ $\mathbf{r}_A = \begin{pmatrix} -4 \\ 22 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{r}_B = \begin{pmatrix} 5 \\ 15 \\ 3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$
Specific behaviours
✓ derives velocity vectors   ✓ position vector for A ✓ position vector for B

(b) Determine an expression for the distance between the two drones at any time t, $t \geq 0$. (3 marks)

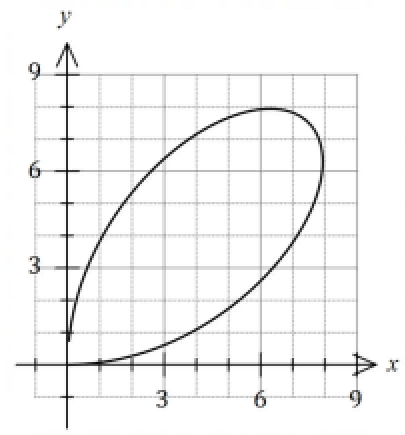
Solution
$\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix} 9 - 7t \\ -7 + t \\ 1 + 4t \end{pmatrix}$ $s = \sqrt{(9 - 7t)^2 + (t - 7)^2 + (1 + 4t)^2}$ $= \sqrt{66t^2 - 132t + 131}$
Specific behaviours
✓ difference of position vectors   ✓ indicates magnitude of vector   ✓ simplified expression

(c) Determine the minimum distance between the drones. (2 marks)

Solution
$\dfrac{ds}{dt} = \dfrac{66(t - 1)}{\sqrt{66t^2 - 132t + 131}}$ $\dot{s} = 0 \Rightarrow t = 1, \quad s(1) = \sqrt{65} \approx 8.06 \text{ m}$
Specific behaviours
✓ time when minimum   ✓ correct distance

Question 13 **(9 marks)**

The path of a particle with position vector $\mathbf{r}(t) = \dfrac{15t}{1+t^3}\mathbf{i} + \dfrac{15t^2}{1+t^3}\mathbf{j}$ metres is shown below, where t is the time in seconds and $t \geq 0$.



(a) Determine the initial velocity of the particle. (2 marks)

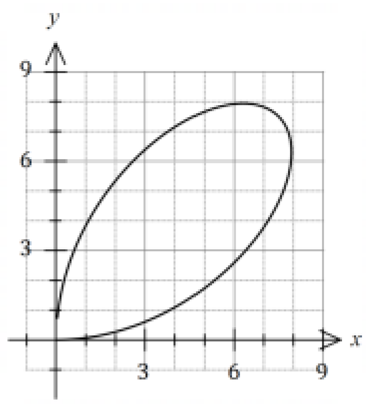
(b) Determine the velocity of the particle at the instant, $t > 0$, when it is moving parallel to the x-axis. (2 marks)

(c) Explain whether the particle will return to its initial position. (2 marks)

(d) Observing that $y - xt = 0$, show that the Cartesian equation for the path of the particle can be expressed in the form $x^3 + y^3 = kxy$ and state the value of the constant k.
(3 marks)

Question 13 **(9 marks)**

The path of a particle with position vector $\mathbf{r}(t) = \dfrac{15t}{1+t^3}\mathbf{i} + \dfrac{15t^2}{1+t^3}\mathbf{j}$ metres is shown below, where t is the time in seconds and $t \geq 0$.



(a) Determine the initial velocity of the particle. (2 marks)

Solution
$\mathbf{v}(t) = \dfrac{15-30t^3}{(1+t^3)^2}\mathbf{i} + \dfrac{30t-15t^4}{(1+t^3)^2}\mathbf{j}$ $\mathbf{v}(0) = 15\mathbf{i}$ m/s
Specific behaviours
✓ expression for velocity   ✓ initial velocity

(b) Determine the velocity of the particle at the instant, $t > 0$, when it is moving parallel to the x-axis. (2 marks)

Solution
Require $\mathbf{j}$-coefficient of velocity to be zero: $30t - 15t^4 = 0 \Rightarrow t = \sqrt[3]{2}$ $\mathbf{v}(\sqrt[3]{2}) = -5\mathbf{i}$ m/s
Specific behaviours
✓ solves for t ✓ velocity

(c) Explain whether the particle will return to its initial position. (2 marks)

Solution
The initial position is at the origin, so the answer is no. The particle will get very close as $t \to \infty$ but neither coefficient of the position vector will ever reach zero, apart from initially.
Specific behaviours
✓ indicates initial position ✓ explains will get close, but never returns

(d) Observing that $y - xt = 0$, show that the Cartesian equation for the path of the particle can be expressed in the form $x^3 + y^3 = kxy$ and state the value of the constant k.

(3 marks)

Solution
$t = \frac{y}{x}$ $x = \frac{15\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^3}$ $x\left(\frac{x^3 + y^3}{x^3}\right) = \frac{15y}{x}$ $x^3 + y^3 = \frac{15yx^3}{x^2}$ $x^3 + y^3 = 15xy \Rightarrow k = 15$
Specific behaviours
✓ expresses x or y term using $\frac{y}{x}$ ✓ correctly obtains expression containing $x^3 + y^3$ ✓ manipulates into required form and states value of k

Question 13 **(9 marks)**

The position vector of a small body is $\mathbf{r}(t) = (1 + 4\sin(t))\mathbf{i} + (1 - 2\cos(2t))\mathbf{j}$ where t is the time in seconds since motion began.

(a) Show that the body is stationary when $t = \frac{\pi}{2}$ and state its position at this instant. (3 marks)

(b) Derive the Cartesian equation of the path of the body. (4 marks)

Question 13 **(9 marks)**

The position vector of a small body is $\mathbf{r}(t) = (1 + 4\sin(t))\mathbf{i} + (1 - 2\cos(2t))\mathbf{j}$ where t is the time in seconds since motion began.

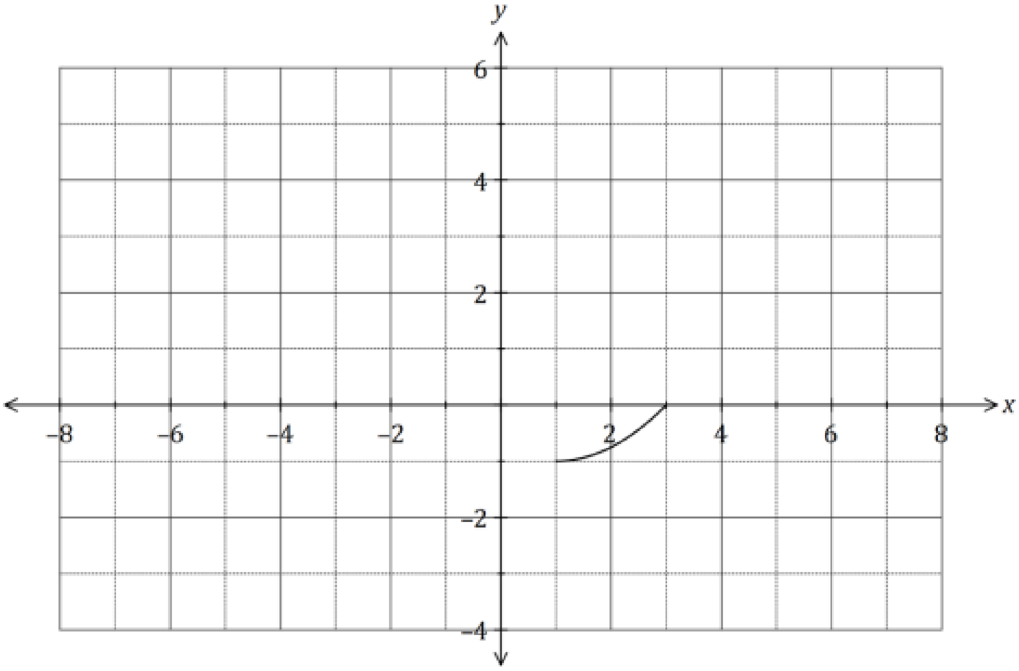
(a) Show that the body is stationary when $t = \frac{\pi}{2}$ and state its position at this instant. (3 marks)

Solution
$\mathbf{v}(t) = 4\cos(t)\,\mathbf{i} + 4\sin(2t)\mathbf{j}$ $\mathbf{v}\left(\frac{\pi}{2}\right) = 4\cos\left(\frac{\pi}{2}\right)\mathbf{i} + 4\sin(\pi)\,\mathbf{j} = \mathbf{0}$ $\mathbf{r}\left(\frac{\pi}{2}\right) = \left(1 + 4\sin\left(\frac{\pi}{2}\right)\right)\mathbf{i} + (1 - 2\cos(\pi))\mathbf{j} = 5\mathbf{i} + 3\mathbf{j}$
Specific behaviours
✓ expression for velocity   ✓ substitutes time and obtains zero vector   ✓ states position

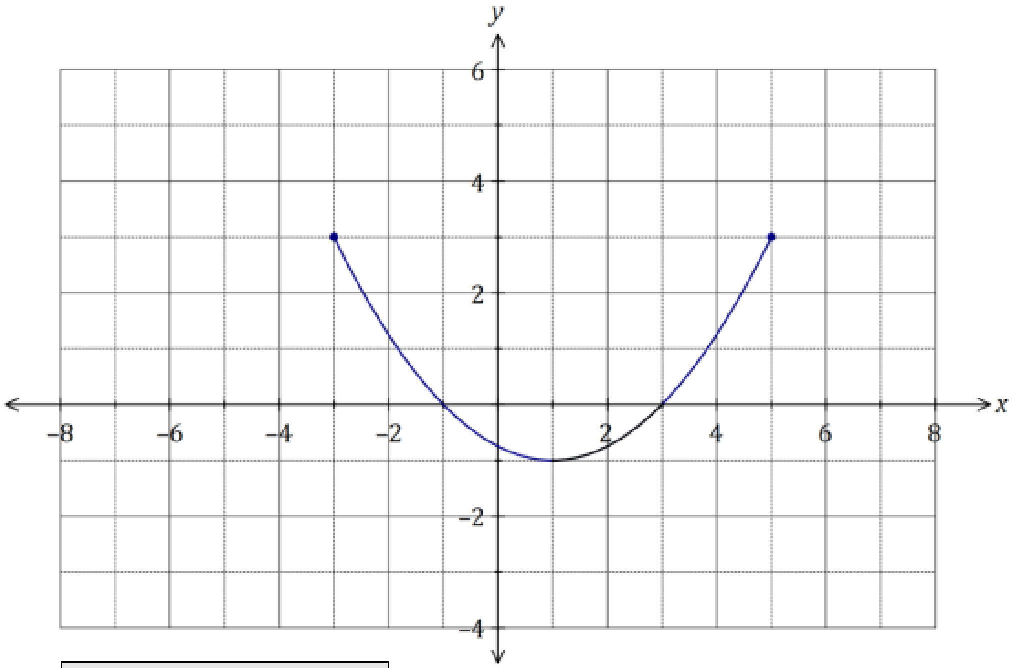
(b) Derive the Cartesian equation of the path of the body. (4 marks)

Solution
$y = 1 - 2\cos(2t) = 1 - 2(1 - 2\sin^2(t)) = 4\sin^2(t) - 1$ $x = 1 + 4\sin(t) \Rightarrow \sin^2(t) = \frac{(x-1)^2}{16}$ $y = \frac{(x-1)^2}{4} - 1 \text{ where } -3 \le x \le 5$
Specific behaviours
✓ expression for y in terms of $\sin^2(t)$ ✓ expression for $\sin^2(t)$ in terms of x ✓ Cartesian equation   ✓ restricts domain or range

(c) Complete the following plot to show the path taken by the body. (2 marks)



(c) Complete the following plot to show the path taken by the body. (2 marks)



Solution
See graph
Specific behaviours
✓ parabola
✓ correct domain

Question 18 **(9 marks)**

A pole and a wall stand vertically on horizontal ground. A small projectile is launched from the pole at a height of 3.16 m above the ground and sometime later hits the wall at a height of 1.79 m above the ground. The projectile has an initial velocity of 32 ms^{-1} at an angle of $36°$ above the horizontal.

Any effects of air resistance and wind can be ignored. Let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors in the horizontal and vertical (upward) directions and the foot of the pole be at $(0, 0)$.

The acceleration acting on the projectile is given by $\mathbf{a}(t) = -9.8\mathbf{j}$ ms^{-2}.

(a) Use the information above to derive vector equations for the velocity $\mathbf{v}(t)$ and displacement $\mathbf{r}(t)$ of the projectile at any time t. (3 marks)

(b) Determine

(i) the time that the projectile takes to travel between the pole and the wall. (2 marks)

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(a) Use the information above to derive vector equations for the velocity $\mathbf{v}(t)$ and displacement $\mathbf{r}(t)$ of the projectile at any time t. (3 marks)

Solution
$\mathbf{v}(t) = (32\cos 36°)\mathbf{i} + (32\sin 36° - 9.8t)\mathbf{j}$ $\mathbf{r}(t) = (32t\cos 36°)\mathbf{i} + (3.16 + 32t\sin 36° - 4.9t^2)\mathbf{j}$
Specific behaviours
✓ integrates correctly twice   ✓ correct expression for $\mathbf{v}(t)$ ✓ correct expression for $\mathbf{r}(t)$

(b) Determine

(i) the time that the projectile takes to travel between the pole and the wall. (2 marks)

Solution
$3.16 + 32t\sin 36° - 4.9t^2 = 1.79$ $t = 3.91$ s
Specific behaviours
✓ equates $\mathbf{j}$ coefficient of displacement to height   ✓ solves for time

(ii) the speed of the projectile at the instant it hits the wall. (2 marks)

(iii) the length of the path taken by the projectile between the pole and the wall. (2 marks)

(ii) the speed of the projectile at the instant it hits the wall. (2 marks)

Solution
$\mathbf{v}(3.91) = 25.89\mathbf{i} - 19.51\mathbf{j}$ $\lvert\mathbf{v}(3.91)\rvert = 32.4\ \mathrm{ms}^{-1}$
Specific behaviours
✓ indicates velocity at correct time ✓ correct speed

(iii) the length of the path taken by the projectile between the pole and the wall. (2 marks)

Solution
$l = \int_0^{3.91} \lvert\mathbf{v}(t)\rvert\, dt$ $= 109.8\ \mathrm{m}$
Specific behaviours
✓ writes integral for total distance ✓ correct distance

Question 17 **(8 marks)**

The velocity vector of a small body at time t seconds is $\mathbf{v}(t) = 2\sin(3t)\,\mathbf{i} - 4\cos(3t)\,\mathbf{j}$ ms^{-1}.
Initially, the body has position vector $\mathbf{i} - 2\mathbf{j}$.

(a) Determine the acceleration vector for the body when $t = \dfrac{2\pi}{9}$. (2 marks)

(b) Show that the maximum speed of the body is 4 ms^{-1}. (3 marks)

(c) Determine the distance the body travels between $t = 0$ and the first instant after this time that the body returns to its initial position, rounding your answer to the nearest cm. (3 marks)

Question 17 **(8 marks)**

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Initially, the body has position vector $\mathbf{i} - 2\mathbf{j}$.

(a) Determine the acceleration vector for the body when $t = \frac{2\pi}{9}$. (2 marks)

Solution
$\mathbf{a}(t) = 6\cos(3t)\,\mathbf{i} + 12\sin(3t)\,\mathbf{j}$ $\mathbf{a}\left(\frac{2\pi}{9}\right) = (-3, 6\sqrt{3})$
Specific behaviours
✓ acceleration vector ✓ acceleration

(b) Show that the maximum speed of the body is 4 ms^{-1}. (3 marks)

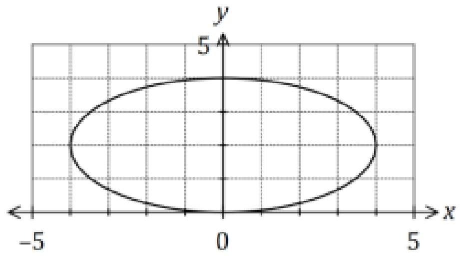
Solution
$s^2 = 4\sin^2(3t) + 16\cos^2(3t)$ $= 4\sin^2(3t) + 4\cos^2(3t) + 12\cos^2(3t)$ $= 4 + 12\cos^2(3t)$ s^2 is max when $\cos^2(3t) = 1$ $s_{max} = \sqrt{4+12} = 4$ m/s
Specific behaviours
✓ expression for s or s^2 ✓ simplifies expression ✓ indicates maximum with justification

(c) Determine the distance the body travels between $t = 0$ and the first instant after this time that the body returns to its initial position, rounding your answer to the nearest cm. (3 marks)

Solution
Period of motion is $\frac{2\pi}{3}$ $d = \int_0^{\frac{2\pi}{3}} \sqrt{4\sin^2(3t) + 16\cos^2(3t)}\, dt$ $d = 6.46$ m
Specific behaviours
✓ period ✓ indicates valid integral ✓ distance (*no rounding penalty*)

Question 20 **(8 marks)**

The position vector of a boat motoring on a lake is given by $\mathbf{r}(t) = 4\sin(2t)\,\mathbf{i} + (4 - 4\cos^2(t))\mathbf{j}$, where t is the time, in hours, after it leaves $(0, 0)$ and distances are in kilometres. The path of the boat is shown below, where the shoreline is represented by the line $y = 0$.



(a) Express the path of the particle as a Cartesian equation. (3 marks)

(b) On the graph above, mark the position of the boat when it is first 3 km from the shoreline and indicate the direction it is travelling. (1 mark)

(c) Determine the speed of the boat when it is first 3 km from the shoreline. (4 marks)

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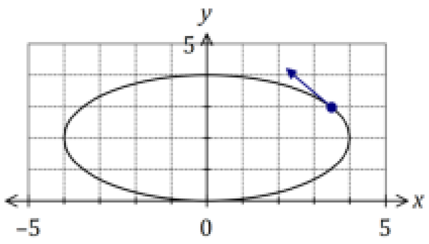
The position vector of a boat motoring on a lake is given by $\mathbf{r}(t) = 4\sin(2t)\,\mathbf{i} + (4 - 4\cos^2(t))\mathbf{j}$, where t is the time, in hours, after it leaves $(0, 0)$ and distances are in kilometres. The path of the boat is shown below, where the shoreline is represented by the line $y = 0$.



Solution (b)
See graph
Specific behaviours
✓ position and direction

(a) Express the path of the particle as a Cartesian equation. (3 marks)

Solution

$$4 - 4\cos^2(t) = 2 - 2\cos(2t)$$

$$\frac{x}{4} = \sin(2t) \text{ and } \frac{y-2}{-2} = \cos(2t)$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

Specific behaviours

✓ uses double angle identity
✓ isolates $\sin(2t)$ and $\cos(2t)$
✓ Cartesian equation

(b) On the graph above, mark the position of the boat when it is first 3 km from the shoreline and indicate the direction it is travelling. (1 mark)

(c) Determine the speed of the boat when it is first 3 km from the shoreline. (4 marks)

Solution

$$4 - 4\cos^2(t) = 3 \Rightarrow t = \frac{\pi}{3}$$

$$\mathbf{v}(t) = 8\cos(2t)\,\mathbf{i} + 8\cos(t)\sin(t)\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{3}\right) = -4\mathbf{i} + 2\sqrt{3}\mathbf{j}$$

$$s = 2\sqrt{7} \approx 5.29 \text{ km/h}$$

Specific behaviours

✓ determines time
✓ obtains velocity vector
✓ velocity vector at time
✓ speed