Question 9 **(6 marks)**

Particles A and B are moving with constant velocities and have initial positions $\begin{pmatrix} -8 \\ 2 \\ 10 \end{pmatrix}$ m and $\begin{pmatrix} 7 \\ 7 \\ -15 \end{pmatrix}$ m respectively. 2 seconds later A is at $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ m.

(a) Determine the velocity of A. (1 mark)

The velocity of B is $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ m/s.

(b) Show that the paths of A and B cross, state the position vector of this point, and explain whether the particles collide. (5 marks)

Question 9 **(6 marks)**

Particles A and B are moving with constant velocities and have initial positions $\begin{pmatrix}-8\\2\\10\end{pmatrix}$ m and $\begin{pmatrix}7\\7\\-15\end{pmatrix}$ m respectively. 2 seconds later A is at $\begin{pmatrix}0\\-2\\4\end{pmatrix}$ m.

(a) Determine the velocity of A. (1 mark)

Solution
$\underset{\sim}{v}_A = \frac{1}{2}\left(\begin{pmatrix}0\\-2\\4\end{pmatrix} - \begin{pmatrix}-8\\2\\10\end{pmatrix}\right) = \begin{pmatrix}4\\-2\\-3\end{pmatrix}$
Specific behaviours
✓ correct velocity

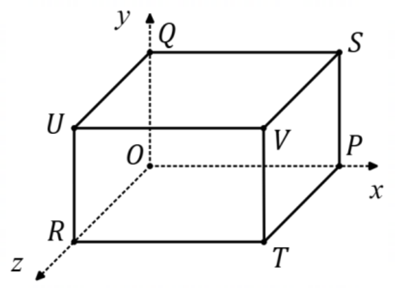
The velocity of B is $\begin{pmatrix}1\\-3\\2\end{pmatrix}$ m/s.

(b) Show that the paths of A and B cross, state the position vector of this point, and explain whether the particles collide. (5 marks)

Solution
$\underset{\sim}{r}_A(t) = \begin{pmatrix}-8\\2\\10\end{pmatrix} + t\begin{pmatrix}4\\-2\\-3\end{pmatrix}, \quad \underset{\sim}{r}_B(s) = \begin{pmatrix}7\\7\\-15\end{pmatrix} + s\begin{pmatrix}1\\-3\\2\end{pmatrix}$ For paths to cross we require $\underset{\sim}{r}_A = \underset{\sim}{r}_B$. Equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ coefficients and solving simultaneously: $4t - 8 = s + 7, \quad 2 - 2t = 7 - 3s \Rightarrow t = 5, s = 5$ Check $\underset{\sim}{k}$ coefficients are equal with these values of t and s: $t = 5 \Rightarrow 10 - 3(5) = -5, \quad s = 5 \Rightarrow -15 + 2(5) = -5$ Because $\underset{\sim}{r}_A(5) = \underset{\sim}{r}_B(5) = \begin{pmatrix}12\\-8\\-5\end{pmatrix}$, their paths cross at this point and because both particles reach this point at the same time they collide.
Specific behaviours
✓ indicates equations for both paths   ✓ forms two equations using different time parameters   ✓ solves equations and checks third coefficient   ✓ correct position vector   ✓ explains why paths cross and whether particles collide

Question 15 **(6 marks)**

The diagram shows a right rectangular prism.



Relative to vertex O, vertices P, Q and R have position vectors $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$.

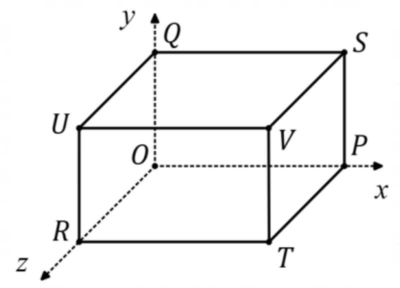
(a) Determine vectors $\overrightarrow{PU}$ and $\overrightarrow{QT}$ in terms of p, q and r. (1 mark)

(b) Use a vector method to show that diagonals PU and QT bisect each other. (3 marks)

(c) Determine the relationship between p, q and r when $\overrightarrow{PU}$ and $\overrightarrow{QT}$ are perpendicular. (2 marks)

Question 15 **(6 marks)**

The diagram shows a right rectangular prism.



Relative to vertex O, vertices P, Q and R have position vectors $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$.

(a) Determine vectors $\overrightarrow{PU}$ and $\overrightarrow{QT}$ in terms of p, q and r. (1 mark)

Solution
$\overrightarrow{PU} = \begin{pmatrix} -p \\ q \\ r \end{pmatrix}, \quad \overrightarrow{QT} = \begin{pmatrix} p \\ -q \\ r \end{pmatrix}$
Specific behaviours
✓ correct vectors

(b) Use a vector method to show that diagonals PU and QT bisect each other. (3 marks)

Solution
Midpoint of line PU:
$\underset{\sim}{m}_1 = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PU} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -p \\ q \\ r \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$
Midpoint of line QT:
$\underset{\sim}{m}_2 = \overrightarrow{OQ} + \frac{1}{2}\overrightarrow{QT} = \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} p \\ -q \\ r \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$
Since $\underset{\sim}{m}_1 = \underset{\sim}{m}_2$ then diagonals are coincident at their midpoints and so bisect each other.
Specific behaviours
✓ develops expression for position vector of one midpoint ✓ develops expression for position vector of one midpoint ✓ shows midpoints are coincident and hence bisect

(c) Determine the relationship between p, q and r when $\overrightarrow{PU}$ and $\overrightarrow{QT}$ are perpendicular. (2 marks)

Solution
For vectors to be perpendicular, require $\begin{pmatrix} -p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} p \\ -q \\ r \end{pmatrix} = 0$.
Hence $-p^2 - q^2 + r^2 = 0$ or $r^2 = p^2 + q^2$.
Specific behaviours
✓ indicates condition for perpendicularity ✓ correct relationship

Question 17 **(7 marks)**

Plane Π has equation $2x - y - 3z = 13$ and point A has coordinates $(1, 5, 4)$.

(a) Determine the coordinates of the point in Π that is closest to A. (4 marks)

Vector $\underset{\sim}{v}$ lies in plane Π, is perpendicular to the line $\dfrac{x+1}{-1} = \dfrac{y-3}{2} = \dfrac{z-1}{2}$ and $\left|\underset{\sim}{v}\right| = \sqrt{26}$.

(b) Let $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$. Determine the value of coefficients a, b and c, given that $a > c$.

(3 marks)

Question 17 **(7 marks)**

Plane Π has equation $2x - y - 3z = 13$ and point A has coordinates $(1, 5, 4)$.

(a) Determine the coordinates of the point in Π that is closest to A. (4 marks)

Solution
Equation of line through $A \perp \Pi$ is $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$. Intersects with plane $2x - y - 3z = 13$ when $2(1 + 2\mu) - (5 - \mu) - 3(4 - 3\mu) = 13 \Rightarrow \mu = 2$ $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ Coordinates are $(5, 3, -2)$.
Specific behaviours
✓ equation of line through point ✓ uses intersection to obtain equation in μ ✓ solves for μ ✓ correct coordinates

Vector $\underset{\sim}{v}$ lies in plane Π, is perpendicular to the line $\dfrac{x+1}{-1} = \dfrac{y-3}{2} = \dfrac{z-1}{2}$ and $\left|\underset{\sim}{v}\right| = \sqrt{26}$.

(b) Let $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$. Determine the value of coefficients a, b and c, given that $a > c$. (3 marks)

Solution
$\underset{\sim}{v}$ lies in plane $\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = 0 \Rightarrow 2\mathrm{a} - \mathrm{b} - 3\mathrm{c} = 0$ $\underset{\sim}{v} \perp$ line $\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0 \Rightarrow -\mathrm{a} + 2\mathrm{b} + 2\mathrm{c} = 0$ $\left|\underset{\sim}{v}\right| = \sqrt{26} \Rightarrow a^2 + b^2 + c^2 = 26$ Solving equations simultaneously: $\underset{\sim}{v} = \pm \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ $a = 4, \quad b = -1, \quad c = 3.$
Specific behaviours
✓ two equations using normals ✓ equation using magnitude ✓ correct set of values

Question 8 **(7 marks)**

Point B lies on a sphere with centre O, radius r and diameter AC.

(a) Let $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Use a vector method to prove that CB is perpendicular to AB.

(4 marks)

(b) If the position vectors of A, B and C are $\begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ -3 \\ 3 \end{pmatrix}$ respectively, determine the value of the constant k.

(3 marks)

Question 8 **(7 marks)**

Point B lies on a sphere with centre O, radius r and diameter AC.

(a) Let $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Use a vector method to prove that CB is perpendicular to AB.

(4 marks)

Solution

$$\overrightarrow{CB} = \mathbf{b} - \mathbf{c}, \qquad \overrightarrow{AB} = \mathbf{c} + \mathbf{b}$$

$$\begin{aligned}\overrightarrow{CB} \cdot \overrightarrow{AB} &= (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{c} + \mathbf{b}) \\ &= \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{b} \\ &= |\mathbf{b}|^2 - |\mathbf{c}|^2\end{aligned}$$

But B, C points on sphere so that $|\mathbf{b}| = |\mathbf{c}| = r$.

Hence $\overrightarrow{CB} \cdot \overrightarrow{AB} = |\mathbf{b}|^2 - |\mathbf{c}|^2 = r^2 - r^2 = 0$ and since $\left|\overrightarrow{CB}\right| \neq 0$ and $\left|\overrightarrow{AB}\right| \neq 0$ we deduce that the angle between CB and AB must be $90°$.

Specific behaviours

- ✓ correct vectors for $\overrightarrow{CB}$ and $\overrightarrow{AB}$
- ✓ forms and expands scalar product
- ✓ uses $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$ explains that $|\mathbf{b}| = |\mathbf{c}| = r$
- ✓ deduces perpendicularity

(b) If the position vectors of A, B and C are $\begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ -3 \\ 3 \end{pmatrix}$ respectively, determine the value of the constant k.

(3 marks)

Solution

$$\overrightarrow{CB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -7 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix}, \qquad \overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 2-k \end{pmatrix}$$

$$\overrightarrow{BA} \cdot \overrightarrow{CA} = \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 5 \\ 2-k \end{pmatrix} = k + 4$$

Hence $k + 4 = 0 \Rightarrow k = -4$.

Specific behaviours

- ✓ vectors for $\overrightarrow{CB}$ and $\overrightarrow{AB}$
- ✓ calculates scalar product
- ✓ correct value of k

Question 11 **(7 marks)**

A small body is moving with constant velocity in space so that initially it is located at $(5, -7, -5)$ and four seconds later it is at $(13, -11, 7)$, where all dimensions are in metres.

(a) Determine a vector equation for the position of the small body at time t seconds. (2 marks)

A laser beam shines along the line with equation $\dfrac{x+1}{3} = \dfrac{y-2}{-2} = \dfrac{z-16}{2}$

(b) Write the vector equation of this line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. (1 mark)

(c) Show that the small body passes through the laser beam and state where this occurs. (4 marks)

Question 11 **(7 marks)**

A small body is moving with constant velocity in space so that initially it is located at $(5, -7, -5)$ and four seconds later it is at $(13, -11, 7)$, where all dimensions are in metres.

(a) Determine a vector equation for the position of the small body at time t seconds. (2 marks)

Solution
$\frac{1}{4}\left[\begin{pmatrix}13\\-11\\7\end{pmatrix}-\begin{pmatrix}5\\-7\\-5\end{pmatrix}\right]=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ $\mathbf{r}=\begin{pmatrix}5\\-7\\-5\end{pmatrix}+t\begin{pmatrix}2\\-1\\3\end{pmatrix}$
Specific behaviours
✓ calculates velocity vector   ✓ correct equation for position of body

A laser beam shines along the line with equation $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z-16}{2}$

(b) Write the vector equation of this line in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. (1 mark)

Solution
$\mathbf{r}=\begin{pmatrix}-1\\2\\16\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$
Specific behaviours
✓ correct vector form

(c) Show that the small body passes through the laser beam and state where this occurs. (4 marks)

Solution
Equating $\mathbf{i}$ and $\mathbf{j}$ coefficients: $5+2t=3\lambda-1$ $-7-t=2-2\lambda$ Solving simultaneously gives $t=15, \lambda=12$.   Using these values, the body is at $\begin{pmatrix}5\\-7\\-5\end{pmatrix}+15\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}35\\-22\\40\end{pmatrix}$ and the laser passes through $\begin{pmatrix}-1\\2\\16\end{pmatrix}+12\begin{pmatrix}3\\-2\\2\end{pmatrix}=\begin{pmatrix}35\\-22\\40\end{pmatrix}$.   Hence as these points are coincident, the small body passes through the laser beam at this point.
Specific behaviours
✓ equates two coefficients   ✓ solves simultaneously   ✓ calculates both $\mathbf{k}$ coefficients or points and states they are same   ✓ states point of coincidence

Question 13 **(6 marks)**

Points A, B and C lie in plane Π with position vectors $\begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ respectively.

(a) Determine the vector equation for plane Π in the form $\mathbf{r} \cdot \mathbf{n} = k$. (3 marks)

The equation of line L is $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$.

(b) Determine, if possible, where line L intersects with plane Π. If not possible, explain why not. (3 marks)

Question 13 **(6 marks)**

Points A, B and C lie in plane Π with position vectors $\begin{pmatrix}7\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\-2\\0\end{pmatrix}$ respectively.

(a) Determine the vector equation for plane Π in the form $\mathbf{r}\cdot\mathbf{n}=k$. (3 marks)

Solution

$$\overrightarrow{BA}=\begin{pmatrix}7\\0\\0\end{pmatrix}-\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}6\\1\\2\end{pmatrix},\qquad \overrightarrow{CA}=\begin{pmatrix}7\\0\\0\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$

$$\begin{pmatrix}4\\2\\0\end{pmatrix}\times\begin{pmatrix}6\\1\\2\end{pmatrix}=\begin{pmatrix}4\\-8\\-8\end{pmatrix}=4\begin{pmatrix}1\\-2\\-2\end{pmatrix}\Rightarrow\mathbf{n}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}$$

$$k=\begin{pmatrix}1\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}7\\0\\0\end{pmatrix}=7$$

Hence equation of Π is $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\-2\end{pmatrix}=7$

Specific behaviours

✓ obtains two vectors in the plane
✓ obtains normal to plane
✓ obtains value of k and states equation of plane

The equation of line L is $\mathbf{r}=\begin{pmatrix}5\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\3\end{pmatrix}$.

(b) Determine, if possible, where line L intersects with plane Π. If not possible, explain why not. (3 marks)

Solution

Substitute equation of line into equation of plane:

$$\begin{pmatrix}5+2\lambda\\2-2\lambda\\-3+3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-2\end{pmatrix}=7$$

Hence

$$5+2\lambda-4+4\lambda+6-6\lambda=7$$
$$7=7$$

Since the equation is true and the solution of the equation is independent of λ then all values of λ are solutions. Hence line L lies in plane Π and all points on L are points of intersection.

Specific behaviours

✓ substitutes equation of line into equation of plane
✓ simplifies
✓ reasons that all points on the line lie in the plane

Question 17 **(8 marks)**

Plane Π has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Show how to deduce that the Cartesian equation of plane Π is $x - 5y - 3z = 1$. (3 marks)

The line through $A(1, 4, 5)$ and point B is perpendicular to Π, and the midpoint of AB lies in Π.

(b) Determine the coordinates of B. (5 marks)

Question 17 **(8 marks)**

Plane Π has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Show how to deduce that the Cartesian equation of plane Π is $x - 5y - 3z = 1$. (3 marks)

Solution
Require $\mathbf{r} \cdot \mathbf{n} = k$. Direction vectors lie in plane, so normal will be: $$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$ And $k = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = 1$ Hence equation of plane is $x - 5y - 3z = 1$
Specific behaviours
✓ indicates two direction vectors lie in plane   ✓ cross product to obtain normal   ✓ dot product with point, derives equation

The line through $A(1, 4, 5)$ and point B is perpendicular to Π, and the midpoint of AB lies in Π.

(b) Determine the coordinates of B. (5 marks)

Solution
Equation of line through A, B is $$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$ Intersection of line and plane when $$\begin{pmatrix} 1+t \\ 4-5t \\ 5-3t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = 1$$ $$1 + t - 20 + 25t - 15 + 9t = 1 \Rightarrow t = 1$$ Since $t = 1$ for midpoint, then $t = 2$ for B: $$\mathbf{r}_B = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$$ $$B(3, -6, -1)$$
Specific behaviours
✓ equation of line   ✓ equation for intersection   ✓ solves for t ✓ value of t for B ✓ coordinates of B

Question 19 **(9 marks)**

The position vectors of particles A and B (in centimetres) at time t seconds, $t \geq 0$, are

$$\mathbf{r}_A = 7\mathbf{i} + 15\mathbf{j} + t(0.5\mathbf{i} - 2\mathbf{j}) \text{ and } \mathbf{r}_B = 5\mathbf{i} + 2\mathbf{j} + t\big((t-6)\mathbf{i} - \mathbf{j}\big).$$

(a) Show that A is moving with constant speed and determine this speed. (2 marks)

(b) Determine the Cartesian path of B. (3 marks)

(c) Determine the position vector of the point where the paths of the particles cross. (4 marks)

Question 19 **(9 marks)**

The position vectors of particles A and B (in centimetres) at time t seconds, $t \geq 0$, are

$$\mathbf{r}_A = 7\mathbf{i} + 15\mathbf{j} + t(0.5\mathbf{i} - 2\mathbf{j}) \text{ and } \mathbf{r}_B = 5\mathbf{i} + 2\mathbf{j} + t\big((t-6)\mathbf{i} - \mathbf{j}\big).$$

(a) Show that A is moving with constant speed and determine this speed. (2 marks)

Solution
$\mathbf{v}_A = \begin{pmatrix} 0.5 \\ -2 \end{pmatrix}$, which is independent of t and hence constant. $$\lvert\mathbf{v}_A\rvert = \frac{\sqrt{17}}{2} \approx 2.06 \text{ cm/s}$$
Specific behaviours
✓ obtains constant velocity vector ✓ correct speed

(b) Determine the Cartesian path of B. (3 marks)

Solution
$y = 2 - t \Rightarrow t = 2 - y$ $x = 5 + t^2 - 6t$ $x = 5 + (2-y)^2 - 6(2-y)$ $x = y^2 + 2y - 3, \quad y \leq 2$ (as $y = 2 - t, t \geq 0$)
Specific behaviours
✓ expressions for x and y in terms of t ✓ eliminates t ✓ simplifies, noting domain

(c) Determine the position vector of the point where the paths of the particles cross. (4 marks)

Solution
Position of A after s seconds: $$\mathbf{r}_A = \begin{pmatrix} 7 + 0.5s \\ 15 - 2s \end{pmatrix}$$ Position of B after t seconds: $$\mathbf{r}_B = \begin{pmatrix} 5 + t^2 - 6t \\ 2 - t \end{pmatrix}$$ Hence require: $$7 + 0.5s = 5 + t^2 - 6t$$ $$15 - 2s = 2 - t$$ Solving simultaneously $(s, t > 0)$: $$s = 10, t = 7$$ Paths intersect at $$\begin{pmatrix} 7 + 0.5(10) \\ 15 - 2(10) \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$
Specific behaviours
✓ positions using different variables for time ✓ equates coefficients ✓ solves for times ✓ determines position vector for intersection

Alternative Solution
Cartesian path of A: $$x = 7 + 0.5t \Rightarrow t = 2x - 14$$ $$y = 15 - 2t = 15 - 2(2x - 14)$$ Solving simultaneously with eqn from (b): $$\{x = 12, y = -5\}, \left\{x = \frac{161}{16}, y = \frac{11}{4}\right\}$$ Ignore second solution since $y = \frac{11}{4} > 2$ using domain restriction from (b). Hence paths intersect at $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$.
Specific behaviours
✓ Cartesian path for A ✓ solves simultaneously ✓ checks for $y \leq 2$ ✓ states position vector of intersection

Question 21 **(4 marks)**

Sphere S of radius 3 has its centre at the origin.

Line L has equation $\mathbf{r} = \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, where k is a positive constant.

Prove that for L to be a tangent to S, then $k = \dfrac{3\sqrt{2}}{2}$.

Question 21 **(4 marks)**

Sphere S of radius 3 has its centre at the origin.

Line L has equation $\mathbf{r} = \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, where k is a positive constant.

Prove that for L to be a tangent to S, then $k = \dfrac{3\sqrt{2}}{2}$.

Solution
$\lvert \mathbf{r} - \mathbf{0} \rvert = 3 \Rightarrow \left\lvert \begin{pmatrix} -k \\ k \\ -k \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\rvert = 3$ $(\lambda - k)^2 + (2\lambda + k)^2 + (-2\lambda - k)^2 = 9$ $3\lambda^2 + 2k\lambda + k^2 - 3 = 0$ For tangent, require single solution for λ and so discriminant of quadratic in λ must be zero: $(2k)^2 - 4(3)(k^2 - 3) = 0$ $4k^2 - 12k^2 + 36 = 0$ $k^2 = \dfrac{9}{2} \Rightarrow k = \dfrac{3\sqrt{2}}{2}$
Specific behaviours
✓ substitutes equation of line into equation for sphere ✓ equation for magnitude, simplified ✓ explains requirement for one solution to quadratic ✓ solves discriminant equation for positive k

Question 10 **(6 marks)**

Consider the following system of equations, where a and b are constants.

$$\begin{aligned} x - 2y + z &= 1 \\ 2x + 2y - z &= 5 \\ 2x + ay + 2z &= b \end{aligned}$$

For each of the following cases, determine the number of solutions that exist for the system and briefly interpret the system geometrically.

(a) $a = 2,\ b = -4.$ (3 marks)

(b) $a = -4,\ b = -2.$ (3 marks)

Question 10 **(6 marks)**

Consider the following system of equations, where a and b are constants.

$$\begin{aligned} x - 2y + z &= 1 \\ 2x + 2y - z &= 5 \\ 2x + ay + 2z &= b \end{aligned}$$

For each of the following cases, determine the number of solutions that exist for the system and briefly interpret the system geometrically.

(a) $a = 2,\ b = -4.$ (3 marks)

Solution
The system has 1 solution.
The system represents three planes that intersect at the point $(2, -1, -3)$.
$\begin{cases} x-2y+z=1 \\ 2x+2y-z=5 \\ 2x+a\times y+2z=b \end{cases} \Big\vert x,y,z$
$\left\{x=2,\ y=\dfrac{b-2}{a+4},\ z=\dfrac{-(a-2\cdot b+8)}{a+4}\right\}$
$\text{ans} \mid a=2 \mid b=-4$
$\{x=2,\ y=-1,\ z=-3\}$
Specific behaviours
✓ number of solutions
✓ interpretation
✓ interpretation includes point of intersection

(b) $a = -4,\ b = -2.$ (3 marks)

Solution
The system has no solutions.
The system represents two parallel planes that are cut by a third non-parallel plane.
$\text{ans} \mid a=-4 \mid b=-2$
$\{x=2,\ \text{Undefined},\ \text{Undefined}\}$
Specific behaviours
✓ number of solutions
✓ interpretation
✓ interpretation refers to parallel planes

Question 15 **(8 marks)**

The position vectors of two particles at time t are given below, where a is a constant.

$$\mathbf{r}_A = 5\mathbf{i} + \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \text{ and } \mathbf{r}_B = a\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} + t(2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

The paths of the particles cross at P but the particles do not meet.

(a) Determine the value of the constant a and the position vector of P. (5 marks)

(b) Show that the point $(0, -8, 1)$ lies in the plane containing the two lines. (3 marks)

Question 15 **(8 marks)**

The position vectors of two particles at time t are given below, where a is a constant.

$$\mathbf{r}_A = 5\mathbf{i} + \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \text{ and } \mathbf{r}_B = a\mathbf{i} - 6\mathbf{j} + 6\mathbf{k} + t(2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

The paths of the particles cross at P but the particles do not meet.

(a) Determine the value of the constant a and the position vector of P. (5 marks)

Solution
$\mathbf{r}_A = \begin{pmatrix} 5+t \\ 1-t \\ -4+3t \end{pmatrix}, \mathbf{r}_B = \begin{pmatrix} a+2s \\ -6+5s \\ 6-4s \end{pmatrix}$
Hence $1 - t = -6 + 5s$ and $-4 + 3t = 6 - 4s \Rightarrow t = 2, s = 1$
Using $\mathbf{i}$ coefficient: $5 + 2 = a + 2(1) \Rightarrow a = 5$
$\mathbf{r}_A(2) \Rightarrow \overrightarrow{OP} = \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}$
Specific behaviours
✓ replaces one t with another variable (e.g. s)
✓ uses $\mathbf{j}$ and $\mathbf{k}$ components to write pair of equations
✓ solves equations for t and s
✓ substitutes into $\mathbf{i}$ components and determines a
✓ uses t or s to find P

ClassPad Solution
Method 1
1
$\begin{cases} 5+t=a+2s \\ 1-t=-6+5s \\ -4+3t=6-4s \end{cases}\Big\vert_{t,s,a}$
{t=2, s=1, a=5}
Method 2
2
solve$\left(\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}+t\times\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}=\begin{bmatrix} a \\ -6 \\ 6 \end{bmatrix}+s\times\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}, \{t,s,a\}\right)$
{t=2, s=1, a=5}
$\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}+t\times\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}\Big\vert t=2$
$\begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$
crossP$\left(\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}\right)$
$\begin{bmatrix} 11 \\ -10 \\ -7 \end{bmatrix}$
dotP$\left(\begin{bmatrix} 11 \\ -10 \\ -7 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}\right)$
73

(b) Show that the point $(0, -8, 1)$ lies in the plane containing the two lines. (3 marks)

Solution
$(2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = (11\mathbf{i} - 10\mathbf{j} - 7\mathbf{k})$
$(11\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}) \cdot (5\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 73$
Equation of plane is $\mathbf{r} \cdot (11\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}) = 73$
$(11\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}) \cdot (-8\mathbf{j} + \mathbf{k}) = 0 + 80 - 7 = 73$
Hence point lies in plane.
Specific behaviours
✓ calculates normal to plane
✓ calculates constant and writes equation of plane
✓ substitutes point, showing equation satisfied

Question 17 **(8 marks)**

Sphere S has diameter PQ, where P and Q have coordinates $(6, -2, -3)$ and $(-2, 4, 1)$ respectively.

(a) Determine the vector equation of the sphere. (3 marks)

(b) Show that the point $(5, 5, 2)$ lies outside the sphere. (2 marks)

(c) Show that the line with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ is tangential to the sphere.

(3 marks)

Question 17 **(8 marks)**

Sphere S has diameter PQ, where P and Q have coordinates $(6, -2, -3)$ and $(-2, 4, 1)$ respectively.

(a) Determine the vector equation of the sphere. (3 marks)

Solution
$\overrightarrow{OC} = \frac{1}{2}(P + Q) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $r = \left\lvert \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rvert = \sqrt{29}$ $\left\lvert \mathbf{r} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rvert = \sqrt{29}$
Specific behaviours
✓ indicates position of centre   ✓ indicates radius   ✓ correct vector equation

(b) Show that the point $(5, 5, 2)$ lies outside the sphere. (2 marks)

Solution
$\left\lvert \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rvert = \sqrt{34}$ Since $\sqrt{34} > \sqrt{29}$, point lies outside sphere.
Specific behaviours
✓ calculates distance   ✓ explains result

(c) Show that the line with equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ is tangential to the sphere. (3 marks)

Solution
$\left\lvert \begin{pmatrix} 5 + 5\lambda \\ 1 \\ 6 + 2\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rvert = \sqrt{29}$ $(3 + 5\lambda)^2 + (0)^2 + (7 + 2\lambda)^2 = 29 \Rightarrow \lambda = -1$ As λ has a unique value, the line only intersects sphere at one point and so it must be a tangent.
Specific behaviours
✓ substitutes line equation into sphere equation   ✓ solves for λ ✓ explains result

Question 20 **(7 marks)**

(a) Point A has coordinates $(6, 0, -7)$ and plane Π has equation $2x - y - 2z = 8$. Determine

(i) a vector equation for the straight line through A perpendicular to Π.

(1 mark)

(ii) the perpendicular distance of A from Π. (3 marks)

(b) Prove that the perpendicular distance from the origin to the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = k$ (where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane) is k. (3 marks)

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(a) Point A has coordinates $(6, 0, -7)$ and plane Π has equation $2x - y - 2z = 8$. Determine

(i) a vector equation for the straight line through A perpendicular to Π.

(1 mark)

Solution
$\mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$
Specific behaviours
✓ correct equation

(ii) the perpendicular distance of A from Π. (3 marks)

Solution
$2(6 + 2\lambda) - (-\lambda) - 2(-7 - 2\lambda) = 8 \Rightarrow \lambda = -2$ $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$ $\lvert\mathbf{r_1}\rvert = 6$
Specific behaviours
✓ substitutes equation of line into equation of plane ✓ determines vector from point to plane ✓ calculates distance

(b) Prove that the perpendicular distance from the origin to the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = k$ (where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane) is k. (3 marks)

Solution
Equation of line perpendicular to plane through origin is $\mathbf{r} = \lambda\hat{\mathbf{n}}$. Line will intersect plane when $\lambda\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = k$. Hence $\lambda = k$ since $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = \lvert\hat{\mathbf{n}}\rvert\lvert\hat{\mathbf{n}}\rvert = 1$. Thus, closest point to origin is $\mathbf{r} = k\,\hat{\mathbf{n}}$ and distance $d = \lvert\mathbf{r}\rvert = k\lvert\,\hat{\mathbf{n}}\,\rvert = k$
Specific behaviours
✓ substitutes equation of line into equation of plane ✓ simplifies $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}$ and uses to obtain closest point to origin ✓ simplifies expression for distance

Alternative solution
An alternative proof could involve using $\lvert\mathbf{r}\rvert\lvert\hat{\mathbf{n}}\rvert\cos\theta$ and explaining why $\theta = 0$.

Question 10 **(5 marks)**

(a) The vector equation of a curve is given by $\mathbf{r}(\mu) = (\mu + 3)\mathbf{i} + (\mu^2 - 1)\mathbf{j}$. Determine the corresponding Cartesian equation for the curve. (2 marks)

(b) A sphere has Cartesian equation $x^2 + y^2 + z^2 - 6x + 4y + 10z = 0$. Determine the vector equation of the sphere. (3 marks)

Question 10 **(5 marks)**

(a) The vector equation of a curve is given by $\mathbf{r}(\mu) = (\mu + 3)\mathbf{i} + (\mu^2 - 1)\mathbf{j}$. Determine the corresponding Cartesian equation for the curve. (2 marks)

Solution
$x = \mu + 3 \Rightarrow \mu = x - 3$ $y = \mu^2 - 1$ $y = (x - 3)^2 - 1$
Specific behaviours
✓ express μ in terms of x ✓ Cartesian equation

(b) A sphere has Cartesian equation $x^2 + y^2 + z^2 - 6x + 4y + 10z = 0$. Determine the vector equation of the sphere. (3 marks)

Solution
$(x - 3)^2 + (y + 2)^2 + (z + 5)^2 = 3^2 + 2^2 + 5^2 = 38$ $\left\lvert \mathbf{r} - \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \right\rvert = \sqrt{38}$
Specific behaviours
✓ completes squares ✓ correct radius ✓ correct vector form

Question 13 **(7 marks)**

A particle, with initial velocity vector $(8, -2, 5)$ ms^{-1}, experiences a constant acceleration for 12 seconds. The velocity vector of the particle at the end of the 12 seconds is $(38, 34, -37)$ ms^{-1}.

(a) Determine the magnitude of the acceleration. (3 marks)

(b) Calculate the change in displacement of the particle over the 12 seconds. (4 marks)

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(a) Determine the magnitude of the acceleration. (3 marks)

Solution
$\Delta\mathbf{v} = \begin{pmatrix} 38 \\ 34 \\ -37 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 30 \\ 36 \\ -42 \end{pmatrix}$ $\mathbf{a} = \frac{\Delta\mathbf{v}}{12} = \begin{pmatrix} 2.5 \\ 3 \\ -3.5 \end{pmatrix}$ $\lvert\mathbf{a}\rvert = \frac{\sqrt{110}}{2} \approx 5.244 \text{ ms}^{-2}$
Specific behaviours
✓ change in velocity   ✓ acceleration vector   ✓ magnitude

(b) Calculate the change in displacement of the particle over the 12 seconds. (4 marks)

Solution
$\mathbf{v} = \begin{pmatrix} 8 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2.5 \\ 3 \\ -3.5 \end{pmatrix}$ $\Delta\mathbf{r} = \int_0^{12} \mathbf{v}\, dt$ $\Delta\mathbf{r} = \left[t\begin{pmatrix} 8 \\ -2 \\ 5 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 2.5 \\ 3 \\ -3.5 \end{pmatrix} \right]_0^{12}$ $\Delta\mathbf{r} = \begin{pmatrix} 276 \\ 192 \\ -192 \end{pmatrix}$
Specific behaviours
✓ velocity vector   ✓ indicates use of integration   ✓ correct displacement vector   ✓ change in displacement

Question 15 **(8 marks)**

The position vectors of bodies L and M at times λ and μ are given by

$$\mathbf{r}_L = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

and

$$\mathbf{r}_M = 7\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k})$$

where a and b are constants, times are in seconds and distances are in metres.

(a) Given that the paths of L and M intersect, show that $a + 4b + 7 = 0$. (4 marks)

(b) Given that the paths of L and M are also perpendicular, determine the values of a and b, and the position vector of the point of intersection of the paths. (4 marks)

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where a and b are constants, times are in seconds and distances are in metres.

(a) Given that the paths of L and M intersect, show that $a + 4b + 7 = 0$. (4 marks)

Solution
From k coefficients: $1 + \lambda = 2 - \mu \Rightarrow \mu = 1 - \lambda$ From j coefficients: $2 + b\lambda = \mu - 1 \Rightarrow b\lambda = 1 - \lambda - 3 \Rightarrow \lambda = -\frac{2}{b+1}$ From i coefficients: $2 + a\lambda = 7 + 3\mu \Rightarrow a\lambda = 5 + 3(1 - \lambda) \Rightarrow \lambda = \frac{8}{a+3}$ Hence $8(b + 1) = -2(a + 3) \Rightarrow 8b + 2a + 14 = 0 \Rightarrow a + 4b + 7 = 0$
Specific behaviours
✓ relates μ and λ from k coefficients   ✓ uses j coefficients to express μ or λ in terms of b ✓ uses i coefficients to express μ or λ in terms of a ✓ equates expressions and simplifies

(b) Given that the paths of L and M are also perpendicular, determine the values of a and b, and the position vector of the point of intersection of the paths. (4 marks)

Solution
$(a\mathbf{i} + b\mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3a + b - 1 = 0$ $3a + b - 1 = 0 \;\;\&\;\; a + 4b + 7 = 0$ $a = 1, \quad b = -2$ $\lambda = 8 \div 4 = 2$ $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + 2(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
Specific behaviours
✓ equates scalar product of directions to 0 ✓ solves for a and b ✓ determines λ ✓ states position vector of point of intersection

Question 19 **(7 marks)**

Points A, B and C have position vectors $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$ respectively, where a, b and c are non-zero, real constants. Point M is the midpoint of B and C. Use a vector method to prove that $\overrightarrow{AM}$ is perpendicular to $\overrightarrow{BC}$ when $|\overrightarrow{OB}| = |\overrightarrow{OC}|$.

Question 19 **(7 marks)**

Points A, B and C have position vectors $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively, where a, b and c are non-zero, real constants. Point M is the midpoint of B and C. Use a vector method to prove that $\overrightarrow{AM}$ is perpendicular to $\overrightarrow{BC}$ when $|\overrightarrow{OB}| = |\overrightarrow{OC}|$.

Solution
$\overrightarrow{BC} = -b\mathbf{j} + c\mathbf{k}$ $\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OB} + \frac{1}{2}(\overrightarrow{BC}) \\ &= -a\mathbf{i} + b\mathbf{j} + \frac{1}{2}(-b\mathbf{j} + c\mathbf{k}) \\ &= -a\mathbf{i} + \frac{1}{2}b\mathbf{j} + \frac{1}{2}c\mathbf{k}\end{aligned}$ $\begin{aligned}\overrightarrow{BC} \cdot \overrightarrow{AM} &= (-b\mathbf{j} + c\mathbf{k}) \cdot \left(-a\mathbf{i} + \frac{1}{2}b\mathbf{j} + \frac{1}{2}c\mathbf{k}\right) \\ &= (-b\mathbf{j}) \cdot \left(\frac{1}{2}b\mathbf{j}\right) + (c\mathbf{k}) \cdot \left(\frac{1}{2}c\mathbf{k}\right) \\ &= \frac{1}{2}(c^2 - b^2) \\ &= 0 \text{ as } \lvert\overrightarrow{OB}\rvert = \lvert\overrightarrow{OC}\rvert\end{aligned}$ Hence $\overrightarrow{BC} \perp \overrightarrow{AM}$ as the scalar product of non-zero vectors is only zero when vectors are perpendicular.
Specific behaviours
✓ vector for $\overrightarrow{BC}$ ✓ indicates method for vector $\overrightarrow{AM}$ ✓ simplifies vector for $\overrightarrow{AM}$ ✓ uses scalar product   ✓ simplifies scalar product   ✓ uses magnitudes   ✓ justifies conclusion