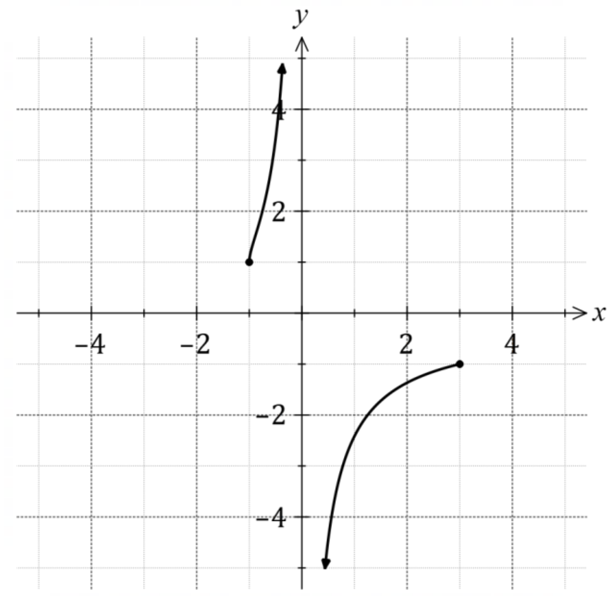
Question 3 **(7 marks)**

The diagram shows the graph of $y = f(x)$, where $f(x) = \dfrac{1}{1-\sqrt{x+1}}$ and the domain of f is restricted to $\{x \in \mathbb{R} \mid -2 \le x \le 2,\ x \ne -1\}$.



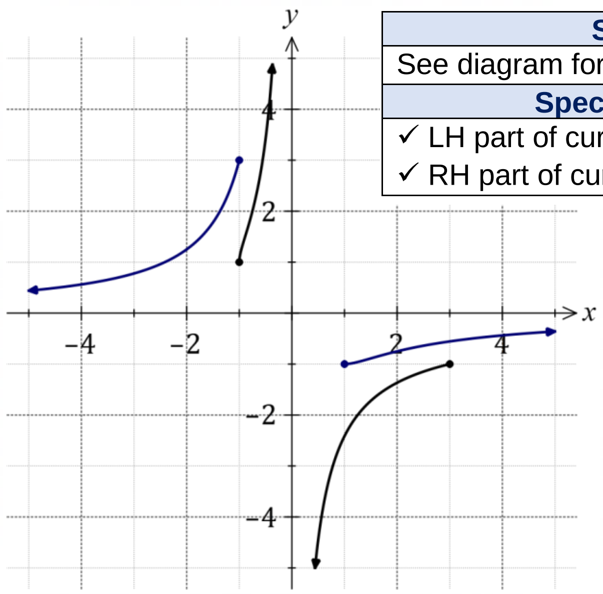
(a) Explain how to use the graph to estimate a solution to the equation $f^{-1}(x) = 2$. (1 mark)

(b) On the same axes, sketch the graph of $y = f^{-1}(x)$. (2 marks)

(c) Determine a simplified rule for $y = f^{-1}(x)$, stating any domain restriction(s). (4 marks)

Question 3 **(7 marks)**

The diagram shows the graph of $y = f(x)$, where $f(x) = \dfrac{1}{1-\sqrt{x+1}}$ and the domain of f is restricted to $\{x \in \mathbb{R} \mid -2 \le x \le 2,\ x \ne -1\}$.



Solution (b)
See diagram for endpoints and curvature
Specific behaviours
✓ LH part of curve from $(-1,3)$ ✓ RH part of curve from $(1,-1)$

(a) Explain how to use the graph to estimate a solution to the equation $f^{-1}(x) = 2$. (1 mark)

Solution
Draw the vertical line $x = 2$ and the y-coordinate of the intersection of this line and the curve will be the solution. *(Do not accept use of graph of inverse function)*
Specific behaviours
✓ correct explanation

(b) On the same axes, sketch the graph of $y = f^{-1}(x)$. (2 marks)

(c) Determine a simplified rule for $y = f^{-1}(x)$, stating any domain restriction(s). (4 marks)

Solution
Range of f, $y \le -1 \cup y \ge 1$, is domain of f^{-1}. $x = \dfrac{1}{1-\sqrt{y+1}}$ $-\sqrt{y+1} = \dfrac{1}{x} - 1$ $y + 1 = \dfrac{1}{x^2} - \dfrac{2}{x} + 1$ $y = f^{-1}(x) = \dfrac{1}{x^2} - \dfrac{2}{x}, \quad \{x \in \mathbb{R} \mid x \le -1 \cup x \ge 1\}$
Specific behaviours
✓ interchanges x, y and cross multiplies ✓ obtains expression for $\sqrt{y+1}$ ✓ obtains defining rule for inverse ✓ states domain restrictions in terms of x for inverse

Question 7 **(7 marks)**

Consider functions $f(x) = \dfrac{x^2 + 7}{2}$ and $g(x) = \sqrt{25 - x^2}$.

(a) Explain why f is not a one-to-one function. (1 mark)

(b) State the domain and range of $g(x)$. (2 marks)

(c) Determine the domain and range of $g\big(f(x)\big)$. (4 marks)

Question 7 **(7 marks)**

Consider functions $f(x) = \dfrac{x^2+7}{2}$ and $g(x) = \sqrt{25-x^2}$.

(a) Explain why f is not a one-to-one function. (1 mark)

Solution
f is a many-to-one function. For example, $f(1) = f(-1) = 4$.
Specific behaviours
✓ states many-to-one or uses examples to show not one-to-one

(b) State the domain and range of $g(x)$. (2 marks)

Solution
$D_g: -5 \le x \le 5, \qquad R_g:\ 0 \le y \le 5.$
Specific behaviours
✓ correct domain ✓ correct range

(c) Determine the domain and range of $g\big(f(x)\big)$. (4 marks)

Solution

$$g\big(f(x)\big) = \sqrt{25 - f(x)^2}$$

Using result from (b) we require $-5 \le f(x) \le 5$ but since the natural range of f is $y \ge \frac{7}{2}$ then for domain of $g \circ f$ we just need the restriction $f(x) \le 5$:

$$\frac{x^2+7}{2} \le 5 \Rightarrow x^2 \le 3 \Rightarrow D_{g\circ f}: -\sqrt{3} \le x \le \sqrt{3}$$

Use $R_f = \left\{\frac{7}{2} \le y \le 5\right\}$ to obtain range of $g \circ f$:

$$g\left(\frac{7}{2}\right) = \sqrt{25 - {}^{49}/_{4}} = \frac{\sqrt{51}}{2}, \qquad g(5) = 0 \Rightarrow R_{g\circ f}:\ 0 \le y \le \frac{\sqrt{51}}{2}$$

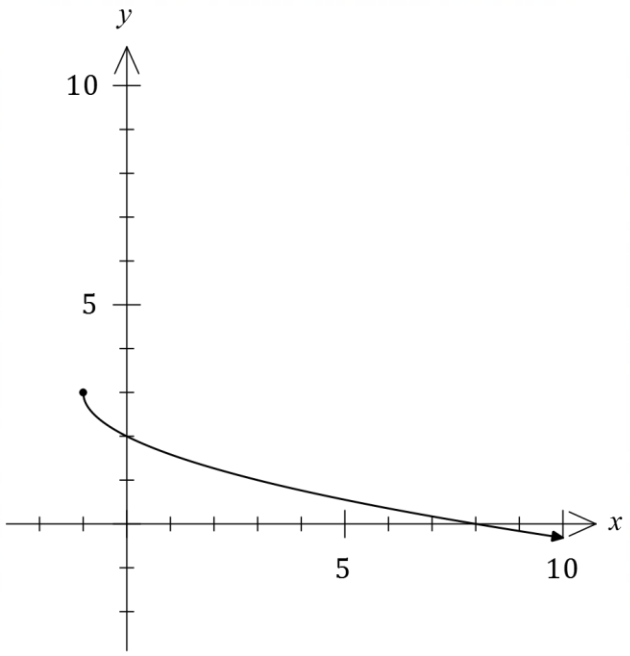
Specific behaviours

✓ indicates that $f(x) \le 5$
✓ correct domain
✓ indicates restricted range of f
✓ correct range

Question 3 **(8 marks)**

Function f is defined as $f(x) = 3 - \sqrt{x+1}$.

The graph of $y = f(x)$ is shown at right.



(a) Sketch the graph of $y = f^{-1}(x)$ on the axes above. (2 marks)

(b) State the domain and range of $f^{-1}(x)$. (2 marks)

Function g is defined as $g(x) = \sqrt{x}$, and $h(x) = g \circ f(x)$.

(c) Write an expression for $h(x)$ and determine the domain and range of $h(x)$. (4 marks)

Question 3 **(8 marks)**

Function f is defined as $f(x) = 3 - \sqrt{x+1}$.

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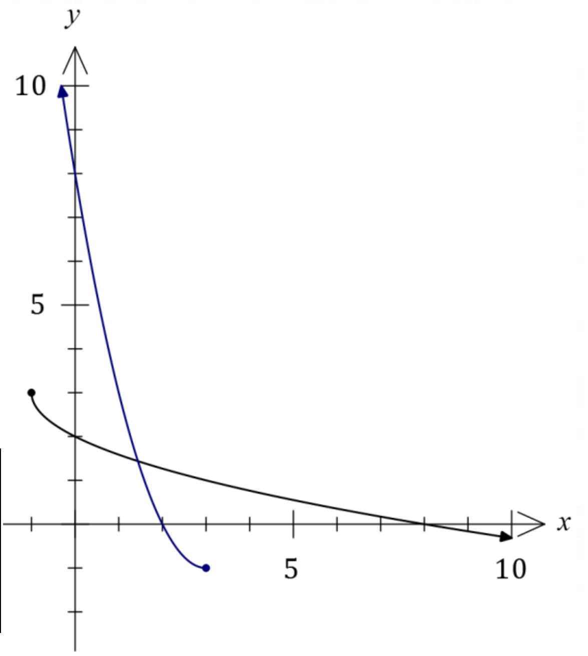


Solution (a)
See graph
Specific behaviours
✓ axis intercepts ✓ endpoint clearly reflection of $f(x)$ in $y = x$

(a) Sketch the graph of $y = f^{-1}(x)$ on the axes above. (2 marks)

(b) State the domain and range of $f^{-1}(x)$. (2 marks)

Solution
$D_{f^{-1}} = \{x: x \in \mathbb{R}, x \leq 3\}$, $R_{f^{-1}} = \{y: y \in \mathbb{R}, y \geq -1\}$
Specific behaviours
✓ correct domain ✓ correct range

Function g is defined as $g(x) = \sqrt{x}$, and $h(x) = g \circ f(x)$.

(c) Write an expression for $h(x)$ and determine the domain and range of $h(x)$. (4 marks)

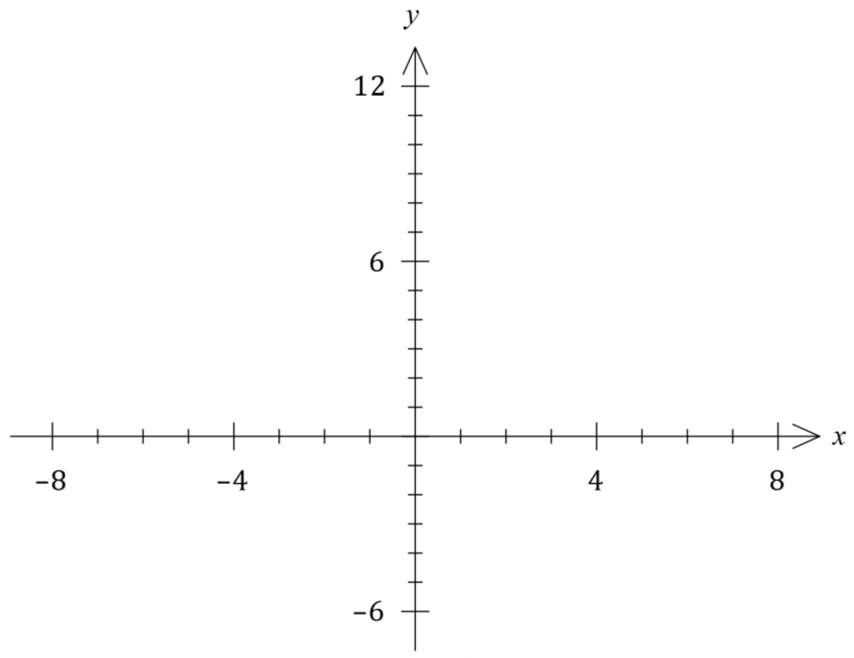
Solution
$$h(x) = \sqrt{3 - \sqrt{x+1}}$$ Domain: $D_f = \{x \geq -1\}$ and $3 - \sqrt{x+1} \geq 0 \rightarrow \sqrt{x+1} \leq 3 \rightarrow x \leq 8$ $$D_h = \{x: x \in \mathbb{R}, -1 \leq x \leq 8\}$$ Range: $h(-1) = \sqrt{3}$, $h(8) = 0$. Hence $$R_h = \{y: y \in \mathbb{R}, 0 \leq y \leq \sqrt{3}\}$$
Specific behaviours
✓ expression for $h(x)$ ✓ uses D_f and indicates $R_f \geq 0$ ✓ correct domain ✓ correct range

Question 7 **(8 marks)**

Consider the function $f(x) = \dfrac{x^2 + bx + c}{ax + d}$, where a, b, c and d are constants.

The graph of $y = f(x)$ has roots at $x = -4$ and $x = 1$, a vertical asymptote $x = 2$ and passes through the point $(3, 7)$.

Sketch the graph of $y = f(x)$, clearly showing the y-intercept and equations of all asymptotes.



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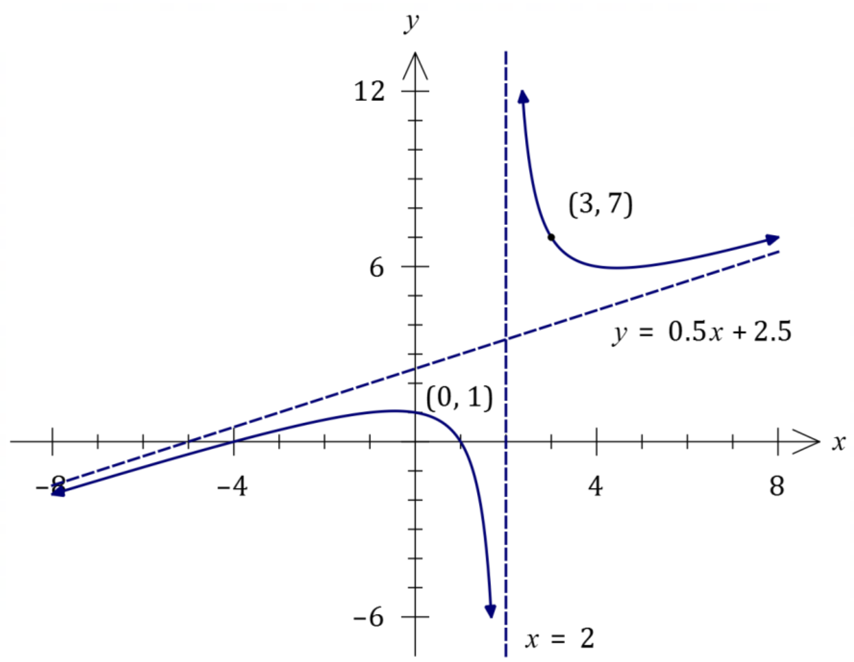


Solution
Use roots to determine numerator: $x^2 + bx + c = (x + 4)(x - 1) = x^2 + 3x - 4$ Use vertical asymptote to eliminate d: $a(2) + d = 0 \rightarrow d = -2a$ Use point $(3, 7)$ to determine a: $$f(x) = \frac{(x + 4)(x - 1)}{a(x - 2)} \rightarrow 7 = \frac{7 \times 2}{a} \rightarrow a = 2$$ Express $f(x)$ as a proper fraction: $$\begin{aligned} f(x) &= \frac{x^2 + 3x - 4}{2(x - 2)} \\ &= \frac{x(x - 2)}{2(x - 2)} + \frac{5(x - 2)}{2(x - 2)} + \frac{6}{2(x - 2)} \\ &= \frac{x}{2} + \frac{5}{2} + \frac{6}{2x - 4} \end{aligned}$$ Hence oblique asymptote is $y = \frac{x}{2} + \frac{5}{2}$ and $f(0) = 1$.
Specific behaviours
✓ uses roots to obtain numerator ✓ uses vertical asymptote to relate a and d ✓ uses point to obtain denominator ✓ expresses $f(x)$ as proper fraction ✓ states correct equation for asymptote ✓ plots roots, y-intercept and both asymptotes ✓ correct curvature of graph to left of vertical asymptote, through roots ✓ correct curvature of graph to right of vertical asymptote, through $(3, 7)$

Question 3 **(6 marks)**

Functions f, g and h are defined as

$$f(x) = x + 3, \qquad g(x) = \sqrt{x}, \qquad h(x) = \frac{4}{2 - x}.$$

(a) Determine

(i) $h \circ g \circ f(6)$. (1 mark)

(ii) the defining rule for $h \circ g \circ f(x)$. (1 mark)

(b) Determine the domain of $h \circ g \circ f(x)$. (2 marks)

(c) Determine the range of $h \circ g \circ f(x)$. (2 marks)

Question 3 **(6 marks)**

Functions f, g and h are defined as

$$f(x) = x + 3, \qquad g(x) = \sqrt{x}, \qquad h(x) = \frac{4}{2 - x}.$$

(a) Determine

(i) $h \circ g \circ f(6)$. (1 mark)

Solution
$h \circ g \circ f(6) = h \circ g(9) = h(3) = -4$
Specific behaviours
✓ correct value

(ii) the defining rule for $h \circ g \circ f(x)$. (1 mark)

Solution
$h \circ g \circ f(x) = h \circ g(x+3)$ $= h(\sqrt{x+3})$ $= \dfrac{4}{2 - \sqrt{x+3}}$
Specific behaviours
✓ correct rule

(b) Determine the domain of $h \circ g \circ f(x)$. (2 marks)

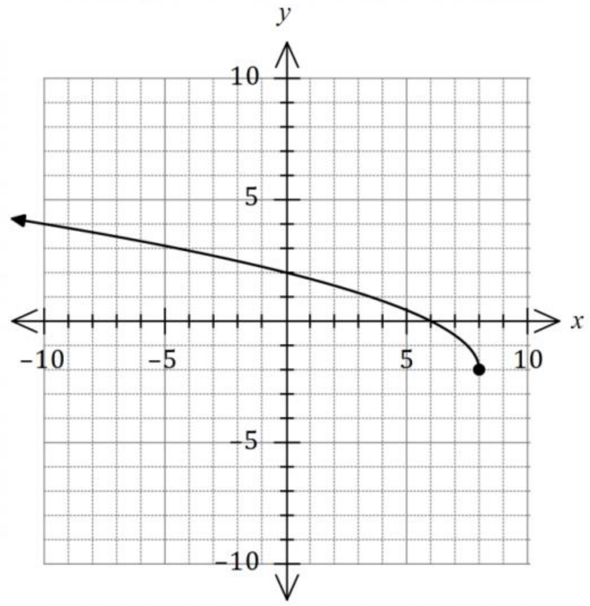
Solution
$x + 3 \geq 0 \Rightarrow x \geq -3$ $\sqrt{x+3} \neq 2 \Rightarrow x \neq 1$ $D_{h \circ g \circ f} = \{x: x \in \mathbb{R}, x \geq -3, x \neq 1\}$
Specific behaviours
✓ states $x \geq -3$ ✓ states $x \neq 1$

(c) Determine the range of $h \circ g \circ f(x)$. (2 marks)

Solution
$-3 < x < 1, \quad y \geq 2$ $x > 1, \quad y < 0$ $R_{h \circ g \circ f} = \{y: y \in \mathbb{R}, y \geq 2 \cup y < 0\}$
Specific behaviours
✓ states $y \geq 2$ ✓ states $y < 0$

Question 4 **(6 marks)**

The graph of $y = f(x)$ is shown below.

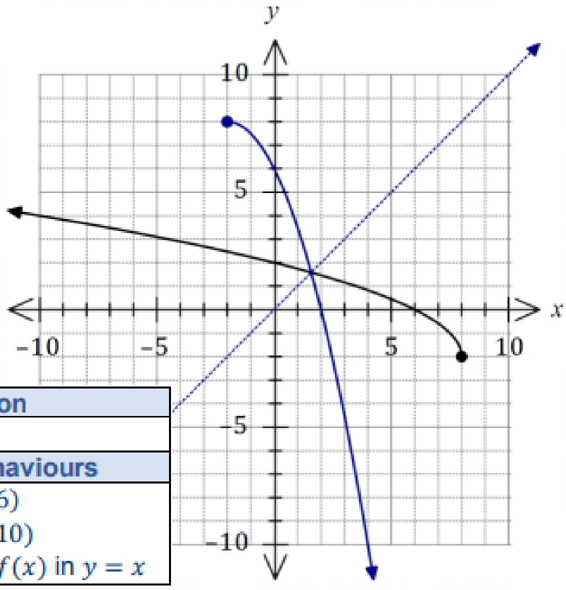


(a) Draw the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)

(b) Given that $f(x) = \sqrt{16 - 2x} - 2$, determine the defining rule for $f^{-1}(x)$. (3 marks)

Question 4 **(6 marks)**

The graph of $y = f(x)$ is shown below.



Solution
See graph
Specific behaviours
✓ $(-2, 8)$ and $(0, 6)$
✓ $(2, 0)$ and $(4, -10)$
✓ symmetry with $f(x)$ in $y = x$

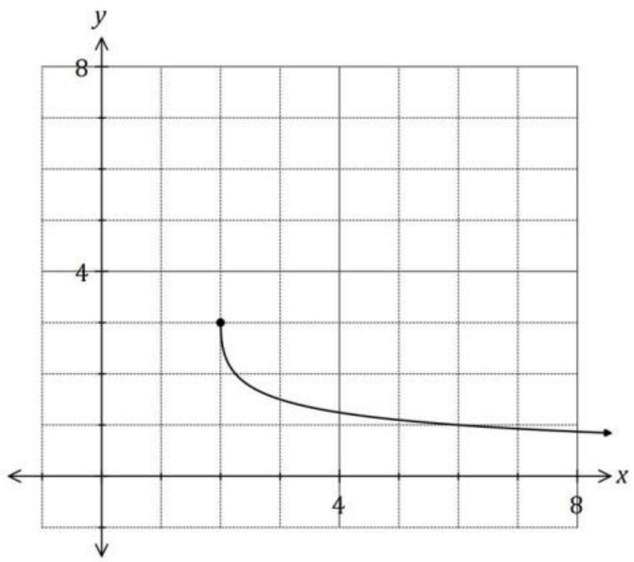
(a) Draw the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)

(b) Given that $f(x) = \sqrt{16 - 2x} - 2$, determine the defining rule for $f^{-1}(x)$. (3 marks)

Solution
$x = \sqrt{16 - 2y} - 2$ $16 - 2y = (x + 2)^2$ $y = 8 - \dfrac{(x + 2)^2}{2}$ $D_{f^{-1}} = R_f : y \geq -2$ $f^{-1}(x) = 8 - \dfrac{(x + 2)^2}{2}, x \geq -2$
Specific behaviours
✓ removes square root from expression   ✓ obtains correct expression for y in terms of x ✓ writes inverse with domain restriction

Question 2 **(6 marks)**

The graph of $y = f(x)$ is shown below, where f is defined by $f(x) = \dfrac{3}{1+\sqrt{x-2}}$.

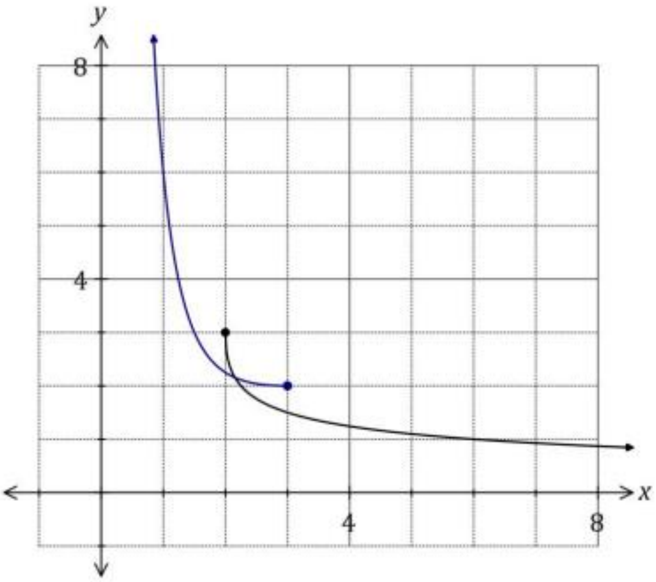


(a) Sketch the graph of $y = f^{-1}(x)$ on the same axes. (2 marks)

(b) Determine the defining rule for $y = f^{-1}(x)$ and state its domain. (4 marks)

Question 2 **(6 marks)**

The graph of $y = f(x)$ is shown below, where f is defined by $f(x) = \dfrac{3}{1+\sqrt{x-2}}$.



(a) Sketch the graph of $y = f^{-1}(x)$ on the same axes. (2 marks)

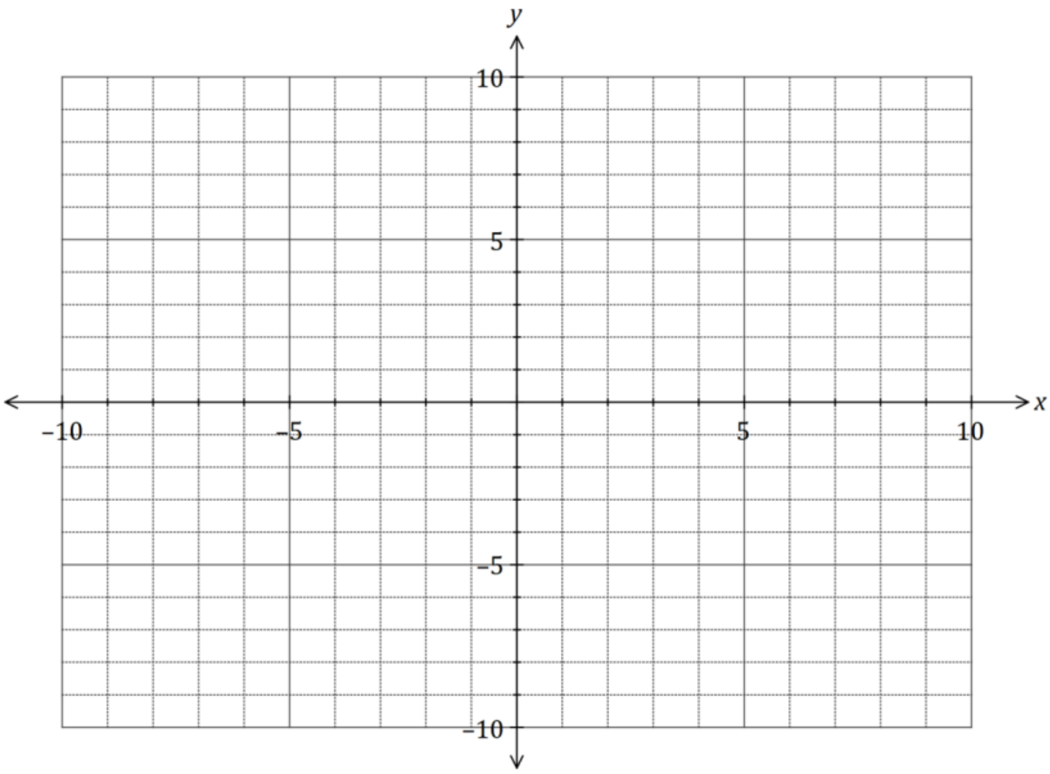
Solution
See graph
Specific behaviours
✓ reflection in $y = x$
✓ starts at $(3,2)$ and through $\approx (2.2, 2.2)$

(b) Determine the defining rule for $y = f^{-1}(x)$ and state its domain. (4 marks)

Solution
$1+\sqrt{y-2} = \dfrac{3}{x}$ $y = f^{-1}(x) = \left(\dfrac{3}{x} - 1\right)^2 + 2$ $D_{f^{-1}}:\ 0 < x \leq 3$
Specific behaviours
✓ inverts variables and starts to rearrange
✓ correct inverse
✓ correct lower bound of domain
✓ correct upper bound of domain

Question 6 **(6 marks)**

The graph of $y = \dfrac{3x^2 - 12}{(x+1)(x-3)}$ has no stationary points. Sketch the graph.



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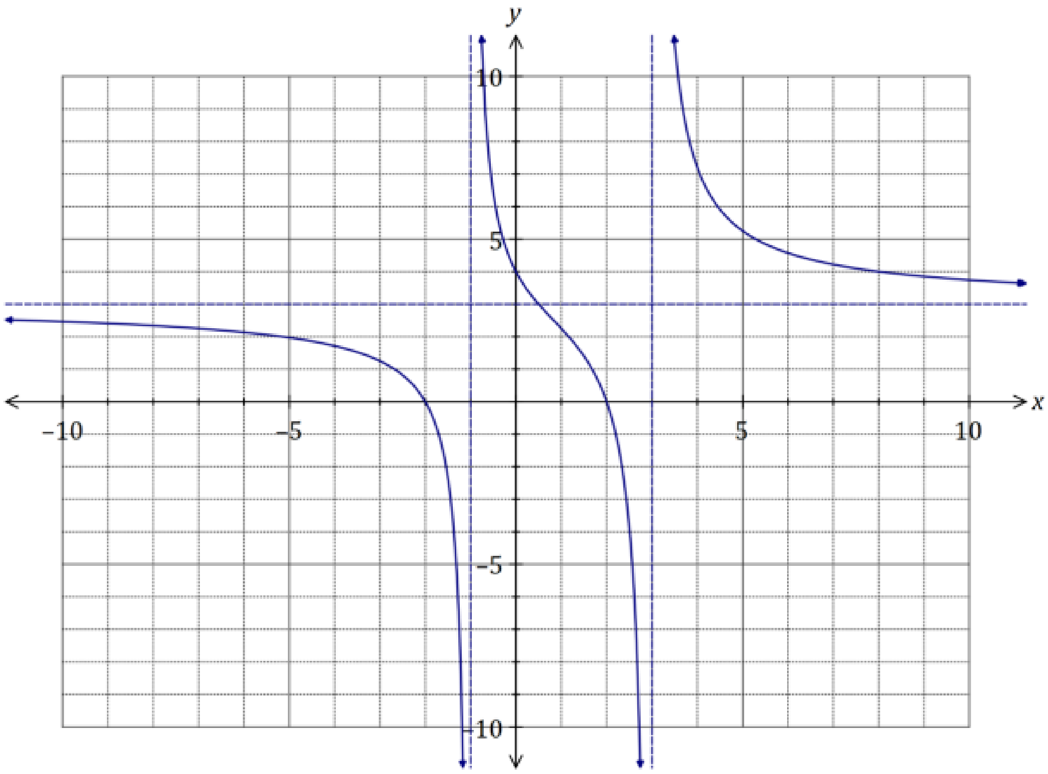


Solution
See graph
Specific behaviours
✓ indicates vertical asymptotes $x = -1, x = 3$ ✓ indicates horizontal asymptote $y = 3$ ✓ through $(-2, 0)$ and correct curvature for $x < -1$ ✓ through $(2, 0)$ and $(0, 4)$ ✓ correct curvature for $-1 < x < 3$ ✓ close to $(5, 5)$ and correct curvature for $x > 3$

Question 8 **(6 marks)**

A function is defined by $f(x) = \dfrac{3 - x}{(2x + 5)(3x - 7)}$.

(a) State the natural domain of $f(x)$. (1 mark)

(b) State the equations of all asymptotes of the graph of $y = x \cdot f(x)$. (2 marks)

(c) The graph of $y = \dfrac{1}{f(x)}$ has an asymptote with equation $y = ax + b$. Determine the values of the constants a and b. (3 marks)

Question 8 **(6 marks)**

A function is defined by $f(x) = \dfrac{3-x}{(2x+5)(3x-7)}$.

(a) State the natural domain of $f(x)$. (1 mark)

Solution
$\left\{x : x \in \mathbb{R}, x \neq -\frac{5}{2}, x \neq \frac{7}{3}\right\}$
Specific behaviours
✓ indicates exceptions

(b) State the equations of all asymptotes of the graph of $y = x \cdot f(x)$. (2 marks)

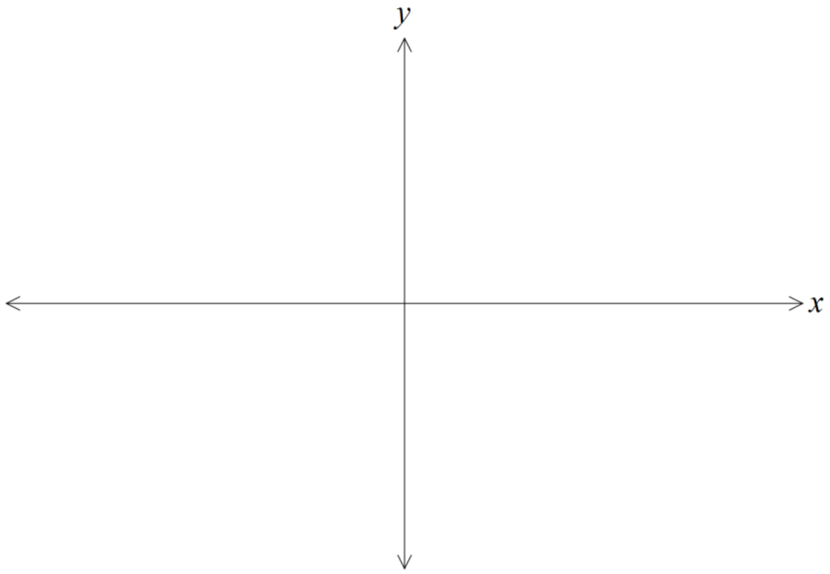
Solution
$x \cdot f(x) = \dfrac{x(3-x)}{(2x+5)(3x-7)}$ Vertical: $x = -\frac{5}{2}, x = \frac{7}{3}$ Horizontal: $x \to \infty, x \cdot f(x) \to -\dfrac{x^2}{6x^2} \to -\dfrac{1}{6} \Rightarrow y = -\dfrac{1}{6}$
Specific behaviours
✓ both vertical asymptotes   ✓ horizontal asymptote   *(penalise if not given as equation)*

(c) The graph of $y = \dfrac{1}{f(x)}$ has an asymptote with equation $y = ax + b$. Determine the values of the constants a and b. (3 marks)

Solution
$\dfrac{1}{f(x)} = \dfrac{(2x+5)(3x-7)}{3-x}$ $= \dfrac{-6x^2 - x + 35}{x-3}$ $= -6x - 19 - \dfrac{22}{x-3}$ $a = -6, b = -19$
Specific behaviours
✓ correct expansion of numerator and denominator   ✓ value of a ✓ value of b

Question 6 **(7 marks)**

(a) Sketch the graph of $y = \dfrac{|x-2|}{2}$ on the axes below. (2 marks)

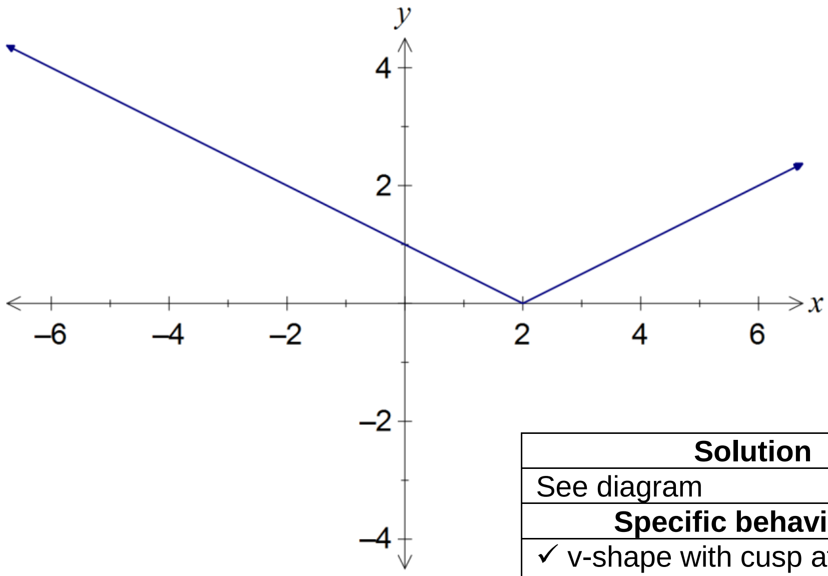


(b) Solve the equation $4|x-8| = 38 - x$. (3 marks)

(c) Solve the inequality $\dfrac{1}{|x+2|} \leq 1$. (2 marks)

Question 6 **(7 marks)**

(a) Sketch the graph of $y = \dfrac{|x-2|}{2}$ on the axes below. (2 marks)



Solution
See diagram
Specific behaviours
✓ v-shape with cusp at (2, 0) ✓ correct y-intercept

(b) Solve the equation $4|x-8| = 38 - x$. (3 marks)

Solution
$x \ge 8 \Rightarrow 4x - 32 = 38 - x \Rightarrow 5x = 70 \Rightarrow x = 14$ $x < 8 \Rightarrow -4x + 32 = 38 - x \Rightarrow 3x = -6 \Rightarrow x = -2$ $x = -2,\ 14$
Specific behaviours
✓ separates into cases ✓ solves first case ✓ solves second case

(c) Solve the inequality $\dfrac{1}{|x+2|} \le 1$. (2 marks)

Solution
$\left.\begin{array}{l} x > -2 \Rightarrow 1 \le x + 2 \Rightarrow x \ge -1 \\ x < -2 \Rightarrow 1 \le -x - 2 \Rightarrow x \le -3 \end{array}\right\} \quad x \le -3,\ x \ge -1$
Specific behaviours
✓ determines correct endpoints ✓ states correct inequalities