Optimal Stopping Problem under Model Ambiguity

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Abstract

In this project, a new framework for optimal stopping under model ambiguity is introduced. Besides model ambiguity, we also take into account the decision maker's *ambiguity attitude*. We study the $\alpha-$maxmin objective function, which has the time-inconsistency issue. To deal with it, we employ the time-consistent equilibrium approach through the fixed point of an iterator. We will also apply the framework on American options with different payoff functions and state processes, and conduct some analysis.

Contents

1 Introduction

Decision making under model ambiguity has been extensively studied, under the best-case and worst-case framework. In real life, however, few people are so pessimistic (or optimistic) about the uncertain environment. In this project, a new method for handling the model ambiguity is introduced: *ambiguity attitude* of a decision maker is incorporated, resulting in a more realistic spectrum of behavior.

We focus on the optimal stopping problem under model ambiguity.
Classically, a person chooses a stopping time τ by maximizing the expected discounted payoff

$$E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] \tag{1.1}$$

In reality, we are usually uncertain about the true probability, which leads to wo types of optimal stopping problem. The first one is a person maximizes the worst-case expected value

$$\inf_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)], \tag{1.2}$$

and the second type is a person maximizes the best-case expected value

$$\sup_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] \tag{1.3}$$

Curley and Yates (1989) asked subjects to compare a risky bet with a winning probability of 0.25 to an ambiguous bet on a ball drawn from an urn containing five winning balls, fifty-five losing balls and forty winning or losing balls in unknown proportion. The authors encountered that majority of subjects prefer the ambiguous act and they are quite optimistic about the distribution of the forty unknown balls. Later, ambiguity seeking preferences have also been studied in Kahn and Sarin (1988), Heath and Tversky (1991), Dominiak and Schnedler (2011) and many others. In particular, Heath and Tversky (1991) demonstrated that peoples reaction can be ambiguity loving if they are experienced, competent or knowledgable on certain events. In an effort to differentiate ambiguity and ambiguity attitude, Ghirardato et al. (2004) recommended the -maxmin preferences via a constant index $\alpha \in [0,1]$ to weight between the worst case and best case objectives. Other more complicated ways to capture heterogeneous ambiguity attitude, such as smooth ambiguity prefer- ence and neo-additive capacity model, can be found in Klibanoff et al. (2005) and Chateauneuf et al. (2007) and etc. To accommodate ambiguity attitude and also ensure our convergence argu- ments hold, we choose to work with the tractable -maxmin preference.

As a result, for our method, a person chooses a stopping time τ by maximizing

$$\alpha \inf_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] + (1-\alpha)\sup_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] \tag{1.4}$$

where $\alpha \in [0,1]$ is the person's *attitude towards ambiguity.*

The challenge in solving (1.4) is *time inconsistency*: An optimal strategy we find today may no longer be optimal at future dates. That is, our future selves may very well deviate from the optimal strategy we set out to employ today. Thereby, the traditional methods used for the best-case and the worst case scenarios are no longer applicable here.

In our framework, a sensible way to deal with *time inconsistency* is *consistent planning*: Knowing the future himself will deviate from now, the person finds the optimal stopping time by taking the future disobedience as a constraint. Assuming every future himself will think reasonably, the resulting strategy will therefore be an equilibrium, from which no future himself will deviate.

The method we introduce is particularly applicable for real option. As real option may suffer from severe model ambiguity, by using our objective function 1.4, multiple plausible values turn into a single one - the convex combination of the best and worst values. This significantly facilitates our decision making process: We just need to compare this single value with the value of immediate stopping, to decide whether the project should be postponed or initiated.

In summary, the focus of this project thesis are following:

(i) In this paper, we resolve the time-inconsistent stopping problem under the $\alpha-$maxmin preference. This allows us to go beyond the standard best-case and worst-case analysis, and study a more practical behavior. Note that in the framework, the prior P is only assumed to be measurable and stability under past condition is not required.

(ii) Our framework provides a new approach for real options valuation. Taking ambiguity attitude into account, using the α-maxmin preference, facilitates decision making can be conducted under model ambiguity, but it also renders the stopping problem time-inconsistent. The methodology we develop particularly resolves this time-inconsistent problem, allowing us to take full advantage of including ambiguity attitude in decision making.

The rest of the thesis is organized as follows. Section 2 introduces the problem set-up, including the objective and the methods to deal with it. Secion 3 is the application of our framework on particular examples. Section 4 is the remark and final conclusions. Literature Reference and Code Reference are also attached.

2 The Set-up

Every $\mathbb{P} \in P(x)$ is believed by the person to be a true description of how the process B will evolve, given its current value. Note that $\mathbb{P} \in P(x)$ is not necessarily continuous. In other words, some elements in $P(x)$ might be mutually singular.

2.1 The $\alpha-$maxmin Objective Function

Consider a payoff function g: $\mathbb{R}^d \longrightarrow \mathbb{R}$. With discount rate $r > 0$, in usual cases, a person intends to maximize $E^{\mathbb{P}}[e^{-r\tau} g(B_\tau)]$ by choosing an appropriate τ, subject to the uncertainty $\mathbb{P} \in P(x)$ at the current state $x \in \mathbb{R}$.

In our framework, by incorporating the *ambiguity attitude* α, this literature is focused on finding

τ that maximizes the objective function

$$\alpha \inf_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] + (1-\alpha)\sup_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)]$$

where $\alpha \in [0,1]$ is the person's *attitude towards ambiguity.* Note that the case that $\alpha = 0$ corresponds to the standard worst case and the case $\alpha = 0$ corresponds to the standard best case.

2.2 Time Inconsistency Issue

While solving

$$\alpha \inf_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)] + (1-\alpha)\sup_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau}g(B_\tau)],$$

the issue of time inconsistency arises. Mathematically speaking, suppose an optimal stopping time $\tilde{\tau_x}$ exists for all $x \in \mathbb{R}^d$. The problem is said to be time-inconsistent if for any $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\tilde{\tau}_x(w) \neq t + \tilde{\tau}_{B_t}(w) \text{ for } w \in \{\tau \geq t\}$$

Note that a critical condition of time consistency is the tower property of conditional expectation. Note that in our scenario, David Schroder (2011) has demonstrated that for $s > t$,

$$\begin{aligned}
&\alpha \inf_{\mathbb{P}\in P} E_t^{\mathbb{P}}[e^{-rt}g(B_t)] + (1-\alpha)\sup_{\mathbb{P}\in P} E_t^{\mathbb{P}}[e^{-rt}g(B_t)] \\
&\neq \alpha \inf_{\mathbb{P}\in P} E_t^{\mathbb{P}}\left[\alpha \inf_{\mathbb{P}\in P} E_s^{\mathbb{P}}[e^{-rs}g(B_s)] + (1-\alpha)\sup_{\mathbb{P}\in P} E_s^{\mathbb{P}}[e^{-rs}g(B_s)]\right] \\
&+ (1-\alpha)\sup_{\mathbb{P}\in P} E_t^{\mathbb{P}}\left[\alpha \inf_{\mathbb{P}\in P} E_s^{\mathbb{P}}[e^{-rs}g(B_s)] + (1-\alpha)\sup_{\mathbb{P}\in P} E_s^{\mathbb{P}}[e^{-rs}g(B_s)]\right]
\end{aligned}$$

To see this, note that the right hand side can be transformed into:

$$\alpha^2 \sup_{\mathbb{P}\in P} E_t^{\mathbb{P}}[e^{-rt}g(B_t)] + 2\alpha(1-\alpha)\sup_{\mathbb{P}\in P} E_t^{\mathbb{P}}\left[\inf_{\mathbb{P}\in P} E_s^{\mathbb{P}}[e^{-rs}g(B_s)]\right] + (1-\alpha)^2 \inf_{\mathbb{P}\in P} E_t^{\mathbb{P}}[e^{-rs}g(B_s)]$$

In our case, the above tower property doesn't hold for the objective function g because of the existence of *inf* and *sup.* Therefore, we have time-inconsistency in our problem.

In our framework, a sensible way to deal with *time inconsistency* is *consistent planning*: Knowing the future himself will deviate from now, the person finds the optimal stopping time by taking the future disobedience as a constraint. Assuming every future himself will think reasonably, the resulting strategy will therefore be an equilibrium, from which no future himself will deviate.

2.3 Problem Formulation and Approach

The majority of the optimal stopping time in the existing work on time consistent robust optimal stopping (with drift uncertainty or volatility uncertainty) in the Markovian framework fit into the type of first hitting time. This motivates us to consider the first hitting time in our time inconsistent setting with model ambiguity. More importantly, the first hitting time allows us to pass the stopping policy to its associated set, which is technically convenient for us to define and verify the convergence of our proposed iteration under model ambiguity. Starting from this point on, let us focus on hitting times to regions in $\mathbb{R}^d$, instead of dealing with all general stopping times.

In the context of time-inconsistent stopping, we implement an iterative approach: equilibrium strategies, formulated as fixed points of an operator, which can be found via fixed-point iterations.

Remember that the objective function of our problem is

$$J(x,R) = \alpha \inf_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau} g(B_\tau)] + (1-\alpha) \sup_{\mathbb{P}\in P} E^{\mathbb{P}}[e^{-r\tau} g(B_\tau)]$$

where R denotes the stopping region.

A person will stop at the moment:

$$\rho_R = \inf\{t > 0 : B_t \in R\}$$

Mathematically, to find the best stopping policy for today, in response to future selves following $R \in U(\mathbb{R}^d)$, the agent simply compares the payoffs $g(x)$ and $J(x,R)$. This leads to

$$\Theta(R) = S_R \cup (I_R \cap R) \tag{2.1}$$

where we define

$$S_R = \{x \in \mathbb{R}^d : g(x) > J(x,R)\}$$

$$I_R = \{x \in \mathbb{R}^d : g(x) = J(x,R)\}$$

$$C_R = \{x \in \mathbb{R}^d : g(x) < J(x,R)\}$$

It is of our interest to see whether the newly obtained region $\Theta(R)$ is again a stopping region. It can be proved that it is indeed an equilibrium stopping region if

$$\Theta(R) = R \tag{2.2}$$

To achieve such stopping region R, using the fixed-point iteration approach, we take:

$$R^* = \lim_{n\to\infty} \Theta^n(R) \tag{2.3}$$

It is noted that the limit of sets in 2.3 should be interpreted in either of two equivalent ways: (i) the upper and lower bounds on the sequence that converge monotonically to the same set and (ii) by convergence of a sequence of indicator functions which are themselves real-valued.

Additionally, we might achieve multiple equilibrium stopping regions R. It is sometimes possible to look for an optimal R_{opt} such that $J(R_{opt})$ dominates all other $J(R)$ where R is an equilibrium policy.

As Huang, Y. and Yu, X. (2020) have proved:

Theorem 2.1. *Fix $\prod \in \mathcal{A}^\infty$ such that $\{(x, \mathcal{P}(x)) : x \in I\} \subseteq I \times \mathfrak{B}(\Omega)$ is universally measurable. Suppose that $g : \bar{I} \mapsto \mathbb{R}$ is continuous and*

$$\lim_{t\to\infty} e^{-rt} g(X_t^{x,b,\sigma}) = 0 \ \ \textit{for } \forall x \in I \ \textit{ and } \ (b, \sigma) \in \prod(x)$$

Then, for any $R \in \mathcal{U}(I)$, let us consider R^* as the limit of convergent sets that

$$R^* = \lim_{n\to\infty} \Theta^n(R).$$

We have that R^* belongs to ϵ. Hence,

$$\epsilon = \{\lim_{n\to\infty} \Theta^n(R) : R \in \mathcal{U}(I)\}.$$

3 Examples and Analysis

In this section, we investigate several examples of American options with different payoff functions and underlying state processes. The time-consistent equilibrium approach will be employed.

Note that we only study American options because only they allow early exercise, therefore, we can study the optimal stopping problem.

We can behold the explicit decision making process through these applications.

3.1 American Put Option on Absolute Value of Brownian Motion

We first consider the underlying state process that is defined by: $X_t^{x,r,\sigma} = |x+0t+\sigma B_t| = |x+\sigma B_t|$.

The stopping time $T_a^{x,r,\sigma} = inf\ \{t > 0 : X_t^{x,r\sigma} = a\}\ (a > 0)$

Note that our uncertainty results from $\sigma \in [\underline{\sigma}, \overline{\sigma}]$

In order to find $E[e^{-rT_a}]$, we need to find the distribution of T_a first.

First,

$$F_{X_t}(m) = P(|x+\sigma B_t| \leq m) = P\left(\frac{-m-x}{\sigma} \leq B_t \leq \frac{m-x}{\sigma}\right) = \int_{\frac{-m-x}{\sigma}}^{\frac{m-x}{\sigma}} \frac{1}{\sqrt{2\pi t}} exp\left(\frac{-s^2}{2t}\right) ds$$

By taking derivative with respect to m, we have:

$$f_{X_t}(m) = \frac{1}{\sqrt{2\pi t\sigma^2}} exp\left(-\frac{m^2+x^2}{\sigma^2 t}\right)\left[exp\left(\frac{(m-x)^2}{2\sigma^2 t}\right) + exp\left(\frac{(m+x)^2}{2\sigma^2 t}\right)\right]$$

Now we intend to find the distribution of T_a

From the formula, we can get: $E[e^{-rT_a}] = exp\left\{-\frac{1}{\sigma}(x-a)\sqrt{2r}\right\}$, as calculated in Borodin and Salminen (2002)

Because we haven't reached the equilibrium yet, we have $a \in (0, x \wedge K)$ and $x \in (a, \infty)$.

Thus, our future payoff function is:

$$\begin{aligned}\Lambda(x,a) &= (K-a)\left(\alpha\ inf_\sigma E^{\mathbb{P}}[e^{-rT_a^{x,r,\sigma}}] + (1-\alpha)\ sup_\sigma E^{\mathbb{P}}[e^{-rT_a^{x,r,\sigma}}]\right) \\ &= (K-a)\left[\alpha exp\left\{-\frac{1}{\sigma_1}(x-a)\sqrt{2r}\right\} + (1-\alpha) exp\left\{-\frac{1}{\sigma_2}(x-a)\sqrt{2r}\right\}\right]\end{aligned}$$

where σ_1 is the smallest and σ_2 is the largest

Our current payoff function is: $(K-x)^+$ .We then study the property of our functions.

For convenience, we denote

$$exp\left\{-\frac{1}{\sigma_1}(x-a)\sqrt{2r}\right\} \text{ as } \tilde{P}_1$$
$$exp\left\{-\frac{1}{\sigma_2}(x-a)\sqrt{2r}\right\} \text{ as } \tilde{P}_2$$

Lemma 3.1. *$x \mapsto \Lambda(x,a)$ is decreasing and convex on (a,∞) with $\lim_{x\to\infty}\Lambda(x,a) = 0$ and $\Lambda(a,a) = K-a$ and $\Lambda(0,a) = (K-a)\left[\alpha exp\left\{\frac{a}{\sigma_1}\sqrt{2r}\right\} + (1-\alpha)exp\left\{\frac{a}{\sigma_2}\sqrt{2r}\right\}\right]$*

Proof.

$$\begin{aligned}\Lambda_x(a,x) = (K-a)\Big[&-\alpha\frac{\sqrt{2r}}{\sigma_1}exp\left\{-\frac{1}{\sigma_1}(x-a)\sqrt{2r}\right\} \\ &-(1-\alpha)\frac{\sqrt{2r}}{\sigma_2}exp\left\{-\frac{1}{\sigma_2}(x-a)\sqrt{2r}\right\}\Big] < 0 \\ \Lambda_{xx}(a,x) = (K-a)\Big[&\alpha\frac{2r}{\sigma_1^2}exp\left\{-\frac{1}{\sigma_1}(x-a)\sqrt{2r}\right\} \\ &+(1-\alpha)\frac{2r}{\sigma_2^2}exp\left\{-\frac{1}{\sigma_2}(x-a)\sqrt{2r}\right\}\Big] > 0\end{aligned}$$

and

$$\lim_{x\to\infty}\Lambda(x,a) = (K-a)[0+0] = 0$$
$$\Lambda(a,a) = K-a[\alpha+1-\alpha] = K-a$$

□

Lemma 3.2. *Suppose the constraint on a is $(K-a)\left[\alpha exp\left\{\frac{a}{\sigma_1}\sqrt{2r}\right\} + (1-\alpha)exp\left\{\frac{a}{\sigma_2}\sqrt{2r}\right\}\right] < K$, denote $a's$ region as $\mathcal{A}$. Then, $(0,a)$ is a stopping region for $a \in [a^*, x\wedge K]\cap\mathcal{A}$ where $a^* = K - \frac{1}{\sqrt{2r}\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}$*

Proof.

$$\lim_{x\to a^*}\Lambda_x(x,a^*) = (K-a^*)\left[-\alpha\frac{\sqrt{2r}}{\sigma_1} - (1-\alpha)\frac{\sqrt{2r}}{\sigma_2}\right] = -1$$
$$\implies a^* = K - \frac{1}{\sqrt{2r}\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}$$

$\lim\limits_{x\to a}\Lambda_x(x,a)$ is an increasing function of a, thus, if $a<a^*$, $\lim\limits_{x\to a}\Lambda_x(x,a)<-1$, thus, $\Lambda_x(x,a)$ will have one intersection with $(K-x)^+$ on (a,K).

If $a>a^*$, $\lim\limits_{x\to a}\Lambda_x(x,a)>-1$, $\Lambda(x,a)>(K-x)^+$ on $x\in(a,\infty)$.

Meanwhile, if $a\in\mathcal{A}$, $\Lambda(0,a)<(K-0)^+$, therefore, we will have $\Lambda(x,a)<(K-x)^+$ for $x\in(0,a)$

Together, $(0,a)$ is a stopping region for $a\in[a^*,x\wedge K]\cap\mathcal{A}$. □

Lemma 3.3. *There exists an optimal stopping region $(0,a_{opt}]$such that $a_{opt}=argmax_a\Lambda(x,a)$ for $a\in[a^*,x\wedge K]\cap\mathcal{A}$. Although a_{opt} depends on the region $\mathcal{A}$, there is one thing we do know, which is $a\mapsto\Lambda(x,a)$ is decreasing on $a\in(a^*,x\wedge K)$*

Proof.

$$\Lambda_a(x,a)=\left[\frac{(K-a)\sqrt{2r}}{\sigma_1}-1\right]\alpha\tilde{P_1}+\left[\frac{(K-a)\sqrt{2r}}{\sigma_2}-1\right](1-\alpha)\tilde{P_2}$$

Let $\dfrac{(K-a)\sqrt{2r}}{\sigma_2}-1=0\Longrightarrow a=K-\dfrac{\sigma_2}{\sqrt{2r}}<a^*$, for $a\in(a^*,x\wedge K)$,

$$\frac{(K-a)\sqrt{2r}}{\sigma_2}-1<0$$

Because $\tilde{P_2}>\tilde{P_1}$

$$\begin{aligned}
\Lambda_a(x,a)&=\left[\frac{(K-a)\sqrt{2r}}{\sigma_1}-1\right]\alpha\tilde{P_1}+\left[\frac{(K-a)\sqrt{2r}}{\sigma_2}-1\right](1-\alpha)\tilde{P_2}\\
&<\left[\frac{(K-a)\sqrt{2r}}{\sigma_1}-1\right]\alpha\tilde{P_1}+\left[\frac{(K-a)\sqrt{2r}}{\sigma_2}-1\right](1-\alpha)\tilde{P_1}\\
&=\tilde{P_1}\left[\frac{(K-a)\sqrt{2r}\alpha}{\sigma_1}+\frac{(K-a)\sqrt{2r}(1-\alpha)}{\sigma_2}-1\right]<0\ \ \text{(because }\ a>a^*)
\end{aligned}$$

□

Lemma 3.4. *$\alpha\mapsto a^*$ is a monotonically increasing function.*

Proof. $a^*=K-\dfrac{\sigma_1\sigma_2}{\sqrt{2r}[\alpha(\sigma_2-\sigma_1)+\sigma_1]}$

Because $\sigma_2>\sigma_1$, $a^*(\alpha)$ is an increasing function.

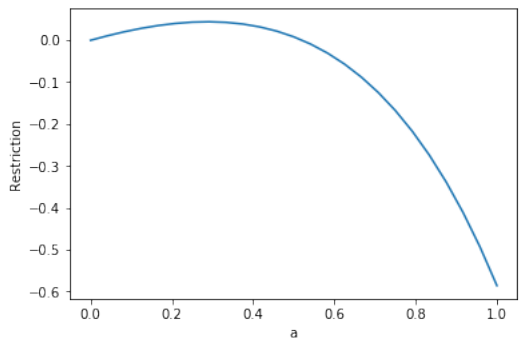
This is financially interpretable, as you become more and more averse to uncertainty, you want to stop eariler, thereby you have a bigger a^*. □

Now, we study an example where $\sigma_1 = 1$, $\sigma_2 = 2$, $r = 1$, $K = 1.2$, $\alpha = 0.5$ and $x = 1$

We have

$$a^* = K - \frac{1}{\sqrt{2r}\left(\frac{\alpha}{\sigma_1} + \frac{1-\alpha}{\sigma_2}\right)} = 0.26$$

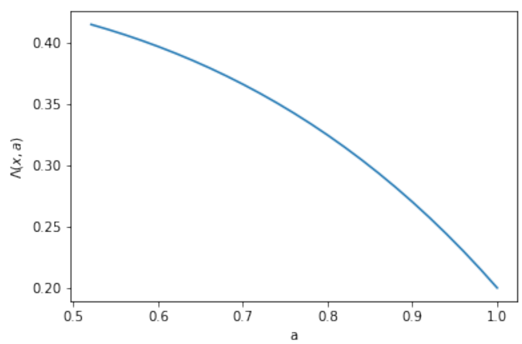
First, we plot out the $(K-a)\left[\alpha exp\left\{\frac{a}{\sigma_1}\sqrt{2r}\right\} + (1-\alpha)exp\left\{\frac{a}{\sigma_2}\sqrt{2r}\right\}\right] - K$ to determine $a's$ restriction region $\mathcal{A}$:



By using the numeric solver, we have $\mathcal{A} = [0.52, 1]$

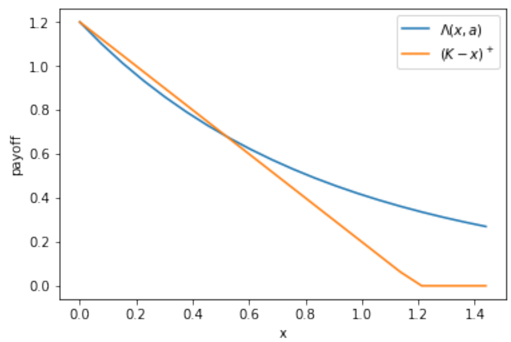
Hence, $a \in [a^*, x \wedge K] \cap \mathcal{A} \Rightarrow a \in [0.52, 1]$

Next, we plot out $a \to \Lambda(x, a)$ w.r.t $a \in [0.52, 1]$ to find the optimal stopping region:



As we can see from the plot, it is strictly decreasing (consistent with theorem 3). Therefore, our optimal $a_{opt} = 0.52$ and the optimal stopping region is $(0, 0.52)$.

The optimal equilibrium stopping region can also be verified through the following plot:



3.2 American Option with Linear Payoff Function on Absolute Value of Brownian Motion

We now consider the underlying state process, which follows $X_t^{x,r,\sigma} = |x + 0t + \sigma B_t| = |x + \sigma B_t|$.

The stopping time $T_a^{x,r,\sigma} = inf\ \{t > 0 : X_t^{x,r\sigma} = a\}\ (a > 0)$

We now consider the linear payoff function $f(x) = x$ (an increasing function), our stopping region should be: (a, ∞). Because we haven't reached the equilibrium yet, we have and $x \in (0, a)$.

From the formula, we can get: $E[e^{-rT_a}] = \dfrac{ch(\frac{x\sqrt{2r}}{\sigma})}{ch(\frac{a\sqrt{2r}}{\sigma})}$

Now we study $\sigma \mapsto f(\sigma) = E[e^{-rT_a}]$ where $ch(x) = \dfrac{e^x + e^{-x}}{2}$

$$f'(\sigma) = \frac{ch(\frac{a\sqrt{2r}}{\sigma})\sigma^2 x\sqrt{2r}[-exp(\frac{x\sqrt{2r}}{\sigma}) + exp(-\frac{x\sqrt{2r}}{\sigma})] - ch(\frac{x\sqrt{2r}}{\sigma})\sigma^2 a\sqrt{2r}[-exp(\frac{a\sqrt{2r}}{\sigma}) + exp(-\frac{a\sqrt{2r}}{\sigma})]}{2ch(\frac{a\sqrt{2r}}{\sigma})^2}$$

For convenience, we denote

$$exp\left(\frac{(a+x)\sqrt{2r}}{\sigma}\right) = \tilde{N}_1$$
$$exp\left(\frac{(a-x)\sqrt{2r}}{\sigma}\right) = \tilde{N}_2$$
$$exp\left(\frac{(x-a)\sqrt{2r}}{\sigma}\right) = \tilde{N}_3$$
$$exp\left(\frac{-(x+a)\sqrt{2r}}{\sigma}\right) = \tilde{N}_4$$

By spliting the ch function, we have $numerator = \dfrac{\sqrt{2r}}{2}\sigma^2(a-x)(\tilde{N}_1 + \tilde{N}_4 - \tilde{N}_2 - \tilde{N}_3)$

We have $\tilde{N}_1\tilde{N}_4 = \tilde{N}_2\tilde{N}_3 = 1$, because $|\tilde{N}_1 - \tilde{N}_4| > |\tilde{N}_2 - \tilde{N}_3|$, we have: $\tilde{N}_1 + \tilde{N}_4 > \tilde{N}_2 + \tilde{N}_3$

Therefore, $f'(\sigma) > 0$ and $\sigma \mapsto f(\sigma) = E[e^{-rT_a}]$ is an increasing function.

Thus, our future payoff function is: $\Lambda(x,a) = a\left[\alpha\frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})} + (1-\alpha)\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})}\right]$

(σ_1 is the smallest and σ_2 the biggest)

Our current payoff function is $f(x) = x$

Now we study their properties.

Lemma 3.5. *$x \mapsto \Lambda(x,a)$ is increasing and convex on $(0,a)$ with $\Lambda(a,a) = a$*

Proof.

$$\begin{aligned}\Lambda_x(x,a) = a\Big[&\alpha\frac{\sqrt{2r}}{2\sigma_1 ch(\frac{a\sqrt{2r}}{\sigma_1})}[exp(\frac{x\sqrt{2r}}{\sigma_1}) - exp(-\frac{x\sqrt{2r}}{\sigma_1})] \\ &+ (1-\alpha)\frac{\sqrt{2r}}{2\sigma_2 ch(\frac{a\sqrt{2r}}{\sigma_2})}[exp(\frac{x\sqrt{2r}}{\sigma_2}) - exp(-\frac{x\sqrt{2r}}{\sigma_2})]\Big]\end{aligned}$$

Because $e^x - e^{-x} > 0$ for $x > 0$, $\Lambda_x(x,a) > 0$

$$\begin{aligned}\Lambda_{xx}(x,a) = a\Big[&\alpha\frac{r}{\sigma_1^2 ch(\frac{a\sqrt{r}}{\sigma_1})}[exp(\frac{x\sqrt{2r}}{\sigma_1}) + exp(-\frac{x\sqrt{2r}}{\sigma_1})] \\ &+ (1-\alpha)\frac{r}{\sigma_2^2 ch(\frac{a\sqrt{2r}}{\sigma_2})}[exp(\frac{x\sqrt{2r}}{\sigma_2}) + exp(-\frac{x\sqrt{2r}}{\sigma_2})]\Big] > 0\end{aligned}$$

□

Lemma 3.6. *Though there is no equilibrium stopping region through one iteration, $(x^*(a), a)$ is a future stopping region for $a > max(x, a^*, K)$ where a^* is the solution of $\Lambda_x(x,a)\Big|_{x=a} = 1$ and $x^*(a)$ is the intersection of $\Lambda(x,a)$ and x*

Proof.

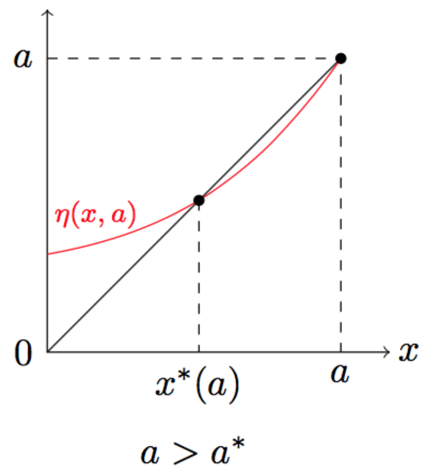
$$\begin{aligned}\Lambda_x(x,a)\Big|_{x=a} = a\Big[&\alpha\frac{\sqrt{2r}}{2\sigma_1 ch(\frac{a\sqrt{2r}}{\sigma_1})}[exp(\frac{a\sqrt{2r}}{\sigma_1}) - exp(-\frac{a\sqrt{2r}}{\sigma_1})] \\ &+ (1-\alpha)\frac{\sqrt{2r}}{2\sigma_2 ch(\frac{a\sqrt{2r}}{\sigma_2})}[exp(\frac{a\sqrt{2r}}{\sigma_2}) - exp(-\frac{a\sqrt{2r}}{\sigma_2})]\Big]\end{aligned}$$

Because $\frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{1+e^{2x}}$ is an increasing function, $\Lambda_x(a,x)\Big|_{x=a}$ is an increasing function of a and $\lim_{a\to 0} \Lambda_x(x,a)\Big|_{x=a} = 0$

Therefore, $\Lambda_x(x,a)\Big|_{x=a} = 1$ will certainly have only one solution, denoted as a^*.

When $a > max(x, a^*, K)$, $\Lambda_x(x,a)\Big|_{x=a} > 1$, thus, for $x \in (x^*(a), a)$, $x > \Lambda(x,a)$, and $(x^*(a), a)$ is indeed a stopping region.

The picture should look like below:



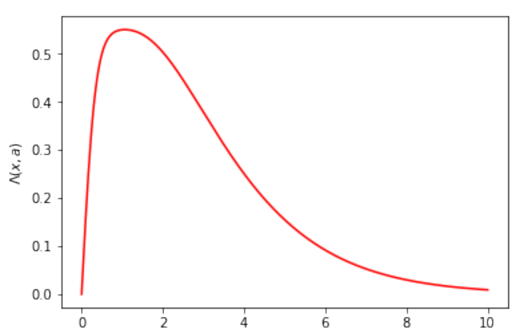
$\square$

Lemma 3.7. *There exists an optimal future stopping region for* $(x^*(a_{opt}, a_{opt})$ *where* $a_{opt} = max(\text{the solution to } \Lambda_a(x, a_{opt}) = 0, a^*, x, K)$ *for* $a > max(x, a^*, K)$.

Proof. Recall

$$\Lambda(x,a) = a\left[\alpha\frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})} + (1-\alpha)\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})}\right]$$

Because $\dfrac{x}{ch(x)}$ is a function that increases first and then decreases, by setting $\sigma_1 = 0.05, \sigma_2 = 0.2$ and $x = 0.5$, $a \mapsto \Lambda(x,a)$ should look like:



Now we study $\Lambda_a(x,a)$

$$\Lambda_a(x,a) = \alpha\frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})} + (1-\alpha)\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})}$$
$$+ a\Big[-\alpha ch(\frac{x\sqrt{2r}}{\sigma_1})ch(\frac{a\sqrt{2r}}{\sigma_1})^{-2}\frac{\sqrt{2r}}{2\sigma_1}\left(exp\left(\frac{a\sqrt{2r}}{\sigma_1}\right) - exp\left(-\frac{a\sqrt{2r}}{\sigma_1}\right)\right)$$
$$-(1-\alpha)ch(\frac{x\sqrt{2r}}{\sigma_2})ch(\frac{a\sqrt{2r}}{\sigma_2})^{-2}\frac{\sqrt{2r}}{2\sigma_2}\left(exp\left(\frac{a\sqrt{2r}}{\sigma_2}\right) - exp\left(-\frac{a\sqrt{2r}}{\sigma_2}\right)\right)\Big]$$

Denote

$$\frac{\sqrt{2r}}{2\sigma_1 ch(\frac{a\sqrt{2r}}{\sigma_1})}[exp(\frac{a\sqrt{2r}}{\sigma_1}) - exp(-\frac{a\sqrt{2r}}{\sigma_1})] \quad \text{as} \quad \tilde{P}_1$$
$$\frac{\sqrt{2r}}{2\sigma_2 ch(\frac{a\sqrt{2r}}{\sigma_2})}[exp(\frac{a\sqrt{2r}}{\sigma_2}) - exp(-\frac{a\sqrt{2r}}{\sigma_2})] \quad \text{as} \quad \tilde{P}_2$$

Then,

$$\Lambda_a(x,a) = \alpha\frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})} + (1-\alpha)\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})} + a\Big[-\alpha\frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})}\tilde{P}_1 - (1-\alpha)\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})}\tilde{P}_2\Big]$$
$$= \frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a\sqrt{2r}}{\sigma_1})}\alpha[1-a\tilde{P}_1] + \frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a\sqrt{2r}}{\sigma_2})}(1-\alpha)[1-a\tilde{P}_2]$$

We know that $a^*[\alpha\tilde{P}_1 + (1-\alpha)\tilde{P}_2] = 1$

Because $\frac{\sqrt{2r}}{2\sigma ch(\frac{a^*\sqrt{2r}}{\sigma})}[exp(\frac{a^*\sqrt{2r}}{\sigma}) - exp(-\frac{a^*\sqrt{2r}}{\sigma})]$ is a decreasing function of σ, we can get:
$\tilde{P}_1 > \frac{1}{a^*} > \tilde{P}_2$

Also, because $x < a^*$, $\frac{ch(\frac{x\sqrt{2r}}{\sigma})}{ch(\frac{a^*\sqrt{2r}}{\sigma})}$ is an increasing function of σ, thus, we have:

$$\frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a^*\sqrt{2r}}{\sigma_2})} > \frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a^*\sqrt{2r}}{\sigma_1})}$$

Therefore:

$$\Lambda_a(x,a^*) = \frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a^*\sqrt{2r}}{\sigma_1})}\alpha[1-a^*\tilde{P}_1] + \frac{ch(\frac{x\sqrt{2r}}{\sigma_2})}{ch(\frac{a^*\sqrt{2r}}{\sigma_2})}(1-\alpha)[1-a^*\tilde{P}_2]$$
$$> \frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a^*\sqrt{2r}}{\sigma_1})}\alpha[1-a^*\tilde{P}_1] + \frac{ch(\frac{x\sqrt{2r}}{\sigma_1})}{ch(\frac{a^*\sqrt{2r}}{\sigma_1})}(1-\alpha)[1-a^*\tilde{P}_2] = 0$$

Thus,

$$\Lambda_a(x,a^*) > 0$$

for a^* where $\Lambda_x(x,a^*)\Big|_{x=a^*} = 1$

Meanwhile, there is a very important finding that

$$\lim_{a\to\infty} \Lambda_a(x,a) = -\infty$$

Combining the above two statements, we can conclude that in order to satisfy the condition $a > max(x,a^*)$, we just need to discuss the optimal stopping region by two cases regarding x:

$$\text{if } \Lambda_a(x,x) > 0\text{: } a_{opt} \text{ is the solution to } \Lambda_a(x,a_{opt}) = 0$$

$$\text{if } \Lambda_a(x,x) \leq 0\text{: } a_{opt} = x$$

We can obtain a_{opt} through numerical methods.

In conclusion, $(x^*(a),a)$ is a future stopping region for $a > a^*$ where a^* is the solution of $\Lambda_x(x,a)\Big|_{x=a} = 1$ and $x^*(a)$ is the intersection of $\Lambda(x,a)$ and x. Also, there exists an optimal future stopping region, where a_{opt} can be the solution to $\Lambda_a(x,a_{opt}) = 0$ or simply x, depending on the value of x. $\square$

Lemma 3.8. *For the case where $a \in [max(x,K),a^*] \neq \emptyset$, $(a,x^*(a))$ is also the future stopping where $x^*(a)$ is the solution of $(x-K)^+ = \Lambda(x,a)$. Still, it is not the equilibrium region as we don't have $\Theta(R) = R$. Furthermore, we could certainly obtain an a_{opt}, given we have a close region of a.*

However, I haven't figured out how to choose the better future stopping region between the cases $a \in [max(x,a^*,K),\infty)$ and $a \in [max(x,K),a^*]$, and for this example, we just focus on the former case.

Now we get numerical results with assigned parameters.

Now we try to obtain the numerical method by letting $r = 0.01, \sigma_1 = 0.05, \sigma_2 = 0.2$ and $\alpha = 0.5$

Then,

$$\Lambda(x,a) = a\Big[0.5\frac{ch(\frac{x\sqrt{0.02}}{0.05})}{ch(\frac{a\sqrt{0.02}}{0.05})} + 0.5\frac{ch(\frac{x\sqrt{0.02}}{0.2})}{ch(\frac{a\sqrt{0.02}}{0.2})}\Big]$$

$$\Lambda_a(x,a) = 0.5\frac{ch(\frac{x\sqrt{0.02}}{0.05})}{ch(\frac{a\sqrt{0.02}}{0.05})} + 0.5\frac{ch(\frac{x\sqrt{0.02}}{0.2})}{ch(\frac{a\sqrt{0.02}}{0.2})}$$
$$+ a\Big[-0.5ch(\frac{x\sqrt{0.02}}{0.05})ch(\frac{a\sqrt{0.02}}{0.05})^{-2}\frac{\sqrt{0.02}}{0.1}\left(exp\left(\frac{a\sqrt{0.02}}{0.05}\right) - exp\left(-\frac{a\sqrt{0.02}}{0.05}\right)\right)$$

$$- 0.5ch(\frac{x\sqrt{0.02}}{0.2})ch(\frac{a\sqrt{0.02}}{0.2})^{-2}\frac{\sqrt{0.02}}{0.4}\left(exp\left(\frac{a\sqrt{0.02}}{0.2}\right) - exp\left(-\frac{a\sqrt{0.02}}{0.2}\right)\right)]$$

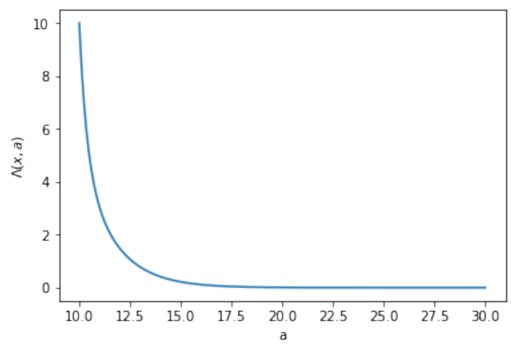
By using $\Lambda_x(x,a^*)\Big|_{x=a^*} = 1$, now that we can have $a^* = 0.6645$

By using the theorem we developed in (1), (2) and (3), we now test on different asset prices.

When $x = 10$

We first have $a > max(x, a^*) = 10$

Then, we plot out $a \mapsto \Lambda(x,a)$ for $a > max(x,a^*)$ to see how it behaves:

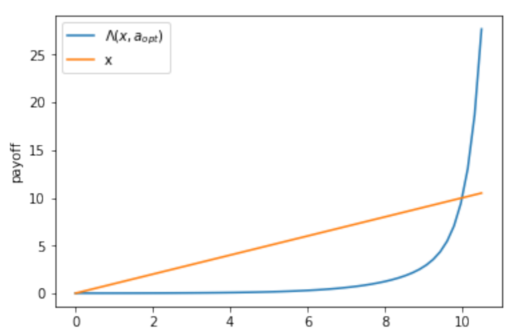


We can see that it is decreasing, therefore, $a_{opt} = x = 10$, which perfectly corresponds to the theorem we developed.

Last, by using numerical method, we can calculate $x^*(a_{opt}) = 0.00849$

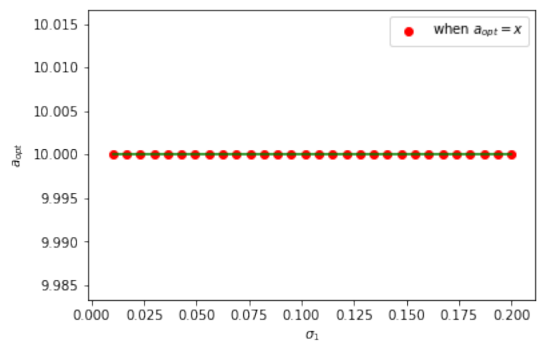
Therefore, our optimal stopping region should be $(0.00849, 10)$ in this case.

Here, we visualize $x \mapsto \Lambda(x, a_{opt})$ and the payoff function x to have a look at what is happening here:

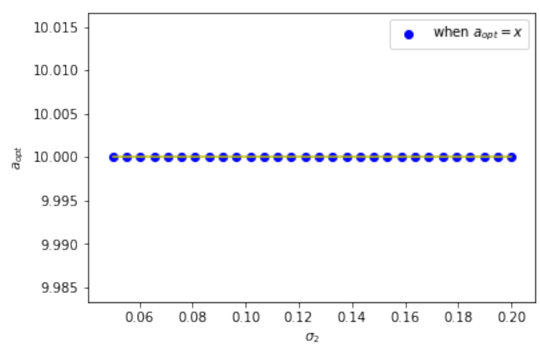


Now, we also want to study the relation between σ's and a_{opt}.

First, we fix $\sigma_2 = 0.2$ and let σ_1 take 30 values from 0.01 to 0.2, and plot the $a_{opt}(\sigma_1)$ out:



Next, we fix $\sigma_1 = 0.05$ and let σ_2 take 30 values from 0.05 to 0.2, and plot the $a_{opt}(\sigma_2)$ out:

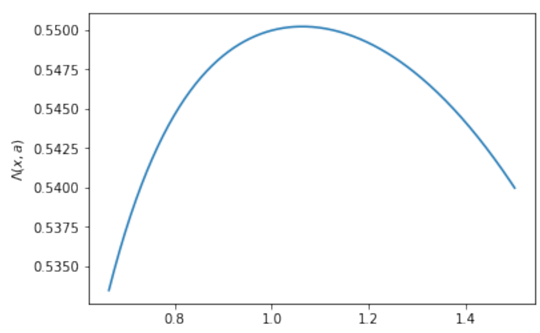


We can see that when $x = 10$, all $a_{opt} = x$ and this case isn't worth studying.

When $x = 0.5$

We first have $a > max(x, a^*) = a^*$

Then, we plot out $a \mapsto \Lambda(x, a)$ for $a > max(x, a^*)$ to see how it behaves:

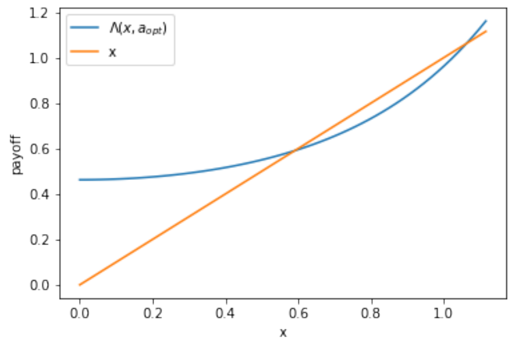


We can see that it is first increasing and then decreasing, which means we can reach a global maximum. By using numerical method, we have $a_{opt} = 1.0627332$, which perfectly corresponds to the theorem we developed in (3).

Last, by using numerical method, we can calculate $x^*(a_{opt}) = 0.59078638$

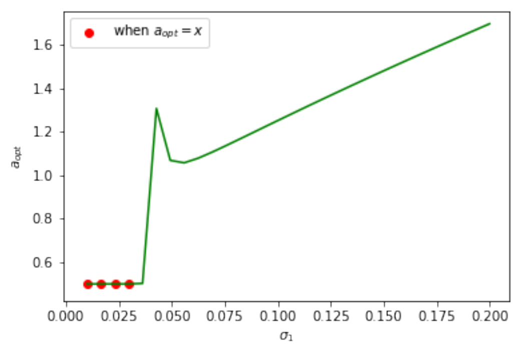
Therefore, our optimal stopping region should be $(0.59078638, 1.0627332)$ in this case.

Here, we visualize $x \mapsto \Lambda(x, a_{opt})$ and the payoff function x to have a look at what is happening here:

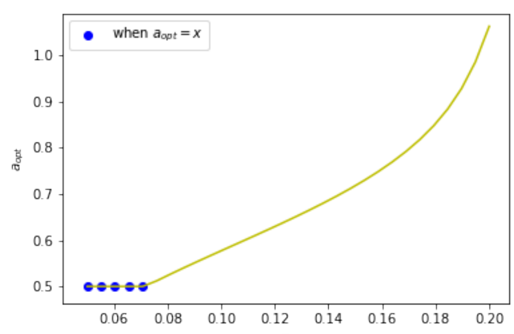


Now, we also want to study the relation between σ's and a_{opt}, where we denote $\underline{\sigma} = \sigma_1$ and $\overline{\sigma} = \sigma_2$

First, we fix $\sigma_2 = 0.2$ and let σ_1 take 30 values from 0.01 to 0.2, and plot the $a_{opt}(\sigma_1)$ out:



Next, we fix $\sigma_1 = 0.05$ and let σ_2 take 30 values from 0.05 to 0.2, and plot the $a_{opt}(\sigma_2)$ out:



We can see that when $x = 0.5$, except the first few points where $a_{opt} = x$, when σ's are increasing, a_{opt} is also increasing.

This intuitively makes sense. As the underlying distribution of the asset becomes more and more volatile, we want to stop earlier because we don't want to risk our money in the uncertainty. Therefore, a_{opt} is larger.

In conclusion, in this example, we are not able to find the equilibrium stopping region through one iteration, but we can find the future stopping region. We can use this method to calculate the optimal future stopping region everytime we obtain an asset price x.

3.3 American Call Option on Geometric Brownian Motion with Both Drift and Volatility Uncertainty

We consider the payoff function as a call option such that $g(x) = (x-K)^+$ for a given constant $K > 0$ and a Geometric Brownian motion with both drift and volatility uncertainty that

$$X_t^\sigma = x + \int_0^t bX_s^\sigma ds + \int_0^t \sigma X_s^\sigma dB_s, \quad \forall t \geq 0, \qquad \mathbb{P}_0\text{-a.s.},$$

in which the negative drift $b < 0$ is an unknown constant that $\underline{b} \leq b \leq \overline{b} < 0$ and σ is an unknown constant that $0 \leq \underline{\sigma} \leq \sigma \leq \overline{\sigma} < \infty$. The interest rate $r > 0$ is a given constant.

The expected payoff $J(x,R)$ in our set up is written as

$$J(x,R) = \alpha \inf_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}[e^{-r\rho_R}(X_{\rho_R}^\sigma - K)^+] + (1-\alpha) \sup_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}[e^{-r\rho_R}(X_{\rho_R}^\sigma - K)^+].$$

For any $a \geq K$ and $x \leq a$, let us define

$$\Lambda(x,a) = \alpha \inf_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_a^{x,\sigma}}(a-K)\right] + (1-\alpha) \sup_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_a^{x,\sigma}}(a-K)\right], \quad \text{for } x \leq a,$$

as well as two auxiliary functions

$$\Lambda_1(x,a) := \inf_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_a^{x,\sigma}}\right]$$

and

$$\Lambda_2(x,a) = \sup_{\substack{b\in[\underline{b},\overline{b}] \\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_a^{x,\sigma}}\right], \text{ for } x \leq a.$$

As Borodin and Salminen (2002) has calculated, we have

$$\Lambda_1(x,a) = \inf_{\substack{b\in[\underline{b},\bar{b}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \left(\frac{x}{a}\right)^{\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}-\frac{b}{\sigma^2}+\frac{1}{2}},$$

and

$$\Lambda_2(x,a) = \sup_{\substack{b\in[\underline{b},\bar{b}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} \left(\frac{x}{a}\right)^{\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}-\frac{b}{\sigma^2}+\frac{1}{2}}.$$

Lemma 3.9. *For the fixed $x \le a$, the function*

$$g(b,\sigma) = \left(\frac{x}{a}\right)^{\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}-\frac{b}{\sigma^2}+\frac{1}{2}}$$

is strictly increasing in b and strictly decreasing in σ.

Proof. Let us denote $k = -\frac{b}{\sigma^2}+\frac{1}{2}$. For the first claim, it is sufficient to show that the exponent $h(k) := \sqrt{k^2+\frac{2r}{\sigma^2}}+k$ is strictly increasing in k. By taking the derivative, we have

$$h'(k) = \frac{k+\sqrt{k^2+\frac{2r}{\sigma^2}}}{\sqrt{k^2+\frac{2r}{\sigma^2}}} > 0,$$

which implies that $g(b,\sigma)$ is strictly increasing in b.

To show that g is strictly increasing in σ, it is sufficient to show that $\kappa(\sigma) := \sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}} - \frac{b}{\sigma^2}+\frac{1}{2}$ is strictly decreasing. By direct calculation,

$$\kappa'(\sigma) = \left(\frac{2}{\sigma^3}\right)\frac{b\left[\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}+\frac{b}{\sigma^2}-\frac{1}{2}\right]+r}{\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}}$$

In this case, we have $b<0$, because

$$\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2} = \frac{1}{4}+\frac{b^2}{\sigma^4}-\frac{b}{\sigma^2}+\frac{2r}{\sigma^2} < (\frac{1}{2}-\frac{b}{\sigma^2}-\frac{r}{b})^2 = \frac{1}{4}+\frac{b^2}{\sigma^4}-\frac{b}{\sigma^2}+\frac{r^2}{b^2}+\frac{2r}{\sigma^2}-\frac{r}{b},$$

We have:

$$\begin{aligned} b\left[\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}+\frac{b}{\sigma^2}-\frac{1}{2}\right]+r &> b\left[\sqrt{(\frac{1}{2}-\frac{b}{\sigma^2}-\frac{r}{b})^2}+\frac{b}{\sigma^2}-\frac{1}{2}\right]+r \\ &= b\left[\frac{1}{2}-\frac{b}{\sigma^2}-\frac{r}{b}+\frac{b}{\sigma^2}-\frac{1}{2}\right]+r = 0 \end{aligned}$$

Therefore we have $\kappa'(\sigma)>0$ and we can further prove $g(b,\sigma)$ is decreasing on σ. □

By Lemma 3.9, Λ_1 and Λ_2 can be computed explicitly using the boundary points that

$$\Lambda_1(x,a) = \left(\frac{x}{a}\right)^{\sqrt{\frac{\underline{b}^2}{\bar{\sigma}^4}+\frac{(2r-\underline{b})}{\bar{\sigma}^2}+\frac{1}{4}}-\frac{\underline{b}}{\bar{\sigma}^2}+\frac{1}{2}} \quad \text{and} \quad \Lambda_2(x,a) = \left(\frac{x}{a}\right)^{\sqrt{\frac{\bar{b}^2}{\underline{\sigma}^4}+\frac{(2r-\bar{b})}{\underline{\sigma}^2}+\frac{1}{4}}-\frac{\bar{b}}{\underline{\sigma}^2}+\frac{1}{2}}$$

Let us denote $m_1 = \sqrt{\frac{\underline{b}^2}{\bar{\sigma}^4}+\frac{(2r-\underline{b})}{\bar{\sigma}^2}+\frac{1}{4}}-\frac{\underline{b}}{\bar{\sigma}^2}+\frac{1}{2} > 0$ and $m_2 = \sqrt{\frac{\bar{b}^2}{\underline{\sigma}^4}+\frac{(2r-\bar{b})}{\underline{\sigma}^2}+\frac{1}{4}}-\frac{\bar{b}}{\underline{\sigma}^2}+\frac{1}{2} > 0$,

with $b < 0$, we have $m_2 > m_1 > 1$.

$\Lambda(x,a)$ can be rewritten as

$$\begin{aligned}\Lambda(x,a) &= \alpha(a-K)\left(\frac{x}{a}\right)^{\sqrt{\frac{\underline{b}^2}{\bar{\sigma}^4}+\frac{(2r-\underline{b})}{\bar{\sigma}^2}+\frac{1}{4}}-\frac{\underline{b}}{\bar{\sigma}^2}+\frac{1}{2}} + (1-\alpha)(a-K)\left(\frac{x}{a}\right)^{\sqrt{\frac{\bar{b}^2}{\underline{\sigma}^4}+\frac{(2r-\bar{b})}{\underline{\sigma}^2}+\frac{1}{4}}-\frac{\bar{b}}{\underline{\sigma}^2}+\frac{1}{2}} \\ &= \alpha(a-K)\left(\frac{x}{a}\right)^{m_1} + (1-\alpha)(a-K)\left(\frac{x}{a}\right)^{m_2}\end{aligned}$$

Lemma 3.10. *The function $x \mapsto \Lambda(x,a)$ is strictly increasing and strictly convex on $(0,a)$ with $\Lambda(a,a) = K - a$ and $\lim_{x\to 0}\Lambda(x,a) = 0$*

Lemma 3.11. *Though there is no equilibrium stopping region through one iteration, $(a, x^*(a))$ is a future stopping region for $a \in [max(x,K), a^*] \neq \emptyset$ where $a^* = \dfrac{m_1\alpha + m_2(1-\alpha)}{m_1\alpha+m_2(1-\alpha)-1}K$ and $x^*(a)$ is the solution to $\Lambda(x,a) = (x-K)^+$.*

Proof. $\Lambda_x(x,a) = \dfrac{a-K}{a}\left[\alpha m_1\left(\dfrac{x}{a}\right)^{m_1-1} + (1-\alpha)m_2\left(\dfrac{x}{a}\right)^{m_2-1}\right]$

$\Lambda_x(x,a)\Big|_{x=a} = \dfrac{a-K}{a}[\alpha m_1 + (1-\alpha)m_2] = 1$

It is an increasing function of a

$\Lambda_x(x,a)\Big|_{x=a^*} \Longrightarrow a^* = \dfrac{m_1\alpha+m_2(1-\alpha)}{m_1\alpha+m_2(1-\alpha)-1}K$

When $a < a^*, \Lambda_x(x,a)\Big|_{x=a} < \Lambda_x(x,a)\Big|_{x=a^*} = 1$, therefore, $(x-K)^+ > \Lambda(x,a)$ on $(a, x^*(a))$

When $a > a^*, \Lambda(x,a) > (x-K)^+$ on (a,∞)

Therefore, we need $a < a^*$ and $(a, x^*(a))$ is the future stopping region. □

Lemma 3.12. $(a^*, x^*(a^*)) = \emptyset$ *is the optimal future stopping region where* $a^* = \dfrac{m_1\alpha+m_2(1-\alpha)}{m_1\alpha+m_2(1-\alpha)-1}K$

Proof.

$$\begin{aligned}\frac{\partial\Lambda(x,a)}{\partial a} &= \alpha\left(\frac{x}{a}\right)^{m_1} + (1-\alpha)\left(\frac{x}{a}\right)^{m_2} - (a-K)\left(m_1\alpha\frac{x^{m_1}}{a^{m_1+1}} + m_2(1-\alpha)\frac{x^{m_2}}{a^{m_2+1}}\right) \\ &= \alpha\left(\frac{x}{a}\right)^{m_1}\left[1 - \frac{(a-K)m_1}{a}\right] + (1-\alpha)\left(\frac{x}{a}\right)^{m_2}\left[1 - \frac{(a-K)m_2}{a}\right]\end{aligned}$$

We conclude that $\dfrac{\partial\Lambda(x,a)}{\partial a}$ is a decreasing function of a

Therefore, for $a \in [max(x,K), a^*]$,

$$\begin{aligned}
&\frac{\partial\Lambda(x,a)}{\partial a} > \frac{\partial\Lambda(x,a)}{\partial a}\Big|_{a=a^*} \\
&= \alpha\left(\frac{x}{a^*}\right)^{m_1}\left[1-\frac{m_1}{m_1\alpha+m_2(1-\alpha)}\right]+(1-\alpha)\left(\frac{x}{a^*}\right)^{m_2}\left[1-\frac{m_2}{m_1\alpha+m_2(1-\alpha)}\right] \\
&> \alpha\left(\frac{x}{a^*}\right)^{m_1}\left[1-\frac{m_1}{m_1\alpha+m_2(1-\alpha)}\right]+(1-\alpha)\left(\frac{x}{a^*}\right)^{m_1}\left[1-\frac{m_2}{m_1\alpha+m_2(1-\alpha)}\right] \\
&= \left(\frac{x}{a^*}\right)^{m_1}\left[\frac{\alpha(\alpha-1)(m_1-m_2)+(1-\alpha)(m_1-m_2)\alpha}{m_1\alpha+m_2(1-\alpha)}\right]=0
\end{aligned}$$

Therefore, $a \mapsto \Lambda(x,a)$ is increasing on $[max(x,K), a^*]$, therefore our optimal future stopping region value $a_{opt} = a^* = \dfrac{m_1\alpha+m_2(1-\alpha)}{m_1\alpha+m_2(1-\alpha)-1}K$.

But the generated future stopping region $(a^*, x^*(a^*)) = \emptyset$, which means we will never stop in the future, and this imight mean we should abandon this case. □

Lemma 3.13. *For the case $a \in [max(x,a^*,K), \infty)$, $(x^*(a), a)$ is also the future stopping region, where $x^*(a)$ is the solution of $(x-K)^+ = \Lambda(x,a)$. Still, it is not the equilibrium region as we don't have $\Theta(R) = R$. Furthermore, we could certainly obtain an a_{opt}, given we have a close region of a.*

However, I haven't figured out how to choose the better future stopping region between the cases $a \in [max(x,K), a^*]$ and $a \in [max(x,a^*,K), \infty)$, and for this example, we just focus on the former case.

Next, we want to study the relation between $a^* = \dfrac{m_1\alpha+m_2(1-\alpha)}{m_1\alpha+m_2(1-\alpha)-1}K$ and σ's.

Lemma 3.14. *$a^*(\underline{\sigma}, \overline{\sigma})$ is an increasing function of σ's.*

Proof. First, we have

$$\frac{\partial a^*}{\partial m_1} = -\frac{\alpha}{(m_1\alpha+m_2(1-\alpha)-1)^2}K \tag{3.1}$$

With

$$m_1 = \sqrt{\frac{\underline{b}^2}{\overline{\sigma}^4}+\frac{(2r-\underline{b})}{\overline{\sigma}^2}+\frac{1}{4}}-\frac{\underline{b}}{\overline{\sigma}^2}+\frac{1}{2}$$

We can also get

$$\frac{dm_1}{d\overline{\sigma}} = -\frac{1}{2\sqrt{\frac{\underline{b}^2}{\overline{\sigma}^4}+\frac{(2r-\underline{b})}{\overline{\sigma}^2}+\frac{1}{4}}}\left[\frac{4b^2}{\overline{\sigma}^5}+\frac{2(2r-b)}{\overline{\sigma}^3}\right]+\frac{2b}{\overline{\sigma}^3} \tag{3.2}$$

Therefore, we can get:

$$\frac{\partial a^*}{\partial \overline{\sigma}} = \frac{\partial a^*}{\partial m_1}\frac{dm_1}{d\overline{\sigma}} = \frac{\alpha K}{(m_1\alpha + m_2(1-\alpha)-1)^2}\left[\frac{1}{2\sqrt{\frac{\underline{b}^2}{\overline{\sigma}^4}+\frac{(2r-\underline{b})}{\overline{\sigma}^2}+\frac{1}{4}}}\left[\frac{4\underline{b}^2}{\overline{\sigma}^5}+\frac{2(2r-\underline{b})}{\overline{\sigma}^3}\right]+\frac{2\underline{b}}{\overline{\sigma}^3}\right] > 0 \tag{3.3}$$

With

$$m_2 = \sqrt{\frac{\overline{b}^2}{\underline{\sigma}^4}+\frac{(2r-\overline{b})}{\underline{\sigma}^2}+\frac{1}{4}} - \frac{\overline{b}}{\underline{\sigma}^2}+\frac{1}{2}$$

Similarly, we have:

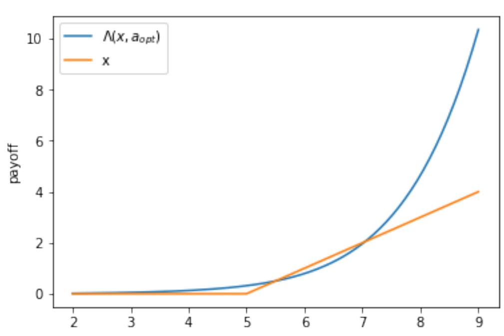
$$\frac{\partial a^*}{\partial \underline{\sigma}} = \frac{\partial a^*}{\partial m_2}\frac{dm_1}{d\underline{\sigma}} = \frac{(1-\alpha)K}{(m_1\alpha + m_2(1-\alpha)-1)^2}\left[\frac{1}{2\sqrt{\frac{\overline{b}^2}{\underline{\sigma}^4}+\frac{(2r-\overline{b})}{\underline{\sigma}^2}+\frac{1}{4}}}\left[\frac{4\overline{b}^2}{\underline{\sigma}^5}+\frac{2(2r-\overline{b})}{\underline{\sigma}^3}\right]+\frac{2\overline{b}}{\underline{\sigma}^3}\right] > 0 \tag{3.4}$$

Therefore, as we can see, both $\frac{\partial a^*}{\partial \overline{\sigma}}$ and $\frac{\partial a^*}{\partial \underline{\sigma}}$ are > 0, implying a^* is generally increasing with σ's. This intuitively makes sense. As σ's increasing, we become more and more uncertain about the trend of the asset, thereby we enjoy more volatility and want to stop later. □

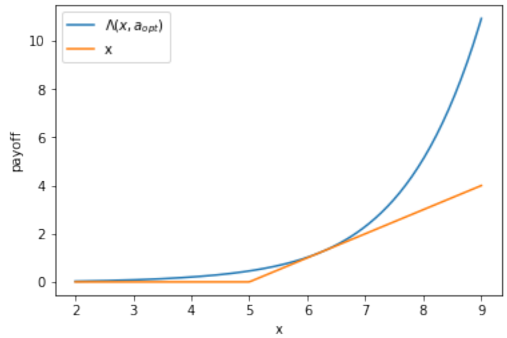
Let's study a numerical example below: Let $r = 0.2, \underline{\sigma} = 0.2, \overline{\sigma} = 0.8$,$\underline{b} = -0.5, \overline{b} = -0.1$, $K = 5$, $\alpha = 0.5$ and $x = 2$.

By applying the above frame work, we have: $a_{opt} = a^* = 6.2276$.

When $a = 5.5 \in (K, a^*)$, the plot is shown below:



When $a = a_{opt}$, the plot is shown below:



We can see that our theorem on the future stopping region is verified. However, we have the future stopping region $\Theta(R) = \emptyset$. This might indicate that we'd better abandon the current case and choose the case $a \in [max(x, a^*, K), \infty)$ to analyze the problem like the previous example.

3.4 American Call Option on Geometric Brownian Motion with Two Barriers

For the call option, we now study the case whereby $b > 0$ and we want to compare two barriers equilibriums such as $[p, q] \cup [l, \infty)$. Now we consider $q < x < l$ where $q > K$ and $l > K$ and $r > 0$ Now, for the given Geometric Brownian motion, we need to consider two barriers hitting time $T_{p,q}$ instead of one barrier hitting time. For any $x \in (q, l)$, let us define $T_{p,q} := \inf\{t \geq 0 : X_t^{x,\sigma} \notin (q, l)\}$. We will have

$$\begin{aligned} J(x, R) =& \alpha \inf_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \left((q - K)\mathbb{E}\left[e^{-rT_{p,q}} \mathbf{1}_{\{X^{x,\sigma}_{T_{q,l}} = q\}} \right] + (l - K)\mathbb{E}\left[e^{-rT_{q,l}} \mathbf{1}_{\{X^{x,\sigma}_{T_{q,l}} = l\}} \right] \right) \\ &+ (1 - \alpha) \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \left((q - K)\mathbb{E}\left[e^{-rT_{p,q}} \mathbf{1}_{\{X^{x,\sigma}_{T_{q,l}} = q\}} \right] + (l - K)\mathbb{E}\left[e^{-rT_{q,l}} \mathbf{1}_{\{X^{x,\sigma}_{T_{q,l}} = l\}} \right] \right) \\ &= \alpha \inf_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \left((q - K)h_1(x, q, l, \sigma) + (l - K)h_2(x, q, l, \sigma) \right) \\ &+ (1 - \alpha) \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \left((q - K)h_1(x, q, l, \sigma) + (l - K)h_2(x, q, l, \sigma) \right), \end{aligned}$$

where we have

$$h_1(x, q, l, \sigma) = \left(\frac{q}{x}\right)^{\frac{b}{\sigma^2} - \frac{1}{2}} \frac{\left(\frac{l}{x}\right)^{\sqrt{(\frac{b}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}} - \left(\frac{x}{l}\right)^{\sqrt{(\frac{b}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}}}{\left(\frac{l}{q}\right)^{\sqrt{(\frac{b}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}} - \left(\frac{q}{l}\right)^{\sqrt{(\frac{b}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}}}, \tag{3.5}$$

$$h_2(x,p,q,\sigma) = \left(\frac{l}{x}\right)^{\frac{b}{\sigma^2}-\frac{1}{2}} \frac{\left(\frac{x}{q}\right)^{\sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}} - \left(\frac{q}{x}\right)^{\sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}}}{\left(\frac{l}{q}\right)^{\sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}} - \left(\frac{q}{l}\right)^{\sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}}}. \tag{3.6}$$

Let us consider the function

$$H(x,p,q,\sigma) := (q-K)h_1(x,q,l,\sigma) + (l-K)h_2(x,q,l,\sigma), \tag{3.7}$$

Then,

$$J(x,R) = \alpha \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]} H(x,q,l,\sigma) + (1-\alpha) \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} H(x,q,l,\sigma)$$

From now on, we study the property of $H(x,q,l,\sigma)$

First, let us define

$$\begin{aligned}
m(\sigma) &= \frac{b}{\sigma^2} - \frac{1}{2} \\
n(\sigma) &= \sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}} > m \\
k_1(\sigma) &= \frac{q^{m(\sigma)} l^{n(\sigma)}}{\left(\frac{l}{q}\right)^{n(\sigma)} - \left(\frac{q}{l}\right)^{n(\sigma)}} > 0 \\
k_2(\sigma) &= \frac{q^{m(\sigma)} l^{-n(\sigma)}}{\left(\frac{l}{q}\right)^{n(\sigma)} - \left(\frac{q}{l}\right)^{n(\sigma)}} > 0 \\
k_3(\sigma) &= \frac{l^{m(\sigma)} q^{-n(\sigma)}}{\left(\frac{l}{q}\right)^{n(\sigma)} - \left(\frac{q}{l}\right)^{n(\sigma)}} > 0 \\
k_4(\sigma) &= \frac{l^{m(\sigma)} q^{n(\sigma)}}{\left(\frac{l}{q}\right)^{n(\sigma)} - \left(\frac{q}{l}\right)^{n(\sigma)}} > 0
\end{aligned}$$

Then we have:

$$h_1(x,q,l,\sigma) = k_1 x^{-m-n} - k_2 x^{n-m}$$

$$h_2(x,q,l,\sigma) = k_3 x^{n-m} - k_4 x^{-m-n}$$

Remark 3.1. *When $b>0$, $m(\sigma)$ is strictly decreasing and $n(\sigma)$ is strictly decreasing when $b > \dfrac{\sigma^2}{2}$.*

Proof. By a direct calculation,

$$m'(\sigma) = \frac{-2b}{\sigma^3} < 0$$
$$n'(\sigma) = \frac{-4\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)\frac{b}{\sigma^3}-\frac{4r}{\sigma^3}}{2\sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}} < 0 \quad \text{when } b > \frac{\sigma^2}{2}$$

□

Lemma 3.15. *When* $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1, (q-K)k_1 - (l-K)k_4 < 0.$

Proof. For $(q-K)k_1 - (l-K)k_4 < 0 \iff \frac{(q-K)k_1}{(l-K)k_4} < 1 \iff \left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1$ □

Lemma 3.16. *When* $b \in (0, r)$, $n(\sigma) > m(\sigma) + 1$.

Proof. For $n > m+1 \iff \sqrt{(\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}} > \frac{b}{\sigma^2}+\frac{1}{2} \iff (\frac{b}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2} > \left(\frac{b}{\sigma^2}+\frac{1}{2}\right)^2 \iff b < r$ □

Lemma 3.17. $H(q, q, l, \sigma) = q - K$ *and* $H(l, q, l, \sigma) = l - K$.

We now study

$$H_x(x,q,l,\sigma) = x^{-m-n-1}(-m-n)[(q-K)k_1 - (1-K)k_4] + x^{n-m-1}(n-m)[(l-K)k_3 - (q-K)k_2]$$
$$H_{xx}(x,q,l,\sigma) = x^{-m-n-2}(m+n+1)(m+n)[(q-K)k_1 - (1-K)k_4] + x^{n-m-2}(n-m-1)(n-m)[(l-K)k_3 - (q-K)k_2]$$

We denote

$$\eta_1(q,l,\sigma) = (q-K)k_1 - (1-K)k_4 \tag{3.8}$$
$$\eta_2(q,l,\sigma) = (l-K)k_3 - (q-K)k_2 \tag{3.9}$$

It is obvious that $\frac{(l-K)k_3}{(q-K)k_2} = \frac{l-K}{q-K}\left(\frac{l}{q}\right)^{m+n} > 1 \Longrightarrow (l-K)k_3 - (q-K)k_2 > 0$
With respect to Lemma 3.15 and Lemma 3.16 and Lemma 3.17, we can begin our discussion.

Lemma 3.18. *When* $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1$ *and* $b \in (0, r)$, $[p, q] \cup [l, \infty]$ *is not an equilibrium region.*

Proof. When $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1 \Longrightarrow \eta_1 < 0$, it's obvious that $H_x(x) > 0$.

When $b \in (0, r) \Longrightarrow n(\sigma) - m(\sigma) - 1 > 0$

Let $H_{xx}(x^*) = 0 \Longleftrightarrow x^* = \left(-\frac{(m+n+1)(m+n)\eta_1}{(n-m-1)(n-m)\eta_2}\right)^{\frac{1}{2n}}$

$H_{xx}(x) < 0$ when $x < x^*$ and $H_{xx}(x) > 0$ when $x > x^*$.

Therefore, we are confident to conclude that there exists an x' such that when $x > x'$, $H(x) > x-K$, thus, (l, ∞) can never be an equilibrium region. □

Lemma 3.19. *When* $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} \geq 1$ *and* $b \in (0, r)$, $[p, q] \cup [l, \infty]$ *is not an equilibrium region.*

Proof. When $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} \geq 1 \Longrightarrow \eta_1 \geq 0$

Let $H_x(x) = 0 \Longrightarrow x^* = \left(\frac{(m+n)\eta_1}{(n-m)\eta_2}\right)^{\frac{1}{2n}}$

$H_x(x) < 0$ when $x < x^*$ and $H_x(x) > 0$ when $x > x^*$

When $b \in (0, r) \Longrightarrow n(\sigma) - m(\sigma) - 1 > 0$, it is obvious that $H_{xx}(x) > 0$

Combining with Lemma 3.17, we have $H(x) > x - K$ for $x > l$, thus, (l, ∞) can never be an equilibrium region. □

Lemma 3.20. *When* $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1$ *and* $b \in (r, \infty)$, $[0, q] \cup [l, \infty]$ *is an equilibrium region.*

Proof. When $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} < 1 \Longrightarrow \eta_1 < 0$, it's obvious that $H_x(x) > 0$.

When $b \in (r, \infty) \Longrightarrow n(\sigma) - m(\sigma) - 1 \leq 0$, it is obvious that $H_{xx}(x) < 0$.

Also, we have Lemma 3.17.

Combining all conditions together, we must have $H(x) < x - K$ for $x \notin (q, l)$. $H(x) > x - K$ for $x \in (q, l)$. It is the same for $J(x, R)$.

Therefore, the equilibrium region is $[0, q] \cup [l, \infty)$ □

Lemma 3.21. *The case where* $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} \geq 1$ *and* $b \in (r, \infty)$ *doesn't exist.*

Proof. $\left(\frac{l}{q}\right)^{n-m}\frac{q-K}{l-K} \geq 1 \Longrightarrow \frac{l^{n-m}}{l-K} \geq \frac{q^{n-m}}{q-K}$

$\Longrightarrow f(x) = \frac{x^{n-m}}{x-K}$ and $f(x)$ is a non-decreasing function on (K, ∞)

$f'(x) = \frac{x^{n-m-1}[(n-m-1)x-(n-m)K]}{(x-K)^2}$

Let $g(x) = (n-m-1)x - (n-m)K$, because $b \in (r, \infty)$, from Lemma 3.16, we have $n - m - 1 < 0$

We have $g(K) = -K < 0$

Therefore, $f(x)$ is strictly decreasing on (K, ∞), which violates our conclusion on $f(x)$.

As a result, the case where $\left(\frac{l}{q}\right)^{n-m} \frac{q-K}{l-K} \geq 1$ and $b \in (r, \infty)$ doesn't exist. □

Remark 3.2. *When* $\left(\frac{l}{q}\right)^{n-m} \frac{q-K}{l-K} < 1$ *and* $b \in (r, \infty)$, *the region for* q *and* l *is simply* $K < q < l$

Proof. $\left(\frac{l}{q}\right)^{n-m} \frac{q-K}{l-K} < 1 \Longrightarrow \frac{l^{n-m}}{l-K} < \frac{q^{n-m}}{q-K}$

$\Longrightarrow f(x) = \frac{x^{n-m}}{x-K}$ and $f(x)$ is a decreasing function on (K, ∞)

$f'(x) = \frac{x^{n-m-1}[(n-m-1)x - (n-m)K]}{(x-K)^2}$

Let $g(x) = (n-m-1)x - (n-m)K$, because $b \in (r, \infty)$, from Remark 3.16, we have $n - m - 1 < 0$

We have $g(K) = -K < 0$

Therefore, $f(x)$ is strictly decreasing on (K, ∞), which means the value region for q and l is simply $K < q < l$, without other restrictions. □

To conclude the above lemma and remakrs, we have

	$\left(\frac{l}{q}\right)^{n-m} \frac{q-K}{l-K} < 1$	$\left(\frac{l}{q}\right)^{n-m} \frac{q-K}{l-K} \geq 1$
$b \in (0, r)$	No Equilibrium	No Equilibrium
$b \in (r, \infty)$	$[0, q] \cup [l, \infty)$ is an Equilibrium where $K < q < l$	Doesn't Exist

Now we study whether there is some optimal value q and l to make the equilibrium optimal for the only case where we have equilibrium region.

To do so, we study the properties of $H'(q)$ and $H'(l)$.

We first calculate the partial derivatives of all $k's$ w.r.t q and l, and we have:

$$k_1'(q) = \frac{q^{m-n-1}[(m+n)l^{2n} + (n-m)q^{2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} > 0$$

$$k_1'(l) = \frac{-2nq^{m+n}l^{-1}}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} < 0$$

$$k_2'(q) = \frac{q^{m+n-1}[(m+n)q^{-2n} + (n-m)l^{-2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} > 0$$

$$k_2'(l) = \frac{-2nq^{m-n}l^{-1}}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} < 0$$

$$k_3'(q) = \frac{2nq^{-1}l^{m-n}}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} > 0$$

$$k_3'(l) = \frac{l^{m+n-1}[(m-n)q^{-2n} - (m+n)l^{-2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} < 0$$

$$k_4'(q) = \frac{2nq^{-1}l^{m+n}}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} > 0$$

$$k_4'(l) = \frac{l^{m-n-1}[(m-n)l^{2n} - (m+n)q^{2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} < 0$$

The properties of $H'(q)$ and $H'(l)$ highly depend on the value of b, r, σ, x, and we can only get the optimal equilibrium region for specific cases. But at least, we can prove for some values of them, the optimal equilibrium region does not exist.

Say one case below:

If l satisfies $l < min\left[\left(\frac{q^{-2n}[(m+n)l^{2n} + (n-m)q^{2n}]}{(m+n)q^{-2n} + (n-m)l^{-2n}}\right)^{\frac{1}{2n}}, \left(\frac{l^{-2n}[(m-n)l^{2n} + (m+n)q^{2n}]}{(m-n)q^{-2n} + (m+n)l^{-2n}}\right)^{\frac{1}{2n}}\right]$ for any $l > K$. There should be no optimal equilibrium region for the first barrier $[0, q]$

Proof.

$$\begin{aligned} h_1'(q) &= (k_1x^{-m-n} - k_2x^{n-m})' \\ &= x^{-m-n}\frac{q^{m-n-1}[(m+n)l^{2n} + (n-m)q^{2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} - x^{n-m}\frac{q^{m+n-1}[(m+n)q^{-2n} + (n-m)l^{-2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} \end{aligned}$$

$h'(q)$ is also a decreasing function of x, by letting it $= 0 \Longrightarrow x^* = \left(\frac{q^{-2n}[(m+n)l^{2n} + (n-m)q^{2n}]}{(m+n)q^{-2n} + (n-m)l^{-2n}}\right)^{\frac{1}{2n}}$

If l satisfies the condition that $l < x^*$, we have $h'(q) > 0$ for $q < x < l$ and $K < q < l$

Simiplarly,

$$\begin{aligned} h_2'(q) &= (k_3x^{n-m} - k_4x^{-m-n})' \\ &= x^{n-m}\frac{l^{m+n-1}[(m-n)q^{-2n} + (m+n)l^{-2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} - x^{-m-n}\frac{l^{m-n-1}[(m-n)l^{2n} + (m+n)q^{2n}]}{\left(\frac{l}{q}\right)^n - \left(\frac{q}{l}\right)^n} \end{aligned}$$

$h'(q)$ is also a decreasing function of x, by letting it $= 0 \Longrightarrow x^* = \left(\frac{l^{-2n}[(m-n)l^{2n} + (m+n)q^{2n}]}{(m-n)q^{-2n} + (m+n)l^{-2n}}\right)^{\frac{1}{2n}}$

If l satisfies the condition that $l < x^*$, we have $h'(q) > 0$ for $q < x < l$ and $K < q < l$

In this case, $H(q) = (q-K)h_1(q) + (l-K)h_2(q)$ is surely an increasing function of q.

Then, for $q > K$, there is no optimal $H(q)$, there is no maximum $J(x, R)$. Our proposition in this case has been proved. $\square$

Now, we consider a numerical example where $\underline{\sigma}^2 = 1, \overline{\sigma}^2 = 2, \underline{b} = \overline{b} = b = 1, r = \frac{1}{4}, K = 50, x = 100$

First, we have:

$$\begin{aligned} n(\underline{\sigma}) &= \frac{\sqrt{3}}{2} \\ m(\underline{\sigma}) &= \frac{1}{2} \\ n(\overline{\sigma}) &= \frac{1}{2} \\ m(\overline{\sigma}) &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} h_1(\underline{\sigma}) &= \frac{q^{\frac{1}{2}} l^{\frac{\sqrt{3}}{2}}}{\left(\frac{l}{q}\right)^{\frac{\sqrt{3}}{2}} - \left(\frac{q}{l}\right)^{\frac{\sqrt{3}}{2}}} x^{-\frac{1}{2}-\frac{\sqrt{3}}{2}} - \frac{q^{\frac{1}{2}} l^{-\frac{\sqrt{3}}{2}}}{\left(\frac{l}{q}\right)^{\frac{\sqrt{3}}{2}} - \left(\frac{q}{l}\right)^{\frac{\sqrt{3}}{2}}} x^{\frac{\sqrt{3}}{2}-\frac{1}{2}} \\ h_2(\underline{\sigma}) &= \frac{l^{\frac{1}{2}} q^{-\frac{\sqrt{3}}{2}}}{\left(\frac{l}{q}\right)^{\frac{\sqrt{3}}{2}} - \left(\frac{q}{l}\right)^{\frac{\sqrt{3}}{2}}} x^{\frac{\sqrt{3}}{2}-\frac{1}{2}} - \frac{l^{\frac{1}{2}} q^{\frac{\sqrt{3}}{2}}}{\left(\frac{l}{q}\right)^{\frac{\sqrt{3}}{2}} - \left(\frac{q}{l}\right)^{\frac{\sqrt{3}}{2}}} x^{-\frac{1}{2}-\frac{\sqrt{3}}{2}} \\ h_1(\overline{\sigma}) &= \frac{l^{\frac{1}{2}}}{\left(\frac{l}{q}\right)^{\frac{1}{2}} - \left(\frac{q}{l}\right)^{\frac{1}{2}}} x^{-\frac{1}{2}} - \frac{l^{-\frac{1}{2}}}{\left(\frac{l}{q}\right)^{\frac{1}{2}} - \left(\frac{q}{l}\right)^{\frac{1}{2}}} x^{\frac{1}{2}} \\ h_2(\overline{\sigma}) &= \frac{q^{-\frac{1}{2}}}{\left(\frac{l}{q}\right)^{\frac{1}{2}} - \left(\frac{q}{l}\right)^{\frac{1}{2}}} x^{\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{\left(\frac{l}{q}\right)^{\frac{1}{2}} - \left(\frac{q}{l}\right)^{\frac{1}{2}}} x^{-\frac{1}{2}} \end{aligned}$$

Therefore, we have:

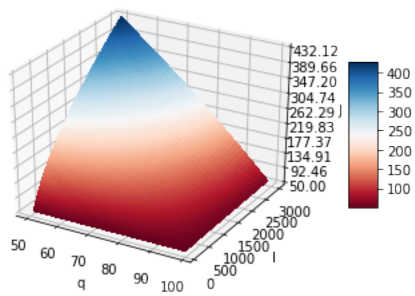
$$H(\underline{\sigma}) = (q-K)h_1(\underline{\sigma}) + (l-K)h_2(\underline{\sigma})$$

$$H(\overline{\sigma}) = (q-K)h_1(\overline{\sigma}) + (l-K)h_2(\overline{\sigma})$$

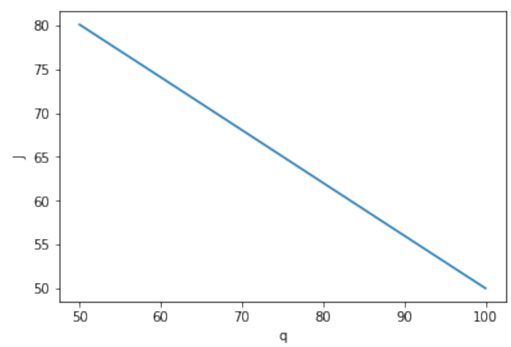
Finally, because we set $\alpha = 0.5$, we don't bother to study inf and sup, we certainly have:

$$J(x, R) = 0.5H(\underline{\sigma}) + 0.5H(\overline{\sigma}) \tag{3.10}$$

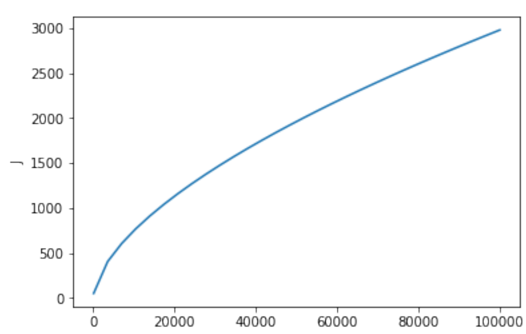
Now, we plug in those pre-set parameters and plot $(q, l) \mapsto J(x, R)$ and $l \mapsto J(x, R)$ to determine the optimal stopping region:



Further more, we fix $l = 200$, plot $q \mapsto J(x, R)$:



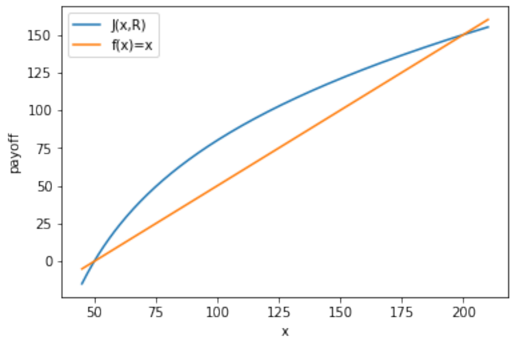
We fix $q = 60$, plot $l \mapsto J(x, R)$



From the above three plots, it's evident that we have an optimal $q_{opt} = K = 50$. But the bigger l is, the better.

Therefore, we don't have an optimal stopping region for $(0, q) \cup (l, \infty)$

To show what's happening here, we take $q = K = 50$ and $l = 200$, and plot $x \mapsto J(x, R)$ together with $f(x) = (x - K)^+$:



From the plot, we can see that $(0, 50) \cup (200, \infty)$ is a stopping region, which perfectly corresponds to the theorem we developed in this example.

4 Conclusion

In the previous work, I studied the optimal stopping problem under model ambiguity under $\alpha-$maxmin preference. To deal with the time-inconsistency issue, I employed the time-consistent equilibrium approach using the fixed point of an operator. I applied the framework on American options with different payoff functions and state processes, including: *American Put Option on Absolute Value of Brownian Motion, American Option with Linear Payoff Function on Absolute Value of Brownian Motion, American Call Option on Geometric Brownian Motion with Both Drift and Volatility Uncertainty, American Call Option on Geometric Brownian Motion with Two Barriers.*

In the previous examples, while implementing the Equilibrium Approach for the Optimal Stopping under Model Ambiguity, we usually have certain pre-conditions on the parameters σ's, K, α, r, that is, if the pre-conditions are violated, the entire approach is not applicable.

For the example 2 and example 3, we have different future stopping regions depending on the chosen region of a. I haven't figured out the choosing policy for a's region, neither do I have the capacity to deal with the case where optimal stopping region = $\emptyset$.

In reality, therefore, in order to make the approach sensible, we always need to first check the conditions. Moreover, we want to carefully study the market to determine the range of these parameters such as b's and σ's.
Given the pre-conditions are well satisfied, our framework can deal with uncertainty to a great extent, and it also has business value for investors.

There are also some other potential future work: Multi-dimensional underlying state process, correlation uncertainty for multi-dimensional process, etc.

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Code Reference

```
from scipy.optimize import fsolve
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import math
from scipy.optimize import minimize
from mpl_toolkits import mplot3d
from numpy import exp,arange
from pylab import meshgrid,cm,imshow,contour,clabel,colorbar,axis,title,show
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import matplotlib.pyplot as plt
```

Section 3.1:

```
K=1.2
s1=1
s2=2
r=1
x=1
alpha=0.5
```

```
a_vec = np.linspace(0,1,25)
restric = lambda a : (K-a) * ( alpha * math.exp(a/s1 * (2*r)**0.5)
        + (1-alpha) * math.exp(a / s2 * (2*r)**0.5)   ) - K
res_vec = [restric(i) for i in a_vec]
plt.plot(a_vec,res_vec)
plt.xlabel('a')
plt.ylabel('Restriction')
plt.show()
a_0 = fsolve(restric,0.5)
a_star = K - 1 / ( (2*r)**0.5*(alpha/s1+(1-alpha)/s2) )
Lambda  = lambda a : (K-a) * (alpha * math.exp(-1/s1* (x-a)*(2*r)**0.5)
        + (1-alpha) * math.exp(-1/s2* (x-a)*(2*r)**0.5)    )
a_vec_2 = np.linspace(a_0,1,25)
Lambda_vec = [Lambda(i) for i in  a_vec_2 ]
plt.plot(a_vec_2,Lambda_vec)
plt.xlabel('a')
plt.ylabel('$\Lambda(x,a)$')
plt.show()
```

Section 3.2:

```
s1=0.05
s2=0.2
def ch(x):
    return (math.exp(x)+math.exp(-x))/2
def Lambda(x,a):
    return a*(0.5*ch(x*math.sqrt(0.02)/s1)/ch(a*math.sqrt(0.02)/s1)
              + 0.5*ch(x*math.sqrt(0.02)/s2)/ch(a*math.sqrt(0.02)/s2))
def Lambda_a(x,a):
```

```
    return 0.5*ch(x*math.sqrt(0.02)/s1)/ch(a*math.sqrt(0.02)/s1)
        +0.5*ch(x*math.sqrt(0.02)/s2)/ ch(a*math.sqrt(0.02)/s2)
        +a * (-0.5*ch(x*math.sqrt(0.02)/s1)*ch(a*math.sqrt(0.02)/s1)**(-2)
        *math.sqrt(0.02)/(2*s1)
        *(math.exp(a*math.sqrt(0.02)/s1)
        -math.exp(-a*math.sqrt(0.02)/s1))
        -0.5*ch(x*math.sqrt(0.02)/s2)*ch(a*math.sqrt(0.02)/s2)**(-2)
        *math.sqrt(0.02)/(2*s2)
        *(math.exp(a*math.sqrt(0.02)/s2)-math.exp(-a*math.sqrt(0.02)/s2)) )
def Lambda_x_at_a(a):
    return a*( 0.5*math.sqrt(0.02)/(2*s1*ch(a*math.sqrt(0.02)/s1))
              *(math.exp(a*math.sqrt(0.02)/s1)
                  -math.exp(-a*math.sqrt(0.02)/s1))
             +0.5* math.sqrt(0.02)/(2*s2*ch(a*math.sqrt(0.02)/s2))
             *(math.exp(a*math.sqrt(0.02)/s2)
             -math.exp(-a*math.sqrt(0.02)/s2)))
def solve_for_a_opt(x):
    def f(a):
        return Lambda_a(x,a)
    res = fsolve(f,x/2)
    return res
def solve_for_x_star(a):
    def f(x):
        return x-Lambda(x,a)
    solution = fsolve(f,a/2)
    return solution
def solve_for_a_star():
```

```
    def f(a):
        return Lambda_x_at_a(a) - 1
    solution=fsolve(f,4)
    return solution
a_star = solve_for_a_star()
a_vec  = np.linspace ( max(x,a_star) , x*3, 100)
lambda_vec = [Lambda(x,i) for i in a_vec]
plt.plot(a_vec,lambda_vec)
plt.xlabel("a")
plt.ylabel("\Lambda(x,a)")
plt.show()
solve_for_x_star(a_opt)
x_vec = np.linspace(0,a_opt*1.05,60)
Lam_vec = [Lambda(i,a_opt) for i in x_vec]
plt.plot(x_vec,Lam_vec,label="$\Lambda(x,a_{opt})$")
plt.plot(x_vec,x_vec,label='x')
plt.xlabel("x")
plt.ylabel("payoff")
plt.legend()
plt.show()
#for sigma1 !!!!!
x = 0.5
s2 = 0.2
s1_vec = np.linspace ( 0.01,s2,30)
a_opt_vec = np.ones(len(s1_vec))
for i in range(len(s1_vec)):
    a_opt_vec[i]=study_sigma(x,s1_vec[i],s2)
```

```
#plot
plt.plot(s1_vec,a_opt_vec,color='g')
plt.scatter(  s1_vec[a_opt_vec==x] ,
              [x for i in range(len(s1_vec[a_opt_vec==x]))] ,
              color = "r", label = "when $a_{opt}=x$")
plt.xlabel('$\sigma_1$')
plt.ylabel('$a_{opt}$')
plt.legend()
plt.show()
#for sigma2 !!!!!!
x = 0.5
s1 = 0.05
s2_vec = np.linspace ( s1,0.2,30)
a_opt_vec = np.ones(len(s2_vec))
for i in range(len(s2_vec)):
    a_opt_vec[i]=study_sigma(x,s1,s2_vec[i])
#plot
plt.plot(s2_vec,a_opt_vec,color = 'y')
plt.scatter(  s2_vec[a_opt_vec==x] ,
        [x for i in range(len(s2_vec[a_opt_vec==x]))] ,
        color = "b", label = "when $a_{opt}=x$")
plt.xlabel('$\sigma_2$')
plt.ylabel('$a_{opt}$')
plt.legend()
plt.show()
```

Section 3.4:

```
#parameters setting
```

```python
s1_sq=1
s2_sq=2
b=1
r=0.25
K=50
x=100
n1=math.sqrt(3)/2
m1=0.5
n2=0.5
m2=0
def J(x,q,l):
    #calculate necessary things
    k1=q**0.5 * l**(math.sqrt(3)/2) /
        ( (l/q)**(math.sqrt(3)/2) - (q/l)**(math.sqrt(3)/2) )
    k2=q**0.5 * l**(-math.sqrt(3)/2) /
        ( (l/q)**(math.sqrt(3)/2) - (q/l)**(math.sqrt(3)/2) )
    k3=l**0.5 * q**(-math.sqrt(3)/2) /
        ( (l/q)**(math.sqrt(3)/2) - (q/l)**(math.sqrt(3)/2) )
    k4=l**0.5 * q**(math.sqrt(3)/2) /
         ( (l/q)**(math.sqrt(3)/2) - (q/l)**(math.sqrt(3)/2) )
    k5=l**0.5 / ( (l/q)**0.5 - (q/l)**0.5 )
    k6=l**(-0.5) / ( (l/q)**0.5 - (q/l)**0.5 )
    k7=q**(-0.5) / ( (l/q)**0.5 - (q/l)**0.5 )
    k8=q**0.5 / ( (l/q)**0.5 - (q/l)**0.5 )
    abbr1=x**(-0.5-math.sqrt(3)/2)
    abbr2=x**(math.sqrt(3)/2-1/2)
    abbr3=x**(-0.5)
```

```
    abbr4=x**(0.5)
    #h's
    h11=k1*abbr1-k2*abbr2
    h21=k3*abbr2-k4*abbr1
    h12=k5*abbr3-k6*abbr4
    h22=k7*abbr4-k8*abbr3
    #H's
    H1=(q-K)*h11+(l-K)*h21
    H2=(q-K)*h12+(l-K)*h22
    #J
    J = 0.5*H1+0.5*H2
    return J
#Maxmize the function!!!
obj_fun = lambda q_l : -J(x,q_l[0],q_l[1])
q_l_init = [(x-K)*0.8+K,x*10**11]
cons = [{'type': 'ineq', 'fun': lambda q_l:  q_l[1] - x } ,
        {'type': 'ineq', 'fun': lambda q_l: x - q_l[0]},
        {'type': 'ineq', 'fun': lambda q_l: q_l[0]-K}]
       #{'type': 'ineq', 'fun': lambda q_l: q_l[1]-q_l[0]} ]
res = minimize(obj_fun, x0=q_l_init, constraints=cons)
print(res)
#plot
q = np.arange(K,x,1)
l = np.arange(x,3*10**3,10**1.5)
Q,L = meshgrid(q,l) # grid of point
Z=J(x,Q,L)
fig = plt.figure()
```

```
ax = fig.gca(projection='3d')
surf = ax.plot_surface(Q, L, Z, rstride=1, cstride=1,
                       cmap=cm.RdBu,linewidth=0, antialiased=False)
ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))
fig.colorbar(surf, shrink=0.5, aspect=5)
ax.set_xlabel("q")
ax.set_ylabel("l")
ax.set_zlabel("J")
plt.show()
#plot 1D
#for q
q_vec = np.linspace(K,x,30)
l = x*2
J_vec = [ J(x,i,l) for  i in q_vec ]
plt.plot(q_vec,J_vec)
plt.xlabel("q")
plt.ylabel("J")
plt.show()
#plot 1D
#for l
l_vec = np.linspace(x,x*1000,30)
q = 60
J_vec = [ J(x,q,i) for  i in l_vec ]
plt.plot(l_vec,J_vec)
plt.xlabel("l")
plt.ylabel("J")
```

```
plt.show()
#plot J(x)
q=K
l=200
x_vec = np.linspace(K-5,l+10,300)
J_vec = [J(i,q,l) for i in x_vec]
plt.plot(x_vec,J_vec,label='J(x,R)')
plt.plot(x_vec,x_vec-K,label='f(x)=x')
plt.xlabel("x")
plt.ylabel("payoff")
plt.legend()
plt.show()
```