Discrete streamfunction method for incompressible Navier-Stokes equation (also known as Exact fractional step method)

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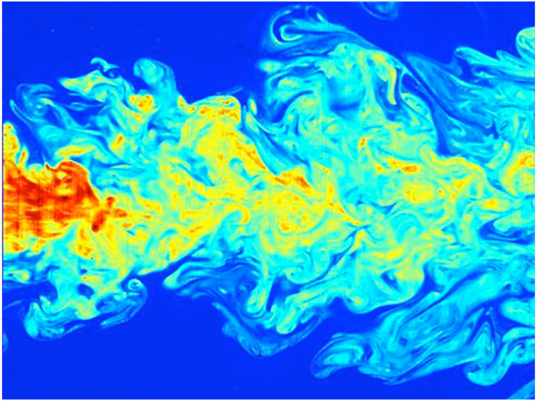
Introduction

Introduction



Introduction

Let $\boldsymbol{v}(x, y, t) = [u(x, y, t), v(x, y, t)]^T$ and $\boldsymbol{p}(x, y, t)$ be solutions to incompressible Navier-Stokes system of equations on 2D domain Ω with Dirichlet Boundary conditions on $\partial\Omega$.

$$\underbrace{\frac{\partial \boldsymbol{v}}{\partial t}}_{\text{transient}} + \underbrace{\boldsymbol{v} \cdot \nabla \boldsymbol{v}}_{\text{advective}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure gradient}} + \underbrace{\frac{\mu}{\rho}\nabla \cdot \nabla \boldsymbol{v}}_{\text{viscous, Laplacian}} \quad \text{(momentum) in } \Omega \tag{1a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{(continuity) in } \Omega, \tag{1b}$$

$$\boldsymbol{v} = \boldsymbol{v}(x, y, t) \quad \text{on } \partial\Omega, \tag{1c}$$

where ρ - density, μ - dynamic viscosity.

Introduction

After introducing the following dimensionless variables (marked with prime $'$):

$$x \to Lx', \quad \boldsymbol{v} \to U_0 \boldsymbol{v}', \quad \nabla \to \frac{1}{L}\nabla', \quad p \to p' \rho U_0^2, \quad t \to \frac{L}{U_0} t',$$

and Reynolds number

$$\mathrm{Re} = \frac{\rho L U_0}{\mu},$$

we obtain the non-dimensional momentum and continuity equations (dropped prime superscript)

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} &= -\nabla p + \frac{1}{\mathrm{Re}} \nabla \cdot \nabla \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} &= 0. \end{aligned} \tag{2}$$

Discretization

Discretization

Denote discrete spatial operators as:

$\hat{L}$: Laplacian.

$\hat{G}$: Gradient.

$\hat{D}$: Divergence.

$\mathbf{H}$: Non-linear advective terms.

Then initial system (2) can be rewritten using these operators as

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{v} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{H}(\boldsymbol{v}) \\ 0 \end{pmatrix} = \begin{bmatrix} \hat{L} & -\hat{G} \\ -\hat{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ p \end{bmatrix} + \mathrm{bc}_{\boldsymbol{v},p}, \quad (3)$$

Discretization (transient)

Attack (3) with the following schemes as in Colonius (2008) (superscript denotes the time step):

Viscous - Implicit trapezoidal - Crank Nicholson scheme (second-order in time).

$$\hat{L}\boldsymbol{v} = \frac{1}{2}\left(\hat{L}\boldsymbol{v}^{n+1} + \hat{L}\boldsymbol{v}^{n}\right) \tag{4}$$

Nonlin. - Explicit Adams-Bashforth (second-order in time).

$$\mathbf{H}(\boldsymbol{v}) = \frac{3}{2}\mathbf{H}(\boldsymbol{v}^{n}) - \frac{1}{2}\mathbf{H}(\boldsymbol{v})^{n-1}. \tag{5}$$

Pressure - Implicit Euler (first-order in time).

$$\hat{G}p = \hat{G}p^{n+1}. \tag{6}$$

Discretization (transient)

The above schemes result in the following time-discretized system:

$$\begin{bmatrix} \frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \\ = \begin{pmatrix} \left[\frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L}\right] \boldsymbol{v}^n - \left[\frac{3}{2}\hat{\mathbf{H}}(\boldsymbol{v}^n) - \frac{1}{2}\hat{\mathbf{H}}(\boldsymbol{v}^{n-1})\right] \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{bc}_1 \\ \hat{bc}_2 \end{pmatrix}, \quad (7)$$

which in short becomes

$$\boxed{\begin{bmatrix} \hat{A} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{r}^n \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{bc}_1 \\ \hat{bc}_2 \end{pmatrix}}. \quad (8)$$

Discretization (spatial)

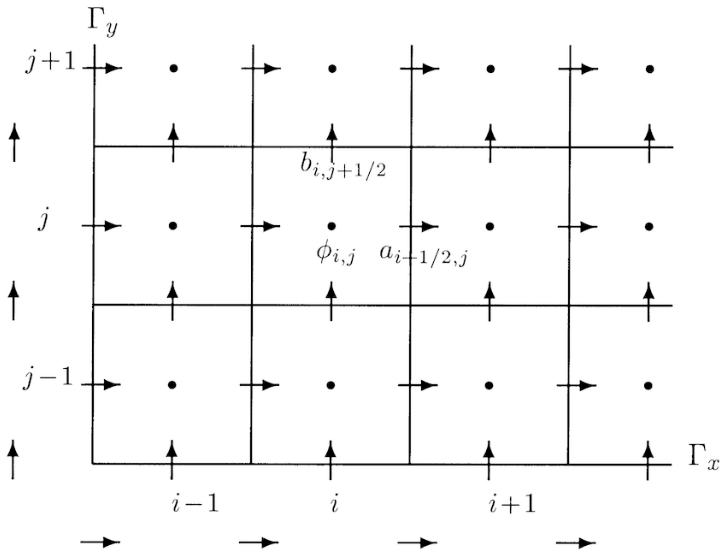


Figure 1: Domain discretization as per Harlow and Welch (1965)

Discretization (spatial Laplacian)

Rewrite Laplacian as a block matrix:

$$\hat{L} = \begin{bmatrix} \hat{L}^u_{xx} + \hat{L}^u_{yy} & 0 \\ 0 & \hat{L}^v_{xx} + \hat{L}^v_{yy} \end{bmatrix}.$$

Use Taylor expansion to derive second derivative approximation

$$\begin{aligned} u_{i-\frac{1}{2}+1,j} = u_{i-\frac{1}{2},j} + & \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right) + \\ & + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right), \end{aligned} \tag{9}$$

$$\begin{aligned} u_{i-\frac{1}{2}-1,j} = u_{i-\frac{1}{2},j} + & \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right) + \\ & + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right). \end{aligned} \tag{10}$$

Discretization (spatial Laplacian)

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \approx \frac{1}{h_c h_e} u_{i-\frac{1}{2}+1} - \frac{2}{h_w h_e} u_{i-\frac{1}{2}} + \frac{1}{h_c h_w} u_{i-\frac{1}{2}-1} + O(\Delta x), \tag{11}$$

where $h_w = x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}$, $h_c = \frac{x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}}{2}$, $h_e = x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}$.

Discretization (spatial Laplacian at boundary)

The exact value on the left boundary ($u_{\frac{1}{2},j}$) is known.

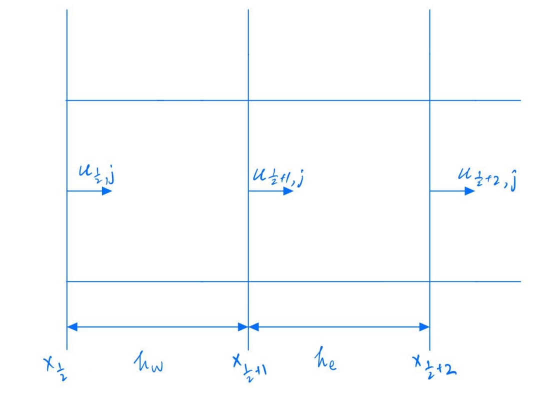


Figure 2: $\hat{L}^u_{xx}$ at the left boundary.

Discretization (spatial Laplacian at boundary)

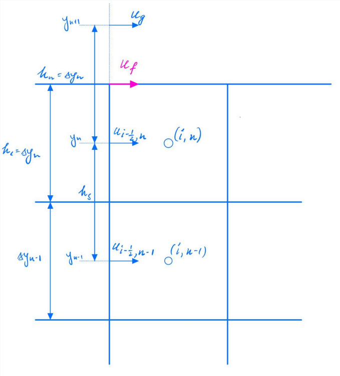


Figure 3: $\hat{L}^{u}_{yy}$ at top boundary.

$$u_f = \frac{u_g + u_{i-\frac{1}{2},N}}{2} \implies u_g = 2u_f - u_{i-\frac{1}{2},N} \implies$$
$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} = \frac{2u_f}{{h_c}^2} + u_{i-\frac{1}{2},N}\left(\frac{-(2h_c+h_s)}{h_c^2 h_s}\right) + u_{i-\frac{1}{2},N-1}\frac{1}{h_s h_c}.$$

Discretization (spatial advection)

Advection in conservative form allows to compute derivatives with simple second order central difference formula $\frac{\partial u}{\partial x}\Big|_i = \frac{u_{i+1}-u_{i-1}}{x_{i+1}-x_{i-1}}$:

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= u\frac{\partial u}{\partial x} + 0 + v\frac{\partial u}{\partial y} \\
&= u\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial u}{\partial y} \\
&= \left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right) + \left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}.
\end{aligned} \tag{12}$$

Discretization (spatial advection)

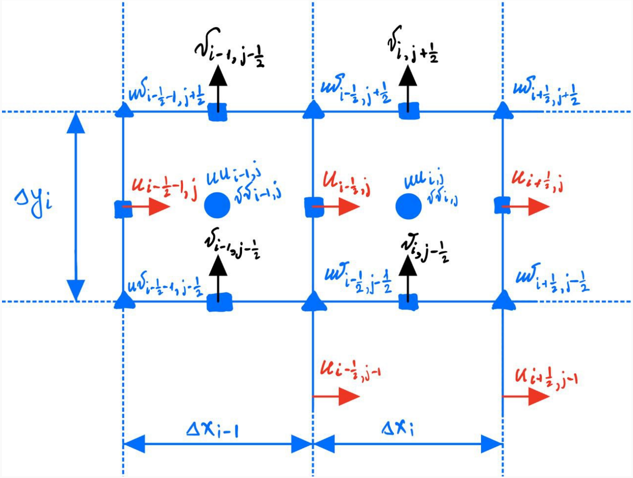


Figure 4: Advection discretization.

Discretization (divergence)

$$\hat{D}\boldsymbol{v} = \hat{bc}_2,$$

$$\begin{bmatrix} \hat{D}_x & \hat{D}_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \hat{bc}_2,$$

$$\frac{1}{\Delta x} D_x u + \frac{1}{\Delta y} D_y v = \hat{bc}_2,$$

$$\frac{1}{\Delta_{xy}} \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix} = \frac{1}{\Delta_{xy}} Dq = \hat{bc}_2,$$

where

$$q = \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix}, \quad \Delta_{xy} = \begin{bmatrix} \frac{1}{\Delta x_1 \Delta y_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta x_2 \Delta y_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\Delta x_M \Delta y_N} \end{bmatrix}. \quad (13)$$

Discretization (divergence)

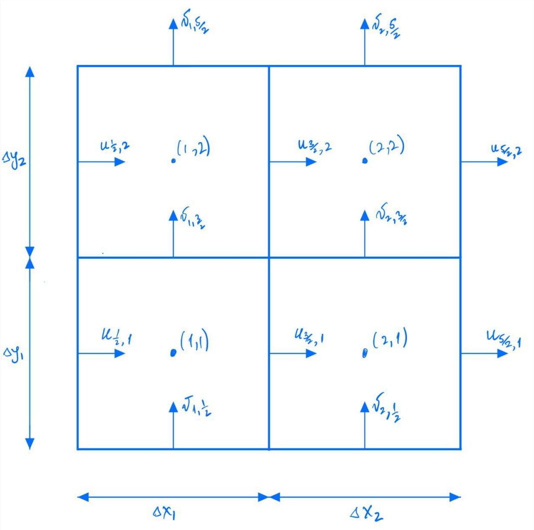


Figure 5: 2×2 grid example for divergence operator.

Discretization (divergence)

$$\Delta_{xy}\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_{\frac{3}{2},1}\Delta y_1 \\ u_{\frac{3}{2},2}\Delta y_2 \\ v_{1,\frac{3}{2}}\Delta x_1 \\ v_{2,\frac{3}{2}}\Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{u_{\frac{1}{2},1}}{\Delta x_1} + \frac{v_{1,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{5}{2},1}}{\Delta x_2} + \frac{v_{2,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{1}{2},2}}{\Delta x_1} - \frac{v_{1,\frac{5}{2}}}{\Delta y_2} \\ \frac{u_{\frac{5}{2},2}}{\Delta x_2} - \frac{v_{2,\frac{5}{2}}}{\Delta y_2} \end{bmatrix}. \tag{14}$$

Discretization (pressure gradient)

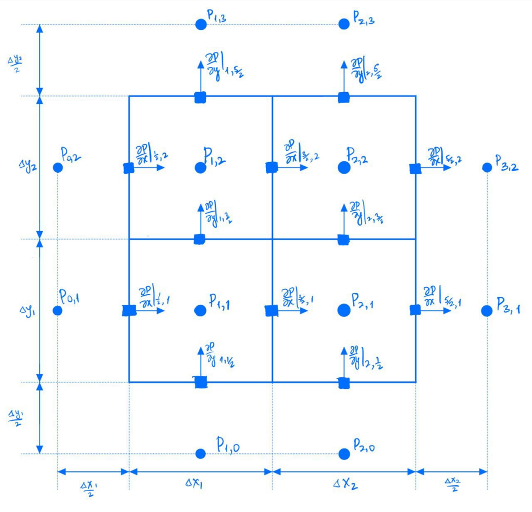


Figure 6: 2×2 grid example for gradient operator.

Discretization (pressure gradient)

Notice, that matrix of coefficients below is very similar to Divergence matrix, i.e. $G = -D^T$.

$$diag\begin{bmatrix} \frac{2}{\Delta x_1+\Delta x_2} \\ \frac{2}{\Delta x_1+\Delta x_2} \\ \frac{2}{\Delta y_1+\Delta y_2} \\ \frac{2}{\Delta y_1+\Delta y_2} \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1,1} \\ p_{2,1} \\ p_{1,2} \\ p_{2,2} \end{bmatrix} + \begin{bmatrix} 0 \\ p_{3,1} \\ 0 \\ p_{3,2} \\ 0 \\ 0 \end{bmatrix} \right). \tag{15}$$

Motivation

Motivation

Stream function $\psi(x, y, t) : \boldsymbol{v} = \nabla \times \psi$ of an incompressible two-dimensional flow:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{16}$$

Vorticity $\omega = \nabla \times \boldsymbol{v}$, in two-dimensional case (x-y-plane) the only non-zero component of ω is z, which leads to

$$\omega = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{17}$$

Motivation

(16) into (17) leads to

$$-\nabla^2\psi = \omega. \tag{18}$$

Apply $(\nabla\times)$ to momentum. Since

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = 0 \tag{19}$$

we get

$$\frac{\partial \omega}{\partial t} - \epsilon\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) = -\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right). \tag{20}$$

Vorticity Streamfunciton Poisson (18) into Vorticity Transport (20):

$$\boxed{-\frac{\partial \nabla^2\psi}{\partial t} + \epsilon\nabla^4\psi = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\nabla^2\psi.} \tag{21}$$

Algorithm