Solving PDE

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Abstract

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1 Problem statement

Let

$$\boldsymbol{v}(x,y,t) = \left(u\left(x,y,t\right), v\left(x,y,t\right)\right)$$

and onsider non-dimensionalized incompressible Navier-Stokes system of equations

$$\begin{aligned}
\text{Momentum: } & \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \epsilon\nabla\cdot\nabla\boldsymbol{v}, && \text{(1.1a)}\\
& \epsilon = \frac{1}{\text{Re}}, 0 \le x \le 1, 0 \le y \le 1, t \ge 0. \\
\text{Continuity: } & \nabla\cdot\boldsymbol{v} = 0, && \text{(1.1b)}\\
& 0 \le x \le 1, 0 \le y \le 1, t \ge 0. \\
\text{Inlet: } & \boldsymbol{v}(t,0,y) = \boldsymbol{v}(t,x,1) = \left(1 + A\cos\left(kx - \omega t + \phi_0\right), v\right), && \text{(1.1c)}\\
& \{A \le 1, k, \omega, \phi_0\} \subset \mathbb{R}. \\
\text{No-slip wall BC: } & \boldsymbol{v}(t,x,0) = (0,0). && \text{(1.1d)}\\
\text{Outlet and freestream BC: } & \text{Artificial boundary conditions.} && \text{(1.1e)}\\
\\
\text{Initial condition: } & \boldsymbol{v}(0,x,y) = (1,0). && \text{(1.1f)}
\end{aligned}$$

We are to determine appropriate boundary conditions for top side of the domain. Taking derivative of BC (1.1c) w.r.t x

$$\frac{\partial u}{\partial x} = -kA\sin\left(kx - \omega t + \phi_0\right), \tag{1.2}$$

and making use of continuity Eq. (1.1b) leads to

$$\frac{\partial v}{\partial y} = kA\sin\left(kx - \omega t + \phi_0\right). \tag{1.3}$$

We can integrate Eq. (1.3) over dy to obtain general expression for vertical velocity component

$$v(x,y,t) = ykA\sin\left(kx - \omega t + \phi_0\right) + g(x,t). \tag{1.4}$$

In case of $g(x,t) = 0$, plugging Eq. (1.4) together with BC (1.1c) into Eq. (1.1a) leads to

$$\frac{\partial p}{\partial x} = 0 \tag{1.5}$$

for x-momentum. If we now take $\frac{\partial}{\partial y}$ of Eq. (1.5) and plug into $\frac{\partial}{\partial x}$ of y-momentum we will obtain

$$\begin{aligned}
&\frac{A\,k^4\,y\,\cos\left(kx + \omega t + \phi_0\right)}{\text{Re}} - A\,k^2\,\omega\,y\,\sin\left(kx + \omega t + \phi_0\right) \\
&-A\,k^3\,y\,\sin\left(kx + \omega t + \phi_0\right)\,\left(A\,\cos\left(kx + \omega t + \phi_0\right) + 1\right) \\
&+A^2\,k^3\,y\,\cos\left(kx + \omega t + \phi_0\right)\,\sin\left(kx + \omega t + \phi_0\right) = 0,
\end{aligned} \tag{1.6}$$

which is obviously not always true for all A, k, ω. Therefore $g(x,t) \ne 0 \quad \forall x, t$.

Keeping $g(x,t)$ term and repeating the above process leads to third order PDE

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2} g\left(x,t\right) - A\,k^3\,y\,\sin\left(kx + \omega t + \phi_0\right)\right)\,\left(A\,\cos\left(kx + \omega t + \phi_0\right) + 1\right) \\
&-\frac{1}{\text{Re}}\left(\frac{\partial^3}{\partial x^3} g\left(x,t\right) - A\,k^4\,y\,\cos\left(kx + \omega t + \phi_0\right)\right) \\
&+\frac{\partial}{\partial x}\frac{\partial}{\partial t} g\left(x,t\right) + A\,k^2\,\cos\left(kx + \omega t + \phi_0\right)\,\left(g\left(x,t\right) + A\,k\,y\,\sin\left(kx + \omega t + \phi_0\right)\right) \\
&-A\,k^2\,w\,y\,\sin\left(kx + \omega t + \phi_0\right) = 0.
\end{aligned} \tag{1.7}$$

We now need to determine boundary conditions for Eq. (1.7). It is known that at the left boundary $v(0, y, t) = 0$. Then Eq. (1.4) at y=1 gives

$$g(x=0,t) = -kA\sin\left(-\omega t + \phi_0\right). \tag{1.8}$$

Outlet BC (1.1e) $\frac{\partial v}{\partial x} = 0$ at $x = 1, y = 1$ with Eq. (1.4) leads to

$$\frac{\partial}{\partial x} g(x=1,t) = -k^2 A\cos\left(k - \omega t + \phi_0\right). \tag{1.9}$$

Initial condition at $t = 0$ we assumed (Eq. (1.1f)) to be $v(x, y = 1, t = 0) = 0$, then Eq. (1.4) provides us

$$g(x,t=0) = -kA\sin\left(kx + \phi_0\right). \tag{1.10}$$

It seems to be possible to obtain pressure values up to a constant if we take $u(x, y = 1, t) = 1 + A\cos(kx + \omega t), v(x, y = 1, t) = -Ak\sin(kx + \omega t)$ and plug into x-momentum Eq. (1.1a). Namely,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1.11}$$

becomes

$$\frac{\partial p}{\partial x} = -A\,w\,\sin\left(k\,x + t\,w\right) + \epsilon A\,k^2\,\cos\left(k\,x + t\,w\right) - A\,k\,\sin\left(k\,x + t\,w\right)\,\left(A\,\cos\left(k\,x + t\,w\right) + 1\right), \tag{1.12}$$

which we can integrate over dx and to get expression for $p(x, y, t)$ up to a constant across the top boundary.