Discrete streamfunction method for incompressible Navier-Stokes equation (also known as Exact fractional step method)

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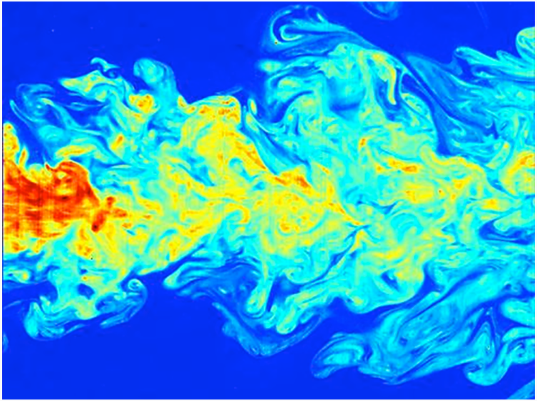
Introduction



Introduction (problem statement)

Let $\boldsymbol{v}(x, y, t) = [u(x, y, t), v(x, y, t)]^T$ and $\boldsymbol{p}(x, y, t)$ be solutions to incompressible Navier-Stokes system of equations on 2D domain Ω with Dirichlet Boundary conditions on $\partial\Omega$.

$$\underbrace{\frac{\partial \boldsymbol{v}}{\partial t}}_{\text{transient}} + \underbrace{\boldsymbol{v} \cdot \nabla \boldsymbol{v}}_{\text{advective}} = \underbrace{-\frac{1}{\rho} \nabla p}_{\text{pressure gradient}} + \underbrace{\frac{\mu}{\rho} \nabla \cdot \nabla \boldsymbol{v}}_{\text{viscous, Laplacian}} \quad \text{(momentum) in } \Omega \tag{1a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{(continuity) in } \Omega, \tag{1b}$$

$$\boldsymbol{v} = \boldsymbol{v}(x, y, t) \quad \text{on } \partial\Omega, \tag{1c}$$

where ρ - density, μ - dynamic viscosity.

Introduction (nondimensionalization)

After introducing the following dimensionless variables (marked with prime $'$):

$$x \to Lx', \quad \boldsymbol{v} \to U_0 \boldsymbol{v}', \quad \nabla \to \frac{1}{L}\nabla', \quad p \to p' \rho U_0^2, \quad t \to \frac{L}{U_0} t',$$

and Reynolds number

$$\mathrm{Re} = \frac{\rho L U_0}{\mu},$$

we obtain the non-dimensional momentum and continuity equations (dropped prime superscript)

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} &= -\nabla p + \frac{1}{\mathrm{Re}} \nabla \cdot \nabla \boldsymbol{v}, \\ \nabla \cdot \boldsymbol{v} &= 0. \end{aligned} \tag{2}$$

Introduction (Projection and other schemes require pressure)

Let $\tilde{\boldsymbol{v}}$ be intermediate value that satisfies

$$\text{Predictor :} \qquad \frac{\widetilde{\boldsymbol{v}} - \boldsymbol{v}^n}{\Delta t} + \boldsymbol{v}^n \cdot \nabla \boldsymbol{v}^* = \nu \nabla \cdot \nabla \boldsymbol{v}^*, \tag{3}$$

$$\text{Corrector :} \qquad \begin{aligned} \Delta p^{n+1} &= \frac{1}{\Delta t} \nabla \cdot \tilde{\boldsymbol{v}}, \\ \frac{\boldsymbol{v}^{n+1} - \widetilde{\boldsymbol{v}}}{\Delta t} &= -\nabla p^*. \end{aligned} \tag{4}$$

Discretization (grid)

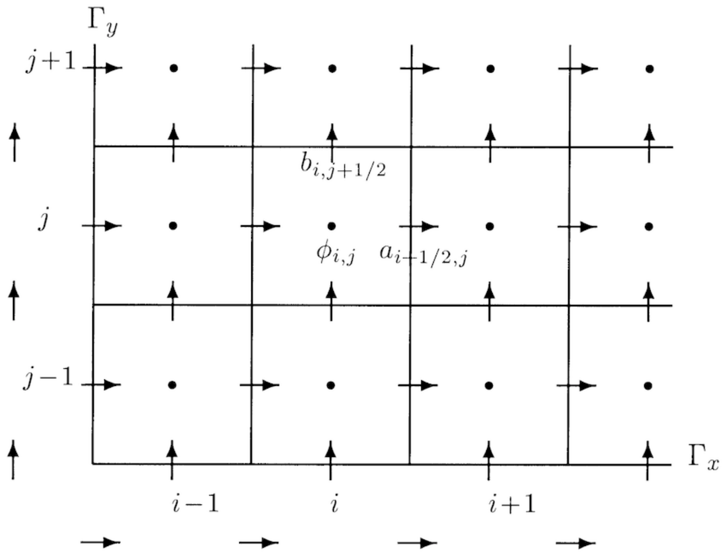


Figure 1: Domain discretization as per Harlow and Welch (1965)

Discretization (discrete operators)

Denote discrete spatial operators as:

$\hat{L}$: Laplacian.

$\hat{G}$: Gradient.

$\hat{D}$: Divergence.

$\mathbf{H}$: Non-linear advective terms.

Then initial system (2) can be rewritten using these operators as

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{v} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{H}(\boldsymbol{v}) \\ 0 \end{pmatrix} = \begin{bmatrix} \hat{L} & -\hat{G} \\ -\hat{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ p \end{bmatrix} + \mathrm{bc}_{\boldsymbol{v},p}, \quad (5)$$

Discretization (transient)

Attack (5) with the following schemes as in Colonius (2008) (superscript denotes the time step):

Viscous - Implicit trapezoidal - Crank Nicholson scheme (second-order in time).

$$\hat{L}\boldsymbol{v} = \frac{1}{2}\left(\hat{L}\boldsymbol{v}^{n+1} + \hat{L}\boldsymbol{v}^{n}\right) \tag{6}$$

Nonlin. - Explicit Adams-Bashforth (second-order in time).

$$\mathbf{H}(\boldsymbol{v}) = \frac{3}{2}\mathbf{H}(\boldsymbol{v}^{n}) - \frac{1}{2}\mathbf{H}(\boldsymbol{v})^{n-1}. \tag{7}$$

Pressure - Implicit Euler (first-order in time).

$$\hat{G}p = \hat{G}p^{n+1}. \tag{8}$$

Discretization (transient)

The above schemes result in the following time-discretized system:

$$
\begin{bmatrix} \frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} =
$$

$$
= \begin{pmatrix} \left[\frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L}\right] \boldsymbol{v}^{n} - \left[\frac{3}{2}\hat{\mathbf{H}}(\boldsymbol{v}^{n}) - \frac{1}{2}\hat{\mathbf{H}}(\boldsymbol{v}^{n-1})\right] \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{bc}_1^{\,n} \\ \hat{bc}_2^{\,n} \end{pmatrix}, \quad (9)
$$

which in short becomes

$$
\boxed{\begin{bmatrix} \hat{A} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{r}^{n} \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{bc}_1^{\,n} \\ \hat{bc}_2^{\,n} \end{pmatrix}}. \qquad (10)
$$

Discretization (spatial Laplacian)

Rewrite Laplacian as a block matrix:

$$\hat{L} = \begin{bmatrix} \hat{L}^u_{xx} + \hat{L}^u_{yy} & 0 \\ 0 & \hat{L}^v_{xx} + \hat{L}^v_{yy} \end{bmatrix}.$$

Use Taylor expansion to derive second derivative approximation

$$u_{i-\frac{1}{2}+1,j} = u_{i-\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right) + \\ + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right), \quad (11)$$

$$u_{i-\frac{1}{2}-1,j} = u_{i-\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right) + \\ + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right). \quad (12)$$

Discretization (spatial Laplacian)

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \approx \frac{1}{h_c h_e} u_{i-\frac{1}{2}+1} - \frac{2}{h_w h_e} u_{i-\frac{1}{2}} + \frac{1}{h_c h_w} u_{i-\frac{1}{2}-1} + O(\Delta x), \tag{13}$$

where $h_w = x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}$, $h_c = \frac{x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}}{2}$, $h_e = x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}$.

Discretization (spatial Laplacian at boundary)

The exact value on the left boundary ($u_{\frac{1}{2},j}$) is known.

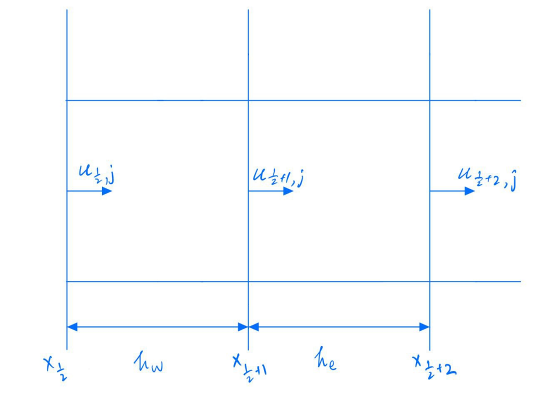


Figure 2: $\hat{L}^u_{xx}$ at the left boundary.

Discretization (spatial Laplacian at boundary)

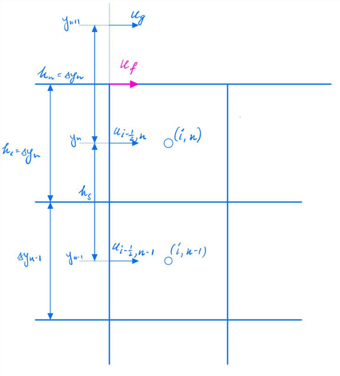


Figure 3: $\hat{L}^{u}_{yy}$ at top boundary.

$$u_f = \frac{u_g + u_{i-\frac{1}{2},N}}{2} \implies u_g = 2u_f - u_{i-\frac{1}{2},N} \implies$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} = \frac{2u_f}{{h_c}^2} + u_{i-\frac{1}{2},N}\left(\frac{-(2h_c+h_s)}{h_c^2 h_s}\right) + u_{i-\frac{1}{2},N-1}\frac{1}{h_s h_c}.$$

Discretization (spatial advection)

$$u_{i+1,j} = u_{i,j} + \left.\frac{\partial u}{\partial x}\right|_{i,j} (x_{i+1} - x_i) + \\ + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (x_{i+1} - x_i)^2 + O\left(\Delta x^3\right)., \quad (14)$$

$$u_{i-1,j} = u_{i,j} + \left.\frac{\partial u}{\partial x}\right|_{i,j} (x_{i-1} - x_i) + \\ + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} (x_{i-1} - x_i)^2 + O\left(\Delta x^3\right). \quad (15)$$

Subtracting (15) from (14) simplifies to

$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{x_{i+1} - x_{i-1}} + O(\Delta x)$, becomes $O(\Delta x^2)$ for uniform grids.

Discretization (spatial advection)

Advection in conservative form allows to compute derivatives with simple second order central difference formula $\frac{\partial u}{\partial x}\Big|_i = \frac{u_{i+1}-u_{i-1}}{x_{i+1}-x_{i-1}}$:

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= u\frac{\partial u}{\partial x} + 0 + v\frac{\partial u}{\partial y} \\
&= u\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial u}{\partial y} \\
&= \left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right) + \left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}. \qquad (16)
\end{aligned}
$$

Discretization (spatial advection)

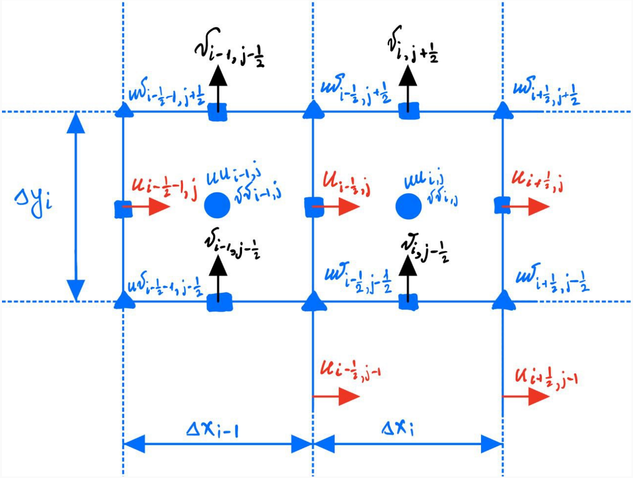


Figure 4: Advection discretization.

Discretization (divergence)

$$\hat{D}\boldsymbol{v} = \hat{bc}_2^n,$$

$$\begin{bmatrix} \hat{D}_x & \hat{D}_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \hat{bc}_2^n,$$

$$\frac{1}{\Delta x} D_x u + \frac{1}{\Delta y} D_y v = \hat{bc}_2^n,$$

$$\frac{1}{\Delta_{xy}} \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix} = \frac{1}{\Delta_{xy}} Dq = \hat{bc}_2^n,$$

where

$$q = \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix}, \quad \Delta_{xy} = \begin{bmatrix} \frac{1}{\Delta x_1 \Delta y_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta x_2 \Delta y_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\Delta x_M \Delta y_N} \end{bmatrix}. \quad (17)$$

Discretization (divergence)

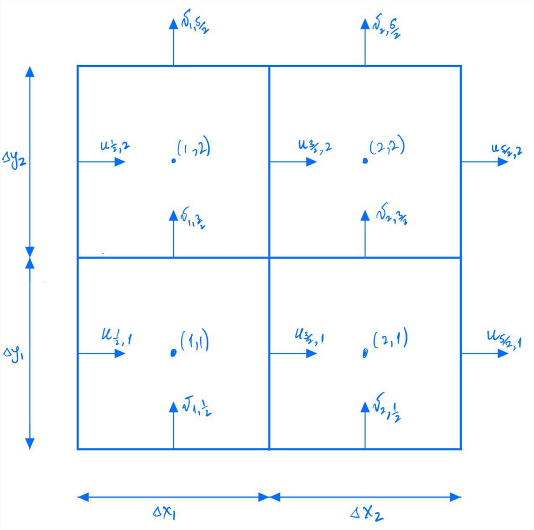


Figure 5: 2×2 grid example for divergence operator.

Discretization (divergence)

$$\Delta_{xy} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{\frac{3}{2},1}\Delta y_1 \\ u_{\frac{3}{2},2}\Delta y_2 \\ v_{1,\frac{3}{2}}\Delta x_1 \\ v_{2,\frac{3}{2}}\Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{u_{\frac{1}{2},1}}{\Delta x_1} + \frac{v_{1,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{5}{2},1}}{\Delta x_2} + \frac{v_{2,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{1}{2},2}}{\Delta x_1} - \frac{v_{1,\frac{5}{2}}}{\Delta y_2} \\ \frac{u_{\frac{5}{2},2}}{\Delta x_2} - \frac{v_{2,\frac{5}{2}}}{\Delta y_2} \end{bmatrix}. \tag{18}$$

Discretization (pressure gradient)

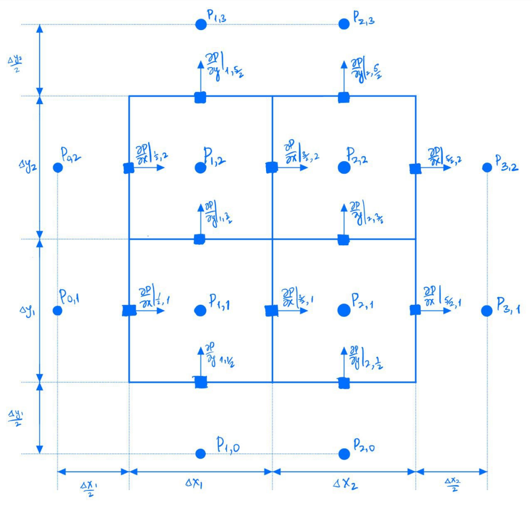


Figure 6: 2×2 grid example for gradient operator.

Discretization (pressure gradient)

Notice, that matrix of coefficients below is very similar to Divergence matrix, i.e. $G = -D^T$.

$$\operatorname{diag}\begin{bmatrix} \frac{2}{\Delta x_1+\Delta x_2} \\ \frac{2}{\Delta x_1+\Delta x_2} \\ \frac{2}{\Delta y_1+\Delta y_2} \\ \frac{2}{\Delta y_1+\Delta y_2} \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1,1} \\ p_{2,1} \\ p_{1,2} \\ p_{2,2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right). \tag{19}$$

Normalization. Symmetrization.

$$R \equiv \begin{bmatrix} \Delta y_j & 0 \\ 0 & \Delta x_i \end{bmatrix},$$
$$\hat{M} \equiv \begin{bmatrix} \frac{1}{2}\left(\Delta x_i + \Delta x_{i-1}\right) & 0 \\ 0 & \frac{1}{2}\left(\Delta y_j + \Delta y_{j-1}\right) \end{bmatrix}.$$

Let $q = R\boldsymbol{v}$, then $\boldsymbol{v} = R^{-1}q$.

Left multiplication by $\hat{M}$ of momentum equation changes $\hat{M}^{-1}G$ to G.

Left multiplication by $\hat{M}$ rescales $\hat{L}$ in one direction, right multiplication by R^{-1} in another. We obtain $\hat{M}\hat{L}R^{-1}$ symmetric, $\hat{M}IR^{-1}$ diagonal, hence, $\hat{M}\hat{A}R^{-1}$ symmetric.

Normalization. Symmetrization.

Resultant system finally becomes symmetric

$$\boxed{\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \end{pmatrix} + \begin{pmatrix} bc_1^n \\ 0 \end{pmatrix}.} \tag{20}$$

We will consider $\hat{bc}_2^n = 0$ for Lid Driven Cavity problem.

Motivation. Is it possible to remove the need to solve for pressure in our the system?

Stream function $\psi(x, y, t) : \boldsymbol{v} = \nabla \times \psi$ of an incompressible two-dimensional flow:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{21}$$

Vorticity $\omega = \nabla \times \boldsymbol{v}$, in two-dimensional case (x-y-plane) the only non-zero component of ω is z, which leads to

$$\omega = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{22}$$

Motivation. Including Poisson equation directly into our system.

(21) into (22) leads to

$$-\nabla^2\psi = \omega. \tag{23}$$

Apply $(\nabla\times)$ to momentum. Since curl of gradient is zero, i.e.:

$$-\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = 0 \tag{24}$$

we get

$$\frac{\partial\omega}{\partial t} - \epsilon\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) = -\left(u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}\right). \tag{25}$$

Vorticity Streamfunciton Poisson (23) into Vorticity Transport (25):

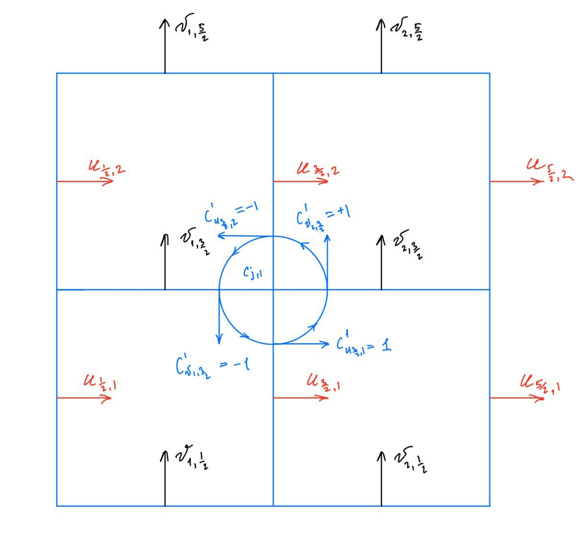
$$\boxed{-\frac{\partial\nabla^2\psi}{\partial t} + \epsilon\nabla^4\psi = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\nabla^2\psi.} \tag{26}$$

Nullspace approach (the main idea). Finding the analogue of $(\nabla\times)$ to eliminate pressure from the system.

Recall, that we previously found out that $G = -D^T$ by construction. Matrix D is wider than tall for systems greater than 2×2. (Number of cells is always less than the number of unknown velocity components.)

Let matrix C be nullspace of D, i.e. $DC \equiv 0$. Then $0 \equiv -(DC)^T = -C^T D^T = C^T G$. Hence, C^T is equivalent to $\nabla\times$.

Nullspace approach (computing matrix C)



$$C = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Algorithm

The motivation is to multiply the system

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \end{pmatrix} + \begin{pmatrix} bc_1^n \\ 0 \end{pmatrix} \tag{27}$$

by C^T in order to eliminate the pressure. If we are to keep system symmetric, then need to make substitution $q = C\psi$, which results in

$$\begin{bmatrix} C^T A & C^T G \\ D & 0 \end{bmatrix} \begin{pmatrix} C\psi^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} C^T r^n \\ 0 \end{pmatrix} + \begin{pmatrix} C^T bc_1^n \\ 0 \end{pmatrix}. \tag{28}$$

Automatically satisfies continuity $DC\psi \equiv 0$.

Algorithm

1. Construct curl matrix C, such that $DC = 0$ and $q_h = C\psi$.
2. Eliminate the pressure terms in the momentum equation.

$$\begin{aligned} Aq^{n+1} &= D^T p^{n+1} + bc_1, && \text{premultiply by } C^T, \\ C^T A q^{n+1} &= C^T bc_1, && \text{use } q^{n+1} = C\psi^{n+1}, \\ C^T A C \psi^{n+1} &= C^T bc_1. && \text{solve symmetric system for } \psi^{n+1}. \end{aligned} \tag{29}$$

3. Obtain $q^{n+1} = C\psi^{n+1}$. Update bc_1 if needed, recompute RHS.
4. Repeat steps (2)-(3) for $n + 2$

Results for Lid Driven Cavity Flow

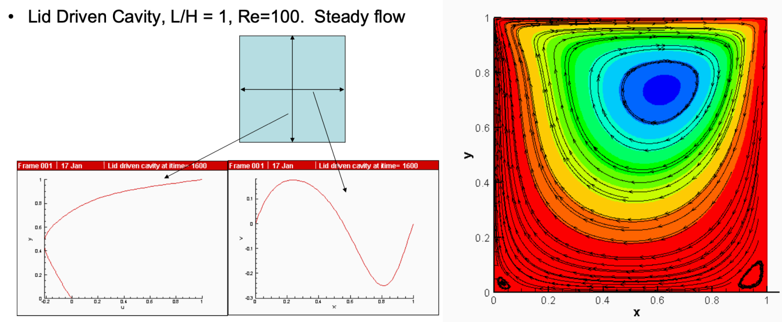


Figure 7: Velocities perpendicular to the the middle axises and streamfunction contour at Re=1600.

Future work

We considered $\hat{bc}_2^{\,n} = 0$ for Lid Driven Cavity problem in continuity equation.

Other types of BCs. Multi-domain approach.