

# Solving PDE

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September 19, 2024

**Abstract**

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# 1 Problem statement

Let

$$\mathbf{v}(x, y, t) = (u(x, y, t), v(x, y, t))$$

and consider non-dimensionalized incompressible Navier-Stokes system of equations

$$\text{Momentum: } \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \epsilon \nabla \cdot \nabla \mathbf{v}, \quad (1.1a)$$

$$\epsilon = \frac{1}{\text{Re}}, 0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.$$

$$\text{Continuity: } \nabla \cdot \mathbf{v} = 0, \quad (1.1b)$$

$$0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.$$

$$\text{Inlet: } \mathbf{v}(t, 0, y) = \mathbf{v}(t, x, 1) = (1 + A \cos(kx - \omega t + \phi_0), v), \quad (1.1c)$$

$$\{A \leq 1, k, \omega, \phi_0\} \subset \mathbb{R}.$$

$$\text{No-slip wall BC: } \mathbf{v}(t, x, 0) = (0, 0). \quad (1.1d)$$

$$\text{Outlet and freestream BC: Artificial boundary conditions.} \quad (1.1e)$$

$$\text{Initial condition: } \mathbf{v}(0, x, y) = (1, 0). \quad (1.1f)$$

We are to determine appropriate boundary conditions for top side of the domain. Taking derivative of [BC \(1.1c\)](#) w.r.t  $x$

$$\frac{\partial u}{\partial x} = -kA \sin(kx - \omega t + \phi_0), \quad (1.2)$$

and making use of continuity [Eq. \(1.1b\)](#) leads to

$$\frac{\partial v}{\partial y} = kA \sin(kx - \omega t + \phi_0). \quad (1.3)$$

We can integrate [Eq. \(1.3\)](#) over  $dy$  to obtain general expression for vertical velocity component

$$v(x, y, t) = ykA \sin(kx - \omega t + \phi_0) + g(x, t). \quad (1.4)$$

In case of  $g(x, t) = 0$ , plugging [Eq. \(1.4\)](#) together with [BC \(1.1c\)](#) into [Eq. \(1.1a\)](#) leads to

$$\frac{\partial p}{\partial x} = 0 \quad (1.5)$$

for  $x$ -momentum. If we now take  $\frac{\partial}{\partial y}$  of [Eq. \(1.5\)](#) and plug into  $\frac{\partial}{\partial x}$  of  $y$ -momentum we will obtain

$$\begin{aligned} & \frac{A k^4 y \cos(kx + \omega t + \phi_0)}{\text{Re}} - A k^2 \omega y \sin(kx + \omega t + \phi_0) \\ & - A k^3 y \sin(kx + \omega t + \phi_0) (A \cos(kx + \omega t + \phi_0) + 1) \\ & + A^2 k^3 y \cos(kx + \omega t + \phi_0) \sin(kx + \omega t + \phi_0) = 0, \end{aligned} \quad (1.6)$$

which is obviously not always true for all  $A, k, \omega$ . Therefore  $g(x, t) \neq 0 \quad \forall x, t$ .

Keeping  $g(x, t)$  term and repeating the above process leads to third order PDE

$$\begin{aligned} & \left( \frac{\partial^2}{\partial x^2} g(x, t) - A k^3 y \sin(kx + \omega t + \phi_0) \right) (A \cos(kx + \omega t + \phi_0) + 1) \\ & - \frac{1}{\text{Re}} \left( \frac{\partial^3}{\partial x^3} g(x, t) - A k^4 y \cos(kx + \omega t + \phi_0) \right) \\ & + \frac{\partial}{\partial x} \frac{\partial}{\partial t} g(x, t) + A k^2 \cos(kx + \omega t + \phi_0) (g(x, t) + A k y \sin(kx + \omega t + \phi_0)) \\ & - A k^2 \omega y \sin(kx + \omega t + \phi_0) = 0. \end{aligned} \quad (1.7)$$

We now need to determine boundary conditions for Eq. (1.7). It is known that at the left boundary  $v(0, y, t) = 0$ . Then Eq. (1.4) at  $y=1$  gives

$$g(x = 0, t) = -kA \sin(-\omega t + \phi_0). \quad (1.8)$$

Outlet BC (1.1e)  $\frac{\partial v}{\partial x} = 0$  at  $x = 1, y = 1$  with Eq. (1.4) leads to

$$\frac{\partial}{\partial x} g(x = 1, t) = -k^2 A \cos(k - \omega t + \phi_0). \quad (1.9)$$

Initial condition at  $t = 0$  we assumed (Eq. (1.1f)) to be  $v(x, y = 1, t = 0) = 0$ , then Eq. (1.4) provides us

$$g(x, t = 0) = -kA \sin(kx + \phi_0). \quad (1.10)$$