

# **Discrete streamfunction method for incompressible Navier-Stokes equation (also known as Exact fractional step method)**

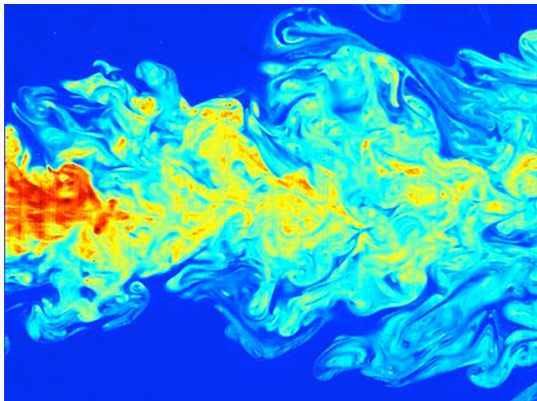
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March 5, 2024

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# Introduction



## Introduction (problem statement)

Let  $\mathbf{v}(x, y, t) = [u(x, y, t), v(x, y, t)]^T$  and  $p(x, y, t)$  be solutions to incompressible Navier-Stokes system of equations on 2D domain  $\Omega$  with Dirichlet Boundary conditions on  $\partial\Omega$ .

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{transient}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{\text{advective}} = \underbrace{-\frac{1}{\rho} \nabla p}_{\text{pressure gradient}} + \underbrace{\frac{\mu}{\rho} \nabla \cdot \nabla \mathbf{v}}_{\text{viscous, Laplacian}} \quad (\text{momentum}) \text{ in } \Omega \quad (1a)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (\text{continuity}) \text{ in } \Omega, \quad (1b)$$

$$\mathbf{v} = \mathbf{v}(x, y, t) \quad \text{on } \partial\Omega, \quad (1c)$$

where  $\rho$  - density,  $\mu$  - dynamic viscosity.

## Introduction (nondimensionalization)

After introducing the following dimensionless variables (marked with prime ' ):

$$x \rightarrow Lx', \quad \mathbf{v} \rightarrow U_0 \mathbf{v}', \quad \nabla \rightarrow \frac{1}{L} \nabla', \quad p \rightarrow p' \rho U_0^2, \quad t \rightarrow \frac{L}{U_0} t',$$

and Reynolds number

$$\text{Re} = \frac{\rho L U_0}{\mu},$$

we obtain the non-dimensional momentum and continuity equations (dropped prime superscript)

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{1}{\text{Re}} \nabla \cdot \nabla \mathbf{v}, \\ \nabla \cdot \mathbf{v} &= 0. \end{aligned} \tag{2}$$

## Discretization (discrete operators)

Denote discrete spatial operators as:

$\hat{L}$  : Laplacian.

$\hat{G}$  : Gradient.

$\hat{D}$  : Divergence.

$\mathbf{H}$  : Non-linear advective terms.

Then initial system (2) can be rewritten using these operators as

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{H}(\mathbf{v}) \\ 0 \end{pmatrix} = \begin{bmatrix} \hat{L} & -\hat{G} \\ -\hat{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ p \end{bmatrix} + \text{bc}_{\mathbf{v},p}, \quad (3)$$

## Discretization (transient)

Attack (3) with the following schemes as in Colonius (2008)  
(superscript denotes the time step):

**Viscous** - Implicit trapezoidal - Crank Nicholson scheme (second-order in time).

$$\hat{L}\mathbf{v} = \frac{1}{2} \left( \hat{L}\mathbf{v}^{n+1} + \hat{L}\mathbf{v}^n \right) \quad (4)$$

**Nonlin.** - Explicit Adams-Bashforth (second-order in time).

$$\mathbf{H}(\mathbf{v}) = \frac{3}{2}\mathbf{H}(\mathbf{v}^n) - \frac{1}{2}\mathbf{H}(\mathbf{v})^{n-1}. \quad (5)$$

**Pressure** - Implicit Euler (first-order in time).

$$\hat{G}p = \hat{G}p^{n+1}. \quad (6)$$

## Discretization (transient)

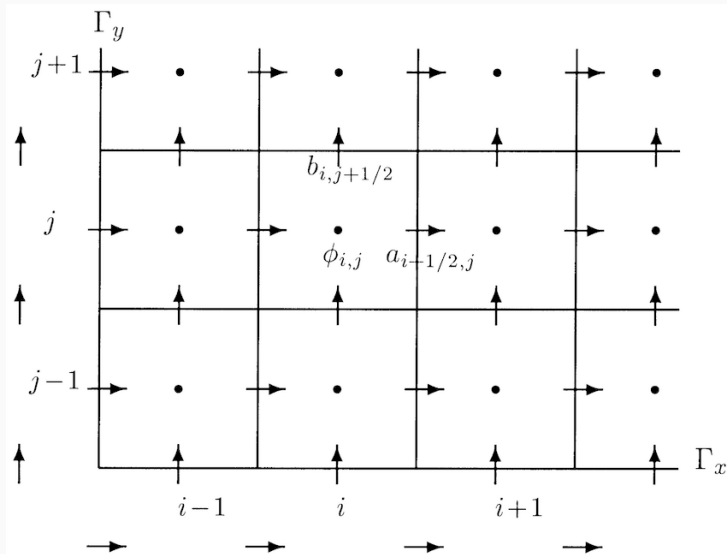
The above schemes result in the following time-discretized system:

$$\begin{bmatrix} \frac{1}{\Delta t} \mathbf{I} - \frac{1}{2} \hat{L} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \left[ \frac{1}{\Delta t} \mathbf{I} - \frac{1}{2} \hat{L} \right] \mathbf{v}^n - \left[ \frac{3}{2} \hat{\mathbf{H}}(\mathbf{v}^n) - \frac{1}{2} \hat{\mathbf{H}}(\mathbf{v}^{n-1}) \right] \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_{c_1}^n \\ \hat{b}_{c_2}^n \end{pmatrix}, \quad (7)$$

which in short becomes

$$\boxed{\begin{bmatrix} \hat{A} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{\gamma}^n \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_{c_1}^n \\ \hat{b}_{c_2}^n \end{pmatrix}}. \quad (8)$$

## Discretization (spatial)



**Figure 1:** Domain discretization as per Harlow and Welch (1965)



## Discretization (spatial Laplacian)

Rewrite Laplacian as a block matrix:

$$\hat{L} = \begin{bmatrix} \hat{L}_{xx}^u + \hat{L}_{yy}^u & 0 \\ 0 & \hat{L}_{xx}^v + \hat{L}_{yy}^v \end{bmatrix}.$$

Use Taylor expansion to derive second derivative approximation

$$\begin{aligned} u_{i-\frac{1}{2}+1,j} &= u_{i-\frac{1}{2},j} + \left. \frac{\partial u}{\partial x} \right|_{i-\frac{1}{2},j} \left( x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}} \right) + \\ &+ \frac{1}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{i-\frac{1}{2},j} \left( x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}} \right)^2 + O(\Delta x^3), \quad (9) \end{aligned}$$

$$\begin{aligned} u_{i-\frac{1}{2}-1,j} &= u_{i-\frac{1}{2},j} + \left. \frac{\partial u}{\partial x} \right|_{i-\frac{1}{2},j} \left( x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}} \right) + \\ &+ \frac{1}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{i-\frac{1}{2},j} \left( x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}} \right)^2 + O(\Delta x^3). \quad (10) \end{aligned}$$

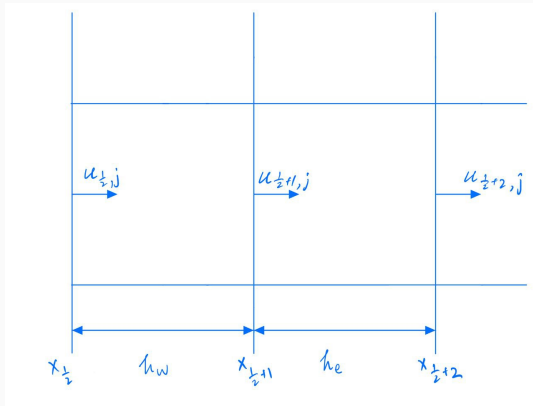
## Discretization (spatial Laplacian)

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{i-\frac{1}{2},j} \approx \frac{1}{h_c h_e} u_{i-\frac{1}{2}+1} - \frac{2}{h_w h_e} u_{i-\frac{1}{2}} + \frac{1}{h_c h_w} u_{i-\frac{1}{2}-1} + O(\Delta x), \quad (11)$$

where  $h_w = x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}$ ,  $h_c = \frac{x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}}{2}$ ,  $h_e = x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}$ .

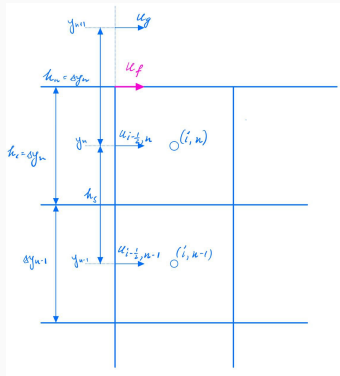
## Discretization (spatial Laplacian at boundary)

The exact value on the left boundary ( $u_{\frac{1}{2},j}$ ) is known.



**Figure 2:**  $\hat{L}_{xx}^u$  at the left boundary.

# Discretization (spatial Laplacian at boundary)



**Figure 3:**  $\hat{L}_{yy}^u$  at top boundary.

$$u_f = \frac{u_g + u_{i-\frac{1}{2}, N}}{2} \implies u_g = 2u_f - u_{i-\frac{1}{2}, N} \implies$$

$$\frac{\partial^2 u}{\partial y^2} \Big|_{i-\frac{1}{2}, N} = \frac{2u_f}{h_c^2} + u_{i-\frac{1}{2}, N} \left( \frac{-(2h_c + h_s)}{h_c^2 h_s} \right) + u_{i-\frac{1}{2}, N-1} \frac{1}{h_s h_c}.$$

## Discretization (spatial advection)

Advection in conservative form allows to compute derivatives with simple second order central difference formula  $\frac{\partial u}{\partial x}|_i = \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}$ :

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= u \frac{\partial u}{\partial x} + 0 + v \frac{\partial u}{\partial y} \\ &= u \frac{\partial u}{\partial x} + u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + v \frac{\partial u}{\partial y} \\ &= \left( u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right) + \left( u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}. \end{aligned} \tag{12}$$

## Discretization (spatial advection)

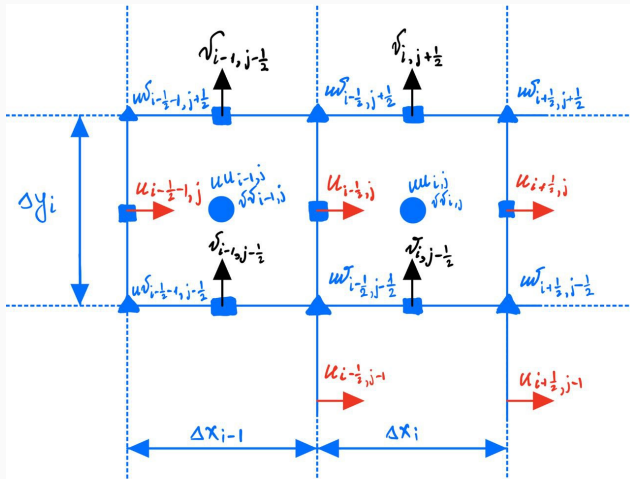


Figure 4: Advection discretization.

## Discretization (spatial advection)

$$u_{i+1,j} = u_{i,j} + \left. \frac{\partial u}{\partial x} \right|_{i,j} (x_{i+1} - x_i) + \frac{1}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{i,j} (x_{i+1} - x_i)^2 + O(\Delta x^3), \quad (13)$$

$$u_{i-1,j} = u_{i,j} + \left. \frac{\partial u}{\partial x} \right|_{i,j} (x_{i-1} - x_i) + \frac{1}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{i,j} (x_{i-1} - x_i)^2 + O(\Delta x^3). \quad (14)$$

Subtracting (14) from (13) simplifies to

$$\left. \frac{\partial u}{\partial x} \right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{x_{i+1} - x_{i-1}} + O(\Delta x), \text{ becomes } O(\Delta x^2) \text{ for uniform grids.}$$

## Discretization (divergence)

$$\hat{D}\mathbf{v} = \hat{b}c_2^n,$$

$$\begin{bmatrix} \hat{D}_x & \hat{D}_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \hat{b}c_2^n,$$

$$\frac{1}{\Delta x} D_x u + \frac{1}{\Delta y} D_y v = \hat{b}c_2^n,$$

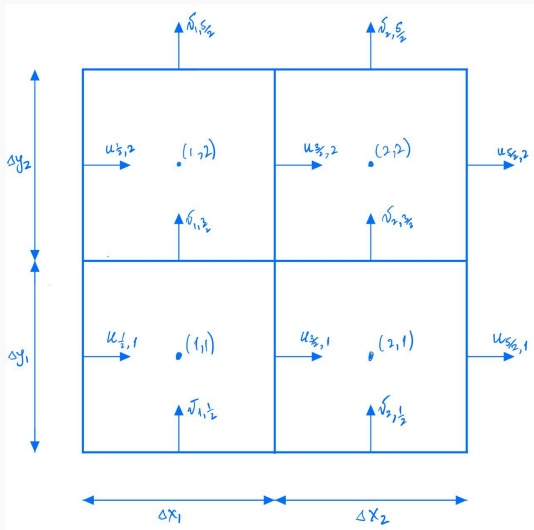
$$\frac{1}{\Delta_{xy}} \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix} = \frac{1}{\Delta_{xy}} Dq = \hat{b}c_2^n,$$

where

$$q = \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix}, \quad \Delta_{xy} = \begin{bmatrix} \frac{1}{\Delta x_1 \Delta y_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta x_2 \Delta y_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\Delta x_M \Delta y_N} \end{bmatrix}. \quad (15)$$



## Discretization (divergence)

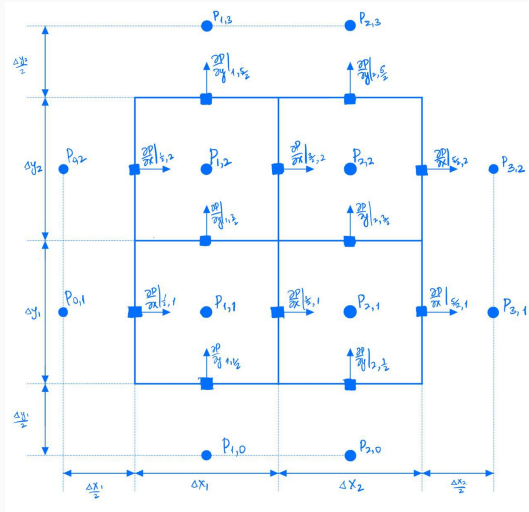


**Figure 5:**  $2 \times 2$  grid example for divergence operator.

## Discretization (divergence)

$$\Delta_{xy} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{\frac{3}{2},1} \Delta y_1 \\ u_{\frac{3}{2},2} \Delta y_2 \\ v_{1,\frac{3}{2}} \Delta x_1 \\ v_{2,\frac{3}{2}} \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{u_{\frac{1}{2},1}}{\Delta x_1} + \frac{v_{1,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{5}{2},1}}{\Delta x_2} + \frac{v_{2,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{1}{2},2}}{\Delta x_1} - \frac{v_{1,\frac{5}{2}}}{\Delta y_2} \\ \frac{u_{\frac{5}{2},2}}{\Delta x_2} - \frac{v_{2,\frac{5}{2}}}{\Delta y_2} \end{bmatrix} . \quad (16)$$

# Discretization (pressure gradient)



**Figure 6:**  $2 \times 2$  grid example for gradient operator.

## Discretization (pressure gradient)

Notice, that matrix of coefficients below is very similar to Divergence matrix, i.e.  $G = -D^T$ .

$$\text{diag} \begin{bmatrix} \frac{2}{\Delta x_1 + \Delta x_2} \\ \frac{2}{\Delta x_1 + \Delta x_2} \\ \frac{2}{\Delta y_1 + \Delta y_2} \\ \frac{2}{\Delta y_1 + \Delta y_2} \end{bmatrix} \left( \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1,1} \\ p_{2,1} \\ p_{1,2} \\ p_{2,2} \end{bmatrix} + \begin{bmatrix} 0 \\ p_{3,1} \\ 0 \\ p_{3,2} \\ 0 \\ 0 \end{bmatrix} \right) . \quad (17)$$

## Normalization. Symmetrization.

$$R \equiv \begin{bmatrix} \Delta y_j & 0 \\ 0 & \Delta x_i \end{bmatrix},$$
$$\hat{M} \equiv \begin{bmatrix} \frac{1}{2} (\Delta x_i + \Delta x_{i-1}) & 0 \\ 0 & \frac{1}{2} (\Delta y_j + \Delta y_{j-1}) \end{bmatrix}.$$

Let  $q = R\mathbf{v}$ , then  $\mathbf{v} = R^{-1}q$ .

Left multiplication by  $\hat{M}$  of momentum equation changes  $\hat{M}^{-1}G$  to  $G$ .

Left multiplication by  $\hat{M}$  rescales  $\hat{L}$  in one direction, right multiplication by  $R^{-1}$  in another. We obtain  $\hat{M}\hat{L}R^{-1}$  symmetric,  $\hat{M}IR^{-1}$  diagonal, hence,  $\hat{M}\hat{A}R^{-1}$  symmetric.

## Normalization. Symmetrization.

Resultant system finally becomes symmetric

$$\boxed{\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \end{pmatrix} + \begin{pmatrix} bc_1^n \\ 0 \end{pmatrix}.} \quad (18)$$

We will consider  $\hat{bc}_2^n = 0$  for Lid Driven Cavity problem.

## Motivation. Is it possible to remove the need to solve for pressure in our the system?

Stream function  $\psi(x, y, t) : \mathbf{v} = \nabla \times \psi$  of an incompressible two-dimensional flow:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (19)$$

Vorticity  $\omega = \nabla \times \mathbf{v}$ , in two-dimensional case (x-y-plane) the only non-zero component of  $\omega$  is z, which leads to

$$\omega = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \quad (20)$$

## Motivation. Including Poisson equation directly into our system.

(19) into (20) leads to

$$-\nabla^2\psi = \omega. \quad (21)$$

Apply  $(\nabla \times)$  to momentum. Since curl of gradient is zero, i.e.:

$$-\frac{\partial}{\partial y} \left( \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial x} \left( \frac{\partial p}{\partial y} \right) = 0 \quad (22)$$

we get

$$\frac{\partial \omega}{\partial t} - \epsilon \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) = - \left( u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} \right). \quad (23)$$

Vorticity Streamfunction Poisson (21) into Vorticity Transport (23):

$$\boxed{-\frac{\partial \nabla^2 \psi}{\partial t} + \epsilon \nabla^4 \psi = \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 \psi.} \quad (24)$$

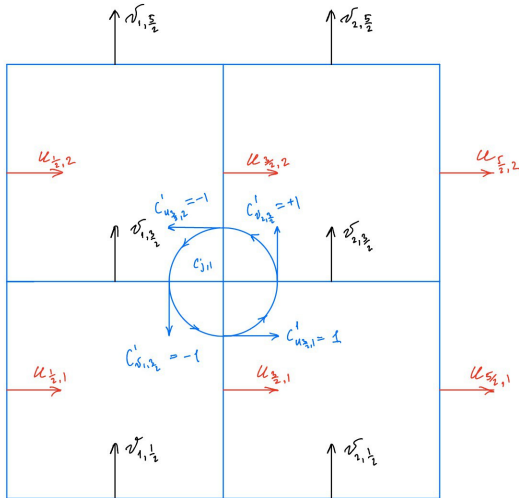


## Nullspace approach (the main idea). Finding the analogue of $(\nabla \times)$ to eliminate pressure from the system.

Recall, that we previously found out that  $G = -D^T$  by construction. Matrix  $D$  is wider than tall for systems greater than  $2 \times 2$ . (Number of cells is always less than the number of unknown velocity components.)

Let matrix  $C$  be nullspace of  $D$ , i.e.  $DC \equiv 0$ . Then  $0 \equiv -(DC)^T = -C^T D^T = C^T G$ . Hence,  $C^T$  is equivalent to  $\nabla \times$ .

# Nullspace approach (computing matrix C)



$$C = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

The motivation is to multiply the system

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \end{pmatrix} + \begin{pmatrix} bc_1^n \\ 0 \end{pmatrix} \quad (25)$$

by  $C^T$  in order to eliminate the pressure. If we are to keep system symmetric, then need to make substitution  $q = C\psi$ , which results in

$$\begin{bmatrix} C^T A & C^T G \\ D & 0 \end{bmatrix} \begin{pmatrix} C\psi^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} C^T r^n \\ 0 \end{pmatrix} + \begin{pmatrix} C^T bc_1^n \\ 0 \end{pmatrix}. \quad (26)$$

Automatically satisfies continuity  $DC\psi \equiv 0$ .

# Algorithm

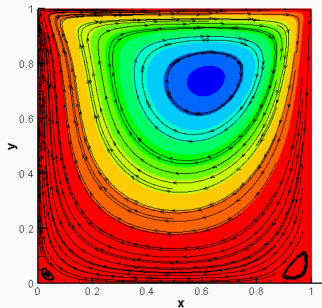
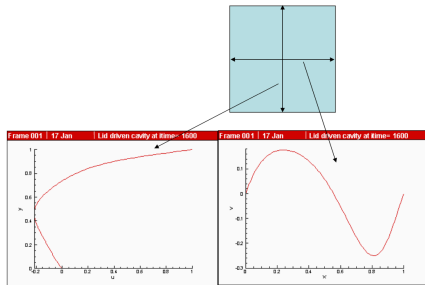
1. Construct curl matrix  $C$ , such that  $DC = 0$  and  $q_h = C\psi$ .
2. Eliminate the pressure terms in the momentum equation.

$$\begin{aligned} Aq^{n+1} &= D^T p^{n+1} + bc_1, && \text{premultiply by } C^T, \\ C^T Aq^{n+1} &= C^T bc_1, && \text{use } q^{n+1} = C\psi^{n+1}, \\ C^T AC\psi^{n+1} &= C^T bc_1. && \text{solve symmetric system for } \psi^{n+1}. \end{aligned} \tag{27}$$

3. Obtain  $q^{n+1} = C\psi^{n+1}$ . Update  $bc_1$  if needed, recompute RHS.
4. Repeat steps (2)-(3) for  $n + 2$

# Results for Lid Driven Cavity Flow

- Lid Driven Cavity,  $L/H = 1$ ,  $Re=100$ . Steady flow



**Figure 7:** Velocities perpendicular to the the middle axes and streamfunction contour at  $Re=1600$ .

## Future work

We considered  $\hat{b}c_2^n = 0$  for Lid Driven Cavity problem in continuity equation.

Other types of BCs. Multi-domain approach.

