Discrete stream function method for the incompressible Navier-Stokes equations with full Dirichlet boundary conditions

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Abstract

The goal of these notes is to present the detailed overview of discrete stream function method for solving incompressible Navier-Stokes equations with simple boundary conditions. We will discuss in detail the scheme formulation, transient and spatial discretizations. After studying these notes one must get a coherent picture of the application of discrete stream function method to incompressible flows and be able to implement the scheme in code.

Contents

1 Introduction and problem statement

Let the velocity vector $\boldsymbol{v}(t,x,y) = (u(t,x,y), v(t,x,y))$ and pressure $p(t,x,y)$ be the solutions to **??** on in rectangular domain $\Omega : 0 \leq x < L, 0 \leq y < H$ together with boundary conditions on $\partial\Omega$:

$$\text{Momentum: } \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla \cdot \nabla \boldsymbol{v} \quad \text{in } \Omega, \tag{1.1a}$$

$$\rho \text{ - density}, \mu \text{ - dynamic viscosity.}$$

$$\text{Continuity: } \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\text{Normal BC: } \boldsymbol{v}(t,x,y) \cdot \hat{\mathbf{n}} = \boldsymbol{v}_{\hat{\mathbf{n}}}(t,x,y) \quad \text{on } \partial\Omega, \tag{1.1c}$$

$$\text{Tangential BC: } \boldsymbol{v}(t,x,y) \cdot \hat{\mathbf{t}} = \boldsymbol{v}_{\hat{\mathbf{t}}}(t,x,y) \quad \text{on } \partial\Omega, \tag{1.1d}$$

$$\text{Initial condition: } \boldsymbol{v}(0,x,y) = g(x,y) \quad \text{in } \Omega, \tag{1.1e}$$

where $\hat{\mathbf{n}}, \hat{\mathbf{t}}$ are normal and tangential unit vectors to the $\partial\Omega$ boundary.

In Appendix B we display how to make the Eqs. (1.1) non-dimensional and show the results in Section 2. Domain discretization is performed in Appendix C. Transient discretization is performed in Section 3, whereas spatial discretizations can be found in Appendices C.1 to C.4 and symmetrization is shown in Appendix D. The way of obtaining matrices which modify the system is displayed in Section 4, whereas the resulting algorithm is shown in Section 5.

2 Problem statement on truncated domain

The initial Eqs. (1.1) from Section 1 on semi-infinite domain now become a system of partial differential equations on a unit square domain

$$\text{Momentum: } \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \epsilon\nabla \cdot \nabla \boldsymbol{v} \quad \text{in } \Omega, \tag{2.1a}$$

$$\epsilon = \frac{1}{\mathrm{Re}}, 0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.$$

$$\text{Continuity: } \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.$$

$$\text{Normal BC: } \boldsymbol{v}(t,x,y) \cdot \hat{\mathbf{n}} = \boldsymbol{v}_{\hat{\mathbf{n}}}(t,x,y) \quad \text{on } \partial\Omega, \tag{2.1c}$$

$$\text{Tangential BC: } \boldsymbol{v}(t,x,y) \cdot \hat{\mathbf{t}} = \boldsymbol{v}_{\hat{\mathbf{t}}}(t,x,y) \quad \text{on } \partial\Omega, \tag{2.1d}$$

$$\text{Initial condition: } \boldsymbol{v}(0,x,y) = g(x,y) \quad \text{in } \Omega, \tag{2.1e}$$

3 Discrete operators

This subsection focuses on the specific operators used in the discretization of Eqs. (2.1), providing the mathematical tools necessary to transform these continuous equations into a form that can be solved numerically.

Denote discrete spatial operators as:

$\hat{L}$: Laplacian.

$\hat{G}$: Gradient.

$\hat{D}$: Divergence.

$\hat{\mathbf{H}}$: Non-linear advective terms.

Then initial system of Eqs. (2.1) can be approximated using such operators as

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{v} \\ p \end{pmatrix} = \begin{bmatrix} \hat{L} & -\hat{G} \\ -\hat{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ p \end{bmatrix} + \begin{pmatrix} -\hat{\mathbf{H}}(\boldsymbol{v}) \\ 0 \end{pmatrix} + \mathrm{bc}_{\boldsymbol{v},p}, \tag{3.1}$$

where boundary conditions are in terms of pressure and velocity. In the following subsections, each of the discrete operators is described individually.

Attack Sys. (3.1) with the following schemes as in Colonius [2] (superscript denotes the time step):

Viscous - Implicit trapezoidal - Crank Nicholson scheme (second-order method in time).

$$\hat{L}\boldsymbol{v} = \frac{1}{2}\left(\hat{L}\boldsymbol{v}^{n+1} + \hat{L}\boldsymbol{v}^{n}\right) \tag{3.2}$$

Nonlinear - Explicit Adams-Bashforth (second-order method in time).

$$\hat{\mathbf{H}}(\boldsymbol{v}) = \frac{3}{2}\hat{\mathbf{H}}(\boldsymbol{v}^n) - \frac{1}{2}\hat{\mathbf{H}}(\boldsymbol{v})^{n-1}. \tag{3.3}$$

Pressure - Implicit Euler, though the pressure variable will be later eliminated in the algorithm (first-order method in time).

$$\hat{G}p = \hat{G}p^{n+1}. \tag{3.4}$$

After applying the discrete operators listed in the following Appendices C.2 to C.4 and **??**, we rewrite Sys. (3.1) as

$$\begin{bmatrix} \frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \left(\begin{matrix} \left[\frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}\hat{L}\right]\boldsymbol{v}^n - \left[\frac{3}{2}\hat{\mathbf{H}}(\boldsymbol{v}^n) - \frac{1}{2}\hat{\mathbf{H}}(\boldsymbol{v}^{n-1})\right] \\ 0 \end{matrix} \right) + \begin{pmatrix} \hat{bc}_1^{\,n} \\ \hat{bc}_2^{\,n} \end{pmatrix}, \tag{3.5}$$

which in short becomes

$$\boxed{\begin{bmatrix} \hat{A} & \hat{G} \\ \hat{D} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{r}^n \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{bc}_1 \\ \hat{bc}_2 \end{pmatrix}.} \tag{3.6}$$

Here, $\hat{r}^n$ includes viscous, non-linear and gradient terms from time steps $n, n-1$ and $\hat{bc}_1, \hat{bc}_2$ are explicit boundary values in terms of velocity and pressure. Appendices C.1 to C.4 display the spatial discretization of $\hat{L}, \hat{\mathbf{H}}, \hat{D}, \hat{G}$ operators. Appendix D shows how to make the system symmetric.

4 Nullspace method and pressure elimination in terms of discrete operators

The goal of this subsection is to show how unknown pressure variables can be eliminated from Sys. (D.1) similarly to vorticity-stream function formulation. Publications of Chang [1] and Hall [3] use the idea that in Sys. (D.1) matrix D is wider than tall for grids larger than 2×2, hence it defines a nullspace. The nullspace of matrix D is the set of all solutions to the homogeneous linear system $Dx = 0$, where x is a vector in the null space of D. Let C be the nullspace matrix containing such vectors x.

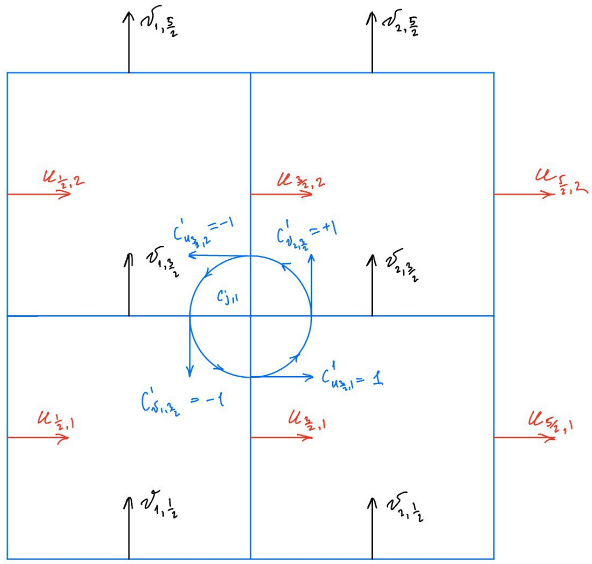

Figure 1: 2×2 example for C matrix.

An intuitive way of constructing the matrix C relies on the utilization of counterclockwise vorticity around the nodes within the domain (Fig. 1). If the direction of the velocity vector on the adjacent face aligns with the vorticity's direction, +1 is assigned to the corresponding row; conversely, -1 is assigned in the case of opposite directions of velocity and vorticity. After applying the above procedure we obtain

$$C = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

The matrix C has dimensions corresponding to unknown velocities times the number of nodes around which these velocities revolve. The desired product then becomes $DC = 0$. $D = -G^T$ (derived in Appendices C.3 and C.4) leads to important property $(DC)^T = C^T D^T = -C^T G = 0$. Premultiplying momentum equation in Sys. (D.1) by C^T creates $C^T G p = 0$ term, which completely eliminates the pressure from our system. New Laplacian operator applied to streamfunction now becomes $C^T C$ with full Dirichlet boundary conditions.

Momentum part of Sys. (3.6) becomes

$$\left(\frac{1}{\Delta t}\hat{M}R^{-1} - \frac{1}{2}L\right) q^{n+1} = RHS, \quad \text{which we premultiply by } C^T \text{ and use } q^{n+1} = C\psi. \tag{4.1}$$

All boundary conditions in this system are Dirichlet.
$\hat{M}R^{-1} = I$ for uniform grids.

$$\left(\frac{1}{\Delta t}C^T\hat{M}R^{-1}C - \frac{1}{2}C^T LC\right)\psi^{n+1} = C^T RHS, \quad \text{has to be solved for } \psi^{n-1}. \tag{4.2}$$

$$\psi^{n+1} = \left(\frac{1}{\Delta t}C^T\hat{M}R^{-1}C - \frac{1}{2}C^T LC\right)^{-1} C^T RHS,$$

We found analytically that

$\mathrm{curl}^T\,\mathrm{curl}\,\psi = -\nabla^2\psi$ and $\mathrm{curl}^T\,\nabla^2\,\mathrm{curl}\,\psi^{n-1} = -\nabla^2\left(\nabla^2\psi\right)$.
However, in discrete form $C^T C = L_\psi \neq L_{\boldsymbol{v}}$,
since L_ψ applies to ψ located at the nodes,
whereas $L_{\boldsymbol{v}}$ applies to $\boldsymbol{v}$ located at faces of each cell.
Finding $\left(C^T\hat{M}R^{-1}C\right)^{-1}$ implies solving Poisson equation
for streamfunciton $\nabla^2\psi = -\omega$.
If we are to use fully explicit scheme on uniform grid, then

$$\left(C^T C\right)\psi^{n+1} = C^T\left(RHS\right) \iff \tag{4.3}$$

$$-\frac{\partial}{\partial t}\omega = \frac{\partial}{\partial t}\left(\nabla^2\psi\right) = \mathrm{curl}\left(-\hat{\mathbf{H}}(\boldsymbol{v}) + \epsilon\nabla^2\boldsymbol{v}\right). \tag{4.4}$$

5 Resulting algorithm

Let us consider the solution to discretized Sys. (D.1)q^{n+1}, where q^{n+1} is a solution of homogeneous continuity equation

$$Dq^{n+1} = 0 \tag{5.1}$$

Taking into account continuous operator identities in vorticity-stream fucntion, the resulting algorithm for discrete Navier-Stokes Sys. (D.1) can be described as follows:

1. Construct curl matrix C, such that $DC = 0$ and $q = C\psi$.
2. Eliminate the pressure terms in the momentum equation.

$$\begin{aligned} Aq^{n+1} &= -Gp^{n+1} + bc_1, & & \\ A(q^{n+1}) &= D^T p^{n+1} + bc_1, & & \text{premultiply by } C^T, \\ C^T Aq^{n+1} &= C^T(bc_1), & & \text{use } q^{n+1} = C\psi^{n+1}, \\ C^T AC\psi^{n+1} &= C^T(bc_1). & & \end{aligned} \tag{5.2}$$

3. Solve resulting Sys. (5.2) for ψ^{n+1}.
4. Obtain $q^{n+1} = C\psi^{n+1}$.
5. Convert q^{n+1} to velocity vector field $\boldsymbol{v}^{n+1} = R^{-1}q^{n+1}$.
6. Transition to the next time instance by repeating Steps (3) to (5).

References

[1] Wang Chang, Francis Giraldo, and Blair Perot. Analysis of an exact fractional step method. *Journal of Computational Physics*, 180(1):183–199, 2002.

[2] Tim Colonius and Kunihiko Taira. A fast immersed boundary method using a nullspace approach and multi-domain far-field boundary conditions. *Computer Methods in Applied Mechanics and Engineering*, 197:2131–2146, April 2008.

[3] C. Hall, T. Porsching, R. Dougall, R. Amit, A. Cha, C. Cullen, George Mesina, and Samir Moujaes. Numerical methods for thermally expandable two-phase flow-computational techniques for steam generator modeling. *NASA STI/Recon Technical Report N*, 1980.

[4] F. Harlow and E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids*, 8:2182–2189, 1965.

A Appendix

B Nondimensionalization

In numerical computations, it is advantageous to maintain variables of comparable magnitude. This ensures that operations such as the multiplication of a large dimensional pressure variable with a small velocity are not performed.

We will truncate the domain and normalize all equations by domain width L and maximum velocity of the boundary condition U_0. After introducing the following dimensionless variables (marked with prime $'$):

$$x \to Lx', \quad \boldsymbol{v} \to U_0 \boldsymbol{v}', \quad \nabla \to \frac{1}{L}\nabla', \quad p \to p'\rho U_0^2, \quad t \to \frac{L}{U_0}t',$$

we obtain

$$\frac{U_0}{\frac{L}{U_0}}\frac{\partial \boldsymbol{v}'}{\partial t'} + U_0\boldsymbol{v} \cdot \left(\frac{1}{L}\nabla'\right) U_0\boldsymbol{v} = -\frac{1}{\rho}\left(\frac{1}{L}\nabla'\right) p'\rho U_0^2 + \frac{\mu}{\rho}\left(\frac{1}{L}\nabla'\right) \cdot \left(\frac{1}{L}\nabla'\right) U_0\boldsymbol{v}'.$$

After multiplying both sides of the equation by $\frac{L}{U_0^2}$ and introducing well-known Reynolds number

$$\mathrm{Re} = \frac{\rho L U_0}{\mu},$$

we obtain the non-dimensional momentum equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla \cdot \nabla \boldsymbol{v},$$

where prime superscript $'$ is suppressed for the dimensionless variables. The continuity equation is nondimensionalized in a similar manner:

$$\left(\frac{1}{L}\nabla'\right) \cdot U_0\boldsymbol{v}' = 0 \iff \nabla' \cdot \boldsymbol{v}' = 0 \implies \nabla \cdot \boldsymbol{v} = 0,$$

where we drop prime superscripts in the last identity as well.

C Domain discretization

Following the nondimensionalization of the governing equations, we now turn our attention to the process of domain discretization. This crucial step involves dividing the computational domain into discrete elements, allowing us to compute variables and their derivatives on a finite set of points called grid. Consider discretizing the area into M intervals horizontally and N intervals vertically forming a mesh of rectangular cells. To enhance memory access it is recommended to organize the data using vectors instead of matrices. The reason behind is memory access time; a single index i works faster than double i, j in most programming languages. In this section, however, for the sake of simplicity, we use two indices i, j (column and row as in Harlow and Welch [4]) to represent coordinates on the grid.

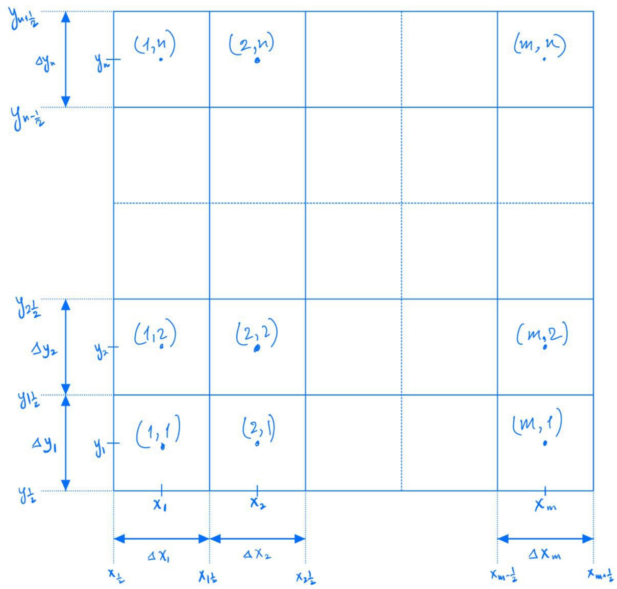

Figure 2: Domain discretization (here $m = M, n = N$)

To eliminate the necessity of solving an additional equation for pressure we will be using a staggered grid arrangement. Details are discussed in Appendices C.3 and C.4 and Section 4. Moreover, the pressure gradients can be evaluated directly using central differences on such grids. This calculation on staggered grids does not require any interpolation, furthermore, it is computationally cheaper and simpler to implement.

The u velocity components are stored at the centres of vertically oriented faces, while the values of v are stored at the centres of horizontal ones. The number of unknown components inside the domain (excluding boundaries) is $M(N-1)$ for v and $(M-1)N$ for u. At the line (row) with coordinate y_1 there are $M-1$ unknown u components located inside the domain with a total of N such y horizontal lines.

Increasing the accuracy closer to the boundary of interest looks advantageous, hence, we will refine the grid using the standard ratio rule $\Delta x_{i+1} = k_x \Delta x_i, \Delta y_{j+1} = k_y \Delta y_j$ with constants k_x, k_y close to the value of 1.

C.1 Laplacian

As per Sys. (3.1) it is required to approximate implicit viscous terms spatially. In order to obtain the discrete operator acting on a velocity vector we will rewrite Laplacian as a block matrix:

$$\hat{L} = \begin{bmatrix} \hat{L}^u_{xx} + \hat{L}^u_{yy} & 0 \\ 0 & \hat{L}^v_{xx} + \hat{L}^v_{yy} \end{bmatrix}.$$

Each pair of the matrices $\hat{L}^u_{xx}, \hat{L}^v_{xx}$ and $\hat{L}^u_{yy}, \hat{L}^v_{yy}$ will be equal on a uniform, but different on non-uniform grids. These matrices are computed similarly. Below we will discuss how the Laplacian matrix is constructed at different parts of the grid.

C.1.1 Inner part

The uniform grid refinement $\Delta x_{i+1} = k_x \Delta x_i, \Delta y_{j+1} = k_y \Delta y_j$ makes the order of the scheme consistent throughout the whole domain. Without loss of generality, let us show how to compute $\hat{L}^u_{xx}$. The other three matrices can be constructed in a similar manner. The power series of of $u_{i-\frac{1}{2}\pm 1,j}$ at nodes $x_{i-\frac{1}{2}\pm 1}$ with respect to $u_{i-\frac{1}{2},j}$ at node $x_{i-\frac{1}{2}}$ are

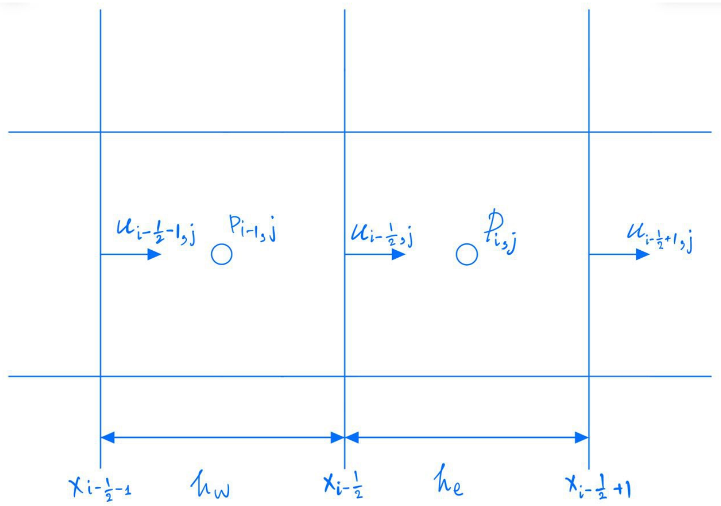

Figure 3: $\hat{L}^u_{xx}$ inner part.

$$u_{i-\frac{1}{2}+1,j} = u_{i-\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right) + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right), \quad \text{(C.1)}$$

$$u_{i-\frac{1}{2}-1,j} = u_{i-\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right) + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} \left(x_{i-\frac{1}{2}-1} - x_{i-\frac{1}{2}}\right)^2 + O\left(\Delta x^3\right). \quad \text{(C.2)}$$

We can combine Eqs. (C.1) and (C.2) by cancelling the first derivative, which will result in

$$\begin{aligned} \left.\frac{\partial^2 u}{\partial x^2}\right|_{i-\frac{1}{2},j} &\approx \frac{u_{i-\frac{1}{2}+1,j}\left(x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}\right) - u_{i-\frac{1}{2},j}\left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}\right) + u_{i-\frac{1}{2}-1,j}\left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right)}{\left(\frac{x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}}{2}\right)\left(x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}\right)\left(x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}\right)} + O(\Delta x) \\ &\approx \frac{1}{h_c h_e} u_{i-\frac{1}{2}+1,j} - \frac{2}{h_w h_e} u_{i-\frac{1}{2},j} + \frac{1}{h_c h_w} u_{i-\frac{1}{2}-1,j} + O(\Delta x), \end{aligned} \quad \text{(C.3)}$$

where $h_w = x_{i-\frac{1}{2}} - x_{i-\frac{1}{2}-1}$, $h_c = \frac{x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}-1}}{2}$, $h_e = x_{i-\frac{1}{2}+1} - x_{i-\frac{1}{2}}$ as in Fig. 3. The coefficients in front of the velocity components are then placed into the $\hat{L}^u_{xx}$ matrix. The order of this approximation becomes 2^{nd} for uniform grids.

C.1.2 Using normal velocity at the boundary for Laplacian

The exact value of the normal velocity to the boundary ($u_{\frac{1}{2},j}$ on Fig. 4) is given as a Dirichlet boundary condition, hence, it is possible to move the corresponding terms together with the coefficients to the right-hand side of the linear system and treat the values explicitly. The values contribute to the vector $\hat{bc}_1$ on the right-hand side of Sys. (3.6).

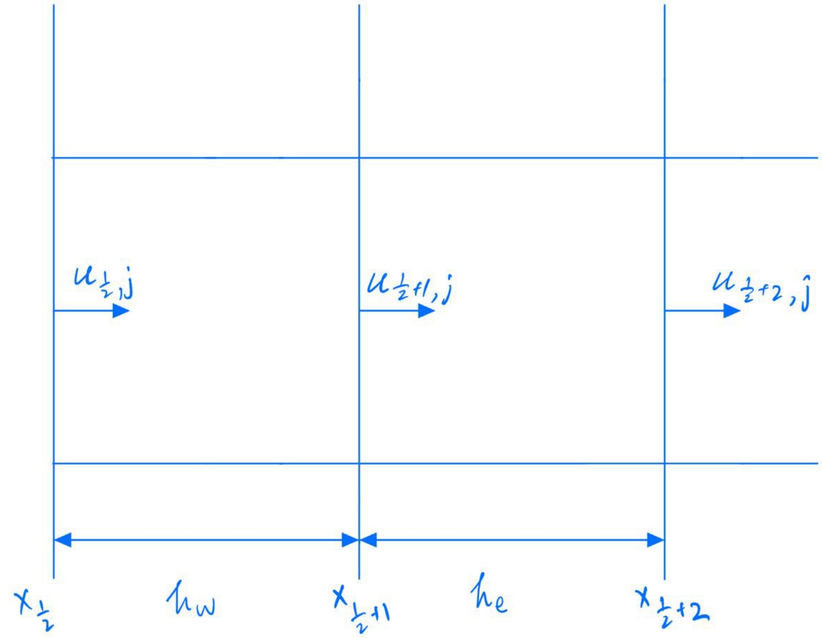

Figure 4: $\hat{L}^u_{xx}$ at the left boundary.

In this particular case on the left side of the domain we have

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{\frac{1}{2}+1,j} \approx \frac{1}{h_c h_e} u_{\frac{1}{2}+2,j} - \frac{2}{h_w h_e} u_{\frac{1}{2}+1,j} + \frac{1}{h_c h_w} u_{\frac{1}{2},j} + O(\Delta x), \tag{C.4}$$

$$\approx \frac{1}{h_c h_e} u_{\frac{1}{2}+2,j} - \frac{2}{h_w h_e} u_{\frac{1}{2}+1,j} + \frac{1}{h_c h_w} \boldsymbol{v}_{\hat{\mathbf{n}}}(t,x,y) + O(\Delta x), \tag{C.5}$$

$$\tag{C.6}$$

then for the corresponding row of a linear system we get Laplacian contribution as

$$\frac{1}{h_c h_e} u_{\frac{1}{2}+2,j} - \frac{2}{h_w h_e} u_{\frac{1}{2}+1,j} = -\frac{1}{h_c h_w} \boldsymbol{v}_{\hat{\mathbf{n}}}(t,x,y), \tag{C.7}$$

where the left side contains unknowns to be solved for, and the right hand side contains explicit boundary condition. Contribution of $\hat{L}^v_{yy}$ is verbatim to the above process - it uses normal velocity component at the horizontal boundaries.

C.1.3 Using tangential velocity at the boundary for Laplacian

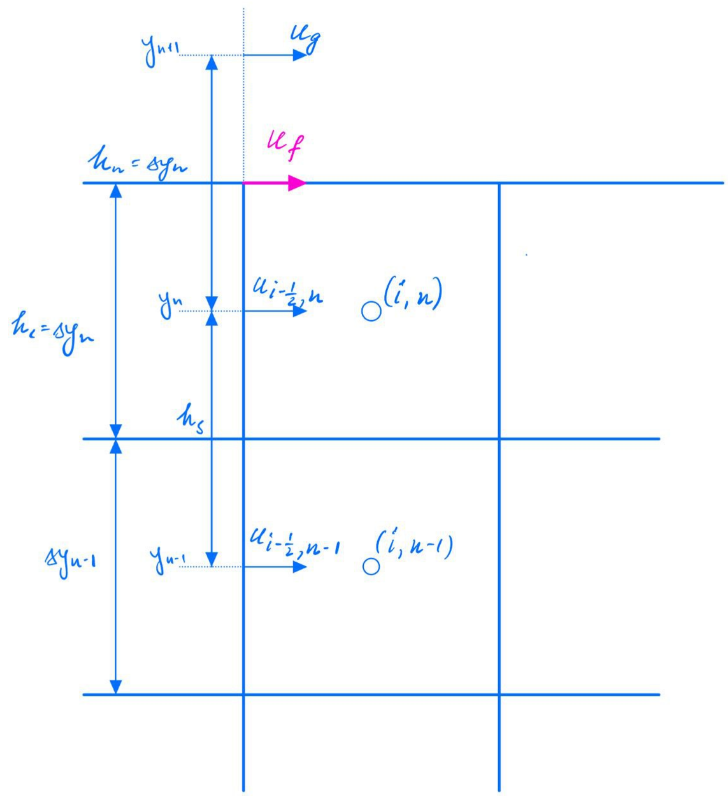

Figure 5: $\hat{L}^u_{yy}$ at top boundary.

We will focus on $\hat{L}^u_{yy}$ in this part; $\hat{L}^v_{xx}$ is computed in a similar manner. Consider the neighbouring cells near the boundary as in Fig. 5. Resultant Eq. (C.3) can be written as

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} = \frac{u_g\,(h_s) + u_{i-\frac{1}{2},N}\,(-2h_c) + u_{i-\frac{1}{2},N-1}\,(h_n)}{h_n h_s h_c}, \tag{C.8}$$

where $h_s = y_n - y_{n-1}, h_n = y_{n+1} - y_n, h_c = \frac{y_{n+1}-y_{n-1}}{2} = \frac{h_s+h_n}{2}$.

It is possible to make the order of the scheme $O(\Delta y^2)$ near the boundary as per Eq. (C.3) if u_g and $u_{i-\frac{1}{2},N-1}$ are set to be equidistant from $u_{i-\frac{1}{2},N}$.The value of u along the top face is known to be freestream velocity u_f, interpolation of freestream velocity $\boldsymbol{v}_{\hat{\mathbf{t}}}(t,x,y) = \boldsymbol{v}_{\hat{\mathbf{t}}} = \frac{u_g+u_{i-\frac{1}{2},N}}{2} \implies u_g = 2\boldsymbol{v}_{\hat{\mathbf{t}}} - u_{i-\frac{1}{2},N}$, which we can plug into Eq. (C.8) from above to obtain

$$\begin{aligned}
\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} &= \frac{2\boldsymbol{v}_{\hat{\mathbf{t}}}h_s}{h_n h_s h_c} + u_{i-\frac{1}{2},N}\left(\frac{-h_s}{h_n h_s h_c} + \frac{-2h_c}{h_n h_s h_c}\right) + u_{i-\frac{1}{2},N-1}\frac{h_n}{u_c h_s h},\\
\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} &= \frac{2\boldsymbol{v}_{\hat{\mathbf{t}}}}{h_c h_c} + u_{i-\frac{1}{2},N}\left(\frac{-(2h_c+h_s)}{h_c h_s h_c}\right) + u_{i-\frac{1}{2},N-1}\frac{1}{h_s h_c},\\
\left.\frac{\partial^2 u}{\partial y^2}\right|_{i-\frac{1}{2},N} &= \frac{2\boldsymbol{v}_{\hat{\mathbf{t}}}}{{h_c}^2} + u_{i-\frac{1}{2},N}\left(\frac{-(2h_c+h_s)}{h_c^2 h_s}\right) + u_{i-\frac{1}{2},N-1}\frac{1}{h_s h_c}.
\end{aligned} \tag{C.9}$$

The first summand from the above equation is treated explicitly, i.e. moved to the vector $\hat{bc}_1$ on the right-hand side in Sys. (3.6), while the other two coefficients are used as elements for the $\hat{L}^u_{yy}$ matrix. The resultant contribution to the corresponding linear equation from Laplace operator becomes

$$u_{i-\frac{1}{2},N}\left(\frac{-(2h_c+h_s)}{h_c^2 h_s}\right) + u_{i-\frac{1}{2},N-1}\frac{1}{h_s h_c} = -\frac{2\boldsymbol{v}_{\hat{\mathbf{t}}}(t,x,y)}{h_c^2}, \tag{C.10}$$

where left hand side contains unknowns with the corresponding coefficients, and the right hand side is explicit boundary value.

C.2 Advection

Since Explicit Adams-Bashforth Eq. (3.3) uses information from time steps n and $n-1$, while we solve the system of equations for time step $n+1$ there is no need for upwinding or even more complicated schemes. For spatial discretization we will try to stick to the central schemes as they are more stable and less dependent on the flow direction. It is convenient to write such schemes for conservative form of the advection, moreover, this also keeps the error low, since there is no product of velocity with acceleration. Hence, let us rewrite advective derivatives in conservative form.

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= u\frac{\partial u}{\partial x} + 0 + v\frac{\partial u}{\partial y} \\
&= u\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial u}{\partial y} \\
&= \left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right) + \left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}. \qquad \text{(C.11)}
\end{aligned}$$

Our goal is to compute advection components at the unknown velocity coordinates. Below we will describe the discretization schemes for Eq. (C.11).

C.2.1 Inner part

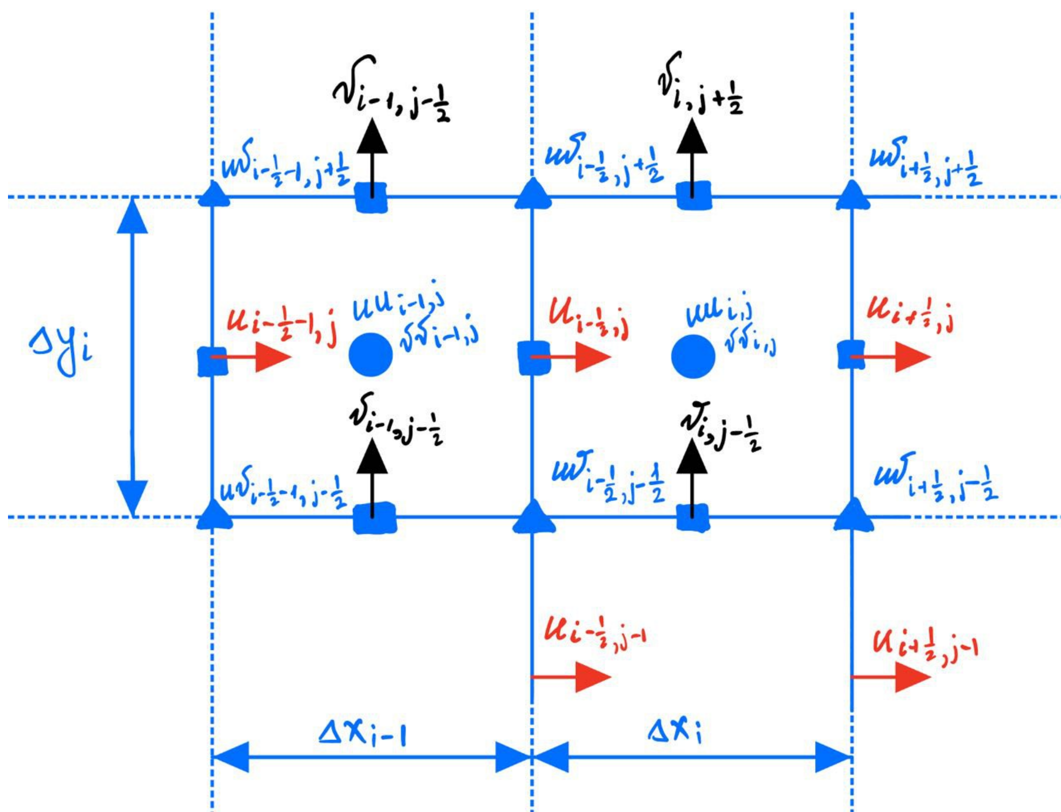

Figure 6: Advection discretization.

Advective component from x-momentum was expressed in conservative form by Eq. (C.11). It is discretized using the standard central differencing schemes [2] as

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)_{i-\frac{1}{2},j}=\frac{(uu)_{i,j}-(uu)_{i-1,j}}{\frac{\Delta x_i+\Delta x_{i-1}}{2}}+\frac{(uv)_{i-\frac{1}{2},j+\frac{1}{2}}-(uv)_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta y_i}, \tag{C.12}$$

which is evaluated at the same points as $u_{i-\frac{1}{2},j}$ (squares in Fig. 6). We need to compute uv at the nodes and uu at cell centres, which are triangles and circles respectively in Fig. 6. Using linear interpolation [2] of velocity values leads to

$$\begin{aligned}
(uu)_{i,j} &= \left(\frac{u_{i+\frac{1}{2},j}+u_{i-\frac{1}{2},j}}{2}\right)^2, \\
(uv)_{i-\frac{1}{2},j-\frac{1}{2}} &= \left(u_{i-\frac{1}{2},j-1}+\frac{\Delta y_{j-1}}{2}\frac{u_{i-\frac{1}{2},j}-u_{i-\frac{1}{2},j-1}}{\frac{\Delta y_{j-1}+\Delta y_j}{2}}\right)\left(v_{i-1,j-\frac{1}{2}}+\frac{\Delta x_{i-1}}{2}\frac{v_{i,j-\frac{1}{2}}-v_{i-1,j-\frac{1}{2}}}{\frac{\Delta x_{i-1}+\Delta x_i}{2}}\right) \\
&= \left(\frac{u_{i-\frac{1}{2},j}\Delta y_{j-1}+u_{i-\frac{1}{2},j-1}\Delta y_j}{\Delta y_{j-1}+\Delta y_j}\right)\left(\frac{v_{i,j-1\frac{1}{2}}\Delta x_{i-1}+v_{i-1,j-\frac{1}{2}}\Delta x_i}{\Delta x_{i-1}+\Delta x_i}\right).
\end{aligned}$$

C.2.2 Using boundary conditions for advection discretization

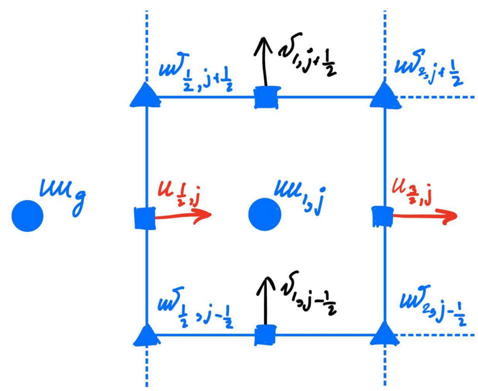

Figure 7: Left boundary advection. Disregard ghost velocity uu_g. Elements $vv_{1,j}$ are located at the same coordinates as $uu_{1,j}$.

Exact values of the $uv_{\frac{1}{2},j+\frac{1}{2}}, uv_{\frac{1}{2},j-\frac{1}{2}}$ located at the boundary (Fig. 7) are known from BCs (2.1c) and (2.1d). The element $vv_{1,j}$ can be expressed directly from computed values $v_{1,j+\frac{1}{2}}, v_{1,j-\frac{1}{2}}$ of previous time steps, whereas $uu_{1,j}$ can be obtained form normal boundary condition $u_{\frac{1}{2},j}$ and previously computed $u_{\frac{3}{2},j}$. The process is verbatim for advection components near the top, right and bottom boundaries.

C.3 Divergence

One of the advantages of the staggered grid is that the discrete divergence D and gradient G operators are equal to the negative transpose of each other. In this section we will show one side of why this identity is true, the other part could be found in Appendix C.4. The second line of Sys. (3.6) reads

$$\begin{gathered}
\hat{D}\boldsymbol{v}=\hat{bc}_2, \\
\begin{bmatrix} \hat{D}_x & \hat{D}_y \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\hat{bc}_2, \\
\frac{1}{\Delta x}D_x u+\frac{1}{\Delta y}D_y v=\hat{bc}_2, \\
\frac{1}{\Delta_{xy}}\begin{bmatrix} D_x & D_y \end{bmatrix}\begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix}=\frac{1}{\Delta_{xy}}Dq=\hat{bc}_2,
\end{gathered}$$

where vector $q = \begin{bmatrix} u\Delta y \\ v\Delta x \end{bmatrix}$ is referred to as velocity flux, and the matrix $D = [D_x D_y]$ without hat is the divergence matrix of integer coefficients. Matrix

$$\Delta_{xy} = \begin{bmatrix} \Delta x_1 \Delta y_1 & 0 & \dots & 0 \\ 0 & \Delta x_2 \Delta y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta x_M \Delta y_N \end{bmatrix} \tag{C.13}$$

is $MN \times MN$ diagonal matrix with entries corresponding to the inverse volumes of cells.

The second-order central difference scheme was employed in the construction of D, the order is consistent with the other spatial schemes. Consider a finite volume $V_{i,j}$, continuity equation for the cell centre is then expressed as

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_{i,j} = 0,$$

which can be discretized at cell centre i, j using central difference schemes into

$$\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x_i} + \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta y_j} = 0.$$

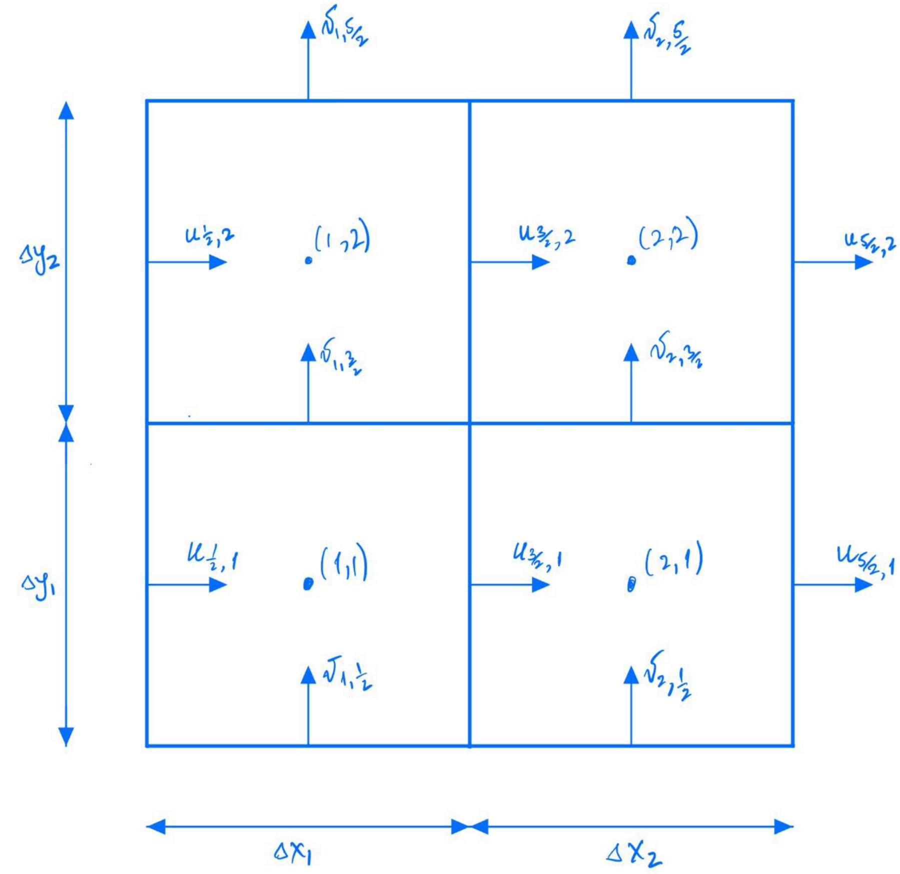

Figure 8: 2×2 grid example for divergence operator.

To provide a good visual example, consider 2×2 grid as in Fig. 8 with the same boundary conditions as in Eqs. (2.1). The partial derivatives of divergence operator evaluated at cell centres are:

$$
\begin{aligned}
V_{1,1} &: \frac{u_{\frac{3}{2},1} - u_{\frac{1}{2},1}}{\Delta x_1} + \frac{v_{1,\frac{3}{2}} - v_{1,\frac{1}{2}}}{\Delta y_1} = 0, \\
V_{1,2} &: \frac{u_{\frac{5}{2},1} - u_{\frac{3}{2},1}}{\Delta x_2} + \frac{v_{2,\frac{3}{2}} - v_{2,\frac{1}{2}}}{\Delta y_1} = 0, \\
V_{2,1} &: \frac{u_{\frac{3}{2},2} - u_{\frac{1}{2},2}}{\Delta x_1} + \frac{v_{1,\frac{5}{2}} - v_{1,\frac{3}{2}}}{\Delta y_2} = 0, \\
V_{2,2} &: \frac{u_{\frac{5}{2},2} - u_{\frac{3}{2},2}}{\Delta x_2} + \frac{v_{2,\frac{5}{2}} - v_{2,\frac{3}{2}}}{\Delta y_2} = 0.
\end{aligned}
$$

After applying normal boundary conditions to the above equations, we obtain the system

$$
\begin{bmatrix} \frac{1}{\Delta x_1 \Delta y_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta x_2 \Delta y_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\Delta x_1 \Delta y_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\Delta x_2 \Delta y_2} \end{bmatrix}
\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}
\begin{bmatrix} u_{\frac{3}{2},1}\Delta y_1 \\ u_{\frac{3}{2},2}\Delta y_2 \\ v_{1,\frac{3}{2}}\Delta x_1 \\ v_{2,\frac{3}{2}}\Delta x_2 \end{bmatrix}
=
\begin{bmatrix} \frac{u_{\frac{1}{2},1}}{\Delta x_1} + \frac{v_{1,\frac{1}{2}}}{\Delta y_1} \\ -\frac{u_{\frac{5}{2},1}}{\Delta x_2} + \frac{v_{2,\frac{1}{2}}}{\Delta y_1} \\ \frac{u_{\frac{1}{2},2}}{\Delta x_1} - \frac{v_{1,\frac{5}{2}}}{\Delta y_2} \\ -\frac{u_{\frac{5}{2},2}}{\Delta x_2} - \frac{v_{2,\frac{5}{2}}}{\Delta y_2} \end{bmatrix}, \tag{C.14}
$$

which in general case is written as $\frac{1}{\Delta_{xy}} Dq = \hat{bc}_2$. The right hand side vector $\hat{bc}_2 = 0$ due to mass conservation **(NEED TO CLARIFY)**. The divergence matrix D contains ± 1, as shown in Sys. (C.14).

C.4 Gradient

The pressure gradients are computed at the same coordinates as the unknown velocities according to Sys. (3.6). Below we will show that the discrete gradient and divergence operators satisfy

$$
G = -D^T \tag{C.15}
$$

on staggered/MAC grids by construction.

As an illustrative example, we may consider a 2×2 grid as in Fig. 9 with the same boundary conditions as in Eqs. (2.1). It is required to determine the pressure gradients across each unknown velocity (squares located at the centre of each face on Fig. 9):

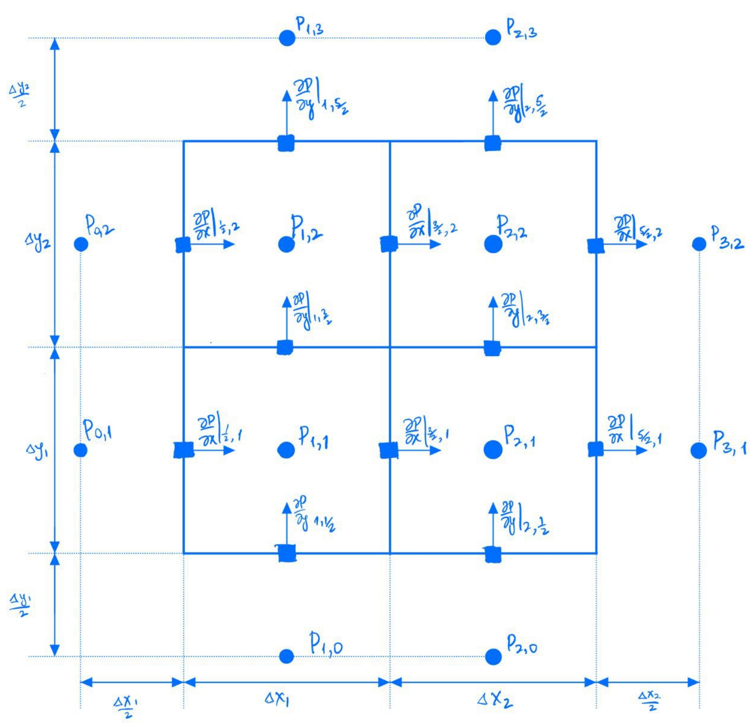

Figure 9: 2×2 grid example for gradient operator.

$$\left.\frac{\partial p}{\partial x}\right|_{\frac{3}{2},1} = \frac{2}{\Delta x_1 + \Delta x_2}\left(p_{2,1} - p_{1,1}\right),$$
$$\left.\frac{\partial p}{\partial x}\right|_{\frac{3}{2},2} = \frac{2}{\Delta x_1 + \Delta x_2}\left(p_{2,2} - p_{1,2}\right),$$
$$\left.\frac{\partial p}{\partial y}\right|_{1,\frac{3}{2}} = \frac{2}{\Delta y_1 + \Delta y_2}\left(p_{1,2} - p_{1,1}\right),$$
$$\left.\frac{\partial p}{\partial y}\right|_{2,\frac{3}{2}} = \frac{2}{\Delta y_1 + \Delta y_2}\left(p_{2,2} - p_{2,1}\right).$$

The next step is to rewrite the above expressions in matrix form using the pressure boundary conditions outside the grid. The resultant system of linear equations then becomes

$$\begin{bmatrix} \frac{2}{\Delta x_1+\Delta x_2} & 0 & 0 & 0 \\ 0 & \frac{2}{\Delta x_1+\Delta x_2} & 0 & 0 \\ 0 & 0 & \frac{2}{\Delta y_1+\Delta y_2} & 0 \\ 0 & 0 & 0 & \frac{2}{\Delta y_1+\Delta y_2} \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{1,1} \\ p_{2,1} \\ p_{1,2} \\ p_{2,2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), \tag{C.16}$$

which can be written in compact format as

$$\nabla p + bc_p \Longleftrightarrow \Delta_{xy} \left(\hat{G} \begin{bmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{mn+(n-1)m} \end{bmatrix} + bc_p \right)$$

where $\hat{M}^{-1}$ is the diagonal matrix containing the distances between neighbouring pressure coordinates, G is the gradient matrix and $[p_1, p_2, ..., p_{(m-1)n+(n-1)m}]^T$ is the vector of pressure values at the cell centres. There is zero contribution from pressure BC values in bc_p since there is no pressure gradient that needs to be computed across the boundary. It can be observed from Sys. (C.14) and (C.16) that Eq. (C.15) holds and the general case is true by construction.

D Normalization

The most effective methods for solving systems of linear equations often work optimally with symmetric matrices, making symmetry a desirable property. In order to achieve a symmetric system, we introduce two matrices - the diagonal scaling matrix $\hat{M}$ and the diagonal flux matrix R. These matrices are defined as

$$R \equiv \begin{bmatrix} \Delta y_j & 0 \\ 0 & \Delta x_i \end{bmatrix},$$
$$\hat{M} \equiv \begin{bmatrix} \frac{1}{2}\left(\Delta x_i + \Delta x_{i-1}\right) & 0 \\ 0 & \frac{1}{2}\left(\Delta y_j + \Delta y_{j-1}\right) \end{bmatrix}.$$

Define matrices

$$\Delta y_j = \mathrm{diag}([\underbrace{\Delta y_1; \Delta y_1; \ldots; \Delta y_1}_{M-1 \text{ times}}; \underbrace{\Delta y_2; \Delta y_2; \ldots; \Delta y_2}_{M-1 \text{ times}}; \ldots; \underbrace{\Delta y_N; \Delta y_N; \ldots; \Delta y_N}_{M-1 \text{ times}}]),$$
$$\Delta x_i = \mathrm{diag}([\underbrace{\Delta x_1; \Delta x_2; \ldots \Delta x_M; \Delta x_1; \Delta x_2; \ldots \Delta x_M; \ldots; \Delta x_1; \Delta x_2; \ldots; \Delta x_M}_{N-1 \text{ blocks of } 1,\ldots,M}]),$$

which are then combined into

$$R = \begin{bmatrix} \Delta y_j & 0 \\ 0 & \Delta x_i \end{bmatrix}.$$

By moving to new variable $q = R\boldsymbol{v}$, we use matrix R, which inverse R^{-1} performs half of the symmetrization process for $\hat{L}$. Here, R is a diagonal matrix of size $(M-1)N \times M(N-1)$. Using the R matrix we can express mass flux $q^{n+1} = R\boldsymbol{v}^{n+1} \implies \boldsymbol{v}^{n+1} = R^{-1}q^{n+1}$. It is required that the difference matrices cancel out $\frac{x_{i+1}-x_{i-1}}{2}$ central term in diffusion discretization to make Laplacian partially symmetric. Define the elements of diagonal mass matrix as

$$\Delta x_i + \Delta x_{i-1} = \mathrm{diag}[\underbrace{\Delta x_2 + \Delta x_1; \Delta x_3 + \Delta x_2; \ldots; \Delta x_M + \Delta x_{M-1}; \ldots; \Delta x_2 + \Delta x_1; \Delta x_3 + \Delta x_2; \ldots \Delta x_M + \Delta x_M}_{N \text{ times for each block of } \Delta x_2 + \Delta x_1 \text{ to } \Delta x_M + \Delta x_{M-1}}]$$

and

$$\Delta y_j + \Delta y_{j-1} = \mathrm{diag}[\underbrace{\Delta y_2 + \Delta y_1; \ldots \Delta y_2 + \Delta y_1}_{M \text{ times}}; \underbrace{\Delta y_3 + \Delta y_2; \ldots \Delta y_3 + \Delta y_2}_{M \text{ times}}; \underbrace{\Delta y_N + \Delta y_{N-1} \ldots \Delta y_N + \Delta y_{N-1}}_{M \text{ times}}],$$

which are then combined into

$$\hat{M} = \begin{bmatrix} \frac{1}{2}\left(\Delta x_i + \Delta x_{i-1}\right) & 0 \\ 0 & \frac{1}{2}\left(\Delta y_j + \Delta y_{j-1}\right) \end{bmatrix},$$

which both removes the M^{-1} in $\hat{G} = \hat{M}^{-1}G$ if left multiplied and completes the symmetrization process of $\hat{L}$.

The identity $q = R\boldsymbol{v}$ implies $\boldsymbol{v} = R^{-1}q$, therefore, Laplacian matrix multiplied by the velocity vector is

$$\hat{L}\boldsymbol{v} = \hat{L}R^{-1}q,$$

premultiplying by $\hat{M}$ we obtain

$$\hat{M}\hat{L}R^{-1}q,$$

where $\hat{M}LR^{-1}$ is symmetric by construction, hence, $\hat{M}\hat{A}R^{-1} = \hat{M}(\frac{1}{\Delta t}\mathbf{I} - \frac{1}{2}L)R^{-1}$ is also symmetric.

Using the above transformations we can modify Sys. (3.6) into

$$\begin{bmatrix} \hat{M}\hat{A}R^{-1} & \hat{M}\hat{G} \\ \hat{D}R^{-1} & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \hat{M}\hat{r}^n \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{M}\widehat{bc}_1 \\ 0 \end{pmatrix},$$

which can be rewritten as

$$\boxed{\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} q^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \end{pmatrix} + \begin{pmatrix} bc_1 \\ 0 \end{pmatrix},} \tag{D.1}$$

where $A = \hat{M}\hat{A}R^{-1} = \frac{1}{\Delta t}\hat{M}R^{-1} - \frac{1}{2}\hat{M}\hat{L}R^{-1}$ is symmetric and $\hat{G} = \hat{M}^{-1}G$ according to Sys. (C.16). It is also possible to transform the divergence operator $\hat{D}$ with non-integer coefficients into D with integer coefficients in continuity part of Sys. (D.1) using Sys. (C.14). Multiplying both sides of continuity equation by Δ_{xy} matrix from Eq. (C.13) leads to

$$\begin{aligned}
\hat{D}\boldsymbol{v}^{n+1} &= 0 \\
(\Delta_{xy})\,\hat{D}\boldsymbol{v}^{n+1} &= (\Delta_{xy})\,0 \\
(\Delta_{xy})\,\frac{1}{\Delta_{xy}}DR\boldsymbol{v}^{n+1} &= (\Delta_{xy})\,\hat{0} \\
(\Delta_{xy})\,\frac{1}{\Delta_{xy}}Dq^{n+1} &= (\Delta_{xy})\,0 \\
Dq^{n+1} &= 0.
\end{aligned}$$

D.1 Transient schemes