Homework 12 - Solution

(p. 462: 9.37)

Let $X_1, X_2, \ldots, X_n$ denote n iid *Bernoulli* random variables such that

$$P(X_i = 1) = p \text{ and } P(X_i = 0) = 1 - p,$$

for each $i = 1, 2, \ldots, n$. Show that $\Sigma_{i=1}^n X_i$ is sufficient for p by using the factorization criterion given in Theorem 9.4.

<u>Solution</u>:

The likelihood function is $L(p) = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$. By theorem 9.4, $\Sigma_{i=1}^n X_i$ is sufficient for p with $\boxed{g(\Sigma_{i=1}^n X_i, p) = p^{\Sigma X_i}(1-p)^{n-\Sigma X_i} \text{ and } h(\mathbf{y}) = 1.}$

(p. 462: 9.38)

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a normal distribution with mean μ and variance σ^2.

(a) If μ is unknown and σ^2 is known, show that $\overline{Y}$ is sufficient for μ.

(b) If is μ known and σ^2 is unknown, show that $\Sigma_{i=1}^n (Y_i - \mu)^2$ is sufficient for σ^2.

(c) If μ and σ^2 are both unknown, show that $\Sigma_{i=1}^n Y_i$ and $\Sigma_{i=1}^n Y_i^2$ are jointly sufficient for μ and σ^2. [Thus, it follows that $\overline{Y}$ and S^2 are also jointly sufficient for μ and σ^2.]

<u>Solution</u>:

For this exercise, the likelihood function is given by

$$L = \frac{1}{(2\pi)^{n/2}\sigma^n}\exp[-\tfrac{\Sigma_{i=1}^n (y_i-\mu)^2}{2\sigma^2}] = (2\pi)^{-n/2}\sigma^{-n}\exp\left[-\tfrac{1}{2\sigma^2}\left(\Sigma_{i=1}^n y_i^{\,2} - 2\mu n\bar{y} + n\mu^2\right)\right]$$

(a) When σ^2 is known, $\overline{Y}$ is sufficient for μ by Theorem 9,4 with

$$\boxed{g(\bar{y},\mu) = \exp\left(\tfrac{2\mu n\bar{y} - n\mu^2}{2\sigma^2}\right) \text{ and } h(\mathbf{y}) = (2\pi)^{-n/2}\sigma^{-n}\exp[-\tfrac{1}{2\sigma^2}\Sigma_{i=1}^n y_i^{\,2}]}$$

(b) When μ is known, use Theorem 9.4 with

$$\boxed{g(\Sigma_{i=1}^n (y_i-\mu)^2, \sigma^2) = (\sigma^2)^{-n/2}\exp[-\tfrac{\Sigma_{i=1}^n (y_i-\mu)^2}{2\sigma^2}] \text{ and } h(\mathbf{y}) = (2\pi)^{-n/2}.}$$

(c) When μ and σ^2 are both unknown, the likelihood can be written in terms of the two statistics $U_1 = \Sigma_{i=1}^n Y_i$ and $U_2 = \Sigma_{i=1}^n Y_i^2$. Let

$$\boxed{g(U_1, U_2, \mu, \sigma^2) = (2\pi\sigma^2)^{-n/2}\exp(-\tfrac{U_2}{2\sigma^2})\exp[-\tfrac{n}{2\sigma^2}(\tfrac{U_1}{n} - \mu)^2] \text{ and } h(\mathbf{y}) = 1}$$

$\Sigma_{i=1}^n Y_i$ and $\Sigma_{i=1}^n Y_i^2$ are jointly sufficient for μ and σ^2.

$\overline{Y}$ and S^2 are also jointly sufficient for μ and σ^2 since they can be written in terms of U_1 and U_2.

(p. 481: 9.80)

Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the Poisson distribution with mean λ.

(a) Find the MLE $\hat{\lambda}$ for λ.

(b) Find the expected value and variance of $\hat{\lambda}$.

(c) Show that the estimator of part (a) is consistent for λ.

(d) What is the MLE for $P(Y = 0) = e^{-\lambda}$?

<u>Solution</u>:

(a) For the Poisson distribution, the likelihood function would be

$$L = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}$$
$$= \frac{e^{-n\lambda}\lambda^{\Sigma_{i=1}^{n} y_i}}{\Pi_{i=1}^{n} y_i!}$$

and the log-likelihood function would be

$$\ln L = \ln\left(\frac{e^{-n\lambda}\lambda^{\Sigma_{i=1}^{n} y_i}}{\Pi_{i=1}^{n} y_i!}\right)$$
$$= -n\lambda + \sum_{i=1}^{n} y_i \ln\lambda - \sum_{i=1}^{n} \ln(y_i!)$$

Now take the derivative of it with respect to λ and set it to 0,

$$\frac{\partial \ln L}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda} \overset{\Delta}{=} 0$$

Solving it for λ, we get

$$\hat{\lambda} = \bar{Y}.$$

Since the second derivative of the log-likelihood function evaluated at $\hat{\lambda} = \bar{Y}$ is less than 0, 1 $\hat{\lambda} = \bar{Y}$ is the MLE for λ.

(b) By sampling distribution $E(\hat{\lambda}) = \lambda$ and $V(\hat{\lambda}) = \lambda / n$.

(c) Since $\hat{\lambda}$ is unbiased and has variance that goes to 0 with increasing n, it is consistent.

(d) By the invariance property, the MLE for $P(Y = 0) = e^{-\lambda}$ is $\boxed{e^{\bar{Y}}}$

(p. 481: 9.81)

Suppose that Y_1, Y_2, ..., Y_n denote a random sample from an exponential distributed population with mean θ. Find the MLE of the population variance θ^2.

<u>Solution</u>:

For the exponential distribution, the likelihood function would be

$$L = \prod_{i=1}^{n} \frac{1}{\theta} e^{-y_i/\theta}$$
$$= \frac{e^{-\Sigma_{i=1}^{n} y_i/\theta}}{\theta^n}$$

Then the log-likelihood function is

$$\ln L = \ln\left(\frac{e^{-\Sigma_{i=1}^{n} y_i/\theta}}{\theta^n}\right)$$
$$= -\Sigma_{i=1}^{n} y_i / \theta - n\ln\theta$$

Now take the derivative of it with respect to θ and set it to 0,

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} y_i}{\theta^2} \stackrel{\Delta}{=} 0$$

Solving it for θ, we get

$$\hat{\theta} = \overline{Y}\ .$$

Since the second derivative of the log-likelihood function evaluated at $\hat{\theta} = \overline{Y}$ is less than 0, the MLE is $\hat{\theta} = \overline{Y}$. By the invariance property of MLEs, the MLE of θ^2 is $\boxed{\overline{Y}^2}$.