# Explicit Constructions of Finite Groups as Monodromy Groups


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## Abstract


In 1963, Greenberg proved that every finite group appears as the monodromy group of some morphism of Riemann surfaces. In this paper, we give two constructive proofs of Greenberg's result. First, we utilize free groups, which given with the universal property and their construction as discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, yield a very natural realization of finite groups as monodromy groups. We also give a proof of Greenberg's result based on triangle groups $\Delta(m,n,k)$. Given any finite group $G$, we make use of subgroups of $\Delta(m,n,k)$ in order to explicitly find a morphism $\pi$ such that $G \simeq \mathrm{Mon}(\pi)$.


## Fuchsian Groups

Given a matrix $h = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$, it acts on an element $\tau$ of the extended upper-half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ as follows:

$$h \circ \tau = \frac{a\tau + b}{c\tau + d}.$$

We call a subgroup $\Gamma$ of $\mathrm{PSL}_2(\mathbb{R})$ a **Fuchsian group**, if

$$\{h \in \Gamma \mid h \circ \tau = \tau\}$$

is finite for every $\tau \in \mathbb{H}^*$. Some examples of Fuchsian groups are: $\mathrm{PSL}_2(\mathbb{Z})$, $\Delta(m,n,k)$ and $\Gamma$, where $\Gamma$ is any subgroup of a Fuchsian group. The quotient space $\Gamma\backslash\mathbb{H}^*$ is defined by the orbits $\Gamma\tau = \{h \circ \tau \mid h \in \Gamma\}$, where $\tau \in \mathbb{H}^*$.

## Riemann Surfaces

The uniformization theorem tells us that all compact connected Riemann surfaces are classified as one of the following:

- Sphere

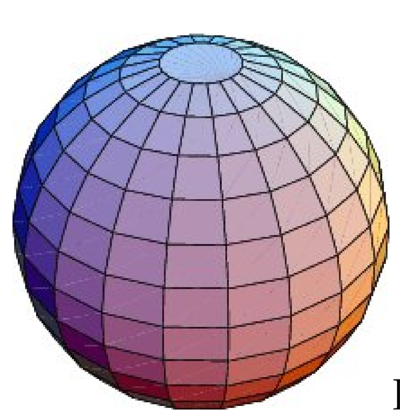

Figure 1

- Torus

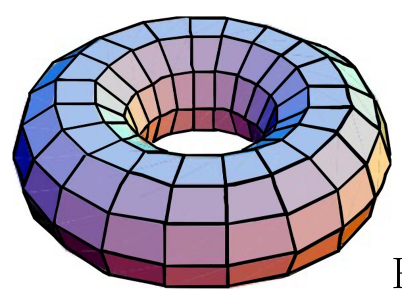

Figure 2

- Riemann surfaces not classified as either toroidal with one hole or spherical take on the form $\Gamma\backslash\mathbb{H}^*$ where $\Gamma$ is a Fuchsian group.

## Greenberg's Theorem

**Theorem (Greenberg 1963):**
If $G$ is a finite group, then $G \simeq \mathrm{Mon}(\pi)$ for some morphism $\pi$ between Riemann surfaces.

## Monodromy

Let $\Gamma'$ and $\Gamma$ be Fuchsian groups, where $\Gamma'$ is a normal subgroup of $\Gamma$. Then the canonical map

$$\pi : \Gamma'\backslash\mathbb{H}^* \to \Gamma\backslash\mathbb{H}^*$$

which sends $\Gamma'\tau$ to $\Gamma\tau$ has **monodromy group**

$$\mathrm{Mon}(\pi) \simeq \Gamma/\Gamma'.$$

## Triangle Groups

Let $m,n,k \in \mathbb{N}$. Then a group with presentation

$$\Delta(m,n,k) = \langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^m = \gamma_1^n = \gamma_\infty^k = \gamma_0\gamma_1\gamma_\infty = 1\rangle$$

is called a **triangle group**. These groups can be classified in 1 of 3 ways:

- The *Euclidean* case, where $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} = 1$

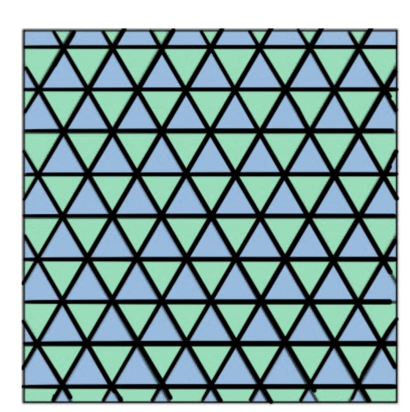

Figure 3: $\Delta(3,3,3)$

- The *spherical* case, where $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} > 1$

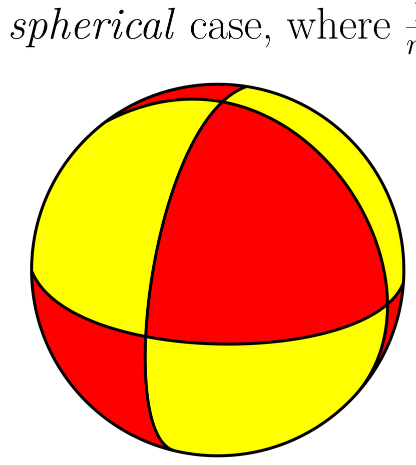

Figure 4: $\Delta(2,2,2)$

- The *hyperbolic* case, where $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} < 1$

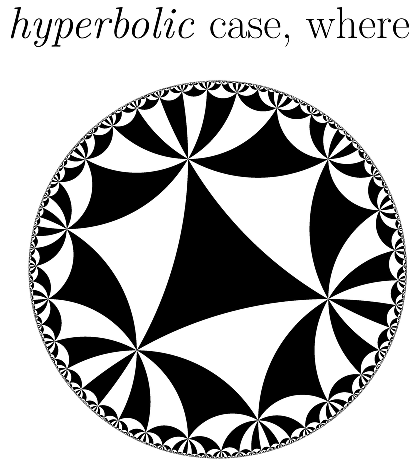

Figure 5: $\Delta(6,6,6)$

## Triangle Group Construction

**Theorem (M. S. T., 2020)**
*Let $G$ be a subgroup of $S_n$* and let

$$\begin{aligned} \varphi\colon \Delta(2,n,n-1) &\to S_n \\ \gamma_0 &\mapsto (1\ 2) \\ \gamma_1 &\mapsto (1\ 2\ \cdots\ n) \\ \gamma_\infty &\mapsto ((1\ 2)(1\ 2\ \cdots\ n))^{-1}. \end{aligned}$$

*Then,*

- *$\varphi$ is a surjective group homomorphism*
- *$\varphi^{-1}(G)$ and $\ker(\varphi)$ are Fuchsian groups*
- *$G \simeq \mathrm{Mon}(\pi)$, where* $\pi\colon \ker(\varphi)\backslash\mathbb{H}^* \to \varphi^{-1}(G)\backslash\mathbb{H}^*$

**Diagram:**

$$\begin{tikzcd} \Delta(2,n,n-1) \arrow[r, "\varphi"] & S_n \\ \varphi^{-1}(G) \arrow[r, "\varphi"] \arrow[u, phantom, sloped, "\subseteq"] & G \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

## Free Groups

A group $G$ is called a **free group** if there exists a generating set $S$ of $G$ such that every non-empty reduced group word on $S$ defines a non-trivial element of $G$. Some examples of free groups are:

- When $S = \{A\}$ we have G as $F_1$ or the free group of rank 1. The integers are $F_1$.
- When $S = \{A, B\}$ we have G as $F_2$. In $F_2$, the words $ABA$, $A^2$ are reduced and $A^{-1}BB^{-1}AA$ is non-reduced.

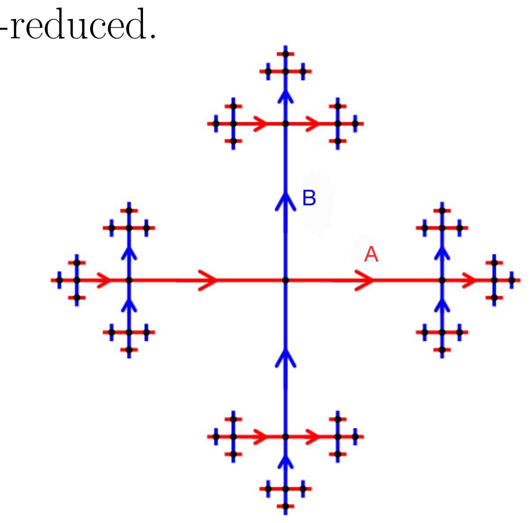


Figure 6: $F_2$

- Let $A = \pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$. When $S = \{X_0, X_1, ..., X_{r-1}\}$, where

$$\begin{aligned} X_0 &= A \\ X_1 &= B^{-1}AB \\ &\vdots \\ X_{r-1} &= B^{-(r-1)}AB^{r-1} \end{aligned}$$

we have $G$ as $F_r$. Moreover, $F_r$ is a subgroup of $F_2$.

## Free Group Construction

**Universal Property of Free Groups**:
Let $G = \langle S\rangle$ be finite and let $\iota\colon S \to G$ be a set map.

**Diagram:**

$$\begin{tikzcd} & F_{|S|} \arrow[d, dashed, "\varphi"] \\ S \arrow[ur] \arrow[r, "\iota"] & G \end{tikzcd}$$

**Surjectivity:**
Since $\varphi$ is an extension of $\iota$ and $F_{|S|}$ is an infinite group, $\varphi$ is a unique surjective group homomorphism.

**Theorem (M. S. T., 2020)**
*Let $G = \langle a_0, a_1, \ldots, a_{r-1}\rangle$ be finite and let*

$$\begin{aligned} \varphi\colon F_r &\to G \\ A &\mapsto a_0 \\ &\vdots \\ B^{-(r-1)}AB^{r-1} &\mapsto a_{r-1} \end{aligned}$$

Then,

- $\ker(\varphi)$ is a free group of rank $1 + |G|(r-1)$
- *$F_r$ and $\ker(\varphi)$ are Fuchsian groups*
- *$G \simeq \mathrm{Mon}(\pi)$, where $\pi\colon \ker(\varphi)\backslash\mathbb{H}^* \to F_r\backslash\mathbb{H}^*$*

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