Understanding Symmetric Representations of Groups of Automorphisms on Bordered Klein surfaces

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1 Introduction

The title of the paper I researched is *On Symmetric Representations of Groups of Automorphisms on Bordered Klein surfaces* and it was written by Czeslaw Bagiński and Grzegorz Gromadzki. Initially this paper interested me because I was always curious about permutation representations especially when they described certain geometric relationships and changes of objects under group action. For me, the topic roots algebra in a much more geometric position which I enjoy.

To understand this paper, I decided to break the title down in this research paper to first examine the nature of Klein surfaces in the context of the paper (with some other outside examples to consider). Second I decided to focus on the automorphisms that can be found on these surfaces. Lastly, I wanted to tie it all together by examining the nature of symmetric representations, as they are called in Bagiński and Gromadzki's paper, representations of automorphism groups on Bordered Klein surfaces.

2 What are Bordered compact Klein Surfaces?

In order to understand the concept of a Bordered Compact Klein surface we must review the idea of the hyperbolic plane, the isometry group of the hyperbolic plane, and then NEC groups. All of these things come together to create the type of Klein surfaces explored in Bagiński and Gromadzki's paper.

2.1 The Hyperbolic Plane

We can conceptualize hyperbolic planes by first considering the Euclidean plane which we are most familiar with. The Euclidean plane is subject to Euclid's 5 postulates. [7]

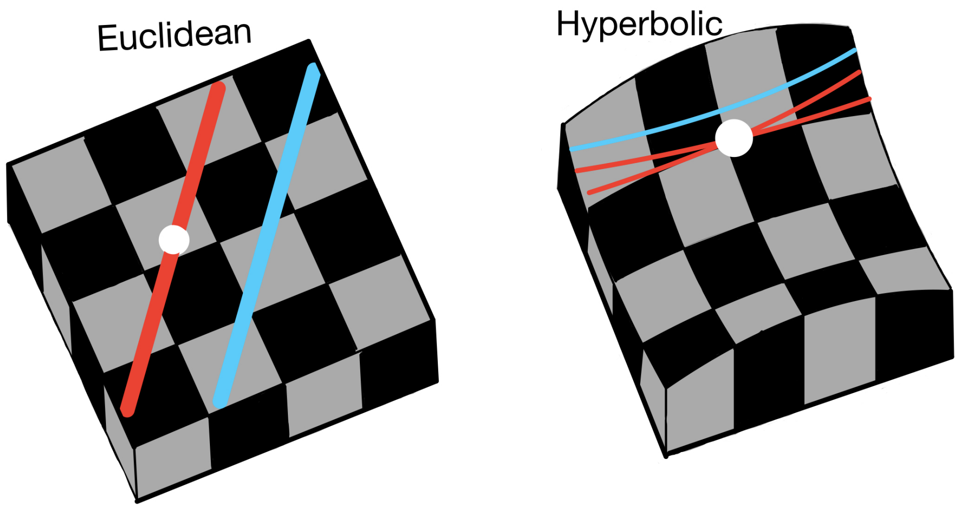
Euclid's 5 Postulates

1. A line Segment can be dawn between any 2 line segments
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the line segment as radius and one of the 2 endpoints as center.
4. All right angles are congruent
5. If 2 lines are drawn which intersect a third in such a way that the sum of the two inner angles is less than the sum of right angles, the two lines will eventually intersect on that particular side.

The last postulate is also referred to as the parallel postulate and its absence is what characterizes non-euclidean geometries such as the Hyperbolic plane and the sphere. Specifically for the hyperbolic plane the fifth postulate is replaced with the following statement:

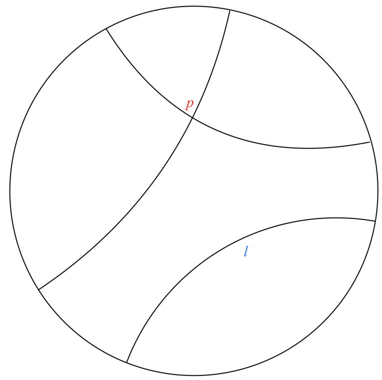
> Given any straight line l and a point p not on l, there are at least 2 lines that intersect p and are parallel l.[8] [4]

This differentiates hyperbolic planes from Euclidean planes since we are familiar with the fact that in the Euclidean plane there is only one distinct line that is parallel to l that passes through p, but as we see in the hyperbolic plane there is more than one such line. This is depicted in the image below.

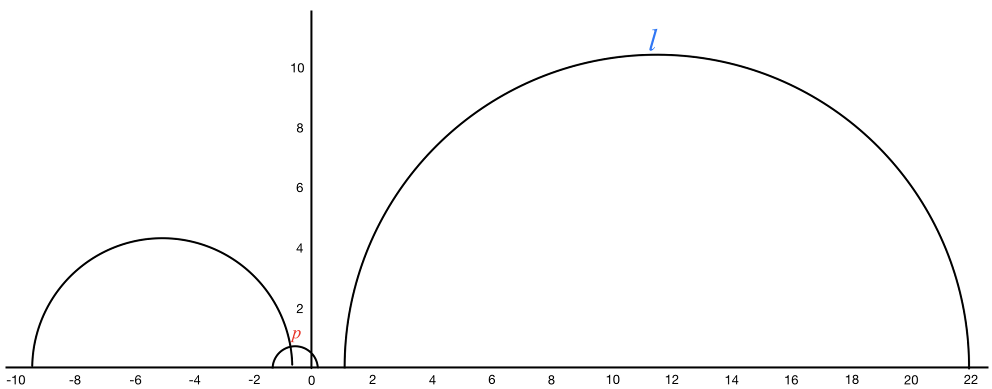


We theoretically conceptualize the hyperbolic plane by its divergence from the Euclidean plane in how it treats parallel lines, but we can also conceptualize the hyperbolic plane visually using models that meet the criteria of violating the fifth postulate. There are various models for the hyperbolic plane; we will examine 2 such models below. [10]

1. First we will consider the Poincare disk model where the entirety of the hyperbolic plane is laid out inside of an open unit disk and straight lines in the plane are expressed as circular arcs that are perpendicular at the endpoints to the unit disk.[10] It can be observed in this model that the modified hyperbolic parallel postulate is present.



2. second we will consider the Poincare upper-half plane. This model consists of the entire upper half of the Cartesian coordinate plane or complex plane, and straight lines in this model are depicted as either semi-circles whose bases are on the x axis or vertical lines moving upward from the x axis. [10] It can be observed in this model as well that the modified hyperbolic parallel postulate is present.



These were two ways of visually conceptualizing the hyperbolic plane and we will consider take them into consideration when defining the isometry group of a hyperbolic plane.

2.2 The Isometry Group

Definition 2.1 (Isometry)**.** *An isometry is a bijective function between two metric spaces that preserves distance between elements in each metric space. Specifically, if X and Y are metric spaces with x_1 , $x_2 \in X$ an isometry $f : X \to Y$ maintains that* $\text{Distance}_{\text{Y}}(f(x_1), f(x_2)) =$

$\text{Distance}_{\text{X}}((x_1), (x_2))$.*[10]*

In this paper, we will generally consider those isometries mapping from metric space X to itself. We will focus on the hyperbolic plane which has a metric we will not discuss in depth. Note that this metric depends on the model of the hyperbolic plane being discussed (Poincare disk uses the hyperbolic metric and the upper-half plane uses the Poincare metric). Additionally, we will use the euclidean plane as a familiar example where the euclidean distance is our metric.

From the Euclidean plane onto itself, isometries take on the form of either translations, orthogonal transformations (which include rotations and reflections across lines passing through the origin), or a combination of translations and orthogonal transformations. [1] All these transformations maintain distance between all points in the plane. An example of a transformation not considered an isometry from the euclidean plane onto itself is dilation as this directly changes the distance between points on the plane.

Definition 2.2 (Isometry Group)**.** *An isometry group of a metric space X is the set of all isometries from that metric space X onto itself. [1]*

The set of all isomotries on the euclidean plane make up an isometry group which we refer to as $E(2)$. Within this group, we have the subgroup of all translations referred to as T and the subgroup of all orthogonal transformations which we refer to as O. It can be shown that $E(2) = TO$.[1]

Remark 2.3. *Now, we will be using the concept of the isometry group to move from the geometric into the algebraic with respect to our focus on the hyperbolic plane.*

Similar to the Euclidean Plane, we can think about the isometry group of the hyperbolic plane. This isometry group depends on the model of the hyperbolic plane being used. For instance if we use the upper-half plane as the model of the hyperbolic group, then the group of all distance preserving transformations on the hyperbolic plane will be $\mathrm{SL}(2, \mathbb{R})$. If we use the Poincare disk model of the hyperbolic plane the the group of all distance preserving transformations on the hyperbolic plane will be $\mathrm{SU}(1, 1)$.[6]

2.3 NEC-Groups

In order to define Non-Euclidean Crystallographic groups (NEC Groups) we must first define discrete subgroups, open covers, compactness, and Cocompact group action.

Definition 2.4 (Discrete Subgroups). *A subgroup Γ of the group of isometries on a topological space X is considered discrete if for any $x \in X$, the orbit $\Gamma x = \{\gamma x | \gamma \in \Gamma\}$ is a discrete subset of X and the* $\text{Stabilizer}_{\Gamma}(x)$ *is finite. [3] [11]*

An example of a familiar topological group is $\mathbb{R}$ which uses the metric topology as it is a metric space with a distance function. A discrete subgroup of $\mathbb{R}$ would be $\mathbb{Z}$.

Definition 2.5 (Open Cover). *Let X be a topological space, and let $C = \{O_\alpha | \alpha \in A\}$ where each O_α is an open subset of X and A is an indexing set be a collection of open sets of X. C is an open cover of X if $X \subseteq \bigcup_{\alpha \in A} O_\alpha$. [9]*

An example of an open cover of the real numbers is $C = \{(-n, n) | n \in \mathbb{N}\}$ where each $(-n, n)$ is an open interval of real numbers and thus an open set.

Definition 2.6 (Compactness). *A Topological space X is considered compact if for any Open cover C, there exists a finite collection of open sets from C that create another open Cover. [9]*

Definition 2.7 (Cocompact Group Action). *If a group G acts on a topological space X (the hyperbolic plane in our case) then if the quotient space X/G is a compact space, we call the group action of G Cocompact.[5]*

Remark 2.8. *By quotient space, we mean the set of all orbits of topological space X under the group action $(\cdot)$ of G on X. An orbit of X containing some $x \in X$ is defined as $G \cdot x = \{g \cdot x | g \in G\}$.*

Consider $\mathbb{R}$ as a topological space and let the group $\mathbb{Z}$ act on $\mathbb{R}$ through addition. Then $\mathbb{R}/\mathbb{Z}$ is compact and this means then that the action of $\mathbb{Z}$ is Cocompact on $\mathbb{R}$.

Definition 2.9 (NEC Group). *We define an NEC group G as a discrete cocompact subgroup of the isometry group on of the hyperbolic plane.[2]*

Each NEC group Γ is assigned a specific signature which determines its algebraic structure.[4][2] These signatures come in the following form:

$$\text{Signature}(\Gamma) = (g; \pm, [m_1...m_r]; \{(n_{11}, ..., n_{1S_1}), ..., (n_{k1}, ..., n_{kS_k})\})$$

This is the list of components in the order they appear from left to right:

1. The first component g is referred to as the genus of the NEC group.
2. The second component is referred to as the sign of the signature and can be either $+$ or $-$.
3. In the third component, the numbers $m_1...m_r$ are referred to as the proper periods of the signature. If $r = 0$ then the fourth component of the signature will simply be $[-]$ meaning an empty set of proper periods.
4. In the fourth component, each $(n_{i1}, ..., n_{1S_i})$ for $1 \leq i \leq k$ is referred to as a period cycle and the numbers $n_{i1}, ..., n_{1S_i}$ are referred to as the periods of that period cycle. If for any period cycle $S_i = 0$ then that period cycle will be considered empty and denoted $(-)$. If $k = 0$ then we consider the entire fourth component an empty set and we denote it as $\{-\}$.

An NEC group is considered a surface group if its signature is of the form

$$\text{Signature}(\Gamma) = (g; \pm, [-]; \{\underbrace{(-), ..., (-)}_{\text{k}}\})$$

If $k > 0$ then we say that Γ is a Bordered surface NEC group.[4]

2.4 Defining Bordered Klein Surfaces

When considering what a Compact Bordered Klein surface is, we can say that for particular Bordered Klein surfaces, they can be represented as the orbit space H/Γ where H

is the hyperbolic plane and Γ is some Bordered surface NEC group.[2] Not all Klein surfaces can be represented in this way.

An example of a Klein surface that isn't represented in this way is the set $\mathbb{H} = \{x+iy \in \mathbb{C} | y > 0\}$ this set is the upper half plane and we considered it earlier in this paper. This Klein surface isn't compact nor is it bordered which is a reason why it cant be represented as the orbit space.

Another example of a Klein surface that isn't represented in the way that we described but is still familiar is the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} | 1 \geq |z|\}$. This surface is compact and bordered but it's not represented by the orbit space of the hyperbolic plane induced by some Bordered surface NEC group.[4]

Baginski and Gromadzki's paper focuses on compact Bordered Klein surfaces that can be defined in this orbit space way. We will not provide a more general definition of Klein surfaces because while one is provided in Baginski and Gromadzki's paper, the inner workings of their paper don't rely on the general definition when considering klein surfaces.

3 Automorphisms on Bordered Klein Surfaces

In this section we talk about what exactly is being represented symmetrically in this paper by Baginski and Gromadzki.

3.1 What are Automorphisms?

In general, we understand automorphisms to be isomorphisms from a mathematical object X to itself. An elementary example may be that provided a set $X = \{1, 2, 3\}$, a map $f : X \to X$ where $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$ is considered an automorphism on the set X as it an isomorphism. X can be any mathematical object from Groups and Rings to topological spaces.

3.2 What About When On Klein Surfaces?

Klein Surfaces are a type of manifold, and manifolds are topological spaces. This means that when we think of automorphisms on Klein surfaces we can conceptualize them as we would automorphisms of some topological space. Take for example the Klein Surface $\mathbb{H}$. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an invertible matrix then it cannot be possible for both c and d to be 0 and we define the following function f on the upper half plane

$$f_{_M}(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } \mathrm{Det}(M) > 0 \\ \frac{a\overline{z}+b}{c\overline{z}+d}, & \text{if } \mathrm{Det}(M) > 0. \end{cases} \tag{1}$$

This function can be shown to be an automorphism by showing is maps surjectively to $\mathbb{H}$ and it is just one familiar example of how we might conceive of these types of maps.[4] Additionally, there are a variety of automorphisms that can act on a particular Klein surface not to mention other categorical objects like groups and sets. In the case of the Klein surface of $\mathbb{H}$, the entire group of matrices $\mathrm{PGL}(2, \mathbb{R})$ can act on the upper half plane in the way that we described above with matrix M.[4] Collections of automorphisms can indeed form entire groups through composition of functions (if satisfying closure), and from this we find the groups of automorphisms that we will be representing symmetrically.

3.3 Boundary Components of a Bordered Klein Surface

The focus of Baginski and Gromadzki's paper is that we can find symmetric representations of groups of automorphism acting on a Klein surface that can be put into the H/Γ form where Γ is a NEC-group of a particular signature. The topic we still have not covered is the fact that symmetric groups represent the permutations of some set of discrete objects and we are not aware of what those objects are in the context of automorphisms on Klein surfaces to justify symmetric groups' relevance. In the abstract of Baginski and Gromadzki's paper, it is provided that automorphisms on Klein surfaces permute the boundary components of a compact bounded Klein surface. This section will be devoted to examining what boundary components are and how they come about.

Definition 3.1 (Boundary of surfaces). *Generally, a surface X is a Hausdorff connected topological space with an atlas $A = \{(U_i, \phi_i)|i \in I\}$. Each map $\phi_i : U_i \to A_i$ is a homeomorphism onto an open subset A_i of either $\mathbb{C}$ or $\mathbb{C}^+ = \{z \in \mathbb{C}|\,\mathrm{imaginary}(z) \geq 0\}$. Define the boundary of surface X as $\mathrm{B}(X) = \{x \in X|\exists\ i \in I$ with $x \in U_i, \phi_i(x) \in \mathbb{R}$ and $\phi_i(U_i) \subseteq \mathbb{C}^+\}$. [4]*

Definition 3.2 (Boundary Components). *Considering the definition of connected components provided by our class textbook, the boundary components of a topological space are simply the connected components of a topological space's boundary.*

Recall from earlier that we said a surface NEC group Γ's signature had k empty period cycles. Provided k is larger than 0 meaning Γ is bordered, this number k determines the number of boundary components that the Klein surface H/Γ has. An automorphism on such a Compact Bordered Klein surface permutes these k boundary components, and again this is what we examine when we begin to consider the use of symmetric representations of groups of automorphisms.[2]

4 Symmetric representations of Groups of automorphisms on Bordered Klein Surfaces

We now have some kind of intuition of how symmetric representations of groups of automorphisms may come into play from the fact that Baginski and Gromadzki tell us at the beginning of their paper that the automorphisms on Bordered Klein surfaces permute the boundary components of a Bordered Klein surface. The paper actually provides a homomorphism that is this symmetric representation which can ultimately be viewed as a permutation representation.

Let G be a group of automorphisms of a Compact Bordered Klein surface X, the boundary components of X consist of $O_1, O_2, ..., O_k$. Let $g \in G$ be an automorphism and define $\rho_X : G \to S_k$ as

$$\rho_X(g)(i) = j \text{ when } g(O_i) = O_j$$

and addionally, Baginski and Gromadzki provide us with the following lemma:

Lemma 4.1. *The mapping ρ_X is a homomorphism whose kernel is either cyclic or dihedral.*

Thus we have defined the representation that the entire Baginski and Gromadzk paper centers around. Much of the remainder of the paper discusses aspects of this representation such as degree of representation, faithfulness, and representation decomposition.

5 Conclusion

To close, I would like to mention a few more aspects of this paper by Baginski and Gromadzki that I didn't discuss in detail and also some aspects of my research that were not directly related to their paper. To begin, I wrote about the hyperbolic plane as a non-euclidean space that violates the fifth Euclidean postulate. The hyperbolic plane isn't the only non-euclidean space as there is also the sphere also referred to as an elliptic geometry that can be considered non-euclidean based on the fifth axiom.

After talking about the hyperbolic plane, I spoke about models of the hyperbolic plane. Another commonly used model in addition to $\mathbb{H}$ and the Poincare disk is the Klein disk which can be thought of as similar to the Poincare disk except all arcs between points on the circle are straightened to be chords on of the circle. I also didn't mention that between the upper half plane and the Poincare disk there is an isometry which makes the pair of models special. This is hinted at by the fact that in both models the angles between the arcs must always be right angle (by definition in the disk and by the fact that only semicircles and vertical lines are used in the half plane).

For isometry groups, many times sources referred to only a specific type of isometric transformation when referring to the isometry group of a manifold. This specific type of isometric transformation is referred to as orientation preserving; additionally there are orientation reversing isometries, and in the context of the paper by Baginski and Gromadzki, they refer to the isometry group of the hyperbolic plane as the collect of all isometric transformations (both orientation preserving and reversing).

I think it is also important that I mention that the theory behind the signatures of NEC groups is much more complex than how I presented in this paper. While I did present the aspects of what makes up a signature of an NEC group, the origin of these aspects and how they impact the nature of the Klein surface was not presented. Particularly, the first to components tell us a lot about the configuration of the Klein surface. More information about NEC-group signatures can be seen here. [4]

Lastly, another thing I didn't mention completely was that these Bordered Klein surfaces that can be put into orbit space form are in fact compact because NEC group are cocompact.

References

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